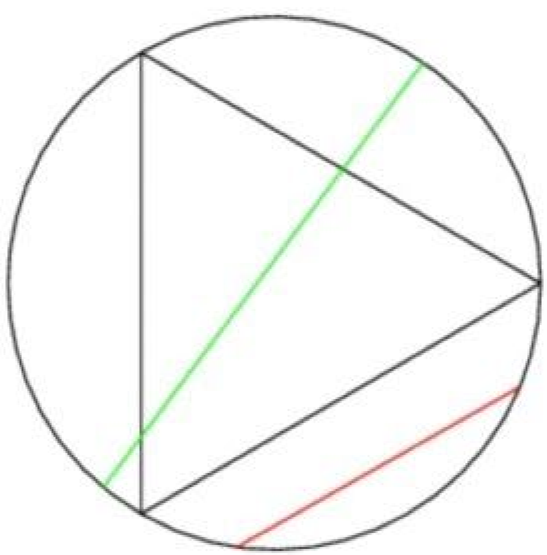
I am well-versed in Probability and statistics. I have a habit of working on probability puzzles. I recently came across Bertrand's Paradox and tried to solve Bertrand's Paradox, and I have framed my work below in this document. It shows my keen interest in the areas of Probability and Statistics.

Tried to Solve Bertrand's Paradox

A couple of days ago, I came across the famous **Bertrand's Paradox**. It is a problem within the *classical interpretation of probability theory.* Joseph Bertrand introduced this paradox in his work ***Calcul des probabilités*** *in*1889.

Bertrand's formulation of the problem

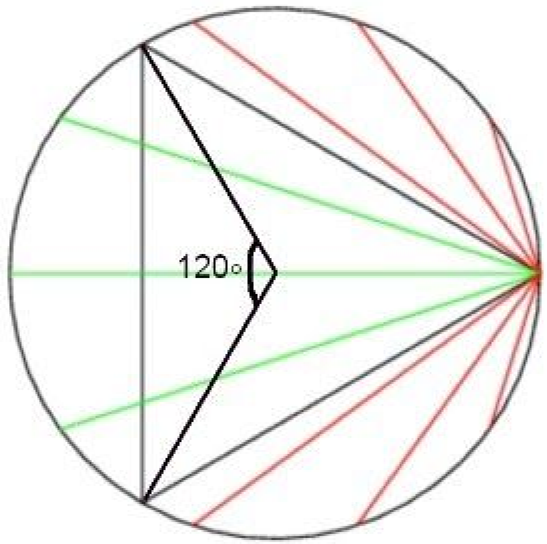
The paradox goes as follows: Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than the length of the side of the triangle?



Bertrand solved this problem with three different methods and got different probabilities each time:

Method 1

We set a point to be stationary and randomly select the other point. Clearly, when the other point is contained in the far 120◦ arc, the chord's length is longer than the triangle's side length (shown in the picture as green), and elsewhere, it is shorter (shown as red). Thus, the probability is 120◦/360◦ = 1/3.



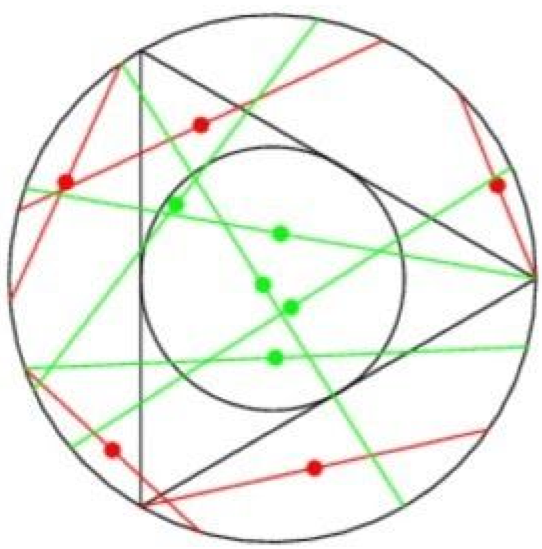
Method 2

We pick a random point inside the circle and draw a chord through it such that the selected point is the midpoint of the chord. Note that whenever the point lies inside the smaller circle, then the chord has a length longer than the triangle's side; otherwise, shorter.

Let the radii of smaller and bigger circles be r and R.

Recall that the length of perpendicular to the triangle's side from the center of the circle is R/2. Therefore r=R/2.

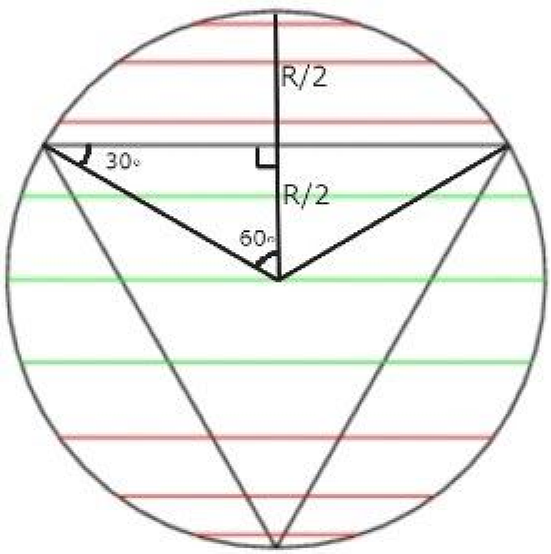
Thus, the two circle's areas' ratio is $(\pi r^2/\pi R^2) = (r^2/R^2) = 1/4$.



Method 3

We randomly choose a point and then draw a horizontal line through it to form a chord in the circle. The probability of the chord being longer than the side length of the triangle is a little harder to figure out but still can be done quickly with the help of some elementary geometry:

The equilateral triangle's side divides the radius of the circle into halves, as shown in the diagram below. Thus, the probability that a random chord is longer than the length of the side of the equilateral triangle is 1/2.



My Solution To Bertrand's Paradox

Is Bertrand's paradox a paradox or just a big blunder! Does this problem actually have an exact solution, or is it truly a paradox?

Here, I am going to figure out the errors in method 2 and method 3.

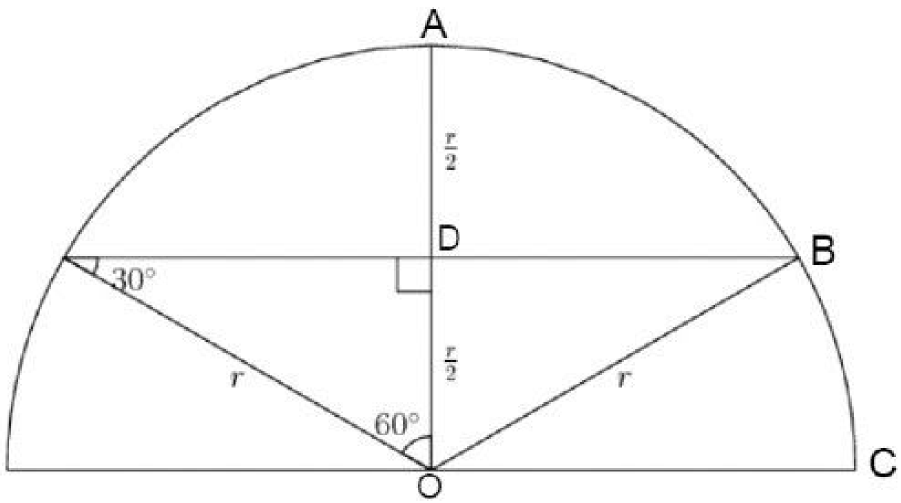
First, look at method 2 by Bertrand:

In method 2, Bertrand calculated the probability by considering areas. And he showed that the likelihood of selecting a chord randomly which is longer than the side of the triangle is 1/4.

But, the center of the circle (both the circles have the same center) is the midpoint of infinitely many chords (i.e., diameters), and except this point, all other points are the midpoints of only one chord. Therefore, this one point inside the smaller circle produces infinitely many chords that are longer than the inscribed equilateral triangle's side. But while calculating area, this point (center) was considered like all other points, i.e., the midpoint of only one chord. Thus, the probability will be greater than 1/4, but we can't calculate the exact probability using this method.

Look at method 3:

In method 3, Bertrand showed that if the chord's distance from the center of the circle is less than R/2 (R be the radius of the circle), it will be longer than the triangle's side. And therefore using this intuition, he calculated that probability is 1/2.



But, the length of arc AB is twice the length of arc BC. Therefore, the number of total chords drawn through the arc AB and parallel to OC is twice the total number of chords drawn through the arc BC and parallel to OC. And the chords whose distance from the center O is less than r/2 are longer than the triangle's side. Thus, if the number of chords drawn through arc BC is x, then the number of the chords drawn through the arc AB will be 2x.

Therefore, the probability of drawing a chord randomly that is longer than the triangle's side is: (no. of chords through BC)/(total no. of chords through AC) = x/3x = 1/3.

And **method 1** seems to be correct, you can also try to solve method 1 by using arc lengths, and it will also give the same probability.

Conclusion: So, according to my solution, the correct probability of selecting a chord randomly inside a circle whose length is greater than the side of the inscribed equilateral triangle should be 1/3.

Reference for Bertrand's Paradox:

https://en.wikipedia.org/wiki/Bertrand_paradox_(probability)