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Problem Set 1

ECON 2140: ECONOMETRIC METHODS

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Problem 1

$$\theta_p = \frac{\mathbb{E}[X|X < \xi_p]}{\mathbb{E}[X]} \iff \theta_p \, \mathbb{E}[X] - \mathbb{E}[X|X < \xi_p] = 0$$

(a)

From theorem in class, we know that

$$\sqrt{n}(\hat{\theta}_p - \theta_p) \to^d N(0, B^{-1}\Omega_0 B^{-1}) \tag{1}$$

We will now derive elements B and Ω_0 of the asymptotic variance.

Let

$$Q_n(\hat{\theta}_p) = \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{\theta}_p \cdot X_i - \frac{X_i \mathbb{1}\{X_i < \xi_p\}}{p}\right]\right)^2 \tag{2}$$

Then,

$$\frac{\partial}{\partial \hat{\theta}_p} Q_n(\hat{\theta}_p) = \frac{1}{n}\sum_{i=1}^{n} X_i \left(\frac{1}{n}\sum_{i=1}^{n}\left[\hat{\theta}_p \cdot X_i - \frac{X_i \mathbb{1}\{X_i < \xi_p\}}{p}\right]\right) = 0 \tag{3}$$

Note that the solution to the above first order condition gives us the desired estimator for θ_p:

$$\hat{\theta}_p = \frac{\frac{1}{np}\sum_i X_i \mathbb{1}\{X_i < \xi_p\}}{\frac{1}{n}\sum_i X_i}$$

Now,

$$\frac{\partial^2}{\partial \hat{\theta}_p \partial \hat{\theta}_p} Q_n(\hat{\theta}_p) = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 \to^p (\mathbb{E}[X])^2 \equiv B$$

Moreover,

$$\sqrt{n}\frac{\partial}{\partial \theta_p} Q_n(\theta_p) = \frac{1}{n}\sum_{i=1}^{n} X_i \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\theta_p \cdot X_i - \frac{X_i \mathbb{1}\{X_i < \xi_p\}}{p}\right]\right) \to^d N(0, \Omega_0)$$

where

$$
\begin{aligned}
\Omega_0 &= \operatorname{Var}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\theta_p \cdot X_i - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i \cdot \mathbb{1}\{X_i < \xi_p\}}{p}\right] \\
&= \frac{\theta_p^2}{n}\cdot n \operatorname{Var}(X) + \frac{1}{np^2}\cdot n \operatorname{Var}(X \cdot \mathbb{1}\{X < \xi_p\}) - 2\frac{\theta_p}{np}\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i \cdot \mathbb{1}\{X_i < \xi_p\}\right) \\
&= \theta_p^2 \operatorname{Var}(X) + \frac{1}{p^2}\operatorname{Var}(X \cdot \mathbb{1}\{X < \xi_p\}) - 2\frac{\theta_p}{p}\operatorname{Cov}(X, X \cdot \mathbb{1}(X < \xi_p))
\end{aligned}
$$

Then, Eq. (1) becomes

$$
\sqrt{n}(\hat{\theta}_p - \theta_p) \to^d N\left(0, \frac{\theta_p^2 \operatorname{Var}(X) + \frac{1}{p^2}\operatorname{Var}(X \cdot \mathbb{1}\{X < \xi_p\}) - 2\frac{\theta_p}{p}\operatorname{Cov}(X, X \cdot \mathbb{1}(X < \xi_p))}{(\mathbb{E}[X])^4}\right)
$$

(b)

$$
\hat{\theta}_{N_p} = \frac{\frac{1}{np}\sum_i X_i \mathbb{1}\{X_i < \hat{\xi}_p\}}{\frac{1}{n}\sum_i X_i}
$$

Problem 2

Problem 3

Problem 4