UNIT - III

3 Reduction Formulae

I) Important formulae

1) $\int_0^{\pi/2} \sin^m x \cos x \, dx = \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}$

2) $\int_0^{\pi/2} \cos^m x \sin x \, dx = \left[-\frac{\cos^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}$

3) $\int_0^{2\pi} \sin^n x \, dx = \begin{cases} 4\int_0^{\pi/2} \sin^n x \, dx & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

4) $\int_0^{2\pi} \cos^n (x) \, dx = \begin{cases} 4\int_0^{\pi/2} \cos^n x \, dx & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

5) $\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\pi/2} \sin^n x \, dx; \ \forall\ n$

6) $\int_0^{\pi} \cos^n x \, dx = \begin{cases} 2\int_0^{\pi/2} \cos^n x \, dx & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

7) $\int_0^{\pi} \sin^m x \cos^n x \, dx = \begin{cases} 2\int_0^{\pi/2} \sin^m x \cos^n x \, dx & \text{if } n \text{ is even and } \forall\ m \\ 0 & \text{if } n \text{ is odd} \end{cases}$

8) $\int_0^{2\pi} \sin^m x \cos^n x \, dx = \begin{cases} 4\int_0^{\pi/2} \sin^m x \cos^n x \, dx & \text{if } m, n \text{ are even} \\ 0 & \text{otherwise} \end{cases}$

II) Reduction Formulae for Sinusoidal Functions

To find reduction formula for $\int \sin^n x \, dx$, where n is a positive integer ≥ 2 and hence evaluate $\int_0^{\pi/2} \sin^n x \, dx$.

Let $\quad I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$

Using rule of integration by parts $\int uv = u\int v - \int \left[\frac{du}{dx}\int v\, dx\right] dx$

Here $u = \sin^{n-1} x$ and $v = \sin x$

We get

$$\begin{aligned} I_n &= \sin^{n-1} x (-\cos x) \\ &\quad -\int (n-1) \sin^{n-2} x \cos x (-\cos x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} (x)(1-\sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx \\ I_n &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1) I_n \end{aligned}$$

$\therefore \quad I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

$\therefore \quad [1+n-1] I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \qquad \dots (1)$$

which is the reduction formula of $\int \sin^n x \, dx$

Now let $\quad U_n = \int_0^{\pi/2} \sin^n x \, dx$

$\therefore$ Using formula (1) we get

$$U_n = \left[-\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$$

$$U_n = 0+\frac{n-1}{n}U_{n-2}$$

$$U_n = \frac{n-1}{n}U_{n-2} \qquad \cdots (2)$$

Applying formula (2) successively we get

$$U_{n-2} = \frac{n-3}{n-2}U_{n-4},\ U_{n-4} = \frac{n-5}{n-4}U_{n-6}$$

$$U_{n-6} = \frac{n-7}{n-6}U_{n-8} \text{ and so on.}$$

If n is even then the last term will be

$$U_2 = \frac{1}{2}U_0 = \frac{1}{2}\int_0^{\pi/2}\sin^0 x\,dx = \frac{1}{2}\left(\frac{\pi}{2}\right)$$

If n is odd, then the last term will be

$$U_3 = \frac{1}{2}U_1 = \frac{1}{2}\int_0^{\pi/2}\sin^1 x\,dx$$

$$= \frac{1}{2}[-\cos x]_0^{\pi/2} = \frac{1}{2}(1)$$

Now, combining all above results, we get

$$\int_0^{\pi/2}\sin^n x\,dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

if n is even

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)(1)$$

if n is odd

Note : We know that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore \quad \int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\sin^n\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi/2}\cos^n x\,dx$$

Thus

$$\int_0^{\pi/2}\cos^n x\,dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

if n is even

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)(1)$$

if n is odd

Thus $$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)$$

$$\left(\frac{\text{subtract by 2}}{\text{subtract by 2}}\right)\cdots\left(\frac{2 \text{ or } 1}{1 \text{ or } 2}\right)\left(\frac{\pi}{2} \text{ if n is even}\right)$$

Q.1 : Find reduction formula for $\int_0^{\pi/3}\cos^n x\,dx$ ***and evaluate*** $\int_0^{\pi/3}\cos^6 x\,dx.$

[SPPU : Dec.-13, Marks 4]

Ans. : Let $I_n = \int\cos^n x\,dx = \int\cos^{n-1}x\cos x\,dx$

By applying integration by parts, Here $u = \cos^{n-1}x$

$v = \cos x$

We get

$$I_n = \cos^{n-1}x(\sin x) - \int(n-1)\cos^{n-2}x(-\sin x)(\sin x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\sin^2 x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x(1-\cos^2 x)\,dx$$

$$I_n = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x - (n-1)\int \cos^n x\, dx$$

$$I_n = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x - (n-1)\, I_n$$

$$I_n + (n-1)\, I_n = \cos^{n-1}x \sin x + (n-1)\, I_{n-2}$$

$$\therefore \quad I_n = \frac{\cos^{n-1}x \sin x}{n} + \frac{(n-1)}{n}\, I_{n-2} \qquad \cdots (1)$$

Now let $U_n = \int_0^{\pi/3} \cos^n x\, dx$

$\therefore$ Using (1) we get

$$U_n = \left[\frac{\cos^{n-1}x \sin x}{n}\right]_0^{\pi/3} + \frac{n-1}{n}\, U_{n-2}$$

$$= \frac{1}{n}\left[\left(\frac{1}{2}\right)^{n-1}\left(\frac{\sqrt{3}}{2}\right)\right] + \frac{n-1}{n}\, U_{n-2}$$

$$U_n = \frac{\sqrt{3}}{n\, 2^n} + \frac{n-1}{n}\, U_{n-2}$$

which is the required reduction formula.

Substituting n = 6, we get

$$U_6 = \frac{\sqrt{3}}{6\,(2^6)} + \frac{5}{6}\, U_4, \quad U_4 = \frac{\sqrt{3}}{4\,(2^4)} + \frac{3}{4}\, U_2 \quad \text{and}$$

$$U_2 = \frac{\sqrt{3}}{2\,(2^2)} + \frac{1}{2}\, U_0 = \frac{\sqrt{3}}{2\,(4)} + \frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$\therefore \quad U_4 = \frac{\sqrt{3}}{64} + \frac{3}{4}\left[\frac{\sqrt{3}}{8} + \frac{\pi}{6}\right] = \frac{\sqrt{3}}{64} + \frac{3\sqrt{3}}{32} + \frac{\pi}{8}$$

$$\therefore \quad U_6 = \frac{\sqrt{3}}{(64)^6} + \frac{5}{6}\left[\frac{\sqrt{3}}{64} + \frac{3\sqrt{3}}{32} + \frac{\pi}{8}\right] = \frac{3\sqrt{3}}{8} + \frac{5\pi}{48}$$

Q.2 : If $I_n = \int_0^{\pi/4} \sin^{2n} x\, dx$ prove that $I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} - \frac{1}{n2^{n+1}}$ and hence find I_3. **[SPPU : Dec.-04, 06, May-07, Marks 4]**

Ans. : We have,

$$I_n = \int_0^{\pi/4} \sin^{2n} x\, dx$$

$$I_n = \int_0^{\pi/4} \sin^{2n-1} x \sin x\, dx$$

$$= [\sin^{2n-1} x(-\cos x)]_0^{\pi/4} - \int_0^{\pi/4} (2n-1)\sin^{2n-2} x \cos x(-\cos x)\, dx$$

$$= -\left(\frac{1}{\sqrt{2}}\right)^{2n} + (2n-1)\int_0^{\pi/4} \sin^{2n-2}(1-\cos^2 x)\, dx$$

$$= -\left(\frac{1}{\sqrt{2}}\right)^{2n} + (2n-1)\int_0^{\pi/4} \sin^{2(n-1)}\, dx - (2n-1)\int_0^{\pi/4} \sin^{2n} x\, dx$$

$$I_n = -\frac{1}{2^n} + (2n-1)\, I_{n-1} - (2n-1)\, I_n$$

$$(1+2n-1)\, I_n = -\frac{1}{2^n} + (2n-1)\, I_{n-1}$$

$$\boxed{I_n = -\frac{1}{n2^{n+1}} + \left(1 - \frac{1}{2n}\right) I_{n-1}}$$

Now, $$I_3 = -\frac{1}{3 \cdot 2^4} + \left(1 - \frac{1}{6}\right) I_2 = -\frac{1}{48} + \frac{5}{6}\left[-\frac{1}{2 \cdot 2^3} + \left(1 - \frac{1}{4}\right) I_1\right]$$

$$= -\frac{1}{48} + \frac{5}{6}\left(-\frac{1}{16}\right) + \frac{15}{24} I_1 = -\frac{7}{96} + \frac{15}{24}\left[-\frac{1}{4} + \left(1 - \frac{1}{2}\right) I_0\right]$$

$$I_3 = -\frac{7}{96} - \frac{15}{96} + \frac{1}{2}\left(\frac{\pi}{4}\right) = -\frac{11}{48} + \frac{\pi}{8}$$

Q.3 : If $I_n = \int_0^{\pi/4} \cos^{2n} x\, dx$ prove that $I_n = \frac{1}{n2^{n+1}} + \frac{2n-1}{2n} I_{n-1}$.

[SPPU : Dec.-14, 16, Marks 4]

Ans. : We have,

$$I_n = \int_0^{\pi/4} \cos^{2n-1} x \cos x\, dx$$

$$I_n = [\cos^{2n-1} x \sin x]_0^{\pi/4} - \int_0^{\pi/4} (2n-1)\cos^{2n-2} x(-\sin x)(\sin x)\, dx$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{2n-1}\left(\frac{1}{\sqrt{2}}\right) + (2n-1)\int_0^{\pi/4} \cos^{2(n-1)} x(1-\cos^2 x)\, dx$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{2n} + (2n-1)\int_0^{\pi/4} \cos^{2(n-1)} x\, dx - (2n-1)\int_0^{\pi/4} \cos^{2n} x\, dx$$

$$I_n = \frac{1}{2^n} + (2n-1)I_{n-1} - (2n-1)I_n$$

$$[1+2n-1]\, I_n = \frac{1}{2^n} + (2n-1)I_{n-1}$$

$$\boxed{I_n = \frac{1}{n2^{n+1}} + \left(\frac{2n-1}{2n}\right) I_{n-1}}$$

Q.4 : Find reduction formula for $\int \tan^n x\, dx$ and hence evaluate $\int_0^{\pi/4} \tan^n x\, dx$ and $\int_0^{\pi/4} \tan^5 x\, dx,\ n \geq 2$.

[SPPU : May-16, 18, Marks 4]

Ans. :

$\Rightarrow$ Let

$$I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \quad \text{... (1)}$$

which is the required reduction formula.

Now consider

$$U_n = \int_0^{\pi/4} \tan^n x\, dx$$

$\therefore$ Using limits equation (1) becomes

$$U_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - U_{n-2}$$

$$U_n = \frac{1}{n-1} - U_{n-2} \quad \text{... (2)}$$

replacing n by n + 1, we get

$$\boxed{U_{n+1} + U_{n-1} = \frac{1}{n}}$$

$\therefore$
$$U_5 = \int_0^{\pi/4} \tan^5 x\, dx = \frac{1}{4-1} - U_3 = \frac{1}{3} - U_3$$

$$U_3 = \frac{1}{2} - U_1 = \frac{1}{2} - \int_0^{\pi/4} \tan x\, dx$$

$$= \frac{1}{2} - \int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx = \frac{1}{2} + [\log \cos x]_0^{\pi/4}$$

$$= \frac{1}{2} + \log\cos\frac{\pi}{4} - \log\cos 0$$

$$= \frac{1}{2} + \log\frac{1}{\sqrt{2}} - \log 1 = \frac{1}{2} + \log 1 - \log\sqrt{2} - 0$$

$$= \frac{1}{2} - \log\sqrt{2}$$

$\therefore$
$$U_5 = \frac{1}{3} - U_3 = \frac{1}{3} - \frac{1}{2} + \log\sqrt{2} = -\frac{1}{6} + \log\sqrt{2}$$

Q.5 : Find reduction formula for $\int \cot^n x\,dx$ and hence evaluate $\int_{\pi/4}^{\pi/2} \cot^n x\,dx$. Find U_6. **[SPPU : May-15, 17, Marks 4]**

Ans. :

Let
$$I_n = \int \cot^n x\,dx$$
$$I_n = \int \cot^{n-2}x \cot^2 x\,dx = \int \cot^{n-2}x(\operatorname{cosec}^2 x - 1)\,dx$$
$$= \int \cot^{n-2}x \operatorname{cosec}^2 x\,dx - \int \cot^{n-2}x\,dx$$
$$\boxed{I_n = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}} \qquad \ldots(1)$$

This is the general reduction formula for $\cot^n x$.

Now, Let,
$$U_n = \int_{\pi/4}^{\pi/2} \cot^n x\,dx$$

$\therefore$ Using (1), we get,
$$U_n = -\left[\frac{\cot^{n-1}x}{n-1}\right]_{\pi/4}^{\pi/2} - U_{n-2}$$
$$U_n = -\frac{1}{n-1}[0-1] - U_{n-2}$$
$$\boxed{U_n = \frac{1}{n-1} - U_{n-2}}$$

Now,
$$U_6 = \int_{\pi/4}^{\pi/2} \cot^6 x\,dx = \frac{1}{5} - U_4 = \frac{1}{5} - \left[\frac{1}{3} - U_2\right]$$
$$U_6 = \frac{1}{5} - \frac{1}{3} + \left[\frac{1}{1} - U_0\right] = -\frac{2}{15} + 1 - \left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{13}{15} - \frac{\pi}{4}$$

Q.6 : Find reduction formula for $\int \sec^n x\,dx$ and hence for $\int_0^{\pi/4} \sec^n x\,dx$ and evaluate $\int_0^{\pi/4} \sec^6 x\,dx$. **[SPPU : May-16, Dec.-18, Marks 4]**

Ans. :

$\Rightarrow$ Let
$$I_n = \int \sec^n x\,dx = \int \sec^{n-2}x \sec^2 x\,dx$$

Using integration by parts, here $u = \sec^{n-2}x$ and $v = \sec^2 x$

$\therefore$ We get
$$I_n = \sec^{n-2}x \tan x - \int (n-2)\sec^{n-3}x\,(\sec x \tan x)\tan x\,dx$$
$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \tan^2 x\,dx$$
$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x(\sec^2 x - 1)\,dx$$
$$= \sec^{n-2}x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2}x\,dx$$
$$I_n = \sec^{n-2}x \tan x - (n-2)\,I_n + (n-2)\,I_{n-2}$$
$$\therefore \quad I_n + (n-2)\,I_n = \sec^{n-2}x \tan x + (n-2)\,I_{n-2}$$
$$I_n = \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \qquad \ldots(1)$$

which is the required reduction formula.

Let
$$U_n = \int_0^{\pi/4} \sec^n x\,dx$$

$\therefore$ Using limits in equation (1) we get
$$U_n = \left[\frac{\sec^{n-2}x \tan x}{n-1}\right]_0^{\pi/4} + \frac{n-2}{n-1} U_{n-2}$$
$$\therefore \quad \boxed{U_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} U_{n-2}} \qquad \ldots(2)$$

Putting n = 6, 4, 2 in equation (2) we get
$$U_6 = \frac{(\sqrt{2})^4}{5} + \frac{4}{5} U_4, \quad U_4 = \frac{(\sqrt{2})^2}{3} + \frac{2}{3} U_2 \text{ and}$$
$$U_2 = \frac{(\sqrt{2})^0}{2-1} + \frac{0}{1} U_0 = \frac{1}{1} = 1$$

$$\therefore \quad U_6 = \frac{2^2}{5}+\frac{4}{5}\left[\frac{2}{3}+\frac{2}{3}(1)\right] = \frac{4}{5}+\frac{4}{5}\left[\frac{4}{3}\right] = \frac{28}{15}$$

Q.7 : Find the reduction formula for $\int \operatorname{cosec}^n x\, dx$ and hence for $\int_{\pi/4}^{\pi/2} \operatorname{cosec}^n x\, dx$.

Ans. : Let $I_n = \int \operatorname{cosec}^n x\, dx = \int \underset{u}{\underset{\downarrow}{\operatorname{cosec}^{n-2} x}}\, \underset{v}{\underset{\downarrow}{\operatorname{cosec}^2 x}}\, dx$

Integration by parts,

$$\begin{aligned} I_n &= \operatorname{cosec}^{n-2} x(-\cot x) \\ &\quad -\int (n-2)\operatorname{cosec}^{n-3} x(-\operatorname{cosec} x \cot x)(-\cot x)\, dx \\ &= -\operatorname{cosec}^{n-2} x \cot x-(n-2)\int \operatorname{cosec}^{n-2} x \cot x^2\, dx \\ &= -\operatorname{cosec}^{n-2} x \cot x-(n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x-1)\, dx \\ &= \operatorname{cosec}^{n-2} x \cot x-(n-2)\int \operatorname{cosec}^{n} x\, dx \\ &\quad +(n-2)\int \operatorname{cosec}^{n-2} x\, dx \\ I_n &= -\operatorname{cosec}^{n-2} x \cot x-(n-2)I_n+(n-2)I_{n-2} \\ [1+(n-2)]I_n &= -\operatorname{cosec}^{n-2} x \cot x+(n-2)I_{n-2} \end{aligned}$$

$$\boxed{I_n = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1}+\left(\frac{n-2}{n-1}\right)I_{n-2}} \quad \ldots(1)$$

This is the required reduction formulae,

$\therefore$ Using limits $\frac{\pi}{4}$ and $\frac{\pi}{2}$ we have, $U_n = \int_{\pi/4}^{\pi/2} \operatorname{cosec}^n x\, dx$

$\therefore$ Equation (1) gives,

$$\begin{aligned} U_n &= \left[-\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}\right]_{\pi/4}^{\pi/2}+\frac{n-2}{n-1}U_{n-2} \\ &= \left[0+\frac{1}{n-1}(\sqrt{2})^{n-2}\right]+\frac{n-2}{n-1}U_{n-2} \end{aligned}$$

$$\boxed{U_n = \frac{1}{n-1}(\sqrt{2})^{n-2}+\frac{n-2}{n-1}U_{n-2}}$$

Q.8 : Find reduction formulae for a) $\int x^n e^{ax} dx$, b) $\int x^m (\log x)^n dx$.

[SPPU : May-19, Marks 4]

Ans. : a) Let $I_n = \int x^n e^{ax} dx$

Integration by parts, we have,

$$I_n = x^n \frac{e^{ax}}{a}-\int n x^{n-1}\frac{e^{ax}}{a}dx = \frac{x^n e^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}dx$$

$$I_n = \frac{x^n e^{ax}}{a}-\frac{n}{a}I_{n-1}$$

This is the required reduction formulae.

b) Let $I_{m,n} = \int \underset{v}{\underset{\downarrow}{x^m}} \underset{u}{\underset{\downarrow}{(\log x)^n}}\, dx$

Integrating by parts,

$$\begin{aligned} I_{m,n} &= (\log x)^n \frac{x^{m+1}}{m+1}-\int n(\log x)^{n-1}\frac{1}{x}\frac{x^{m+1}}{m+1}dx \\ &= \frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}\int x^m(\log x)^{n-1}dx \\ I_{m,n} &= \frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}I_{m,n-1} \end{aligned}$$

This is the required reduction formula.

Q.9 : If $I_{m,n} = \int \cos^m x \sin nx\, dx$ then show that $(m+n)I_{m,n} = -\cos^m x \cos nx + m\, I_{m-1,n-1}$ and hence evaluate $\int_0^{p/2} \cos^5 x \sin 3x\, dx$

Ans. : Given that $I_{m,n} = \int \cos^m x \sin nx\, dx$

Using integration by parts, here

$$u = \cos^m x, \quad v = \sin nx$$

$\therefore$ We get

$$I_{m,n} = -\frac{\cos nx}{n}\cos^m x - \int\left(-\frac{\cos nx}{n}\right)(m\cos^{m-1}x)(-\sin x)\,dx$$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1}x(\cos nx\sin x)\,dx$$

We know that $\cos nx\sin x = \sin nx\cos x - \sin(n-1)x$

$\therefore$ We get

$$I_{m,n} = -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^{m-1}x\,[\sin nx\cos x - \sin(n-1)x]\,dx$$

$$= -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}\int \cos^m x\sin nx\,dx + \frac{m}{n}\int \cos^{m-1}x\sin(n-1)x\,dx$$

$$I_{m,n} = -\frac{\cos^m x\cos nx}{n} - \frac{m}{n}I_{m,n} + \frac{m}{n}I_{m-1,n-1}$$

$$\therefore \quad I_{m,n} + \frac{m}{n}I_{m,n} = -\frac{\cos^m x\cos nx}{n} + \frac{m}{n}I_{m-1,n-1}$$

$$\therefore \quad I_{m,n} = -\frac{\cos^m x\cos nx}{m+n} + \frac{m}{m+n}I_{m-1,n-1} \quad \dots (1)$$

which is the required result

Let $\quad U_{m,n} = \int_0^{\pi/2}\cos^m x\sin nx\,dx$

$\therefore$ Using limits in equation (1) we get

$$U_{m,n} = \frac{1}{m+n} + \frac{m}{m+n}U_{m-1,n-1} \quad \dots (2)$$

To find $U_{5,3} = \int_0^{\pi/2}\cos^5 x\sin 3x\,dx$ putting $m = 5$ and $n = 3$ in equation (2), we get,

$$U_{5,3} = \frac{1}{5+3} + \frac{5}{5+3}U_{4,2}$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{4+2} + \frac{4}{4+2}U_{3,1}\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{12}U_{3,1}$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{12}\int_0^{\pi/2}\cos^3 x\sin x\,dx$$

$$U_{5,3} = \frac{1}{8} + \frac{5}{48} + \frac{5}{12}\left[\frac{1}{4}\right] = \frac{1}{3}$$

Q.10 : If $I_n = \int_0^{\pi/2}\cos^n x\cos nx\,dx$ prove that $I_n = \frac{1}{2}I_{n-1} = \frac{\pi}{2^{n+1}}$

Ans. : We have

$$I_n = \int_0^{\pi/2}\underset{\downarrow\,u}{\cos^n x}\;\underset{\downarrow\,v}{\cos nx}\,dx$$

$\therefore$ By integration by parts we get

$$I_n = \left[\cos^n x\frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} n\cos^{n-1}x(-\sin x)\frac{\sin nx}{n}\,dx$$

$$= 0 + \frac{n}{n}\int_0^{\pi/2}\cos^{n-1}x\sin x\sin nx\,dx$$

($\because$ $\sin nx\sin x = \cos(n-1)x - \cos nx\cos x$)

$$\therefore \quad I_n = \int_0^{\pi/2}\cos^{n-1}x\,[\cos(n-1)x - \cos nx\cos x]\,dx$$

$$I_n = \int_0^{\pi/2}\cos^{n-1}x\cos(n-1)x^{dx} - \int_0^{\pi/2}\cos^n x\cos nx\,dx$$

$$I_n = I_{n-1} - I_n \Rightarrow 2I_n = I_{n-1} \Rightarrow I_n = \frac{1}{2}I_{n-1}$$

$$\therefore \quad I_n = \frac{1}{2}I_{n-1} = \frac{1}{2}\left(\frac{1}{2}I_{n-2}\right) = \frac{1}{2^2}I_{n-2}$$

$$= \frac{1}{2^2}\left[\frac{1}{2} I_{n-3}\right] = \frac{1}{2^3} I_{n-3}$$

$\therefore$ In general

$$I_n = \frac{1}{2^n} I_{n-n} = \frac{1}{2^n} I_0 = \frac{1}{2^n}\left(\frac{\pi}{2}\right) = \frac{\pi}{2^{n+1}}$$

Q.11 : If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ prove that $I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$ and hence find I_5.

Ans. : Given that

$$I_n = \int_0^{\pi/2} x \cos^n x \, dx = \int_0^{\pi/2} (\underset{u}{\underset{\downarrow}{x \cos^{n-1} x}}) \underset{v}{\underset{\downarrow}{\cos x}} \, dx$$

$\therefore$ By integration by parts, we get

$$I_n = [x \cos^{n-1} x (\sin x)]_0^{\pi/2}$$

$$- \int_0^{\pi/2} [(1) \cos^{n-1} x + x (n-1) \cos^{n-2} x (-\sin x)] (\sin x)$$

$$= [x \cos^{n-1} x (\sin x)]_0^{\pi/2} - \int_0^{\pi/2} \cos^{n-1} x \sin x \, dx$$

$$+ \int_0^{\pi/2} x (n-1) \cos^{n-2} x \sin^2 x \, dx$$

$$= 0 + \left[\frac{\cos^n x}{n}\right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} x \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$I_n = -\frac{1}{n} + (n-1) \int_0^{\pi/2} x \cos^{n-2} x \, dx - (n-1) \int_0^{\pi/2} x \cos^n x \, dx$$

$$I_n = -\frac{1}{n} + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -\frac{1}{n} + (n-1) I_{n-2}$$

$$n I_n = -\frac{1}{n} + (n-1) I_{n-2}$$

$$\therefore \quad I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$$ which is the required result.

Now putting $n = 5$, we get

$$I_5 = -\frac{1}{5^2} + \frac{4}{5} I_3 = -\frac{1}{25} + \frac{4}{5}\left[-\frac{1}{9} + \frac{2}{3} I_1\right]$$

$$= -\frac{1}{25} - \frac{4}{45} + \frac{8}{15} \int_0^{\pi/2} x \cos x \, dx$$

$$= -\frac{1}{5}\left[\frac{1}{5} + \frac{4}{9}\right] + \frac{8}{15} [x \sin x + \cos x]_0^{\pi/2}$$

$$= -\frac{1}{5}\left(\frac{29}{45}\right) + \frac{8}{15}\left[\frac{\pi}{2} - 1\right]$$

$$I_5 = -\frac{29}{5 \times 45} + \frac{8}{15}\frac{\pi}{2} - \frac{8}{15} = \frac{4\pi}{15} - \frac{149}{225}$$

Q.12 : Evaluate $\int_0^{\pi/6} \cos^6 3x \sin^2 6x \, dx$.

Ans. : Let $I = \int_0^{\pi/6} \cos^6 3x \sin^2 6x \, dx$

Put $3x = \theta$, $3 \, dx = d\theta$

where $x = 0 \Rightarrow \theta = 0$

$$x = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{2}$$

$$\therefore \quad I = \int_0^{\pi/2} \cos^6\theta \sin^2 2\theta \frac{d\theta}{3} = \frac{1}{3} \int_0^{\pi/2} \cos^6\theta (2 \sin\theta \cos\theta)^2 \, d\theta$$

$$= \frac{2^2}{3} \int_0^{\pi/2} \cos^8\theta \sin^2\theta \, d\theta = \frac{4}{3} \frac{[1][7 \cdot 5 \cdot 3 \cdot 1]}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2}$$

$$\therefore \quad I = \frac{7\pi}{384}$$

Q.13 : If $I_n = \int_0^{\pi/2} x^n \sin x\, dx$ then such that

$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

Ans. : Given that

$$I_n = \int_0^{\pi/2} \underset{u}{x^n} \underset{v}{\sin x}\, dx$$

Using the rule of integration by parts, we get

$$I_n = [x^n(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)\, dx$$

$$= 0 + n\int_0^{\pi/2} \underset{u}{x^{n-1}} \underset{v}{\cos x}\, dx$$

Again using integration by parts, we get

$$I_n = n[x^{n-1}(\sin x)]_0^{\pi/2} - n\int_0^{\pi/2} (n-1)\, x^{n-2} \sin x\, dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$$

$\therefore \quad I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$ which is the required result.

Q.14 : Find the reduction formula for $I_n = \int_0^{\infty} e^{-px} \sin^n x\, dx$; $n \geq 2$ and $p > 0$ by using I_{n-2} and hence find I_4. **[SPPU : May-14, Marks 4]**

Ans. : Given that $I_n = \int_0^{\infty} \underset{v}{e^{-px}} \underset{u}{\sin^n x}\, dx$

Using rule of integration by parts, we get

$$I_n = \left[\sin^n x\left(\frac{e^{-px}}{-p}\right)\right]_0^{\infty} - \int_0^{\infty} n \sin^{n-1} x \cos x\left(-\frac{e^{-px}}{p}\right) dx$$

$$= 0 + \frac{n}{p}\int_0^{\infty} \underset{v}{e^{-px}}\, (\underset{u}{\sin^{n-1} x \cos x})\, dx$$

$$\therefore \quad I_n = \frac{n}{p}\left\{\left[\sin^{n-1} x \cos x\left(-\frac{e^{-px}}{p}\right)\right]_0^{\infty} - \int_0^{\infty} [\sin^{n-1} x(-\sin x) + (n-1)\sin^{n-2} x \cos x \cos x]\left(\frac{e^{-px}}{-p}\right) dx\right\}$$

$$= \frac{n}{p^2}\int_0^{\infty} [-\sin^n x + (n-1)\sin^{n-2} x \cos x]\, e^{-px}\, dx$$

$$= -\frac{n}{p^2} I_n + \frac{n(n-1)}{p^2}\int_0^{\infty} \sin^{n-2} x(1-\sin^2 x)\, e^{-px}\, dx$$

$$\therefore \quad I_n = -\frac{n}{p^2} I_n + \frac{n(n-1)}{p^2} I_{n-2} - \frac{n(n-1)}{p^2} I_n$$

$$\therefore \quad \left[1 + \frac{n(n-1)}{p^2}\right] I_n + \frac{n}{p^2} I_n = \frac{n(n-1)}{p^2} I_{n-2}$$

$$(p^2 + n^2)\, I_n = \frac{n(n-1)}{p^2} I_{n-2}$$

$\Rightarrow I_n = \frac{n(n-1)}{p^2(p^2+n^2)} I_{n-2}$ which is the required reduction formula.

Putting $n = 0 \Rightarrow I_0 = \int_0^{\infty} e^{-px} \sin^0 x\, dx = \left[\frac{e^{-px}}{-p}\right]_0^{\infty} = \frac{1}{p}$

$$I_4 = \frac{4(4-1)}{p^2(p^2+4^2)} I_2$$

$$= \frac{12}{p^2(p^2+16)}\left[\frac{2(1)}{p^2(p^2+4)} I_0\right]$$

$$I_4 = \frac{24}{p^2(p^2+4)(p^2+16)}$$

Q.15 : Evaluate $\int_0^{\infty} \frac{x^8 - x^5}{(1+x^3)^3} dx$

Ans. : Let $I = \int_0^{\infty} \frac{x^8 - x^5}{(1+x^3)^5} dx$

Put $x^3 = \tan^2\theta$

$\therefore 3x^2 dx = 2\tan\theta \sec^2\theta \, d\theta$

and

x	0	∞
θ	0	$\pi/2$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\tan^4\theta - \tan^2\theta}{(1+\tan^2\theta)^5} \frac{2}{3} \tan\theta \sec^2\theta \, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} (\tan^5\theta - \tan^3\theta) \cos^8\theta \, d\theta$$

$$= \frac{2}{3}\left[\int_0^{\pi/2} \sin^5\theta \cos^3\theta \, d\theta - \int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta\right]$$

$$I = 0$$

Q.16 : If $f(m, n) = \int x^m (1-x)^n dx$ then prove that $f(m, n) = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} f(m, n-1)$

Ans. : Given that

$$f(m, n) = \int \underset{v}{\underset{\downarrow}{x^m}} \underset{u}{\underset{\downarrow}{(1-x)^n}} \, dx$$

Using integration by parts, we get

$$f(m, n) = \frac{(1-x)^n x^{m+1}}{m+1} - \int n(1-x)^{n-1}(-1)\frac{x^{m+1}}{m+1} dx$$

$$= \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1}\int (1-x)^{n-1} x^m x \, dx$$

$$= \frac{(1-x)^n x^{m+1}}{m+1} \quad \frac{n}{m+1}\int (1-x)^{n-1} x^m (1-x+1) \, dx$$

$$= \frac{(1-x)^n x^{m+1}}{m+1}$$

$$-\frac{n}{m+1}\int (1-x)^n x^m dx \quad \frac{n}{m+1}\int (1-x)^{n-1} x^m \, dx$$

$$\therefore \quad f(m, n) = \frac{(1-x)^n x^{m+1}}{m+1} - \frac{n}{m+1} f(m, n) + \frac{n}{m+1} f(m, n-1)$$

$$\therefore \; f(m+n) + \frac{n}{m+1} f(m, n) = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} f(m, n-1)$$

$$f(m, n) = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} f(m, n,-1)$$

which is the required reduction formula.

Q.17 : Evaluate $\int_0^{\pi} x \sin^7 x \cos^4 x \, dx$

Ans. :

Let $\quad I = \int_0^{\pi} x \sin^7 x \cos^4 x \, dx \quad \left(\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right)$

$$\therefore \quad I = \int_0^{\pi} (\pi - x) \sin^7(\pi - x) \cos^4(\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \sin^7 x \cos^4 x \, dx$$

$$= \int_0^{\pi} \pi \sin^7 x \cos^4 x \, dx - \int_0^{\pi} x \sin^7 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^7 x \cos^4 x \, dx - I$$

$$\therefore \quad 2I = \pi \, 2 \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx$$

$$\therefore \quad I = \pi \frac{[6 \cdot 4 \cdot 2][3 \cdot 1]}{[11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1]}(1) = \frac{16\pi}{1155}$$

Q.18 : Evaluate $\int_4^6 \sin^4 \pi x \cos^2 2\pi x \, dx$.

Ans. : Let $I = \int_4^6 \sin^4 \pi x \cos^2 2\pi x \, dx$

Put $\pi x = t + 4\pi \quad \therefore \pi \, dx = dt$

x	4	6
t	0	2π

and $\sin(4\pi + t) = \sin t, \ \cos(8\pi + 2t) = \cos 2t$

$\therefore$ Integral becomes

$$I = \frac{1}{\pi}\int_0^{2\pi} \sin^4 t \cos^2 2t \, dt = \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t \cos^2 2t \, dt$$

As $\quad \cos 2t = (1 - 2\sin^2 t)$

$$\therefore \quad I = \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t \, [1 - 2\sin^2 t]^2 \, dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t \, [1 - 4\sin^2 t + 4\sin^4 t] \, dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} [\sin^4 t - 4\sin^6 t + 4\sin^8 t] \, dt$$

$$= \frac{4}{\pi}\left\{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 4\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\frac{\pi}{2} + 4\cdot\frac{7}{8}\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\}$$

$$I = \frac{7}{16}$$

Q.19 : Evaluate $\int_0^{2a} x^3 \sqrt{2ax - x^2} \, dx$

Ans. : Let $I = \int_0^{2a} x^3 \sqrt{2ax - x^2} \, dx$

Put $x = 2a \sin^2\theta$

$\therefore dx = 4a \sin\theta \cos\theta \, d\theta$

x	0	2a
θ	0	$\frac{\pi}{2}$

$$\therefore \ I = \int_0^{\pi/2} (2a \sin^2\theta)^3 \sqrt{2a\,(2a\sin^2\theta - (2a\sin^2\theta)^2} \times (4a \sin\theta \cos\theta \, d\theta)$$

$$\therefore \ I = \int_0^{\pi/2} 8a^3 \sin^6\theta \, 2a \sin\theta \cos\theta \, (4a \sin\theta \cos\theta) \, d\theta$$

$$= 64a^5 \int_0^{\pi/2} \sin^8\theta \cos^2\theta \, d\theta$$

$$I = 64a^5 \frac{[7\cdot5\cdot3\cdot1]\,[1]}{[10\cdot8\cdot6\cdot4\cdot2]}\cdot\frac{\pi}{2} = \frac{7\pi a^5}{8}$$

END...

UNIT - IV

4 Gamma and Beta Functions

4.1 : Gamma Functions

1) The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \text{ where } n > 0$$

This integral is also known as Euler's integral of the second kind.

$$\therefore \quad \Gamma(n+1) = \int_0^\infty e^{-x}x^{n}dx$$

$$\Gamma(2) = \int_0^\infty e^{-x}\, x\, dx$$

and

$$\int_0^\infty e^{-x}x^{9}dx = \Gamma(10)$$

2) Property 1 : $\Gamma(1) = 1$

Proof : We have $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$

Put $n = 1$

$$\therefore \quad \Gamma(1) = \int_0^\infty e^{-x}x^{0}dx = \int_0^\infty e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$\Gamma(1) = [-e^{-\infty} - (-e^{-0})] = [-0+1] = 1$$

$$\therefore \quad \boxed{\Gamma(1) = 1}$$

Property 2 : Reduction formula for Gamma function

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \text{ or } \Gamma(n+1) = n\Gamma(n)$$

Proof : We have $\Gamma(n+1) = \int_0^\infty e^{-x}x^{n}dx$

Applying integrating by parts

$$\Gamma(n+1) = \left\{x^n\left(\frac{e^{-x}}{-1}\right)\right\}_0^\infty - \int_0^\infty \left(\frac{e^{-x}}{-1}\right)(n x^{n-1})dx$$

$$= 0 + n\int_0^\infty e^{-x}x^{n-1}dx = n\Gamma(n)$$

Hence $\boxed{\Gamma(n+1) = n\Gamma(n)}$

If n is a natural number i.e. 1, 2, 3, ...

then

$$\Gamma(n+1) = n\Gamma(n) \qquad (\because \Gamma(n) = (n-1)\Gamma(n-1))$$

$$= n(n-1)\Gamma(n-1) \qquad (\because \Gamma(n-1) = (n-2)\Gamma(n-2))$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\Gamma(n-3)$$

In general

$$\Gamma(n+1) = n(n-1)(n-2)(n-3) \qquad \ldots 3.2.\Gamma(1)$$

$$= n(n-1)(n-2)(n-3) \qquad \ldots 3.2.1$$

$$\Gamma(n+1) = n!$$

In particular

$$\Gamma(4) = \Gamma(3+1) = 3! = 3 \times 2 \times 1 = 6$$

$$\Gamma(1) = \Gamma(0+1) = 0! = 1$$

$$\Gamma(7/2) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{5}{3}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{5}{3}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}+1\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

Property 3 : $\Gamma(0) = \infty$

Proof : We know that $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

put $n = 0$

$$\therefore \quad \Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$$

$$\boxed{\Gamma(0) = \infty}$$

Property 4 : Alternate form of Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x^2} x^{2n-1} dx$$

Proof : We know that $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$

put $t = x^2 \therefore dt = 2x\,dx$ and

t	0	∞
x	0	∞

$$\therefore \quad \Gamma(n) = \int_0^\infty e^{-x^2} (x^2)^{n-1} 2x\,dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

Property 5 : $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof : We have $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

put $n = \frac{1}{2}$

$$\therefore \quad \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} x^{2\left(\frac{1}{2}\right)-1} dx = 2\int_0^\infty e^{-x^2} dx$$

$$\Gamma\left(\frac{1}{2}\right) = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

Property 6 : $I = \int_0^\infty e^{-kx} x^{n-1} dx = \dfrac{\Gamma(n)}{k^n}$

Proof : Put $kx = t \Rightarrow x = \frac{t}{k} \Rightarrow dx = \frac{dt}{k}$ and

x	0	∞
t	0	∞

$$I = \int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{n-1} \frac{dt}{k}$$

$$= \frac{1}{k^n}\int_0^\infty e^{-k} t^{n-1} dt$$

$$\boxed{\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}}$$

For example

$$\int_0^\infty e^{-3x} x^9 dx = \frac{\Gamma(10)}{3^{10}} \text{ (as } k = 3 \text{ and } n = 10)$$

and $$\int_0^\infty e^{-2x} x^4 dx = \frac{\Gamma(5)}{2^5} = \frac{4!}{32} = \frac{24}{32} = \frac{3}{4}$$

Property 7 : $\Gamma(P)\,\Gamma(1-P) = \dfrac{\pi}{\sin P\pi}$ where $0 < p < 1$

For example

1) $$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} = \pi\sqrt{2}$$

2) $$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin \pi/2} = \pi$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Property 8 : $\int_0^\infty e^{-x^m} x^{mn-1}\, dx = \frac{\Gamma(n)}{m}$

Proof : We have $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$

Put $t = x^m \Rightarrow dt = m x^{m-1} dx$

$$\therefore \quad \Gamma(n) = \int_0^\infty e^{-x^m} (x^m)^{n-1} (m x^{m-1} dx)$$

$$= m\int_0^\infty e^{-x^m} x^{mn-m} x^{m-1} dx$$

$$\Gamma(n) = m\int_0^\infty e^{-x^m} x^{mn-1} dx$$

$$\therefore \int_0^\infty e^{-x^m} x^{mn-1} dx = \frac{\Gamma(n)}{m}$$

For example

$$\int_0^\infty e^{-x^3} x^{14} dx = \int_0^\infty e^{-x^3} x^{3(5)-1} dx = \frac{\Gamma(5)}{3} = \frac{4!}{3} = 8$$

Type I : Examples reducible to standard form

If the form of integral is $\int_0^\infty e^{-a x^m} x^n dx$ then substitute $ax^m = t$ and reduce integral to $\int_0^\infty e^{-t} (t)^k dx$ and use definition of gamma function.

Q.1 : Prove that $\int_0^\infty e^{-x^n} dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$

Ans. :

Let $I = \int_0^\infty e^{-x^n} dx$, put $x^n = t \Rightarrow x = t^{\frac{1}{n}}$ $dx = \frac{1}{n} t^{\frac{1}{n}-dt}$

and

x	0	∞
t	0	∞

$$\therefore \quad I = \int_0^\infty e^{-t} \frac{1}{n} t^{\frac{1}{n}-1} dt = \frac{1}{n}\int_0^\infty e^{-t} t^{\frac{1}{n}-1} dt = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$$

Thus $$\int_0^\infty e^{-x^5} dx = \frac{1}{5}\Gamma\left(\frac{1}{5}\right), \int_0^\infty e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Q.2 : Evaluate $\int_0^\infty x^9 e^{-2x^2} dx$ [SPPU : Dec.-18, Marks 4]

Ans. :

Let $$I = \int_0^\infty x^9 e^{-2x^2} dx \quad \ldots(1)$$

Put $2x^2 = t \Rightarrow x^2 = \frac{t}{2} \therefore 2x\, dx = \frac{dt}{2}$

$$\Rightarrow \quad dx = \frac{dt}{4x} = \frac{dt}{\frac{4\sqrt{t}}{\sqrt{2}}} = \frac{\sqrt{2}}{4}\frac{1}{\sqrt{t}} dt$$

x	0	∞
t	0	∞

$$\therefore \quad I = \int_0^\infty e^{-t}\left(\frac{t}{2}\right)^{9/2} \frac{\sqrt{2}}{4\sqrt{t}} dt = \frac{\sqrt{2}}{2^{9/2} 4}\int_0^\infty e^{-t} t^4 dt$$

2) $$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin \pi/2} = \pi$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Property 8 : $\int_0^\infty e^{-x^m} x^{mn-1}\, dx = \frac{\Gamma(n)}{m}$

Proof : We have $\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$

Put $t = x^m \Rightarrow dt = m x^{m-1} dx$

$$\therefore \qquad \Gamma(n) = \int_0^\infty e^{-x^m} (x^m)^{n-1} (m x^{m-1} dx)$$

$$= m\int_0^\infty e^{-x^m} x^{mn-m} x^{m-1} dx$$

$$\Gamma(n) = m\int_0^\infty e^{-x^m} x^{mn-1} dx$$

$$\therefore \int_0^\infty e^{-x^m} x^{mn-1} dx = \frac{\Gamma(n)}{m}$$

For example

$$\int_0^\infty e^{-x^3} x^{14} dx = \int_0^\infty e^{-x^3} x^{3(5)-1} dx = \frac{\Gamma(5)}{3} = \frac{4!}{3} = 8$$

Type I : Examples reducible to standard form

If the form of integral is $\int_0^\infty e^{-a x^m} x^n dx$ then substitute $ax^m = t$ and reduce integral to $\int_0^\infty e^{-t} (t)^k dx$ and use definition of gamma function.

Q.1 : Prove that $\int_0^\infty e^{-x^n} dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$

Ans. :

Let $I = \int_0^\infty e^{-x^n} dx$, put $x^n = t \Rightarrow x = t^{\frac{1}{n}}$ $dx = \frac{1}{n} t^{\frac{1}{n}-dt}$

and

x	0	∞
t	0	∞

$$\therefore \qquad I = \int_0^\infty e^{-t} \frac{1}{n} t^{\frac{1}{n}-1} dt = \frac{1}{n}\int_0^\infty e^{-t} t^{\frac{1}{n}-1} dt = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$$

Thus $$\int_0^\infty e^{-x^5} dx = \frac{1}{5}\Gamma\left(\frac{1}{5}\right), \int_0^\infty e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Q.2 : Evaluate $\int_0^\infty x^9 e^{-2x^2} dx$

[SPPU : Dec.-18, Marks 4]

Ans. :

Let $$I = \int_0^\infty x^9 e^{-2x^2} dx \qquad \ldots(1)$$

Put $2x^2 = t \Rightarrow x^2 = \frac{t}{2} \therefore\ 2x\, dx = \frac{dt}{2}$

$$\Rightarrow \qquad dx = \frac{dt}{4x} = \frac{dt}{\frac{4\sqrt{t}}{\sqrt{2}}} = \frac{\sqrt{2}}{4}\frac{1}{\sqrt{t}} dt$$

x	0	∞
t	0	∞

$$\therefore \qquad I = \int_0^\infty e^{-t}\left(\frac{t}{2}\right)^{9/2} \frac{\sqrt{2}}{4\sqrt{t}} dt = \frac{\sqrt{2}}{2^{9/2}4}\int_0^\infty e^{-t} t^4 dt$$

$$I = \frac{1}{64}\overline{|5} = \frac{4!}{64} = \frac{24}{64} = \frac{3}{8}$$

Q.3 : Evaluate $\int_0^\infty x^7 e^{-2x^2} dx$ **[SPPU : Dec.-18, Marks 4]**

Ans. :

Let $$I = \int_0^\infty x^7 e^{-2x^2} dx \quad \ldots(1)$$

Put $$2x^2 = t \Rightarrow x^2 = \frac{t}{2} \Rightarrow 2x\,dx = \frac{dt}{2}$$

$\therefore$ $$x\,dx = \frac{dt}{4}$$

x	0	∞
t	0	∞

$\therefore$ $$I = \int_0^\infty e^{-2x^2} x^6 x\,dx = \int_0^\infty e^{-t}\frac{t^3}{2^3}\frac{dt}{4}$$

$$I = \frac{1}{32}\int_0^\infty e^{-t}t^3 dt = \frac{1}{32}\overline{|4} = \frac{3!}{32} = \frac{6}{32} = \frac{3}{16}$$

Q.4 : Evaluate $\int_0^\infty 4\sqrt{x}\, e^{-\sqrt{x}} dx$ **[SPPU : Dec.-14, Marks 4]**

Ans. :

Let $$I = \int_0^\infty 4\sqrt{x}\, e^{-\sqrt{x}} dx \quad \ldots(1)$$

Put $$\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$$

x	0	∞
t	0	∞

$\therefore$ $$I = \int_0^\infty e^{-t}(t^2)^{\frac{1}{4}} 2t\,dt = 2\int_0^\infty e^{-t} t^{\frac{3}{2}}\,dt$$

$$I = 2\overline{\left|\frac{3}{2}+1\right.} = 2\,\frac{3}{2}\overline{\left|\frac{3}{2}\right.} = 3\overline{\left|\frac{1}{2}+1\right.} = 3\,\frac{1}{2}\overline{|1/2} = \frac{3}{2}\sqrt{\pi}$$

Q.5 : Evaluate $\int_0^\infty e^{-\sqrt{y}}\sqrt{y}\,dy$ **[SPPU : Dec.-17, Marks 4]**

Ans. :

Let $$I = \int_0^\infty e^{-\sqrt{y}}\sqrt{y}\,dy \quad \ldots(1)$$

Put $$\sqrt{y} = t \Rightarrow y = t^2 \Rightarrow dy = 2t\,dt$$

and

y	0	∞
t	0	∞

$\therefore$ $$I = \int_0^\infty e^{-t}t(2t)dt = 2\int_0^\infty e^{-t}t^2 dt = 2\overline{|3}$$

$$I = 2 \times 2! = 4$$

Q.6 : Show that $\int_0^\infty e^{-x^2}(x^4+1)^3 dx = \frac{3989}{2}\sqrt{\pi}$

Ans. :

Let $$I = \int_0^\infty e^{-x^2}(x^4+1)^3 dx \quad \ldots(1)$$

Put $$x^2 = t \Rightarrow x = t^{\frac{1}{2}} \Rightarrow dx = \frac{1}{2}t^{-\frac{1}{2}}dt$$

x	0	∞
t	0	∞

$\therefore$ $$I = \int_0^\infty e^{-t}(t^2+1)^3\frac{1}{2}t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}(t^6+3t^4+3t^2+1)t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\left[t^{1\frac{1}{2}}+3t^{7/2}+3t^{3/2}+t^{-\frac{1}{2}}\right]dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}t^{1\frac{1}{2}}dt+\int_0^\infty e^{-t}3t^{7/2}dt+\int_0^\infty 3e^{-t}t^{3/2}dt+\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\left\{\Gamma\left(\frac{11}{2}+1\right)+3\Gamma\left(\frac{7}{2}+1\right)+3\Gamma\left(\frac{3}{2}+1\right)+\Gamma\left(\frac{1}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{11}{2}\Gamma\left(\frac{9}{2}+1\right)+3\frac{7}{2}\Gamma\left(\frac{5}{2}+1\right)+3\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)+\sqrt{\pi}\right\}$$

$$= \frac{1}{2}\left\{\frac{11}{2}\frac{9}{2}\Gamma\left(\frac{7}{2}+1\right)+\frac{21}{2}\frac{5}{2}\Gamma\left(\frac{3}{2}\right)+\frac{9}{2}\frac{\sqrt{\pi}}{2}+\sqrt{\pi}\right\}$$

$$= \frac{1}{2}\left\{\frac{99}{4}\cdot\frac{35}{8}\sqrt{\pi}+\frac{105\sqrt{\pi}}{8}+\frac{9\sqrt{\pi}}{4}+\sqrt{\pi}\right\} = \frac{3989}{2}\sqrt{\pi}$$

Type II : Examples involving (constant) variable

If the form of integral is $\int_0^\infty a^{bx^2}x^n dx$ then put $a^{bx^2} = e^{-t}$ and processed further.

Q.7 : $\int_0^\infty \frac{x^a}{a^x}dx \quad (a > 0)$

Ans. :

Let $\quad I = \int_0^\infty \frac{x^a}{a^x}dx$

Put $\quad a^x = e^t \Rightarrow x \log a = t \Rightarrow dx = \frac{dt}{\log a}$

x	2	∞
t	0	∞

$$\therefore \quad I = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a \frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}t^a dt$$

$$I = \frac{1}{(\log a)^{a+1}}\Gamma(a+1)$$

Q.8 : Evaluate $\int_0^\infty \frac{x^n}{a^{x^m}}dx$.

Ans. :

Let $\quad I = \int_0^\infty \frac{x^n}{a^{x^m}}dx$

Put $\quad a^{x^m} = e^t \Rightarrow x^m \log a = t \Rightarrow x$

$$= \left(\frac{t}{\log a}\right)^{\frac{1}{m}}$$

$$\therefore \quad dx = \frac{1}{(\log a)^{\frac{1}{m}}}\frac{1}{m}t^{\frac{1}{m}-1}dt$$

x	0	∞
t	0	∞

$$\therefore \quad I = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^{\frac{n}{m}}\frac{1}{(\log a)^{\frac{1}{m}}}\cdot\frac{1}{m}t^{\frac{1}{m}-1}dt$$

$$I = \frac{1}{m(\log a)^{\frac{n+1}{m}}}\int_0^\infty e^{-t}t^{\frac{n+1}{m}-1}dt$$

$$= \frac{1}{m(\log a)^{\frac{n+1}{m}}}\Gamma\left(\frac{n+1}{m}\right)$$

Type III : Examples involving log ax

If the form of integral is $\int_0^1 (\log ax)^m x^n dx$ then put $\log ax = -t$ and proceed further.

Q.9 : Evaluate $\int_0^1 x^m (\log x)^m dx$.

Ans. :

Let $$I = \int_0^1 x^m (\log x)^m dx$$

Put $$\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t} dt$$

At $$x = 0, -t = \log 0 = -\infty$$

x	0	1
t	∞	0

$$\therefore \quad I = \int_\infty^0 (e^{-t})^m (-t)^m (-e^{-t}) dt$$

$$= \int_0^\infty (e^{-mt})(-1)^m t^m e^{-t} dt = (-1)^m \int_0^\infty (e^{-(m+1)t} t^m dt$$

Put $$(m+1)t = y \Rightarrow dt = \frac{1}{m+1} dy$$

$$\therefore \quad I = (-1)^m \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^m \frac{1}{m+1} dy$$

$$I = \frac{(-1)^m}{(m+1)^{m+1}} \int_0^\infty e^{-y} y^m dy = \frac{(-1)^m \overline{|m+1}}{(m+1)^{m+1}}$$

Q.10 : Show that $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \overline{|n}$.

Ans. : Put $\log \frac{1}{x} = t \Rightarrow \frac{1}{x} = e^t \Rightarrow x = e^{-t} \Rightarrow dx$

$$= -e^{-t} dt$$

and

x	0	1
t	∞	0

$$\therefore \quad I = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$$

$$= \int_\infty^0 t^{n-1} (-e^{-t}) dt$$

$$\therefore \quad I = \int_0^1 e^{-t} t^{n-1} dt = \overline{|n}$$

Type IV : Examples involving sin ax or cos ax

Q.11 : Show that $\int_0^\infty \cos\left(ax^{1/n}\right) dx = \frac{\overline{|n+1}}{a^n} \cos \frac{n\pi}{2}$.

Ans. : We know that $\cos\left(ax^{1/n}\right) = \text{Real } e^{-i\left(ax^{1/n}\right)}$

$$\left[\because e^{-iax^{1/n}} = \cos\left(ax^{1/n}\right) + i \sin\left(ax^{1/n}\right)\right]$$

Thus the integral becomes

$$I = \text{Real} \int_0^\infty e^{-iax^{1/n}} dx$$

Put $$iax^{1/n} = t$$

$$\therefore \quad x^{1/n} = \frac{t}{ia}, \; x = \frac{t^n}{(ia)^n}$$

$$dx = \frac{n t^{n-1} dt}{(ia)^n}$$

Limits :

x	0	∞
t	0	∞

$$\therefore \int_0^\infty \cos\left(ax^{1/n}\right) dx = \text{Real} \int_0^\infty e^{-t} \cdot \frac{n\, t^{n-1}\, dt}{(i\,a)^n}$$

$$= \text{Real}\ \frac{n}{a^n}\left(\frac{1}{i}\right)^n \int_0^\infty e^{-t}\, t^{\,n-1} dt$$

$$= \text{Real}\ \frac{n}{a^n}(-i)^n \,\overline{|n}$$

$$= \text{Real}\ \frac{n\overline{|n}}{a^n}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^n$$

$$= \frac{\overline{|n+1}}{a^n} \cdot \cos\frac{n\pi}{2}$$

4.2 : Beta Functions

1) Definition : A Beta function of m, n is denoted by $\beta(m, n)$ and defined as,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad (m > 0,\ n > 0)$$

The Beta function is also called as Euler's integral of the first kind.

We have $\beta(3,6) = \int_0^1 x^2(1-x)^5 dx$

2) Properties of Beta Function

Property 1) $\beta(m, n) = \beta(n, m)$

Proof : Let $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ ...(1)

Put $1 - x = t$, $-dx = dt$ in equation (1)

x	0	1
t	1	0

$$\beta(m,n) = \int_1^0 (1-t)^{m-1}\, t^{n-1}\, (-dt)$$

$$= \int_0^1 t^{n-1}(1-t)^{m-1}\, dt$$

$$\boxed{\beta(m,n) = \beta(n,m)}$$

Property 2 :

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\ d\theta$$

Proof : Let $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ ...(1)

Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$ in equation (1)

x	0	1
θ	0	π/2

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\ d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta$$

Thus $\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta}$

Substituting $m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

We get, practical formula as

$$\boxed{\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

Property 3 :

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Proof : Let $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \frac{t}{1+t}$, $dx = \frac{dt}{(1+t)^2}$

x	0	1
t	0	∞

Substituting we get

$$\beta(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\cdot\frac{dt}{(1+t)^2}$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

As the variable is immaterial in definite Integral, hence we get the result.

$$\boxed{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx}$$

We can use this result as another definition of Beta function.

Property 4 :

Relation between Beta and Gamma functions

$$\boxed{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}}$$

1) $\beta(2,3) = \frac{\Gamma 2\,\Gamma 3}{\Gamma 5} = \frac{1!\,2!}{4!} = \frac{2}{24} = \frac{1}{12}$

2) $\beta(1,1) = \frac{\Gamma 1\,\Gamma 1}{\Gamma 2} = \frac{0!\,0!}{1!} = 1$

3) $\beta\left(\frac{7}{2},-\frac{1}{2}\right) = \frac{\Gamma\frac{7}{2}\,\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}-\frac{1}{2}\right)} = \frac{\Gamma\frac{7}{2}\,\Gamma\left(-\frac{1}{2}\right)}{\Gamma 3} = \frac{1}{2!}\Gamma\frac{7}{2}\,\Gamma\left(-\frac{1}{2}\right)$

$\Rightarrow \quad \beta\left(\frac{7}{2},-\frac{1}{2}\right) = \frac{1}{2}\Gamma\frac{7}{2}\,\Gamma\frac{-1}{2}$

But we know that $\Gamma n = \frac{\Gamma(n+1)}{n}$ and $\Gamma(n+1) = n\Gamma n$

$\therefore \quad \Gamma\frac{7}{2} = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2} = \frac{15}{4}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$$\Gamma\frac{7}{2} = \frac{15}{8}\sqrt{\pi}$$

and $\quad \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\frac{1}{2} = 2\sqrt{\pi}$

$\therefore \quad \beta\left(\frac{7}{2},-\frac{1}{2}\right) = \frac{1}{2}\left(\frac{15}{8}\sqrt{\pi}\right)(-2\sqrt{\pi}) = -\frac{15}{8}\pi$

4) $\Gamma\left(-\frac{7}{2}\right) = \frac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}} = -\frac{2}{7}\Gamma\left(-\frac{5}{2}\right)$

$$= -\frac{2}{7}\,\frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = \frac{4}{35}\Gamma\left(-\frac{3}{2}\right)$$

$$= \frac{4}{35}\,\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{8}{105}\Gamma\left(-\frac{1}{2}\right)$$

$$= -\frac{8}{105}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{16}{105}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{7}{2}\right) = \frac{16\sqrt{\pi}}{105}$$

Property 5 :

$$\Gamma(1/2) = \sqrt{\pi}$$

Proof : We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \; d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Put p = q = 0

$$\int_0^{\pi/2} d\theta = \frac{1}{2}\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)}$$

$$\frac{\pi}{2} = \frac{1}{2}(\Gamma(1/2))^2$$

$$\therefore \quad \Gamma(1/2) = \sqrt{\pi}$$

Property 6 : Legendre's duplication formula :

$$\boxed{\Gamma(m)\,\Gamma(m+1/2) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$

Proof : We know that

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \; d\theta$$

Put q = p

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \int_0^{\pi/2} (\sin\theta \cos\theta)^p \; d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^p d\theta$$

Put $2\theta = u$, $2\,d\theta = du$

Limits :

θ	0	π/2
u	0	π

$$\therefore \frac{1}{2}B\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \frac{1}{2^p}\int_0^{\pi} \sin^p u \cdot \frac{du}{2}$$

$$= \frac{1}{2\cdot 2^p}\int_0^{\pi} \sin^p u \; du$$

$$\left\{ \because \quad \int_0^{2a} f(x)\,dx = \int_0^{a} [f(x) + f(2a-x)]\,dx \right\}$$

$$= \frac{1}{2\cdot 2^p}\int_0^{\pi/2} [\sin^p u + \sin^p(\pi - u)] \; du$$

$$= \frac{1}{2\cdot 2^p}\int_0^{\pi/2} 2\sin^p u \, du \qquad \{\because \sin(\pi - u) = \sin u\}$$

$$\frac{1}{2}\beta\left(\frac{p+1}{2}\ \frac{p+1}{2}\right) = \frac{2}{2\cdot 2^p}\int_0^{\pi/2} \sin^p u \cdot \cos^0 u \; du$$

Using the formula we get,

$$\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = \frac{1}{2^p}\cdot\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{0+1}{2}\right)$$

Put $\frac{p+1}{2} = m$

$\therefore \; p = 2m - 1$

$$\beta(m, m) = \frac{1}{2^{2m-1}}\beta\left(m, \frac{1}{2}\right)$$

$$\therefore \quad \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\frac{\Gamma(m)\,\Gamma(1/2)}{\Gamma(m+1/2)}$$

$$\therefore \quad \Gamma(m)\,\Gamma(m+1/2) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

1) Put $m = \frac{1}{2}, \; \Gamma\frac{1}{2}\,\Gamma 1 = \frac{\sqrt{\pi}}{2^0}\Gamma 1$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

2) $$\Gamma 3\;\Gamma\frac{7}{2} = \frac{\sqrt{\pi}}{2^{2(3)-1}}\Gamma 2(3) = \frac{\sqrt{\pi}}{2^5}\Gamma 6 = \frac{\sqrt{\pi}}{32}\,5!$$

$$= \frac{\sqrt{\pi}}{32}(120) = \frac{15\sqrt{\pi}}{4}$$

Q.12 : Evaluate $\int_0^{2a} x\sqrt{2ax - x^2}\,dx$ **[SPPU : Dec.-17, Marks 4]**

Ans. :

Let $$I = \int_0^{2a} x\sqrt{2ax-x^2}\,dx = \int_0^{2a} x\sqrt{x}\sqrt{2a-x}\,dx$$

$$I = \int_0^{2a} x^{3/2}\sqrt{2a-x}\,dx \qquad \ldots(1)$$

Put $x = 2at \Rightarrow dx = 2adt$ and

x	0	2a
t	0	1

$\therefore$ $$I = \int_0^1 (2a\,t)^{3/2}\sqrt{2a-2a\,t}\,(2a\,dt)$$

$$= (2a)^3\int_0^1 t^{3/2}(1-t)^{\frac{1}{2}}\,dt$$

$$= 8a^3\beta\left(\frac{5}{2},\frac{3}{2}\right) = 8a^3\frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}$$

$$= \frac{8a^3\,\frac{3}{2}\,\Gamma\frac{3}{2}\left(\frac{1}{2}\right)\Gamma\frac{1}{2}}{\Gamma 4}$$

$$I = \frac{8a^3}{3!}\left(\frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}\right)\left(\frac{1}{2}\,\Gamma\frac{1}{2}\right)$$

$$= \frac{8a^3}{6}\left(\frac{3}{8}\right)\sqrt{\pi}\sqrt{\pi} = \frac{a^3\pi}{3}$$

Q.13 : Evaluate $\int_0^1 x^m(1-x^n p)\,dx$. **[SPPU : Dec.-13, 16, Marks 4]**

Ans. :

Let $$I = \int_0^1 x^m(1-x^n p)\,dx \qquad \ldots(1)$$

Put $$x^n = y \Rightarrow x = y^{1/n}$$

$$dx = \frac{1}{n}y^{\frac{1}{n}-1}dy$$

x	0	1
y	0	1

$\therefore$ $$I = \int_0^1\left(y^{\frac{1}{n}}\right)^m(1-yp)\frac{y^{\frac{1}{n}-1}}{n}\,dy$$

$$I = \frac{1}{n}\int_0^1 y^{\frac{m}{n}+\frac{1}{n}-1}(1-yp)\,dy$$

$$= \frac{1}{n}\int_0^1 y^{\left(\frac{m+1}{n}\right)-1}(1-yp)\,dy$$

$$I = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

Q.14 : Prove that $\int_0^1 \left(1-x^{1/n}\right)^m dx = \frac{m!\,n!}{(m+n)!}$, m, n $\in$ N.

[SPPU : May-05, Dec.-11, Marks 4]

Ans. :

Let
$$I = \int_0^1 \left(1 - x^{1/n}\right)^m dx$$

Put $x^{1/n} = t$, $x = t^n$, $dx = n\,t^{n-1}\,dt$

Limits :

x	0	1
t	0	1

$$\therefore \quad I = \int_0^1 (1-t)^m \cdot n \cdot t^{n-1}\,dt$$
$$= n\int_0^1 t^{n-1}(1-t)^m\,dt$$
$$= n\,\beta(n, m+1)$$
$$= n\frac{\Gamma n\,\Gamma m+1}{\Gamma n+m+1}$$
$$= \frac{\Gamma n+1\,\Gamma m+1}{\Gamma m+n+1} = \frac{n!\,m!}{(m+n)!}$$

Q.15 : Show that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

[SPPU : May-08, Dec.-17, Marks 4]

Ans. :

Let
$$I = \int_0^\infty \frac{dx}{1+x^4} \qquad \ldots(1)$$

Put
$$x^2 = \tan\theta,\ x = \sqrt{\tan\theta},$$
$$dx = \frac{1}{2}(\tan\theta)^{-1/2} \cdot \sec^2\theta\,d\theta$$

Limits :

x	0	∞
θ	0	$\pi/2$

Thus the integral becomes

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta}\,\frac{1}{2}\cdot\tan^{-1/2}\theta\cdot\sec^2\theta\cdot d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \tan^{-1/2}\theta\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \cot^{1/2}\theta\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,\cos^{1/2}\theta\,d\theta$$

Use, $\int_0^{\pi/2} \sin^p\theta\,\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$\therefore \quad I = \frac{1}{2}\cdot\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$
$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4}\frac{\Gamma 1/4\,\Gamma 3/4}{\Gamma 1} = \frac{1}{4}\Gamma 1/4\,\Gamma 1-1/4$$
$$= \frac{1}{4}\frac{\pi}{\left(\sin\frac{\pi}{4}\right)} \qquad \left(\because \Gamma p\,\Gamma 1-p = \frac{\pi}{\sin p\pi}\right)$$
$$= \frac{1}{4}\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)}$$
$$\therefore \quad \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Q.16 : Show that $\int_0^{\pi/2} \tan^n x\, dx = \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right)$.

Ans. :

Let
$$I = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx \quad \ldots(1)$$
$$= \frac{1}{2}\beta\left(\frac{1+n}{2}, \frac{1-n}{2}\right)$$
$$= \frac{1}{2}\frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{\Gamma 1}$$
$$= \frac{1}{2}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(1-\left(\frac{1-n}{2}\right)\right)$$
$$= \frac{1}{2}\frac{\pi}{\sin\left(\frac{1+n}{2}\right)\pi} \qquad \because \left(p = \frac{1+n}{2}\right)$$
$$= \frac{\pi}{2\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{\pi}{2\cos\frac{n\pi}{2}}$$
$$I = \frac{\pi}{2}\sec\frac{n\pi}{2}$$

Q.17 : Evaluate $\int_2^5 (x-2)^3(5-x)^2\, dx$.

Ans. :

Let
$$I = \int_2^5 (x-2)^3(5-x)^2\, dx \quad \ldots(1)$$

Here $a = 2, b = 5$

Put $x - 2 = (5-2)t = 3t \Rightarrow dx = 3dt$

x	2	5
t	0	1

$$\therefore \quad I = \int_0^1 (3t)^3(5-3t-2)^2 3dt$$
$$= \int_0^1 3^3 t^3 3^2(1-t)^2 3dt = 3^6\int_0^1 t^3(1-t)^2\, dt$$
$$I = 3^6\beta(4,3) = 3^6\frac{\Gamma 4\,\Gamma 3}{\Gamma 4+3}$$
$$= 3^6 \times \frac{3!\times 2!}{6!} = \frac{24^3}{20}$$

Q.18 : Prove that $\int_{-1}^1 (1+x)^m(1-x)^n\, dx = 2^{m+n+1}\frac{m!\,n!}{(m+n+1)!}$ **where m, n are positive integers.** **[SPPU : Dec.-03, May-09]**

Ans. : Let
$$I = \int_{-1}^1 (1+x)^m(1-x)^n\, dx$$
$$a = -1, \quad b = 1$$

Put $1 + x = 2t$, $dx = 2\, dt$

x	– 1	1
t	0	1

$$\therefore \quad I = \int_0^1 (2t)^m(2-2t)^n \cdot 2\,dt$$
$$= 2^{m+n+1}\int_0^1 t^m(1-t)^n\, dt$$
$$= 2^{m+n+1}\beta(m+1, n+1)$$
$$= 2^{m+n+1}\frac{\Gamma m+1\,\Gamma n+1}{\Gamma m+n+2}$$
$$= 2^{m+n+1}\frac{m!\,n!}{(m+n+1)!}$$

Q.19 : Evaluate $\int_a^b (x-a)^m (b-x)^n \, dx$.

Ans. : Let $I = \int_a^b (x-a)^m (b-x)^n \, dx$

Put $\quad x - a = (b-a)t \Rightarrow dx = (b-a)dt$

x	a	b
t	0	1

$$\therefore \quad I = \int_0^1 [(b-a)t]^m [b-(b-a)t-a]^n (b-a)dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^{m+n} dt$$

$$I = (b-a)^{m+n+1} \beta(m+1, n+1)$$

Q.20 : Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b^m} \beta(m,n)$

[SPPU : May-2000, 12]

Ans. : Let $\quad I = \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

Put $bx = at$, $\therefore \quad x = \frac{at}{b}$, $dx = \frac{a}{b} dt$

x	0	∞
t	0	∞

$$\therefore \quad I = \int_0^\infty \frac{\left(\frac{at}{b}\right)^{m-1}}{(a+at)^{m+n}} \cdot \frac{a}{b} dt$$

$$= \int_0^\infty \frac{1}{a^{m+n}} \frac{a^{m-1+1}}{b^{m-1+1}} \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \frac{1}{a^n b^m} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt = \frac{1}{a^n b^m} \beta(m,n)$$

Q.21 : Evaluate $\int_0^\infty \frac{x^9(1-x^5)}{(1+x)^{25}} dx$.

[SPPU : May-16, Marks 4]

Ans. : Let $\quad I = \int_0^\infty \frac{x^9(1-x^5)}{(1+x)^{25}} dx = \int_0^\infty \frac{x^9 - x^{14}}{(1+x)^{25}} dx$

$$= \int_0^\infty \frac{x^9}{(1+x)^{25}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{25}} dx$$

$$= \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+15}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+10}} dx$$

$$= \beta(10,15) - \beta(15,10)$$

$$I = 0$$

Q.22 : Prove that $\beta(m,m) = 2^{1-2m} \beta\left(m, \frac{1}{2}\right)$.

Ans. : We know that

$$\beta(m,m) = \frac{\Gamma m \, \Gamma n}{\Gamma 2m} \quad \ldots(1)$$

We have, Legendre's duplication formula

$$\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\, \Gamma 2m}{2^{2m-1}}$$

$$\frac{\Gamma m}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1} \Gamma\left(m+\frac{1}{2}\right)} = \frac{\Gamma\frac{1}{2}}{2^{2m-1} \Gamma\left(m+\frac{1}{2}\right)} = \frac{2^{1-2m} \Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \quad \beta(m,m) = \Gamma m \left(\frac{\Gamma m}{\Gamma 2m}\right)$$

$$= \Gamma m \frac{2^{1-2m}\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{1-2m}\frac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)} = 2^{1-2m}\beta\left(m,\frac{1}{2}\right)$$

Memory Map

I) Gamma Function

1) $$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n\, dx$$

2) $$\Gamma 1 = 1,\ \Gamma(n+1) = n\,\Gamma n,\ \Gamma(n+1) = n!$$

if n is natural number

3) $$\Gamma 0 = \infty;\ \Gamma\frac{1}{2} = \sqrt{\pi}$$

4) $$\Gamma n = \int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$$

5) $$\Gamma P\,\Gamma(1-P) = \frac{\pi}{\sin P\pi} \text{ if } 0 < P < 1$$

II) Beta Function

1) $$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$$

2) $$\beta(m,n) = \beta(n,m)$$

3) $$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

4) $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

5) $$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$$

END...

UNIT - III

5 Differentiation under Integral Signs

5.1 : Leibnitz's Rule for Integrals with Constant Limits

1) Statement : Let $I(\alpha) = \int_a^b f(x, \alpha)\, dx$ where α is a parameter and a and b are constants.

Then

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx$$

This is also known as Leibnitz's Rule I.

Proof : Let $\qquad I(\alpha) = \int_a^b f(x, \alpha)\, dx$

Now by using the definition of derivatives by first principle.

$$I'(\alpha) = \lim_{\delta\alpha \to 0} \frac{I(\alpha + \delta\alpha) - I(\alpha)}{\delta\alpha}$$

Substituting $I(\alpha)$ and $I(\alpha + \delta\alpha)$ we get,

$$I'(\alpha) = \lim_{\delta\alpha \to 0} \frac{I}{\delta\alpha}\left[\int_a^b f(x, \alpha + \delta\alpha)\, dx - \int_a^b f(x, \alpha)\, dx\right]$$

$$= \lim_{\delta\alpha \to 0} \int_a^b \frac{f(x, \alpha + \delta\alpha) - f(x, \alpha)}{\delta\alpha}\, dx$$

$$= \lim_{\delta\alpha \to 0} \int_a^b \frac{\delta}{\delta\alpha} f(x, \alpha)\, dx$$

$$I'(\alpha) = \int_a^b \frac{\delta}{\delta\alpha} f(x,\alpha)\ dx$$

Thus

$$\boxed{\frac{d}{d\alpha}\int_a^b f(x,\alpha)\ dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\ dx}$$

i.e. if a and b are constants then the derivative with respect to parameter α outside the definite integral, becomes partial derivative inside the integral.

Q.1 Show that $\int_0^1 \frac{x^a - 1}{\log x} dx = \log(a+1), (a \geq 0)$ **[SPPU : May-04, 13, Dec.-18, Marks 4]**

Ans. :

Let, $$I(a) = \int_0^1 \frac{x^a - 1}{\log x}\ dx \quad \ldots(1)$$

Here a is a parameter and is a variable.

Differentiate both sides w.r.t. a.

$$I'(a) = \frac{d}{da}\int_0^1 \frac{x^a - 1}{\log x}\ dx$$

Apply DUIS. rule, we get

$$I'(a) = \int_0^1 \frac{\partial}{\partial a}\frac{x^a - 1}{\log x}\ dx \qquad \left(\because \frac{\partial}{\partial a} x^a = x^a \log x\right)$$

$$I'(a) = \int_0^1 \frac{x^a \log x}{\log x}\ dx$$

$$I'(a) = \int_0^1 x^a\ dx$$

$$I'(a) = \left[\frac{x^{a+1}}{a+1}\right]_0^1$$

$$I'(a) = \frac{1}{a+1} - 0 = \frac{1}{a+1}$$

Integrating w.r.t. a we get,

$$I(a) = \log(a+1) + c \quad \ldots(2)$$

Substitute suitable value of a to find c.

Put a = 0, in equation (2).

$$I(0) = \log(1) + c$$

$$\int_0^1 \frac{x^0 - 1}{\log x}\ dx = \log 1 + c$$

$$0 = 0 + c$$

$$\Rightarrow \quad c = 0$$

Substituting c in equation (2) we get the value of the integral.

$$\boxed{I(a) = \log(a+1)}$$

Q.2 Prove that

$\int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x} dx = \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$ **[SPPU : Dec.-10, May-15, 19, Marks 4]**

Ans. :

Let, $$I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x}\ dx \quad \ldots(1)$$

Here a is a parameter and x is a variable.

Differentiate both sides w.r.t.a

$$I'(a) = \frac{d}{da}\int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x}\ dx$$

Applying DUIS Rule I we get

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x} - e^{-ax}}{x \sec x}\right) dx$$

$$I'(a) = \int_0^\infty \frac{0 - e^{-ax}(-x)}{x \sec x} dx$$

$$I'(a) = \int_0^\infty e^{-ax} \cos x \, dx$$

$$I'(a) = \left\{\frac{e^{-ax}}{a^2+1}[-a\cos x + \sin x]\right\}_0^\infty$$

$$I'(a) = \left\{0 - \frac{1}{a^2+1}[-a+0]\right\}$$

$$I'(a) = \frac{a}{a^2+1}$$

Integrate w.r.t. a.

$$I(a) = \frac{1}{2}\log\left(a^2+1\right) + c \qquad \ldots(2)$$

Substitute suitable value of a to find c.

Put $a = 1$, in equation (2)

$$I(1) = \frac{1}{2}\log 2 + c$$

$$\int_0^\infty \frac{e^{-x} - e^{-x}}{x \sec x} dx = \frac{1}{2}\log 2 + c$$

$$0 = \frac{1}{2}\log 2 + c$$

$$\Rightarrow \qquad c = \frac{-1}{2}\log 2$$

Substituting c in equation (2), we get the value of the integral.

$$\therefore \qquad I(a) = \frac{1}{2}\log\left(a^2+1\right) - \frac{1}{2}\log 2$$

$$\boxed{I(a) = \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)}$$

Q.3 Evaluate $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ and deduce $\int_0^\infty \frac{\sin x}{x} dx$.

[SPPU : Dec.13, Marks 4]

Ans. :

Let $$I(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} dx$$

x = Variable and a = Parameter

Differentiate w.r.t. a,

$$I'(a) = \frac{d}{da}\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$$

Apply DUIS rule I, we get

$$= \int_0^\infty \frac{\partial}{\partial a} e^{-ax} \frac{\sin x}{x} dx$$

$$= \int_0^\infty e^{-ax}(-x)\frac{\sin x}{x} dx$$

$$= -\int_0^\infty e^{-ax} \sin x \, dx$$

$$= -\left[\frac{e^{-ax}}{a^2+1}(-a\sin x - \cos x)\right]_0^\infty$$

$$= -\left[0 - \frac{1}{a^2+1}(0-1)\right]$$

$$I'(a) = \frac{-1}{a^2+1}$$

Integrating w.r.t. a,

$$I(a) = -\tan^{-1} a + c$$

Let $a \to \infty$,

$$\lim_{a\to\infty} I(a) = 0 = -\frac{\pi}{2} + c$$

$$\therefore \quad \frac{\pi}{2} = c \qquad \left\{ \text{As } \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2} \right\}$$

Thus

$$\boxed{\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - \tan^{-1} a}$$

Put a = 0.

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$

Q.4 Evaluate $\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x} e^{-ax}\right) dx$

[SPPU : May-02]

Ans. :

Let $\quad I(a) = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x} e^{-ax}\right) dx$

x = Variable, a = Parameter,

$$\therefore \quad I'(a) = \frac{d}{da} \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x} e^{-ax}\right) dx$$

∴ By DUIS,

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a} \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x} e^{-ax}\right) dx$$

Differentiating partially w.r.t. a,

$$I'(a) = \int_0^\infty \frac{e^{-x}}{x}\left(1 - 0 + \frac{1}{x} e^{-ax}(-x)\right) dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(1 - e^{-ax}\right) dx$$

By above example

$$I'(a) = \log(a+1)$$

Integrating w.r.t. a, $\left(\because \int \log x \, dx = x \log x - x + c\right)$

$$I(a) = (a+1)\log(a+1) - (a+1) + c$$

Put a = 0. $\quad 0 = 0 - (0+1) + c$

$$\Rightarrow \quad c = 1$$

$$\therefore \quad I(a) = (a+1)\log(a+1) - a$$

Q.5 Show that $\int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2} e^{-2a}$**,**

(a > 0) **[SPPU : May-2000]**

Ans. : Let $\quad I(a) = \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx \quad \ldots (1)$

x = Variable, a = Parameter

$$\therefore \quad I'(a) = \int_0^\infty \frac{\partial}{\partial a} e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx$$

$$= \int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} \cdot \frac{-2a}{x^2} \, dx$$

Put $\frac{a}{x} = t$, $\therefore \frac{-a}{x^2} dx = dt$

x	0	∞
t	∞	0

$$I'(a) = \int_\infty^0 e^{-\left(\frac{a^2}{t^2} + t^2\right)} \cdot 2\,dt$$

$$= -2 \int_0^\infty e^{-\left(t^2 + \frac{a^2}{t^2}\right)} dt$$

$$= -2\, I(a) \qquad \ldots \text{From equation (1)}$$

Thus $\dfrac{I'(a)}{I(a)} = -2$

Integrate w.r.t. a,

$$\log I(a) = -2a + c$$

$\therefore$

$$I(a) = e^{-2a+c}$$

$$I(a) = e^{-2a} \cdot e^{c}$$

Let $e^{c} = A$ say

$$I(a) = A\, e^{-2a} \qquad \dots (2)$$

Put a = 0.

$$I(0) = A$$

Put a = 0 in equation (1).

$$\int_0^{\infty} e^{-x^2}\, dx = A = \frac{\sqrt{\pi}}{2} \qquad \left(\text{Since } \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\right)$$

Thus from equation (2),

$$I(a) = \frac{\sqrt{\pi}}{2}\, e^{-2a}$$

Q.6 Prove that

$\displaystyle\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$ **where n is positive integer.**

[SPPU : May-98, 08, Dec.-08]

Ans. : Let $\displaystyle I(m) = \int_0^1 x^m\, dx$

x = Variable, m = Parameter

$\therefore$

$$\int_0^1 x^m\, dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1$$

$$\int_0^1 x^m\, dx = \frac{1}{m+1}$$

Differentiate w.r.t. m and apply DUIS,

$$\int_0^1 \frac{\partial}{\partial m} x^m\, dx = \frac{-1}{(m+1)^2}$$

$$\int_0^1 x^m \log x\, dx = \frac{-1}{(m+1)^2}$$

Again differentiating w.r.t. m and DUIS,

$$\int_0^1 \frac{\partial}{\partial m} x^m \log x\, dx = \frac{(-1)(-2)}{(m+1)^3}$$

$$\int_0^1 x^m (\log x)^2\, dx = \frac{(-1)^2 \cdot 2!}{(m+1)^3}$$

Hence applying DUIS n times we get,

$$\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n \cdot (n)!}{(m+1)^{n+1}}$$

Q.7 Prove that $\displaystyle\int_0^{\infty} \frac{1}{x^2} \log(1+ax^2)\,dx = \pi\sqrt{a}$ **and hence deduce that** $\displaystyle\int_0^{\infty} \frac{\log(1+x^2)\,dx}{x^2} = \pi.$

[SPPU : Dec.-06, 08, May-10]

Ans. : Let $\displaystyle \phi(a) = \int_0^{\infty} \frac{1}{x^2} \log(1+ax^2)\,dx$

Differentiating w.r.t. a and applying DUIS, we get,

$$\phi'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\log(1+ax^2)}{x^2}\right] dx$$

$$= \int_0^{\infty} \frac{1}{x^2(1+ax^2)}\, x^2\, dx$$

$$= \int_0^{\infty} \frac{1}{1+ax^2}\, dx = \frac{1}{a}\int_0^{\infty} \frac{1}{x^2 + \left(\frac{1}{\sqrt{a}}\right)^2}\, dx$$

$$\phi'(a) = \frac{1}{a}\frac{1}{1/\sqrt{a}}\left[\tan^{-1}(x\sqrt{a})\right]_0^{\infty} = \frac{1}{\sqrt{a}} - \frac{\pi}{2}$$

Integrating, we get

$$\phi(a) = \pi\sqrt{a} + C \quad \cdots (1)$$

To find C, put a = 0 in equation (1) we get,

$$\phi(0) = 0 + c \Rightarrow c = \phi(0) = 0$$

$$\therefore \quad \phi(a) = \int_0^{\infty} \frac{1}{x^2}\log(1+ax^2)dx = \pi\sqrt{a}$$

Put a = 1, we get

$$\phi(1) = \int_0^{\infty} \frac{\log(1+x^2)}{x^2}dx = \pi$$

Q.8 Show that $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}dx = \log\frac{b}{a}$ **[SPPU : Dec.-17, Marks 4]**

Ans. : Let

$$I(a) = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}dx \quad \ldots(1)$$

Here, x is a variable and a and b are two parameters out of these two parameter treat any one as parameter and requiring as constant. Without loss of generality we assume that a is parameter and b is constant.

Differentiate w.r.t. a and apply DUIS,

$$I'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\frac{e^{-ax} - e^{-bx}}{x}dx$$

$$= \int_0^{\infty} \frac{e^{-ax}(-x) - 0}{x}dx$$

$$= -\int_0^{\infty} e^{-ax}dx = -\left[\frac{e^{-ax}}{-a}\right]_0^{\infty}$$

$$= -\left[0 + \frac{1}{a}\right]$$

$$I'(a) = \frac{-1}{a}$$

Integrate w.r.t. a,

$$I(a) = -\log a + c$$

Put a = b.

$$0 = -\log b + c$$

$$\Rightarrow \quad c = \log b$$

Thus $\quad I(a) = -\log a + \log b = \log\frac{b}{a}$

Q.9 Prove that $\int_0^1 \frac{x^a - x^b}{\log x}dx = \log\left[\frac{a+1}{b+1}\right]$, $a > 0, b > 0$.

[SPPU : May-09, 17, Marks 4]

Ans. : Let

$$I(a) = \int_0^1 \frac{x^a - x^b}{\log x}dx \quad \ldots(1)$$

Here x is a variable and assume that a is parameter and b is constant.

$\therefore$ Differentiating w.r.t. a and applying DUIS rule we get,

$$I'(a) = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a - x^b}{\log x}\right]dx$$

$$= \int_0^1 \frac{x^a \log x}{\log x}dx = \int_0^1 x^a dx$$

$$I'(a) = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$$

Integrating we w.r.t. a, we get

$$I(a) = \log(a+1) + c \quad \ldots (1)$$

Putting a = b, we get

$$I(0) = \log(b+1) + c$$

$$\Rightarrow \quad c = -\log(b+1)$$

$\therefore$ Equation (1) becomes

$$I(a) = \log(a+1) - \log(b+1)$$

$$I(a) = \log\left(\frac{a+1}{b+1}\right)$$

5.2 : Leibnitz's Rule with Limits as Functions of Parameter Statement

Rule II : Let $I(\alpha) = \int_a^b f(x,\alpha)\, dx$

where α is a parameter and a and b are functions of parameter α i.e. $a = f_1(\alpha)$ and $b = f_2(\alpha)$.

Then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\, dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

Proof : Let $I(\alpha) = \int_a^b f(x,\alpha)\, dx$

As a and b are functions of α.

$\therefore I \to a, b, \alpha \to \alpha$

$$\therefore \quad \frac{dI}{d\alpha} = \frac{\partial I}{\partial\alpha}\cdot\frac{d\alpha}{d\alpha} + \frac{\partial I}{\partial a}\cdot\frac{da}{d\alpha} + \frac{\partial I}{\partial b}\cdot\frac{db}{d\alpha} \quad \ldots (1)$$

$\frac{\partial I}{\partial\alpha}$ is obtained by treating a, b constants.

Thus $$\frac{\partial I}{\partial\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\, dx$$

Let $$\int f(x,\alpha)\, dx = \phi(x,\alpha)$$

$$\therefore \quad \frac{\partial}{\partial x}\phi(x,\alpha) = f(x,\alpha) \quad \ldots (2)$$

Thus $$I(\alpha) = \int_a^b f(x,\alpha)\, dx$$

$$= [\phi(x,\alpha)]_a^b = \phi(b,\alpha) - \phi(a,\alpha)$$

Thus $$\frac{\partial I}{\partial a} = \frac{\partial}{\partial a}[\phi(b,\alpha) - \phi(a,\alpha)]$$

$$= 0 - \frac{\partial}{\partial a}\phi(a,\alpha) = -f(a,\alpha)$$

... From equation (2)

and $$\frac{\partial I}{\partial b} = \frac{\partial}{\partial b}[\phi(b,\alpha) - \phi(a,\alpha)]$$

$$= \frac{\partial}{\partial b}\phi(b,\alpha) - 0 = f(b,\alpha)$$

... From equation (2)

Substituting the values of I(a), $\frac{\partial I}{\partial a}, \frac{\partial I}{\partial b}$ in equation (1)

$$\boxed{\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\, dx + \frac{db}{d\alpha}\, f(a,\alpha) - \frac{da}{d\alpha}\, f(b,\alpha)}$$

Notes :

1) If a and b are functions of α then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \left[\begin{matrix}\text{Partial derivative} \\ \text{of integrated w.r.t } \alpha\end{matrix}\right] dx + \left[\begin{matrix}\text{Value of the function} \\ \text{at upper limit b}\end{matrix}\right]\cdot\left[\begin{matrix}\text{Derivative of upper} \\ \text{limit b w.r.t } \alpha\end{matrix}\right] - \left[\begin{matrix}\text{Value of the function} \\ \text{at lower limit a}\end{matrix}\right]\cdot\left[\begin{matrix}\text{Derivative of lower} \\ \text{limit a w.r.t } \alpha\end{matrix}\right]$$

2) If a is a function of parameter of b is a constant then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\, dx - \frac{da}{d\alpha} f(a,\alpha)$$

3) If a is constant and b is a function of α then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\, dx + f(b,\alpha)\frac{db}{d\alpha}$$

4) If a and b are constants then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{d}{d\alpha}[f(x,\alpha)]dx$$

Thus : Rule (1) is the particular case of rule II

5) To verify Leibnitz's rule of DUIS for

$$I(\alpha) = \int_a^b f(x,\alpha)]dx =$$

Consider the following steps :

Step 1 : Find $\frac{dI}{d\alpha}$ by actual integration and differentiation.

Step 2 : Find $\frac{dI}{d\alpha}$ by DUIS rule.

Step 3 : Show that $\frac{dI}{d\alpha}$ from step 1 and step 2 are equal.

Q.10 Verify Leibnitz rule of DUIS for the integral $\int_a^{a^2} \frac{dx}{x+a}$.

[SPPU : Dec.-01, 09]

Ans. : Step 1 : By direct method :

Let
$$I(a) = \int_a^{a^2} \frac{dx}{x+a} = [\log(x+a)]_a^{a^2}$$

$$= \log\left(a+a^2\right) - \log 2a = \log\frac{a\,(a+1)}{2a}$$

$$I(a) = \log\frac{a+1}{2}$$

Thus
$$I'(a) = \frac{1}{a+1} \quad \text{... (1)}$$

Now again
$$I(a) = \int_a^{a^2} \frac{dx}{x+a}$$

Step 2 : By DUIS Rule : Here x = Variable and a is parameter.

$$I'(a) = \int_a^{a^2} \frac{\partial}{\partial a}\frac{1}{x+a}\,dx + \left(\frac{d}{da}a^2\right)\cdot\left(\frac{1}{a+a^2}\right) - \left(\frac{d}{da}a\right)\cdot\left(\frac{1}{a+a}\right)$$

$$= \int_a^{a^2} \frac{-1}{(x+a)^2}\,dx + \frac{2a}{a+a^2} - \frac{1}{2a}$$

$$= \left(\frac{1}{x+a}\right)_a^{a^2} + \frac{2}{a+1} - \frac{1}{2a}$$

$$= \frac{1}{a^2+a} - \frac{1}{2a} + \frac{2}{a+1} - \frac{1}{2a}$$

Simplifying we get,

$$= \frac{1}{a+1} \quad \text{... (2)}$$

Step 3 : From equation (1) and (2), DUIS is verified.

Q.11 Verify the rule of differentiation under integral sign for the integral $\int_0^{a^2} \tan^{-1}\frac{x}{a}\cdot dx$.

Ans. :

Let
$$I(a) = \int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx$$

Step 1 : By direct method :

$$I(a) = \int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right)\cdot 1\cdot dx$$

x is variable and a is parameter

Use integration by parts.

$$= \left[x\cdot\tan^{-1}\frac{x}{a}\right]_0^{a^2} - \int_0^{a^2} \frac{a}{x^2+a^2}\cdot x\,dx$$

$$= \left(a^2 \tan^{-1} a - 0\right) - \frac{a}{2}\left[\log\left(x^2 + a^2\right)\right]_0^{a^2}$$

$$= a^2 \tan^{-1} a - \frac{a}{2}\log\left(\frac{a^4 + a^2}{a^2}\right)$$

$$I(a) = a^2 \tan^{-1} a - \frac{a}{2}\log\left(a^2 + 1\right)$$

Differentiating w.r.t. a we get,

$$I'(a) = \frac{-1}{2}\log\left(a^2 + 1\right) + 2a \tan^{-1} a \quad \text{... (1)}$$

Step 2 : By DUIS rule

$$I'(a) = \int_0^{a^2} \frac{\partial}{\partial a}\tan^{-1}\left(\frac{x}{a}\right) dx$$

$$+ \frac{d}{da}\left(a^2\right)\cdot \tan^{-1}\left(\frac{a^2}{a}\right) - 0$$

$$= \int_0^{a^2} \frac{1}{1+\left(\frac{x}{a}\right)^2}\left(\frac{-x}{a^2}\right) dx + 2a \tan^{-1} a$$

$$= -\int_0^{a^2} \frac{x}{x^2 + a^2} dx + 2a \tan^{-1} a$$

$$= \frac{-1}{2}\left[\log\left(x^2 + a^2\right)\right]_0^{a^2} + 2a \tan^{-1} a$$

$$= \frac{-1}{2}\left[\log\left(\frac{a^4 + a^2}{a^2}\right)\right] + 2a \tan^{-1} a$$

Thus $$I'(a) = \frac{-1}{2}\log\left(a^2 + 1\right) + 2a \tan^{-1} a \quad \text{... (2)}$$

Step 3 : From equations (1) and (2) the rule of differentiation under integral sign is verified.

Q.12 If $f(x) = \int_0^x (x - t)^2\, G(t)\, dt$ then prove that $\frac{d^3 f}{dx^3} = 2G(x)$.

[SPPU : May-18, Marks 4]

Ans. : Given that,

$$f(x) = \int_0^x (x - t)^2\, G(t)\, dt \quad \text{... (1)}$$

Here, t is variable and x is parameter,

$$\frac{df}{dx} = \frac{d}{dx}\int_0^x (x - t)^2\, G(t)\, dt$$

By DUIS,

$$= \int_0^x \frac{\partial}{\partial x}(x-t)^2 G(t)\, dt$$

$$+ \frac{dx}{dx}(0) - \frac{d}{dx}(0)\cdot t^2 G(t)\, dt$$

$$\frac{df}{dx} = \int_0^x 2(x - t)\, G(t)\, dt$$

Again by DUIS,

$$\frac{d^2 f}{dx^2} = \int_0^x \frac{\partial}{\partial x} 2(x-t)G(t)\, dt + 0 - 0$$

$$\frac{d^2 f}{dt^2} = \int_0^x 2\, G(t)\, dt$$

Again by DUIS,

$$\frac{d^3 f}{dx^3} = \int_0^x \frac{\partial}{\partial x} 2G(t)\, dt + \frac{dx}{dx}\cdot 2G(x) - 0$$

$$= 0 + 2\, G(x) - 0$$

$$\therefore \quad \frac{d^3 f}{dx^3} = 2\, G(x)$$

Q.13 If $y = \int_0^x f(t) \sin a(x-t)\, dt$. Show that $\frac{d^2y}{dx^2} + a^2 y = a \cdot f(x)$

[SPPU : May-99, 08, Dec.-2000]

Ans. : We have,

$$y = \int_0^x f(t) \cdot \sin a(x-t)\ dt$$

Here t is variable and x is parameter.

Applying DUIS,

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x} f(t) \sin a(x-t)\, dt + \left(\frac{dx}{dx}\right) \cdot f(x) \sin a(x-x) - \left(\frac{d0}{dx}\right) \cdot f(0) \cdot \sin a(x-0)$$

$$\frac{dy}{dx} = \int_0^x a \cdot f(t) \cos a(x-t)\, dt + 0 - 0$$

Differentiating again and applying DUIS,

$$\frac{d^2y}{dx^2} = \int_0^x \frac{\partial}{\partial x} a \cdot f(t) \cdot \cos a(x-t)\, dt + \left(\frac{dx}{dx}\right) \cdot a \cdot f(x) \cdot \cos a(x-x) - \left(\frac{d0}{dx}\right) \cdot a \cdot f(0) \cdot \cos a\ (x-0)$$

$$= \int_0^x -a^2 f(t) \sin a(x-t)\, dt + a\, f(x) - 0$$

i.e.
$$\frac{d^2y}{dx^2} = -a^2 y + a\, f(x)$$

i.e.
$$\frac{d^2y}{dx^2} + a^2 y = a\, f(x)$$

Q.14 Assuming $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ prove that $k^2 F(y) - \frac{d^2F}{dy^2} = \frac{\pi}{2}$ where, $F(y) = \int_0^\infty \frac{\sin xy}{x(k^2+x^2)} dx$

[SPPU : Dec.-05]

Ans. : Given that $F(y) = \int_0^\infty \frac{\sin xy}{x(k^2+x^2)} dx$

Here x is variable and y is parameter.

Differentiating w.r.t. y and using DUIS, we get

$$\frac{dF}{dy} = F'(y) = \int_0^\infty \frac{1}{x(k^2+x^2)} \frac{\partial}{\partial y} (\sin xy)\, dx$$

$$= \int_0^\infty \frac{1}{x(k^2+x^2)} x \cos xy\ dx$$

$$\frac{dF}{dx} = \int_0^\infty \frac{\cos (xy)}{k^2+x^2} dx, \text{ applying DUIS,}$$

$$\frac{d^2F}{dx^2} = \int_0^\infty \frac{1}{k^2+x^2} \frac{\partial}{\partial y} (\cos xy)\, dx$$

$$= \int_0^\infty \frac{-x \sin xy}{k^2+x^2} dx = \int_0^\infty \frac{-x^2}{x(k^2+x^2)} \sin xy\, dx$$

$$= \int_0^\infty \frac{-(x^2+k^2-k^2) \sin xy}{x(k^2+x^2)} dx$$

$$= \int_0^\infty \frac{\sin xy}{x} dx + k^2 \int_0^\infty \frac{\sin xy}{x(k^2+x^2)} dx = -\frac{\pi}{2} + k^2 F(y)$$

$$\Rightarrow \quad \frac{d^2F}{dx^2} - k^2 F(y) = -\frac{\pi}{2}$$

$$\Rightarrow \quad k^2 F(y) - \frac{d^2F}{dx^2} = \frac{\pi}{2}$$

Which is the required result.

Memory Map

1) If a and b are functions of α then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \begin{bmatrix}\text{Partial derivative} \\ \text{of integrated w.r.t } \alpha\end{bmatrix} dx + \begin{bmatrix}\text{Value of the function} \\ \text{at upper limit b}\end{bmatrix} \cdot \begin{bmatrix}\text{Derivative of upper} \\ \text{limit b w.r.t } \alpha\end{bmatrix} - \begin{bmatrix}\text{Value of the function} \\ \text{at lower limit a}\end{bmatrix} \cdot \begin{bmatrix}\text{Derivative of lower} \\ \text{limit a w.r.t } \alpha\end{bmatrix}$$

2) If a is a function of parameter of b is a constant then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\,dx - \frac{da}{d\alpha} f(a,\alpha)$$

3) If a is constant and b is a function of α then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha}[f(x,\alpha)]\,dx + f(b,\alpha)\frac{db}{d\alpha}$$

4) If a and b are constants then

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{d}{d\alpha}[f(x,\alpha)]dx$$

Thus : Rule (1) is the particular case of rule II.

5) To verify Leibnitz's rule of DUIS for

$$I(\alpha) = \int_a^b f(x,\alpha)]dx$$

Step 1 : Find $\frac{dI}{d\alpha}$ by actual integration and differentiation.

Step 2 : Find $\frac{dI}{d\alpha}$ by DUIS rule

Step 3 : Show that $\frac{dI}{d\alpha}$ from step 1 and step 2 are equal.

END...

UNIT - III

6 Error Functions

6.1 : Error Function

An error function of x is denoted by erf(x) and defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du \quad \ldots(1)$$

It is used in normal probability distribution.

For examples : $\operatorname{erf}(3) = \frac{2}{\sqrt{\pi}}\int_0^3 e^{-u^2}\,du$

$$\operatorname{erf}(-5) = \frac{2}{\pi}\int_0^{-5} e^{-u^2}\,du$$

6.2 : Complementary Error Function

A complementary error function of x is defined as

$$\operatorname{erf}_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du \quad \ldots(2)$$

For examples : $\operatorname{erf}_c(4) = \frac{2}{\sqrt{\pi}}\int_4^\infty e^{-u^2}\,du$

$$\operatorname{erf}_c(0) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du$$

6.3 : Properties of Error Functions

Property 1 : Alternate forms

Put $u^2 = t$ in equation (1), we get

$$u = \sqrt{t},\ du = \frac{1}{2\sqrt{t}}dt$$

$$\therefore \quad erf(x) = \frac{2}{\sqrt{\pi}}\int_0^{x^2} e^{-t}\frac{1}{2\sqrt{t}}dt = \frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}t^{-\frac{1}{2}}dt$$

and

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}\frac{1}{2\sqrt{t}}dt = \frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}t^{-\frac{1}{2}}dt$$

Property 2 : i) $erf(0) = 0,\ erf_c(\infty) = 0$

Proof we have, $erf(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-u^2}du = 0$

$$erf_c(\infty) = \frac{2}{\sqrt{\pi}}\int_{\infty}^{\infty} e^{-u^2}du = 0$$

Property 3 : $erf(\infty) = 1$ and $erf_c(0) = 1$

Proof : We have by property I

$$erf(x) = \frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}t^{-\frac{1}{2}}dt$$

$$\therefore \quad erf(\infty) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt = \frac{1}{\sqrt{\pi}}\left\lceil -\frac{1}{2}+1\right. = \frac{1}{\sqrt{\pi}}\left\lceil\frac{1}{2}\right. = \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$erf(\infty) = 1$$

Now, $erf_c(0) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt = 1$

Property 4 : $erf(x) + erf_c(x) = 1$

Proof : Consider L.H.S. $= erf(x) + erf_c(x) = 1$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du + \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}du = erf(\infty) = 1$$

Hence, $erf(x) + erf_c(x) = 1$

Property 5 : erf(x) is an odd function

i.e. $erf(-x) = -erf(x)$

Proof :

We have, $$erf(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-u^2}du \qquad \ldots(1)$$

Put, $u = -v \Rightarrow du = -dv$

u	0	− x
v	0	x

$$\therefore \quad erf(-x) = \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-v^2}(-dv)$$

$$erf(-x) = -\frac{2}{\sqrt{\pi}}\int_0^{x} e^{-v^2}dv = -erf(x)$$

Thus, error function is an odd function.

Property 6 : Expansion of erf(x) in powers of x

$$erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]$$

Proof : We known that,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$$

$$\therefore \quad e^{-u^2} = 1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \frac{u^8}{4!} - \frac{u^{10}}{5!} + \ldots$$

Now, $$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x\left[1-u^2+\frac{u^4}{2}-\frac{u^6}{6}+\frac{u^8}{24}-\ldots\right]du$$

$$= \frac{2}{\sqrt{\pi}}\left[u-\frac{u^3}{3}+\frac{u^5}{10}-\frac{u^7}{42}+\frac{u^9}{216}-\ldots\right]_0^x$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\left[x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}+\frac{x^9}{216}-\ldots\right]$$

Property 7 : Prove that, $\int_{-a}^{a} \text{erf}(t)dt = 0$

Proof : We know that

$$\int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is an odd function}$$

$$= 2\int_0^a f(x)dx \qquad \text{if } f(x) \text{ is an even function}$$

We have, erf(x) is an odd function

$$\therefore \qquad \int_{-a}^{a} \text{erf}(x)\,dx = 0$$

Property 8 : Graphs of error functions :

a) Graph of error function

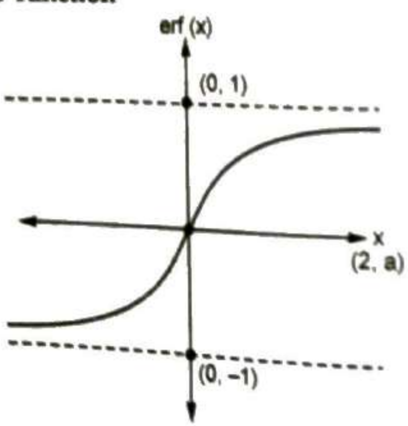


Fig. 1

b) Graph of complementary error function

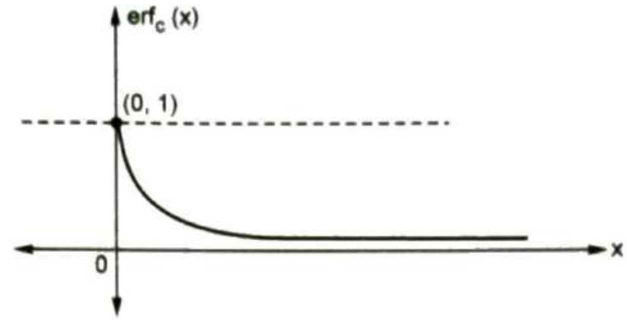


Fig. 2

Q.1 : Show that

$$\int_a^b e^{-x^2}dx = \frac{\sqrt{p}}{2}\,[erf\,(b) - erf\,(a)]$$

[SPPU : May-07, 14, Marks 4]

Ans. : We know that erf $(\infty) = 1$

$$\therefore \qquad 1 = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx$$

$$\therefore \qquad 1 = \frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-x^2}dx\right]$$

Splitting the above integral in three parts we get

$$1 = \frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-x^2}dx + \int_a^b e^{-x^2}dx + \int_b^\infty e^{-x^2}dx\right]$$

Using the definitions, we get,

$$1 = \text{erf}(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}dx + \text{erf}_c(b)$$

$$[1-\text{erf}_c(b)] = \text{erf}(a)+\frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}\,dx$$

$$\text{erf}(b) = \text{erf}(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}\,dx$$

$$\therefore \quad \text{erf}(b) - \text{erf}(a) = \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}dx$$

$$\therefore \quad \int_a^b e^{-x^2}dx = \frac{\sqrt{\pi}}{2}[\text{erf}(b) - \text{erf}(a)]$$

Q.2 : Show that
$\int_0^\infty e^{-x^2 - 2ax}dx = \frac{\sqrt{\pi}}{2} e^{a^2} [1 - erf(a)]$

[SPPU : Dec.-90, 95, 02, 09, 15, May-92, 94, 98, 01, 04, 05, 12, Marks 4]

Ans. :

$$\text{L.H.S.} = \int_0^\infty e^{-x^2 - 2ax - a^2 + a^2}dx$$

$$= e^{a^2}\int_0^\infty e^{-(x+a)^2}dx$$

Put $\quad x + a = u, dx = du$

x	0	∞
u	a	∞

$$\therefore \quad \text{L.H.S.} = e^{a^2}\int_a^\infty e^{-u^2}\,du = e^{a^2}\,\frac{\sqrt{\pi}}{2}\,[\text{erf}_c(a)]$$

$$= e^{a^2}\,\frac{\sqrt{\pi}}{2}\,[1 - \text{erf}(a)]$$

$$(\because\ \text{erf}_c(x) + \text{erf}(x) = 1)$$

Q.3 : If $\alpha(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2/2}dt$ **show that** $erf(x) = \alpha\,[x\sqrt{2}]$.

[SPPU : Dec.-03, 06, 10]

Ans. : From the definition of $\alpha(x)$, we can write $\alpha(x\sqrt{2})$

$$\therefore \quad \alpha(x\sqrt{2}) = \sqrt{\frac{2}{\pi}}\int_0^{x\sqrt{2}} e^{-t^2/2}\,dt$$

Put $\quad \frac{t^2}{2} = u^2,\ t = \sqrt{2}\,u, \quad dt = \sqrt{2}\,du$

t	0	$x\sqrt{2}$
u	0	x

$$\therefore \quad \alpha(x\sqrt{2}) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2}\sqrt{2}\,du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = \text{erf}(x)$$

Q.4 : Prove that $erf_c(-x) + erf_c(x) = 2$

[SPPU : Dec.-91, 93, 04, 07, 08, May-2000]

Ans. : We know that

$$\text{erf}(x) + \text{erf}_c(x) = 1 \quad \text{... (1)}$$

The above property is true for $-x$ also.

$$\therefore \quad \text{erf}(-x) + \text{erf}_c(-x) = 1$$

but $\quad \text{erf}(-x) = -\text{erf}(x)$

$$\therefore \quad -\text{erf}(x) + \text{erf}_c(-x) = 1 \quad \text{... (2)}$$

Adding equation (1) and (2) we get

$$\text{erf}_c(-x) + \text{erf}_c(x) = 2$$

Q.5 : Show that $\frac{d}{dx}\,erf(ax) = \frac{2a}{\sqrt{\pi}}\,e^{-a^2x^2}$ **and evaluate** $\int_0^t erf(ax)\,dx$.

[SPPU : May-07]

Ans. :

$$\text{erf}(ax) = \frac{2}{\sqrt{\pi}}\,\frac{d}{dx}\int_0^{ax} e^{-u^2}du$$

$$\frac{d}{dx}\operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax} e^{-u^2}du, \text{ (using DUIS)}$$

$$= \frac{2}{\sqrt{\pi}}\left\{\int_0^{ax}\frac{\partial}{\partial x}e^{-u^2}du + \left(\frac{d}{dx}ax\right) e^{-a^2x^2} - \left(\frac{d}{dx}0\right)e^{-0}\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + a\cdot e^{-a^2x^2} - 0\right\}$$

$$= \frac{2a}{\sqrt{\pi}}e^{-a^2x^2} \qquad \dots (1)$$

Now $\int_0^t \operatorname{erf}(ax)\cdot 1\cdot dx$

Use integration by parts.

$$= \left\{[x\operatorname{erf}(ax)]_0^t - \int_0^t\left[\frac{d}{dx}\operatorname{erf}(ax)\right]\cdot x\cdot dx\right\}$$

$$= \left\{[t\operatorname{erf}(at) - 0] - \int_0^t\left[\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}\right]\cdot x\,dx\right\}$$

$$= \left\{[t\operatorname{erf}(at)] + \frac{1}{a\sqrt{\pi}}\int_0^t e^{-a^2x^2}(-2a^2x)dx\right\}$$

$$\left[\because \int e^{f(x)}f'(x)\,dx = e^{f(x)}\right]$$

$$= t\operatorname{erf}(at) + \frac{1}{a\sqrt{\pi}}\left[e^{-a^2x^2}\right]_0^t$$

$$= t\operatorname{erf}(at) + \frac{1}{a\sqrt{\pi}}\left[e^{-a^2t^2} - 1\right]$$

Q.6 : Find $\frac{d}{dx}\operatorname{erf}(ax^n)$

[SPPU : Dec.-06, 07, 08, May-04]

Ans. : $$\frac{d}{dx}\operatorname{erf}(ax^n) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax^n}e^{-u^2}du$$

By rule of DUIS,

$$= \frac{2}{\sqrt{\pi}}\left\{\int_0^{ax^n}\frac{\partial}{\partial x}e^{-u^2}du + \left(\frac{d}{dx}ax^n\right)e^{-a^2x^{2n}} - \left(\frac{d}{dx}0\right)e^{-0}\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + n\,a\,x^{n-1}\,e^{-a^2x^{2n}} - 0\right\}$$

$$= \frac{2an\,x^{n-1}}{\sqrt{\pi}}e^{-a^2x^{2n}}$$

Q.7 : Show that $\frac{d}{dt}(\operatorname{erf}\sqrt{t}) = \frac{e^{-t}}{\sqrt{\pi t}}$ and hence evaluate $\int_0^\infty e^{-t}\operatorname{erf}(\sqrt{t})\,dt$.

[SPPU : May-10]

Ans. : We have,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}du$$

Differentiating w.r.t. t and using DUIS, we get

$$\frac{d}{dt}(\operatorname{erf}\sqrt{t}) = \frac{2}{\sqrt{\pi}}\left[\int_0^{\sqrt{t}}\frac{\partial}{\partial t}(e^{-u^2})du + \frac{d}{dt}(\sqrt{t})e^{-(\sqrt{t})^2} - 0\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + \frac{1}{2\sqrt{t}}e^{-t}\right] = \frac{e^{-t}}{\sqrt{\pi t}}$$

Now, let $I = \int_0^\infty e^{-t}\operatorname{erf}(\sqrt{t})\,dt$

Integrating by parts, we get

$$I = \left[\operatorname{erf}\sqrt{t}\,\frac{e^{-t}}{-1}\right]_0^\infty - \int_0^\infty \frac{e^{-t}}{-1}\frac{d}{dt}(\operatorname{erf}\sqrt{t})\,dt$$

$$= 0 + \int_0^\infty e^{-t}\frac{e^{-t}}{\sqrt{\pi t}}\,dt$$

$$= \int_0^\infty \frac{e^{-2t}\, t^{-1/2}\, dt}{\sqrt{\pi}}, \text{ put } y = 2t,\ dy = 2\ dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} \left(\frac{y}{2}\right)^{-\frac{1}{2}} \frac{dy}{2} = \frac{1}{\sqrt{2}\sqrt{\pi}} \int_0^\infty e^{-y}\, y^{-1/2} dy$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} \overline{\left|\frac{1}{2}\right.} = \frac{1}{\sqrt{2}\sqrt{\pi}} \sqrt{\pi} = \frac{1}{\sqrt{2}}$$

Q.8 : Show that $\int_0^\infty e^{-st}\, erf\sqrt{t}\ dt = \frac{1}{s\sqrt{s+1}}$ **[SPPU : May-03, 05]**

Ans. : Solving the integral by parts

$$\text{L.H.S.} = \left(\text{erf}\sqrt{t} \cdot \frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{d}{dt}\ \text{erf}\ \sqrt{t} \cdot \frac{e^{-st}}{-s}\ dt$$

$$= 0 + \frac{1}{s} \int_0^\infty \frac{e^{-t}}{\sqrt{\pi t}} \cdot e^{-st} dt \qquad \left\{\text{As } \frac{d}{dt}\ \text{erf}\sqrt{t} = \frac{e^{-t}}{\sqrt{\pi t}}\right\}$$

$$= \frac{1}{s} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(s+1)t}}{\sqrt{t}}\ dt$$

Put $(s+1)\ t = u,\ t = \frac{u}{s+1}$

$$dt = \frac{du}{s+1} = \frac{1}{s} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{\frac{u}{s+1}}} \cdot \frac{du}{s+1}$$

$$= \frac{1}{\sqrt{\pi}\ s\ \sqrt{s+1}} \int_0^\infty e^{-u} u^{-1/2} du$$

$$= \frac{1}{\sqrt{\pi}\ s\ \sqrt{s+1}} \overline{|1/2} \quad \text{... (Using Gamma function)}$$

$$= \frac{1}{s\ \sqrt{s+1}} \qquad (\text{As } \overline{|1/2} = \sqrt{\pi}.)$$

Q.9 : Prove that $\frac{1}{x} \frac{d}{da} erf_c(ax) = \frac{-1}{a} \frac{d}{dx} erf(ax)$ **[SPPU : May-08]**

Ans. : We know that

$$\frac{d}{da}\ \text{erf}_c\ (ax) = \frac{2}{\sqrt{\pi}} \frac{d}{da} \int_{ax}^\infty e^{-u^2}\ du.$$

By rule of DUIS,

$$= \frac{2}{\sqrt{\pi}} \left\{ \int_{ax}^\infty \frac{\partial}{\partial a} e^{-u^2} du + \left(\frac{d}{da} \infty\right) \cdot e^{-\infty} - \left(\frac{d}{da} ax\right) e^{-a^2x^2} \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{0 + 0 - x\ e^{-a^2x^2}\right\} = \frac{-2x}{\pi}\ e^{-a^2x^2}$$

$$\therefore \quad \text{L.H.S.} = \frac{1}{x} \cdot \frac{d}{da}\ \text{erf}_c(ax) = \frac{-2}{\sqrt{\pi}}\ e^{-a^2x^2} \quad \text{... (1)}$$

Also $\frac{d}{dx} \text{erf}(ax) = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^{ax} e^{-u^2} du$

By rule of DUIS,

$$= \frac{2}{\sqrt{\pi}} \left\{ \int_0^{ax} \frac{\partial}{\partial x} e^{-u^2} du + \left(\frac{d}{dx} ax\right) \cdot e^{-a^2x^2} - \left(\frac{d}{dx} 0\right) e^0 \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{0 + a\ e^{-a^2x^2} - 0\right\} = \frac{2a}{\sqrt{\pi}}\ e^{-a^2x^2}$$

$$\therefore \quad \text{R.H.S.} = \frac{-1}{a} \frac{d}{dx}\ \text{erf}(ax)$$

$$= \frac{-1}{a} \cdot \left[\frac{2a}{\sqrt{\pi}}\ e^{-a^2x^2}\right]$$

$$-\frac{1}{a} \frac{d}{dx} \text{erf}\ (ax) = \frac{-2}{\sqrt{\pi}}\ e^{-a^2x^2} \quad \text{... (2)}$$

From equations (1) and (2) L.H.S. = R.H.S.

Memory Map

1) $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{D}^{x} e^{-u^2}\, du$$

2) $$\operatorname{erf}_c(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2}\, du$$

3) $$\operatorname{erf}(0) = 0, \operatorname{erf}_c(\infty) = 0$$

4) $$\operatorname{erf}(\infty) = 1, \operatorname{erf}_c(O) = 1$$

5) $$\operatorname{erf}(x) + \operatorname{erf}_c(x) = 1$$

6) erf (x) is an odd function.

7) $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]$$

8) $$\int_{-a}^{a} \operatorname{erf}(t)\, dt = 0$$

END...

UNIT - IV

7 Curve Tracing

7.1 : Basic Definitions

1. Monotonic functions :

Let f(x) be function defined on an interval I and let $x_1, x_2 \in I$. Then

i) If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.

ii) If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

iii) A function which is either increasing or decreasing on an interval I is called **monotonic function** on I.

2. Monotonic fuctions by derivatives :

Let f be continuous on [a, b] and differentiable on (a, b) then,

i) A function f is said to be **increasing** on [a, b] if $f'(x) > 0$; $\forall$ $x \in (a, b)$.

ii) A function f is said to be **decreasing** on [a, b] if $f'(x) < 0$; $\forall$ $x \in (a, b)$.

3. Concave up :

- The graph of a differentiable function y = f(x) is said to be **concave up** on an open interval I if $f'(x)$ is increasing on I or $f''(x) > 0$; $\forall$ $x \in I$. OR
- The curve is said to be concave up at A. If the portion of the curve on both sides of A lies above the tangent at A.
- It is also known as **convex down**.

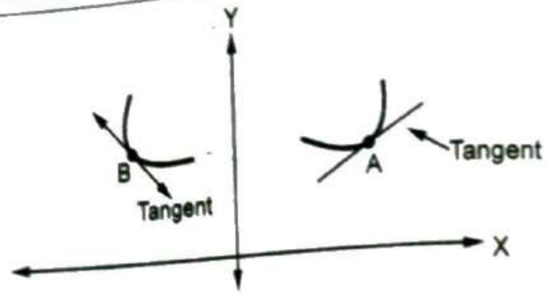


Fig. 7.1

4. Concave down :

- The graph of a differentiable function $y = f(x)$ is said to be **concave down** on an open interval I if $f'(x)$ is decreasing on I or $f''(x) < 0, \forall\ x \in I$. OR
- The curve is said to be concave down at A, If the portion of the curve on both sides of A lies below the tangent at A. It is also known as **convex up**

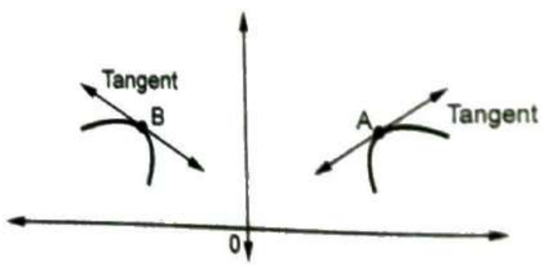


Fig. 7.2

5. Point of inflection :

- A point where the graph of a function has tangent line and where the graph changes form concavity up to concavity down or vice versa is known as a **point of inflection.**

OR

- A point on a curve where $f''(x)$ is positive on one side and negative on the other side is known as **point of inflection**. At such point $f''(x)$ is either zero or undefined.

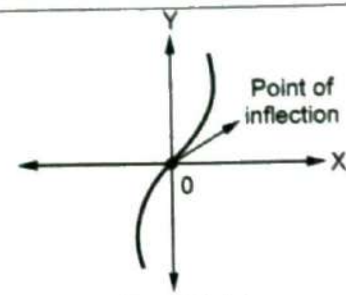


Fig. 7.3 (a)

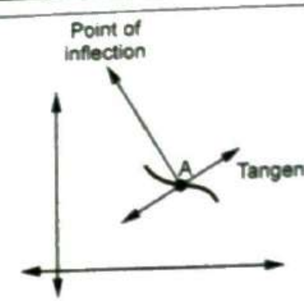


Fig. 7.3 (b)

Note :

i) An inflection point may not exist where $y'' = 0$ e.g. If $y = mx + c$ then $y'' = 0 \therefore x = 0$ is not an inflection point of y because y'' does not change sign.

ii) An inflection point may occur where y'' does not defined

6. Multiple point :

- A point through which more than one branches of curve pass, is called a **multiple point** of the curve.
- A point on a curve is called a **double point**, if two branches of the curve pass through it.
- If three branches of curve pass through a point, then such point is known as **triple point**.
- If r branches pass through a point, then such point is called a **multiple point of order r**.

7. Node :

A double point is called **a node** if distinct branches have distinct real tangents.

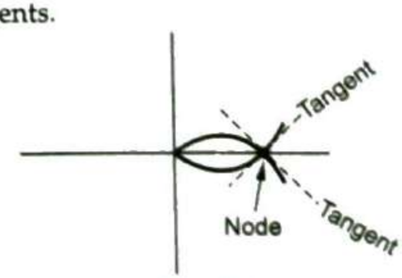


Fig. 7.4

8. Cusp :

A double point is called **a cusp** point if two branches have a common tangent at that point.

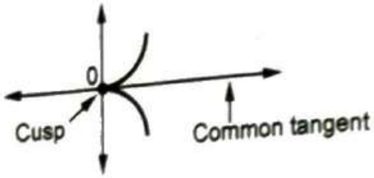


Fig. 7.5

9. Isolated point (Conjugate point) :

A point P is called a **isolated point or conjugate point** if the co-ordinates of P satisfies the equation of the curve but branches do not pass through P.

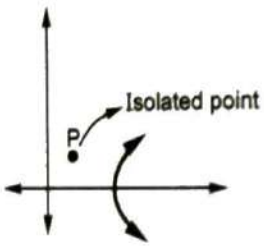


Fig. 7.6

10. Method for finding double points :

- The necessary and sufficient conditions for any point (x, y) on $f(x, y) = 0$ to be a double point are that $f_x(x, y) = 0$, $f_y(x, y) = 0$.
- Thus double points are obtained by solving

$$f_x(x, y) = 0, \; f_y(x, y) = 0, \; f(x, y) = 0$$

The slope of tangents at double point are the roots of

$$f_{yy}\left(\frac{dy}{dx}\right)^2 + 2 f_{xy} \frac{dy}{dx} + f_{xx} = 0$$

Thus a double point (x, y) will be

i) a node if $(f_{xy})^2 - f_{xx} f_{yy} > 0$

ii) a cusp if $(f_{xy})^2 - f_{xx} f_{yy} = 0$

iii) a conjungate or isolated point if $(f_{xy})^2 - f_{xx} f_{yy} < 0$

7.2 : Rules for Tracing of Cartesian Curves

The following rules will be helpful in tracing of cartesian curves together with definitions stated in section 7.1.

Rule 1 : Symmetry of the curve

a) **Symmetry about X-axis :** If the equation of the curve remains unchanged when y is replaced by – y i.e. all the powers of y are even in the given equation , then the curve is symmetrical about the X-axis.

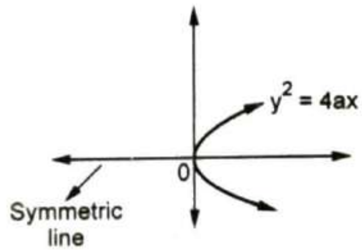


Fig. 7.7 (a)

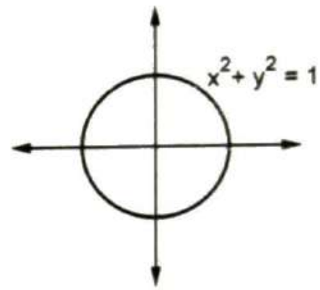


Fig. 7.7 (b)

e.g. $y^2 = 4ax$, $a > 0$ is symmetrical about X-axis, as power of y is even.

Also $x^2 + y^2 = 1$ is symmetric about x-axis

b) **Symmetry about Y-axis :** If the equation of the curve remains unchanged when x is replaced by – x i.e. all the powers of x are even in the given equation, then the curve is symmetrical about Y-axis.

e.g. $x^2 = 4ay$, $a > 0$ is symmetrical about Y-axis as power of x is even :

and $x^2 + y^2 = 1$ is symmetric about x-axis

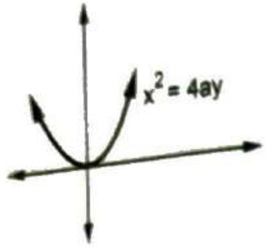


Fig. 7.8

c) **Symmetry about both X and Y-axes :** If the equation of the curve contains only even powers of x and y, then the curve is symmetrical about both the axes.

e.g. $x^2 + y^2 = r^2$ is symmetrical about both the axes as powers of x and y are even.

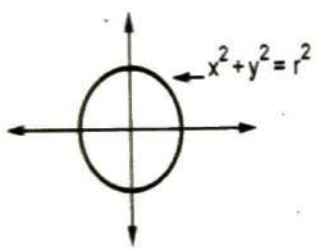


Fig. 7.9

$x^2y^2 = x^2 + y^2$ is symmetric about with axes.

d) **Symmetry about the origin or symmetry in the opposite quadrants :**

If the equation of curve remains unchanged when x and y are replaced by – x and – y respectively, then curve is symmetrical about the origin.

e.g. $y = x^3$ is symmetrical about origin as $-y = (-x)^3$
$\Rightarrow -y = -x^3$

$$\Rightarrow y = x^3$$

$x^2 + y^2 = 1$ is symmetric about origin.

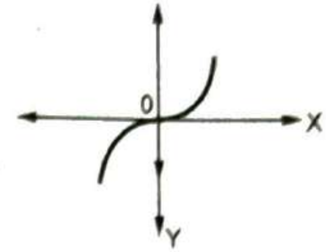


Fig. 7.10

e) **Symmetry about the line y = x :** If the equation of curve remains unchanged when x and y are replaced by y and x respectively, then the curve is symmetrical about the line y = x.

e.g. $xy = a^2$, is symmetrical about the line y = x.

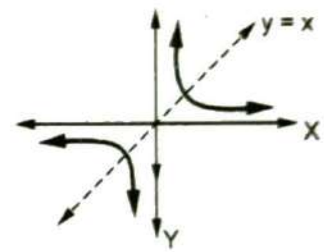


Fig. 7.11

f) **Symmetry about the line y = – x :** If the equation of curve remains unchanged when x and y are replaced by – y and – x respectively, then the curve is symmetrical about the line y = – x.

e.g. $xy = a^2 \Rightarrow (-y)(-x) = xy = c^2$

∴ The curve is symmetrical about the line y = – x

Rule 2 : Intercepts with co-ordinate axes and symmetric lines

a) **Intersection with co-ordinate axes :** To find intersection with X-axis, put y = 0

To find intersection with Y-axis, put x = 0

b) Intersection with the line $y = \pm x$

If the curve is symmetrical about the line $y = x$ or $y = -x$, then the points of intersection can be obtained by putting $y = x$ or $y = -x$ in the given equation respectively.

Rule 3 : Nature of tangents

a) At origin : If (0, 0) lies on the curve then the tangent at (0, 0) can be obtained by equating to zero, the lowest degree terms taken together in the equation of the curve.

e.g. In the equation $x^3 + y^3 = 3axy$, the lowest degree term is $3axy \therefore 3axy = 0 \Rightarrow x = 0, y = 0$.

Thus, the tangents at (0, 0) are $x = 0$ and $y = 0$

b) At any other point : To find the nature of the tangent at any point P, first find $\frac{dy}{dx}$ at point P.

Consider following possibilities.

i) If $\left(\frac{dy}{dx}\right)_P = 0$, then the tangent at P is parallel to X-axis.

ii) If $\left(\frac{dy}{dx}\right)_P = \infty$, then the tangent at P is parallel to Y-axis.

iii) If $\left(\frac{dy}{dx}\right)_P$ = Positive, then the tangent at P makes acute angle with X-axis.

iv) If $\left(\frac{dy}{dx}\right)_P$ = Negative, then the tangent at P makes obtuse angle with X-axis.

Note : Find nature of tangents at multiple points.

Rule 4 : Asymptotes

Asymptotes are the tangents to the curve at infinity. Find the asymptotes and the position of the curve with respect to them.

a) Asymptotes parallel to X-axis : Asymptotes parallel to X-axis are obtained by equating to zero the coefficient of highest degree term in x of the given equation.

b) Asymptotes parallel to Y-axis : Asymptotes parallel to Y-axis are obtained by equating to zero, the coefficient of highest degree term in y of the given equation.

e.g. $xy^2 = a^2(x+a)$ has two asymptotes

$y = \pm a$ are asymptotes parallel to X-axis

and $x = 0$ is asymptote parallel to Y-axis

c) Oblique asymptote : Asymptotes which are not parallel to co-ordinate axes, are called as **oblique asymptotes**. If the curve is not symmetric about X or Y-axes then we check for oblique asymptote. We can obtain the equation of oblique asymptote by following methods.

Method 1 : Let $y = mx + c$ be the equation of oblique asymptotes. To find m and c substitute this y in the given equation of curve $f(x, y) = 0$

$\therefore$ We get the points of intersection with the curve i.e. $f(x, mx + c) = 0$

Now, equating to zero, two successive highest powers of x, we get two equations in m and c. Solving these equations we get particular values of m and c

e.g. Let $x^3 + y^3 = 3a(xy)$ ... (1)

Assume that $y = mx + c$ be the equation of an oblique asymptote of equation (1).

$\therefore$ Substituting y in equation (1) we get

$$x^3 + (mx+c)^3 = 3ax(mx+c)$$

$$x^3 + m^3x^3 + 3m^2x^2c\,(+3mxc^2) + c^3 = 3amx^2 + 3ac\,x$$

Equating coefficients of x^3 and x^2 from both sides we get

$1 + m^3 = 0$ and $3\,m^2c = 3\,a\,m$

$\Rightarrow m = -1$ and $c = -a$

$\therefore y = mx + c = -x - a$ i.e. $x + y = -a$ is the oblique asymptote.

Method 2 : Let $y = mx + c$ be the oblique asymptote

i) Consider the highest degree (n) term of the equation substitute $x = 1$ and $y = m$ in that term and call it as $\phi_n(m)$.

e.g. For $x^3 + y^3 = 3axy$, $\phi_n = 1 + m^3$

ii) Similarly find $\phi_{n-1}(m)$

e.g. For $x^3 + y^3 = 3axy$, $\phi_{n-1}(m) = -3am$

iii) Solve $\phi_n(m) = 0$ to find m (here $m = -1$)

iv) Find $\phi'_n(m)$ and use it to find c

$$\therefore \quad c = \frac{-\phi_{n-1}(m)}{\phi'_n(m)}$$

Here, $\phi'_n(m) = 3m^2$, $\therefore c = \dfrac{-(-3am)}{3m^2} = -a$

Thus $y = mx + c = -x - a$ is the oblique asymptote.

Note :

1) If $y \to \infty$ as $x \to a$ then $x = a$ must be an asymptote parallel to Y-axis.
2) If $x \to \infty$ as $y \to b$, then $y = b$ must be an asymptote parallel to X-axis.
3) If $y \to \infty$ as $x \to \infty$ and if there is approximately linear relation between x and y then find oblique asymptote.

Rule 5 : Region of absence of the curve

i) If possible, express the given equation as $y^2 = f(x)$ then we can find some values of x where y^2 becomes negative, which is not possible. Hence the curve does not exist in that region.

e.g. For the curve $y^2 = 4x$

If x is negative then $y^2 < 0$. So curve does not exist for $x < 0$.

ii) If possible, express the given equation as $x^2 = f(y)$ then we can find some values of x where x^2 becomes negative, which is not possible. Hence, the curve does not exist in that region.

e.g. For the curve $x^2 = 2y$

If $y < 0$ then $x^2 < 0$ $\therefore$ The curve does not exist for $y < 0$.

By considering asymptotes, tangents and symmetric lines, divide plane into desired number of strips, or subregions say $R_1, R_2, \ldots$

Now verify whether curve exists in these regions or not by selecting an arbitrary point of each region.

Step 6 : Points of curve

Find few points of the curve, which will help for tracing of the curve $(x_1, y_1)(x_2, y_2), \ldots\ldots$

Rule 7 : By considering above all points, draw the rough sketch of the curve.

Note :

1. If there are two points on the line of symmetry then there is always a loop between that two points.
2. As asymptone is the tangent to curve at infinity, so if the curve is finite i.e. x is finite ; $\forall$ y and y is finite $\forall$ x, then asymptote does not exist.
3. While solving examples, we need to write only symmetries which exists.

Q.1 : Trace the curve $y^2(2a - x) = x^3$; $a > 0$. **[SPPU : Dec.-17, Marks 4]**

Ans. : This curve is known as the **"Cissoid of Diocle."**

The given equation of curve can be written as

$$y^2 = \frac{x^3}{2a - x} \; ; \; a > 0$$

Consider the following rules

Rule 1 : Symmetry of the curve

The curve is symmetrical about X-axis, because it's equation contains only even power of y.

Rule 2 : Intercepts

a) Intersection with co-ordinate axes :

To find points of intersection with X-axis, put $y = 0$ in given equation of the curve

$\therefore$ We get $\qquad x^3 = 0 \Rightarrow x = 0$

$\therefore$ The curve meets X-axis at (0, 0) only.

To find points of intersection with Y-axis, put $x = 0$ in the given equation of the curve

$\therefore$ We get $\qquad y = 0$

Thus the curve meets co-ordinate axes only at (0, 0)

Rule 3 : Nature of tangents :

a) At origin : The given curve passes through origin. $\therefore$ The tangents at (0, 0) are obtained by equating to zero, the lowest degree terms of the given equation. Here lowest degree term is $2ay^2$.

$\therefore \qquad 2ay^2 = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$ is a double point

Therefore we have two tangents at (0, 0) which are coincident. Thus (0, 0) is a cusp.

Rule 4 : Asymptotes

a) Asymptotes parallel to co-ordinate axes :

The coefficient of highest power of x is constant

$\therefore$ There does not exist any asymptote parallel to X-axis.

The asymptote parallel to Y-axis is obtained by equating to zero, the coefficient of the highest power in y.

i.e. $2a - x = 0 \Rightarrow x = 2a$ is asymptote parallel to Y-axis.

Rule 5 : Region of absence of the curve

We have $\qquad y^2 = \dfrac{x^3}{2a-x}$

As curve is symmetric about X-axis, therefore Y-axis and asymptote divide plane into three regions say R_1, R_2 and R_3 as shown in Fig. Q.1.1. Now select any one point from each region and find y^2.

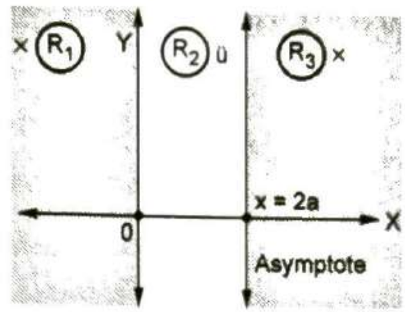


Fig. Q.1.1

1) In R_1,

At $\qquad x = -a$, then $y^2 = \dfrac{(-a)^3}{2a-(-a)} = \dfrac{a^2}{3} < 0$

$\Rightarrow y^2 < 0 \;\therefore$ The curve does not exists in R_1

2) In R_2. At $x = a$, $y^2 = \dfrac{a^3}{2a-a} > 0$

$\therefore$ The curve exists in R_2.

3) In R_2 at $x = 3a$, $\quad y^2 < 0$

$\therefore$ The curve does not exists in R_3

Rule 6 : Points of curve

At $\qquad x = a$, $y^2 = \dfrac{a^3}{2a-a} = a^2$

$\therefore \qquad y^2 = a^2 \Rightarrow y = \pm a$

$\therefore$ The curve passes through (a, a) and (a, –a)

Rule 7 : Thus taking into consideration of all above rules, the rough sketch of the curve is given below.

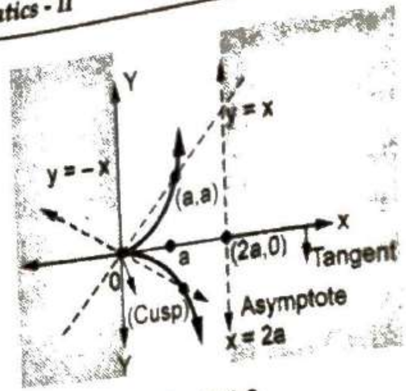


Fig. Q.1.2

Q.2 : Trace the curve $y^2(a^2 - x^2) = a^3x$. [SPPU : May-17, 19, Marks 4]

Ans. : The given equation of curve can be written as

$$y^2 = \frac{a^3x}{a^2 - x^2} \quad \ldots(1)$$

Consider the following rules.

Rule 1 : Symmetry of curve

As all even powers of y are present in the equation of curve, so the curve is symmetric about X-axis.

Rule 2 : Intercepts

a) Intersection with co-ordinate axes

To find intersection with X-axis, put $y = 0$ in equation (1), we get $x = 0$

$\therefore$ The curve meets X-axis at (0, 0)

To find intersection with Y-axis, put $x = 0$ in equation (1) we get $y = 0$

$\therefore$ The curve intersects Y-axis at (0, 0) only.

Rule 3 : Nature of tangents

a) At (0, 0) : The given curve passes through (0, 0) and the lowest degree term is a^3x. Therefore the tangent at (0, 0) is given by $a^3x = 0 \Rightarrow x = 0$ i.e. Y-axis.

b) At any other point : The curve does not intersect co-ordinate axes at other points.

Rule 4 : Asymptote

Here, the highest degree term in y is y^2 and it's coefficient is $a^2 - x^2$.

$\therefore$ $x^2 - a^2 = 0 \Rightarrow x = \pm a$ are asymptotes parallel to Y-axis.

The highest degree term in x is x^2 and it's coefficient is $-y^2$.

$\therefore$ $-y^2 = 0 \Rightarrow y = 0$ is the asymptote parallel to X-axis.

Rule 5 : Region of absence of the curve

We have, $$y^2 = \frac{a^3x}{a^2 - x^2}$$

Here the curve is symmetric about X-axis and there are 3 asymptotes. So Y-axis and asymptotes divide the plane into four regions say R_1, R_2, R_3, R_4 as shown in Fig. Q.2.1.

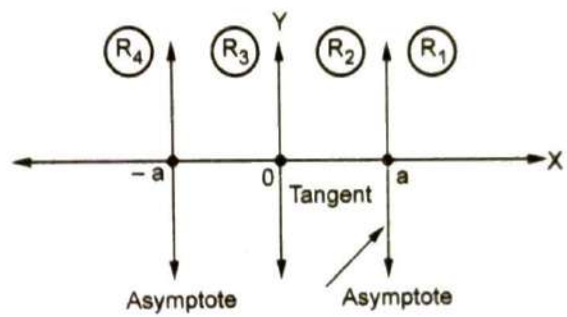


Fig. Q.2.1

i) In R_1, at $x = 2a$, $y^2 = \dfrac{a^3 2a}{-3a^2} < 0$

$\therefore$ The curve does not exist in R_1 i.e. for $x > a$.

ii) In R_2, at $x = \frac{a}{2}$, $y^2 = \dfrac{a^4}{2} \dfrac{1}{a^2/2} > 0$

$\therefore$ The curve exists in R_2 i.e. for $0 < x < a$.

iii) In R_3, at $x = -\frac{a}{2}$, $y^2 = -\frac{a^4}{2}\frac{1}{a^2/2} < 0$

$\therefore$ The curve does not exists in R_3 i.e. for $-a < x < 0$.

iii) In R_4, at $x = -2a$, $y^2 = -\frac{2a^4}{-3a^2} > 0$

$\therefore$ The curve exists in R_4 i.e. for $x < -a$.

Rule 6 : Points of curve

x	0	$\frac{a}{2}$	$-2a$
y	0	a^2	$\frac{2}{3}a^2$

Rule 7 : Thus taking into consideration of all above rules the rough sketch of the curve is given below :

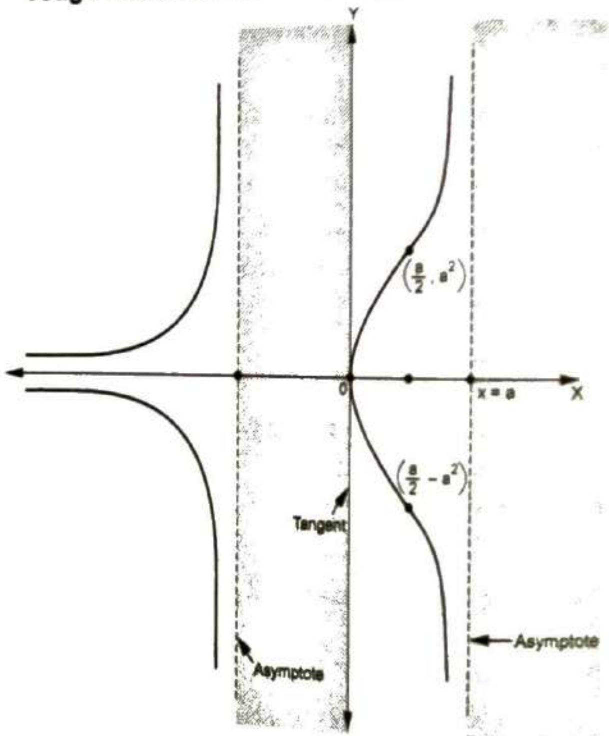


Fig. Q.2.2

Q.3 : Trace the curve $9ay^2 = x(x-3a)^2$; $a > 0$. [SPPU : Dec.-08]

Ans. : The given equation of curve can be written as

$$y^2 = \frac{1}{9a}x(x-3a)^2;\ a>0$$

Consider following rules

Rule 1 : Symmetry of the curve

The curve is symmetrical about X-axis because it's equation contains only even power of y.

Rule 2 : Intercepts

a) Intersection with co-ordinate axes :

To find intersection with X-axis, put $y = 0$ in given equation of the curve.

$\therefore$ $y = 0 \Rightarrow x(x-3a)^2 = 0 \Rightarrow x = 0$ and $x = 3a$

$\therefore$ The curve meets X-axis at (0, 0) and (3a, 0)

Now, put $x = 0 \Rightarrow y = 0$

Thus, the curve meets co-ordinate axes at (0, 0) and (3a, 0) only.

Rule 3 : Nature of tangents

a) At (0, 0) :

The given curve passes through (0, 0) $\therefore$ The tangent at (0, 0) is given by

$$9a^2x = 0 \Rightarrow x = 0$$

i.e. $x = 0$ is the tangent at (0, 0)

b) At any other point : We have

$$y = \frac{3a-x}{3}\sqrt{\frac{x}{a}} \quad \therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{ax}}(a-x)$$

$$\left(\frac{dy}{dx}\right)_{(3a,\,0)} = \text{Negative}$$

$\therefore$ The tangent at point (3a, 0) makes obtuse angle.

And $\left(\frac{dy}{dx}\right)_{\left(a,\frac{2a}{3}\right)} = 0$

$\therefore$ The tangent at $\left(a, \frac{2a}{3}\right)$ is parallel to X-axis

Rule 4 : Asymptotes

a) Asymptotes parallel to co-ordinate axes :

As coefficients of highest powers of x and y are constants $\therefore$ There is no asymptote parallel to co-ordinate axes.

Rule 5 : Region of absence of the curve

We have $y^2 = \frac{1}{9a} x(x-3a)^2$; $a > 0$

The curve is symmetric about X-axis. Therefore Y-axis divides plane into two regions R_1 and R_2

i) In R_1, at $x = a$, $y^2 = \frac{1}{9a}(a)(a-3a)^2 > 0$

$\therefore$ The curve exist for $x > 0$ or in R_1

ii) In R_2, at $x = -a$, $y^2 = \frac{1}{9a}(-a)(-a-3a)^2 < 0$

$\therefore$ The curve does not exists in R_2 i.e. for $x < 0$

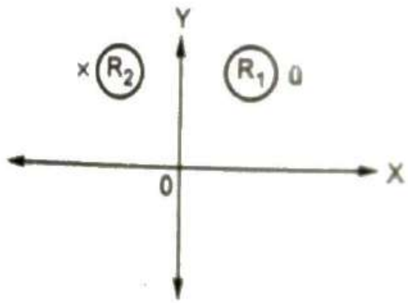


Fig. Q.3.1

Rule 6 : Points of curve

At $x = a$, $y^2 = \frac{1}{9a} a(a-3a)^2 = \frac{1}{9} 4a^2 \Rightarrow y = \pm\frac{2}{3} a$

At $x = 4a$, $y^2 = \frac{1}{9a} 4a(4a-3a)^2 = \frac{4}{9}a^2 \Rightarrow y = \pm\frac{2}{3}a$

At $x = \infty$, $y \to \infty$

$\therefore$ The curve passes through

$\left(a, \frac{2}{3}a\right)\left(a, -\frac{2}{3}a\right)\left(4a, \frac{2}{3}a\right)\left(4a, \frac{2}{3}a\right)$

Rule 7 : Concave nature of the curve

For $0 < x < 3a$, $\frac{d^2y}{dx^2} = -\frac{1}{2a} < 0$ at $x = a$

$\therefore$ The curve is concave down when $x = a$ and $y > 0$

and For $x > 3a$, $\frac{d^2y}{dx^2} = \frac{x+a}{4x\sqrt{ax}} > 0 \ \forall \ x > 3a$

$\therefore$ The curve is concave up when $x > 3a$

Rule 8 : Thus by considering above all steps, the rough sketch of the curve is given below

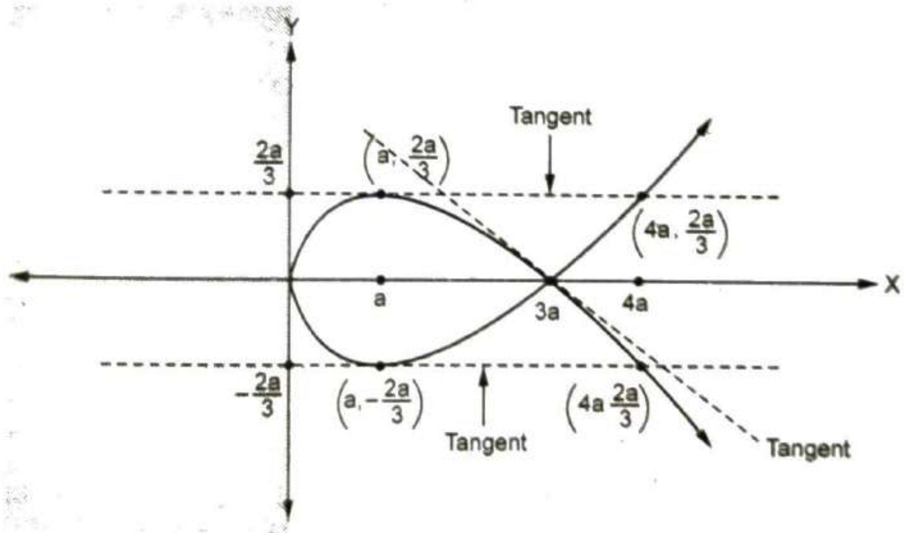


Fig. Q.3.2

Q.4 : Trace the curve $y = x + \frac{1}{x}$ **[SPPU : Dec.-08]**

Ans. : Given equation of curve is $y = x + \frac{1}{x}$

Consider following rules

Rule 1 : Symmetry of the curve

The equation of the curve remains unchanged after replacing x by − x and y by − y, so the curve is symmetrical about origin.

Rule 2 : Intercepts

a) To find intersection with co-ordinate axis put

$y = 0 \Rightarrow x + \frac{1}{x} = 0 \Rightarrow \frac{x^2+1}{x} = 0 \Rightarrow x^2 = -1$

and $x = 0 \Rightarrow y = \infty$

Thus the curve does not meets x-axis but meets Y-axis at infinity.

b) To find points of intersection with the line $y = \pm x$ put $y = \pm x$ in given equation of curve.

$\therefore \quad y = x \Rightarrow x = x + \frac{1}{x} \Rightarrow \frac{1}{x} = 0 \Rightarrow x = \infty$

and $y = -x \Rightarrow -x = x + \frac{1}{x} \Rightarrow \frac{1}{x} = -2x \Rightarrow x^2 = -\frac{1}{2}$ Not possible

Thus the line y = x meets curve at infinity.

Rule 3 : Nature of tangents

a) The curve does not pass through origin

∴ No tangents at (0, 0)

b) There does not exist other special points

∴ No tangents at other points.

Rule 4 : Asymptotes

a) As coefficients of highest powers of x and y are constants, But as $x \to 0$, $y \to \pm\infty$ ∴ y-axis is an asymptote.

Let y = mx + c be an oblique asymptote.

∴ Putting y = mx + c in given equation, we get

$$mx + c = \frac{x^2+1}{x}$$

$$\Rightarrow \quad mx^2 + cx = x^2 + 1$$

Now, equating coefficients of x^2 and x from both sides, we get m = 1 and c = 0

Thus y = x is an oblique asymptote.

Rule 5 : Region of absence of the curve

y is defined for all values of x, except x = 0

Thus the region of curve is $(-\infty, 0)$ and $(0, \infty)$

Rule 6 : Points of curve

At x = 1, $y = \frac{3}{2}$

x	$\frac{1}{3}$	$\frac{1}{5}$	1	1.5	2	3
y	$\frac{10}{3}$	$\frac{26}{5}$	$\frac{3}{2}$	$\frac{13}{6}$	$\frac{5}{2}$	$\frac{10}{3}$

Rule 7 : Thus by considering above all points, the rough sketch of the curve is given below.

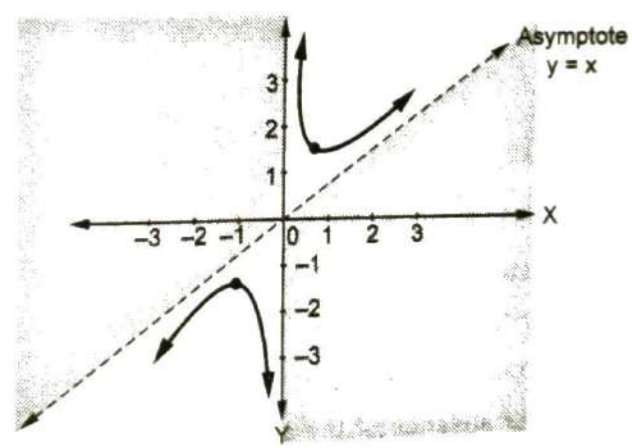


Fig. Q.4.1

Q.5 : Trace the curve $y = \dfrac{(x+1)^2}{1+x^2}$.

[SPPU : Dec.-04, 09, May-17, Marks 4]

Ans. : Given equation of curve is $y = \dfrac{(x+1)^2}{1+x^2}$

Consider following rules,

Rule 1 : Symmetry of the curve

There are no symmetries about either axes or origin.

Rule 2 : Intercepts

Putting $y = 0$ we get $(x+1)^2 = 0 \Rightarrow x + 1 = 0$

$$\Rightarrow x = -1 \therefore y = 0$$

and $x = 0 \Rightarrow y = 1$

Thus the curve meets co-ordinate axes at $(-1, 0)$ and $(0, 1)$

Rule 3 : Nature of tangents

We have $\dfrac{dy}{dx} = \dfrac{2(1-x^2)}{(1+x^2)}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(-1,0)} = 0$$

$\therefore$ The tangent at $(-1, 0)$ is parallel to X-axis

At $x = 1, y = 2$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,2)} = 0$$

$\therefore$ The tangent at $(1, 2)$ is parallel to X-axis.

Rule 4 : Asymptotes

Here $y \to 1$ as $x \to \pm\infty$

Thus $y = 1$ is asymptote of the given curve

Rule 5 : Region of absence of the curve

The curve exists for all values of x.

Rule 6 : Increasing and decreasing nature of the curve

We have, $\dfrac{dy}{dx} = \dfrac{2(1-x^2)}{(1+x^2)^2}$

$\therefore \dfrac{dy}{dx} < 0$ for $x \in (-\infty, -1)$ i.e. $-\infty < x < -1$ or $1 < x < \infty$

and $\dfrac{dy}{dx} > 0$ for $-1 < x < 1$

$\therefore \dfrac{dy}{dx} < 0$ for $-1 < x < 1$

Thus, the curve is increasing in $-1 < x < 1$ and decreasing in $-\infty < x < -1$ or $1 < x < \infty$.

Rule 7 : Concave nature of the curve

We have $\dfrac{dy}{dx} = \dfrac{2(1-x^2)}{(1+x^2)^2}$

$$\frac{d^2y}{dx^2} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

$$\left(\frac{d^2y}{dx^2}\right) = 0 \text{ at } x = \pm\sqrt{3}$$

$\therefore \quad x = \pm\sqrt{3}$ is the inflection point

Rule 8 : By considering above all points, the rough sketch of the curve is given below.

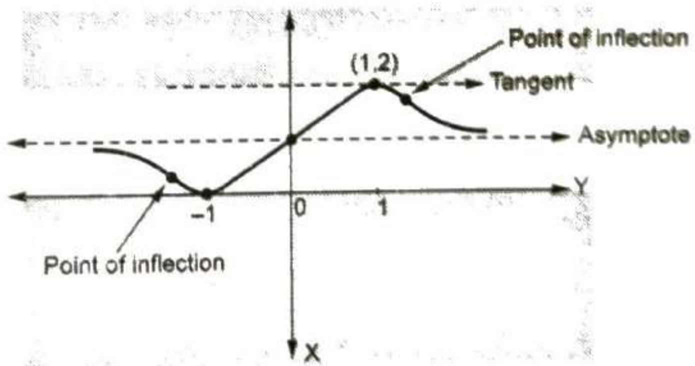


Fig. Q.5.1

Q.6 : Trace the curve $x^3 + y^3 = 3axy$.

Ans. : It is known as "Foulium of Descarts" Consider following rules.

Rule 1 : Symmetry of the curve : The equation of the curve remains unchanged after the replacement of x by y and y by x, so the curve is symmetrical about the line y = x.

Rule 2 : Intercepts

Putting $x = 0$ we get $y = 0$

and $y = 0 \Rightarrow x = 0$

Thus the curve meets co-ordinate axes at(0, 0) only

Substituting $y = x$, we get

$$x^3 + x^3 = 3ax^2 \Rightarrow 2x^3 = 3ax^2$$

$$\Rightarrow \quad x^3(2x - 3a) = 0 \Rightarrow x = 0 \text{ or } x = \frac{3a}{2}$$

Thus the curve meets the line y = x

at (0, 0) and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ only.

Rule 3 : Nature of tangents

a) The tangent at (0, 0) is given by

$$3axy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

$\therefore$ We have two distinct tangents at (0, 0)

Thus (0, 0) is the node.

b) We have

$$3x^2 + 2y^2 \frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$(2y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{3ay - 3x^2}{2y^2 - 3ax}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} < 0$$

$\therefore$ The tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ makes obtuse angle with X-axis.

Rule 4 : Asymptotes

Let y = mx + c be an oblique asymptote

$\therefore$ Substituting y = mx + c in given equation we get

$$x^3 + (mx + c)^3 = 3ax(mx + c)$$

$$x^3 + m^3x^3 + 3m^2x^2c + 3mxc^2 + c^3 = 3max^2 + 3acx$$

Equating the coefficients of x^3 and x^2 from both sides, we get

$$1 + m^3 = 0 \text{ and } 3cm^2 = 3ma$$

$$\Rightarrow \quad m = -1 \text{ and } c = -a$$

$\therefore$ y = – x – a i.e. x + y = – a is an oblique asymptote of the curve.

Rule 5 : Region of absence of the curve.

The curve exists for all values of x.

Rule 6 : Thus by considering above all point, the rough sketch of the curve is given below.

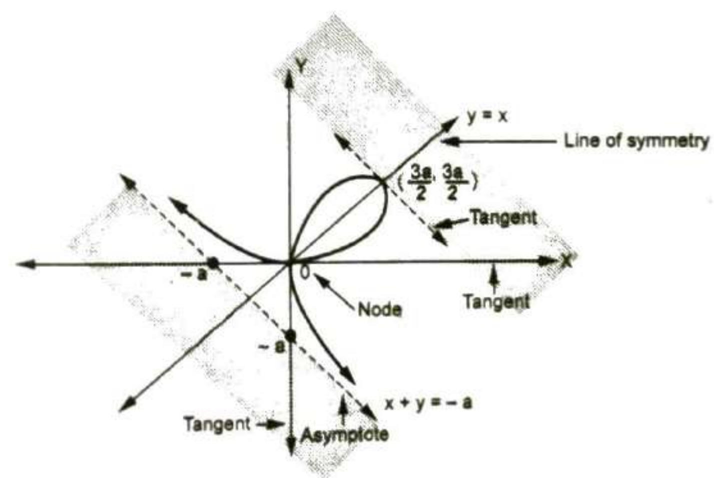


Fig. Q.6.1

Q.7 : Trace the curve $y^2(a+x) = x^2(a-x)$ [SPPU : Dec.-18, Marks 4]

Ans. : The given equation of curve can be written as

$$y^2 = \frac{x^2(a-x)}{a+x} \quad \ldots(1)$$

Consider the following rules.

Rule 1 : Symmetry of curve

As powers of y are even verywhere, so the curve is symmetric about X-axis.

Rule 2 : Intercepts

At $x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$

At $y = 0 \Rightarrow x^2(a-x) = 0 \Rightarrow x = 0$ or $x = a$

$\therefore$ The curve intersects co-ordinate axes at (0, 0) and (a, 0)

Rule 3 : Nature of tangents

a) **At (0, 0) :** The lowest degree term in equation (1) is

$ay^2 = ax^2 \Rightarrow y = \pm x$ are tangents to the curve at (0, 0)

b) **At (a, 0) :** We have

$$2y\frac{dy}{dx}(a+x)+y^2 = 2x(a-x)-x^2$$

$$2y(a+x)\frac{dy}{dx} = 2ax-3x^2-y^2$$

$$\frac{dy}{dx} = \frac{2ax-3x^2-y^2}{2y(a+x)}$$

At (a, 0), $\quad \frac{dy}{dx} = \infty$

$\therefore$ The tangent at (a, 0) is parallel to Y-axis

Rule 4 : Asymptotes

Here, the highest degree term in y is y^2 and the coefficient of y^2 is a + x.

$\therefore$ $a + x = 0 \Rightarrow x = -a$ is the asymptote parallel to Y-axis

Here the highest degree term in x is x^3 and its coefficient is 1.

$\therefore$ There is no any asymptote parallel to X-axis.

Rule 5 : Region of absence of curve

We have,

$$y^2 = \frac{x^2(a-x)}{a+x}$$

The curve has one tangent and one asymptote which

$\therefore$ divide plane into 3 regions say $R_1 - R_2$ and R_3

i.e. $x > a$, $-a < x < a$ and $x < -a$ respectively.

In R_1, at $x = 2a$, $y^2 = -ve$,

$\therefore$ The curve does not exist in R_1 i.e. $x > a$

In R_2, at $x = 0$, $y^2 = 0$,

$\therefore$ The curve axists in R_2 i.e. $-a < x < a$

In R_3, at $x = -2a$, $y^2 = -ve$,

$\therefore$ The curve does not exist in R_3 i.e. $x > -a$

Rule 6 : Points of curve

x	0	a	$\frac{a}{2}$	$\frac{-a}{2}$
y	0	0	$\frac{a^2}{12}$	$\frac{3a^2}{4}$

Rule 7 : Thus by considering all above rule, the rough sketch of the curve is given below.

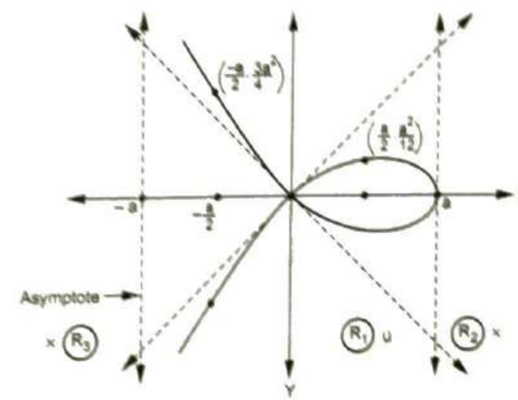


Fig. Q.7.1

Q.8 : Trace the curve $y^2 = x^5(2a - x)$ [SPPU : May-15, Marks 4]

Ans. : We have

$$y^2 = x^5(2a-x)$$

Rule 1 : Symmetry of curve

Even power of y are present everywhere.

$\therefore$ Curve is symmetric about X-axis

Rule 2 : Intercepts

At $x = 0,\ y = 0$

and At $y = 0 \Rightarrow x^5(2a-x) = 0 \Rightarrow x = 0,\ x = 2a$

$\therefore$ The curve intersects co-ordinate axes at (0, 0) (0, 2a)

Rule 3 : Tangents

a) **At (0, 0) :** Lowest degree term $= y^2 = 0$

$\Rightarrow y = 0$ is the tangent to the curve at (0, 0)

b) **At (20, 0) :** We have,

$$2y\frac{dy}{dx} = 5x^4(2a-x)+x^5(-1)$$

$$\frac{dy}{dx} = \frac{10ax^4-6x^5}{2y} = \frac{5ax^4-3x^5}{y}$$

$$\frac{dy}{dx} = \frac{x^4}{y}(5a-3x)$$

$\therefore$ At (2a, 0) $\dfrac{dy}{dx} = \dfrac{1}{0} = \infty$

$\therefore$ The tangent at (2a, 0) is parallel to Y-axis.

Rule 4 : Asymptotes

Here, the coefficients of highest degree terms of x or y are constants. So there is no any asymptote.

Rule 5 : Region of absence fo curve

We have $\quad y^2 = x^5(2a-x)$

The curve has two tangents, so divided plane into 3 regions say R_1, R_2 and R_3.

i.e. $x > 2a$, $0 < x < 2a$ and $x < 0$ respectively

In R_1, at $x = 3a \Rightarrow y^2 < 0$ $\therefore$ The curve does not exist in R_1

In R_2, at $x = a \Rightarrow y^2 > 0$ $\therefore$ The curve exists in R_2 i.e. $0 < x < 2a$.

In R_3, at $x = -a \Rightarrow y^2 < 0$

$\therefore$ The curve does not exists in R_3 i.e. $x < 0$

Rule 6 : Points of curve

x	0	a	$-a$	$\frac{a}{2}$
y	0	a^6	a^6	$\frac{3a^6}{64}$

Rule 7 : By considering all above rules, the rough sketch of the curve is given below.

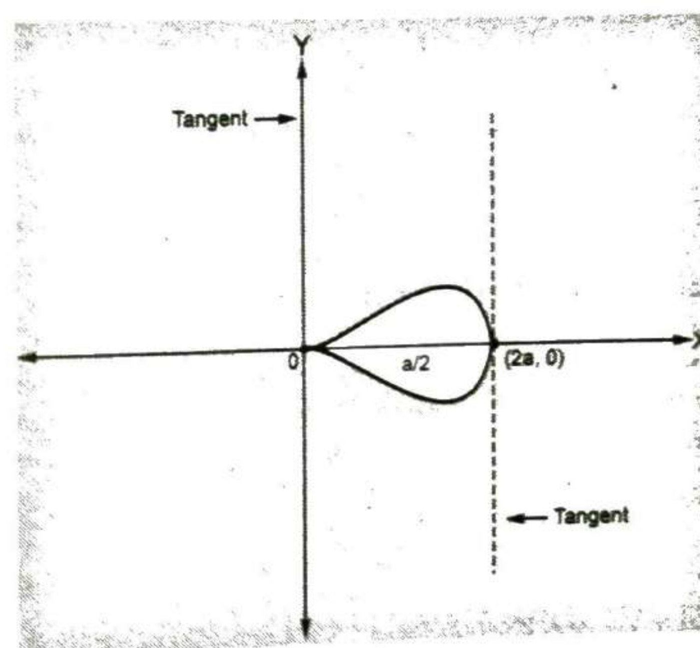


Fig. Q.8.1

7.3 : Tracing of Curves in Parametric Form

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of curve, where t is a parameter. Consider the following rules for tracing of curves in parametric form.

Rule 1 : Symmetry of the curve

a) **About X-axis :** If x remains unchanged after the replacement of t by $-t$ and y changes the sign, then the curve is symmetric about X-axis.

i.e. x is even and y is odd then the curve is symmetric about x axis.

Note : A function f(x) is said to be an even if

$$f(-x) = f(x)$$

and odd if $\quad f(-x) = -f(x)$

e.g. If $x = at^2$ and $y = 2at$, then the curve is symmetrical about X-axis.

Fig. 7.12

b) **About y-axis :** The curve is symmetrical about Y-axis if $x = f(t)$ is an odd and $y = g(t)$ is even i.e. $f(-t) = -f(t)$ and $g(-t) = g(t)$ OR For trigonometric equations, if by replacing t by $\pi - t$, y remains unchanged and x changes sign then the curve is symmetrical about Y-axis.

Fig. 7.13

e.g If $x = 2at$ and $y = at^2$ then the curve is symmetrical about Y-axis.

c) **About Origin (Opposite Quadrants) :** The curve is symmetrical about origin if both x and y are odd functions of t.

e.g. If $x = t$ and $y = t^3$, then the curve is symmetrical about origin.

Fig. 7.14

Rule 2 : Points of intersection with co-ordinate axes

a) If both x and y become zero for some real value of t then the curve passes through origin.

b) To find intersection with X-axis, first find values of t for which $y = g(t) = 0$ then substitute these values of t in $x = f(t)$.

c) To find intersection with Y-axis, first find values of t for which $x = f(t) = 0$, then substitute these values of t in $y = g(t)$.

Rule 3 : Nature of tangents :

We have $\quad \dfrac{dy}{dx} = \dfrac{(dy/dt)}{(dx/dt)}$

If $\dfrac{dy}{dx} = 0$ for some value of t, then the tangent is parallel to X-axis at point (x, y)

If $\left(\dfrac{dy}{dx}\right) = \infty$, for some value of t, then the tangent is parallel to Y-axis at point (x, y) ,

Rule 4 : Asymptotes : If $\lim\limits_{t \to t_1} x \to \infty$ and $\lim\limits_{t \to t_1} y \to \infty$ then, $t = t_1$ is the asymptote to the curve.

Rule 5 : Region of absence of the curve.

a) Find the region where curve does not exist or where x and y are imaginary.

b) If possible, find the least and greatest values of x and y and draw the lines parallel to co-ordinate axes corresponding to these values. Therefore the curve lies in the region determined by these lines.

Rule 6 : Variation of x and y w.r.t. t :

Construct the table for values of x and y by using different values of t.

Rule 7 : By considering above all points, draw the rough sketch of curve

Notes :

1) If we get simple equation of curve (i.e. powers of x and y are integers only) in cartesian form by eliminating parameter t from $x = f(t)$ and $y = g(t)$, then trace the curve by using cartesian form.

2) There are some equations of curves in cartesian form in which powers of x or y are not integer, then convert to parametric form for tracing of curve.

e.g. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1,$

Here powers of x and y are 2/3

$\therefore$ Convert to parametric form and then trace the curve.

Q.9 : Trace the curve : $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$, where θ is a paramter. [SPPU : Dec.-12, May-16, Marks 4]

Ans. : Given equation of curve is

$x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$, θ = parameter

i.e. $x = f(\theta) = a(\theta + \sin\theta)$, $y = g(\theta) = a(1+\cos\theta)$

consider following rules

Rule 1 : Symmetry of the curve

Here $f(-\theta) = a(-\theta + \sin(-\theta)) = a(-\theta - \sin\theta)$

$= a(\theta + \sin\theta) = -f(\theta)$

$\therefore$ x is an odd

and $g(-\theta) = a(1+\cos(-\theta)) = g(\theta)$

$\therefore$ y is even

Thus the curve is symmetrical about Y-axis

Rule 2 : Points of intersection with co-ordinate axes :

a) There does not exist value of θ for which both x and y become zero. $\therefore$ the curve does not pass through origin.

b) The curve meets X-axis where $y = 0$

$\Rightarrow a(1+\cos\theta) = 0 \Rightarrow 1 + \cos\theta = 0 \Rightarrow \cos\theta = -1$

$\Rightarrow \theta = \pm\pi, \pm 3\pi, \pm 5\pi$ .....

$\therefore$ At $\theta = \pi$, $x = (a)(\theta + \sin\theta) = a(\pi + \sin\pi) = a\pi$

θ	$\pm\pi$	$\pm 3\pi$	$\pm 5\pi$ ...
x	$\pm a\pi$	$\pm 3a\pi$	$\pm 5a\pi$ ...

$\therefore$ The curve meets X-axis at $x = \pm a\pi, \pm 3a\pi, \pm 5a\pi$

c) The curve meets Y-axis where $x = 0$

$\Rightarrow a(\theta + \sin\theta) = 0 \Rightarrow \theta + \sin\theta = 0$

$\Rightarrow \sin\theta = -\theta \Rightarrow \theta = 0$

At $\theta = 0$, $y = a(1+\cos\theta) = a(1+\cos 0) = a(1+1) = 2a$

$\therefore$ The curve meets Y-axis at $y = 2a$ only

Rule 3 : Nature of tangents

We have $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{-a\sin\theta}{a(1+\cos\theta)}$

$\therefore \quad \frac{dy}{dx} = -\tan\left(\frac{\theta}{2}\right)$

At $\theta = 0$, $\frac{dy}{dx} = 0$ $\therefore$ The tangent is parallel to X-axis at $\theta = 0$.

At $\theta = -\pi, \pi, \frac{dy}{dx} = \infty$

$\therefore$ The tangents are parallel to Y-axis at $\theta = -\pi, \pi$.

Rule 4 : Asymptotes

For any finite value of θ, no asymptote exists.

Rule 5 : Region of absence of the curve

Form $y = a(1+\cos\theta) \le a(1+1) = 2a$

$\therefore y \le 2a$ and y is always non negative i.e. $y \ge 0$.

$\therefore 0 \le y \le 2a$. $\therefore$ The curve lies between $y = 0$ and $y = 2a$.

For any real θ, x is unbounded. y is periodic function.

Rule 6 : Variation of x and y w.r.t θ :

Consider following table.

θ	$-\pi$	0	π	2π
x	$-a\pi$	0	$a\pi$	$2a\pi$
y	0	2a	0	0

Rule 7 : By considering above rules, the rough sketch of the curve is given below.

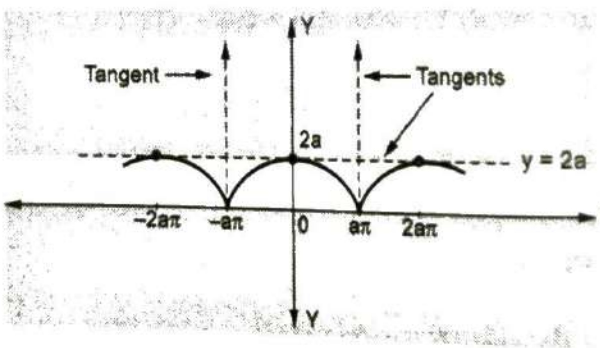


Fig. Q.9.1

Q.10 : Trace the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$.

[SPPU : May-14, Dec. -16, Marks 4]

Ans. : Let, $x = f(t) = a(t + \sin t)$

$y = g(t) = a(1 - \cos t)$

Consider the following rules.

Rule 1 : Symmetry of the curve

As x is odd and y is even, the curve is symmetric about Y-axis.

Rule 2 : Intercepts

At $t = 0$, $x = 0$ and $y = 0$

$\therefore$ The curve passes through (0, 0)

For all t, $0 \le y \le 2a$ and y is periodic function.

t	0	π	2π
x	0	$a\pi$	$2a\pi$

Rule 3 : Nature of tangents

We have $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\sin t}{a(1+\cos t)} = \tan\left(\frac{t}{2}\right)$$

At $t = 0$ and 2π, $\frac{dy}{dx} = 0$

$\therefore$ Tangents are parallel to X-axis.

At $t = \pi$, $\frac{dy}{dx} = \infty$

$\therefore$ Tangents is parallel to Y-axis.

Rule 4 : Asymptote

For any finite value of t, there is no asymptote.

Rule 5 : Region of absence fo curve

we have $0 \le y \le 2a$ and $-\infty < x < \infty$

$\therefore$ The curve exists in strip $0 \le y \le 2a$

Rule 6 : Variation of x and y

t	0	π	2π
x	0	$a\pi$	$2a\pi$
y	0	2a	0
$\frac{dy}{dx}$	0	∞	0
Nature of tangent	Parallel to X-axis	Parallel to Y-axix	Parallel to X-axis

Rule 7 : By considering all above rules, The rough sketch of the curve is given below.

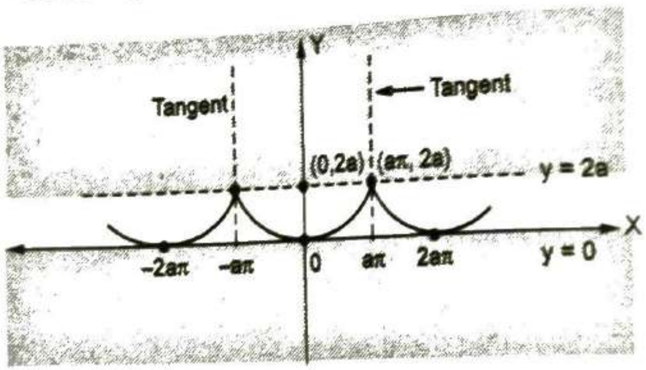


Fig. Q.10.1

Q.11 : Trace the curve $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

[SPPU : Dec.-07, May-08, 10, 13, Marks 4]

Ans. : Given that $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

It is known as **astroid** (or star shaped curve). Here power of x is rational i.e. 2/3. ∴ Convert to parametric form. The parametric equations of the given curve are $x = a\cos^3\theta$, $y = b\sin^3\theta$ where θ is a parameter.

Consider following rules.

Rule 1 : Symmetry of the curve

a) **About X-axis** : Here x is even and y is odd

∴ The curve is symmetrical about X-axis.

b) **About Y-axis** : We have $\cos^3(\pi-\theta)=-\cos^3\theta$ and $\sin^3(\pi-\theta)=\sin^3\theta$

∴ For θ and $\pi-\theta$, y has same values but x has opposite values.

∴ The curve is symmetrical about Y-axis.

Rule 2 : Intercepts

a) There does not exist value of θ for which both x and y become zero. ∴ The curve does not pass through origin.

b) The curve meets X-axis where y = 0

$$\Rightarrow \quad b\sin^3\theta = 0 \Rightarrow \sin^3\theta = 0 \Rightarrow \theta = 0, \pi, -\pi, \ldots..$$

∴ We get

θ	0	π	$-\pi$
x	a	– a	– a

∴ The curve meets X-axis at points (a, 0) and (– a, 0)

c) The curve meets Y-axis where $x = 0 \Rightarrow a\cos^3\theta = 0$

$$\Rightarrow \cos^3\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots.$$

∴ We get

θ	$\pi/2$	$3\pi/2$
y	a	– a

∴ The curve meets Y-axis at points (0, b) and (0, – b)

Rule 3 : Nature of tangents

We have
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3b\sin^2\theta\cos\theta}{3a\cos^2\theta\,(-\sin\theta)} = -\frac{b}{a}\tan\theta$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ for } \theta = 0 \text{ and } \frac{dy}{dx} = \infty \text{ for } \theta = \frac{\pi}{2}$$

$\therefore$ The tangent of the curve at $\theta = 0$ is parallel to X-axis and at $\theta = \frac{\pi}{2}$, tangent is parallel to Y-axis.

Rule 4 : Asymptotes : For any θ, x and y are finite

$\therefore$ No asymptote exists.

Rule 5 : Region of absence of the curve

We have $x = a\cos^3\theta \therefore |x| = |a\cos^3\theta| \le a$

$\Rightarrow -a \le x \le a$

And $y = b\sin^3\theta \therefore |y| = |b\sin^3\theta| \le b$

$\Rightarrow -b \le y \le b$

$\therefore$ The curve exist within $-b \le y \le b$ and $-a \le x \le b$

Rule 6 : Variation of x and y w.r.t. θ :

Consider following table

θ	0	$\pi/2$	π	$3\pi/2$
x	a	0	$-a$	0
y	0	b	0	$-b$

Rule 7 : By considering above all points, the rough sketch of the curve is given below.

Q.12 : Trace the curve $x = t^2$, $y = t - \frac{t^3}{3}$.

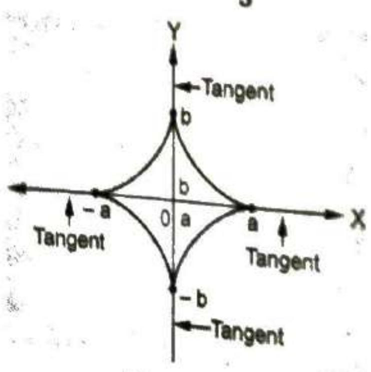


Fig. Q.11.1

Ans. : Given equation of curve is $x = t^2$, $y = t - \frac{t^3}{3}$

Consider following rules.

Rule 1 : Symmetry of the curve

Here $x = t^2$ is even and $y = t - \frac{t^3}{3}$ is an odd function

$\therefore$ The curve is symmetrical about X-axis

Rule 2 : Intercepts

a) For $t = 0$, both x and y become zero $\therefore$ The curve passes through (0, 0).

b) Putting $y = 0$ we get $t\left(1 - \frac{t^2}{3}\right) = 0 \Rightarrow t = 0$ or $1 - \frac{t^2}{3} = 0$

$\therefore t = 0$ and $t = \pm\sqrt{3}$

For $t = 0$, $x = 0$, for $t = \pm\sqrt{3}$, $x = 3$

$\therefore$ The curve meets x-axis at (3, 0) and (0, 0)

c) Putting $x = 0$ We get $t^2 = 0 \Rightarrow t = 0 \Rightarrow y = 0$

$\therefore$ The curve meets Y-axis at (0, 0) only

Rule 3 : Nature of tangents : We have

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{1 - t^2}{2t}$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ for } t = \pm 1$$

$\therefore$ The tangent to the curve at $t = \pm 1$ is parallel to X-axis.

$$\frac{dy}{dx} = \infty \text{ for } t = 0$$

$\therefore$ The tangent to the curve at $t = 0$ is parallel to Y-axis.

Rule 4 : Asymptotes : Asymptote does not exist for any value of t.

Rule 5 : Region at absence of the curve

We have $x = t^2$ $\therefore x < 0$ for $t^2 < 0 \Rightarrow t$ is imaginary but t is a real parameter.

$\therefore$ The curve does not exist for $x < 0$

Rule 6 : Increasing and decreasing nature of the curve

We have, $\dfrac{dy}{dt} = \dfrac{1-t^2}{2t}$

$\therefore$ In first quadrant

$$\frac{dy}{dx} > 0 \text{ for } x \in (0, 1)$$

and $$\frac{dy}{dx} < 0 \text{ for } x \in (1, 3)$$

Thus the curve is increasing for $0 < x < 1$ and decreasing for $1 < x < 3$.

Rule 7 : Variation of x and y w.r.t.t :

Consider following table

t	0	1	$\sqrt{3}$	2
x	0	1	3	4
y	0	2/3	0	–2/3

Rule 8 : By considering above all points, the rough sketch of the curve is given below.

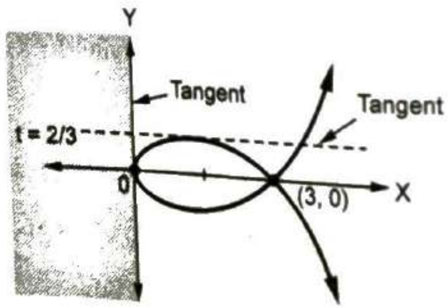


Fig. Q.12.1

7.4 : Tracing of Curves in Polar Form

A) Basic Terms :

i) Let P(x, y) be any point in the cartesian co-ordinate system. Draw perpendicular from point P on X-axis and denote foot of perpendicular by M.

$\therefore$ $$OM = x, PM = y$$

$$OP = r \text{ and } \angle XOP = \angle MOP = \theta$$

In Δ MOP, $\cos\theta = \dfrac{x}{r}, \sin\theta = \dfrac{y}{r}$

$\therefore$ $x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2,\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

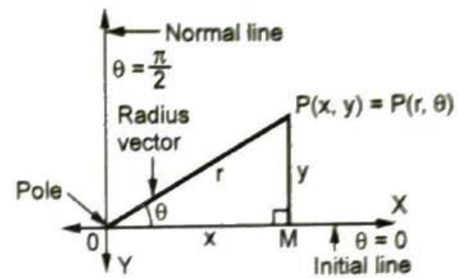


Fig. 7.15

Then, the pair P(r, θ) is called the polar co-ordinates of P. 0 is called pole, $\vec{OX}$ is called the initial line and OP = r is called the radius vector.

$$\theta = \frac{\pi}{2} \text{ (positive Y-axis) is called normal line.}$$

ii) The polar curves are given by polar co-ordinates (r, θ) and in explicit form it can be written as $r = f(\theta)$ or $\theta = f(r)$ and in implicit form it is written as $f(r,\theta) = 0$.

iii) In polar form $r^2 = x^2 + y^2$ which is a circle with centre at (0, 0) and radius r.

$\therefore$ r = Constant, represents a family of concentric circles with centre at origin.

And θ = Constant, represents a family of straight lines passing through origin.

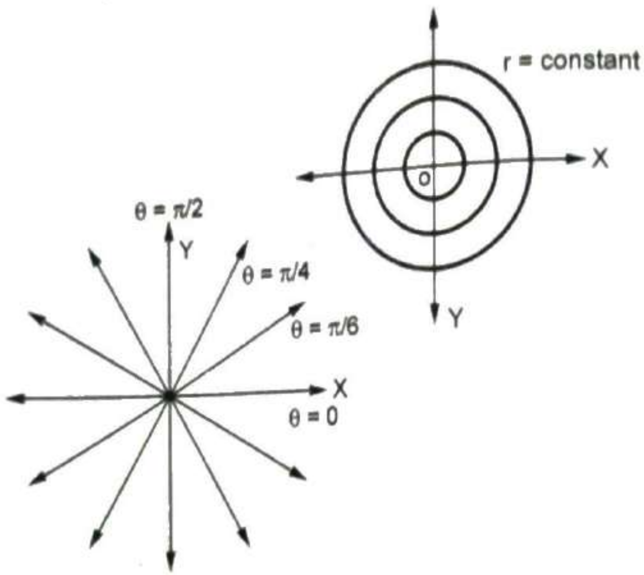


Fig. 7.16

The following rules give the guideline for tracing of curves in polar form :

Rule 1 : Symmetry of the curve

a) **About initial line :** If the equation of the curve remains unchanged by replacing θ by $-\theta$, then the curve is symmetrical about inital line.

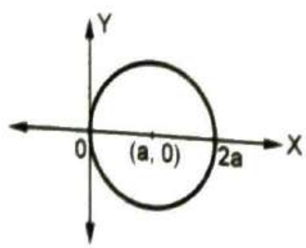


Fig. 7.17

e.g. $r = 2a \cos\theta$ is symmetrical about initial line.

b) **About pole (origin) :** If the equation of the curve remains unchanged by replacing r by $-r$, then the curve is symmetrical about the pole.

e.g. $r^2 = a^2 \cos 2\theta$ is symmetrical about pole.

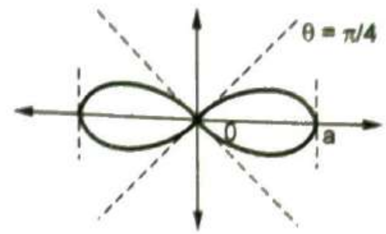


Fig. 7.18

c) **About Y-axis (Normal line)** : If the equation of the curve remains unchanged by replacing r by $-r$ and θ by $-\theta$, then the curve is symmetrical about Y-axis or if replacing θ by $\pi - \theta$, equation remains unchanged, then the curve is symmetrical about Y-axis.

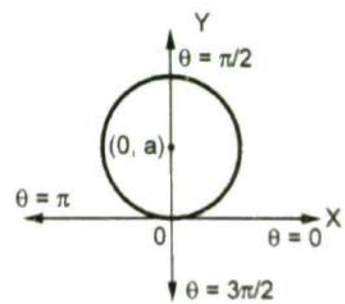


Fig. 7.19

e.g. $x^2 + y^2 = 2ay$, its polar form is $r = 2a \sin\theta$ is symmetrical about Y-axis.

d) **About the line $\theta = \frac{\pi}{4}$** : If the equation of the curve remains unchanged by replacing θ by $\frac{\pi}{2} - \theta$, then the curve is symmetrical about the line $\theta = \frac{\pi}{4}$.

e) **About the line $\theta = \frac{3\pi}{4}$** : If the equation of the curve remains unchanged by replacing θ by $\frac{3\pi}{2} - \theta$ then the curve is symmetrical about the line $\theta = 3\pi/4$.

Rule 2 : Region of absence :

a) If for $\alpha < \theta < \beta$, r^2 becomes negative, then the curve does not exist for $\alpha < \theta < \beta$.

b) Find the least and greatest values of r, to get bounds of the curve. If the least value of r is a and greatest value is b, then the curve exists in the annulus region $a < r < b$ only.

e.g. The curve $r = a(1+\cos\theta)$ lies within a circle $r = 2a$.

Rule 3 : Pole and tangents at pole

a) If for some value of θ, r becomes zero then the pole lies on the curve.

b) To find tangents at pole, put r = 0 in the given equation of curve, then the values of θ gives the tangents at pole.

e.g. $r^2 = a^2\cos\theta$ for $\theta = \frac{\pi}{2}$, r becomes zero

$\therefore$ The curve passes through pole

And $r = 0 \Rightarrow a^2 \cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{(2n+1)\pi}{2}$

i.e. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ are tangents to the curve at pole

Rule 4 : Points of intersection :

Find the points of intersection of the curve with initial line and the line $\theta = \frac{\pi}{2}$.

Make the table showing values of r for different values of θ.

Rule 5 : Direction of tangents : Find the angle between radius vector and tangent ϕ by using formula

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

Find the values of θ for which $\phi = 0$ or $\frac{\pi}{2}$. The values of θ for which $\phi = 0$, then the tangent will coinside with radius vector and the values of θ for which $\phi = \frac{\pi}{2}$, then the tangent will be perpendicular to radius vector.

Rule 6 : Asymptotes of the curve

If $\lim_{\theta\to\theta_1} r \to \infty$ then the asymptote to the curve exist and it is given by $r\sin(\theta-\theta_1) = f'(\theta_1)$

where θ_1 is the solution of $\frac{1}{f(\theta)} = 0$

Rule 7 : By considering above all rules, draw the curve.

Note :

1) For the curves having $\sin n\theta$ or $\cos n\theta$, divide each quadrant into n equal parts.

2) Sometimes, it is convenient to convert the polar equations into cartesian form or cartesian to polar.

3) If we know the curve of $r^n = a^n\cos n\theta$ for any n, then the curve of $r^n = a^n \sin n\theta$ can be obtained by rotating the plane through $\frac{\pi}{2n}$.

Q.13 : Trace the curve $r = a(1+\cos\theta)$: $a > 0$.

[SPPU : Dec.-15, May-17, Marks 4]

Ans. : It is known as "Cardioid"

Consider the following rules,

Rule 1 : Symmetry of the curve : The equation of the curve remains same by replacing θ by $-\theta$. Therefore the curve is symmetrical about initial line.

Rule 2 : Region of absence

a) The curve exists for all values of θ.

b) We have $r = a(1+\cos\theta)$ and $-1 \le \cos\theta \le 1$

$\therefore 0 \le r \le 2a$

Hence the curve lies within a circle of radius 2a. and centre at (0, 0).

Rule 3 : Pole and tangents at pole

a) If $\theta = \pm\pi$, r becomes 0 $\therefore$ The curve passes through pole.

Therefore the tangent to the curve at pole is $\theta = \pm \pi$.

Thus the curve passes through pole and tangents at pole are coincident $\therefore$ Pole is a cusp.

Rule 4 : Points of intersection

The curve meets initial line at (2a, 0) and (0, 0) and meets the line $\theta = \frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right)$ and $\left(a, \frac{3\pi}{2}\right)$

Consider the following table.

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	$2\pi/3$	π
r	2a	1.7 a	$\frac{3a}{2}$	a	$\frac{\sqrt{2}-1}{\sqrt{2}}$	0.7 a	0

Rule 5 : Direction of tangents

We have
$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$
$$= -\cot(\theta/2) = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$
$$\Rightarrow \qquad \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

When $\theta = 0$, $\phi = \frac{\pi}{2}$ $\therefore$ The tangent is perpendicular to radius.

Rule 6 : Asymptotes of the curve : As r is finite for any value of θ $\therefore$ No asymptote exists.

Rule 7 : Increasing and decreasing nature of the curve

We have $$\frac{dr}{d\theta} = -a\sin\theta$$

We know that $\sin\theta$ is increasing in $0 < \theta < \pi/2$ and decreasing in $\frac{\pi}{2} < \theta < \pi$.

Thus the given curve decreases in $0 < \theta < \pi/2$ and increases in $\frac{\pi}{2} < \theta < \pi$.

Rule 8 : By considering above all points, the rough sketch of the curve is given below.

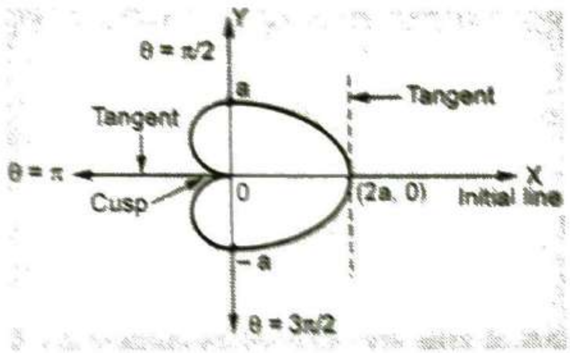


Fig. Q.13.1

Q.14 : Trace the curve $\boxed{r^2 = a^2\cos 2\theta}$ It is known as Bernoulli's is lamiscate. [SPPU : Jan.-10]

Ans. : Given equation of curve is $r^2 = a^2\cos 2\theta$

Consider the following steps

Step 1 : Symmetry of the curve

The equation of the curve remains unchanged by replacing θ by $-\theta$ or r by $-$ r or θ by $\pi-\theta$ so the curve is symmetrical about initial line, pole and the line $\theta = \frac{\pi}{2}$.

Rule 2 : Region of absence

a) We have $\qquad r^2 = a^2\cos 2\theta$

Cosine curve is positive in $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow r^2 = a^2 \cos 2\theta$ is positive in $-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

i.e. $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$

Cosine curve is negative in $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$\Rightarrow r^2 = a^2 \cos 2\theta$ is negative in $\frac{\pi}{2} < 2\theta < \frac{3\pi}{2}$

i.e. $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$

Thus the curve does not exist for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

b) We have $|\cos\theta| \le 1 \therefore |r^2| = r^2 \le a^2$

Thus the curve lies entirely within the circle $r = a$

Rule 3 : Pole and tangents at pole

a) For $\theta = \frac{\pi}{4}$, r becomes zero $\therefore$ The curve passes through pole.

b) The tangents at pole are obtained by putting $r = 0$ in the given equation.

$\therefore$ We get $r^2 = a^2 \cos 2\theta = 0 \Rightarrow \cos 2\theta = 0$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{2}, \ldots$$

$$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$ are tangents to the curve at pole.

Rule 4 : Points of intersection

Here $\theta = 0 \Rightarrow r^2 = a^2 (\cos 0) \Rightarrow r^2 = a^2 \Rightarrow r = \pm a$

$\therefore$ The curve meets initial line at (a, 0) and (– a, 0)

Putting $\theta = \frac{\pi}{2}$, we get $r^2 = 0 \Rightarrow r = 0$

$\therefore$ The curve meets the line $\theta = \frac{\pi}{2}$ at (0, 0)

Consider the following table,

θ	0	π/4	3π/4	π	5π/4	7π/4	2π
1	a	0	0	– a	0	0	a

Rule 5 : Direction of tangent

We have
$$\tan\phi = r\frac{d\theta}{dr} = \frac{a\sqrt{\cos 2\theta}}{\frac{a}{2\sqrt{\cos 2\theta}}(-2\sin 2\theta)}$$
$$= -\cot 2\theta$$
$$\therefore \quad \tan\phi = \tan\left(\frac{\pi}{2} + 2\theta\right) \Rightarrow \phi = \frac{\pi}{2} + 2\theta$$

$\therefore$ When $\theta = 0$, $\phi = \frac{\pi}{2}$ $\therefore$ The tangent is perpendicular to radius vector.

Rule 6 : Asymptotes of the curve

Asymptote does not exist because r is finite for any value of θ.

Rule 7 : Increasing and decreasing nature of the curve

We have
$$\frac{dr}{d\theta} = \frac{-a\sin 2\theta}{\sqrt{\cos 2\theta}}$$
$$\frac{dr}{d\theta} < 0 \text{ for } 0 < \theta < \frac{\pi}{4}$$
$$> 0 \text{ for } \frac{3\pi}{4} < \theta < \pi$$

Thus the curve increases in $\frac{3\pi}{4} < \theta < \pi$ and decreases in $0 < \theta < \frac{\pi}{4}$.

Rule 8 : By considering above all points, the rough sketch of the curve is given below.

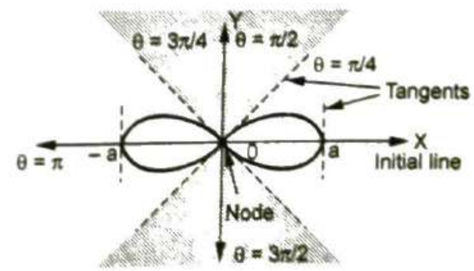


Fig. Q.14.1

Note : The curve $\boxed{r^2 = a^2 \sin 2\theta} = a^2 \cos 2\left(\frac{\pi}{4} - \theta\right)$ can be obtained by rotating the curve $r^2 = a^2 \cos 2\theta$ by an angle $\frac{\pi}{4}$ in anticlockwise direction.

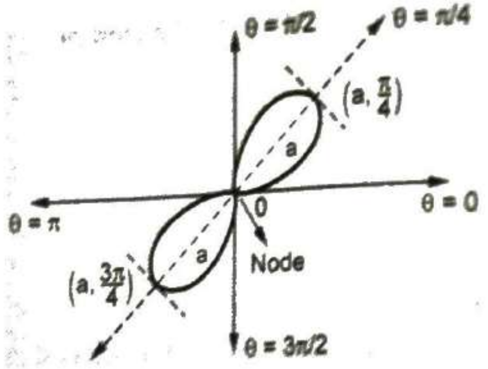


Fig. Q.14.2

Q.15 : Trace the curve r = a cos 2 θ.

[SPPU : Dec.-13, May-17, 18, Marks 4]

Ans. : Note : The curve $r = a \cos 2\theta$ can be obtained from $r = a \sin 2\theta$ by rotating the curve through an angle $\frac{\pi}{4}$ in anticlockwise direction.

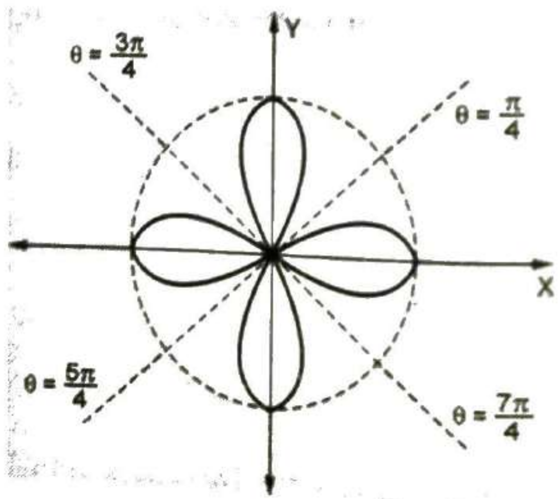


Fig. Q.15.1

Q.16 : Trace the curve $r = a \cos 3\theta$.

[SPPU : May-16, Marks 4]

Ans. : It is known as 'Rose Curve', $r = a \cos 3\theta$

Rule 1 : Symmetty of the curve

The equation of cure remains same after replacing θ by $-\theta$

$\therefore$ The curve is symmetric about X-axis or intial line.

Rule 2 : Region of absence of curve

We know that,

$$-1 \le \cos 3\theta \le 1$$

$$\therefore \quad -a \le a \cos 3\theta \le a \Rightarrow -a \le r \le a$$

$$\therefore \quad 0 \le r \le a$$

$\therefore$ The curve lies in $0 \le r \le a$

Rule 3 : Poles and Tangents at pole

At $\theta = \frac{\pi}{6}$, $r = 0 \Rightarrow$ The curre passes through pole.

Now $r = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\therefore \quad \theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots$ are tangents at pole

Rule 4 : Points and Intersection

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{4\pi}{3}$	$\frac{5\pi}{6}$
r	a	0	–a	0	a	0

$\therefore$ r is negative for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$

Rule 5 : Direction of tangents

We have $$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = \frac{a \cos 3\theta}{-3a \sin 3\theta}$$

$$= -\frac{1}{3}\frac{\cos 3\theta}{\sin 3\theta}$$

$$\tan\phi = \infty \text{ at } \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3} \ldots\ldots$$

$$\therefore \quad \phi = \frac{\pi}{2} \text{ at } \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3} \ldots\ldots$$

Rule 6 : Asymptotes : Asymptote does not exist as r is finite.

Rule 7 : By considering all above rules, the rough sketch of the curve is given below.

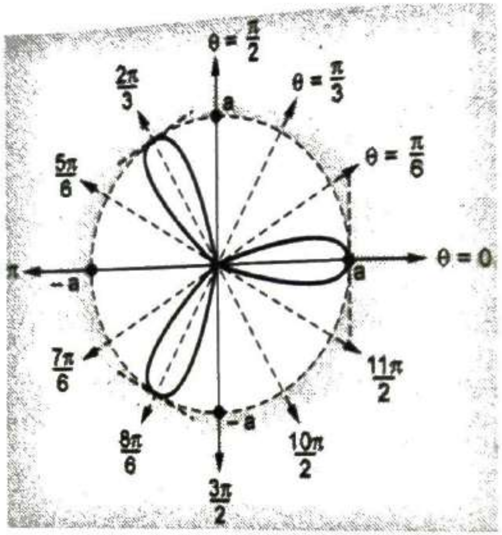


Fig. Q.16.1

Q.17 : Trace the curve r = a sin 3θ

[SPPU : May-14, Dec.-14, Marks 4]

Ans. : The curve, $r = a\sin 3\theta$ can be obtained by rotating $r = a\cos 3\theta$ though an angle $\frac{\pi}{6}$ in anticlockwise direction. as

$$r = a\sin 3\theta = a\cos 3\left(\frac{\pi}{6} - \theta\right)$$

$\therefore$ The rough sketch of the curve is given below.

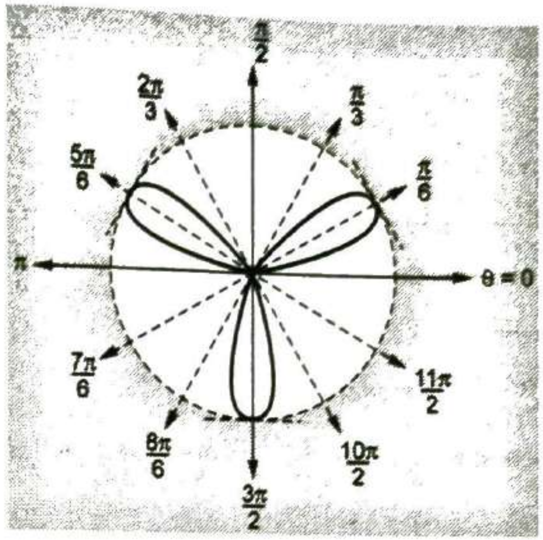


Fig. Q.17.1

Q.18 : Trace the curve $r = a\left(\frac{\sqrt{3}}{2} + \cos\frac{\theta}{2}\right)$.

Ans. :

1) Symmetry - About initial line.
2) Passes through pole.
3) Tangents at the pole is

$$\frac{\sqrt{3}}{2} + \cos\frac{\theta}{2} = 0 \Rightarrow \cos\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \pi - \frac{\theta}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\theta}{2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{5\pi}{6}$$

4)

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	2π
r	$a\left(\frac{\sqrt{3}}{2}+1\right)$	$a\left(\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\right)$	$a\frac{\sqrt{3}}{2}$	$a\left(\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\right)$	0	$-a\left(1-\frac{\sqrt{3}}{2}\right)$

i) Here we observe that as θ increases from 0 to $\frac{5\pi}{3}$, r decreases continuously.

ii) For $\frac{5\pi}{3} < \theta < 2\pi$, r is negative.

$\therefore$ Curve reflects through the origin in opposite quadrants.

5) Angle ϕ

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a\left(\frac{\sqrt{3}}{2}+\cos\frac{\theta}{2}\right)}{-\frac{a}{2}\sin\frac{\theta}{2}}$$

$$= \infty \quad \text{at } \theta = 0, 2\pi$$

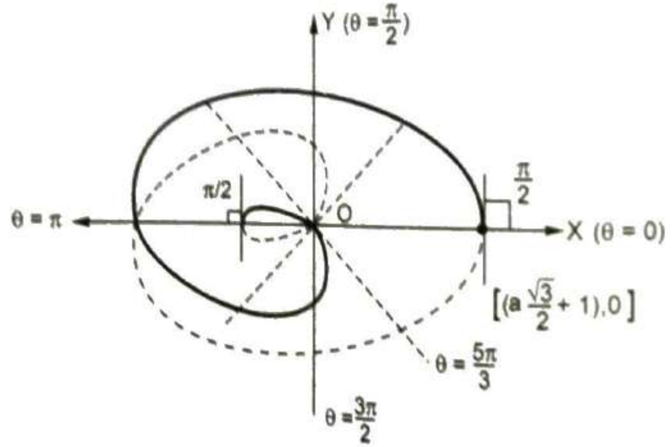


Fig. Q.18.1

Memory Map

	Cartesian Curves	Paramatric Curves	Polar Curves
Rule 1	Symmetry of the curve (X - axis, Y - axis X and Y axes, about origin, y = ± x)	Symmetry of the curve • X - axis • Y - axis • Origin	Symmetry of the curve • About initial line • Pole • Y - axis • $\theta = \frac{\pi}{4}$
Rule 2	Intercepts with co-ordinate axes and symmetric lines (X - axis, Y - axis and y = ± x lines)	Points of intersection with co-ordinate axes : • X - axis • Y - axis	Region of absence of the curve
Rule 3	Nature of tangents a) At origin b) At any other points	Nature of tangents	Poles and tangents at poles
Rule 4	Asymptotes a) Parallel to X - axis b) Parallel to Y - axis c) Oblique asymptote	Asymptotes	Points of intersection
Rule 5	Region of absence of the curve	Region of absence of the curve	Direction of tangents
Rule 6	Points of curve	Variation of x an y w.r.t. t	Asymptotes of the curve
Rule 7	Rough sketch of the curve	Rough sketch of the curve	Rough sketch of the curve.

END...

UNIT - IV

8 Rectification of Curves

Formulae of Rectification of Curve

Sr. No.	Equation of curve	$S = \int ds$
1.	$y = f(x)$	$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
2.	$x = f(y)$	$\int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$
3.	$x = f_1(t)$ $y = f_2(t)$	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
4.	$r = f(\theta)$	$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
5.	$\theta = f(r)$	$\int_{r_1}^{r_2} \sqrt{1+r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr$
6.	$r = f_1(\alpha)$ $\theta = f_2(\alpha)$ α is parameter	$\int_{\alpha_1}^{\alpha_2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + \left(\frac{d\theta}{d\alpha}\right)^2}\, d\alpha$

Type I - Examples of Cartesian Form

Q.1 : Find the circumference of circle of radius a where a is constant.

Ans. : Equation of circles of various 'a' with centre at (0, 0) is $x^2 + y^2 = a^2$

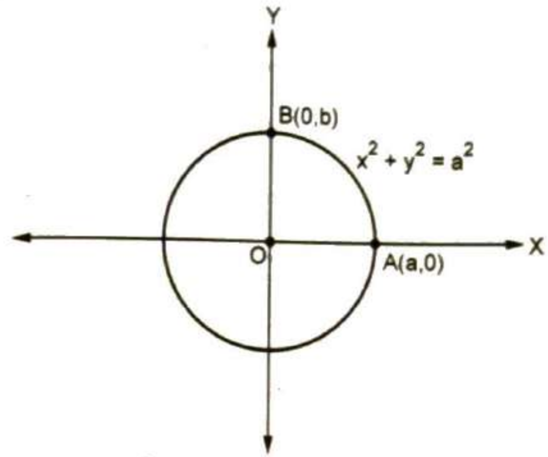


Fig. Q.1.1

$$\therefore \quad y = \sqrt{a^2 - x^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{a^2 - x^2} = \frac{a^2 - x^2 + x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}$$

$$S_{AB} = \text{Length of arc AB} = \int_A^B \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$S_{AB} = \int_0^a \sqrt{\frac{a^2}{a^2 - x^2}}\, dx = \int_0^a \frac{a}{\sqrt{a^2 - x^2}}\, dx$$

$$= a\left[\sin^{-1}\frac{x}{a}\right]_0^a = a\left[\sin^{-1}\frac{a}{a} - \sin^{-1}0\right]$$

$$S_{AB} = a\left[\frac{\pi}{2} - 0\right] = \frac{\pi a}{2}$$

$$S = \text{The complete arc length of circle}$$

$$S = \text{Circumference} = 4 \times \frac{\pi a}{2} = 2\pi a$$

Q.2 : Show that the length of arc of parabola $y^2 = 4ax$ cut-off by the line $3y = 8x$ is a $\left(\log 2 + \frac{15}{16}\right)$ and that cut-off by the latus rectum is $2a[\sqrt{2} + \log(1 + \sqrt{2})]$.

Ans. : The rough sketch of the curve is given below :

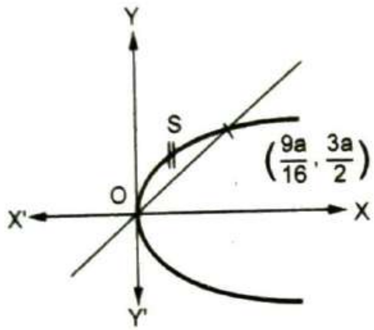


Fig. Q.2.1

Given that $y^2 = 4ax$, $3y = 8x$

$\therefore \quad y = \frac{8x}{3}$

$\therefore \quad y^2 = 4xa$

$\Rightarrow \quad \frac{64x^2}{9} = 4xa$

$$64x^2 - 36xa = 0$$

$$16x^2 - 9xa = 0$$

$$x(16x - 9a) = 0$$

$$x = 0,\ x = \frac{9a}{16}$$

$\therefore$ At $\quad x = 0 \Rightarrow y = 0$

At $\quad x = \frac{9a}{16},\ y = \frac{3a}{2}$

$\therefore$ we get two points (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$.

Thus, $y_1 = 0$ and $y_2 = \frac{3a}{2}$

$$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$\therefore$

$$S = \int_0^{3a/2} \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}\, dy$$

$$= \int_0^{3a/2} \sqrt{1 + \frac{y^2}{4a^2}}\, dy \quad = \frac{1}{2a}\int_0^{3a/2} \sqrt{(4a^2 + y^2)}\, dy$$

$$= \frac{1}{2a}\left[\frac{y\sqrt{y^2 + 4a^2}}{2} + \frac{4a^2}{2}\log\{y + \sqrt{y^2 + 4a^2}\}\right]_0^{3a/2}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\left(\frac{5a}{2}\right) + 2a^2 \log\frac{\frac{3a}{2} + \frac{5a}{2}}{2}\right] - \frac{1}{2a}[0 + 2a^2 \log 2a]$$

$\therefore$

$$\boxed{S = a\left(\frac{15}{16} + \log 2\right)}$$

Now, for second part

$\therefore$

$$S = 2\int_0^{2a} \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}\, dy = 2\int_0^{2a} \sqrt{\left(1 + \frac{y^2}{4a^2}\right)}\, dy$$

$$= 2 \cdot \frac{1}{2a}\left[\frac{y\sqrt{y^2 + 4a^2}}{2} + 2a^2 \log[y + \sqrt{y^2 + 4a^2}]\right]_0^{2a}$$

$$= \frac{1}{a}\left[\frac{2a}{2}\sqrt{4a^2 + 4a^2} + 2a^2 \log\{2a + \sqrt{4a^2 + 4a^2}\} - 2a^2 \log 2a\right]$$

$$= \frac{1}{a}\left[a(a)\,2\sqrt{2} + 2a^2 \log\frac{2a + 2a\sqrt{2}}{2a}\right]$$

$$\therefore \quad \boxed{S_1 = 2a\,[\sqrt{2} + \log(1 + \sqrt{2})]}$$

Q.3 : Find length of arc of parabola $y^2 = 4ax$ from vertex to one extremity of latus rectum.

Ans. : The rough sketch of the curve is shown in Fig. Q.3.1.

Latus rectum is the chord passing through the focus perpendicular to the axis i.e. for $y^2 = 4ax$, $x = a$ is the latus rectum. Substituting $x = a$ in equation of parabola we get $y^2 = 4a^2$.

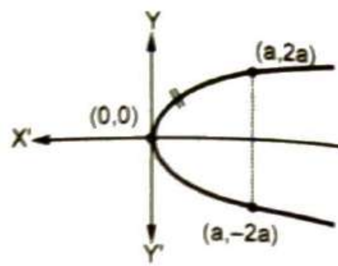


Fig. Q.3.1

i.e. $y = \pm 2a \therefore y_1 = 0,\ y_2 = 2a$.

Hence required length is of arc OA. Here we use formula,

$$\therefore \quad S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$S = \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x = \frac{y^2}{4a}$$

$$\therefore \quad \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2} = \frac{4a^2 + y^2}{4a^2}$$

$$S = \int_0^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy = \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + y^2}\, dy$$

$$S = \frac{1}{2a}\left[\frac{y\sqrt{4a^2 + y^2}}{2} + \frac{4a^2}{2}\log\{y + \sqrt{4a^2 + y^2}\}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a\sqrt{4a^2 + 4a^2}}{2} + 2a^2\log\{2a + \sqrt{4a^2 + 4a^2}\}\right] - \left[2a^2\log\{\sqrt{4a^2}\}\right]$$

$$= \frac{1}{2a}\left[2a\sqrt{2a^2} + 2a^2\log\frac{2a + 2a\sqrt{2}}{2a}\right]$$

$$S = a\,[\sqrt{2} + \log(1 + \sqrt{2})]$$

Q.4 : Show that the length of the arc of the curve $ay^2 = x^3$ from origin to the point whose abscissa is b is $\frac{1}{27\sqrt{a}}(9b + 4a)^{3/2} - \frac{8a}{27}$.

[SPPU : May-09]

Ans. : Curve is symmetrical about X-axis and P is the point on the curve whose abscissa (X co-ordinate) is b.

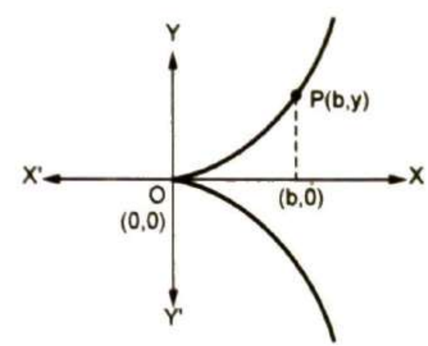


Fig. Q.4.1

Given that $\quad ay^2 = x^3 \quad$ ...(1)

Hence we integrate between the limits 0 and b using formula, we get

$$S = \int_0^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{... (2)}$$

Differentiating the equation of the curve w.r.t. x, we get

$$2a\frac{dy}{dx}y = 3x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9x^4}{4a^2y^2} = \frac{9x^4}{4ax^3} = \frac{9x}{4a}$$

$\therefore$ Equation (2) becomes

$$S = \int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^b \sqrt{1+\frac{9x}{4a}}\,dx$$

$$= \frac{1}{2\sqrt{a}}\int_0^b \sqrt{4a+9x}\,dx$$

$$S = \frac{1}{2\sqrt{a}}\left[\frac{(4a+9x)^{3/2}}{\frac{3}{2}\cdot 9}\right]_0^b$$

$$= \frac{1}{27\sqrt{a}}\left[(4a+9b)^{3/2} - (4a)^{3/2}\right]$$

$$= \frac{(4a+9b)^{3/2}}{27\sqrt{a}} - \frac{8a^{3/2}}{27\sqrt{a}}$$

$$S = \frac{(4a+9b)^{3/2}}{27\sqrt{a}} - \frac{8a}{27}$$

Q.5 : Find the length of the loop of the given curve $3ay^2 = x(a-x)^2$. **[SPPU : Dec.-03, May-06]**

Ans. : The given curve is,

$$3ay^2 = x(a-x)^2 \quad \dots (1)$$

Differentiating w.r.t. x,

$$6ay\frac{dy}{dx} = x\cdot 2(a-x)(-1) + (a-x)^2$$

$$= (a-x)(a-3x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{(a-x)(a-3x)}{6ay}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(a-x)^2(a-3x)^2}{36a^2y^2}$$

$$= 1+\frac{(a-x)^2(a-3x)^2(3a)}{36a^2x(a-x)^2}$$

... Using equation (1)

$$= 1+\frac{(a-3x)^2}{12ax}$$

$$= \frac{12ax + a^2 - 6ax + 9x^2}{12ax}$$

$$= \frac{(a+3x)^2}{12ax} \quad \dots (2)$$

We have

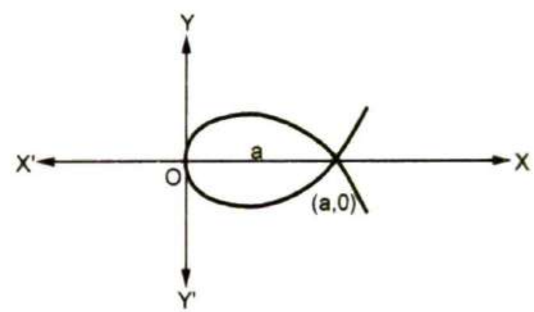


Fig. Q.5.1

Loop is symmetrical about X-axis with x = 0 to x = a hence length of the loop is given by (by symmetry).

$$S = 2\int_{x_1}^{x_2} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$S = 2\int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= 2\int_0^a \sqrt{\frac{(a+3x)^2}{12ax}}\, dx$$

... From equation (2)

$$= 2\frac{1}{2\sqrt{3a}}\int_0^a \left(ax^{-1/2} + 3x^{1/2}\right) dx$$

$$= \frac{1}{\sqrt{3a}}\left[\frac{ax^{1/2}}{1/2} + \frac{3x^{3/2}}{3/2}\right]_0^a$$

$$S = \frac{2}{\sqrt{3a}}\left[a^{3/2} + a^{3/2}\right] = \frac{4}{\sqrt{3}}a$$

Type - II Examples of Parametric Form

Q.6 : Evaluate $\int xy\, ds$ along the arc of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant. [SPPU : Dec.-12, Marks 4]

Ans. : Parametric equations of ellipse are $x = a\cos\theta$, $y = b\sin\theta$. Along the arc in the positive quadrant θ varies from 0 to $\frac{\pi}{2}$.

$$\int xy\, ds = \int xy\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \qquad ...(1)$$

Substituting $x = a\cos\theta$, $y = b\sin\theta$, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$ and as we have to find the length in the first quadrant integrating between the limits $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$I = \int_0^{\pi/2} xy\, ds$$

$$= \int_0^{\pi/2} a\cos\theta \cdot b\sin\theta \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2(1-\sin^2\theta)}\, [2\sin\theta\cos\theta]\, d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2} \sqrt{b^2 + \left(a^2 - b^2\right)\sin^2\theta}\, d(\sin^2\theta)$$

Integrating w.r.t. $\sin^2\theta$

$$I = \frac{ab}{2}\left[\frac{[b^2 + (a^2 - b^2)\sin^2\theta]^{3/2}}{\frac{3}{2}\cdot\left(a^2 - b^2\right)}\right]_0^{\pi/2}$$

$$= \frac{ab}{3\left(a^2 - b^2\right)}\left[\left(a^2\right)^{3/2} - \left(b^2\right)^{3/2}\right]$$

$$= \frac{ab}{3\left(a^2 - b^2\right)}[a^3 - b^3]$$

$$= \frac{ab}{3\left(a^2 - b^2\right)}(a-b)\left(a^2 + ab + b^2\right)$$

$$I = \frac{1}{3}\left[\frac{ab\left(a^2 + ab + b^2\right)}{(a+b)}\right] \qquad \text{... Ans.}$$

Q.7 : Find the length of the loop of the curve $x = t^2$; $y = t\left(1 - \frac{t^2}{3}\right)$ [SPPU : Dec.-03]

Ans. : To trace the curve, it is advisable in this problem to convert the equation into Cartesian form.

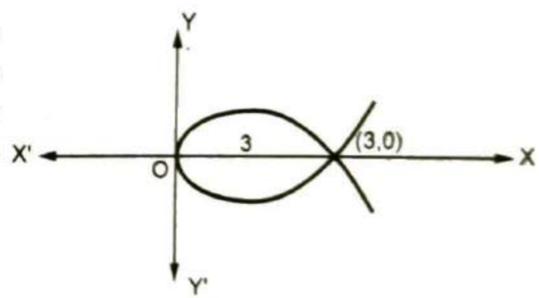


Fig. Q.7.1

$$y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2$$

or

$$y^2 = x\left(1 - \frac{x}{3}\right)^2$$

Loop is symmetrical about X-axis between $x = 0$ and $x = 3$ corresponding to $x = 0$ and $x = 3$ we have $t = 0$ and $t = \sqrt{3}$. $\left(\because x = t^2\right)$

To obtain length of loop, we use formula

$$S_1 = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

which will give length of upper half of the loop.

$x = t^2,\ \frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 1 - t^2$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 1 - 2t^2 + t^4 = \left(1 + t^2\right)^2$$

$$\therefore \qquad S = \int_0^{\sqrt{3}} \sqrt{\left(1 + t^2\right)^2}\, dt = \int_0^{\sqrt{3}} \left(1 + t^2\right) dt$$

As loop is symmetric about X axis

Total length is

$$S = 2\int_0^{\sqrt{3}} [1 + t^2]\, dt = 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}}$$

$$S = 2\left[\sqrt{3} + \frac{3\sqrt{3}}{3}\right] = 4\sqrt{3}$$

Q.8 : Show that the length of an arc of the curve $x = \log(\sec\theta + \tan\theta) - \sin\theta$, $y = \cos\theta$ from $\theta = 0$ to $\theta = t$ is $\log(\sec t)$.

[SPPU : May-13, Marks 4]

Ans. : We have,

$$\frac{dx}{d\theta} = \frac{1}{\sec\theta + \tan\theta}(\sec\theta\tan\theta + \sec^2\theta) - \cos\theta$$

$$= \sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1 - \cos^2\theta}{\cos\theta}$$

$$\frac{dx}{d\theta} = \frac{\sin^2\theta}{\cos\theta} \quad \text{and} \quad \frac{dy}{d\theta} = -\sin\theta$$

$$\therefore \qquad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{\sin^4\theta}{\cos^2\theta} + \sin^2\theta = \sin^2\theta\,[\tan^2\theta + 1]$$

$$= \sin^2\theta\,[\sec^2\theta]$$

$$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

$$\therefore \qquad S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^t \tan\theta\, d\theta$$

$$S = [\log(\sec\theta)]_0^t = \log(\sec t) - 0 = \log(\sec t)$$

Q.9 : Find the complete arc length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

[SPPU : Dec.-18, May-16, Marks 4]

Ans. : Step 1 : Parametric equations of $x^{2/3} + y^{2/3} = a^{2/3}$ are

$$x = a\cos^3\theta,\ y = a\sin^3\theta$$

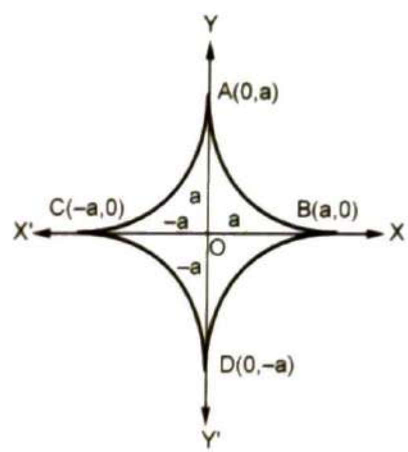


Fig. Q.9.1

As we are using parametric equations and curve is symmetrical in all the four quadrants total length of astroid is given by the formula,

$$S = 4 \int_{AB} \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]}\, d\theta \qquad \ldots(1)$$

We have $\quad x = a\cos^3\theta$

$\therefore \quad \dfrac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$

$y = a\sin^3\theta \Rightarrow \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Consider,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\cos^2\theta\sin^2\theta\,[\cos^2\theta + \sin^2\theta]$$

$$= 9a^2\left(\frac{\sin 2\theta}{2}\right)^2$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\left(\frac{\sin 2\theta}{2}\right)^2$$

In first quadrant limits are from 0 to $\pi/2$. Hence arc length is given by,

$$\therefore \quad S = 4\int_0^{\pi/2} 3a\,\frac{\sin 2\theta}{2}\,d\theta$$

(to get the whole length in all the four quadrants we multiply by 4)

$$S = \frac{12a}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta \quad = 6a\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{6a}{2}[1+1]$$

$$S = 6a$$

Q.10 : Find the arc of length of cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ from one cusp to another cusp. If S is the length of the arc from origin to point P (x, y). Show that $S^2 = 8ay$.

[SPPU : May-17, Marks 4]

Ans. : The rough sketch of the curve is given below in Fig. Q.10.1.

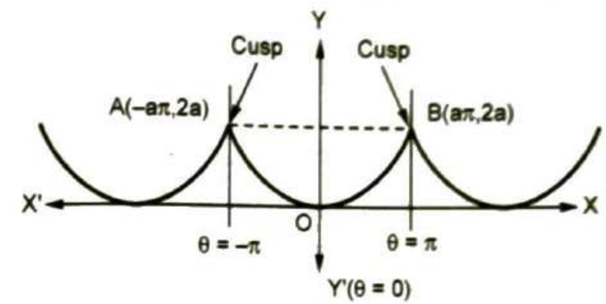


Fig. Q.10.1

Given that

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= 2a^2(1+\cos\theta) = 2a^2\left(2\cos^2\frac{\theta}{2}\right)$$

$$= 4a^2\cos^2\frac{\theta}{2}$$

Part I :

$$S_1 = \text{Length of arc AB} = 2 \text{ length OB}$$

$$= 2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$S_1 = 2\int_0^{\pi} 2a\cos\frac{\theta}{2}\cdot d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

which is the required length from one cusp to another cusp.

Part II

$$S = \int_0^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{\theta} 2a\cos\frac{\theta}{2}\,d\theta = 2a\left[2\sin\frac{\theta}{2}\right]_0^{\theta}$$

$$S = 4a\sin\frac{\theta}{2}$$

$$S^2 = 16a^2\sin^2\frac{\theta}{2} = 16a^2\left(\frac{1-\cos\theta}{2}\right)$$

$$8a\left(a\left(1-\cos\theta\right)\right) = 8ay \quad\Rightarrow\quad \boxed{S^2 = 8\,ay}$$

Q.11 : Find the length of the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ between $\theta = 0$ to $\theta = 2\pi$.

[SPPU : May-15, Marks 4]

Ans. : We have,

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$$

$$= a^2(2-2\cos\theta) = 2a^2(1-\cos\theta)$$

$$= 2a^2(2\sin^2\theta/2) = 4a^2\sin^2(\theta/2)$$

Now, The required arc length is

$$S = \int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_0^{2\pi} 2a\sin\theta/2\,d\theta = 2a\left[-\frac{\cos\theta/2}{1/2}\right]_0^{2\pi}$$

$$= -4a\,[\cos\pi - \cos 0]$$

$$S = -4a[-1-1] = 8a$$

Q.12 : Find the length of the are of the curve $x = e^{\theta}\cos\theta$, $y = e^{\theta}\sin\theta$ from $\theta = 0$ to $\frac{\pi}{2}$.

[SPPU : Dec.-16, Marks 4]

Ans. : Given that $x = e^{\theta}\cos\theta,\ y = e^{\theta}\sin\theta$

$$\therefore \quad \frac{dx}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta = e^{\theta}(\cos\theta-\sin\theta)$$

$$\frac{dy}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta = e^{\theta}(\sin\theta+\cos\theta)$$

Consider

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right) = e^{2\theta}(\cos^2\theta - 2\cos\theta\sin\theta+\sin^2\theta)$$

$$+ e^{2\theta}(\sin^2\theta + 2\sin\theta\cos\theta+\cos^2\theta)$$

$$= e^{2\theta}+e^{2\theta} = 2e^{2\theta}$$

$$\text{Now, Arc length} = S = \int_{\theta_1}^{\theta_2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$\therefore \quad S = \int_0^{\pi/2}\sqrt{2}\,e^{\theta}\,d\theta = \sqrt{2}\,[e^{\theta}]_0^{\pi/2} = \sqrt{2}\,[e^{\pi/2}-e^{0}]$$

$$\boxed{S = \sqrt{2}\left(e^{\pi/2}-1\right)}$$

Type - III : Examples of Polar Form

Q.13 : Find perimeter of cardiod $r = a(1+\cos\theta)$ and show that a line $\theta = \frac{\pi}{3}$ divides upper half of the cardiod.

[SPPU : Dec.-13, 18, May-18, 19, Marks 4]

Ans. : The rough sketch of the curve is given below in Fig. Q.13.1

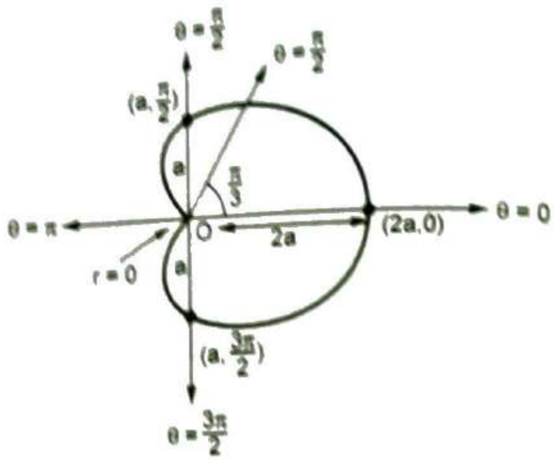


Fig. Q.13.1

The curve is symmetrical about initial line OX. For upper half of the arc θ varies from $\theta = 0$ to $\theta = \pi$. Since the curve is given in polar form we use formula with appropriate limits to find arc length,

$$S = \text{Perimeter of cardiod}$$

$$S = 2[\text{Perimeter of cardiod above x - axis}]$$

$$\therefore \quad S = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \dots (1)$$

Now, given that

$$r = a\,(1 + \cos\theta)$$

$$\therefore \quad \frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\left(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\right)$$

$$= 2a^2\,(1 + \cos\theta)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}\,a\,\sqrt{1 + \cos\theta}$$

Substituting in equation (1), we get,

$$S = 2\int_0^{\pi} \sqrt{2}\,a\,\sqrt{1 + \cos\theta}\, d\theta$$

$$= 2\sqrt{2}\,a\int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta = 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

$\therefore$ The complete arc length of the cardioid is 8a.

To prove second part, we integrate equation (1) between the limits $\theta = 0$ to $\theta = \frac{\pi}{3}$.

$$S_1 = \sqrt{2}\,a\int_0^{\pi/3} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 2a\int_0^{\pi/3} \cos\frac{\theta}{2}\, d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3}$$

$$= 4a\sin\frac{\pi}{6} = 4a\left(\frac{1}{2}\right) = 2a$$

which shows that the line $\theta = \frac{\pi}{3}$ divides upper half.

Q.14 : Find the length of the arc of cardioid $r = a(1 - \cos\theta)$ which lies outside of the circle $r = a\cos\theta$.
[SPPU : Dec.-14, Marks 4]

Ans. : The rough sketch of the curve is given below in Fig. Q.14.1.

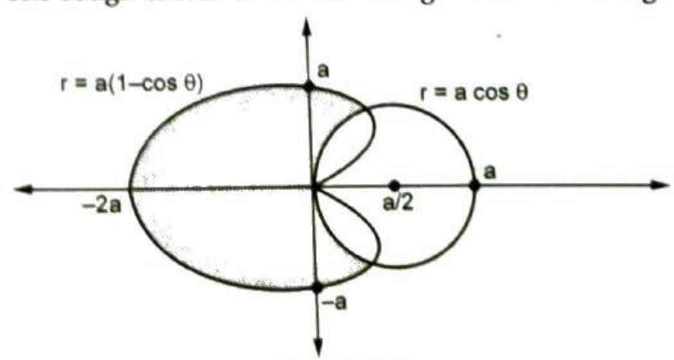


Fig. Q.14.1

We have $x = r\cos\theta,\ y = r\sin\theta$

$$\Rightarrow \quad x^2+y^2 = r^2$$

$$\therefore\ r = a\cos\theta \Rightarrow \quad r^2 = ar\cos\theta$$

$$x^2+y^2 = ax \Rightarrow x^2-ax+y^2 = 0$$

$$\Rightarrow \quad \left(x-\frac{a}{2}\right)^2+(y-0)^2 = \frac{a^2}{4}$$

$\therefore r = a\cos\theta$ is a circle with centre at $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$.

Now, we have intersection points of two curves are

$$r = a(1-\cos\theta) = a\cos\theta$$

$$\Rightarrow \quad 1-\cos\theta = \cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

$\therefore$ We have, $\quad r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = +a\sin\theta$$

$$\therefore \quad r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2(1-\cos\theta)^2+a^2\sin^2\theta$$

$$= a^2[1-2\cos\theta+\cos^2\theta+\sin^2\theta]$$

$$= a^2 2[1-\cos\theta] = 2a^2[2\sin^2\theta/2]$$

$$= 4a^2\sin^2\theta/2$$

$\therefore$ The required arc length $= S = 2\int_{\pi/3}^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$S = 2\int_{\pi/3}^{\pi} 2a\sin\theta/2\,d\theta = 4a\left[-\frac{-\cos\theta/2}{1/2}\right]_{\pi/3}^{\pi}$$

$$= -8a\left[0-\cos\frac{\pi}{6}\right] = -8a\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{S = 4a\sqrt{3}}$$

Q.15 : Find the length of cardiod $r = a(1+\cos\theta)$, which lies, outside the circle $r + a\cos\theta = 0$. **[SPPU : May-04, 10, Dec.-11]**

Ans. : The rough sketch of the curve is given below in Fig. Q.15.1.

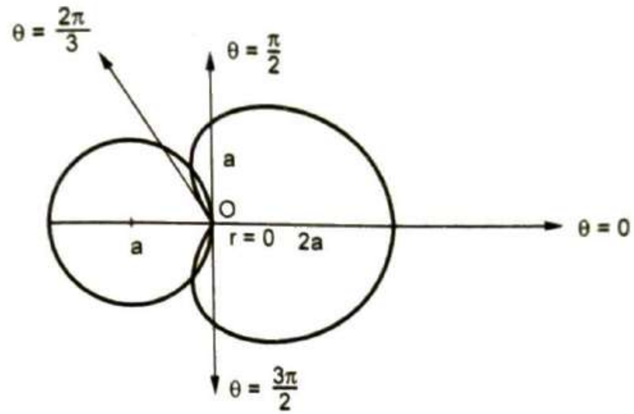


Fig. Q.15.1

The point of intersection of the curve is given by $r = -a\cos\theta$, $r = a(1+\cos\theta)$

$$\Rightarrow \quad a(1+\cos\theta) = -a\cos\theta$$

$$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

The arc length outside the circle is twice the arc length BA.

Required length $= L = \int dS = \int \frac{dS}{d\theta}\,d\theta$

$$\left(\frac{dS}{d\theta}\right)^2 = r^2+\left(\frac{dr}{d\theta}\right)^2$$

$$= a^2[1+2\cos\theta+\cos^2\theta+\sin^2\theta]$$

$$= 2a^2(1+\cos\theta)$$

$$= 2a^2 \cdot 2\cos^2\frac{\theta}{2} = 4a^2\cos^2\frac{\theta}{2}$$

$$L = 2\int_0^{2\pi/3} 2a\cos\frac{\theta}{2}\,d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$\therefore \quad L = 8a\left(\sin\frac{\pi}{3}\right) = 4\sqrt{3}\,a$$

Q.16 : Find the length of the upper arc of one loop of Lemiscate $r^2 = a^2\cos 2\theta$. **[SPPU : May-14, Marks 4]**

Ans. : The rough shech of curve is as follows

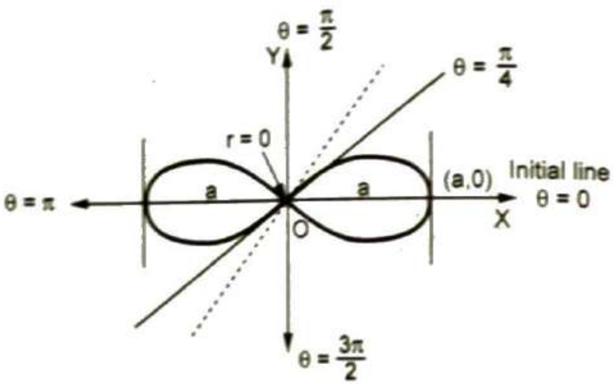


Fig. Q.16.1

For upper arc of the curve, θ varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

$$S = \int_0^{\pi/4}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \text{... (1)}$$

$$r = a\sqrt{\cos 2\theta}$$

$$\frac{dr}{d\theta} = a \cdot \frac{1\,(-2\sin 2\theta)}{2\sqrt{\cos 2\theta}}$$

$$\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta} = \frac{a^2}{\cos 2\theta}$$

$$S = \int_0^{\pi/4}\frac{a}{\sqrt{\cos 2\theta}}\,d\theta \qquad \text{... From equation (1)}$$

Put $2\theta = t \quad \therefore d\theta = \frac{1}{2}dt$

When $\theta = 0,\ t = 0$ and $\theta = \frac{\pi}{4},\ t = \frac{\pi}{2}$

$$S = \frac{a}{2}\int_0^{\pi/2}\frac{dt}{\sqrt{\cos t}} = \frac{a}{2}\int_0^{\pi/2}\sin^0 t\cos^{-1/2} t\,dt$$

$$S = \frac{a}{4}B\left(\frac{0+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{a}{4}\,\frac{\Gamma(1/2)\,\Gamma(1/4)}{\Gamma(3/4)} = \frac{a}{4}\,\frac{\sqrt{\pi}\,\left(\Gamma(1/4)\right)^2}{\Gamma(1/4)\,\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{a}{4}\,\frac{(\sqrt{\pi})\left(\Gamma(1/4)\right)^2}{\pi}\sin\frac{\pi}{4} = \frac{a}{4\sqrt{\pi}}\left(\Gamma(1/4)\right)^2$$

Q.17 : Find the arc length of the curve $r = 2a\cos\theta$. **[SPPU : Dec.-15, Marks 4]**

Ans. : We have $r = 2a\cos\theta$

$$r^2 = 2ar\cos\theta$$

but $x^2 + y^2 = r^2$ and $x = r\cos\theta$

$$\Rightarrow \quad x^2 + y^2 = 2ax$$

$$x^2 - 2ax + y^2 = 0$$

$$(x-a)^2 + (y-0)^2 = a^2$$

This is the circle with centre at (a, 0) and radius a.

The curve is symmetric about x - axis and Limits are $\theta \le 0 \le \pi/2$.

$$r = 2a\cos\theta$$

$$\frac{dr}{d\theta} = -2a\sin\theta$$

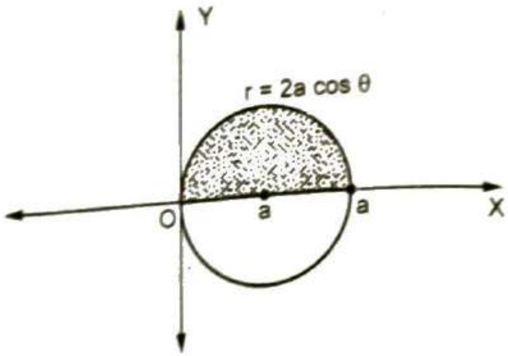


Fig. Q.17.1

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2(\sin^2\theta + \cos^2\theta) = 4a^2$$

$\therefore$ Required arc length = S = 2

[Arc length above X - axis]

$$S = 2\int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi/2} 2a\, d\theta$$

$$S = 4a\frac{\pi}{2} = 2\pi a$$

Memory Map

Sr. No.	Equation of curve	$S = \int ds$
1.	$y = f(x)$	$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
2.	$x = f(y)$	$\int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

3.	$x = f_1(t)$ $y = f_2(t)$	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
4.	$r = f(\theta)$	$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
5.	$\theta = f(r)$	$\int_{r_1}^{r_2} \sqrt{1+r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr$
6.	$r = f_1(\alpha)$ $\theta = f_2(\alpha)$ α is parameter	$\int_{\alpha_1}^{\alpha_2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + \left(\frac{d\theta}{d\alpha}\right)^2}\, d\alpha$

END...

UNIT - V

9 Co-ordinate Systems

9.1 : The Cartesian Co-ordinate System

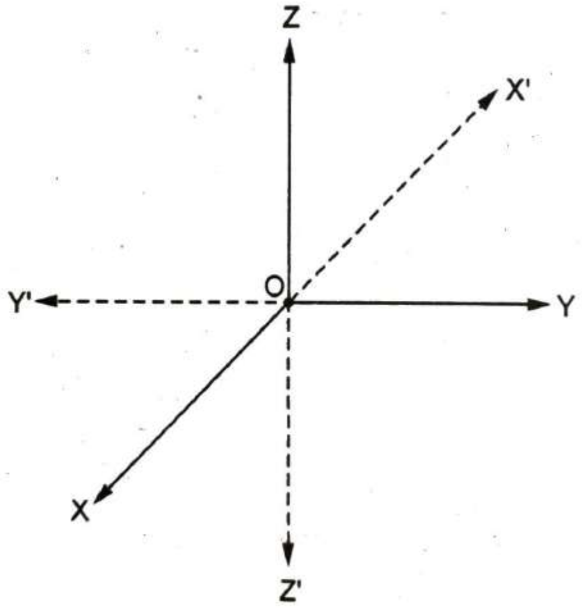


Fig. 9.1

In three dimensional geometry we have three mutually perpendicular lines X'OX, Y'OY, Z'OZ intersecting at O. These lines are known as co-ordinate axes and

OX indicates positive direction of X-axis.

OY indicates positive direction of Y-axis.

OZ indicates positive direction of Z-axis.

OX' indicates negative direction of X-axis.

OY' indicates negative direction of Y-axis.

OZ' indicates negative direction of Z-axis

The plane in which X-axis, Y-axis lies is XY plane.

The plane in which Y-axis, Z-axis lies is YZ plane.

The plane in which Z-axis, X-axis lies is ZX plane.

Equation of XY plane or XOY plane is Z = O.

Equation of YZ plane or YOZ plane is X = O.

Equation of ZX plane or ZOX plane is Y = O.

The above three planes are known as co-ordinate planes, these planes divide the entire space in 8 equal parts called as **octants**. The octant in which X, Y, Z are positive is known as **positive octant**. Similar to two dimensions the position of a point P in three dimensions is denoted by three real numbers (x, y, z). These are the distances respectively from the origin to the intersection of the perpendicular dropped from point P to X, Y, Z planes respectively.

Draw perpendicular from point P(x, y, z) on z = 0 plane and denote foot of perpendicular by M (x, y, 0). Draw X axis and denote it by A (x, 0, 0) and on Y axis and denoted by B (0, y, 0). Draw perpendicular from O on Z axis and denote point by C (0, 0, z).

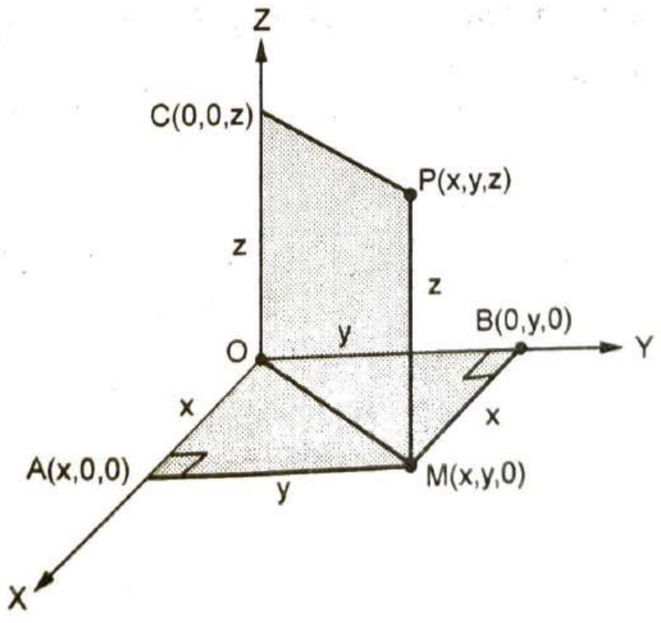


Fig. 9.2

For example : Let P(1, 2, 3) be any point in space.

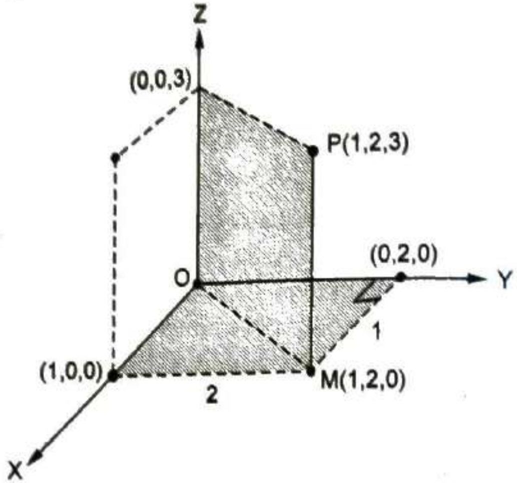


Fig. 9.3

9.2 : Spherical Polar Co-ordinate System

Let P (r, θ, ϕ) be any point in space. Draw perpendicular form P on z = 0 plane and denote foot of perpendicular by Q.

Join OP and OQ.

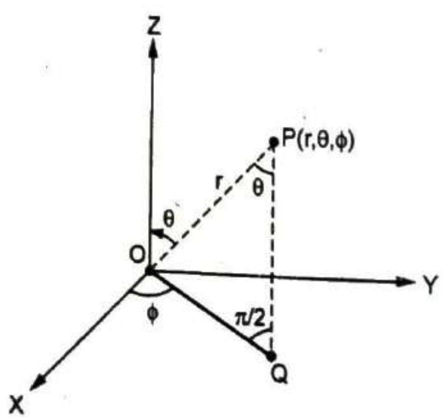


Fig. 9.4

Here

$r = l\,(OP),\ 0 \le r \le \infty$

θ = Angle between OP and positive Z-axis.

$\therefore \quad 0 \le \theta \le \pi$

ϕ = Angle between OQ with positive X-axis.

$\therefore \quad 0 \le \theta \le 2\pi$

Therefore any point of a space is uniquely denoted by (r, θ, ϕ). This representation of D is known as the spherical co-ordinate system.

9.3 : Relation between Cartesian System and Spherical Polar Co-ordinate System

Let (x, y, z) be Cartesian co-ordinates of point P. From Fig. 9.6, OP = Distance of point P from origin.

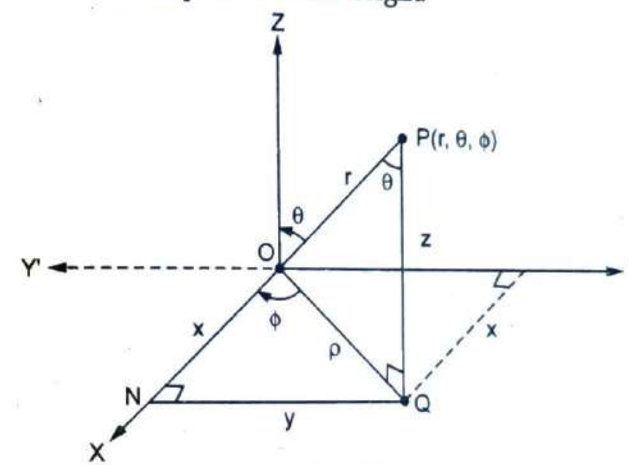


Fig. 9.5

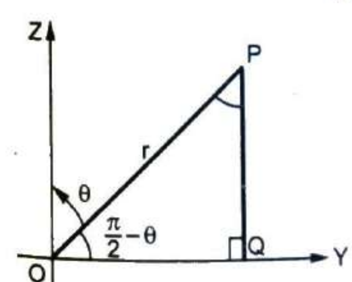


Fig. 9.6

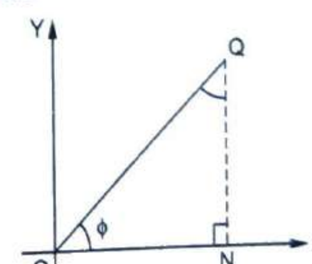


Fig. 9.7

i.e. OP = Distance between (0,0,0) and (x, y, z)

$$= \sqrt{x^2 + y^2 + z^2} = r$$

From Δ OPQ (Fig. 9.6)

$$OP = r$$

$$PQ = r \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore \quad z = r \cos\theta \qquad \ldots (1)$$

$$OQ = r \cos\left(\frac{\pi}{2} - \theta\right)$$

$$= r \sin\theta$$

From Δ ONQ (Fig. 9.7)

$$ON = OQ \cos\phi$$

$$\therefore \quad x = r \sin\theta \cos\phi \qquad \ldots (2)$$

$$QN = OQ \sin\phi$$

$$y = r \sin\theta \sin\phi \qquad \ldots (3)$$

Thus, from equations (1), (2) and (3)

$$\boxed{\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}}$$

This is the relation between cartesian and spherical co-ordinate systems.

Also from equations (1), (2) and (3), if x, y, z are positive.

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left|\frac{\sqrt{x^2 + y^2}}{z}\right|$$

$$\phi = \tan^{-1}\left|\frac{y}{x}\right|$$

Note :

1) For spherical polar co-ordinates

i) If $z > 0$, then $0 \le \theta \le \pi/2$

ii) If $z < 0$, then $\pi/2 \le \theta \le \pi$

iii) If $x > 0$, $y > 0$ then $0 \le \phi \le \pi/2$

iv) If $x < 0$, $y > 0$ then $\pi/2 \le \phi \le \pi$

v) If $x < 0$, $y < 0$ then $\pi \le \phi \le 3\pi/2$

vi) If $x > 0$, $y < 0$ then $3\pi/2 \le \phi \le 2\pi$

2) If z co-ordinate is + ve then θ is acute i.e

$$\theta = \tan^{-1}\left|\frac{\sqrt{x^2 + y^2}}{z}\right|$$

If z co-ordinate is – ve then θ is obtuse i.e

$$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{x^2 + y^2}}{z}\right|$$

3) The quadrant of the point depends on + ve, and – ve values of x and y, thus φ can be calculated using Fig. 9.8.

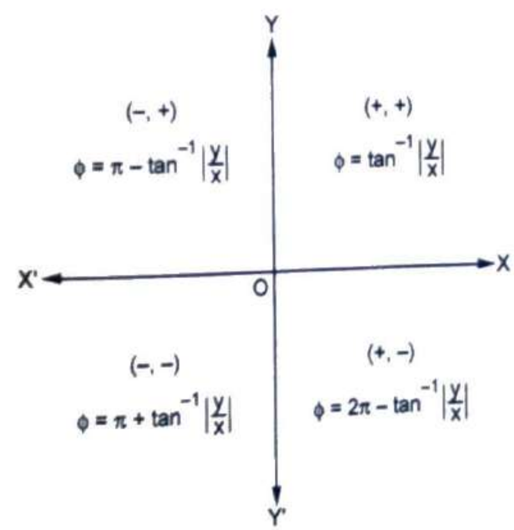


Fig. 9.8

9.4 : Cylindrical Polar Co-ordinates

From Fig. 9.5 let OQ = ρ

$\therefore$ PQ = z

from triangle OQN

$$ON = OQ \cos\phi$$

$$NQ = OQ \sin\phi$$

$$\therefore \quad \boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}}$$

where $\rho = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\left|\frac{y}{x}\right|$ (if x, y are + ve)

Note :

1) The quadrant of the point depends on + ve and –ve values of x and y, thus ϕ can be calculated using Fig. 9.8.

9.5 : Solved Examples

Q.1 : Find the cartesian co-ordinates of the point $\left(3, \frac{2\pi}{3}, \frac{\pi}{6}\right)$.

Ans. : Given spherical polar co-ordinates are $r = 3, \theta = \frac{2\pi}{3}$, $\phi = \frac{\pi}{6}$.

We have

$$x = r\sin\theta\cos\phi = 3\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$$

$$y = r\sin\theta\sin\phi = 3\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $z = r\cos\theta = 3\cos\left(\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$

Thus the cartesian co-ordinates of given point are

$$(x, y, z) = \left(\frac{9}{4}, \frac{3\sqrt{3}}{4}, \frac{-3}{2}\right)$$

Q.2 : Find the spherical polar and cylindrical co-ordinates of a point (1, 1, 1).

Ans. : Given cartesian co-ordinates are x = 1, y = 1, z = 1.

$$\therefore \quad r_2 = \sqrt{x^2+y^2+z^2} = \sqrt{3}$$

$$\theta = \tan^{-1}\left|\frac{\sqrt{x^2+y^2}}{z}\right| = \tan^{-1}\left(\frac{\sqrt{2}}{1}\right) = 54.74°$$

and $\phi = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}1 = 45°$

$\therefore$ The spherical polar co-ordinates are $(\sqrt{3}, 54.74°, 45°)$

Q.3 : Find the spherical polar and cylindrical co-ordinates of (– 3, – 4, – 5).

Ans. : $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 16 + 25} = 5\sqrt{2}$

As z is – ve.

$$\therefore \quad \theta = \pi - \tan^{-1}\left|\frac{\sqrt{x^2+y^2}}{z}\right|$$

$$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{9+16}}{5}\right|$$

$$\theta = \pi - \tan^{-1}(1)$$

$$\theta = \pi - \frac{\pi}{4}$$

$$\theta = \frac{3\pi}{4}$$

$$\theta = 135°$$

As x, y both are – ve

$$\therefore \quad \phi = \pi + \tan^{-1}\left|\frac{y}{x}\right|$$

$$\phi = \pi + \tan^{-1}\frac{4}{3}$$

$$\phi = 233^\circ 8'$$

(Note : Don't write degrees using decimal's)

Thus $(r, \theta, \phi) = \left(5\sqrt{2},\ 135^\circ,\ 233^\circ\ 8'\right)$

are the spherical polar co-ordinates.

Now to find cylindrical co-ordinates.

$$\rho = \sqrt{x^2 + y^2} = \sqrt{9 + 16} = 5$$

$$\phi = \pi + \tan^{-1}\left|\frac{y}{x}\right| \qquad \text{(Same as above)}$$

$$= 233^\circ 8'$$

$$z = z = -5$$

$\therefore$ $(\rho, \phi, z) = (5,\ 233^\circ 8',\ -5)$

are the cylindrical polar co-ordinates.

Q.4 : Find Cartesian co-ordinates of $\left(2, \frac{5\pi}{6}, \frac{-3\pi}{4}\right)$.

Ans. : Given spherical polar co-ordinates.

$$r = 2, \quad \theta = \frac{5\pi}{6}, \quad \phi = \frac{-3\pi}{4}$$

$\therefore$

$$x = r \sin\theta \cos\phi$$

$$= 2 \sin\left(\frac{5\pi}{6}\right) \cdot \cos\left(\frac{-3\pi}{4}\right) = 2 \cdot \left(\frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}$$

$$y = r \sin\theta \sin\phi$$

$$= 2 \sin\left(\frac{5\pi}{6}\right) . \sin\left(\frac{-3\pi}{4}\right)$$

$$= 2 \cdot \left(\frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$$

$$z = r \cos\theta$$

$$= 2 \cos\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

Thus $(x, y, z) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, -\sqrt{3}\right)$

Q.5 : Find 1) polar 2) cylindrical equations of the right circular cone whose cartesian equation is given by $x^2 + y^2 = z^2 \tan^2 a$.

Ans. : 1) We have polar transformations

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$y = r \cos\theta$$

$\therefore$ Given equation of cone becomes

$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 = (r \cos\theta)^2 \tan^2 a$$

$$\therefore r^2 \sin^2\theta\,(\cos^2\phi + \sin^2\phi) = r^2 \cos^2\theta \tan^2 a$$

$$\therefore \tan^2\theta = \tan^2 a$$

$\therefore$ $\theta = a$ which is required polar equation.

2) We have cylindrical transformations

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

$\therefore$ Given equation of cone becomes

$$(\rho\cos\phi)^2 + (\rho\sin\phi)^2 = z^2 \tan^2 a$$

$\therefore$

$$\boxed{\rho = z \tan a}$$

which is required equation.

9.6 : Important Formulae and Definitions

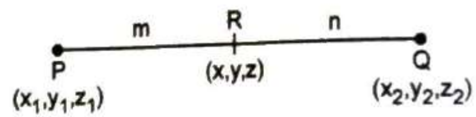
A) Distance formula :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points then distance between; them is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

B) Section formula :

i) Internal Division :



If the point R divides the line segment PQ internally in the ratio m : n then the co-ordinates of R are given by

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}, \quad z = \frac{mz_2 + nz_1}{m + n}$$

ii) External Division : If R divides PQ externally in the ratio m : n then the co-ordinates of R are

$$x = \frac{mx_2 - nx_1}{m - n}, \quad y = \frac{my_2 - ny_1}{m - n}, \quad z = \frac{mz_2 - nz_1}{m - n}$$

iii) Mid Point Formula : If R is the mid point of line segment PQ then it divides PQ in the ratio 1 : 1

$\therefore$ the co-ordinates of R are

$$x = \frac{x_2 + x_1}{2}, \quad y = \frac{y_2 + y_1}{2}, \quad z = \frac{z_2 + z_1}{2}$$

C) Direction cosines of a line :

If α, β, γ are the angles made by the given line with + ve x, y, and z axes respectively then $\cos\alpha, \cos\beta, \cos\gamma$ are known as direction cosines of a line (OR d.c's of a line).

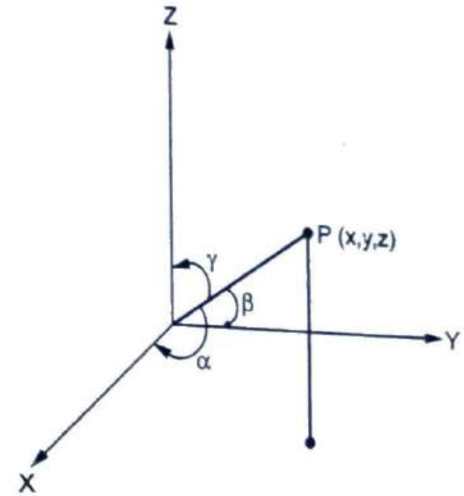


Fig. 9.9

i) As X-axis makes angles of 0, 90°, 90° with X, Y, Z axes respectively $\therefore$ cos 0, cos 90, cos 90 are dc's of X-axis.

$\therefore$ (1, 0, 0) are dc's of X-axis.

Similarly

(0, 1, 0) are dc's of Y-axis.

(0, 0, 1) are dc's of Z-axis.

ii) If l, m, n are dc's of a given line OP and (x, y, z) are co-ordinates of point P where OP = r then

$$x = r\cos\alpha = lr$$
$$y = r\cos\beta = mr$$
$$z = r\cos\gamma = nr$$

$$\therefore \quad l^2 + m^2 + n^2 = 1 \text{ i.e. } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

iii) If l_1, m_1, n_1 and l_2, m_2, n_2 are dc's of two lines then

a) Angle between them is given by

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

b) Two lines are parallel if

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \text{Constant}$$

c) Two lines are perpendicular if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

D) Direction ratio's of a line :

The numbers a, b, c which are proportional to dc's l, m, n are known as direction ratio's OR dr's of a line.

i.e. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{1}{\sqrt{a^2+b^2+c^2}}$

i) We can calculate dc's from dr's a, b, c by using

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

ii) Let A (x_1, y_1, z_1), B (x_2, y_2, z_2). The dr's of line AB are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

iii) If a_1, b_1, c_1 and a_2, b_2, c_2 are dr's of two lines then

a) Angle between two lines is given by

$$\boxed{\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}}$$

b) Two lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ = Constant.

c) Two lines are perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

E) Projection formula of a line segment :

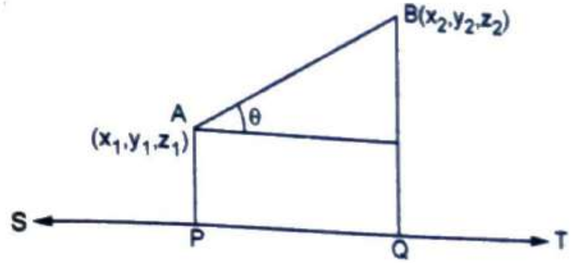


Fig. 9.10

Let l, m, n be dc's of line ST. Let AB be the given line segment its projection on line ST is given by PQ.

where $PQ = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

9.7 : The Plane

1) Equations of plane :

a) General form :

$ax + by + cz + d = 0$ where constants a, b, c are dr's of normal to the plane.

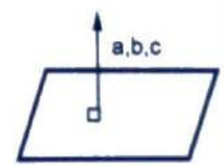


Fig. 9.11

b) Passing through origin :

$$ax + by + cz = 0$$

c) Equation of a plane passing through (x_1, y_1, z_1) and having a, b, c as dr's of normal is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

d) Intercept form : The plane which makes intercepts a, b, c on co-ordinate axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

e) **Normal form** : If l, m, n are dc's of normal to the plane and 'p' is the length of perpendicular from origin to the plane, then its equation is given by

$$lx + my + nz = p$$

f) The equation of plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d_1 = 0$

Note :

i) From any equation of plane the coefficients of x, y, z gives the dr's of normal to the plane.

ii) Two planes are parallel if their normals are parallel.

iii) Two planes are perpendicular if their normals are perpendicular.

iv) Angle between two planes $a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x+b_2y+c_2z+d_2 = 0$ is the angle between their normals having dr's a_1, b_1, c_1 and a_2, b_2, c_2 respectively.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

2) Length of perpendicular :

a) The length of perpendicular from a point (x_1, y_1, z_1) to the plane

$ax + by + cz + d = 0$ is

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Length of perpendicular from (0, 0, 0) to the plane

$ax + by + cz + d = 0$ is

$$p = \left|\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

b) Equation of the plane passing through the intersection of two planes

$a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x+b_2y+c_2z+d_2 = 0$ is

$(a_1x+b_1y+c_1z+d_1) + \lambda\,(a_2x+b_2y+c_2z+d_2) = 0$

where λ is a parameter.

c) The equation of plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

3) Equations of a line :

i) As straight line is the intersection of two planes.

ii) Two point formula : The line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

iii) Symmetrical form : The line having dr's a, b, c and passing through (x_1, y_1, z_1) is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

I) Coplanirity of two lines :

Two lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and

$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are

coplanar if $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

END...

UNIT - V

10 The Sphere

10.1 : Equations of Sphere in Different Forms

1. Centre and radius form : Let P(x, y, z) be any point on the sphere and A (a, b, c) be the centre of the sphere and r be the radius.

$$\therefore \quad (AP)^2 = r^2$$

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2 = r^2} \quad \dots (1)$$

Particular case :

i) If the centre of the sphere is at O (0,0,0) then equation (1) becomes

$$\boxed{x^2+y^2+z^2 = r^2} \quad \dots (2)$$

This is called the **standard form** of the sphere.

ii) If the radius of the sphere is 1, then equation (1) becomes

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2 = 1} \quad \dots (3)$$

This is known as the unit sphere.

2. General form : The equation of the sphere in centre and radius form is

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$$

$$\Rightarrow \quad x^2+y^2+z^2-2ax-2by-2zc\,(a^2+b^2+c^2-r^2) = 0$$

Put $a = -u$, $b = -v$, and $c = -w$

$$a^2+b^2+c^2-r^2 = d$$

and

$$\boxed{x^2+y^2+z^2+2ux+2vy+2wz+d = 0} \quad \dots (4)$$

This is known as the **general form** of the sphere. This general form is obtained by using the centre and radius form.

Can we obtain centre and radius from the general form of the sphere ? (Yes).

The centre of the sphere is at A (a, b, c) and $a = -u$, $b = -v$ and $c = -w$.

$$\therefore \quad \boxed{\text{Centre is at } A(-u, -v, -w)}$$

And its radius is,

$$r^2 = a^2+b^2+c^2-d$$

$$= u^2+v^2+w^2-d$$

$$\therefore \quad \boxed{r = \sqrt{u^2+v^2+w^2-d}} \quad \boxed{r \geq 0}$$

Important note :

i) If $u^2+v^2+w^2-d > 0$ i.e. $r > 0$, equation (4) represents a sphere with centre at (–u, –v, –w) and real radius.

ii) If $u^2+v^2+w^2-d = 0$ i.e. $r = 0$, equation (4) represents a sphere with centre at (–u, –v, –w) and radius zero i.e. the sphere coincides with the centre. Such sphere is know as the point **sphere**.

iii) If $u^2+v^2+w^2-d < 0$ i.e. $r < 0$, then we can't draw sphere.

iv) Equation (4) is a second degree equation in x, y, z with coefficients of x^2, y^2, z^2 are and terms xy, yz, zx are absent.

It contains four arbitrary constants u,v,w and d.

So to define unique sphere, we require four points of that sphere.

e.g. a) The general form of the sphere is,

$$x^2+y^2+z^2-3x+5y-6z+8 = 0.$$

Comparing this equation with equation (4)

We get 2 u = –3, 2v = 5, 2w = – 6 and d = + 8,

$u = -\frac{3}{2}, \ v = \frac{5}{2}, \ w = -\frac{6}{2}$

$\therefore$ Centre is at $(-u,-v,-w) = \left(\frac{3}{2}, -\frac{5}{2}, 3\right)$

and

$$\text{radius} = \sqrt{u^2+v^2+w^2-d}$$
$$= \sqrt{\frac{9}{4}+\frac{25}{4}+9-8}$$
$$\text{Radius} = \sqrt{\frac{38}{4}}$$

3. Intercept Form : To find the equation of the sphere which cuts off x, y, and z axes.

To define sphere, we require four points.

Let us take fourth point as 0 (0, 0, 0).

Let the equation of required sphere passing through A(a, 0, 0), B(0, b, 0), C(0, 0, c) and 0(0, 0, 0).

Consider the sphere.

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0 \quad \ldots(5)$$

This sphere passes through (0,0,0) $\Rightarrow \boxed{d = 0}$

The sphere equation (5) passes through A (a, 0, 0).

$\therefore$ We get

$$a^2+0+0+2ua+0+0+0 = 0$$
$$a^2+2ua = 0$$
$$\Rightarrow \quad a(a+2u) = 0$$

But $a \neq 0 \quad \therefore \ a + 2u = 0 \Rightarrow 2u = -a$

Similarly $\boxed{2v = -b}$ and $\boxed{2w = -c}$

$\therefore$ We get

$$\boxed{x^2+y^2+z^2-ax-by-cz = 0}$$

This is the equation of sphere in intercept form.

e.g. The equation of sphere passing through (2, 0, 0) (0, 5, 0) (0, 0, –3) and (0, 0, 0) is

$$x^2+y^2+z^2-2x-5y+3z = 0$$

4. Diameter form : Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the extremities of the diameter of the sphere.

Let P(x, y, z) be any point on the sphere.

Join A & P and B & P.

$\therefore$ From Fig. 10.1 AP is perpendicular to BP;

$\angle APB = 90°$

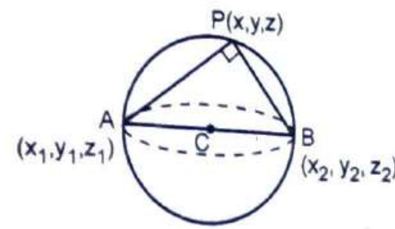


Fig. 10.1

The direction ratio's of AP are $x-x_1, y-y_1, z-z_1$.

The direction ratio's of BP are $x-x_2, y-y_2, z-z_2$. As AP $\perp$ BP, by using the condition of perpendicularity, we get

$$\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2) = 0}$$

This is the equation of sphere in diameter form.

Note : The expansion of above equation leads to the general form of the equation of sphere.

We have,

$$x^2-(x_1+x_1)x+x_1x_2+y^2-(y_1+y_2)y+y_1y_2+z^2 - (z_1+z_2)z+z_1z_2 = 0$$

$$\therefore \quad x^2+y^2+z^2-(x_1+x_2)x-(y_1+y_2)y-(z+z_2)z+x_1x_2+y_1y_2+z_1z_2=0$$

⇒ Comparing this with the general form of sphere we get,

$$2u = -(x_1+x_2),\ 2v=-(y_1+y_2)$$

$$2w = -(z_1+z_2),$$

and $$d = x_1x_2+y_1y_2+z_1z_2$$

⇒ $$-u = \frac{x_1+x_2}{2},\ -v=\frac{y_1+y_2}{2},$$

$$-w = \frac{z_1+z_2}{2}$$

Thus, the centre of the sphere is at

$$(-u,\ -v,\ -w) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$ i.e. at the

mid point of AB. Moreover radius

$$l(CA) = l(CB)$$

where C is the center of the sphere. The equation of the sphere whose diameter is AB where A = (1, 2, 3) and B = (2, –1, 4) is,

$$x^2+y^2+z^2-(1+2)x-(2-1)y-(3+4)z+2-2+12 = 0$$

$$x^2+y^2+z^2-3x-y+7z+12 = 0$$

10.2 : Touching Spheres

We know that two curves touch each other means they have only one common point and tangents at that point are same for both curves.

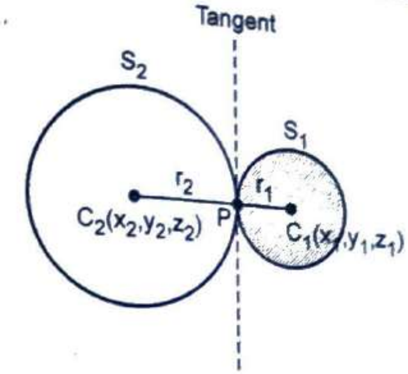


Fig. 10.2

a) **Touch externally** : Two spheres touch externally if the distance between their centers is equal to the sum of their radii. Please refer Fig. 10.2.

i.e. $$d = r_1+r_2 = d(c_1c_2)$$

The point of contact P divides C_1C_2 internally in the ratio $r_1:r_2$

Let, $P(x,y,z)$, $C_1(x_1,y_1,z_1)$ and $C_2(x_2,y_2,z_2)$

then, $$x = \frac{r_2x_1+r_1x_2}{r_1+r_2},\ y=\frac{r_2y_1+r_1y_2}{r_1+r_2},$$

$$z = \frac{r_2z_1+r_1z_2}{r_1+r_2}$$

b) **Touch internally** : Two spheres touch internally if the distance between their centers is equal to the positive difference of their radii.

i.e. $$d(C_1C_2) = |r_1-r_2|$$

The point of contact $P(x,y,z)$ divides C_1C_2 externally in the ratio $r_1:r_2$

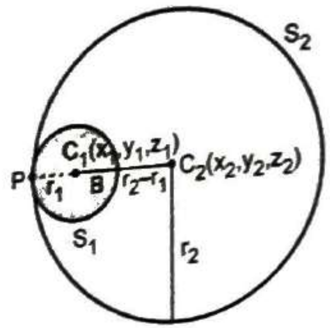


Fig. 10.3

Q.1 : Show that the spheres $x^2+y^2+z^2 = 25$ and $x^2+y^2+z^2-18x-24y-40z+225 = 0$ touch externally and find their point of contact. **[SPPU : Dec.-18, Marks 05]**

Ans. : Let, $S_1 = x^2+y^2+z^2-18x-24y-40z+225 = 0$

$$S_2 = x^2+y^2+z^2-25 = 0$$

Comparing these spheres to the general form

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

The center of sphere S_1 is at $C_1(-u, -v, -w)$

$\therefore$ $C_1(9, 12, 20)$

And, $\text{radius} = r_1 = \sqrt{9^2+12^2+20^2-225} = 20$

The center of sphere S_2 is at $C_2(0, 0, 0)$

And, $\text{radius} = r_2 = \sqrt{u^2+v^2+w^2-d}$

$$= \sqrt{0+0+0+25} = 5$$

Now, $d(C_1C_2) = \sqrt{(9-0)^2+(12-0)^2+(20-0)^2} = \sqrt{625} = 25$

and, $r_1 + r_2 = 20+5 = 25$

Thus, $d(c_1c_2) = r_1 + r_2$

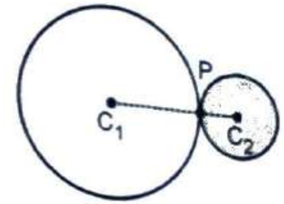


Fig. Q.1.1

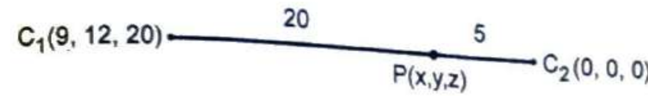


Fig. Q.1.1 (a)

Hence, S_1 and S_2 touch externally. The point of contact $P(x, y, z)$ divides C_1C_2 in the ratio 20 : 5 externally.

$$\therefore \quad x = \frac{9\times 5+0\times 20}{20+5} = \frac{45}{25} = \frac{9}{5}$$

$$y = \frac{12\times 5+0\times 20}{20+5} = \frac{60}{25} = \frac{12}{5}$$

$$z = \frac{20\times 5+0\times 20}{20+5} = \frac{100}{25} = 4$$

Therefore, the point of contact is

$$\boxed{P\left(\frac{9}{5}, \frac{12}{5}, 4\right)}$$

Q.2 : Find the equation of sphere passing through (1, 0, 0), (0, 1, 0), (0, 0, 1) and having least possible radius. **[SPPU : Dec.-10]**

Ans. : Let

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

be the equation of sphere.

This passes through (1, 0, 0)

$$\Rightarrow \quad 1 + 2u + d = 0$$

i.e. $u = \frac{(d+1)}{-2}$

Similarly, $v = \frac{1+d}{-2},\ w = \frac{1+d}{-2}$

We know that

$$r^2 = u^2 + v^2 + w^2 - d$$

$$\therefore \quad r^2 = \frac{3(d+1)^2}{4} - d$$

$$= \frac{3\left(d^2 + 2d + 1\right) - 4d}{4}$$

$$r^2 = \frac{3d^2 + 2d + 3}{4} = f(d) \text{ say}$$

For least possible radius $f'(d) = 0$

i.e. $6d + 2 = 0$

$$\therefore \quad d = -\frac{1}{3}$$

$$\therefore \quad 2u = 2v = 2w = -\frac{2}{3}$$

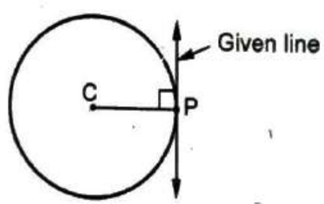
Hence the required equation of sphere is

$$x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$$

Q.3 : Find the equation of the sphere which has its centre at C(2, 3, –1) and touches the line $\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4}$

[SPPU : Dec.-10]

Ans. : Let $\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4} = t$ say



$$\therefore \quad x = -5t - 1,\ y = 3t + 8,\ z = 4t + 4$$

are the co-ordinates of point of contact P.

$\therefore$ dr's of CP are $-5t - 1 - 2,\ 3t + 8 - 3,\ 4t + 4 + 1$

i.e. $-5t - 3,\ 3t + 5,\ 4t + 5$

Also dr's of touching line are $-5, 3, 4$

as these two lines are perpendicular

$$\therefore \quad -5(-5t - 3) + 3(3t + 5) + 4(4t + 5) = 0$$

$$\therefore \quad 25t + 15 + 9t + 15 + 16t + 20 = 0$$

$$50t = -50$$

$$t = -1$$

$\therefore$ Co-ordinates of 'P' are (4, 5, 0) and distance CP is

i.e. $\sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2} = 3$

$\therefore$ Using centre radius form

$$(x-2)^2 + (y-3)^2 + (z+1)^2 = 3^2$$

i.e. $x^2 + y^2 + z^2 - 4x - 6y + 2z + 5 = 0$

is the required sphere

Q.4 : Find the equation of the sphere which touches the co-ordinate axes, whose centre is in the positive octant and has radius 4. **[SPPU : Dec.-16, Marks 5]**

Ans. : Let the equation of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{... (1)}$$

As the radius is 4.

$$u^2 + v^2 + w^2 - d = 16 \quad \text{... (2)}$$

As equation of x-axis is $y = 0, z = 0$

$\therefore$ from equation (1) we get

$$x^2 + 2ux + d = 0$$

As x-axis touches the sphere the two roots of this equation must be equal i.e. discriminant = 0

$\therefore \quad 4u^2 - 4d = 0$

$\therefore \quad u^2 = d$

Similarly $\quad v^2 = d, \; w^2 = d$

Substituting in equation (2) we get

$$d = 8$$

$\therefore \quad u^2 = v^2 = w^2 = 8$

$\therefore \quad u = v = w = \pm 2\sqrt{2}$

As the centre is in positive octant

$\therefore \quad (-u, -v, -w) = \left(2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}\right)$

$\therefore$ The equation of sphere is

$$x^2 + y^2 + z^2 - 4\sqrt{2}(x + y + z) + 8 = 0$$

Q.5 : A sphere of radius r passes through the origin and meets the axes in A, B, C show that the locus of centroid of triangle ABC is $9\left(x^2 + y^2 + z^2\right) = 4r^2$. **[SPPU : May-07]**

Ans. : Let

$A(x_1, 0, 0), B(0, y_1, 0), C(0, 0, z_1)$ be the co-ordinates.

Then equation of sphere OABC in intercept form is

$$x^2 + y^2 + z^2 - xx_1 - yy_1 - zz_1 = 0 \quad \dots (1)$$

It's centre is $\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2}\right)$

and $\quad$ Radius $= \sqrt{\frac{x_1^2}{4} + \frac{y_1^2}{4} + \frac{z_1^2}{4}} = r$ (constant)

$$\therefore \quad x_1^2 + y_1^2 + z_1^2 = 4r^2 \quad \dots (2)$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be the point on the locus i.e. $\bar{x}, \bar{y}, \bar{z}$ be the centroid of A, B, C

$\therefore \bar{x} = \frac{x_1 + 0 + 0}{3}, \quad \bar{y} = \frac{0 + y_1 + 0}{3}, \quad \bar{z} = \frac{0 + 0 + z_1}{3}$

$\therefore x_1 = 3\bar{x}, \quad y_1 = 3\bar{y}, \quad z_1 = 3\bar{z}$

Substituting in equation (2) we get

$$9\left(\bar{x}^2 + \bar{y}^2 + \bar{z}^2\right) = 4r^2$$

Replacing $\bar{x}\ \bar{y}\ \bar{z}$ by x, y, z we get

$$9\left(x^2 + y^2 + z^2\right) = 4r^2$$

Q.6 : A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the foot of the perpendicular from origin to the plane ABC is given by $\left(x^2 + y^2 + z^2\right)^2 \left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right] = 4r^2$ **[SPPU : Dec.-04]**

Ans. : Let

$A(x_1, 0, 0), B(0, y_1, 0), C(0, 0, z_1)$ be the co-ordinates of A, B, C.

$\therefore$ Equation of plane passing through A, B, C in intercept form is

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1 \quad \dots (1)$$

and equation of sphere OABC is

$$x^2 + y^2 + z^2 - xx_1 - yy_1 - zz_1 = 0 \quad \dots (2)$$

whose radius is constant

$\therefore \quad \sqrt{\frac{x_1^2}{4} + \frac{y_1^2}{4} + \frac{z_1^2}{4}} = r$

i.e. $\quad x_1^2 + y_1^2 + z_1^2 = 4r^2 \quad \dots (3)$

Let $P(\bar{x}, \bar{y}, \bar{z})$ be the foot of the perpendicular from origin to the plane ABC

$\therefore$ dr's of OP (Normal to the plane ABC)

are $\bar{x} - 0, \; \bar{y} - 0, \; \bar{z} - 0$ i.e. $\bar{x}, \bar{y}, \bar{z}$

We know that $P(\bar{x}, \bar{y}, \bar{z})$ is one point on the plane

$\therefore$ Equation of plane ABC is

$$\bar{x}(x - \bar{x}) + \bar{y}(y - \bar{y}) + \bar{z}(z - \bar{z}) = 0$$

$$x\bar{x} + y\bar{y} + z\bar{z} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 = H \text{ (say)}$$

$$\therefore \frac{x}{(H/\bar{x})} + \frac{y}{(H/\bar{y})} + \frac{z}{(H/\bar{z})} = 1 \quad \dots (4)$$

Comparing with equation (1)

$$x_1 = \frac{H}{\bar{x}},\ y_1 = \frac{H}{\bar{y}},\ z_1 = \frac{H}{\bar{z}}$$

Substituting in equation (3)

$$H^2\left[\frac{1}{(\bar{x})^2} + \frac{1}{(\bar{y})^2} + \frac{1}{(\bar{z})^2}\right] = 4r^2$$

Substituting H and replacing $\bar{x}$, $\bar{y}$, $\bar{z}$ by x y z we get

$$\left(x^2 + y^2 + z^2\right)^2 \left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right] = 4r^2$$

10.3 : Tangent Plane of The Sphere

The tangent plane to a sphere at a given point is the plane that "Just Touches" the sphere at that point.

1) If the plane is tangent to the sphere at point then CP is the normal vector to the plane at the point P. The point of contact P lies on the sphere, plane and the normal vector to the plane.
2) If the plane is tangent to the sphere at the point P then the distance between the point and the center of the sphere is equal to the radius of the sphere

d(Plane, Center of sphere) = Radius of sphere

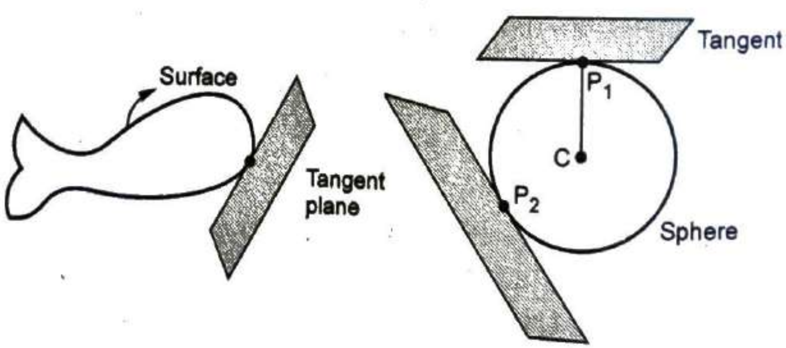


Fig. 10.4

Theorem :

The equation of tangent plane to the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$

at $(x_1\ y_1\ z_1)$ is

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

Proof :

From Fig. 10.5 dr's of CP are

$x_1 + u,\ y_1 + v,\ z_1 + w.$

CP is normal to the plane at P.

$\therefore$ Equation of tangent plane is

$$x(x_1 + u) + y(y_1 + v) + z(z_1 + w) = k \text{ (say)} \quad \dots (1)$$

As (x_1, y_1, z_1) lies on equation (1)

$\therefore x_1(x_1 + u) + y_1(y_1 + v) + z_1(z_1 + w) = k$

$\therefore k = x_1^2 + y_1^2 + z_1^2 + x_1u + y_1v + z_1w$

Substituting in equation (1)

$$x(x_1 + u) + y(y_1 + v) + z(z_1 + w) = x_1^2 + y_1^2 + z_1^2 + x_1u + y_1v + z_1w$$

i.e. $xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1 \quad \dots (2)$

As (x_1, y_1, z_1) lies on the sphere

$\therefore\ x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0.$

i.e. $x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1 = -ux_1 - vy_1 - wz_1 - d$

$\therefore$ From equation (2)

$$xx_1 + yy_1 + zz_1 + ux + vy + wz = -ux_1 - vy_1 - wz_1 - d$$

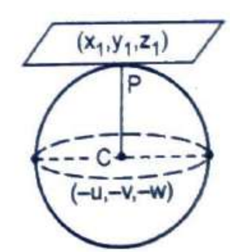


Fig. 10.5

$$\therefore \quad \boxed{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

is the required equation of tangent plane to the given sphere at point (x_1, y_1, z_1).

Note :

i) To find the equation of tangent plane to the sphere at (x_1, y_1, z_1)

Replace x^2 by xx_1

Replace y^2 by yy_1

Replace z^2 by zz_1

Replace $2x$ by $x + x_1$

Replace $2y$ by $y + y_1$

Replace $2z$ by $z + z_1$ in the equation of sphere.

ii) **Tangent plane property :** If a plane touches a sphere, then length of perpendicular from the centre of the sphere to the plane must be equal to **radius** of the sphere.

iii) To find the point of contact, determine the point of intersection of the line perpendicular from the centre of the sphere to the tangent plane.

Q.7 : Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact.

[SPPU : Dec.-13, 18, May-16, Marks 5]

Ans. : Given that the sphere is

$$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0 \quad \ldots(1)$$

$$\therefore \quad 2u = -2,\ 2v = -4,\ 2w = 2 \text{ and } d = -3$$

$\therefore$ It's center is at $c(-u, -v, -w) = c(1, 2, -1)$

and, $\quad$ radius $= r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1 + 4 + 1 + 3} = 3$

The perpendicular distance from $c(1,2,-1)$ to the plane $2x - 2y + z + 12 = 0$ is

$$L = \left| \frac{2(1) - 2(2) + (-1) + 12}{\sqrt{2^2 + (-2)^2 + (1)^2}} \right| = \left| \frac{9}{3} \right| = 3 = \text{Radius of sphere}$$

$\therefore$ The given plane touches to the given sphere. Let the point of contact CP is normal to the given plane. Therefore, coefficient of x,y,z in the equation of plane are direction ratio's of normal to the plane

$\therefore$ Direction ratio's of CP are 2,–2,1 and it passes through C(1, 2, –1)

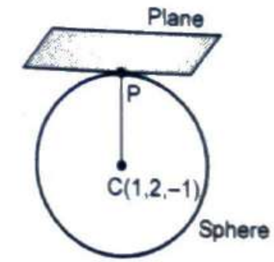


Fig. Q.7.1

$\therefore$ The equation of CP is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

$$\therefore \quad \frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = k \text{ (say}$$

$$\therefore \quad x = 2k + 1,\ y = -2k + 2,\ z = k - 1$$

These are the co-ordinates of any point of CP. So, assume that these are co-ordinates of P. But, the point P lies on $2x - 2y + z + 12 = 0$

$\therefore \quad 2(2k+1) - 2(-2k+2) + (k-1) + 12 = 0$

$9k + 9 = 0 \Rightarrow k = -1$

$\therefore x = 2(-1) + 1 = -1,\ y = -2(-1) + 2 = 4,\ z = -1 - 1 = -2$

Therefore, the point of contact is $(-1, 4, -2)$

Q.8 : Show that the plane $4x - 3y + 6z - 35 = 0$ is tangential to the sphere. $x^2 + y^2 + z^2 - y - 2z - 14 = 0$ and find the point of contact. **[SPPU : May-15, 17, Marks 5]**

Ans. : From $x^2 + y^2 + z^2 - y - 2z - 14 = 0$

Co-ordinates of centre C are $\left(0, \frac{1}{2}, 1\right)$

radius $CC_1 = \sqrt{0 + \frac{1}{4} + 1 + 14} = \frac{\sqrt{61}}{2}$

Now perpendicular distance from the centre $\left(0, \frac{1}{2}, 1\right)$ to the plane

$4x - 3y + 6z - 35 = 0$ is

$$p = \left| \frac{0 - 3\left(\frac{1}{2}\right) + 6(1) - 35}{\sqrt{4^2 + 3^2 + 6^2}} \right| = r \quad \therefore \; r = \frac{30.5}{\sqrt{61}}$$

$$= \frac{\sqrt{61}}{2} = \text{Radius the sphere}$$

Thus the given plane is tangential to the sphere.

To find the point of contact (C_1) write the equation of normal line CC_1. From equation of plane the coefficients of (x, y, z) gives dr's of normal.

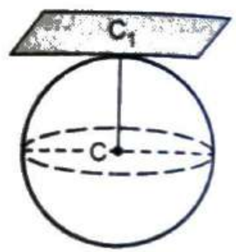


Fig. Q.8.1

$\therefore$ (4, – 3, 6) are dr's of normal

Co-ordinates of C are $\left(0, \frac{1}{2}, 1\right)$.

$$\therefore \quad \frac{x-0}{4} = \frac{y - 1/2}{-3} = \frac{z-1}{6} = t \text{ (Say)}$$

$$\therefore \quad x = 4t, \; y = -3t + \frac{1}{2}, \; z = 6t + 1$$

Substituting in $4x - 3y + 6z - 35 = 0$

We get $4(4t) - 3\left(-3t + \frac{1}{2}\right) + 6(6t + 1) - 35 = 0$

$$\Rightarrow \quad t = \frac{1}{2}$$

Hence substituting in (x, y, z) we get (2, – 1, 4) as the point of contact.

Q.9 : Find the equation of the sphere tangential to the plane $x - 2y - 2z = 7$ at (3,–1,–1) and passing through the point (1,1,–3).
[SPPU : Dec.-14, Marks 5]

Ans. : Given that the given plane is tangent to the sphere at P(3,–1,–1).

Let C be the center of the required sphere.

The line CP is normal to the plane.

$\therefore$ CP passes through (3,–1,–1) having direction ratio's 1,–2,–2.

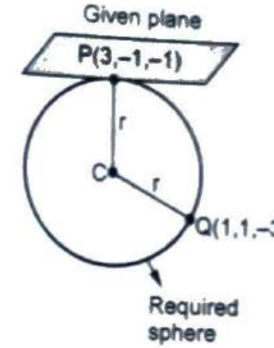


Fig. Q.9.1

$\therefore$ The equation of CP is

$$\frac{x-3}{1} = \frac{y+1}{-2} = \frac{z+1}{-2} = k \text{ (say)}$$

$\therefore x = k+3, \; y = -2k-1, \; z = -2k-1$

This (x, y, z) is the general point on CP.

Assume that C (x, y, z) is the center of the sphere P and Q lie on the sphere.

$$\therefore \quad (CP)^2 = (CQ)^2$$

$$(k+3-3)^2 + (-2k-1+1)^2 + (-2k-1+1)^2 = (k+3-1)^2 + (-2k-1-1)^2 + (-2k-1+3)^2$$

$$k^2 + 4k^2 + 4k^2 = (k+2)^2 + (-2k-2)^2 + (-2k+2)^2$$

$$9k^2 = k^2 + 4k + 4 + 4(k^2 + 2k + 1) + 4k^2 - 8k + 4$$

$$4k = -12$$

$$\Rightarrow \quad \boxed{k = -3}$$

$\therefore$ The center is C (0, 5, 5) and radius

$$= (CP) = \sqrt{9k^2} = \sqrt{81} = 9$$

Thus, the required equation of the sphere is

$$(x-0)^2 + (y-5)^2 + (z-5)^2 = 9^2$$

$$x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$$

Q.10 : Find the equation of the sphere which has it's center at (2, 3, −1) and touches the line $\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4}$.

[SPPU : May-16, Marks 5]

Ans. : We have, given line is tangent to the required sphere at P(x, y, z).

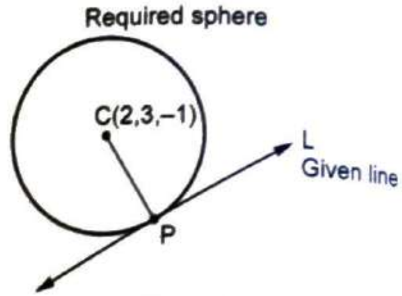


Fig. Q.10.1

Equation of line L is

$$\frac{x+1}{-5} = \frac{y-8}{3} = \frac{z-4}{4} = k$$

$$\Rightarrow x = -5k-1,\ y = 3k+8,\ z = 4k+4$$

Assume that these are co-ordinates of point P.

∴ The direction ratio's of L are −5, 3, 4.

∴ Direction ratios of CP are

$-5k-1-2,\ 3k+8-3,\ 4k+4+1$

i.e. $-5k-3,\ 3k+5,\ 4k+5$

CP is perpendicular to the line L i.e. $CP \perp L$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow -5(-5k-3) + 3(3k+5) + 4(4k+5) = 0$$

$$50k + 50 = 0$$

$$\Rightarrow \boxed{k = -1}$$

∴ The co-ordinates of P are

$$x = -5k-1 = 4,\ y = 5,\ z = 0$$

Therefore, $(CP) = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$

Radius $CP = \sqrt{9} = 3$

Therefore the required equation of the sphere with center at C(2,3,−1) and radius = 3 is

$$(x-2)^2 + (y-3)^2 + (z+1)^2 = 9$$

$$x^2 + y^2 + z^2 - 4x - 6y + 2z + 5 = 0$$

Q.11 : Find the equation of the sphere which passes through the point (1, 0, 0) and touches the plane 2x − y − 2z = 4 at the point (1, 2, −2).

[SPPU : May-16, Marks 5]

Ans. : Given plane is

$$2x - y - 2z = 4$$

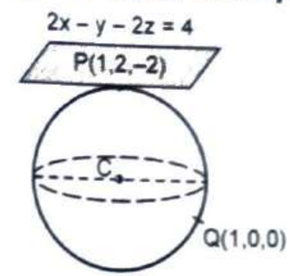


Fig. Q.11.1

∴ Coefficients of (x, y, z) i.e. (2, − 1, 2) are the dr's of the normal (i.e. CP).

Co-ordinates of P are (1, 2, − 2).

Thus equation of CP is

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+2}{-2} = t \text{ (Say)}$$

$$\therefore \quad x = 2t+1,\ y = -t+2,\ z = -2t-2.$$

From Fig. Q.11.1,

$$CP^2 = CQ^2$$

$$(2t+1-1)^2 + (-t+2-2)^2 + (-2t-2+2)^2$$

$$= (2t+1-1)^2 + (-t+2)^2 + (-2t-2)^2$$

$$\Rightarrow 4t^2 + t^2 + 4t^2 = 4t^2 + t^2 - 4t + 4 + 4t^2 + 8t + 4$$

$$\Rightarrow t = -2$$

∴ Co-ordinates of 'C' are (− 3, 4, 2)

Also $CQ^2 = (-3-1)^2 + (4-0)^2 + (2-0)^2$

$$= 16 + 16 + 4$$

$$= 36$$

$\therefore$ We use centre radius form

$$(x+3)^2 + (y-4)^2 + (z-2)^2 = 36$$

$$\Rightarrow x^2 + y^2 + z^2 + 6x - 8y - 4z - 7 = 0$$

Q.12 : Find the equation of the sphere which touches the plane $4x + 3y = 47$ at (8, 5, 4) and touches the sphere $x^2 + y^2 + z^2 = 1$ internally. [SPPU : May-04, Dec.-98]

Ans. :

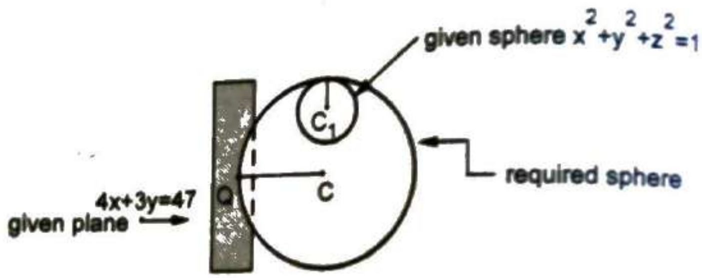


Fig. Q.12.1

From equation of plane $4x + 3y = 47$, the coefficients of x, y, z gives dr's of the normal to the plane.

$\therefore$ The dr's of QC are 4, 3, 0.

As co-ordinates of Q are (8, 5, 4).

$\therefore$ Equation of QC passing through (8, 5, 4) having dr's 4, 3, 0 is

$$\frac{x-8}{4} = \frac{y-5}{3} = \frac{z-4}{0} = k \text{ (say)}$$

$\therefore$ Co-ordinates of C are $(4k+8, 3k+5, 4)$. Using distance formula for QC.

$$QC = \sqrt{(4k+8-8)^2 + (3k+5-5)^2 + (4-4)^2} = \sqrt{25k^2}$$

As the required sphere touches internally to the given sphere

$x^2 + y^2 + z^2 = 1$ ($\because C_1$ is (0 0 0) and radius = 1)

$$\therefore \quad QC = CC_1 + 1$$

$$\therefore \quad 5k = \sqrt{(4k+8)^2 + (3k+5)^2 + (4)^2} + 1$$

$$\therefore \quad (5k-1)^2 = (4k+8)^2 + (3k+5)^2 + 4^2$$

Solving we get

$$k = -1$$

$\therefore$ Co-ordinates of C are (4, 2, 4) and radius = 5.

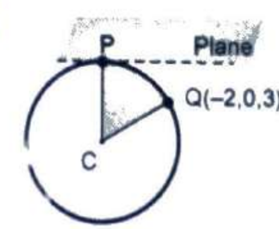
$\therefore$ The required equation is

$$(x-4)^2 + (y-2)^2 + (z-4)^2 = 0$$

i.e. $x^2 + y^2 + z^2 - 8x - 4y - 8z + 11 = 0$

Q.13 : Find the equation of the sphere which is tangential to the plane $2x - 2y - z + 16 = 0$ at (– 3,4,2) and passing through (–2, 0, 3). [SPPU : May-10]

Ans. : Given plane is tangent to the required sphere at point P(– 3, 4, 2) CP is normal to the plane



$$2x - 2y - z + 16 = 0$$

$\Rightarrow$ dr's of CP are 2, – 2, – 1

So, equation of CP is

$$\frac{x+3}{2} = \frac{y-4}{-2} = \frac{z-2}{-1} = k$$

$\Rightarrow x = 2k - 3,\ y = -2k + 4,\ z = -k + 2$

Thus co-ordinates of centre C are $(2k-3, -2k+4, -k+2)$

$$\Rightarrow \quad (CP)^2 = 4k^2 + 4k^2 + k^2 = 9k^2$$

$$\text{But} \quad (CP)^2 = (CQ)^2$$

$$9k^2 = (2k-3+2)^2 + (-2k+4-0)^2 + (-k+2-3)^2$$

$$= (2k-1)^2 + (-2k+4)^2 + (-k-1)^2$$

$$9k^2 = 4k^2 - 4k + 1 + 4k^2 - 16k + 16 + k^2 + 2k + 1$$

$\Rightarrow \qquad -18k + 18 = 0 \Rightarrow k = 1$

Thus centre is (-1,2,1) and radius is $\sqrt{9k^2} = \sqrt{9} = 3$.

Thus, the required equation of sphere is

$$(x+1)^2 + (y-2)^2 + (z-1)^2 = 9$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 2x - 4y - 2z - 3 = 0$$

10.4 : Section of a Sphere by a Plane, Intersection of Two Spheres, Equation of a Circle, Sphere through a Circle

I) Section of a Sphere by a Plane

Consider the equation of sphere in general form

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \ldots(1)$$

Let the equation plane bc

$$ax + by + cz + h = 0 \qquad \ldots(2)$$

Claim : Show that the section of a sphere by a plane is a circle.

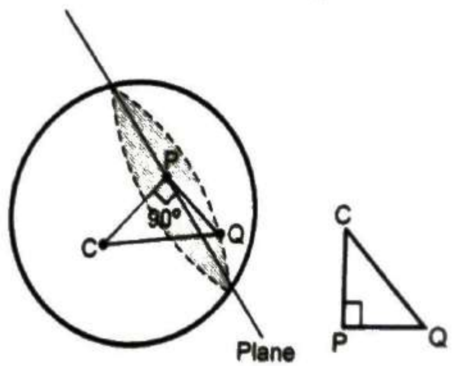


Fig. 10.6

Draw perpendicular from the centre of the sphere on the section and denote foot of perpendicular by P.

For a given sphere and a plane C and P are the fixed points and also CP is constant.

Let Q be any point on the locus of the section i.e. Q lies on plane as well as sphere.

$\therefore \qquad CQ = $ Radius of the sphere = constant

In $\Delta CPQ \qquad \angle CPQ = 90°$

Therefore $\qquad (CQ)^2 = (PQ)^2 + (CP)^2$

$\therefore \qquad (PQ)^2 = (CQ)^2 - (CP)^2$

$\therefore \qquad PQ = \sqrt{(CQ)^2 - (CP)^2} = $ Constant

($\because$ CQ and CP are constant)

where $\qquad$ CR = Radius of sphere

and $\qquad$ CP = Length of the perpendicular from centre of the sphere on the intersecting plane.

Therefore PQ is constant and P is fixed point. Thus the locus of Q is a circle with centre at P and radius PQ which is the section of a sphere by a plane.

Note :

i) To find the centre of this circle. write an equation of CP

$\dfrac{x+u}{a} = \dfrac{y+v}{b} = \dfrac{z+w}{c} = k$ $(-u, -v, -w)$ is the centre of the sphere and a, b, c, are the coefficients of x, y, z in the equation of given plane, which are the drs of CP.

Substituting these x, y, z in the equation of plane, we get the value of k and hence the co-ordinates of P.

ii) To find the radius of the circle use

$$(PQ)^2 = (CQ)^2 - (CP)^2$$

and $\qquad$ CP = Distance between C $(-u, -v, -w)$ and the given plane.

$$CP = \left| \frac{-au - bv - cw + h}{a^2 + b^2 + c^2} \right|$$

iii) Great circle

The section of a sphere by a plane through the centre of the sphere is called the great circle. It's centre and radius are same as that of the sphere

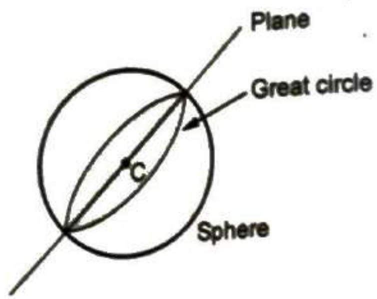


Fig. 10.7

II) Intersection of Two Spheres

Let the equations of two spheres be

$$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = 0 \quad \ldots(1)$$

$$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2 = 0 \quad \ldots(2)$$

Consider

$$S_1 - S_2 = 2(u_1-u_2)x + 2(v_1-v_2)y + 2(w_1-w_2)z + d_1 - d_2$$

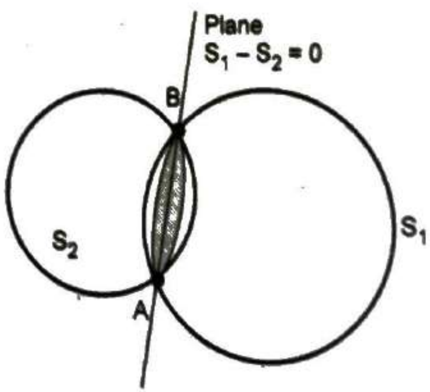


Fig. 10.8

Which is the first degree equation in x, y, z and it represents a plane known as Radical Plane. Therefore the point of intersection of two spheres is same as the intersection of any sphere and the Radical Plane $S_1 - S_2 = 0$ Therefore, by section 10.6, such intersection is a circle. Thus, the intersection of two spheres is a circle.

III) Equation of a Circle

i) We know that the section of a sphere by a plane is a circle. Let $S = 0$ and $U = 0$ be the equations of sphere and plane respectively.

Therefore $\underline{S = 0 \text{ and } U = 0}$ together represents a circle.

ii) We know that intersection of two spheres is a circle, Therefore $\underline{S_1 = 0 \text{ and } S_2 = 0}$ these two.

Spheres together represents a circle.

Note : In three dimensional geometry, circle will not be represented by a single equation. The circle is presented by two equations.

IV) Sphere through a Circle

1) Consider, the equation of circle as

$$\left.\begin{aligned} S &= x^2+y^2+z^2+2ux+2vy+2wz+d = 0 && \ldots(A) \\ U &= ax+by+cz+h = 0 && \ldots(B) \end{aligned}\right] \text{(i)}$$

Multiplying equation (B) by λ and adding to (A), we get

$$S + \lambda U = 0$$

$$x^2+y^2+z^2+2ux+2vy+2wz+d+\lambda(ax+by+cz+h) = 0$$

$$x^2+y^2+z^2+(2u+\lambda a)x+(2v+\lambda b)y+(2w+\lambda c)z+(d+\lambda h) = 0 \quad \ldots(ii)$$

Equation (ii) represents a sphere.

Now, co-ordinates of points, which satisfy equation A and B both i.e. the points lying on the circle (i). Clearly satisfy equation (ii)

Hence (ii) represents sphere passing through the circle (i) and corresponding to different values of λ we get, different spheres. Thus equation (ii) represents a family of spheres passing through the circle.

2) Consider the equation of circle as

$$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = 0$$

$$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2 = 0$$

Similarly, $S_1+\lambda S_2 = 0$ represent a family of spheres passing through a circle $S_1=0$ and $S_2=0$

Q.14 : Find the centre and radius of the circle which is an intersection of the sphere $x^2+y^2+z^2-2y-4z-11 = 0$ by the plane $x+2y+2z = 15$. **[SPPU : Dec.-16, May-17, Marks 5]**

Ans. : Given that equation of circle is

$$x^2+y^2+z^2-2y-4z-11=0. \quad \ldots(1)$$

and $x+2y+2z-15=0$ ...(2)

∴ The centre of sphere is at C(−u, −v, −w) i.e. C(0, 1, 2)

and it's radius is

$$CQ = \sqrt{0+1+4+11} = 4$$

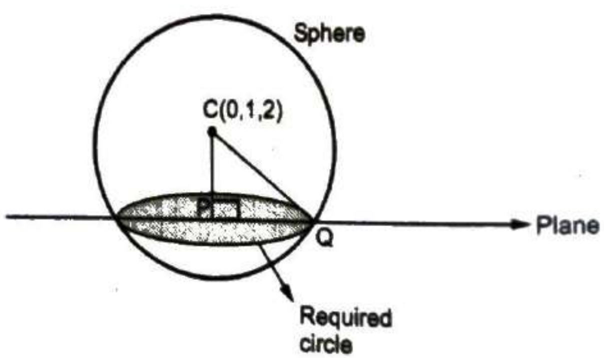


Fig. Q.14.1

Draw perpendicular from the centre of the sphere to the plane and denote foot of perpendicular by P.

Join CP and CQ.

CP is normal to the plane $x+2y+2z-15=0$

∴ The direction ratio's of CP are 1, 2, 2.

Therefore,the equation of CP passing through C(0,1,2) with drs 1, 2, 2, is

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2} = k(\text{say})$$

$$\therefore \quad x = k,\ y = 2k+1,\ z = 2k+2$$

Assume that x, y, z are the co-ordinates of the point P for some particular value of k.

But P(x, y, z) lies on the given plane.

$$\therefore \ k+2(2k+1)+2(2k+2)-15 = 0$$

$$9k-9 = 0 \Rightarrow \boxed{k=1}$$

Therefore the co-ordinates of 'P' are $x = 1$, $y = 3$, $z = 4$, i.e. P(1, 3, 4)

Thus the centre of the circle is at P(1,3, 4)

Now, $CP = \sqrt{(1-0)^2+(3-1)^2+(4-2)^2} = \sqrt{9} = 3$

and CQ = Radius of the sphere = 4

In ΔCPQ, $(CQ)^2 = (CP)^2+(PQ)^2$

$$\therefore \quad PQ = \sqrt{(CQ)^2-(CP)^2}$$

$$= \sqrt{16-9} = \sqrt{7} = \text{Radius of circle}$$

Thus the centre and radius of the required circle is P(1, 3, 4) and radius = $\sqrt{7}$

Remarks : Alternate Method of Find CP.

Distance CP can be obtained by finding the length of the perpendicular between centre (0, 1, 2) and the given plane.

$$\therefore \quad CP = \left|\frac{0+2(1)+2(2)-15}{\sqrt{1+4+4}}\right| = \left|\frac{-9}{3}\right| = 3$$

Q.15 : Find the equation of the circle which is a section of the sphere $x^2 + y^2 + z^2 + 6y - 6z - 21 = 0$ and has its centre at the point (2, -1, 2)

Ans. : The equation of the circle is given by S = 0, U = 0.

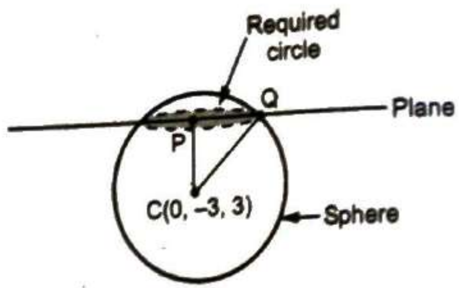


Fig. Q.15.1

We know S = 0 i.e. equation of sphere. To find U = 0 i.e. equation of plane.

Given co-ordinates of P = (2, – 1, 2)

From sphere co-ordinates of C = (0, – 3, 3)

$$\therefore \quad \text{dr's of CP} = (0-2), (-3+1), (3-2)$$
$$= -2, -2, 1$$

Thus eqaution of plane passing through P(2, –1, 2) having dr's of normal (–2, –2, 1) is

$$-2(x-2) - 2(y+1) + 1(z-2) = 0$$

i.e. $\quad -2x - 2y + z = 0$

Thus $\quad x^2 + y^2 + z^2 + 6y - 6z - 21 = 0$

and $\quad -2x - 2y + z = 0$

together represents a circle.

Q.16 : Find the centre and radius of the circle $x^2 + y^2 + z^2 + 4x - 6y + 2z - 10 = 0,\ 3x - y + 3z - 7 = 0.$

[SPPU : May-02, Marks 5]

Ans. : The centre C of the given sphere is (–2, 3, – 1) and sphere $R = \sqrt{4+9+1+10} = \sqrt{24}$

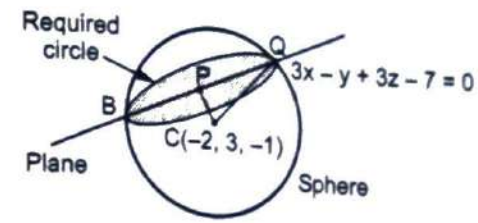


Fig. Q.16.1

Let P be the centre of the circle.

Now CP is normal to the plane $3x - y + 3z - 7 = 0$

$\therefore$ dr's of CP are 3, – 1, 3

Equation of CP is $\dfrac{x+2}{3} = \dfrac{y-3}{-1} = \dfrac{z+1}{3} = k$

P is $(3k - 2, -k + 3, 3k - 1)$ for some values of k

P lies on the plane

$$3(3k-2) - (-k+3) + 3(3k-1) - 7 = 0$$
$$\therefore \quad 19k - 19 = 0$$
$$k = 1$$

$\therefore$ P is (1, 2, 2) which is the centre of

$$C(P)^2 = (1+2)^2 + (2-3)^2 + (2+1)^2$$
$$= 9 + 1 + 9 = 19 \quad \therefore CP = \sqrt{19}$$

From Δ CPQ

$$(PQ)^2 = (CQ)^2 - (CP)^2$$
$$CQ = 24 - 19 = 5$$
$$\therefore \quad \text{Radius} = PQ = \sqrt{5}$$

Q.17 : Find the centre and radius of the circle $x^2+y^2+z^2=2x+4y+2z-6=0$, $x+2y+2z-4=0$ and Also find the orthogonal projection of the area of the circle in yz plane.
[SPPU : Dec.-08, 17, May-19, Marks 5]

Ans. : Given plane cuts the sphere in a circle whose centre is P. From Fig. Q.17.1 as shown in below CP ⊥ PQ. ∴ The dr's of CP are 1, 2, 2.

∴ The equation of CP is $\frac{x-1}{1}=\frac{y+2}{2}=\frac{z+1}{2}=k$.

⇒ Any point on the line CP is given by

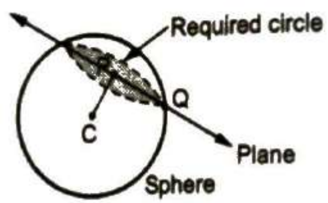


Fig. Q.17.1

$x=k+1,\ y=2k-2,\ z=2k-1$ ... (1)

Substituting in the given equaiton of plane , we get

$(k+1)+2(2k-2)+2(2k-1)-4=0 \Rightarrow k=1$

Putting $k=1$ in equation (1) we get $x=2,\ y=0,\ z=1$

$$\therefore \quad CP = \sqrt{(2-1)^2+(0+2)^2+(1+1)^2} = \sqrt{9} = 3$$

CQ = Radius of the given sphere

$$CQ = \sqrt{1+4+1+6} = \sqrt{12}$$

∴ Radius of given circle $= PQ = \sqrt{CQ^2-CP^2} = \sqrt{12-9} = \sqrt{3}$

Thus area of circle $= \pi(\sqrt{3})^2 = 3\pi$

Orthogonal projection of area of circle in yz plane is $= 3\pi\cos\theta$ direction ratio's of normal to the plane of circle are 1, 2, 2 and dr's of normal to yz plane are 1, 0, 0.

$$\therefore \quad \cos\theta = \frac{1+0+0}{\sqrt{1+4+4}\sqrt{1+0+0}} = \frac{1}{3}$$

∴ Orthogonal projection of area of circle

$$= 3\pi\cos\theta = 3\pi\left(\frac{1}{3}\right) = \pi$$

Q.18 : Find the equation of sphere which passes through (3, 1, 2) and meets XOY plane in a circle of radius 3 units with the centre at (1, −2, 0).
[SPPU : Dec.-04,05,15, May-08, Marks 5]

Ans. :

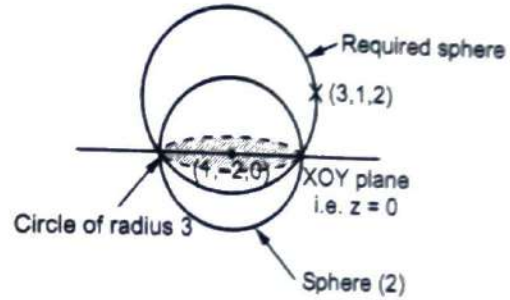


Fig. Q.18.1

From Fig. Q.18.1 equation of circle in XOY plane whose centre is (1,−2,0) and radius 3 is

$$(x-1)^2+(y+2)^2 = 9 \quad \text{... (1)}$$

Thus equation of sphere whose intersection with $z=0$ is the great circle (1) is given by,

$(x-1)^2+(y+2)^2+z^2=9$ ... (Note this step)

$\therefore x^2+y^2+z^2-2x+4y-4=0$ ... (2)

Thus the required sphere is

$$S+\lambda U = 0$$

$x^2+y^2+z^2-2x+4y-4+\lambda z=0$ ... (3)

which passes through (3, 1, 2)

$\therefore \quad 9+1+4-6+4-4+2\lambda=0$

$\Rightarrow \quad \lambda = -4$

$\therefore$ Substituting λ in equation (3)

$x^2 + y^2 + z^2 - 2x + 4y - 4z - 4 = 0$

is the required equation of sphere.

Q.19 : Find the sphere through the circle $x^2 + y^2 + z^2 = 4$, $z = 0$, meeting the plane $x + 2y + 2z = 0$ in a circle of radius 3.

[SPPU : May-04, 07,18, Marks 5]

Ans. :

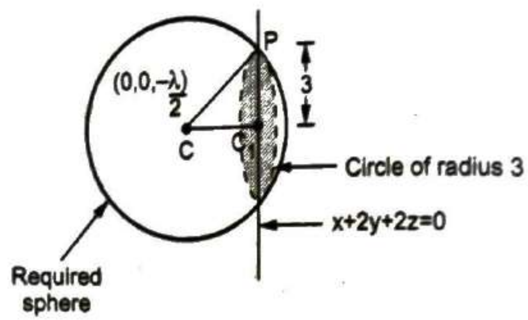


Fig. Q.19.1

Required sphere is $S + \lambda U = 0$

i.e. $x^2 + y^2 + z^2 - 4 + \lambda z = 0$ ... (1)

centre C is $\left(0,0,\frac{-\lambda}{2}\right)$ and radius R is $\sqrt{\frac{\lambda^2}{4} + 4}$

CC_1 = Perpendicular distance from $\left(0,0,\frac{-\lambda}{2}\right)$ on

$$x + 2y + 2z = 0$$

$$= \left|\frac{0 + 0 - \lambda}{\sqrt{1 + 4 + 4}}\right| = \left|-\frac{\lambda}{3}\right|$$

From $\Delta\, CC_1P$

$$CP^2 = CC_1^2 + C_1P^2$$

$$\frac{\lambda^2}{4} + 4 = \frac{\lambda^2}{9} + 9$$

$$\Rightarrow \quad \lambda = \pm 6$$

$\therefore$ Substituting in equation (1) we get the required sphere

$x^2 + y^2 + z^2 \pm 6z - 4 = 0$

Q.20 : Find the equation of the sphere through the circle $x^2 + y^2 = 9$, $z = 0$ and the point (α, β, γ)

[SPPU : May-17, Marks 5]

Ans. : Given that the sphere is

$$S \equiv x^2 + y^2 + z^2 - 9 = 0$$

and the plane is $U \equiv z = 0$

The equation of the sphere passing through the circle is

$$S + \lambda U = 0$$

$$(x^2 + y^2 + z^2 - 9) + \lambda(z) = 0 \quad \text{...(1)}$$

But, the sphere (1) Passes through the point (α, β, γ), we get,

$$\alpha^2 + \beta^2 + \gamma^2 - 9 + \lambda\gamma = 0$$

$$\therefore \quad \lambda = -\left(\frac{\alpha^2 + \beta^2 + \gamma^2 - 9}{\gamma}\right)$$

$\therefore$ Equation (1) becomes

$$x^2 + y^2 + z^2 - 9 = -\left(\frac{\alpha^2 + \beta^2 + \gamma^2 - 9}{\gamma}\right) z = 0$$

$$\therefore \gamma(x^2 + y^2 + z^2 - 9) = (\alpha^2 + \beta^2 + \gamma^2 - 9)z$$

This is the required equation of the sphere.

Q.21 : Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and point (1, 2, 3).

[SPPU : May-11, Dec.-15, Marks 5]

Ans. : Equation of the circle is given by

$$S = x^2 + y^2 + z^2 - 9 = 0$$

and

$$U = 2x + 3y + 4z - 5 = 0$$

Equation of the sphere passing through this circle is

$$S + \lambda U = 0$$

i.e. $x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0$ ... (1)

It passes through (1, 2, 3).

$\therefore 1 + 4 + 9 - 9 + \lambda (2 + 6 + 12 - 5) = 0$

$$5 + \lambda 15 = 0$$

$$\therefore \qquad \lambda = -\frac{1}{3}$$

Substituting in equation (1), we get

Hence the equation of the sphere is

$$3\left(x^2 + y^2 + z^2\right) - 2x - 3y - 4z - 22 = 0$$

Q.22 : Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point (1, 1, −1) and passes through the point (0, 0, 3).

[SPPU : May-11, Dec.-07, 08, 15, Marks 5]

Ans. :

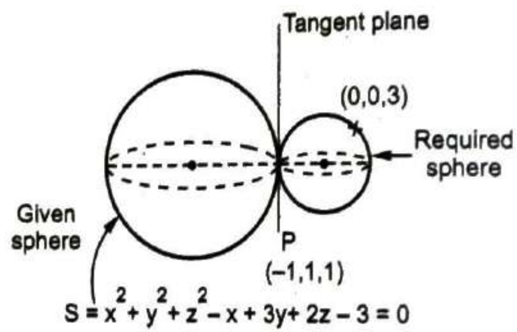


Fig. Q.22.1

Given sphere

$$S = x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$$

Equation of the tangent plane at (1, 1, − 1) to the sphere S = 0 is

$$x + y - z - \frac{1}{2}(x + 1) + \frac{3}{2}(y + 1) + (z - 1) - 3 = 0$$

i.e. $$x + 5y - 6 = 0$$

The required sphere passes through the point circle of intersection of S = 0 and U = 0 i.e the required sphere is $S + \lambda U = 0$.

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 + \lambda(x + 5y - 6) = 0 \qquad \text{... (1)}$$

(0, 0, 3) lies on the sphere

$$9 + 6 - 3 + \lambda (6) = 0$$

$$12 - 6\lambda = 0$$

$$\Rightarrow \qquad \lambda = 2$$

Substituting in equation (1)

$\therefore$ The required sphere is

$$x^2 + y^2 + z^2 + x + 13y + 2z - 15 = 0$$

10.5 : Orthogonal Spheres, Tangent Line to a Sphere, Tangent to a Circle at a Point (x_1, y_1, z_1)

I) Orthogonal Spheres

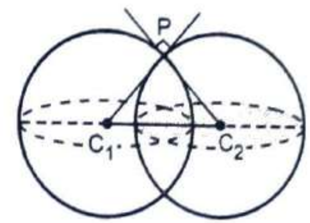


Fig. 10.9

Two spheres are said to be Orthogonal if the tangent planes to the two spheres at the points of intersection are at right angles. i.e. the normals to the two tangent planes at the point of intersection through C_1 and C_2 are perpendicular. In ΔC_1PC_2, we have,

i.e. $$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2 \qquad \text{... (1)}$$

If $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

and

$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$

then $(C_1C_2)^2 = (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2$

$$(C_1P)^2 = u_1^2 + v_1^2 + w_1^2 - d_1$$

$$(C_2P)^2 = u_2^2 + v_2^2 + w_2^2 - d_2$$

Substituting in equation (1)

$$(u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2$$

$$= u_1^2 + v_1^2 + w_1^2 - d_1 + u_2^2 + v_2^2 + w_2^2 - d_2$$

We get the condition of orthogonality on simplification,

$$\boxed{2(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2}$$

II) Tangent Line to a Sphere

A line is said to be a tangent to a sphere if it cuts the sphere at only one point or two coincident points.

Length of the tangent line

Let A (x_1, y_1, z_1) be any point outside the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Then the length of the tangent line from A to the sphere is given by

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

Students can easily derive it since

$$AT^2 = CA^2 - CT^2$$

$$CT = \text{Radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

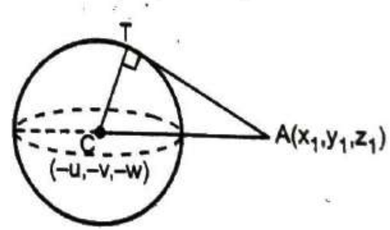


Fig. 10.10

III) Tangent to a Circle at a Point (x_1, y_1, z_1)

Let S = 0 and P = 0 be the given equation of the circle.

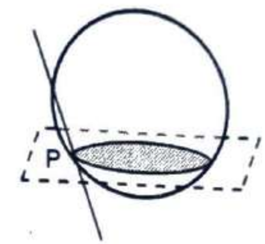


Fig. 10.11

The tangent line to the circle (P = 0 and S = 0) is given by the line of intersection of the tangent plane of the sphere at (x_1, y_1, z_1) and the plane P = 0.

Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$P = lx + my + nz - p = 0$

be the circle. Then equation of the tangent line to the circle is

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

and $lx + my + nz - p = 0$

Q.23 : Find the equation of sphere passing through the circle $x^2 + y^2 + z^2 = 1$, $2x + 3y + 4z = 5$ and which intersects the sphere $x^2 + y^2 + z^2 + 3(x - y + z) - 56 = 0$ orthogonally.

[SPPU : May-15, Marks 5]

Ans. : The euqation of sphere passing through given circle is,

$$x^2 + y^2 + z^2 - 1 + \lambda\,(2x + 3y + 4z - 5) = 0 \quad \dots (1)$$

The sphere equation (1) intersects the sphere

$x^2 + y^2 + z^2 + 3x - 3y + 3z - 56 = 0$ orthogonally,

$\therefore$ Using condition of orthogonality, we get

$$2(\lambda)\left(\frac{3}{2}\right) + 2\left(\frac{3\lambda}{2}\right)\left(\frac{-3}{2}\right) + 2\left(\frac{4\lambda}{2}\right)\left(\frac{3}{2}\right) = -1 - 5\lambda - 56$$

$$\frac{6\lambda}{2} - \frac{9\lambda}{2} + \frac{12\lambda}{2} = -5\lambda - 57$$

$$6\lambda - 9\lambda + 12\lambda = -10\lambda - 114$$

$$\Rightarrow \qquad 19\lambda = -114 \Rightarrow \lambda = -6$$

Putting $\lambda = -6$ in equation (1) we get,

$$x^2 + y^2 + z^2 - 1 + (-6)(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 12x - 18y - 24z + 29 = 0$$

which is the required equation of sphere.

Q.24 : Find the equation of the sphere which touches plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and also cuts the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ Orthogonally.

[SPPU : Dec.-99]

Ans. : $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$

Centre C_2 s $(2, -3, 0)$ and

$$\text{Radius} = C_2A = \sqrt{4 + 9 - 4} = 3$$

C_1M is perpendicular to $3x + 2y - z + 2 = 0$ where C_1 is the centre of required sphere and M is $(1, -2, 1)$ direction ratios of C_1M are 3, 2, − 1.

Equation of C_1M is $\dfrac{x-1}{3} = \dfrac{y+2}{2} = \dfrac{z-1}{-1} = k$

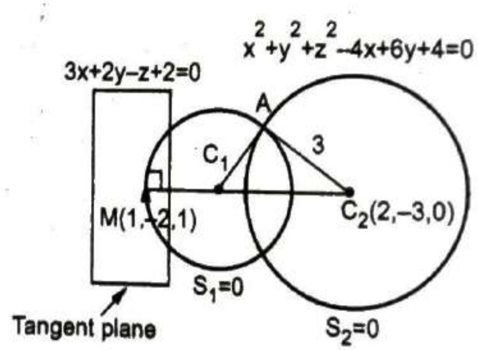


Fig. Q.24.1

C_1 is $(3k + 1, 2k - 2, -k + 1)$ for some value of k.

$C_1A = C_1M =$ Radius of required sphere

$$= \left[9k^2 + 4k^2 + k^2\right]^{1/2} = \sqrt{14}\,k$$

The sphere $S_1 = 0$, $S_2 = 0$ intersect orthogonally.

$$\therefore \qquad C_1A^2 + AC_2^2 = C_1C_2^2$$

$$14k^2 + 9 = (3k-1)^2 + (2k+1)^2 + (1-k)^2$$

$$= 9k^2 + 4k^2 + k^2 - 4k + 3$$

$$-4k = 6$$

$$k = -\frac{3}{2}$$

C_1 is $\left(-\dfrac{7}{2}, -\dfrac{10}{2}, \dfrac{5}{2}\right)$ and $C_1A^2 = 14\,\dfrac{9}{4} = \dfrac{63}{2}$

$\therefore$ Equation of sphere is

$$\left(x + \frac{7}{2}\right)^2 + \left(y + \frac{10}{2}\right)^2 + \left(z - \frac{5}{2}\right)^2 = \frac{63}{2}$$

$$\Rightarrow x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$$

Q.25 : Find the equation of the sphere that passes through the circle. $x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$ and $3x - 4y + 5z - 15 = 0$ and cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

[SPPU : May-10, Dec.-11]

Ans. : Equation of the sphere passing through the circle.

$$(x^2 + y^2 + z^2 - 2x + 3y - 4z + 6) + \lambda(3x - 4y + 5z - 15) = 0 \qquad \dots (1)$$

$$x^2 + y^2 + z^2 + x(3\lambda - 2) + y(3 - 4\lambda) + z(5\lambda - 4) + 6 - 15\lambda = 0$$

The above sphere is orthogonal to the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$$

If two spheres are orthogonal then,

$$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2 \qquad \dots (2)$$

Here $\quad 2u_1 = 3\lambda - 2,\ 2v_1 = 3 - 4\lambda$

$\quad 2w_1 = 5\lambda - 4,\ d_1 = 6 - 15\lambda$

$\quad u_1 = 1,\ v_2 = 2,\ w_2 = -3,\ d_2 = 11$

Substituting in equation (2) we get

$$3\lambda - 2 + 6 - 8\lambda - 15\lambda + 12 = 17 - 15\lambda$$

$$-5\lambda = 1$$

$$\lambda = -\frac{1}{5}$$

Substituting λ in equation (1) we get

$$5\left(x^2 + y^2 + z^2\right) - 13x + 19y - 25z + 45 = 0$$

Q.26 : Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle.

[SPPU : May-18, Marks 5]

Ans. : Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + k\,(2x + 2y + 4z - 8) = 0$$

i.e. $x^2 + y^2 + z^2 + 2kx + (7 + 3k)\,y + (-2 + 4k)\,z + 2 - 8k = 0$ ... (1)

$$\text{Centre } C\left(-k, \frac{7 + 3k}{-2}, \frac{4k - 2}{-2}\right)$$

The given circle is a great circle of this sphere

$\therefore$ The centre of the circle and the sphere is the same

$$\text{i.e. } C\left(-k, \frac{7 + 3k}{-2}, -2k + 1\right)$$

Now C lies in

$$2x + 3y + 4z - 8 = 0$$

$$-2k - \frac{3}{2}(7 + 3k) + 4(-2k + 1) - 8 = 0$$

Solving we get $k = -1$

Substituting k in equation (1) we get

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

Q.27 : A sphere S has points (1, -2, 3), (4, 0, 6) as opposite ends of a diameter. Find the equations of the sphere having the intersection of S with plane $x + y - 2z + 6 = 0$ as great circle.

[SPPU : Dec.-14, Marks 5]

Ans. : The equation of sphere S with points (1, - 2, 3), (4, 0, 6) as diameter is

$$(x - 1)(x - 4) + (y + 2)\,y + (z - 3)(z - 6) = 0$$

i.e. $x^2 + y^2 + z^2 - 5x + 2y - 9z + 22 = 0$

Equation of the sphere passing through the circle of intersection of

$$x^2 + y^2 + z^2 - 5x + 2y - 9z + 22 = 0$$

$$x + y - 2z + 6 = 0$$

$$x^2 + y^2 + z^2 - 5x + 2y - 9z + 22 + k(x + y - 2z + 6) = 0$$

$$x^2 + y^2 + z^2 + x(k - 5) + y(k + 2) - z(9 + 2k) + (22 + 6k) = 0$$ ... (1)

As the given circle of intersection is a great circle of this sphere (1)

$\therefore$ The centre of sphere (1) lies in the plane

$$x + y - 2z + 6 = 0$$

$$\text{Centre C is } \left(\frac{k - 5}{-2}, \frac{k + 2}{-2}, \frac{-(9 + 2k)}{-2}\right)$$

C lies on $x + y - 2z + 6 = 0$

$$\frac{-(k - 5)}{2} - \frac{k + 2}{2} - \frac{2(9 + 2k)}{2} + 6 = 0$$

$$-k + 5 - k - 2 - 18 - 4k + 12 = 0$$

$$-6k - 2 = 0$$

$$k = -\frac{1}{2}$$

$$x^2 + y^2 + z^2 - \frac{11}{2}x + \frac{3}{2}y - 8z + 19 = 0$$

$$2\left(x^2 + y^2 + z^2\right) - 11x + 3y - 16z + 38 = 0$$

Memory Map

1) Equations of Sphere in different forms :

a) Centre and Radius form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

and $\quad x^2 + y^2 + z^2 = r^2$

b) General form :

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

centre = $(-u, -v, -w)$ and

radius = $\sqrt{u^2+v^2+w^2-d} \geq 0$

c) Intercept Form : $x^2+y^2+z^2-ax-by-cz=0$

d) Diameter form :

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

2) Touching Spheres :

a) Touch externally : d(centres) = sum of Radii

b) Touch Internally : d(dentres) = $|r_1 - r_2|$

3) Tangent plane of the sphere

iff d(plane, centre of sphere) = Radius of sphere

$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

4) Section of a sphere by a plane

a) Intersection of a sphere and a plane is circle

b) Intersection of two spheres is a circle

c) Equation of sphere passing through the circle $s=0$ and $v=0$ is $s+\lambda u=0$

and $s_1=0$ and $s_2=0$ is $s_1+\lambda s_2=0$

5) Orthogonal spheres : $2(u_1u_2+v_1v_2+w_1w_2)=d_1+d_2$

END...

UNIT - V

11 The Cone

11.1 : The Cone

A **cone** is a surface generated by a straight line which passes through a fixed point and satisfies one more condition like, it passes through a given curve of surface or touches a given surface.

The fixed point is called the **vertex or apex.** The given curve is called the **guiding curve** of the cone . Any straight line through the vertex and guiding curve is called a generator of the cone.

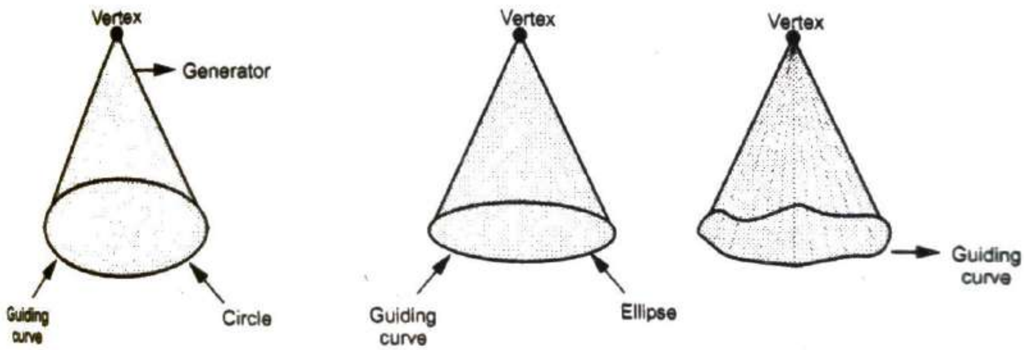


Fig. 11.1

The guiding curves of the cone may be circle, an ellipse, triangle or any type of curve. Cone whose equation is of second degree in x, y, z is called Quadratic cone. In this chapter we are going to study only quadratic cones.

11.2 : Right Circular Cone

A **right circular cone** is the surface generated by a straight line which passes through a fixed point and makes constant angle with fixed line through the fixed point.

The fixed point is called the **vertex** and the fixed line as the **axis** and the constant angle is the **semi vertical angle.**

The section of a right circular cone by a plane perpendicular to axis is a circle.

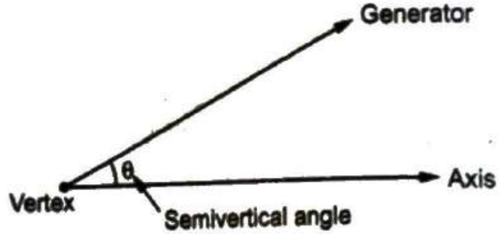


Fig. 11.2 Necessary data of RCC

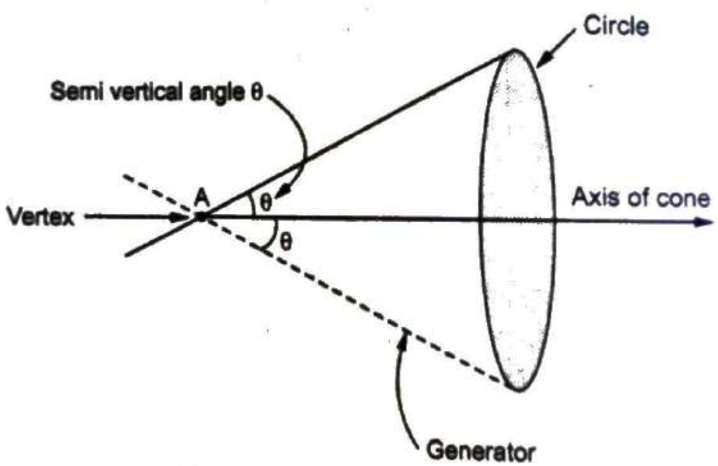


Fig. 11.3 Right circular cone

Necessary data :

i) The vertex is at A (a, b, c)

ii) dr's of axis are l, m, n

iii) Semi vertical angle is θ

If the above necessary data is not given then first find it and then proceed as given below

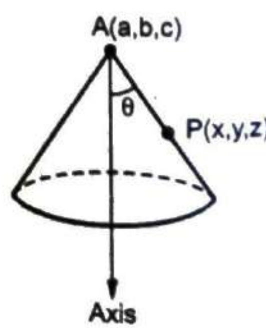


Fig. 11.4

Step 1 : Let P (x, y, z) be any point on generator of the cone

Step 2 : The vertex is at A (a, b, c)

∴ dr's of line AP are x – a, y – b, z – c

Step 3 : Consider dr's of axis l, m, n.

Step 4 : Use the formula for cos θ.

$$\cos\theta = \frac{l(x-a)+m(y-b)+n(z-c)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

Simplifying we get the required equation of cone.

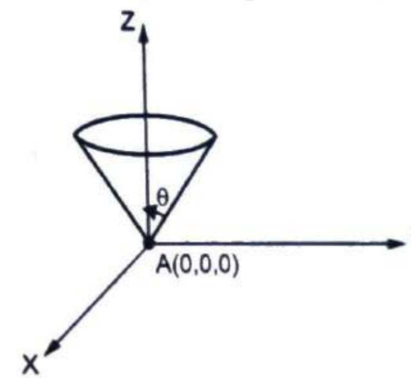


Fig. 11.5

∴ The general equation of RC cone is

$$l(x-a)+m(y-b)+n(z-c) = \cos\theta\sqrt{l^2+m^2+n^2}\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} \quad \text{... (1)}$$

Particular cases :

1) **The equation of RCC with vertex at origin :** It is obtained by substituting a = b = c = 0 in equation (1), we get,

$$lx+my+nz = \cos\theta\sqrt{l^2+m^2+n^2}\sqrt{x^2+y^2+z^2} \quad \text{... (2)}$$

2) **The equation of RCC with vertex at origin and axis of cone along z-axis :**

Put l = 0, n = 0, n = 1 in equation (2) we get,

$$z = \cos\theta\sqrt{1}\sqrt{x^2+y^2+z^2}$$

Squaring both sides, we get

$$z^2 = (x^2+y^2+z^2)\cos^2\theta$$

$$\therefore \quad x^2+y^2 = z^2(\sec^2\theta-1) = z^2+\tan^2\theta$$

$$\boxed{x^2+y^2 = z^2\tan^2\theta}$$

Similarly about Y-axis is

$$\boxed{x^2+z^2 = y^2\tan^2\theta}$$

and about x axis is,

$$\boxed{y^2+z^2 = x^2\tan^2\theta}$$

Examples of Right Circular Cone

Necessary Data :

1) Vertex : Either directly given or need to find by given data.
2) Axis : Axis of RCC is a straight line and to define any straight line, we need direction ratio's and one point or two points of that line.
3) Semi vertical Angle : It is the angle between Axis and generator i.e. angle between two straight lines.

The semi vertical angle is either given directly or drs of axis and generator are given then use formula.

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

Q.1 : Find the equation of right circular cone whose vertex is at (1,2,3) and axis is given by $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{4}$ and semi vertical angle is 60 °.

[SPPU : Dec.-14, May 17, Marks 04]

Ans. : Necessary Data

i) Vertex is at A(1, 2, 3)

ii) Direction ratios of axis are (2, –1,4)

iii) Semi vertical angle is $\theta = 60°$

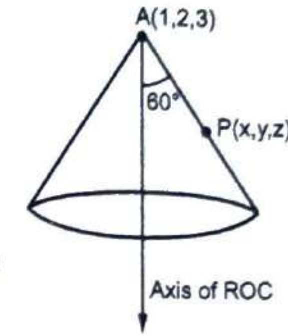


Fig. Q.1.1

Consider the following steps

Step 1 : Let P(x,y,z) be any point on the generator of the right circular cone.

Step 2 : The vertex is at A(1,2,3)

$\therefore$ The direction ratio's of AP are $x-1, y-2, z-3$

Step 3 : Given that direction ratio's of axis are 2, –1, 4.

Step 4 : If θ is the symmetrical angle then

$$\cos\theta = \cos 60°$$

$$= \frac{2(x-1)-1(y-2)+4(z-3)}{\sqrt{4+1+16}\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}$$

$$\frac{1}{2} = \frac{2x-y+4z-12}{\sqrt{25}\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}$$

Squaring both sides, we get

$$25[(x-1)^2+(y-2)^2+(z-3)^2] = 4[2x-y+4z-12]^2$$

$$25[x^2-2x+1+y^2-4y+4+z^2-6z+9] = 4[4x^2+y^2+16z^2+144 -4xy+16xz-48x-8yz-24y-96z]$$

$$9x^2+21y^2-39z^2+16xy-64xz+32yz+142x-4y+234z-226 = 0$$

This is the required equation of the cone.

Q.2 : Find the equation of the right circular cone whose vertex is at origin with axis $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and has a semivertical angle 30°.

[SPPU : Dec.-15, Marks 04]

Ans. : Necessary Data

i) Vertex is at A(0,0,0)

ii) Direction Ratio's of axis are 1, 2, 3

iii) Semi vertical angle is $\theta = 30^\circ$

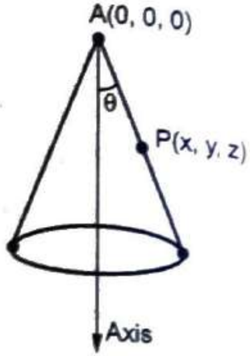


Fig. Q.2.1

Consider the following steps :

Step 1 : Let P(x,y,z) be any point on the generator of the right circular cone.

Step 2 : The vertex is at A (0,0,0)

$\therefore$ The direction ratios of AP are $x-0, y-0, z-0$ i.e. x, y, z.

Step 3 : Given that the direction ratio's of axis are 1, 2, 3.

Step 4 : We have $\theta = 30^\circ$

$$\cos\theta = \frac{1(x)+2(y)+3(z)}{\sqrt{1+4+9}\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{x+2y+3z}{\sqrt{14}\sqrt{x^2+y^2+z^2}}$$

Squaring both sides, we get

$$3(14)(x^2+y^2+z^2) = 4(x+2y+3z)^2$$

$$21(x^2+y^2+z^2) = 2(x^2+4y^2+9z^2+4xy+6xz+12yz)$$

$$\therefore \; 19x^2+13y^2+3z^2-8xy-2xz-24yz = 0$$

This is the required equation of the right circular cone.

Q.3 : Find the equation of the cone with vertex at (1, 2, – 3), semi vertical angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and the axis is $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+1}{-1}$.

[SPPU : Dec.-16, May 16, Marks 04]

Ans. : Necessary Data

i) Vertex is at A(1,2,–3)

ii) Direction ratios of axis 1, 2,–1

iii) Semi vertical angle $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \theta$

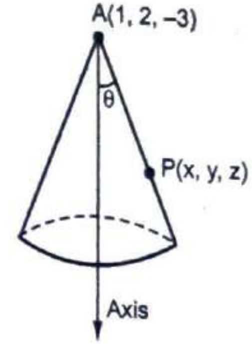


Fig. Q.3.1

Consider the following steps :

Step 1 : Let P(x,y,z) be any point of the generator of the right circular cone.

Step 2 : The vertex is at A(1,2,–3)

$\therefore$ The direction ratios of AP are $x-1, y-2, z+3$

Step 3 : Given that, the drs of axis are 1,2,–1

Step 4 : If θ is the semi vertical angle then

$$\cos(\theta) = \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1(x-1)+2(y-2)-1(z+3)}{\sqrt{1+4+1}\sqrt{(x-1)^2+2(y-2)^2+(z+3)^2}}$$

$$\frac{1}{\sqrt{3}} = \frac{x+2y-z-8}{\sqrt{6}\sqrt{(x-1)^2+2(y-2)^2+(z+3)^2}}$$

Squaring both sides, we get,

$$2[(x-1)^2+(y-2)^2+(z+3)^2] = (x+2y-2-8)^2$$

$$2[x^2-2x+1+y^2-4y+4+z^2+6z+9]$$

$$= x^2+4y^2+z^2+64+4xy-2xz-4yz-16x-32y+16z$$

$$\therefore \; x^2-2y^2+z^2-4xy+2xz+4yz+12x+24y-4z-36 = 0$$

This is the required at of the right circular cone.

Q.4 : Find the equation of right circular cone whose vertex is at (1,1,1) axis is $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ and semi vertical angle $\frac{\pi}{4}$.

[SPPU : May-18, Marks 04]

Ans. : Necessary Data :

i) Vertex is at A(1,1,1)

ii) drs of axis are 1,2,3

iii) Semi vertical angle is $\frac{\pi}{4}$

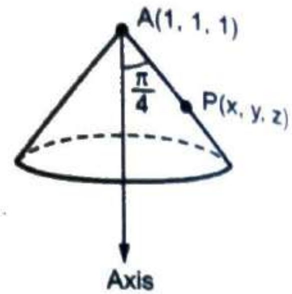


Fig. Q.4.1

Consider the following steps.

Step 1 : Let P(x,y,z) be any point on the generator of the right circular cone.

Step 2 : The vertex is at A(1,1,1)

$\therefore$ drs of AP are $x-1, y-1, z-1$

Step 3 : drs of axis are 1,2,3

Step 4 :

$$\therefore \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$= \frac{(x-1)+2(y-1)+3(z-1)}{\sqrt{1+4+9}\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}}$$

Squaring both sides

$$14[(x-1)^2+(y-1)^2+(z-1)^2] = 2[x+2y+3z-6]^2$$

$$7[x^2-2x+1+y^2-2y+1+z^2-2z+1]$$

$$= 2[x^2+4y^2+9z^2+36+4xy+6xz+12yz-12x-24y-36z]$$

$$\therefore 5x^2-y^2-11z^2-8xy-12xz-24yz+10x+34y+58z-51 = 0$$

This is the required equation of RCC.

Q.5 : Find the equation of the right circular cone whose vertex is (1, –1, 2) and axis is the line $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{-2}$ and semi vertical angle 45°. ☞ [SPPU : Dec-05,17, May-09,11,15, Marks 5]

Ans. : Necessary data :

i) vertext is at A(1, –1, 2)

ii) drs of axis are 2, 1, –2.

iii) $\theta = 45°$

Step 1 : Let P (x, y, z) be any point on generator.

Step 2 : Given vertex A (1, – 1, 2)

$\therefore$ drs of AP are $x-1, y+1, z-2$

Step 3 : dr's of axis 2, 1, – 2.

Step 4 : Use formula for $\cos\theta$.

$$\cos 45° = \frac{2(x-1)+1(y+1)-2(z-2)}{\sqrt{4+1+4}\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}}$$

Squaring both sides we get

$$9[(x-1)^2+(y+1)^2+(z-2)^2] = x[2x+y-2z+3]^2$$

$$5x^2+8y^2+13z^2-4xy+4yz+8xz-30x+12y-24z+45 = 0$$

Q.6 : Find the equation of right circular cone which passes through the point (1, 1, 2) and has its axis as the line 6x = – 3y = 4z and vertex origin. ☞ [SPPU : May-04,17, Dec.-06,11,14, Marks 4]

Ans. : The axis of the cone is ,

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$$

$\therefore$ dr's of axis are 2, – 4, 3.

The direction ratio's of line joining A (0, 0, 0) and given point Q (1, 1, 2) are 1, 1, 2.

$\therefore$ The semi vertical angle of the cone is the angle between the axis and line AQ.

$\therefore$ Using dr's of axis 2, – 4, 3 and dr's of AQ are 1, 1, 2.

$$\therefore \quad \cos\theta = \frac{2(1)-4(1)+3(2)}{\sqrt{4+16+9}\sqrt{1+1+4}}$$

$$\cos\theta = \frac{4}{\sqrt{174}}$$

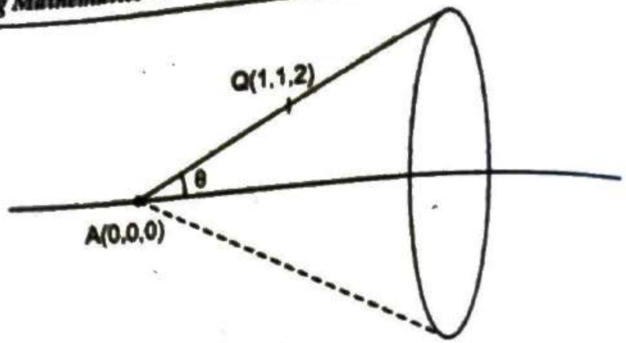


Fig. Q.6.1

Necessary data,

i) Co-ordinates of vertex A (0, 0, 0)

ii) dr's of axis 2, – 4, 3

iii) Semi vertical angle.

$$\theta = \cos^{-1}\frac{1}{\sqrt{174}}$$

Consider the following steps :

Step 1 : Let P (x, y, z) be any point on generator.

Step 2 : Given vertex is A (0, 0, 0)

$\therefore$ dr's of AP are x – 0, y – 0, z – 0

Step 3 : dr's of axis are (2, – 4, 3).

Step 4 : Use formula for cos θ.

$$\cos\theta = \frac{2x-4y+3z}{\sqrt{4+16+9}\sqrt{x^2+y^2+z^2}}$$

$$\frac{4}{\sqrt{174}} = \frac{2x-4y+3z}{\sqrt{29}\sqrt{x^2+y^2+z^2}}$$

Squaring both sides we get

$$3(2x-4y+3z)^2 = 8\left(x^2+y^2+z^2\right)$$

i.e. $4x^2 + 40y^2 + 19z^2 - 48xy - 72yz + 36zx = 0$

is the required equation of the cone.

Q.7 : Find the equation of right circular cone whose vertex is at (0, 0, 0), semi vertical angle $\frac{\pi}{4}$ and axis along the line x = –2y = z.

[SPPU : Dec.-07, 18, Marks 4]

Ans. : Necessary Data

i) Vertex is at A(0,0,0)

ii) Axis is x = – 2y = z

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$$

$\therefore$ drs of axis are 2, –1, 2.

iii) Semi vertical angle is $\theta = \frac{\pi}{4}$

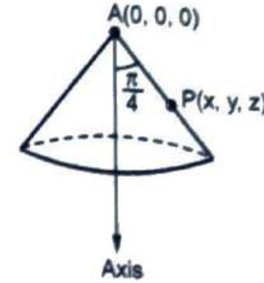


Fig. Q.7.1

Consider the following steps

Step 1 : Let P(x, y, z) be any point on the generator of the right circular cone.

Step 2 : The vertex is at A(0,0,0).

$\therefore$ drs of AP are $x-0, y-0, z-0$ i.e. x, y, z

Step 3 : The direction ratio's of axis are 2, –1, 2

Step 4 : Semi vertical angle is $\theta = \frac{\pi}{4}$

$$\therefore \quad \cos\theta = \cos\frac{\pi}{4} = \frac{2x-y+2z}{\sqrt{4+1+4}\sqrt{x^2+y^2+z^2}}$$

$$\frac{1}{\sqrt{2}} = \frac{2x-y+2z}{3\sqrt{x^2+y^2+z^2}}$$

Squaring both sides, we get

$$9(x^2+y^2+z^2) = 2(2x-y+2z)^2$$

$$9x^2+9y^2+9z^2 = 2(4x^2+y^2+4z^2-4xy+8xz-4yz)$$

$$\therefore \quad x^2+7y^2+z^2+8xy-16xz+8yz = 0$$

This is the required equation of the right circular cone.

Q.8 : Find the equation of right circular cone with vertex at (0, 0, 2), direction ratio of the generator are 0, 3, – 2 and the axis is z-axis. [SPPU : May-19, Marks 4]

Ans. : Necessary Data :

i) Vertex is at A(0, 0, 2)

ii) Axis of RCC is z axis and drs of z-axis are 0, 0, 1

$\therefore$ Drs of axis are 0, 0, 1

iii) The semi vertical angle θ is the angle between generator AQ and axis having drs 0, 3, –2 and 0, 0, 1 respectively.

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\cos\theta = \frac{0(0)+0(3)+1(-2)}{\sqrt{0+0+1}\sqrt{0+9+4}} = \frac{-2}{\sqrt{13}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-2}{\sqrt{13}}\right)$$

Consider the following steps

Step 1 : Let P(x, y, z) be any point on the generator of RCC.

Step 2 : The vertex is at A(0, 0, 2) and drs of AP are

$x-0, y-0, z-2$ i.e. $x, y, z-2$

Step 3 : drs of axis are 0, 0, 1

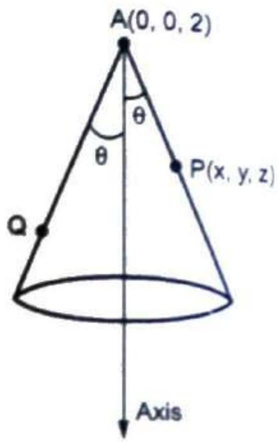


Fig. Q.8.1

Step 4 :

$$\therefore \quad \cos\theta = \frac{0(x)+0(y)+1(z-2)}{\sqrt{0+0+1}\sqrt{x^2+y^2+(z-2)^2}}$$

$$-\frac{2}{\sqrt{13}} = \frac{z-2}{3\sqrt{x^2+y^2+(z-2)^2}}$$

Squaring both sides, we get

$$4[x^2+y^2+(z-2)^2] = 13(z-2)^2$$

$$\Rightarrow \quad 4[x^2+y^2+z^2-4z+4] = 13(z^2-4z+4)$$

$$4x^2+4y^2-9z^2+36z-36 = 0$$

This is the required equation of the RCC.

Q.9 : Find equation of right circular cone which has its vertex at the point (0, 0, 12) whose intersection with the plane z = 0 is a circle of diameter 10. [SPPU : May-06, Dec.-10,18]

Ans. : Given that the vertex of right circular cone is at point A (0, 0, 12).

$\therefore$ Z-axis is the axis of the cone with dr's 0, 0, 1.

Let B the any point on a circle

$\therefore$ OB = radius of circle = 5.

The Δ AOB is a right angle triangle.

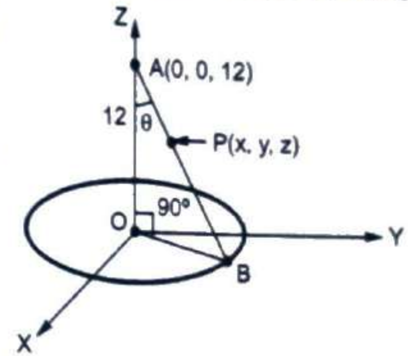


Fig. Q.9.1

$$\therefore \quad (AB)^2 = (AO)^2 + (OB)^2 = 144 + 25$$

$$(AB)^2 = 169$$

$$\therefore \quad AB = 13$$

Let θ be the semi vertical angle.

$$\therefore \quad \cos\theta = \frac{12}{13} \quad \therefore \theta = \cos^{-1}\left(\frac{12}{13}\right)$$

Thus we have

i) Vertex of RCC = A (0, 0, 12)

ii) dr's of axis of RCC are 0, 0, 1

iii) Semi vertical angle = $\cos^{-1}\left(\frac{12}{13}\right)$

∴ Follow our standard procedure.

1) Let P (x, y, z) be any point on generator of cone.

2) Given vertex is at A (0, 0, 12) dr's of AP we x, y, z – 12.

3) dr's of axis of cone are 0, 0, 1.

4) ∴ We have

$$\cos\theta = \frac{(x)(0)+y(0)+(z-12)(1)}{\sqrt{0+0+1}\sqrt{x^2+y^2+(z-12)^2}}$$

$$\therefore \quad \frac{12}{13} = \frac{(z-12)}{\sqrt{x^2+y^2+(z-12)^2}}$$

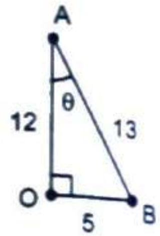


Fig. Q.9.2

Squaring both sides.

$$\frac{144}{169} = \frac{(z-12)^2}{\sqrt{(x^2+y^2+(z-12)^2}}$$

$$144x^2+144y^2-25z^2+600z-3600 = 0$$

which is required equation of cone.

Q.10 : The axis of a right circular cone whose vertex is origin 'O' makes equal angles with the co-ordinate axes and the cone passes through the line drawn form 0 with direction cosines proportional to 1, – 2, 2. Find the equation of the cone.

[SPPU : May-10,15, Marks 4]

Ans. : Given that

i) Vertex is at origin O(0, 0, 0)

ii) Axis makes equal angles with co-ordinate axes

∴ Equation of axis is $\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$

i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

∴ dr's of axis are 1, 1, 1.

iii) dr's of generator of cone are 1, – 2, 2.

$$\therefore \quad \cos\theta = \frac{1-2+2}{\sqrt{3}\sqrt{1+4+4}} = \frac{1}{3\sqrt{3}}$$

where θ is semivertical angle.

Now consider the following steps.

Step 1 : Let p(x, y, z) be any point on generator of cone.

Step 2 : Given O(0, 0, 0) ∴ dr's of OP are x, y, z.

Step 3 : dr's of axis are 1, 1, 1.

Step 4 : Use formula for cos θ.

$$\cos\theta = \frac{x+y+z}{\sqrt{1^2+1^2+1^2}\sqrt{x^2+y^2+z^2}}$$

$$\frac{1}{3\sqrt{3}} = \frac{x+y+z}{\sqrt{3}\sqrt{x^2+y^2+z^2}}$$

Squaring both sides, we get

$$x^2+y^2+z^2 = 9(x+y+z)^2$$

$$\therefore \quad 8x^2+8y^2+8z^2+18xy+18xz+18yz = 0$$

$$4(x^2+y^2+z^2)+9(xy+xz+yz) = 0$$

which is the required equation of RCC.

Q.11 : Find equation of right circular cone with vertex at (1, – 1, 1), semivertical angle is 45° and it's axis is perpendicular to the plane 2x + y – 2z + 1 = 0.

[SPPU : Dec.-10]

Ans. : Given that

i) Vertex is at A(1, – 1, 1)

ii) Axis of RCC is perpendicular to the plane

2x + y – 2z + 1 = 0

∴ dr's of axis are 2, 1, – 2.

iii) Semivertical angle is 45°.

Consider the following steps.

Step 1 : Let $P(x, y, z)$ be any point one the generator of RCC.

Step 2 : Given vertex $A(1, -1, 1)$

$\therefore$ dr's of AP are $x-1, y+1, z-1$.

Step 3 : dr's of axis are $2, 1, -2$.

Step 4 : We have

$$\cos 45^\circ = \frac{2(x-1)+1(y+1)+(-2)(z-1)}{\sqrt{4+1+4}\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}$$

$$\frac{1}{\sqrt{2}} = \frac{2x+y-2z+1}{3\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}$$

Squaring both sides, we get

$$\frac{1}{2} = \frac{(2x+y-2z+1)^2}{9[(x-1)^2+(y+1)^2+(z-1)^2]}$$

$$9[x^2-2x+1+y^2+2y+1+z^2-2z+1]$$

$$= 2(2x+y-2z+1)^2$$

$$9x^2 = 18x+9+9y^2+18y+9+9z^2-18z+9$$

$$= 2[4x^2+y^2+4z^2+1+4xy-8xz+4x$$

$$-4yz+2y-4z]$$

$$\Rightarrow x^2+7y^2+z^2-8xy+16xz+8yz-26x+14y+10z+25 = 0$$

which is the required equation of RCC.

Q.12 : Find the equation at the right circular cone with vertex at $(-1, 0, 0)$, semi vertical angle 60° and axis is x-axis.

[SPPU : May 18, Marks 04]

Ans. : Necessary data :

i) Vertex is at $A(-1,0,0)$

ii) Axis of RCC is x-axis having drs 1,0,0

iii) Semi vertical angle $\theta = 60^\circ$

Consider the following steps :

Step 1 : Let $P(x,y,z)$ be any point on the generator of RCC.

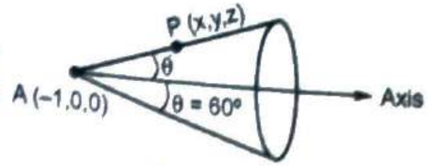


Fig. Q.12.1

Step 2 : The vertex is at $A(-1,0,0)$.

$\therefore$ drs of AP are $x+1, y, z$

Step 3 : drs of axis are 1,0,0.

Step 4 : Therefore,

$$\therefore \quad \cos 60^\circ = \frac{1(x+1)+0(y)+0(z)}{\sqrt{1+0+0}\sqrt{(x+1)^2+y^2+z^2}} = \frac{\sqrt{3}}{2}$$

Squaring both sides, we get

$$3[(x+1)^2+y^2+z^2] = 4(x+1)^2$$

$$3[x^2+2x+1+y^2+z^2] = 4(x^2+2x+1)$$

$$-x^2+3y^2+3z^2-2x-1 = 0$$

$$\Rightarrow x^2-3y^2-3z^2+2x+1 = 0$$

Q.13 : Find the equation of right circular cone with vertex at origin axis is the y-axis and semi vertical angle is 30°.

[SPPU : Dec.-18, Marks 04]

Ans. : The vertex is at $A(0,0,0)$

drs of axis are 0,1,0 and $\theta = 30^\circ$

drs of AP are x,y,z

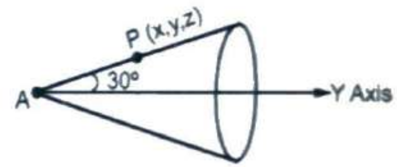


Fig. Q.13.1

$$\therefore \quad \cos 30^\circ = \frac{0(x)+1(y)+0(z)}{\sqrt{0+1+0}\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{3}}{2}$$

Squaring both sides, we get

$$3(x^2+y^2+z^2) = 4(y)^2$$

$$3x^2-y^2+3z^2 = 0$$

Q.14 : Obtain the equation of a right circular cone which passes through the point (2,1,3) and vertex at (1,1,2) and axis is parallel to line $\frac{x-2}{2} = \frac{y-1}{-4} = \frac{z+2}{3}$. [SPPU : Dec.-17, Marks 04]

Ans. : Vertex is at AC(1, 1, 2)

drs of AQ are $2-1, 1-1, 3-2$ i.e. 1, 0, 1

drs of axis are 2, –4, 3

$$\therefore \quad \cos\theta = \frac{2+0+3}{\sqrt{1+0+1}\sqrt{4+16+9}} = \frac{5}{\sqrt{2}\sqrt{29}}$$

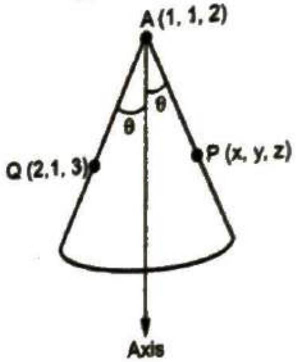


Fig. Q.14.1

$$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{\sqrt{2}\sqrt{29}}\right)$$

Now drs of AP are $x-1, y-1, z-2$ and axis are 2, –4, 3.

$$\therefore \quad \cos\theta = \frac{2(x-1)-4(y-1)+3(z-2)}{\sqrt{4+16+9}\sqrt{(x-1)^2+(y-1)^2+(z-2)^2}}$$

$$\left(\frac{5}{\sqrt{2}\sqrt{29}}\right) = \frac{2x-4y+3z-4}{\sqrt{29}\sqrt{(x-1)^2+(y-1)^2+(z-2)^2}}$$

$\therefore$ Squaring both sides we get

$$25[(x-1)^2+(y-1)^2+(z-2)^2] = 2[2x-4y+3z-4]^2$$

$$25[x^2+y^2+z^2-2x-2y-4z+6] = 2[4x^2+16y^2+9z^2+16-16xy +12xz-24yz-16x+32y-242]$$

$$\therefore \quad 17x^2-7y^2+7z^2+32xy-24xz+48yz-18x-114y+2z+118 = 0$$

Q.15 : Find the equation of right circular cone whose vertex is at (1, –1,1) and axis is parallel to $x = -\frac{y}{2} = -z$ and one of it's generators has direction cosines proportional to (2,2,1). [SPPU : Dec.-15, Marks 04]

Ans. : The vertex is at A(1,–1, 1)

drs of generator AQ are 2,2,1

drs of axis are 1,–2,–1

$$\cos\theta = \frac{2-4-1}{\sqrt{4+4+1}\sqrt{1+4+1}} = \frac{-3}{3\sqrt{6}}$$

Now drs of AP are $x-1, y+1, z-1$

$$\therefore \quad \cos\theta = \frac{-3}{3\sqrt{6}}$$

$$= \frac{(x-1)-2(y+1)-(z-1)}{\sqrt{1+4+1}\sqrt{(x-1)^2+(y+1)^2+(z-1)^2}}$$

Squaring both sides, we get

$$0x^2-2x+1+y^2+2y+1+z^2-2z+1 = [x-2y-z-2]^2$$

$$\therefore 0x^2-3y^2+0z^2+4xy+2xz-4yz+2x-6y-6z-1 = 0$$

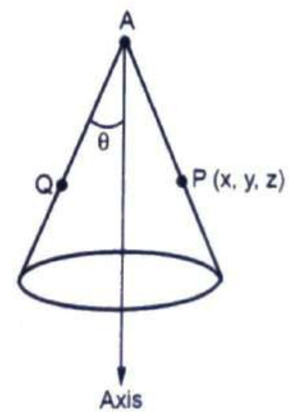


Fig. Q.15.1

Memory Map

1) Equation of cone is $ax^2+by^2+cz^2+2fyz+2gxz+2hxy+2ux+2vy+2wz+d=0$
2) Equation of the quadratic cone is $fyz+gzx+hxy=0$
3) Right circular cone
   i) The vertex is at A (a, b, c)
   ii) dr's of axis are l, m, n
   iii) Semi vertical angle is θ

If the above necessary data is not given then first find it and then proceed as given below

Step 1 : Let P (x, y, z) be any point on generator of the cone

Step 2 : The vertex is at A (a, b, c)

$\therefore$ dr's of line AP are $x-a$, $y-b$, $z-c$

Step 3 : Consider dr's of axis l, m, n.

Step 4 : Use the formula for $\cos\theta$.

$$\cos\theta = \frac{l(x-a)+m(y-b)+n(z-c)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

Simplifying we get the required equation of cone.

$\therefore$ The general equation of RC cone is

$$l(x-a)+m(y-b)+n(z-c) = \cos\theta\sqrt{l^2+m^2+n^2}\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} \quad \text{... (1)}$$

END...

UNIT - V

12 The Cylinder

12.1 : The Cylinder

A **cylinder** is a surface generated by straight line which is parallel to a fixed line and satisfies one more geometrical condition like intersecting a given curve or directrix or touches to a given surface.

The fixed straight line is called the **axis** and the given curve or surface is called as the **guiding curve** of the cylinder. The moving straight line is called the **generator** of the cylinder. Refer Fig. 12.1 (a),(b),(c).

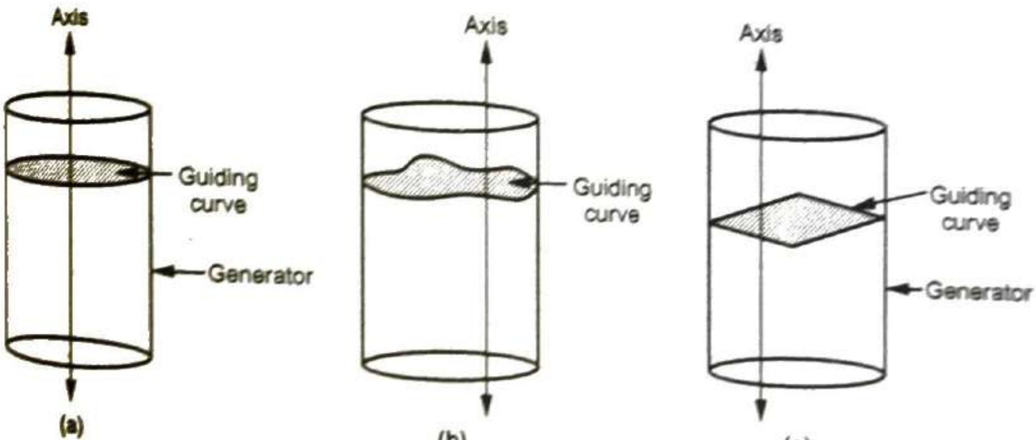


Fig. 12.1

Q.1 : Find the equation of the cylinder whose guiding curve is $ax^2 + by^2 = 2z$ and $lx + my + nz = p$ and generators parallel to the z axis.

[SPPU : May-05, Dec.-10]

Ans. : Since the generators are parallel to the z axis we have to eliminate z from the above two equations of the guiding curve

$$ax^2 + by^2 = 2z$$

and
$$lx + my + nz = p$$

i.e.
$$lx + my + n\frac{\left(ax^2 + by^2\right)}{2} = p$$

or the required equation is

$$n\left(ax^2 + by^2\right) + 2lx + 2my - 2p = 0$$

Q.2 : Find the equation of the cylinder whose generators are parallel to $\frac{x}{3} = \frac{y}{1} = \frac{z}{\sqrt{6}}$ and whose guiding curve is $x^2 + y^2 = 25$, $z = 0$. [SPPU : May-08, Dec.-03, Marks 4]

Ans. : Step 1 : Let (x_1, y_1, z_1) be any point on the generator of the cylinder. As 3, 1, $\sqrt{6}$ are the dr's of the generator

$\therefore$ Equation of the generator is

$$\frac{x - x_1}{3} = \frac{y - y_1}{1} = \frac{z - z_1}{\sqrt{6}} \quad \ldots (1)$$

Step 2 : This generator meets the plane $z = 0$.

$$\therefore \quad \frac{x - x_1}{3} = \frac{y - y_1}{1} = \frac{z - 0}{\sqrt{6}}$$

Step 3 : Find x, y in terms of x_1, y_1, z_1.

$$x = x_1 - \frac{3z_1}{\sqrt{6}} \quad \text{and} \quad y = y_1 - \frac{z_1}{\sqrt{6}}$$

Step 4 : This point lies on $x^2 + y^2 = 25$.

$\therefore$ Substituting x and y.

$$\left(x_1 - \frac{3z_1}{\sqrt{6}}\right)^2 + \left(y_1 - \frac{z_1}{\sqrt{6}}\right)^2 = 25$$

$$\left(\sqrt{6}\,x_1 - 3z_1\right)^2 + \left(\sqrt{6}\,y_1 - z_1\right)^2 = 150$$

Simplifying and replacing (x_1, y_1, z_1) by (x, y, z) we get

$$6x^2 + 6y^2 + 10z^2 - 2\sqrt{6}\,yz - 6\sqrt{6}\,xz - 150 = 0$$

which is the required locus of (x_1, y_1, z_1) i.e. the equation of cylinder.

12.2 : Right Circular Cylinder

Right circular cylinder is a surface generated by straight line which is parallel to a fixed line and is at a constant distance from it.

The fixed line is called as the **axis** and the constant distance is the **radius** of the cylinder.

The section of a right circular cylinder by any plane perpendicular to axis is a circle.

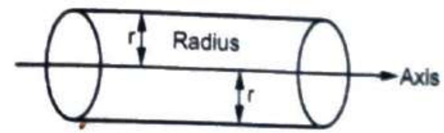


Fig. 12.2 (a) Rigth circular cylinder

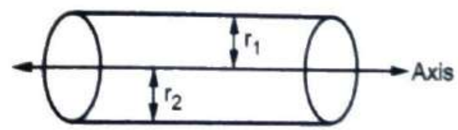


Fig. 12.2 (b) Cylinder but not right circular cylinder $r_1 \neq r_2$

12.3 : Equation of a Right Circular Cylinder

General Form :

Q.3 : To find the equations of the right circular cylinder whose radius is r and axis is the line $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$.

Ans. : Let A be the point (α, β, γ) and AB the axis whose equations are

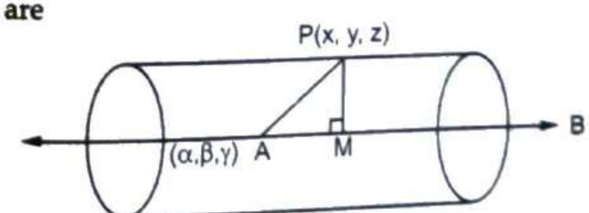


Fig. 12.3

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

The dr's of the line are l, m, n

∴ The dc's of the line are

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$$

Let P (x, y, z) be any point on the cylinder and PM perpendicular to AB.

Then PM = Radius = r

$$PA = \sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}$$

AM = Projection of AP on line AB

$$AM = \frac{(x-\alpha)\, l}{\sqrt{l^2+m^2+n^2}} + \frac{(y-\beta)\, m}{\sqrt{l^2+m^2+n^2}} + \frac{(z-\gamma)\, n}{\sqrt{l^2+m^2+n^2}}$$

From Δ APM

$$AP^2 = AM^2 + PM^2$$

i.e. $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$

$$= r^2 + \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2} \quad \text{... (1)}$$

is the required equation of the cylinder.

Note : 1) The equation of right circular cylinder whose equation of axis is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is obtained by putting $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ in equation (1).

Thus the equation is

$$x^2+y^2+z^2 = r^2 + \frac{[lx+my+nz]^2}{l^2+m^2+n^2} \quad \text{... (2)}$$

2) The equation of RCC whose axis is z-axis is obtained by putting $l = 0$, m = 0 and n = 0 in equation (2).

∴ It's equation is

$$x^2+y^2+z^2 = r^2+0+z^2$$

i.e. $$x^2+y^2 = r^2$$

3) Similarly the equation of RCC whose axis is x axis is $y^2+z^2=r^2$.

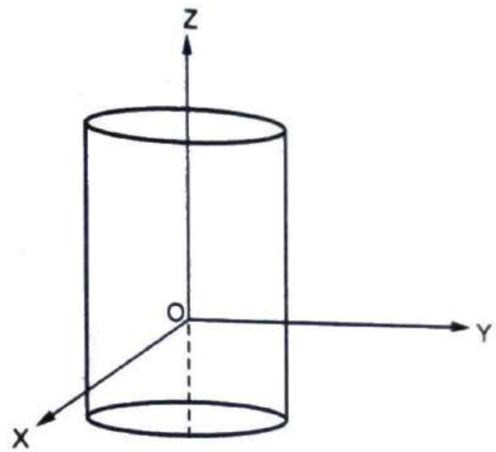


Fig. 12.4

12.4 : Enveloping Cylinder

The locus of the tangent lines to a given surface and parallel to a given line is a cylinder and is called the enveloping of the surface.

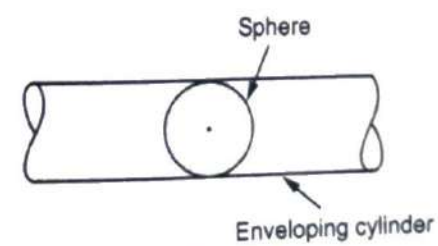


Fig. 12.5

Equation of the enveloping cylinder

To find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ whose generators are parallel to line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Let $P(x_1, y_1, z_1)$ be any point on the tangent to the sphere $x^2 + y^2 + z^2 = a^2$ and parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. Then the equation of the tangent is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{... (1)}$$

$$\therefore \quad x = x_1 + lr$$

$$y = y_1 + mr$$

$$z = z_1 + nr$$

Substituting in $x^2 + y^2 + z^2 = a^2$

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2 \left(l^2 + m^2 + n^2\right) r^2$$

$$+ 2r(lr + my_1 + nz_1) + x_1^2 + y_1^2 + z_1^2 - a^2 = 0 \quad \text{... (2)}$$

Since line (1) is a tangent line, the line should touch the sphere in only one point and therefore equation (2) has two equal roots or the discriminant of the equation (2) will be zero.

$$4(lx_1 + my_1 + nz_1)^2 - 4(l^2 + m^2 + n^2)(x_1^2 + y_1^2 + z_1^2 - a^2) = 0$$

$\therefore$ The locus of the point (x, y, z) is

$$(lx + my + nz)^2 - (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2) = 0$$

Note : Enveloping cylinder of a sphere is always right circular.

Examples on right circular cylinder

1) The necessary data for the right circular cylinder is
   i) Radius of RCC
   ii) Equation of axis : To define axis or straight line, we need two points of that line or one point and its direction ratios.

2) For solving the problems on right circular cylinder the following three things are necessary
   i) one point on axis
   ii) dr's of axis
   iii) radius of cylinder

If these three things are known then we can follow the following standard procedure for solving the problems of RCC.

Q.4 : Find the equation of right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$.

[SPPU : May-19, Marks 4]

Ans. : Necessary Data : Equation of axis is $\frac{x-1}{2} = \frac{y-0}{3} = \frac{z-3}{1}$

$\therefore$ Axis passes through the point (1, 0, 3) having drs 2, 3, 1.

Therefore,

i) One point on Axis is at A(1, 0, 3)

ii) Direction ratios of axis are 2, 3, 1.

iii) Radius of cylinder is 2.

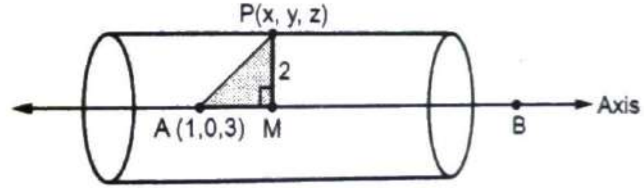


Fig. Q.4.1

Consider the following steps.

Step 1 : Let P(x, y, z) be any point on cylinder and A(1, 0, 3) is one point of axis. Draw perpendicular from P on axis and denote foot of perpendicular by M. Join AP.

Step 2 :

$$AP = \sqrt{(x-1)^2 + (y-0)^2 + (z-3)^2}$$

$$PM = 2 \text{ Radius}$$

Direction ratio's of axis are 2, 3, 1

$\therefore$ Direction cosines of axis are

$$\frac{2}{\sqrt{4+9+1}}, \frac{3}{\sqrt{4+9+1}}, \frac{1}{\sqrt{4+9+1}} \text{ i.e. } \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$$

Direction ratio's of Above x–1, y, z–3.

$$AM = \text{Projection of AP on axis.}$$

$$= \frac{2}{\sqrt{14}}(x-1)+\frac{3}{\sqrt{17}}(y)+\frac{1}{\sqrt{14}}(z-3) = \frac{1}{\sqrt{14}}(2x+3y+z-5)$$

Step 3 : In right angle triangle ΔAMP

$$(AP)^2 = (PM)^2 + (AM)^2$$

$$(x-1)^2+y^2+(z-3)^2 = 2^2+\frac{1}{14}(2x+3y+z-5)^2$$

$$14(x^2-2x+1+y^2+z^2-6z+9)$$

$$= 56+(4x^2+9y^2+z^2+25+12xy$$
$$+4xz+6yz-20x-30y-10z)$$

$$\therefore\ 10x^2+5y^2+13z^2-12xy-4xz-6yz-8x+30y-74+130 = 0$$

This is the required equation of the right circular cylinder.

Q.5 : Find the equation of a right circular cylinder having it's radius as 03 units and equation of axis is $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{3}$

[SPPU : May-16,18, Marks 4]

Ans. : Necessary Data :

i) One point on Axis is at A(1, –1, 2)

ii) drs at axis are 2, –1, 3

iii) Radius of cylinder is 3.

Consider the following steps.

Step 1 : Let P(x, y, z) be any point on the cylinder and A(1, –1, 2) is one point on the axis. Draw perpendicular from point P on axis and denote foot of perpendicular by M. Join AP.

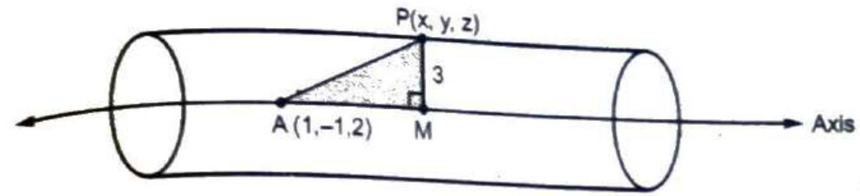


Fig. Q.5.1

Step 2 :

$$(AP)^2 = (x-1)^2+(y+1)^2+(z-2)^2$$

$$PM = 3$$

drs of AP are $x-1,\ y+1,\ z-2$

drs of axis are 2, –1, 3.

$\therefore$ direction cosines of axis are $\frac{2}{\sqrt{4+1+9}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

$\therefore$

$$AM = \text{Projection of AP on Axis,}$$

$$= \frac{2}{\sqrt{14}}(x-1)-\frac{1}{\sqrt{14}}(y+1)+\frac{3}{\sqrt{14}}(z-2)$$

$$AM = \frac{1}{\sqrt{14}}(2x-y+32-8)$$

Step 3 : In ΔAMP

$$(AP)^2 = (PM)^2+(AM)^2$$

$$(x-1)^2+(y+1)^2+(z-2)^2 = 9+\frac{1}{14}(2x-y+3z-8)^2$$

$$14(x^2-2x+1+y^2+2y+1+z^2-4z+4)$$

$$= 126+(4x^2+y^2+9z^2+64-4xy$$
$$+12xz-6yz-32x+16y-48z)$$

$$\therefore\ 10x^2+13y^2+5z^2+4xy-12xz+6yz+4x+12y-8z-134 = 0$$

This is the required equation of the RCC.

Q.6 : Find the equation of a right circular cylinder having radius 04 units and axis is $\frac{x+1}{1} = \frac{y+1}{-1} = \frac{z+1}{1}$.

[SPPU : May-18, Marks 4]

Ans. : Necessary Data :

i) One point on Axis is at A(–1, –1, –1)

ii) drs of axis are (1, –1, 1)

iii) Radius of cylinder is 4.

Consider the following steps

Step 1 :

Let P(x, y, z) be any point on the cylinder and A(–1, –1, –1) is one point on Axis. Draw Perpendicular from point P on axis and denote foot of perpendicular by M. Join AP.

Step 2 : PM = 4

$$(AP)^2 = (x+1)^2+(y+1)^2+(z+1)^2$$

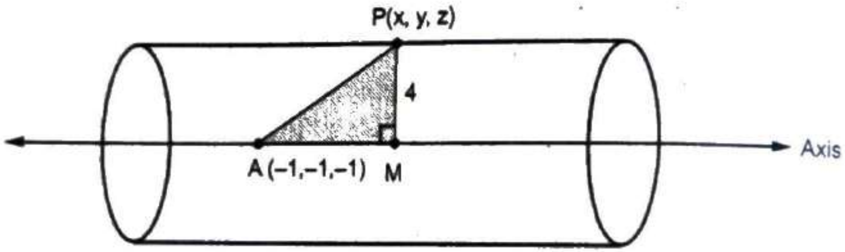


Fig. Q.6.1

dr's of AP are $x+1,\ y+1,\ z+1$

dr's of axis are 1, –1, 1.

direction cosines of axis are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$\therefore$ AM = Projection of AP on axis.

$$= \frac{1}{\sqrt{3}}(x+1)-\frac{1}{\sqrt{3}}(y+1)+\frac{1}{\sqrt{3}}(z+1) = \frac{1}{\sqrt{3}}(x-y+z+1)$$

Step 3 : In ΔAMP

$$(AP)^2 = (PM)^2+(AM)^2$$

$$(x+1)^2+(y+1)^2+(z+1)^2 = 16+\frac{1}{3}(x-y+z+8)^2$$

$$3(x^2+2x+1+y^2+2y+1+z^2-2z+1)$$

$$= 48+(x^2+y^2+z^2+1$$
$$-2xy+2xz-2yz+2x-2y-2z)$$

$$\therefore \quad 2x^2+2y^2+2z^2+2xy-2xz+2yz+4x+8y-4z-40 = 0$$

$$\therefore \quad x^2+y^2+z^2+xy-xz+yz+2x+4y-2z-20 = 0$$

This is the required equation of the RCC.

Q.7 : Find the equation of right circular cylinder of radius 2 and equation of axis is $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{6}$.

[SPPU : May-15,17, Marks 4]

Ans. : Necessary data : i) One point on axis = A(1, 2, 3)

ii) dr's of axis = 2, – 3, 6

iii) Radius of cylinder = 2

Step 1 : Let P (x, y, z) be any point on cylinder and A (1, 2, 3) be one point on axis. Draw perpendicular from point P on axis and denote foot of perependicular by M.

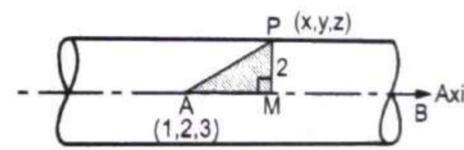


Fig. Q.7.1

Step 2 :

$$\text{Distance AP} = \sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$$

$$\text{Distance PM} = \text{Radius} = 2$$

and distance AM = Projection of AP on axis AB

Now from dr's (2, – 3, 6) we can find dc's of axis.

$\therefore$ dc's of AB are $\left(\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}\right)$

$$\therefore \quad AM = \frac{2}{7}(x-1)-\frac{3}{7}(y-2)+\frac{6}{7}(z-3)$$

$$AM = \frac{2x - 3y + 6z - 14}{7}$$

Step 3 : From Δ APM

$$AP^2 = AM^2 + MP^2$$

Substituting AP, AM, MP we get

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = \frac{(2x - 3y + 6z - 14)^2}{49} + 4$$

$$49[x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9]$$

$$= (4x^2 + 9y^2 + 36z^2 + 196 - 12xy + 24xz - 36yz$$

$$-56x + 84y - 168z + 196 + 196)$$

$$45x^2 + 40y^2 + 13z^2 + 12xy + 36yz - 24xz - 42x - 280y - 126z + 294 = 0$$

Q.8 : Find the equation of right circular cylinder of radius 2, whose axis passes through (1, 2, 3) and has direction cosines proportional to 2, 1, 2. [SPPU : Dec.-15,18, Marks 4]

Ans. : Given that : i) One point on axis = (1,2,3)

ii) dr's of axis are 2, 1, 2

iii) Radius of cylinder = 2

Step 1 : Let P (x, y, z) be any point on cylinder and A (1, 2, 3) be one point on axis and M be the perpendicular from P on axis.

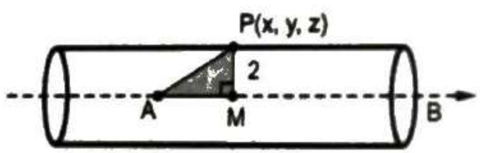


Fig. Q.8.1

Step 2 : Distance $AP = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$

PM = 2

AM = Projection of AP on axis AB dc's of AB are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

$$\therefore \quad AM = \frac{2}{3}(x-1) + \frac{1}{3}(y-2) + \frac{2}{3}(z-3)$$

$$AM = \frac{2x + y + 2z - 10}{3}$$

Step 3 : From Δ APM, $AP^2 = AM^2 + MP^2$

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = \frac{1}{9}[2x + y + 2z - 10]^2 + 4$$

$$9[x^2 - 2x + 1 + y^2 - 4y + 4 + z^3 - 6z + 9] = 4x^2 + y^2 + 4z^2 + 100 + 4xy$$

$$+8xz - 40x + 4yz - 20y - 40z + 4$$

$$5x^2 + 8y^2 + 5z^2 + 22x - 16y - 14z - 4xy - 8xz - 4yz - 23 = 0$$

which is required equation of RCC.

Q.9 : Find equation of RCC of radius a having its axis on the line $x = y = -z$. And show that the foot of perpendicular from the point (α, β, γ) on line $x = y = -z$ is $\left(\frac{\alpha + \beta - \gamma}{3}, \frac{\alpha + \beta - \gamma}{3}, \frac{-\alpha - \beta + \gamma}{3}\right)$

Ans. : Given that the equation of axis is

$$x = y = -z$$

$$\therefore \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{-1} = k \quad \dots (1)$$

$$\therefore \quad x = k, \; y = k, \; z = -k$$

Let M be the foot of perpendicular from point A on the axis.

$\therefore$ The co-ordinates of M are $x = k, y = k, z = -k$.

$\therefore$ The dr's of AM are $\alpha - k, \beta - k, \gamma + k$ and dr's of axis are 1, 1, – 1.

AM is perpendicular to the axis of cylinder.

$\therefore$ $(1)(\alpha - k) + (1)(\beta - k) + (-1)(\gamma + k) = 0$

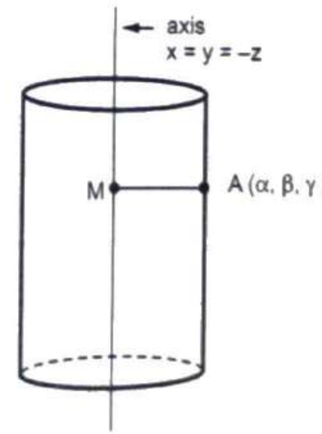


Fig. Q.9.1

$$\therefore \quad k = \frac{\alpha + \beta - \gamma}{3}$$

$\therefore$ Thus the foot of the perpendicular from the point (α,β,γ) on axis is $\left(\frac{\alpha+\beta-\gamma}{3}, \frac{\alpha+\beta-\gamma}{3}, \frac{-\alpha-\beta+\gamma}{3}\right)$

Let AM = a = Radius of cylinder.

$$\therefore \quad (AM)^2 = a^2$$

By distance formula

$$\left(\alpha-\frac{\alpha+\beta-\gamma}{3}\right)^2+\left(\beta-\frac{\alpha+\beta-\gamma}{3}\right)^2+\left(\gamma-\frac{-\alpha-\beta+\gamma}{3}\right)^2 = a^2$$

$$\therefore \quad 2(\alpha^2+\beta^2+\gamma^2)-2\alpha\beta+2\beta\gamma+2\gamma\alpha = 3a^2$$

Hence locus of A (α,β,γ) is obtained by replacing α,β,γ by x, y, z.

$$2(x^2+y^2+z^2)-2xy+2yz+2zx = 3a^2$$

which is required equation of RCC.

Q.10 : (Find the equation of right circular cylinder whose axis is $x = 2y = -z$ and radius is 4.) Prove that the area of the section of this cylinder by the plane $z = 0$ is 24π.

[SPPU : Dec.-11, May-16, Marks 4]

Ans. : Given that : i) One point on cylinder = (0, 0, 0)

ii) dr's of axis are 2, 1, – 2

iii) Radius of cylinder is 4

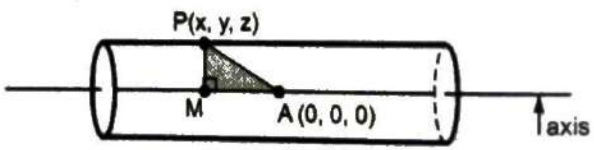


Fig. Q.10.1

Step 1 : Let P (x, y, z) be any point on cylinder and A (0, 0, 0) be one point on axis and M be the foot of perpendicular from P on axis.

Step 2 : Distance AP = $\sqrt{x^2+y^2+z^2}$

distance PM = radius = 4. dr's of AP are x, y, z.

dc's of axis are $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$.

AM = Projection of AP on axis

$$\therefore \quad AM = \frac{2}{3}x+\frac{1}{3}y-\frac{2}{3}z = \frac{2x+y-2z}{3}$$

Step 3 : From Δ APM

$$AP^2 = AM^2+PM^2$$

$$x^2+y^2+z^2 = \left(\frac{2x+y-2z}{3}\right)^2+16$$

$$9(x^2+y^2+z^2)-(2x+y-2z)^2-144 = 0$$

$$5x^2+8y^2+5z^2-4xy+4yz+8xz-144 = 0 \quad \text{... (1)}$$

which is required equation of R.C.C.

The section of RC cylinder by the plane z = 0 gives

$$5x^2+8y^2-4xy-144 = 0$$

$$\frac{5}{144}x^2+\frac{1}{18}y^2-\frac{1}{36}xy = 1 \quad \text{... (2)}$$

{Note the equation $Ax^2+2Hxy+By^2=1$ represents an ellipse if

$$AB-H^2 > 0$$

and if a, b are semiaxes, then squares of a and b are the roots of the equation

$(AB-H)^2 r^4-(A+B)r^2+1=0$ }

From equation (2) we get $A = \frac{5}{144}$, $H = -\frac{1}{18}$, $B = \frac{1}{18}$

and $AB-H^2 > 0$

and $\quad r^4-52r^2+576 = 0$

$$\therefore \quad (r^2-36)(r^2-16) = 0$$

$r^2 = 16, 36, \quad r = 4, 6$

$\therefore$ a = 6, b = 4

We have area of an ellipse = $\pi ab = \pi(6)(4)$

$= 24\pi$ sq. unit.

Q.11 : Find the equation of a right circular cylinder whose axis is the line $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$ axis which passes through the point (0, 0, 3) [Dec.-17, 16, 14, Marks 4]

Ans. : Given that axis is $\frac{x-2}{2} = \frac{y-1}{1} = \frac{2-0}{3}$

$\therefore$ Axis passes through (2, 1, 0) i.e. point on XY plane. The required cylinder passes through (0, 0, 3)

$\therefore$ The distance OQ = radius of the required cylinder.

dr's of AP are $x-2,\ y-1,\ z$

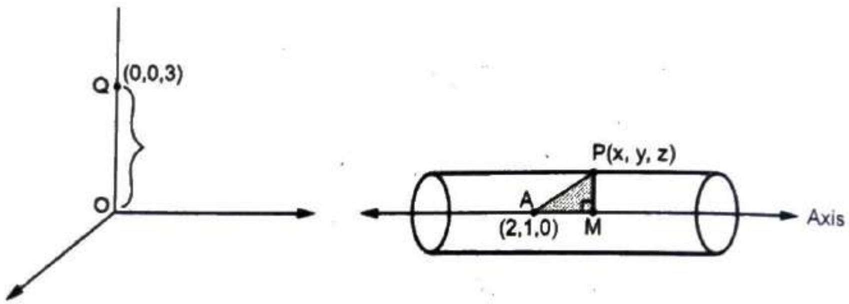


Fig. Q.11.1

dr's of axis are 2, 1, 3.

direction cosines of axis are $\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

$$\therefore \quad AM = \text{Projection of AP on axis.}$$

$$= \frac{2}{\sqrt{14}}(x-2)+\frac{1}{\sqrt{14}}(y-1)+\frac{3}{\sqrt{14}}(z) = \frac{1}{\sqrt{14}}(2x+y+3z-5)$$

In ΔAMP

$$(AP)^2 = (PM)^2+(AM)^2$$

$$(x-2)^2+(y-1)^2+z^2 = 9+\frac{1}{14}(2x+y+3z-5)^2$$

$$14(x^2-4x+4+y^2-2y+1+z^2) = 126+(4x^2+y^2+9z^2$$
$$+25+4xy+12xz+6yz-20x-10y-30z) = 0$$

$$\therefore 10x^2+13y^2+5z^2-4xy-12xz-6yz-36x-18y+30z-81 = 0$$

This is the required equation of the RCC.

Q.12 : Find equation of RCC whose generator passes through (0, 0, 5) and axis passes through (1, 1, 3) and perpendicular to z-axis.

Ans. : Given that Q (0, 0, 5) lies on generator and RCC and B (1, 1, 3) lies on axis of cylinder. The axis of RCC is perpendicular to z-axis.

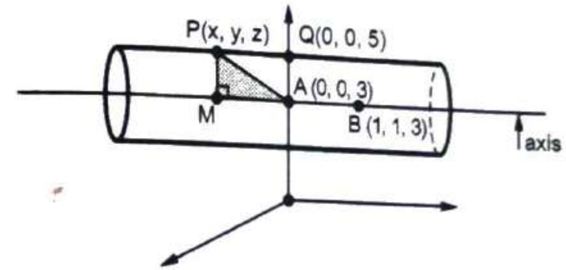


Fig. Q.12.1

$\therefore$ The intersection of axis of cylinder and RCC is A (0, 0, 3)

From Fig. Q.12.1.

Radius of RCC = distance AQ = 2

$\therefore$ dr's of AB are 1, 1, 0

$\therefore$ dr's of axis of RCC are 1, 1, 0

$\therefore$ Its dc's are $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$

We have

i) One point on axis A (0, 0, 3)

ii) dr's of axis 1, 1, 0

iii) radius = 2

Thus

Step 1 : Let P (x, y, z) be any point on generator and A (0, 0, 3) is a point on axis of RCC.

Let M be the foot of perpendicular from P on axis.

Step 2 : $$(AP) = \sqrt{x^2+y^2+(z-3)^2}$$

$$PM = 2$$

$$AM = \text{Projection of AP on axis is}$$

$$= \frac{1}{\sqrt{2}}(x-0)+\frac{1}{\sqrt{2}}(y-0)+0\,(z-3) = \frac{x+y}{\sqrt{2}}$$

Step 3 : In Δ APM, $$AP^2 = AM^2+PM^2$$

$$x^2+y^2+(z-3)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2+4$$

$$2\,(x^2+y^2+z^2-6z+9)-(x+y)^2-8 = 0$$

$$x^2+y^2+2\,z^2-2\,xy-12\,z+10 = 0$$

which is required equation of RCC.

Q.13 : Find the equation of right circular cylinder of radius 3 whose axis is the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$.

[SPPU : May-17, Dec.-14,18, Marks 4]

Ans. : We have, i) One point on axis = A (1, 3, 5)

ii) Directions of ratios axis 2, 2, –1.

iii) Radius of cylinder is 3.

Consider the following steps.

Step 1 : Let P (x, y, y) be any point on generator of RCC. A (1, 3, 5) be one point on axis and M be the foot of perpendicular from P on axis.

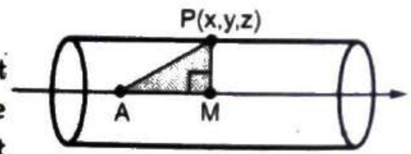


Fig. Q.13.1

Step 2 : $$(AP)^2 = (x-1)^2+(y-3)^2+(z-5)^2$$

PM = 3, direction cosines of axis are $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

$$AM = \text{Projection of AP on axis}$$

$$= \frac{2}{3}(x-1)+\frac{2}{3}(y-3)-\frac{1}{3}(z-5) = \frac{1}{3}(2x+2y-z-3)$$

Step 3 : From Δ APM, $(AP)^2 = (AM)^2+(PM)^2$

$$(x-1)^2+(y-3)^2+(z-5)^2 = \frac{1}{9}(2x+2y-z-3)^2+9$$

$$9[x^2-2x+1+y^2-6y+9+z^2-10z+25]$$

$$= 4x^2+4y^2+z^2+9+8xy-4xz-12x-4yz-12y+6z+9$$

$$\therefore\ 5x^2+5y^2+8z^2-8xy+4\,xz+4yz-6x-42\,y-96\,z+225 = 0.$$

which is the required equation of RCC.

Q.14 : Find the equation of RCC cylinder whose axis is the line 2(x-1) = y + 2 = z and radius is 3.

[SPPU : Dec.-09]

Ans. : Given that equation of axis of R.C.C is

$$\frac{x-1}{1} = \frac{y-(-2)}{2} = \frac{z-0}{2}$$

$\therefore$ We have, i) One point on axis A (1, –2, 0)

ii) dr's of axis are 1, 2, 2.

iii) Radius of cylinder = 3

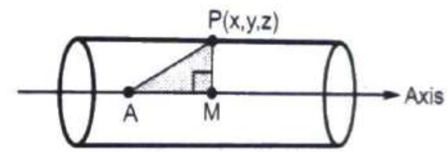


Fig. Q.14.1

Consider the following steps.

Step 1 : Let P (x, y, z) be any point on RCC.

A (1, –2, 0) be one point on axis and

M be the foot of perpendicular on axis from point P.

Step 2 : Distance $(AP)^2 = (x-1)^2+(y+2)^2+(z-0)^2$

PM = 3, direction ratios of axis are 1, 2, 2.

Direction cosines of axis are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$.

$$AM = \text{Projection of AP on axis}$$

$$= \frac{1}{3}(x-1) + \frac{2}{3}(y+2) + \frac{2}{3}(z-0) = \frac{1}{3}(x + 2y + 2z + 3)$$

Step 3 : From Δ AMP, $(AP)^2 = (AM)^2 + (MP)^2$

$$(x-1)^2 + (y+2)^2 + (z-0)^2 = \frac{1}{9}(x + 2y + 2z + 3)^2 + 9$$

$$9(x^2 - 2x + 1 + y^2 + 4y + 4 + z^2) = (x^2 + 4y^2 + 4z^2 + 9 + 4xy + 4xz + 6x + 8yz + 12y + 12z) + 9$$

$$\therefore\ 8x^2 + 5y^2 + 5z^2 - 4xy + 4xz + 8yz - 24x + 24y - 122 - 90 = 0$$

which is the required equation of RCC.

Q.15 : Find the equation of right circular cylinder whose guiding curve is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$. [SPPU : Dec.-16, Marks 4]

Ans. :

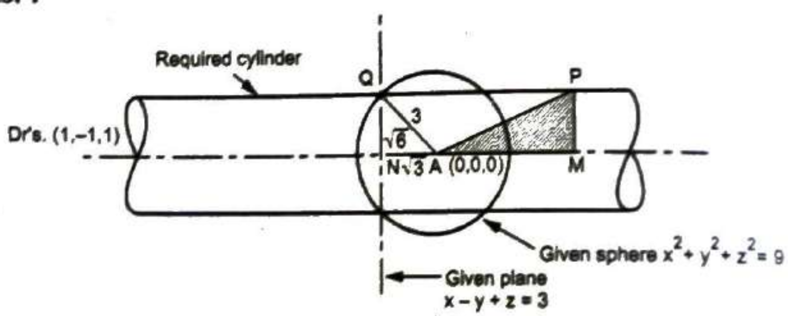


Fig. Q.15.1

Given sphere $x^2 + y^2 + z^2 = 9$

$\therefore$ radius of sphere = 3

Centre of sphere = (0, 0, 0)

$\therefore$ AQ = 3 and A is (0, 0, 0)

In Δ AQN

AN = Perpendicular distance from (0, 0, 0) to

$$x - y + z = 3$$

$$= \left|\frac{0 + 0 + 0 - 3}{\sqrt{1 + 1 + 1}}\right| = \sqrt{3}$$

From Δ AQN

$$AQ^2 = AN^2 + NQ^2$$

$$9 = 3 + NQ^2$$

$$\therefore\quad NQ = \sqrt{6}$$

Thus $\quad PM$ = Radius of cylinder = $\sqrt{6}$

From equation of plane $x - y + z = 3$ the coefficients of x, y, z i.e. 1, – 1, 1 gives the direction ratios of normal to the plane. The axis of the cylinder is normal to the plane

$\therefore$ 1, – 1, 1 are dr's of axis of cylinder

Now we know that

i) (0, 0, 0) one point on axis

ii) 1, – 1, 1 dr's of axis

iii) $\sqrt{6}$ radius of cylinder

Thus

Step 1 : Let p (x, y, z) be any point on generator A (0, 0, 0) is one point on axis and M be the foot of the perpendicular from P on axis.

Step 2 : $\therefore$

$$AP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$PM = \text{Radius of cylinder} = \sqrt{6}$$

$$AM = \text{Projection of AP on axis}$$

Now dc's of axis are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$$\therefore\ AM = \frac{1}{\sqrt{3}}(x-0) - \frac{1}{\sqrt{3}}(y-0) + \frac{1}{\sqrt{3}}(z-0) = \left(\frac{x - y + z}{\sqrt{3}}\right)$$

Step 3 : From Δ APM $\quad AP^2 = AM^2 + PM^2$

Substituting AP, AM, PM

$$x^2 + y^2 + z^2 = \frac{(x - y + z)^2}{3} + 6$$

$$3\left(x^2+y^2+z^2\right)-(x-y+z)^2 = 18$$

$$x^2+y^2+z^2+xy+yz-zx = 9$$

Q.16 : Find the equation of the right circular cylinder which passes through the section of the sphere $x^2+y^2+z^2=25$, $x+2y+2z=0$ **[SPPU : May-08, 11]**

Ans. :

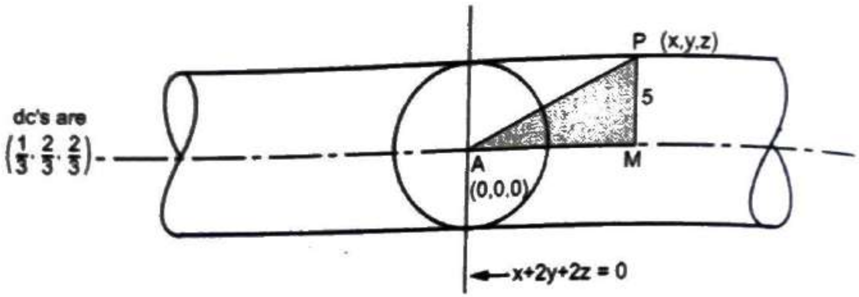


Fig. Q.16.1

Given sphere $x^2+y^2+z^2=25$

$\therefore$ radius of sphere = 5 and centre of sphere = (0, 0, 0)

Note that this centre satisfies $x+2y+2z=0$.

Thus radius of the cylinder = radius of sphere = 5

i.e. the required cylinder is the enveloping cylinder of the given sphere

Now we know that

i) (0, 0, 0) one point on axis

ii) (1, 2, 2) dr's of axis ($\because$ the axis is normal to the plane)

iii) radius of cylinder = 5

Thus

Step 1 : Let P (x, y, z) be any point on the cylinder A (0, 0, 0) is one point on axis.

Let M be the foot of the perpendicular from point P on the axis.

Step 2 : $AP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$

PM = 5 (radius of cylinder)

AM = Projection of AP on axis

Now $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$ are dc's of axis

$$\therefore \quad AM = \frac{1}{3}(x-0)+\frac{2}{3}(y-0)+\frac{2}{3}(z-0) = \frac{x+2y+2z}{3}$$

Step 3 : From Δ APM

$$AP^2 = AM^2+PM^2$$

$$x^2+y^2+z^2 = \frac{(x+2y+2z)^2}{9}+25$$

Step 4 : Simplify.

$$9\left(x^2+y^2+z^2\right)-(x+2y+2z)^2 = 225$$

$$8x^2+5y^2+5z^2-4xy-4zx-8yz-225 = 0$$

is the required equation of cylinder.

Memory Map

1. The cylinder

   Necessary data : Axis and guiding curve

2. Right circular cylinder

   Necessary data :

   i) One point on axis

   ii) Direction ratio's of axis

   iii) Radius of cylinder

3. Enveloping cylinder.

END...

UNIT - V

13 Multiple Integrals

13.1 : Double Integrals

I) In the application of calculus of integration, we know that integration $\int_a^b f(x)$ is defined as a limit of approximating sums as $n \to \infty$

i.e. $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\,\delta x_r$$

We can extend same logic for the definition of double integrals. Let the function $z = f(x, y)$ be continuous and defined over the region R bounded by some closed curve 'C'. Divide the region R into subregions $R_1, R_2, \ldots R_n$ of areas $\delta A_1, \delta A_2, \ldots, \delta A_n$ respectively. Let P (x_r, y_r) be any point inside r^{th} subregions of area δA_r. Adding these areas together, we get

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_n, y_n)\,\delta A_n$$
$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

Thus, the double integral of the function f(x, y) over the region R is

$$\iint_R f(x, y)\,dA = \lim_{\substack{n \to \infty \\ \delta A \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

II) If f(x, y) is continuous on R and $f(x, y) \geq 0$ then $\iint_R f(x, y)\,dA$ gives the volume of the solid bounded by the function over region R.

III) **Mean value theorem for double integrals** : If f(x, y) is continuous over closed region R, then there exist at least one point (a, b) in R such that

$$\iint_R f(x, y)\,dx\,dy = f(a, b) \times A \text{ where } A = \iint_R dA \text{ is the area of R.}$$

IV) **Properties of double integrals :**

a) $\iint_R [f(x, y) \pm g(x, y)]\,dxdy = \iint_R f(x, y)\,dxdy \pm \iint_R g(x, y)\,dxdy$

b) $\iint_R k\,f(x, y)dxdy = k \iint_R f(x, y)\,dxdy;\ k \neq 0$ constant.

c) If $f(x, y) \geq 0\ \forall\ (x, y) \in R$ then $\iint_R f(x, y)\,dxdy \geq 0$.

d) If $f(x, y) \geq g(x, y)\ ;\ \forall\ (x, y) \in R$ then

$$\iint_R f(x, y)\,dx\,dy \geq \iint_R g(x, y)\,dx\,dy$$

V) **Determination of limits of double integrals** : we have

$$\iint_R f(x, y)\,dx\,dy = \int \left[\int f(x, y)\,dx\right] dy = \int \left[\int f(x, y)\,dy\right] dx$$

Integral inside of bracket is always evaluated first, which is known as **inner integral.** The integral outside of bracket is evaluated last, which is known as **outer integral.** These integrals are known as repeated integrals.

Consider following cases of region.

Case i) When R is a rectangle :

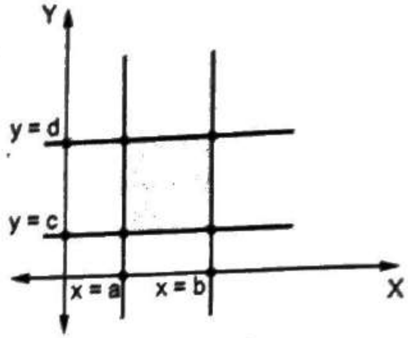


Fig. 13.1

$$\therefore \qquad R = \{(x, y) \,/\, a \le x \le b,\ c \le y \le d\}$$

And f(x, y) is continuous on R, then

$$\iint_R f(x, y)\, dxdy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$$

$$= \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

i.e. If limits of integration are constants, then the order of integration is immaterial.

Case ii) When R is non rectangular region

a) If inner limits are functions of x :

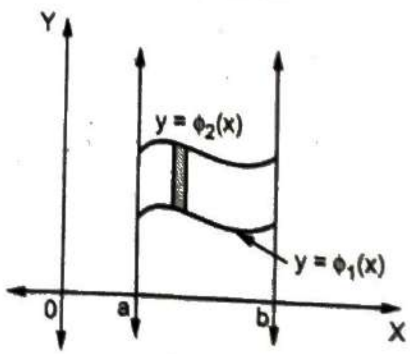


Fig. 13.2

In this case, draw strip parallel to y-axis within region R. The lower end of the strip touches the curve $y = \phi_1(x)$ and upper end touches the curve $y = \phi_2(x)$.

Thus $\phi_1(x) \le y \le \phi_2(x)$ and $a \le x \le b$,

which are limits of y and x respectively. Here limits of x are constants $\therefore$ These limits must be limits of outer integral.

Thus,

$$\iint_R (x, y)\, dx\, dy = \int_{x=a}^{x=b} \left[\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y)\, dy\right] dx$$

b) If inner limits are functions of y :

In this case, draw strip to x-axis within region R. The lower end of the strip touches the curve $x = \phi_1(y)$ and upper end touches the curve $x = \phi_2(y)$. Thus $\phi_1(y) \le x \le \phi_2(y)$ and $c \le y \le d$ which are limits of x and y respectively. As limits of y are constant, these limits must be limits of outer integral.

$$\text{Thus,} \quad \iint_R f(x, y)\, dx\, dy = \int_{y=c}^{d} \left[\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x, y)\, dx\right] dy$$

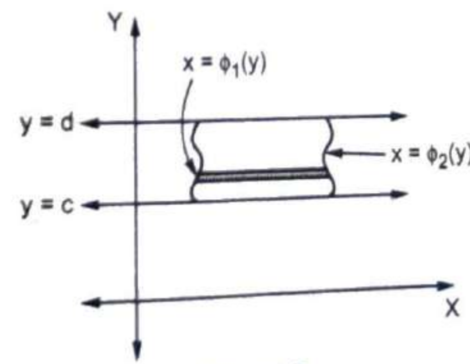


Fig. 13.3

Q.1 : Evaluate $\int\limits_0^1 \int\limits_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ **[SPPU : May-19, Marks 6]**

Ans. : Let

$$I = \int\limits_{x=0}^{1} \int\limits_{y=0}^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$$

As limits of inner integral are functions of x, therefore integrate first w.r.t. y and then x.

$$= \int\limits_0^1 \left[\int\limits_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1+x^2+y^2)} \right] dx$$

$$= \int\limits_0^1 \left[\int\limits_{y=0}^{a} \frac{dy}{a^2+y^2} \right] dx \qquad \text{Let } a = \sqrt{1+x^2} = \text{constant}$$

$$= \int\limits_0^1 \left[\frac{1}{a} \tan^{-1} \frac{y}{a} \right]_0^a dx = \int\limits_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) dx$$

$$= \frac{\pi}{4} \int\limits_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} [\log (x + \sqrt{1+x^2})]_0^1$$

$$\boxed{I = \frac{\pi}{4} \log (1+\sqrt{2})} \qquad \text{... Ans.}$$

Q.2 : Evaluate $\int\limits_0^1 \int\limits_0^x (x^2+y^2)\,dy\,dx$ **[SPPU : Jan.-09, Marks 5]**

Ans. : As limits of inner integral are functions of x. Therefore integrate first w.r.t. y and then x.

Let

$$I = \int\limits_0^1 \left[\int\limits_{y=0}^{y=x} (x^2+y^2)\,dy \right] dx$$

$$= \int\limits_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=x} dx$$

$$= \int\limits_0^1 \left[x^3 + \frac{x^3}{3} \right] dx = \int\limits_0^1 \frac{4x^3}{3}\,dx$$

$$I = \left[\frac{4x^4}{3x^4} \right]_0^1 = \frac{1}{3} [x^4]_0^1 = \frac{1}{3}$$

Q.3 : Evaluate $\int\limits_0^1 dx \int\limits_1^\infty e^{-y} y^x \log y\,dy.$

[SPPU : Dec.-15, Marks 6]

Ans. : Let

$$I = \int\limits_0^1 dx \int\limits_1^\infty e^{-y}\, y^x \log y\,dy \qquad \text{...(1)}$$

As limits of integrals are constants, so order of integration is immaterial. We know that $\frac{d}{dx}(y^x) = y^x \log y.$

$$\therefore \quad I = \int\limits_{y=1}^{\infty} e^{-y}\,dy \int\limits_{x=0}^{1} y^x \log y\,dx$$

$$= \int\limits_{y=1}^{\infty} e^{-y}\,dy \left[y^x \right]_{x=0}^{1} = \int\limits_{y=1}^{\infty} e^{-y}\,[y-1]\,dy$$

$$= \left[(y-1)\left(\frac{e^{-y}}{-1} \right) - (1)(e^{-y}) \right]_1^\infty$$

$$= (0-0) - [0 - e^{-1}] = \frac{1}{e}$$

$$\therefore \quad \boxed{I = \frac{1}{e}}$$

Q.4 : Evaluate $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \frac{dx\,dy}{(1+e^y)\sqrt{1-x^2-y^2}}$ **[SPPU : Dec.-06, Marks 5]**

Ans. : Given limits :

$$y = 0$$

$$y = \sqrt{1-x^2} \Rightarrow x^2+y^2=1$$

$$x = 0, \; x = 1$$

The region of integration is as shown below :

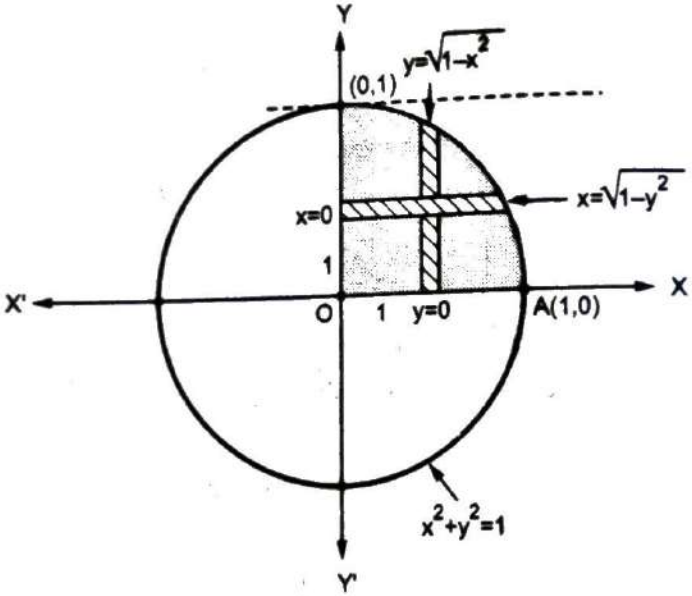


Fig. Q.4.1

Changing the order of integration we get,

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{dx\,dy}{(1+e^y)\sqrt{(1-y^2)-x^2}}$$

$$= \int_0^1 \frac{1}{1+e^y} \left(\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right)_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \left(\frac{\pi}{2}-0\right) \frac{1}{1+e^y}\, dy$$

$$= \frac{\pi}{2} \int_0^1 \frac{e^{-y}}{e^{-y}+1}\, dy$$

$$= -\frac{\pi}{2} \left[\log(1+e^{-y})\right]_0^1$$

$$= -\frac{\pi}{2} \left[\log(e^{-1}+1) - \log 2\right]$$

$$= \frac{\pi}{2} \log\left(\frac{2e}{1+e}\right)$$

13.2 : Evaluation of Double Integrals if Limits are not given

If the limits of integral are not given then we find limits by sketching the region of integral for the given curves.

Suppose the region of integration is bounded by the lines $x = 0$, $y = 0$ and $x + y = 2$.

Line $x + y = 2$ passes through $(0, 2)$ and $(2, 0)$

The region of integration OABO.

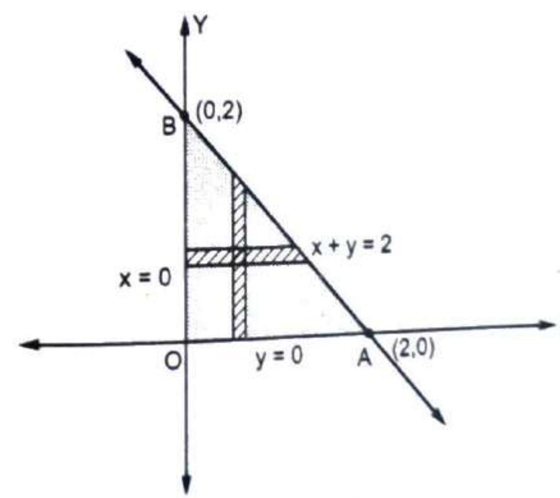


Fig. 13.4

Depending upon the nature of the integral, we require particular order of integral.

a) First w.r.t. x and y : Draw strip parallel to x-axis in the region OABO. Lower end (Left side) of the strip lies on $x = 0$ and upper end (Right side) of the strip lies on $x + y = 2$ i.e. $x = 2 - y$.

$\therefore$ The limits of x are $x = 0$ to $x = 2 - y$

To cover complete region, we need to move strip vertically from $y = 0$ to $y = 2$.

$\therefore$ The limits of y are $y = 0$ to $y = 2$.

$\therefore$ The limits of inner integral are $x = 0$ to $x = 2 - y$ and outer integral are $y = 0$ to $y = 2$.

$$\therefore \qquad I = \int_{y=0}^{y=2} \left[\int_{x=0}^{x=2-y} f(x, y)\, dx \right] dy$$

b) First w.r.t. y and then y : Draw strip parallel to y axis. Lower end of the strip lies on $y = 0$ and upper end lies on $y = 2 - x$. The limits of y are $y = 0$ to $y = 2 - x$. And the limits of x are $x = 0$ to $x = 2$ (i.e. the smallest and greatest values of x in the region)

$$\therefore \qquad I = \int_{x=0}^{x=2} \left[\int_{y=0}^{y=2-x} f(x, y)\, dy \right] dx$$

Note : Depending upon the nature of integral we need to decide to order of the intgeral and accordingly draw strip.

Q.5 : Evaluate $\iint_R \frac{xy\, dx\, dy}{\sqrt{1 - y^2}}$ over the positive quadrant of the circle $x^2 + y^2 = 1$. **[SPPU : Dec.-01, Marks 5]**

Ans. : Let

$$I = \iint_R \frac{xy\, dx\, dy}{\sqrt{1 - y^2}} \qquad \text{... (1)}$$

where R is the region bounded in the positive quadrant of the circle $x^2 + y^2 = 1$ as shown in Fig. Q.5.1.

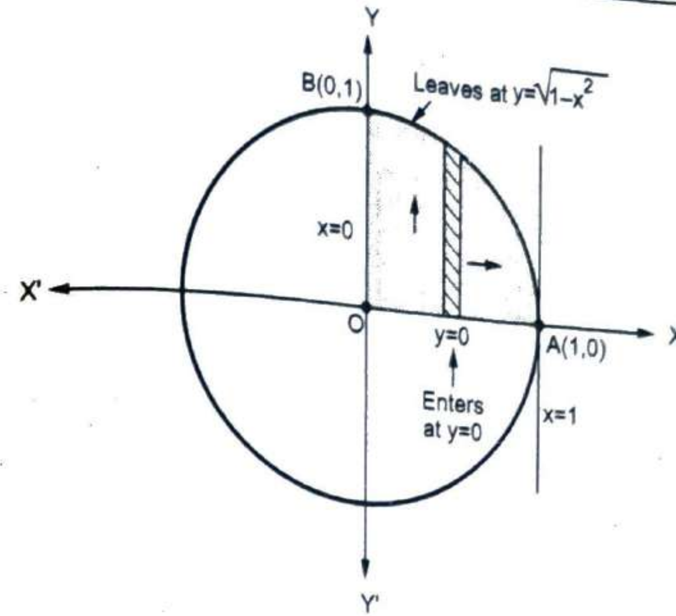


Fig. Q.5.1

To find limits for x and y, consider an elementary strip of width dx parallel to Y-axis over which y varies from $y = 0$ to $y = \sqrt{1 - x^2}$

Now, moving the strip horizontally from $x = 0$ to $x = 1$, we get the complete region of integration.

$\therefore$ From integral (1), we have

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \frac{xy\, dx\, dy}{\sqrt{1 - y^2}}$$

The order of integral is first w.r.t y and then x.

$$= \int_0^1 x \left[\int_{y=0}^{\sqrt{1-x^2}} \frac{y\, dy}{\sqrt{1 - y^2}} \right] dx$$

$$I = \int_0^1 x [I_1]\, dx \qquad \text{... (2)}$$

Where $$I_1 = \int_{y=0}^{\sqrt{1-x^2}} \frac{y\,dy}{\sqrt{1-y^2}}$$

Put $1 - y^2 = u \therefore -2y\,dy = du$ or $y\,dy = \frac{-du}{2}$

Limits :

y	u
0	1
$\sqrt{1-x^2}$	x^2

$$\therefore \quad I_1 = \int_1^{x^2} \frac{\left(-\frac{du}{2}\right)}{\sqrt{u}} = -\frac{1}{2}\int_1^{x^2} u^{-1/2}\,du$$

$$= -\frac{1}{2}\left[\frac{u^{-1/2+1}}{-\frac{1}{2}+1}\right]_1^{x^2}$$

$$I_1 = -[\sqrt{u}]_1^{x^2} = -[\sqrt{x^2} - 1]$$

$$\boxed{I_1 = 1 - x} \quad \text{.. (3)}$$

Now, from equation (2), we get

$$I = \int_0^1 x(1-x)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

$$\boxed{I = \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}} \quad \text{.... Ans.}$$

Q.6 : Evaluate $\iint_R y\,dx\,dy$ over the region bounded by $x = 0$, $y = x^2$ and $x + y = 2$ in the first quadrant. [SPPU : May-08, Marks 5]

Ans. : Let $$I = \iint_R y\,dx\,dy \quad \text{... (1)}$$

where R is region bounded by $y = x^2$ (PARABOLA), $x = 0$, $x + y = 2$ as shown in the Fig. 13.7.

To find limits for x and y, consider a vertical strip (parallel to Y-axis) in the region of integration over which y varies from $y = x^2$ to $y = 2 - x$.

Now, moving (sliding) the strip from $x = 0$ to $x = 1$ (horizontally), we get complete region of integration.

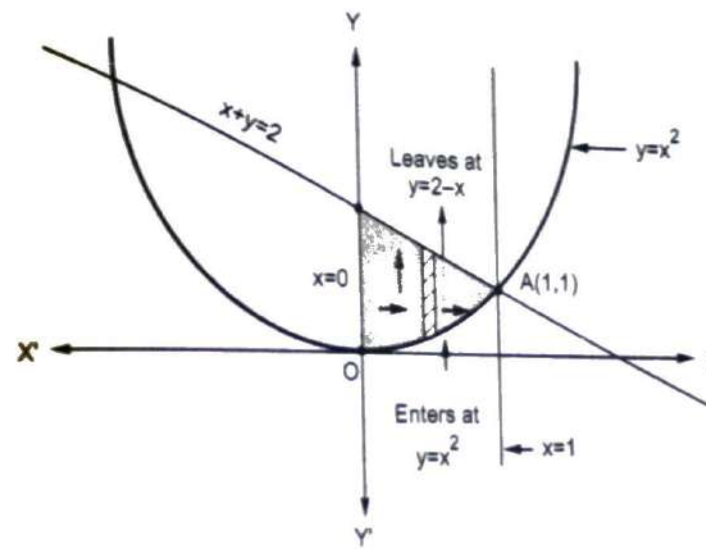


Fig. Q.6.1

$\therefore$ From equation (1),

$$I = \int_{x=0}^{1} \int_{y=x^2}^{y=2-x} y\,dx\,dy$$

(w.r.t. 'y' first keeping 'x' as constant)

$$= \int_x^1 \left(\int_{x^2}^{2-x} y\,dy\right) dx$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 [(2-x)^2 - x^4]\, dx$$

$$= \frac{1}{2}\int_0^1 \left(4 - 4x + x^2 - x^4\right) dx$$

$$= \frac{1}{2}\left(4x - 4\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5}\right)_0^1$$

$$= \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right] - \frac{1}{2}[0]$$

$$\boxed{I = \frac{16}{15}}$$ .... Ans.

Q.7 : Evaluate $I = \iint \frac{1}{x^4 + y^2}\, dx\, dy$ over the region $y \geq x^2$ and $x \geq 1$.

[SPPU : May-16, Marks 6]

Ans. : We have $I = \iint \frac{1}{x^4 + y^2}\, dx\, dy$ ... (1)

The integrand involves x^4 at demoninator, so it is difficult to integrate w.r.t. x first. Therefore the order of the intgration is first w.r.t. y and then x.

we have $y \geq x^2$ $\therefore y = x^2$ represents a parabola with symmetric about y-axis.

Now, $x = 1 \Rightarrow y = 1$ $\therefore$ The given two curves intersect at (1, 1).

The shaded unbounded region is the region of integration.

Draw strip parallel to Y-axis.

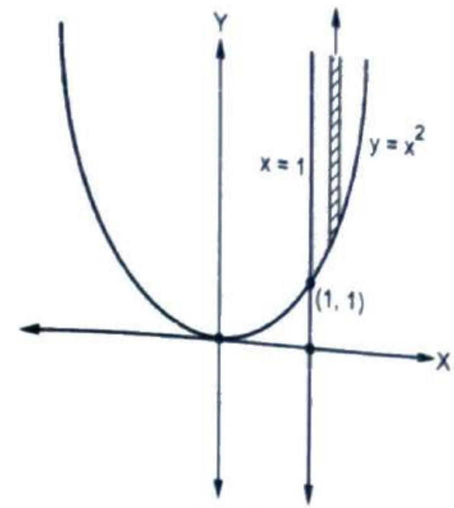


Fig. Q.7.1

$\therefore$ Limits are

$$y = x^2 \text{ to } y = \infty \text{ [unbounded]}$$
$$x = 1 \text{ to } x = \infty \text{ [unbounded]}$$

$$\therefore \quad I = \int_{x=1}^{x=\infty}\left(\int_{y=x^2}^{y=\infty} \frac{1}{x^4 + y^2}\, dy\right) dx \qquad \left\{\because \int \frac{1}{a^2 + y^2}\, dy = \frac{1}{a}\tan^{-1}\frac{1}{a}\right\}$$

$$I = \int_{x=1}^{\infty} \frac{1}{x^2}\left[\tan^{-1}\frac{y}{x^2}\right]_{y=x^2}^{\infty} dx = \int_{x=1}^{\infty} \frac{1}{x^2}[\tan^{-1}\infty - \tan^{-1} 1]\, dx$$

$$= \int_{x=1}^{\infty} x^{-2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] dx\ \frac{\pi}{4}\left[\frac{x^{-2+1}}{-2+1}\right]_1^{\infty} = \frac{\pi}{4}\left[-\frac{1}{x}\right]_1^{\infty}$$

$$= \frac{\pi}{4}[0 - 1] = \frac{\pi}{4}$$

$$\Rightarrow \quad \boxed{I = \frac{\pi}{4}}$$

Q.8 : Evaluate $\iint\limits_R \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy$, where R is the region bounded by $y^2 = 4ax$ and $y = x$. **[SPPU : May-09, Marks 5]**

Ans. : Here the limits are

$x = \frac{y^2}{4a}$, $x = y$, $y = 0$, $y = 4a$

or $\quad y^2 = 4ax,\ x = y,\ y = 0,$

$\quad y = 4a$

The region of integration is as shown in the Fig. Q.8.1

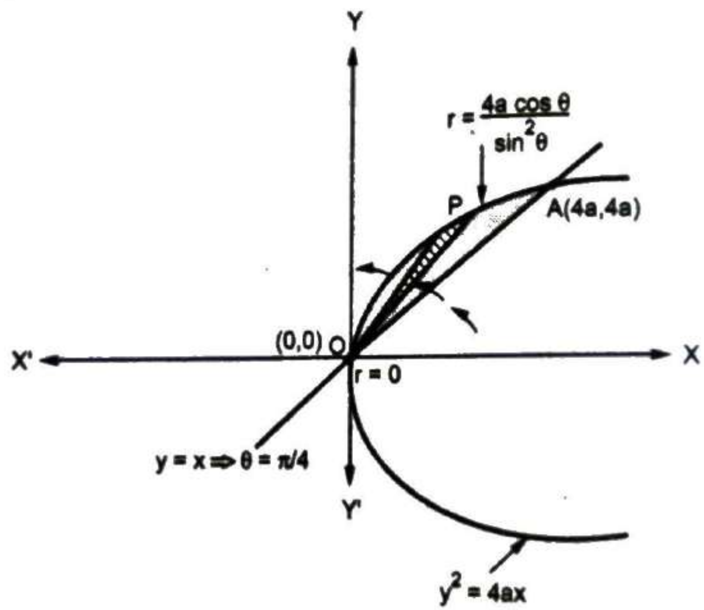


Fig. Q.8.1

$y^2 = 4ax,\ \ y = 4a$

At A, $(4a)^2 = 4ax$

$\Rightarrow\ 16\,a^2 = 4ax \Rightarrow 4a\,(4a - x) = 0$

$\Rightarrow\ x = 4a$

$\Rightarrow\ y = 4a$

It is convenient to transform the double integral into polar form.

$\therefore$ Put $x = r\cos\theta,\ y = r\sin\theta,$

$\therefore\ dxdy = r\, d\theta\, dr$

and $x^2 + y^2 = r^2$

$x^2 - y^2 = r^2\,(\cos^2\theta - \sin^2\theta)$

'r' varies from $r = 0$ to $r = \frac{4\cos\theta}{\sin^2\theta}$ and θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Given integral will have the form

$$I = \int\limits_{\theta=\pi/4}^{\pi/2} \int\limits_{r=0}^{4\frac{\cos\theta}{\sin^2\theta}} \frac{r^2\,(\cos^2\theta - \sin^2\theta)}{r^2}\, r\, d\theta\, dr$$

$$= \int\limits_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int\limits_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\frac{16\,a^2\cos^2\theta}{\sin^4\theta}\, d\theta$$

$$= 8a^2\int\limits_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\, d\theta$$

$$= 8a^2\int\limits_{\pi/4}^{\pi/2} (\cot^2\theta\,\mathrm{cosec}^2\theta - 2\,\mathrm{cosec}^2\theta + 2)\, d\theta$$

$$= 8a^2\left\{\frac{-\cot^3\theta}{3} + 2\cot\theta + 2\theta\right\}_{\pi/4}^{\pi/2}$$

$$= 8a^2\left[0 + 0 + \pi + \frac{1}{3} - 2 - \frac{\pi}{2}\right]$$

$$\boxed{I = 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]}$$

13.3 : Evaluation of Double Integrals by Changing the Order of Integration

It is observed that some problems are very difficult to solve one way changing the order makes them simpler.

If the integrand f(x, y) in the double integral $\int\left[\int f(x,y)\,dy\right]dx$ is difficult or even impossible to integrate w.r.t. y first. But can be integrated easily w.r.t. x first. In such cases it becomes necessary to change the order of double integrals i.e. $\int\left[\int f(x,y)\,dx\right]dy$ and vice versa.

Q.9 : Evaluate by changing the order of integration $\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$.

[SPPU : Dec.-16, Marks 6]

Ans. : Let

$$I = \int_0^\infty \left[\int_0^x x e^{-x^2/y}\,dy\right]dx \qquad \ldots(1)$$

It is difficult to integrate w.r.t. y first. Therefore change the order of integral.

Step 1 : Region of Integral :

In the given integral limits are y = 0 and y = x. x = 0 to x = ∞

The region of integral is as shown in the figure.

The old strip was parallel to y-axis.

Step 2 : New limits : Draw new strip parallel to x-axis.

∴ Limits are x = y to x = ∞, y = 0 to y = ∞.

Step 3 : Equation (1) becomes

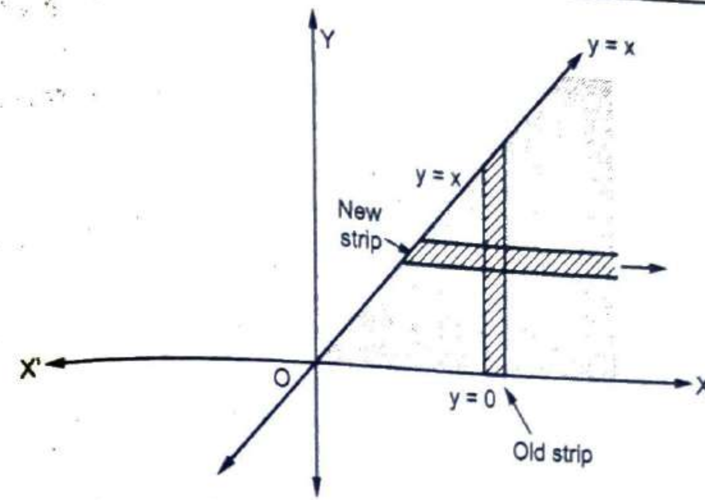


Fig. Q.9.1

$$I = \int_{y=0}^{\infty}\int_{x=y}^{\infty} x e^{-x^2/y}\,dx\,dy$$

$$= \int_{y=0}^{\infty}\left(-\frac{y}{1}\right)\left[\int_{x=y}^{\infty} e^{-x^2/y}\left(\frac{-2x}{y}\,dx\right)\right]dy$$

$$\left\{\text{use} \int e^f f'\,dx = e^f\right\}$$

$$= \int_{y=0}^{\infty}\left[e^{-x^2/y}\right]_{x=y}^{\infty}\left(-\frac{y}{2}\right)dy$$

$$= \int_{y=0}^{\infty}\left(-\frac{y}{2}\right)[e^{-\infty} - e^{-y}]\,dy = \frac{1}{2}\int_0^\infty e^{-y}\,y\,dx$$

$$= \frac{1}{2}\left[y(-e^{-y}) - (1)(e^{-y})\right]_0^\infty$$

$$I = \frac{1}{2}[0-0-0+1] = \frac{1}{2}$$

$$\therefore \quad \boxed{I = \frac{1}{2}}$$

Q.10 : Evaluate by changing the order of integration $\int_0^{\pi/2}\int_0^{y} \cos 2y\sqrt{1-a^2\sin^2 x}\,dx\,dy$. **[SPPU : May-09, Dec.-13, Marks 6]**

Ans. : Let

$$I = \int_0^{\pi/2}\int_0^{y} \cos 2y\sqrt{1-a^2\sin^2 x}\,dx\,dy \qquad \ldots(1)$$

It is difficult to integrate w.r.t. x first. Therefore change the order of integral.

Step 1 : Region of Integral : The limits of given integral are $x = 0$ to $x = y$ and $y = 0$ to $y = \pi/2$.

The region of integral is OABO. The old strip was parallel to x-axis.

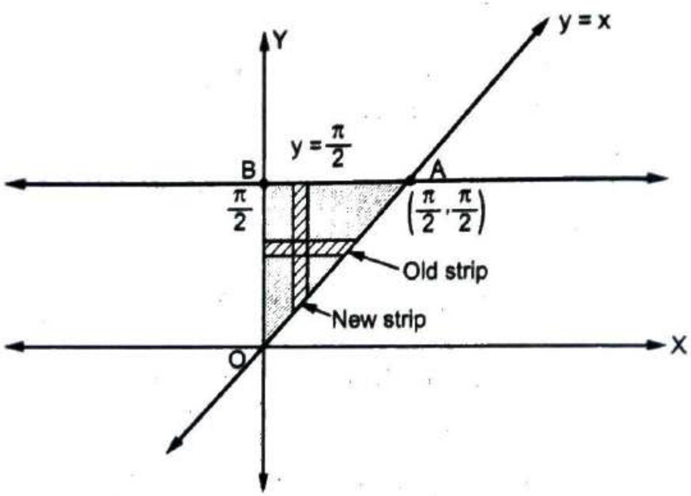


Fig. Q.10.1

Step 2 : New limits : Draw new strip parallel to Y-axis.

$\therefore$ Limits are $y = x$ to $y\ \pi/2$ and $x = 0$ to $x = \frac{\pi}{2}$

Step 3 : Therefore equation (1) becomes

$$I = \int_{x=0}^{\pi/2}\left(\int_{y=x}^{\pi/2}\cos 2y\sqrt{1-a^2\sin^2 x}\,dy\right)dx$$

$$= \int_{x=0}^{\pi/2}\sqrt{1-a^2\sin^2 x}\,dx\left[\frac{\sin 2y}{2}\right]_x^{\pi/2}$$

$$= \int_{x=0}^{\pi/2}\sqrt{1-a^2\sin^2 x}\left[0-\frac{\sin 2x}{2}\right]dx$$

$$= -\frac{1}{2}\int_0^{\pi/2}\sin 2x\sqrt{1-a^2\sin^2 x}\,dx$$

$$= -\frac{1}{2}\int_0^{\pi/2}\sqrt{1-a^2\sin^2 x}\,(2\sin x\cos x)\,dx$$

Put $\sin^2 x = t \quad \Rightarrow \quad 2\sin x\cos x\,dx = dt$

x	0	$\pi/2$
t	0	1

$\therefore$
$$I = -\frac{1}{2}\int_0^1 (1-a^2 t)^{1/2}\,dt = -\frac{1}{2}\left[\frac{(1-a^2 t)^{\frac{3}{2}}}{\frac{3}{2}(-a^2)}\right]_0^1$$

$$I = \frac{1}{3}a^2\left[(1-a^2)^{3/2}-1\right]$$

Q.11 : Show that $\int_0^a\int_{y^2/a}^{y}\frac{y\,dx\,dy}{(a-x)\sqrt{(ax-y^2)}} = \frac{\pi a}{2}$. **[SPPU : Dec.-05, May-13, Marks 6]**

Ans. : It is difficult to integrate w.r.t. x first. Therefore change the order of integral.

Step 1 : Region of Integral : The limits of the given integral are $x = \frac{y^2}{a}$ to $x = y$ and $y = 0$ to $y = c$.

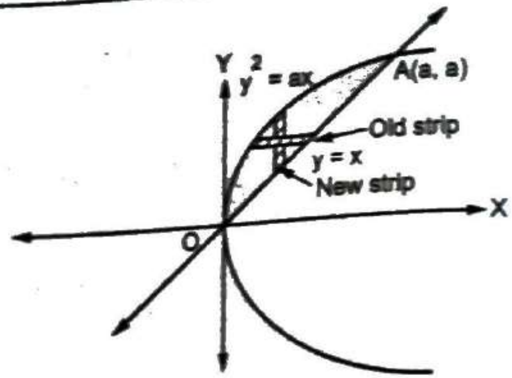


Fig. Q.11.1

The region of integral is as shown in the figure.

The old strip was parallel to X-axis.

Step 2 : New limits : Draw new strip parallel to Y-axis.

$\therefore$ Limits are $y = x$ to $y = \sqrt{ax}$ and $x = 0$ to $x = a$.

Step 3 :

$$I = \int_{x=0}^{a} \frac{1}{a-x} dx \int_{y=x}^{\sqrt{a}\sqrt{x}} \frac{y\, dy}{\sqrt{ax-y^2}}$$

$$\left\{ \text{Use} \int \frac{f'}{\sqrt{f}} dy = 2\sqrt{f} \text{ Here } f = ax - y^2 \right\}$$

$$I = \int_{x=0}^{a} \frac{1}{a-x} dx \left(-\frac{1}{2}\right) \int_{y=x}^{\sqrt{a}\sqrt{x}} \frac{-2y\, dy}{\sqrt{ax-y^2}}$$

$$= \int_{x=0}^{a} \frac{1}{a-x} dx \left(-\frac{1}{2}\right) \left[2\sqrt{ax-y^2}\right]_{y=x}^{y=\sqrt{a}\sqrt{x}}$$

$$= \int_{x=0}^{a} \frac{(-1)}{a-x} \left[\sqrt{ax-ax} - \sqrt{ax-x^2}\right] dx$$

$$I = \int_{x=0}^{a} \frac{(-1)}{a-x} - \left[-\sqrt{x}\sqrt{a-x}\right] dx$$

$$= \int_{0}^{a} \sqrt{x}(a-x)^{-\frac{1}{2}} dx$$

Put $x = at,\ dx = adt$

x	0	a
t	0	1

$$I = \int_{0}^{1} \sqrt{at}(a-at)^{-\frac{1}{2}} adt$$

$$= a\int_{0}^{1} t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt = a\ \beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= a \frac{\Gamma\frac{3}{2} + \Gamma\frac{1}{2}}{\Gamma\frac{3}{2} + \frac{1}{2}} = a \frac{\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\sqrt{2}}$$

$$= \frac{a}{2} \frac{\sqrt{\pi}\sqrt{\pi}}{1!}$$

$$I = \frac{a\pi}{2}$$

Q.12 : Evaluate $\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} \frac{\cos^{-1} x\, dx\, dy}{\sqrt{(1-x^2-y^2)(1-x^2)}}$

[SPPU : Dec.-17, Marks 6]

Ans. : Let

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{\cos^{-1} x\, dx\, dy}{\sqrt{1-x^2-y^2}\sqrt{1-x^2}} \quad \text{... (1)}$$

The integrand in equation (1) is complicated to integrate w.r.t. x first, but easy to integrate w.r.t y, therefore, it is required to change the order of integration.

Step 1 : Region of integral

The limits are $x = 0, x = \sqrt{1-y^2}$ or $x^2 = 1 - y^2$

$\Rightarrow x^2 + y^2 = 1$ (circle) centre = (0, 0) Radius = 1 and y = 0, y = 1.

Region of integration is as shown in the Fig. Q.12.1.

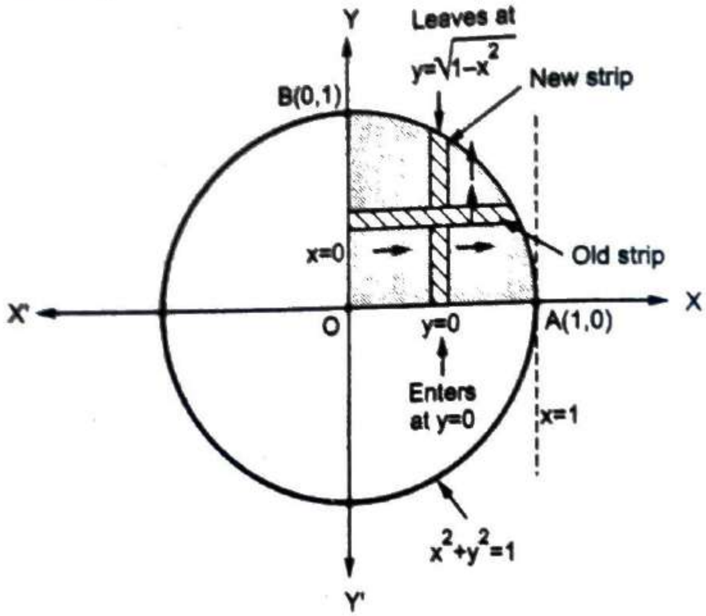


Fig. Q.12.1

Step 2 : New Limits : Draw strip parallel to Y axis.

Vertical strip in the region of integration over which y varies from $y = 0$ to $y = \sqrt{1-x^2}$ and moving the strip from $x = 0$ to $x = 1$ we get complete shaded quadrant of the circle (Region of integration).

Step 3 : Therefore

$$I = \int_{x=0}^{1} \frac{\cos^{-1}x}{\sqrt{1-x^2}} \left[\int_{y=0}^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}} \right] dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}} (I_1)\, dx \quad \ldots (2)$$

where
$$I_1 = \int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}} = \int_0^m \frac{dy}{\sqrt{m^2-y^2}}$$

(Assume $m = \sqrt{1-x^2}$)

$$I_1 = \left(\sin^{-1}\frac{y}{m}\right)_0^m = \sin^{-1}(1) - \sin^{-1}0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \ldots (3)$$

From equation (2)

$$I_1 = \frac{\pi}{2}\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\, dx$$

Put $\cos^{-1}x = u$

$\therefore$
$$\frac{-dx}{\sqrt{1-x^2}} = dy$$

Limits :

x	0	1
u	$\frac{\pi}{2}$	0

$$= \frac{\pi}{2}\int_{\pi/2}^{0} -u\, du$$

$$= \frac{\pi}{2}\int_0^{\pi/2} u\, du$$

$$= \frac{\pi}{2}\left(\frac{u^2}{2}\right)_0^{\pi/2}$$

$$\boxed{I = \frac{\pi^3}{16}} \quad \ldots \text{Ans.}$$

Q.13 : Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dx\, dy$ **[SPPU : Dec.-14, Marks 6]**

Ans. : Let $$I = \int_0^{\infty}\left[\int_{y=x}^{\infty}\frac{e^{-y}}{y}\,dy\right]dx \quad \dots (1)$$

The integral is difficult to integrate w.r.t. y first but easy to integrate w.r.t. x. Therefore, we change the order of integration.

Step 1 : Region of integral

Limits are $\left.\begin{matrix} y = x, & y = \infty \\ x = 0, & x = \infty \end{matrix}\right\}$

for which the region is as shown in Fig. Q.13.1.

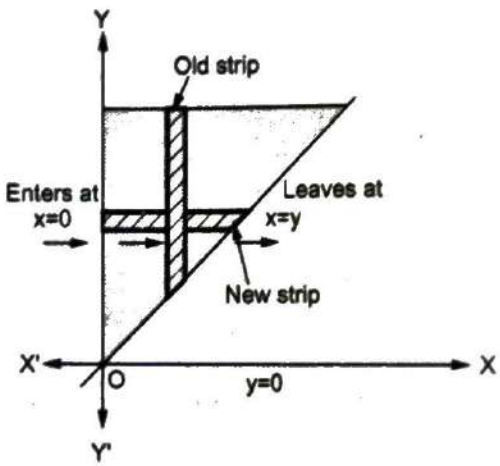


Fig. Q.13.1

Step 2 : New Limits

To find new limits, draw a horizontal strip (parallel to X-axis) over which x varies from **x = 0 to x = y** and moving the strip vertically from **y = 0 to y = ∞** we get shaded region.

Step 3 :

∴ From equation (1),

$$I = \int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^{\infty}\frac{e^{-y}}{y}\left[\int_{x=0}^{y}dx\right]dy$$

$$= \int_0^{\infty}\frac{e^{-y}}{y}(y)\,dy = \left(\frac{e^{-y}}{-1}\right)_0^{\infty} = 1$$

$$\boxed{I = 1}$$

... Ans.

Q.14 : Change the order of integral $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx\,dy$

[SPPU : Dec.-14, Marks 6]

Ans. : Let $$I = \int_{y=0}^{a}\left[\int_{x=\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx\right]dy \quad \dots(1)$$

Step 1 : Region of integral :

The Limits of the given integral are

$x = \sqrt{a^2-y^2} \Rightarrow x^2 + y^2 = a^2$ to $x = y + a$

and y = 0 to y = a

The region at integral is ABCA. The old strip is parallel to X-axis.

Step 2 : New Limits : To change the order of integral draw strip parallel to Y-axis, but the lower end of this strip lies on two curves. So we need two strips as shown in Fig. Q.14.1. (shown on next pg.)

Let R_1 and R_2 be the regions ADCA and ABDA respectively.

In R_1, limits are $y = \sqrt{a^2-x^2}$ to y = a

x = 0 to x = a

In R_2, limits are y = x − a to y = a and x = a to x = 2a

Step 3 : Therefore, the equation (1) becomes.

$$I = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

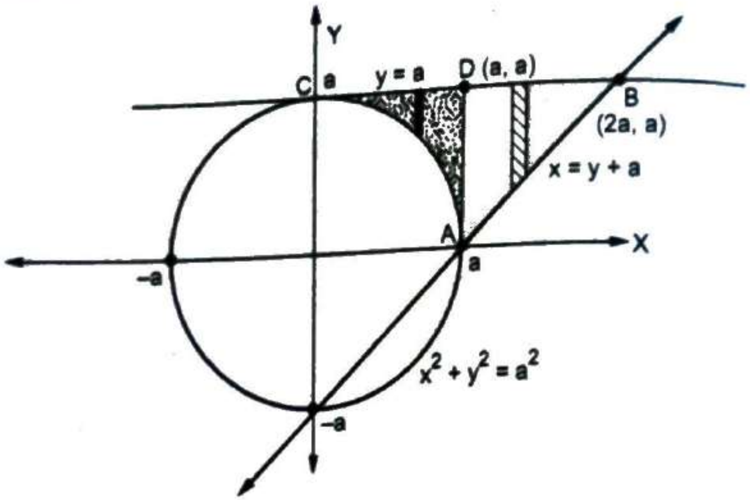


Fig. Q.14.1

$$I = \int_{x=a}^{b}\int_{y=\sqrt{a^2-x^2}}^{a} f(x, y)\,dy\,dx + \int_{x=a}^{2a}\int_{y=x-a}^{a} f(x, y)\,dy\,dx$$

Q.15 : Express as single integral and hence evaluate

$$\underbrace{\int_0^1\int_0^y \left(x^2+y^2\right)dx\,dy}_{I_1} + \underbrace{\int_1^2\int_0^{2-y}\left(x^2+y^2\right)dx\,dy}_{I_2}$$

[SPPU : Dec.-99, 08, 10, May-06]

Ans. : Step 1 : Region of integral :

The limits for I_1 area I_2 are

x = 0,	x = y
y = 0,	y = 1

x = 0	x = 2 – y
	∴ x + y = 2
y = 1	y = 2

The region of integration for these limits is as shown in the Fig. Q.15.1. The old strips were parallel to x-axis.

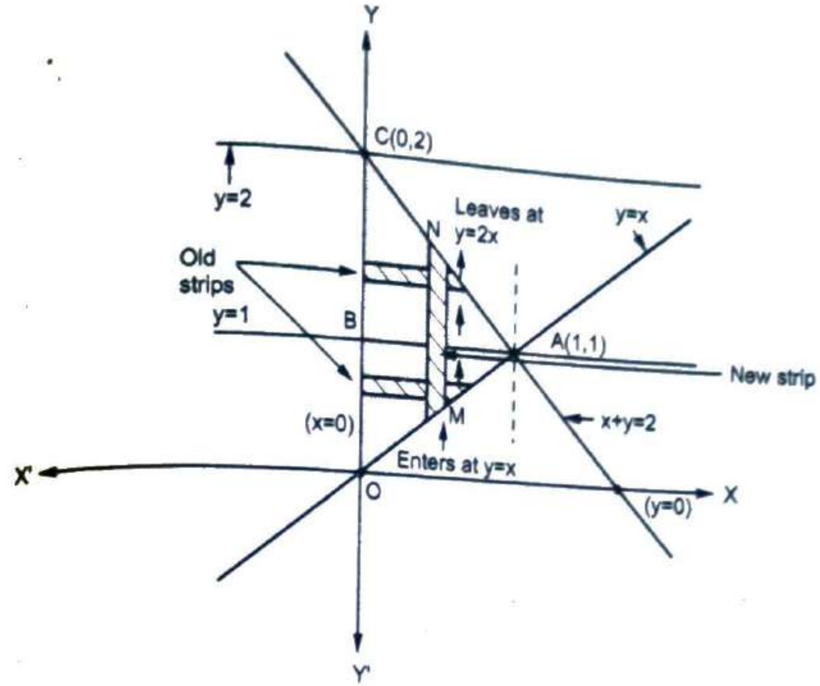


Fig. Q.15.1

Step 2 : New Limits :

To change the order of integration, consider a vertical strip over which y varies from **y = x to y = 2 – x,**

Now, moving the strip from **x = 0 to x = 1** we get entire region covered.

Step 3 :

∴ From equation (1)

$$I = \int_{x=0}^{1}\int_{y=x}^{2-x}\left(x^2+y^2\right)dx\,dy \text{(w.r.t. y first keeping x as constant)}$$

$$= \int_{x=0}^{1}\left[\int_{x}^{2-x}\left(x^2+y^2\right)dy\right]dx$$

$$= \int_{0}^{1}\left(x^2y+\frac{y^3}{3}\right)_{x}^{2-x}dx$$

$$= \int_{0}^{1}\left[\left\{x^2(2-x)+\frac{(2-x)^3}{3}\right\}-\left\{x^3+\frac{x^3}{3}\right\}\right]dx$$

$$= \int_{0}^{1}\left(-\frac{8}{3}x^3+4x^2-4x+\frac{8}{3}\right)dx$$

$$= \left[-\frac{8}{3}\frac{x^4}{4}+4\frac{x^3}{3}-4\frac{x^2}{2}+\frac{8}{3}x\right]_0^1$$

$$= -\frac{2}{3}+\frac{4}{3}-2+\frac{8}{3}$$

$$\boxed{I=\frac{4}{3}}$$ ... Ans.

Q.16 : Change the order of integration $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx\,dy$.

[SPPU : Dec.-99, 08, 10, May-06, Marks 5]

Ans. : Let

$$I = \int_{y=0}^{a}\int_{x=\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx\,dy \qquad \text{... (1)}$$

Step 1 : Region of integral

The limits are $x=\sqrt{a^2-y^2},\ x=y+a$

or $x^2+y^2=a^2$, $x-y=a$

and $y=0,\ y=a$

The region of integration for the above limits is as shown in the Fig. Q.16.1. The old strip was parallel to x-axis.

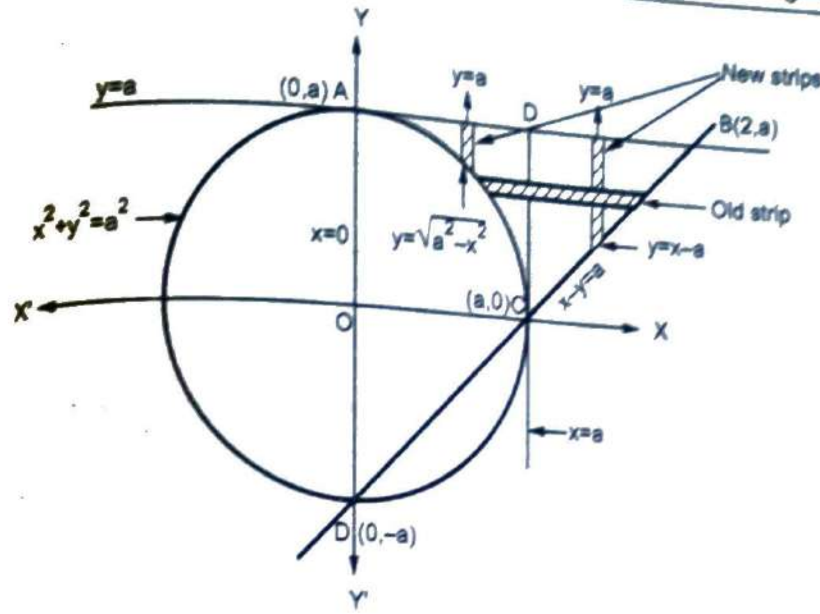


Fig. Q.16.1

Step 2 : New Limits

To change the order of integration (to consider vertical strip) the region of integration is to be divided into two parts ADC and BCD.

i) For the region ADC,

The limits over the vertical strip are $y=\sqrt{a^2-x^2}$ and $y=a$ and moving strip from $x=0$ to $x=a$.

ii) For the region BCD

The limits over the vertical strip are $y=x-a$, $y=a$ and moving the strip in the region BCD from $x=a$ to $x=2a$.

Step 3 :

Therefore, from integral (1)

$$I = \int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dx\,dy$$

$$\boxed{I = \int_{x=0}^{a} \int_{\underset{(ADC)}{y=\sqrt{a^2-x^2}}}^{a} f(x,y)\,dx\,dy + \int_{x=0}^{a} \int_{\underset{(BCD)}{y=x-a}}^{a} f(x,y)\,dx\,dy}$$

... Ans.

Q.17 : **Express as single integral and evaluate** $\underbrace{\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy}_{I_1} + \underbrace{\int_1^3 dy \int_{-1}^{1} dx}_{I_2}$. **[SPPU : Dec.-99, 08, 10, May-06, Marks 5]**

Ans. : Step 1 : Region in integral

Limits for I_1

$$x = -\sqrt{y}, x = \sqrt{y} \Rightarrow x^2 = y$$
$$y = 0, y = 1 \Rightarrow y = 0, y = 1$$

Limits for I_2 $x = -1, x = 1$

$$y = 1, \quad y = 3$$

The region of integration is as shown in the Fig. Q.17.1.

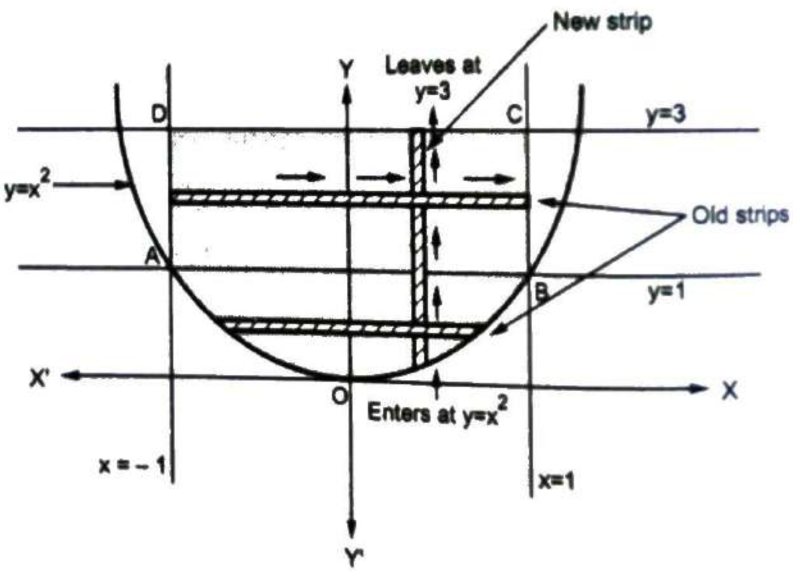


Fig. Q.17.1

Step 2 : New limits

To express as single integral, we change the order of integration by considering vertical strip in the region of integration over which y varies from $y = x^2$ to $y = 3$ and moving the strip from $x = -1$ to $x = 1$.

Step 3 :

Given integral can be written as

$$I = \int_{x=-1}^{1} \int_{y=x^2}^{y=3} dx\,dy$$

$$= \int_{-1}^{1} \left[\int_{x^2}^{3} dy\right] dx$$

$$= \int_{-1}^{1} (y)_{x^2}^{3}\, dx$$

$$= \int_{-1}^{1} \left(3 - x^2\right) dx$$

$$= 2\int_{0}^{1} \left(3 - x^2\right) dx$$

$$= 2\left(3x - \frac{x^3}{3}\right)_0^1$$

$$\boxed{I = 2\left(3 - \frac{1}{3}\right) = \frac{16}{3}}$$

... Ans.

13.4 : Transformation of Double Integral into Polar Co-ordinates (r, θ)

In many cases it is convenient to transform the double integral

$$I = \iint_R f(x, y)\, dx\, dy \qquad \dots (13.1)$$

into polar co-ordinates

1) If the integrand involves the term $x^2 + y^2$

i.e $\dfrac{x^2}{x^2 + y^2}, \left(x^2 + y^2\right)^{\frac{n}{2}}, \log\left(x^2 + y^2\right)$ or $\cos\left(x^2 + y^2\right)$ etc.

2) The region of integration is circular or elliptical boundaries viz. $x^2 + y^2 = a^2,\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ (x - a)^2 + y^2 = a^2$ etc. by putting $x = r\cos\theta,\quad y = r\sin\theta \quad \therefore\ x^2 + y^2 = r^2$ and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ and

$$dx\,dy = |J|\, d\theta\, dr = \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| d\theta\, dr = r\, d\theta\, dr$$

$\therefore$ From equation (13.1), we have

$$I = \iint_R f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$= \iint_R F(r,\theta)\, r\, d\theta\, dr \qquad \dots (13.2)$$

To find corresponding limits for r, θ. Draw a radial strip OPQ in the region of integration R from the pole (r = 0) over which 'r' varies from $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ (for region R)

To sweepout the complete region of integration. Rotate the strip OPQ from CD to AB (In anticlockwise direction from $\theta = \alpha$ to $\theta = \beta$)

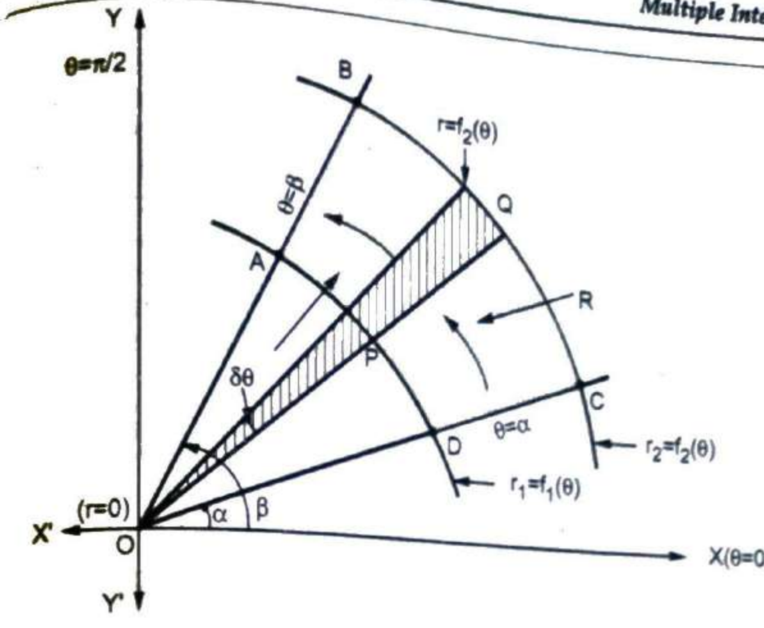


Fig. 13.5

$\therefore$ From equation (13.2)

$$I = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r,\theta)\, r\, d\theta dr \qquad \dots (13.3)$$

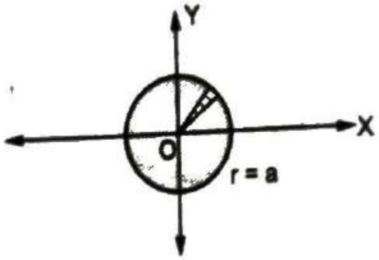
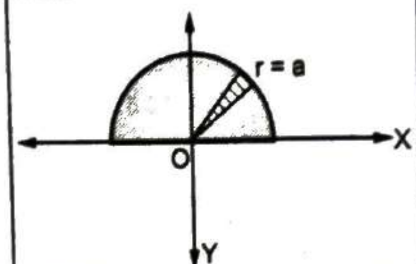
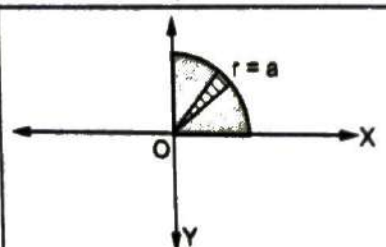
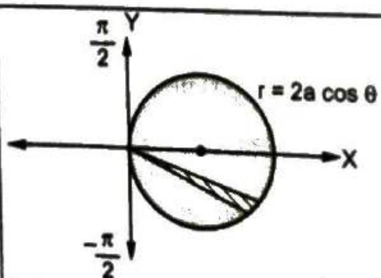
It is clear that the integrand to be integrated w.r.t. r first over the limits $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ (keeping θ constant)

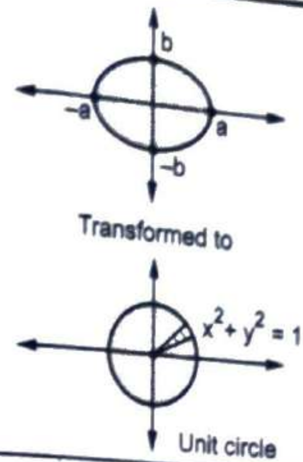
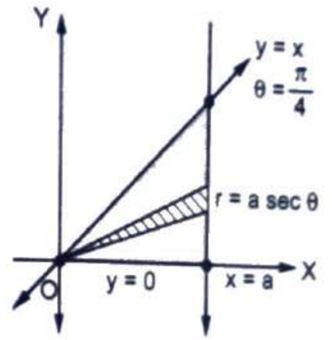
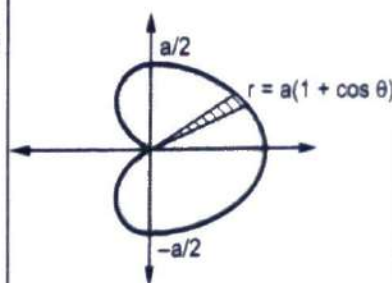
$$\therefore \qquad I = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r,\theta)\, r\, dr\ \ d\theta \qquad \dots (13.4)$$

Similarly $\qquad I = \displaystyle\int_{r=\alpha}^{r=\beta} \int_{\theta=f_1(r)}^{f_2(r)} f(r,\theta)\, r\, dr\, d\theta$

Note : Let $I = \iint f(r,\theta)\, r\, dr\, d\theta$

Consider the following Table which gives the curves and limits in polar form.

Sr. No.	Equation of curve	Graph of curve	Limits in Polar Form Put $x = r\cos\theta$ $y = r\sin\theta$ $dxdy = r\,dr\,d\theta$ $x^2 + y^2 = r^2$
1	Circle $x^2 + y^2 = a^2$		$r = 0$ to $r = a$ and $\theta = 0$ to $\theta = 2\pi$
2	Semicircle $x^2 + y^2 = a^2$ and $y \geq 0$		$r = 0$ to $r = a$ and $\theta = 0$ to $\theta = \pi$
3	Positive Quadrant of circle $x^2 + y^2 = a^2$ and $x \geq 0$, $y \geq 0$		$r = 0$ to $r = a$ and $\theta = 0$ to $\theta = \frac{\pi}{2}$
4	$x^2 + y^2 = 2ax$ i.e. $(x-a)^2 + (y-0)^2 = a^2$ OR $r = 2a\cos\theta$		$r = 0$ to $r = 2a$ $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

5	Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$		Put $x = ar\cos\theta$ $y = br\sin\theta$ $dxdy = abr\,dr\,d\theta$ $\therefore r = 0$ to 1 and $\theta = 0$ to $\theta = 2\pi$
6	Triangle $y = 0$, $x = a$ $x = y$		We have $x = r\cos\theta$ $\Rightarrow a = r\cos\theta$ $r = a\sec\theta$ $r = 0$ to $r = a\sec\theta$ $\theta = 0$ to $\theta = \frac{\pi}{4}$
7	Cardioid $r = a(1 + \cos\theta)$		$r = 0$ to $r = a(1 + \cos\theta)$ $\theta = 0$ to $\theta = 2\pi$

Q.18 : Evaluate $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log_e(x^2 + y^2)\,dx\,dy$

[SPPU : May-16, Dec.-18, Marks 6]

Ans. : The limits of the given integral are $x = y$ to $x = \sqrt{a^2 - y^2}$ i.e. $x^2 = a^2 - y^2 \Rightarrow x^2 + y^2 = a^2$ and $y = 0$ to $y = \frac{a}{\sqrt{2}}$

To evaluate the given integral, it is easy to transform it to polar coordinates

$\therefore$ Put $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$.

Now, Draw radial vector from (0, 0) to $r = a$

The lower end of this vector is at $r = 0$ and upper end is at $r = a$.

The region of integral is 0ABO

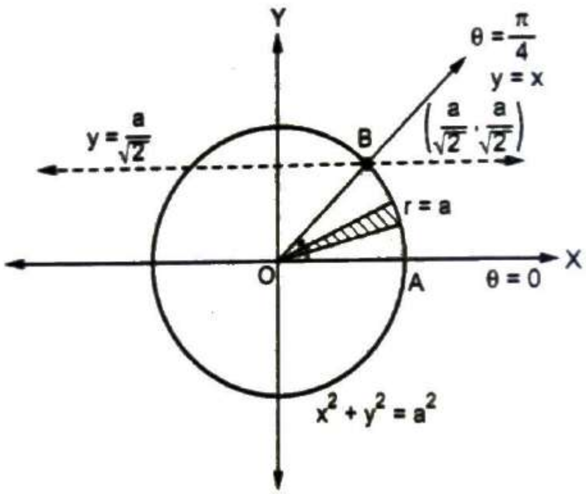


Fig. Q.18.1

$\therefore$ Limits are $r = 0$ to $r = a$ and $\theta = 0$ to $\theta = \frac{\pi}{4}$

$\therefore$ We get,

$$I = \int_0^{a\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log_e(x^2 + y^2)\,dx\,dy$$

$$= \int_{\theta=0}^{\pi/4} \int_{r=0}^{a} \log r^2\, r\,dr\,d\theta$$

$$= \left[\int_{\theta=0}^{\pi/4} d\theta\right]\left[\int_{r=0}^{a} 2\log r\,(r\,dr)\right]$$

$$= [\theta]_0^{\pi/4}\; 2\left\{\left[(\log r)\left(\frac{r^2}{2}\right)\right]_0^a - \int_0^a \frac{1}{r}\frac{r^2}{2}\,dr\right\}$$

$$= \frac{\pi}{4}(2)\left[\left\{\frac{a^2}{2}\log a - 0\right\} - \frac{1}{2}\left(\frac{r^2}{2}\right)_0^a\right] \qquad \left(\because \lim_{r\to 0}\frac{r^2}{2}\log r = 0\right)$$

$$I = \frac{\pi}{4}\left[a^2\log a - \frac{a^2}{2}\right] = \frac{\pi}{4}a^2\left[\log a - \frac{1}{2}\right]$$

Q.19 : Evaluate $\iint_R x^2y^2\,dx\,dy$ over the positive of quadrant of $x^2 + y^2 = 1$.

[SPPU : May-18, Marks 6]

Ans. : Let $$I = \iint_R x^2y^2\,dx\,dy \qquad \ldots(1)$$

where R is the positive quadrant of $x^2 + y^2 = 1$

Put $$x = r\cos\theta,\ y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta$$

The region of integral is 0 ABO

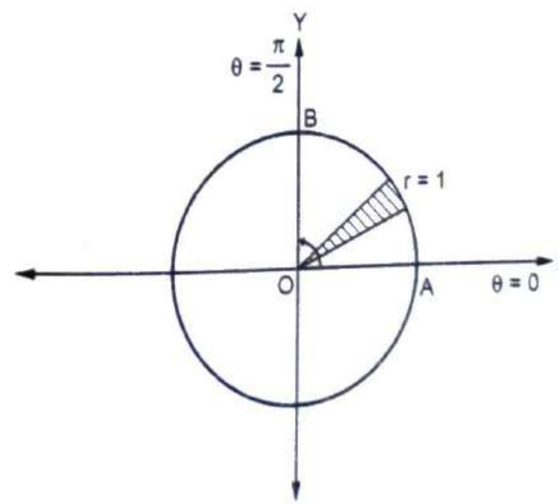


Fig. Q.19.1

$\therefore$ Limits are $r = 0$ to $r = 1$ and and $\theta = 0$ to $\theta = \pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\cos^2\theta\, r^2\sin^2\theta\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\cos^2\theta\sin^2\theta\,d\theta \int_{r=0}^{1} r^5\,dr$$

$$I = \frac{1\times1}{4\times2}\times\frac{\pi}{2}\cdot\left[\frac{r^6}{6}\right]_0^1 = \frac{\pi}{16}\cdot\frac{1}{6} = \frac{\pi}{96}$$

Q.20 : Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dx\,dy.$ **[SPPU : May-15, Marks 6]**

Ans. : Limits of the given integral are

$y = 0$ to $y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$ and $x = 0$ to $x = a$

The region of interaction is 0ABO as shown in Fig. Q.20.1.

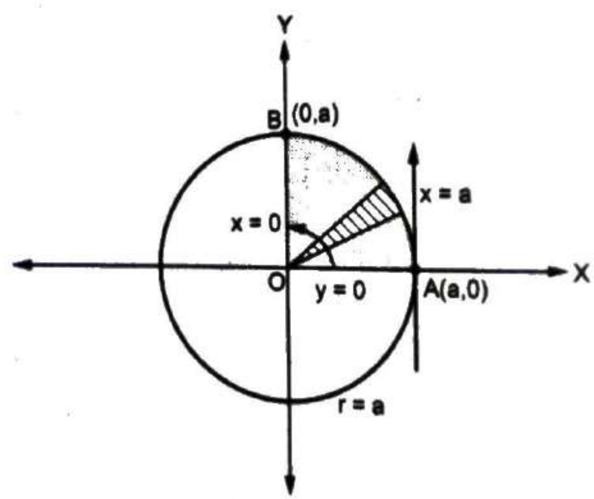


Fig. Q.20.1

Now put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Limits are $r = 0$ to $r = a$, $\theta = 0$ to $\theta = \pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} e^{-r^2}\, r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left(\frac{-1}{2}\right)\left(e^{-r^2}\right)_0^a d\theta$$

$$= \frac{-1}{2}\int_{\theta=0}^{\pi/2}(e^{-a^2}-1)\,d\theta = \frac{-1}{2}(e^{-a^2}-1)[\theta]_0^{\pi/2}$$

$$I = \frac{\pi}{4}(e^{-a^2}-1) = \frac{\pi}{4}(1-e^{-a^2})$$

Q.21 : Evaluate $\iint_R \sin(x^2+y^2)\,dx\,dy$ **where R is a circle** $x^2+y^2 = a^2$. **[SPPU : Dec.-14, Marks 6]**

Ans. : We have

$$x^2+y^2 = a^2$$

$$r = a \text{ and } dx\,dy = r\,dr\,d\theta$$

Limits are, $r = 0$ to $r = a$ and $\theta = 0$ to $\theta = 2\pi$

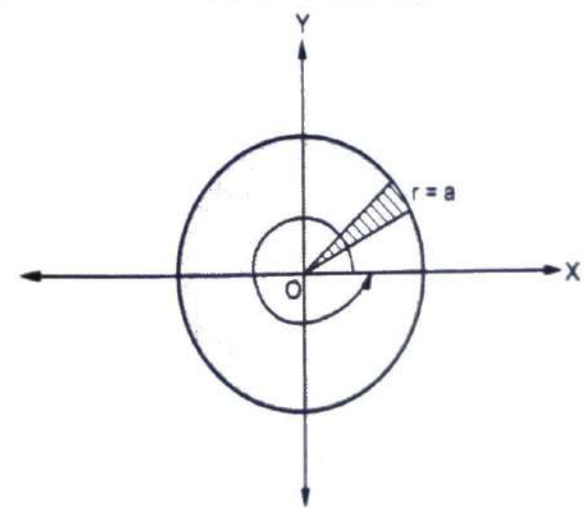


Fig. Q.21.1

$$\therefore \quad I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\sin^{r^2} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\sin(r^2)(2r\,dr)\,d\theta$$

$$\left(\because \int \sin f\, f'\,dx = -\cos f\right)$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}[-\cos r^2]_0^a\,d\theta$$

$$= \frac{1}{2}[-\cos a^2 + 1]\int_{\theta=0}^{2\pi} d\theta$$

$$I = \frac{1}{2}[1-\cos a^2][2\pi] = \pi(1-\cos a^2)$$

Q.22 : Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \sin\left\{\frac{\pi}{a^2}(a^2-x^2-y^2)\right\} dx\,dy.$

[SPPU : Dec.-15, Marks 6]

Ans. : Limits of the given integral are

$y = 0$ to $y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$ and $x = 0$ to $x = a$.

The region of integral is OABO

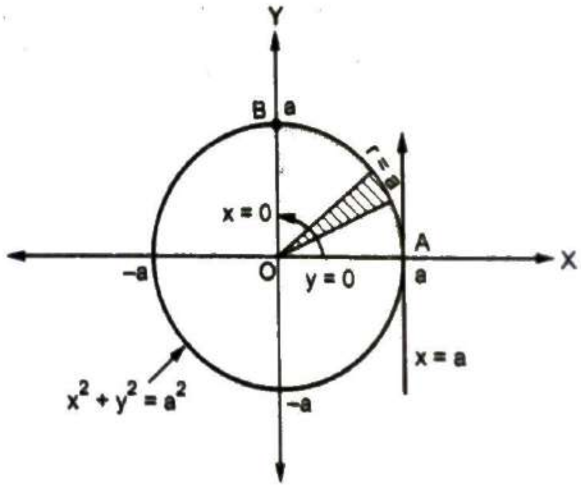


Fig. Q.22.1

Put $x = r\cos\theta,\ y = r\sin\theta,\ x^2+y^2 = r^2$

$$dx\,dy = r\,dr\,d\theta$$

$\therefore$ Limits are $\quad r = 0$ to $r = a$ and $\theta = 0$ to $\theta = \pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\sin\left\{\frac{\pi}{a^2}(a^2-r^2)\right\} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} d\theta\left(\frac{a^2}{-2\pi}\right)\int_{r=0}^{a}\sin\left\{\frac{\pi}{a^2}(a^2-r^2)\right\}\left(-2r\cdot\frac{\pi}{a^2}\right)dr$$

$$= \left[\frac{\pi}{2}-0\right]\left(-\frac{a^2}{2\pi}\right)\left[-\cos\left\{\frac{\pi}{a^2}(a^2-r^2)\right\}\right]_0^a$$

$$= \frac{-a^2}{4}[-\cos 0+\cos\pi] = \frac{-a^2}{4}[-1-1] = +\frac{a^2}{2}$$

$$\therefore \quad I = \frac{a^2}{2}$$

Q.23 : Evaluate $\iint (x+y)^2\,dx\,dy$ **over the area bounded by an ellipse** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1.$

[SPPU : May-14, Marks 6]

Ans. : Let $I = \iint_R (x+y)^2\,dx\,dy$

$$= \iint_R (x^2+y^2+2xy)\,dx\,dy \qquad \ldots(1)$$

Put $\quad x = ar\cos\theta,\ y = br\sin\theta$

$\therefore \quad dx\,dy = ab\ r\,dr\,d\theta\,;$

By these substitutions an ellipse is transformed to unit circle.

Therefore limits are $r = 0$ to $r = 1$ and $\theta = 0$ to $\theta = 2\pi$

$$\therefore I = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}[a^2r^2\cos^2\theta + b^2r^2\sin^2\theta + 2ab\,r^2\sin\theta\cos\theta]\,ab\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}[a^3b\cos^2\theta + ab^3\sin^2\theta + 2a^2b^2\sin\theta\cos\theta][r^3]\,dr\,d\theta$$

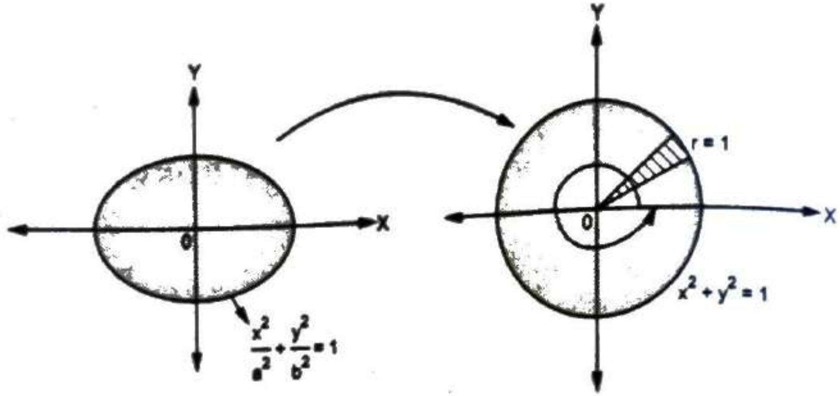


Fig. Q.23.1

$$= \left\{\int_{\theta=0}^{2\pi}[a^3b\cos^2\theta + ab^3\sin^2\theta]\,d\theta + \int_{\theta=0}^{2\pi}2a^2b^2\sin\theta\cos\theta\,d\theta\right\}\left[\frac{r^4}{4}\right]_0^1$$

$$= \frac{1}{4}\left\{4a^3b\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]+4ab^3\left[\frac{1}{2}\frac{\pi}{2}\right]+0\right\}$$

$$I = \frac{\pi}{4}ab[a^2+b^2]$$

Q.24 : Evaluate $\iint\limits_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$, where R is annulus between $x^2+y^2=4$ and $x^2+y^2=9$ **[SPPU : May-17, Marks 6]**

Ans. : Let

$$I = \iint\limits_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy \qquad \dots (1)$$

The integrand is difficult to integrate w.r.t. x or w.r.t. y and it becomes handy if we transform into polar co-ordinates.

$\therefore$ Put $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,d\theta\,dr$$

and $$x^2+y^2 = r^2$$

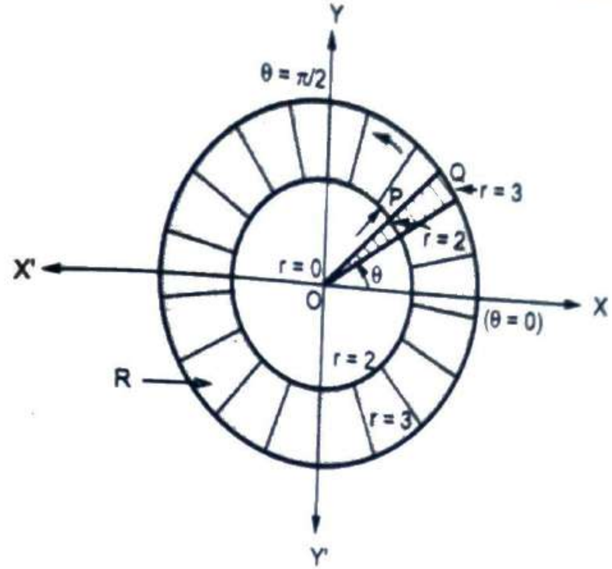


Fig. Q.24.1

$$x^2+y^2 = 4 \Rightarrow r = 2$$

$$x^2+y^2 = 9 \Rightarrow r = 3$$

To find limits for r and θ, draw a strip OPQ in the region of integration over which 'r' varies from **r = 2 to r = 3** (Since at P, r = 2 and at Q, r = 3)

Now rotating the strip from '**θ = 0 to θ = 2π**,' we get entire shaded region R which is region of integration.

$\therefore$ From equation (1),

$$I = \int_{\theta=0}^{2\pi}\int_{r=2}^{3}\frac{r^2\cos^2\theta\; r^2\sin^2\theta}{r^2}\,r\,d\theta\,dr$$

$$= \int_{0}^{2\pi}\int_{r=2}^{3} r^3\sin^2\theta\cos^2\theta\,d\theta\,dr$$

$$= \int_{0}^{2\pi}\sin^2\theta\cos^2\theta\left[\int_{r=2}^{3} r^3\,dr\right]d\theta$$

$$= \int_0^{2\pi} \sin^2\theta \cos^2\theta \left(\frac{r^4}{4}\right)_2^3 d\theta$$

$$= 4\int_0^{\pi/2} \sin^2\theta \cos^2\theta \left[\frac{(3)^4}{4} - \frac{(2)^4}{4}\right] d\theta$$

$$= 65 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$\boxed{I = 65\left[\frac{(2-1)(2-1)}{4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{65\pi}{16}}$$ ... Ans.

Q.25 : Evaluate $\displaystyle\int_0^1 \int_{y=\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xy\, e^{-x^2-y^2}}{x^2+y^2}\, dx\, dy$

[SPPU : Dec.-05, 11, Marks 6]

Ans. : Here the limits are

$y = \sqrt{x - x^2}$, $y = \sqrt{1 - x^2}$, x = 0 and x = 1

$\Downarrow \qquad\qquad \Downarrow$

$x^2 + y^2 = x$, $x^2 + y^2 = 1$, x = 0 and x = 1

$r^2 = r\cos\theta$, $r^2 = 1$

$r = \cos\theta$, $r = 1$

The region of integration is as shown in the Fig. 13.43.

Transforming into polar by putting

$x = r\cos\theta$, $y = r\sin\theta$

$dx\, dy = r\, d\theta$

'r' varies from $r = \cos\theta$ (At P) to $r = 1$ (At Q) and 'θ' varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

'r' varies from $r = 0$ to $r = \frac{4\cos\theta}{\sin^2\theta}$ and θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

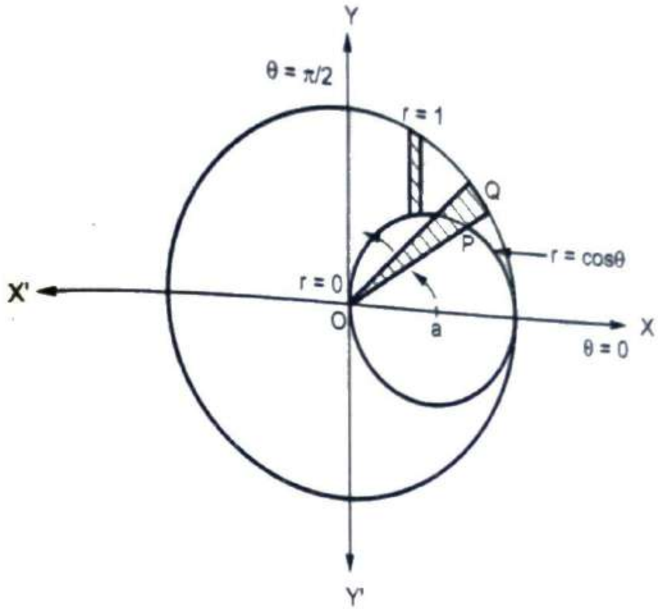


Fig. Q.25.1

Given integral will have the form

$$I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{4\frac{\cos\theta}{\sin^2\theta}} \frac{r^2\left(\cos^2\theta - \sin^2\theta\right)}{r^2}\, r\, d\theta\, dr$$

$$= \int_{\pi/4}^{\pi/4} \left(\cos^2\theta - \sin^2\theta\right)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \left(\cos^2\theta - \sin^2\theta\right)\frac{16a^2\cos^2\theta}{\sin^4\theta}\, d\theta$$

From given integral :

$$I = \int_{\theta=0}^{\pi/2} \int_{r=\cos\theta}^{1} \frac{r\cos\theta\, r\sin\theta\, e^{-r^2}}{r^2}\, r\, d\theta\, dr$$

$$= \int_0^{\pi/2} \sin\theta \cos\theta \left[\int_{r = \cos\theta}^{1} r\ e^{-r^2} dr \right] d\theta$$

$$= \int_0^{\pi/2} \sin\theta \cos\theta \left(-\frac{e^{-r^2}}{2} \right)_{\cos\theta}^{1} d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \sin\theta \cos\theta \left[e^{-1} - e^{-\cos^2\theta} \right] d\theta$$

$$= -\frac{1}{2e} \int_0^{\pi/2} \sin\theta \cos\theta\ d\theta$$

$$+ \frac{1}{2} \int_0^{\pi/2} \sin\theta \cos\theta\ e^{-\cos^2\theta} d\theta$$

$$= -\frac{1}{2e} \left(\frac{\sin 2\theta}{2} \right)_0^{\pi/2} + \frac{1}{2} \left[\frac{e^{-\cos^2\theta}}{2} \right]_0^{\pi/2}$$

$$= -\frac{1}{2e} \left[\frac{1}{2} - 0 \right] + \frac{1}{4} \left[e^0 - e^{-1} \right]$$

$$\boxed{I = \frac{1}{4} \left(1 - \frac{2}{e} \right)}$$ .... Ans.

Q.26 : Evaluate $\iint\limits_R \frac{\sqrt{x^2 + y^2}}{x^2} dx\, dy$ where R is the region enclosed by the curves $x^2 + y^2 = 2x$ and $y = x$ and $y = 0$ in the first quadrant. **[SPPU : May-10, Marks 5]**

Ans. : Given region is bounded by the curves $x^2 + y^2 = 2x$, $y = x$ and $y = 0$

Let $$I = \iint\limits_R \frac{\sqrt{x^2 + y^2}}{x^2} dx\, dy$$

$$= \int_0^{\pi/4} \int_0^{2\cos\theta} \frac{r}{r^2 \cos^2\theta}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} \sec^2\theta\, d\theta \left(\int_0^{2\cos\theta} dr \right)$$

$$= \int_0^{\pi/4} \sec^2\theta\, (2\cos\theta\, d\theta) = \int_0^{\pi/4} 2 \sec\theta\, d\theta$$

$$= 2\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= 2 \log(\sqrt{2} + 1)$$

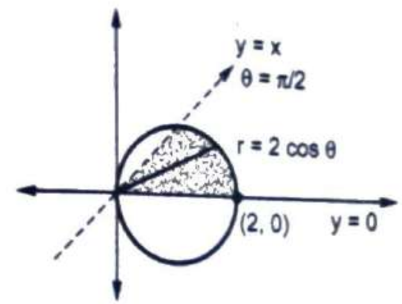


Fig. Q.26.1

13.5: Triple Integrals

Definition :

I) Triple Integrals : The triple integral of a function f(x, y, z) over a solids in xyz space is defined in the same manner as a double integral. In this case instead of considering xy plane we will consider solid in xyz-space. And instead of dividing the region of x-y plane into small rectangles, we will divide the solid into small rectangular solids, or unit cubes, as shown in figure.

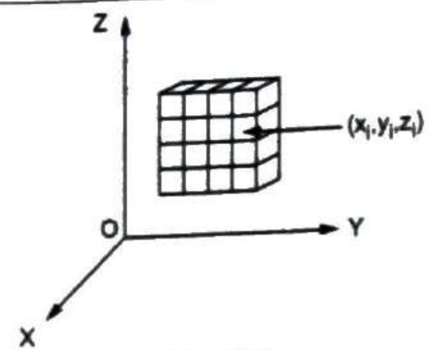


Fig. 13.6

The volume of the solid is the number of unit cubes of solid.

Proceeding as in the case of the double integrals, we can obtain sums of the form.

$$\delta = \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta x_i\,\delta y_i\,\delta z_i$$

where (x_i, y_i, z_i) is any point of solid R.

The triple integral of f(x, y, z) over the region R is the limiting value to which $\delta x_i, \delta y_i, \delta z_i \to 0$.

Thus

$$\iiint_R f(x, y, z)\,dv = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta x_i\,\delta y_i\,\delta z_i$$

II) Mean value theorem for triple integrals : Let f(x, y, z) be a continuous on the closed solid R then $\exists$ at least one point (a, b, c) in R such that

$$\iiint_R f(x, y, z)\,dv = f(a, b, c)\cdot V$$

where $$V = \iiint_R dx\,dy\,dz \text{ is the volume of R.}$$

III) Evaluation of triple integrals : It is very difficult to evaluate triple integral directly from it's definition (limit form). So, we will evaluate triple integral by repeated single integrations.

a) If R is the solid region such that

$a \le x \le b,\ c \le y \le d,\ e \le z \le f$, then

$$\iiint_R f(x, y, z)\,dv = \int_a^b \left[\int_c^d \left(\int_e^f f(x, y, z)\,dz\right) dy\right] dx$$

$$= \int_c^d \left[\int_a^b \left(\int_e^f f(x, y, z)\,dz\right) dx\right] dy$$

$$= \int_e^f \left\{\int_a^b \left[\int_c^d f(x, y, z)\,dy\right] dx\right\} dz$$

Hence, if limits of integration are constants, then the order of integration is immaterial.

b) If the solid region is bounded by some curves then,

$$\int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} \left(\int_{z=\phi_1(x, y)}^{z=\phi_2(x, y)} f(x, y, z)\,dz\right) dy\right] dx$$

Q.27 : Evaluate the integral $\displaystyle\int_0^{2a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

[SPPU : May-15, Marks 7]

Ans. : Let

$$I = \int_0^{2a}\int_0^{x}\int_0^{x+y} e^x e^y e^z\,dz\,dy\,dx$$

$$= \int_0^{2a}\int_0^{x} e^y\, e^x \left(\int_0^{x+y} e^z\,dz\right) dy\,dx$$

$$= \int_0^{2a}\int_0^{x} e^x e^y\,[e^z]_0^{x+y}\,dy\,dx$$

$$= \int_0^{2a}\int_0^{x} e^x\,e^y\,(e^{x+y} - 1)\,dy\,dx$$

$$= \int_0^{2a} e^x \int_0^x [e^{x+2y} - e^y]\, dy\, dx$$

$$= \int_0^{2a} e^x \left[\frac{e^{x+2y}}{2} - e^y\right]_0^x dx$$

$$= \int_0^{2a} e^x \left[\frac{e^{3x}}{2} - e^x - \frac{e^x}{2} + 1\right] dx$$

$$= \int_0^{2a} \left(\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^x\right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3}{2}\frac{e^{2x}}{2} + e^x\right]_0^{2a}$$

$$= \left(\frac{e^{8a}}{8} - \frac{3}{4}e^{4a} + e^{2a}\right) - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$I = \frac{e^{8a}}{8} - \frac{3}{4}e^{4a} + e^{2a} - \frac{3}{8}$$

Q.28 : Evaluate $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$ **[SPPU : May-19, Marks 7]**

Ans. : Let

$$I = \int_{x=0}^{\log 2}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} [e^{x+y+z}]_0^{x+y}\, dy\, dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} [e^{x+y+x+y} - e^{x+y}]\, dy\, dx$$

$$= \int_{x=0}^{\log 2}\int_{y=0}^{x} [e^{2x}e^{2y} - e^x e^y]\, dy\, dx$$

$$= \int_{x=0}^{\log 2} \left[e^{2x}\frac{e^{2y}}{2} - e^x \frac{e^y}{2}\right]_0^x dx$$

$$= \int_{x=0}^{\log 2} \left[e^{2x}\frac{e^{2x}}{2} - e^x\frac{e^x}{2} - \frac{e^{2x}}{2} + \frac{e^x}{2}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{\log 2} [e^{4x} - 2e^{2x} + e^x]\, dx$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{4} - \frac{2e^{2x}}{2} + e^x\right]_0^{\log 2}$$

$$= \frac{1}{2}\left[e^{4\log 2} - e^{2\log 2} + e^{\log 2} - \frac{1}{4} + 1 - 1\right]$$

$$I = \frac{1}{2}\left[2^4 - 2^2 + 2 - \frac{1}{4}\right] = \frac{1}{2}\left[14 - \frac{1}{4}\right] = \frac{55}{8}$$

Q.29 : Evaluate $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$ **[SPPU : Dec.-17, Marks 7]**

Ans. : Let

$$I = \int_{-1}^{1}\int_0^z \left[\int_{y=x-z}^{y=x+z} (x+y+z)\, dy\right] dx\, dz \qquad \text{(w.r.t. y first)}$$

$$= \int_{-1}^{1}\int_0^z \left[(x+z)y + \frac{y^2}{2}\right]_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1}\int_0^z \left[(x+z)^2 + \frac{(x+z)^2}{2} - (x^2 - z^2) - \frac{(x-z)^2}{2}\right] dx\, dz$$

$$= \int_{-1}^{1}\int_0^z \left[\frac{3}{2}(x+z)^2 - x^2 + z^2 - \frac{(x-z)^2}{2}\right] dx\, dz$$

$$= \int_{-1}^{1} \left[\frac{3}{2}\frac{(x+z)^3}{3} - \frac{x^3}{3} + z^2 x - \frac{(x-z)^3}{6}\right]_0^z dz$$

$$= \int_{-1}^{1}\left[4z^3 - \frac{z^3}{3} + z^3 - \frac{z^3}{2} - \frac{z^3}{6}\right]dz$$

$$= \int_{-1}^{1} 4z^3\,dz = 4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$\boxed{I = 4\,[1-1] = 0} \quad \text{... Ans.}$$

Q.30 : Evaluate $\int_0^1 \int_0^{\sqrt{2}} \int_0^{2\pi} (r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz$

[SPPU : Dec.-17, Marks 7]

Ans. : Let

$$I = \int_0^1 \int_0^{\sqrt{2}} \int_0^{2\pi} [(r^3\cos^2\theta + r\,z^2)d\theta]\,dr\,dz$$

$$= \int_0^1 \int_0^{\sqrt{2}} \int_0^{2\pi} \left[r^3\left(\frac{1+\cos 2\theta}{2}\right) + rz^2\right]d\theta\,dr\,dz$$

$$= \int_0^1 \int_0^{\sqrt{2}} \left[\frac{r^3}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + rz^2\theta\right]_0^{2\pi} dr\,dz$$

$$= \int_0^1 \int_0^{\sqrt{2}} (r^3\pi + 2\pi\,z^2 r)\,dr\,dz = \int_0^1 \left[\frac{r^4\pi}{4} + 2\pi\,z^2\,\frac{r^2}{2}\right]_0^{\sqrt{z}} dz$$

$$= \int_0^1 \left[\frac{\pi}{4}(\sqrt{z})^4 + \pi\,z^2\left(\sqrt{z}\right)^2\right]dz$$

$$= \int_0^1 \left[\frac{\pi}{4}z^2 + \pi\,z^3\right]dz = \left[\frac{\pi}{4}\frac{z^3}{3} + \pi\frac{z^4}{4}\right]_0^1$$

$$I = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$$

Q.31 : Evaluate $\iiint_R 2x\,dv$ where R is the solid region under the plane $2x + 3y + z = 6$ that lies in the first octant.

[SPPU : Dec.-17, Marks 7]

Ans. :

Let $$I = \iiint_R 2x\,dv$$

The projection of R on xy plane is a triangle.

We have $z = 0$ to $z = 6 - 2x - 3y$.

$0 \le x \le 3$ and $0 \le y \le \frac{6-2x}{3}$

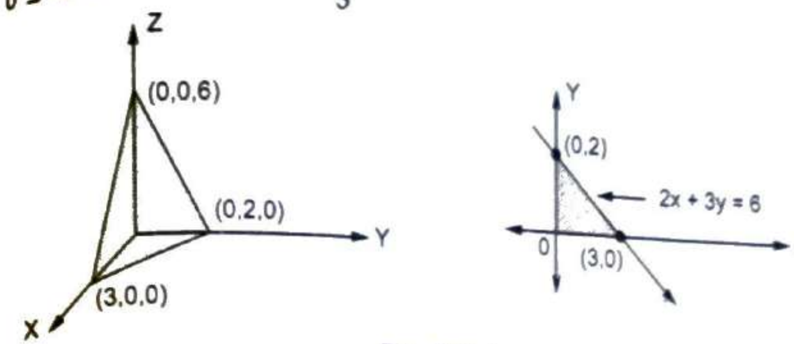


Fig. Q.31.1

$$\therefore \quad I = \int_{x=0}^{3} \int_{y=0}^{6-\frac{2x}{3}} \int_{z=0}^{6-2x-3y} 2x\,dz\,dy\,dx$$

$$= \int_0^3 2x \int_0^{6-\frac{2x}{3}} [6 - 2x - 3y - 0]\,dy\,dx$$

$$= \int_0^3 2x\left[6y - 2xy - \frac{3y^2}{2}\right]_0^{6-\frac{2x}{3}} dx$$

$$= \int_0^3 \left(\frac{4}{3}x^3 - 8x^2 + 12x\right)dx$$

$$= \left[\frac{x^4}{3} - \frac{8x^3}{3} + 6x^2\right]_0^3 = 9$$

13.6 : Transformation of Triple Integrals by Jacobians

I) Cartesian to spherical polar :

Put $x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\phi$

$x^2 + y^2 + z^2 = r^2$

and $dx\,dy\,dz = |J|\, d\theta\, d\phi\, dr$

$$= \left|\frac{\partial(x\,y\,z)}{\partial(r,\theta,\phi)}\right| d\theta\; d\phi\; dr$$

$$= r^2 \sin\theta\; d\theta\; d\phi\; dr$$

$$\therefore \quad I = \iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_V F(r,\theta,\phi) r^2 \sin\theta\, d\theta\, d\phi\, dr$$

II) Some standard limits :

i) For complete sphere $x^2 + y^2 + z^2 = a^2$

r varies from $r = 0$ to $r = a$

θ varies from $\theta = 0$ to $\theta = \pi$

ϕ varies from $\phi = 0$ to $\phi = 2\pi$

ii) For hemisphere $x^2 + y^2 + z^2 = a^2 \quad (z \geq 0)$

r varies from $r = 0$ to $r = a$

θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

ϕ varies from $\phi = 0$ to $\phi = 2\pi$

iii) For ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Put $x = a\, r \sin\theta \cos\phi$,

$y = b\, r \sin\theta \sin\phi$,

$z = c\, r \cos\theta$

$dx\, dy\, dz = abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

r varies from $r = 0$ to $r = 1$

θ varies from $\theta = 0$ to $\theta = \pi$

ϕ varies from $\phi = 0$ to $\phi = 2\pi$

III) Cylindrical polar co-ordinates :

Put $x = \rho \cos\phi$

$y = \rho \sin\phi$

$z = z$

$\therefore \quad x^2 + y^2 = \rho^2$

and $dx\, dy\, dz = |J|\, d\rho\, d\phi\, dz$

$$= \left|\frac{\partial(x\,y\,z)}{\partial(\rho,\phi,z)}\right| d\rho\; d\phi\; dz$$

$$dx\, dy\, dz = \rho\, d\rho\, d\phi\, dz$$

IV) Dirichlet's theorem :

i) For two variables x and y,

$$\iint x^{a-1} y^{b-1}\, dx\, dy = \frac{\Gamma a \cdot \Gamma b}{\Gamma a+b+1} \quad \text{where } x + y \leq 1$$

ii) For three variables x, y, z

$$\iiint x^{a-1}, y^{b-1}, z^{c-1}\, dx\, dy\, dz = \frac{\Gamma a \cdot \Gamma b \cdot \Gamma c}{\Gamma 1 + a + b + c}$$

where $x + y + z \leq 1$

iii) For n variables $x_1, x_2, \ldots, x_n$

$$\iiint \cdots \int x_1^{a_1-1} \cdot x_2^{a_2-1} \cdot x_3^{a_3-1} \ldots x_n^{a_n-1} dx_1\, dx_2 \ldots dx_n$$

$$= \frac{\Gamma a_1 \Gamma a_2 \Gamma a_3 \ldots \Gamma a_n}{\Gamma 1 + a_1 + a_2 + \ldots + a_n}$$

(where $x_1 + x_2 + x_3 + \ldots + x_n \leq 1$)

V) Let R and S be the two solid regions in xyz - space and uvw - space respectively. Let $x = g_1(u, v, w)$, $y = g_2(u, v, w)$, $z = g_3(u, v, w)$ be the transformations such that each point in R has the unique image in S.

If f(x, y, z) is continuous on R and $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0$.

Then $\iiint\limits_R f(x, y, z)\, dx\, dy\, dz$

$$= \iiint\limits_S f[g_1(u, v, w), g_2(u, v, w), g_3(u, v, w)]$$

$|J|\, du\, dv\, dw$

Q.32 : Evaluate $\iiint\limits_R \dfrac{dv}{\sqrt{1-x^2-y^2-z^2}}$ taken over the volume of $x^2+y^2+z^2=1$ in the I[st] of octant.

[SPPU : Dec.-13, May-16,18, Marks 7]

Ans. : Let

$$I = \iiint\limits_R \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz$$

We have $x^2 + y^2 + z^2 = 1$

$\therefore$ The projection of sphere on xy-plane is a circle of radius 1 i.e. $x^2+y^2 = 1$

$\therefore 0 \leq x \leq 1$

$0 \leq y \leq \sqrt{1-x^2}$

and $0 \leq z \leq \sqrt{1-x^2-y^2}$

$$\therefore \quad I = \iiint\limits_R \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{1}{\sqrt{1-r^2}}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[-\frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1} r\right]_0^1 d\theta\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta\, d\theta\, d\phi = \frac{\pi}{4} \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} d\phi = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{8}$$

Q.33 : Evaluate $\iiint\limits_V \dfrac{dx\, dy\, dz}{(1+x^2+y^2+z^2)^2}$ over the entire positive octant of the space.

[SPPU : Dec.-15, Marks 7]

Ans. : Transforming the integral into spherical polar form using

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \cos\phi$

$z = r\cos\theta$;

$x^2 + y^2 + z^2 = r^2$

$\therefore dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

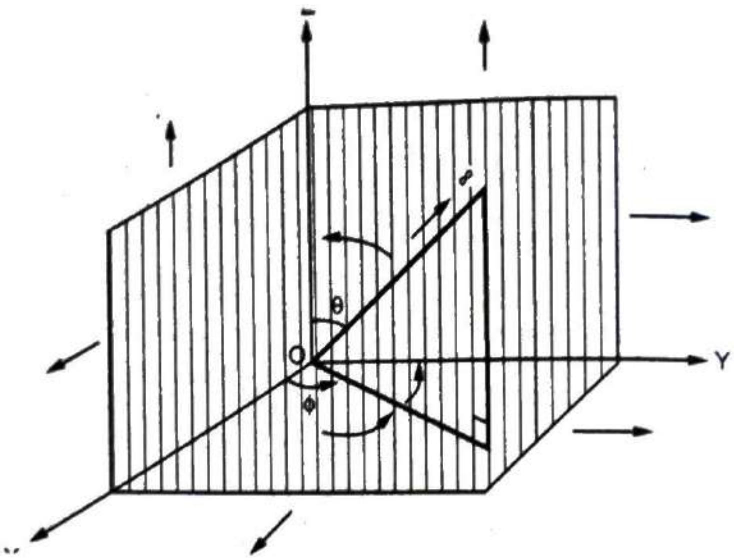


Fig. Q.33.1

Over the positive octant :

r varies from r to 0 to $r = \infty$

θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

ϕ varies from $\phi = 0$ to $\phi = \frac{\pi}{2}$

From the given integral

$$I = \iiint_V \frac{dx\, dy\, dz}{\left(1 + x^2 + y^2 + z^2\right)^2}$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{r^2 \sin\theta\, d\theta\, d\phi\, dr}{\left(1 + r^2\right)^2}$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \left[\int_{r=0}^{\infty} \frac{r^2 dr}{\left(1+r^2\right)^2}\right] d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \left[I_1\right] d\theta\, d\phi \quad \ldots (1)$$

where $$I_1 = \int_0^{\infty} \frac{r^2\, dr}{\left(1 + r^2\right)^2}$$

Put $r^2 = u$

$\therefore$ $2r\,dr = du$

$r\,dr = \frac{du}{2}$

$r^2 dr = \frac{\sqrt{u}\, du}{2}$

Limits :

r	0	∞
u	0	∞

$$\therefore \quad I_1 = \int_0^{\infty} \frac{1}{(1 + u)^2} \frac{u^{1/2} du}{2}$$

$$= \int_0^{\infty} \frac{u^{3/2}}{(1 + u)^{3/2 + 1/2}}\, du$$

$$= \frac{1}{2}\beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma 3/2\, \Gamma 1/2}{\Gamma 2}$$

$$= \frac{1}{2} \frac{\frac{1}{2}\Gamma 1/2\, \Gamma 1/2}{1} = \frac{\pi}{4} \quad \ldots (2)$$

From equation (1),

$$I = \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left(\frac{\pi}{4}\right) d\theta\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\left[\int_0^{\pi/2} \sin\theta\, d\theta\right] d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, d\phi$$

$$\boxed{I = \frac{\pi^2}{8}} \quad \text{... Ans.}$$

Q.34 : Evaluate $\iiint (x^2y^2 + y^2z^2 + z^2x^2)\, dx\, dy\, dz$ throughout the volume of sphere $x^2 + y^2 + z^2 = a^2$

[SPPU : Dec.-15, May-17, Marks 7]

Ans. : Let

$$I = \iiint (x^2y^2 + y^2z^2 + z^2x^2)\, dx\, dy\, dz \quad \text{... (1)}$$

Put

$$\frac{x^2}{a^2} = u, \therefore x = au^{1/2},\ dx = \frac{a}{2} u^{-1/2} du$$

$$\frac{y^2}{a^2} = v,\ y = a\, v^{1/2},\ dy = \frac{a}{2}\ v^{-1/2}\ dv$$

$$\frac{z^2}{a^2} = w,\ z = aw^{1/2},\ dz = \frac{a}{2}\ w^{-1/2} dw$$

where $u + v + w \le 1$

By considering 8 times positive octant of sphere, From equation (1),

$$I = 8\iiint (a^4uv + a^4vw + a^4wu)\frac{a^3}{8}$$

$$u^{-1/2}v^{-1/2}w^{-1/2} du\, dv\, dq$$

$$= a^7\iiint \begin{pmatrix} u^{1/2}v^{1/2}w^{-1/2} + u^{-1/2}v^{1/2}w^{1/2} \\ + u^{1/2}v^{-1/2}w^{1/2} \end{pmatrix} du\, dv\, dw$$

$$= a^7 \cdot 3\left(\frac{\Gamma 3/2\ \Gamma 3/2\ \Gamma 3/2}{\Gamma 1 + 3/2 + 3/2 + 3/2}\right) \text{(Using Dirichlet's theorem)}$$

$$= 3 \cdot a^7 \frac{\frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}$$

$$\boxed{I = \frac{4a^7\pi}{35}} \quad \text{– Ans.}$$

Q.35 : Evaluate $\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz$ throughout the volume of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

[SPPU : May-15, Marks 7]

Ans. : Using $x = a\, r \sin\theta \cos\phi$,

$y = b\, r \sin\theta \sin\phi$, $z = c\, r \cos\theta$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$,

$dx\, dy\, dz = abc\ r^2 \sin\theta\ dr\, d\theta\, d\phi$ ellipsoid gets transformed to a unit sphere $r = 1$.

r varies from $r = 0$ to $r = 1$

θ varies from $\theta = 0$ to $\theta = \pi$

ϕ varies from $\theta = 0$ to $\phi = 2\pi$

From the given integral

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} \sqrt{1 - r^2}\ abc\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= abc \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\left[\int_{r=0}^{1}\sqrt{1 - r^2}\ r^2 dr\right] d\theta\, d\phi$$

Put $r = \sin t$, $dr = \cos t\ dt$

$$= abc \int_{\phi=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \times \int_{t=0}^{\pi/2} \cos t \sin^2 t \cos t \, dt$$

$$= abc(2)(2\pi)\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi^2 abc}{4}}$$ ... Ans.

Q.36 : Evaluate $\iiint_V \sqrt{x^2 + y^2} \, dx \, dy \, dz$ over the volume bounded by the right circular cone $x^2 + y^2 = z$, $(z > 0)$ and the planes $z = 0$ and $z = 1$. **[SPPU : May-13, Dec.-16, Marks 7]**

Ans. : Transforming the given integral to cylindrical polar co-ordinates by putting $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$ and $dx \, dy \, dz = \rho \, d\rho \, d\phi \, dz$

ρ varies from $\rho = 0$ to $\rho = 1$

ϕ varies from $\phi = 0$ to $\phi = 2\pi$

z varies from $z = \rho$ to $z = 1$

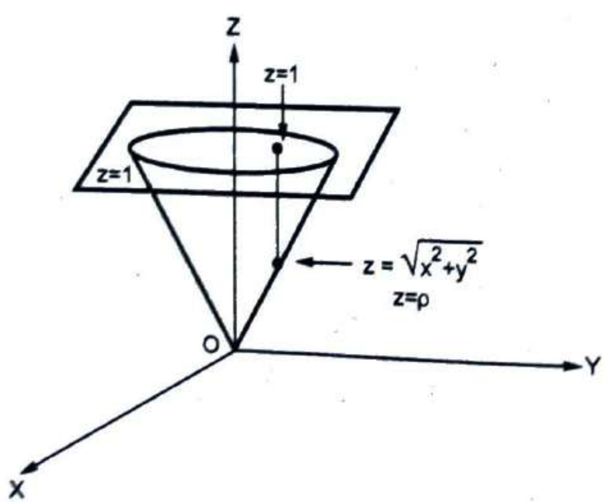


Fig. Q.36.1

$$\therefore \quad I = \iiint \sqrt{\rho^2} \, \rho \, d\rho \, d\phi \, dz$$

$$= \int_0^1 \rho^2 d\rho \cdot \int_0^{2\pi} d\phi \cdot \int_{\rho}^{1} dz$$

$$= \int_0^1 \rho^2 d\phi \, (2\pi)(1 - \rho)$$

$$= 2\pi \left[\frac{\rho^3}{3} - \frac{\rho^4}{4} \right]_0^1$$

$$\boxed{I = 2\pi \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{\pi}{6}}$$

Q.37 : Evaluate $\iiint z(x^2 + y^2) \, dx \, dy \, dz$ over the valume of the cylinder $x^2 + y^2 = 1$ intercepted by the planes $z = 2$ nd $z = 3$ **[SPPU : May-14, Marks 7]**

Ans. : Let

$$I = \iiint z(x^2 + y^2) \, dx \, dy \, dz$$

Substituting cylinderical co-ordinates, we get

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 2$

$dx \, dy \, dz = r \, dr \, d\theta \, dz$

$$\therefore \quad I = \iint (x^2 + y^2) dx \, dy \int_{z=2}^{3} z \, dz$$

$$= \iint (x^2 + y^2) dx \, dy \left[\frac{z^2}{2} \right]_2^3$$

$$= \iint (x^2 + y^2) dx \, dy \left[\frac{9}{2} - \frac{4}{2} \right]$$

$$= \frac{5}{2} \iint (x^2 + y^2) dx \, dy$$

$$= \frac{5}{2} \times 4 \int_{0}^{\pi/2} \int_{0}^{1} r^2 \, r \, dr \; d\theta$$

$$= 10 \int_{0}^{\pi/2} \left[\frac{r^4}{4} \right]_0^1 d\theta$$

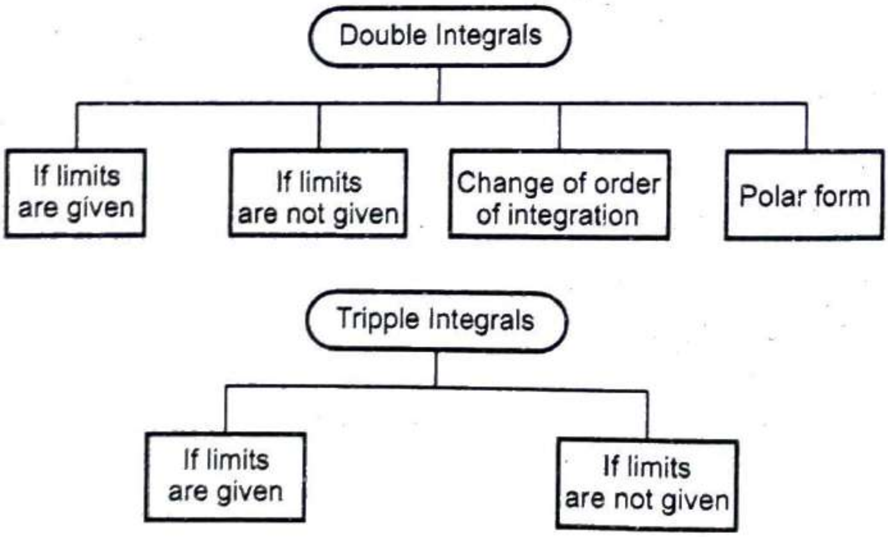
$$= \frac{10}{4} [\theta]_0^{\pi/2} = \frac{10}{4} \left[\frac{\pi}{2} \right] = \frac{5\pi}{4}$$

Memory Map

1) $\iint\limits_R f(x, y) \, dx \, dy = \lim\limits_{\substack{n \to \infty \\ \delta A \to 0}} \sum\limits_{r=1}^{n} f(x_i, y_i) \, \delta A_R$

2) $\int\limits_{x=4}^{b} \int\limits_{y=c}^{d} f(x, y) \, dx \, dy = \int\limits_{y=c}^{d} \int\limits_{x=a}^{b} f(x, y) \, dx \, dy$

3)



END...

UNIT - VI

14 Applications of Multiple Integrals

14.1 : Change of Variables in Double Integrals by Jacobians

Sometimes the problems of double integrals can be solved easily by change of independent variables by using Jacobian.

Theorem : Let R and S be the regions of xy-plane and uv-plane respectively. Let $x = g(u, v)$ and $y = h(u, v)$ be two transformations such that each point in R has the unique image in S. If $f(x, y)$ is continuous on R and

$$J = \frac{\partial (x, y)}{\partial (u, v)} \neq 0 \text{ on R then}$$

$$\iint_R f(x,y)\, dx\, dy = \iint_S f\,[g(u, v), h(u, v)]\; |J|\; du\, dv$$

where $$dx\, dy = |J|\; du\, dv = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| dx\, dy$$

Q.1 : Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\, dx\, dy$ by applying transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating over an appropriate region in the uv-plane.

Ans. : Let $$I = \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy \qquad \ldots (1)$$

and limits are $x = \frac{y}{2}$, $x = \frac{y}{2} + 1$, $y = 0$ to $y = 4$

Now $u = \frac{2x-y}{2} = x - \frac{y}{2}$ and $v = \frac{y}{2}$

$\therefore$ $x = u - \frac{y}{2} = u + v$ and $y = 2v$ ... (2)

Now $x = y/2 \Rightarrow u + v = v \Rightarrow u = 0$

$$x = \frac{y}{2} + 1 \Rightarrow u + v = v + 1 \Rightarrow u = 1$$

$$y = 0 \Rightarrow v = 0$$

$$y = 4 \Rightarrow v = 2$$

Here
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Thus
$$I = \int_{v=0}^{2} \int_{u=0}^{1} u\,|J|\,du\,dv$$

$$= \int_0^2 \int_0^1 2u\,du\,dv$$

$$= 2\int_0^2 \left[\frac{u^2}{2}\right]_0^1 dv = 2\int_0^2 \frac{1}{2}\,dv$$

$$= (2 - 0) = 2$$

14.2 : Area

Area in Cartesian Co-ordinate System

Let 'R' be the area enclosed by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, $x = a$ and $x = b$ as shown in the Fig. 14.1.

Area of rectangle PQRS = $\delta x\, \delta y$

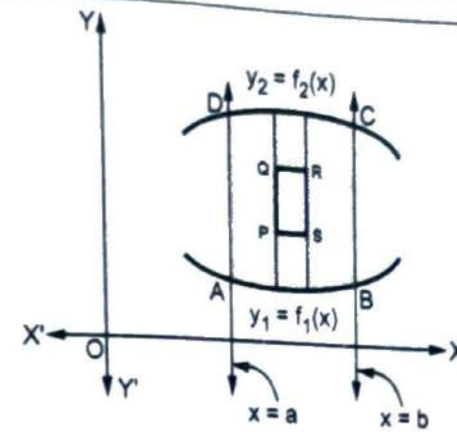


Fig. 14.1

Area of vertical strip $= \lim_{\delta x \to 0} \sum_{y_1}^{y_2} \delta x\, \delta y = \delta x \int_{y_1}^{y_2} dy$

Adding all such strips from x = a to x = b we get,

$$\text{Area ABCD} = \lim_{\delta x \to 0} \sum_a^b \delta x \int_{y_1}^{y_2} dy$$

$$= \int_a^b dx \int_{y_1}^{y_2} dy$$

OR
$$\iint_R dx\,dy = \int_{x=a}^{x=b} \int_{y_1 = f_1(x)}^{y_2 = f_2(x)} dx\,dy \quad \text{... (1)}$$

Similarly, if x having variable limits i.e.

$x = \phi_1(y)$ and $x = \phi_2(y)$, then,

$$\text{Area} = \int_{y=c}^{d} \int_{x_1 = \phi_1(y)}^{x_2 = \phi_2(y)} dy\,dx \quad \text{... (2)}$$

To be integrated w.r.t. x first over the limits $x_1 = \phi_1(y)$ to $x_2 = \phi_2(y)$

Note :

1) The area bounded by the curve $y = f(x)$, the X-axis and the lines $x = a$, $x = b$ is given by

$$\text{Area} = \int_{x=a}^{b} y\,dx = \int_{a}^{b} f(x)\,dx$$

2) The area bounded by the curve $x = \phi(y)$, Y-axis and the lines $y = c$, $y = d$, is given by,

$$\text{Area} = \int_{y=c}^{d} x\,dy = \int_{c}^{d} f(y)\,dy$$

Area in Polar Form

$$\text{Area} = \iint_R dx\,dxy, \text{ .... (Cartesian form)} \quad \text{... (3)}$$

Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$

Find corresponding limits for r and θ

$$\therefore \quad \boxed{\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \int_{r_1 = f_1(\theta)}^{r_2 = f_2(\theta)} r\,d\theta\,dr \quad \text{.....(Polar form)}}$$

Integral (2) to be evaluated w.r.t. r first over the limits $r = f_1(\theta)$ to $r = f_2(\theta)$ and then w.r.t. θ over the limits $\theta = \alpha$ to $\theta = \beta$.

Area in Parametric Form

If $x = f(t)$ and $y = g(t)$, $t \in [a,b]$ be the equation of curve in parametric form bounding a region of the Cartesian plane, then the area of this region is

$$\text{Area} = \int y\,dx = \int_{t=a}^{b} y(t)\,x'(t)\,dt$$

$$\text{Area} = \int x\,dy = \int_{a}^{b} x(t)\,y'(t)\,dt$$

Type I : Area by Cartesian curves

$$\boxed{\text{Area} = \iint dx\,dy}$$

Q.2 : Find the area lying between $y = x^2$ and $y = x$.

[SPPU : May-19, Marks 6]

Ans. : Given that $y = x^2$ and $y = x$

Now,
$$y = x^2$$
$$x = x^2$$
$$\Rightarrow \quad x^2 - x = 0$$
$$\Rightarrow \quad x(x-1) = 0$$
$$\Rightarrow \quad x = 0,\ x = 1$$

⇒ (0, 0) and (1, 1) are the points of intersection of given curves.

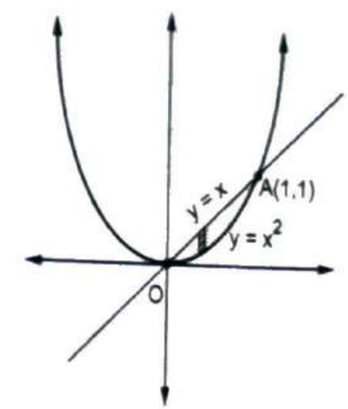


Fig. Q.2.1

The region of integral is OAO

Draw strip parallel to Y-axis

∴ Limits are $y = x^2$ to $y = x$

and, $x = 0$ to $x = 1$

$$\therefore \quad \text{Area} = \iint dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} dy\,dx$$

$$= \int_{x=0}^{1} [y]_{x^2}^{x}\,dx = \int_{0}^{1} [x - x^2]\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Q.3 : Find area between the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.

[SPPU : May-16, Marks 6]

Ans. :
$$\text{Area} = \iint_A dx\,dy \quad \text{... (1)}$$

Where A is area between $y^2 = 4x$, and $2x - 3y + 4 = 0$ as shown in the Fig. Q.3.1.

To find the point of intersection B and C.

$$y^2 = 4x,\ 2x - 3y + 4 = 0.$$

$$y^2 = 2(2x)$$

$$= 2(-4 + 3y)$$

$$\Rightarrow y^2 - 6y + 8 = 0$$

$$\Rightarrow \quad y = 2,\ y = 4$$

When $y = 2,\ x = 1$

$y = 4,\ x = 4,$

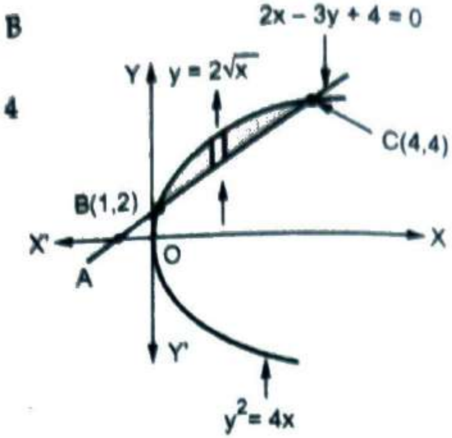


Fig. Q.3.1

$\therefore$ B (1, 2) and C(4, 4).

Limits over the strip $y = \frac{2x+4}{3}$ to $y = 2\sqrt{x}$ and moving the strip from $x = 1$ to $x = 4$.

We get shaded area required.

$$\therefore \quad \text{Area} = \int_{x=1}^{4} \left[\int_{y=\frac{2x+4}{3}}^{y=2\sqrt{x}} dy \right] dx$$

$$= \int_{1}^{4} \left[2\sqrt{x} - \left(\frac{2x+4}{3} \right) \right] dx$$

$$= \left[2\frac{2}{3} x^{3/2} - \frac{x^2 + 4x}{3} \right]_1^4$$

$$= \left[\frac{32}{3} - \frac{32}{3} \right] - \left[\frac{4}{5} - \frac{5}{3} \right]$$

$$\boxed{\text{Area} = \frac{1}{3}}$$

Q.4 : Find the area bounded by $y = 2 - x$ and $y^2 = 2x + 4$.

Ans. : Given that $y = 2 - x$ and $y^2 = 2x + 4$.

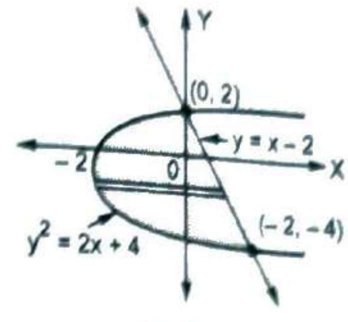


Fig. Q.4.1

Draw strip parallel to X-axis.

$\therefore$ Limits are

$\frac{y^2 - 4}{2} \le x \le 2 - y$ and $-4 \le y \le 2$.

$$\therefore \quad \text{Area} = \iint dx\, dy = \int_{-4}^{2} \int_{\frac{y^2-4}{2}}^{2-y} dx\, dy$$

$$= \int_{-4}^{2} [x]_{\frac{y^2-4}{2}}^{2-y}\, dy$$

$$= \int_{-4}^{2} \left[2 - y - \frac{y^2}{2} + 2 \right] dy$$

$$= \left[4y - \frac{y^2}{2} - \frac{y^3}{6} \right]_{-4}^{2}$$

$$\text{Area} = 18$$

Q.5 : Find area bounded by the curves $y^2 = 4\,ax$ and $x^2 = 4\,ay$.

Ans. : $$\text{Area} = \iint_A dx\, dy \quad \text{... (1)}$$

Where A ($y^2 = 4$ ax, $x^2 = 4$ ay) as shown in the Fig. Q.5.1.

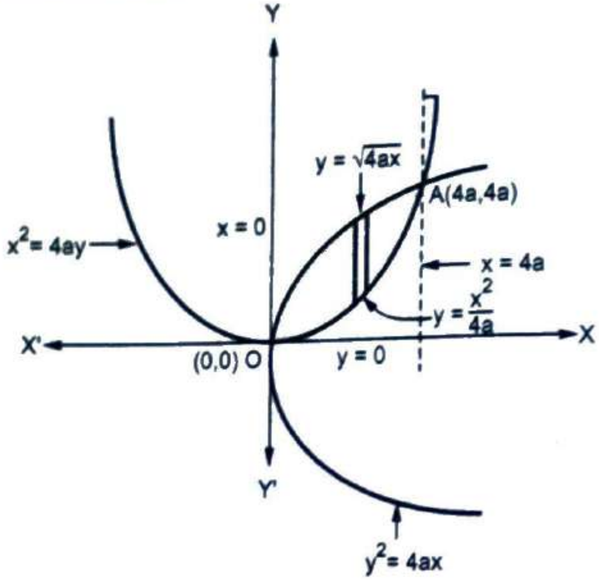


Fig. Q.5.1

At A, $$y^2 = 4\,ax,\ \frac{x^4}{16\,a^2} = y^2 = 4\,ax$$

$$\frac{x^4}{16\,a^2} - 4\,ax = 0 \Rightarrow x\left(x^3 - 64\,a^3\right) = 0$$

$$\Rightarrow \quad x = 0,\ x^3 = 64a^3$$

$$\Rightarrow \quad x = 0,\ x = 4a$$

$\therefore$ O (0, 0) and A (4a, 4a) are the point of intersection of two curves.

In the shaded area, over the strip y varies from

$$y = \frac{x^2}{4a} \text{ to } y = \sqrt{4\,ax}$$

and x varies from

$$x = 0 \text{ to } x = 4a \qquad \text{from (1)}$$

$$\therefore \qquad \text{Area} = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dx\,dy$$

$$= \int_0^{4a} (y)_{\frac{x^2}{4a}}^{\sqrt{4\,ax}} dx = \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[\sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\,\frac{x^3}{3}\right]_0^{4a}$$

$$= \left[\sqrt{4a}\,(4a)^{3/2}\,\frac{2}{3} - \frac{1}{4a \times 3}\,(4a)^3\right] = \frac{32\,a^2}{3} - \frac{16\,a^2}{3}$$

$$\boxed{\text{Area} = \frac{16\,a^2}{3}}$$

... Ans.

Q.6 : Find the area between the curve $y^2x = 4a^2(2a - x)$ and it's asymptote. **[SPPU : May-15, Marks 6]**

Ans. : x = 0 i.e. Y-axis an asymptote to the curve.

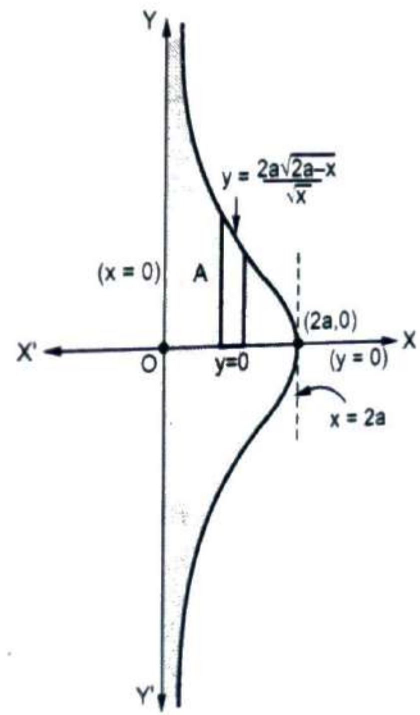


Fig. Q.6.1

Put $y = 0 \Rightarrow 2a - x = 0$

$$\Rightarrow x = 2a$$

(2a, 0) is the point where, the curve meet X-axis.

By symmetry,

$$\text{Total area} = 2\iint\limits_A dx\,dy$$

$$= 2 \times \text{Area of upper half}$$

$$= 2 \times \int_{x=0}^{2a} \int_{0}^{2a\sqrt{\frac{2a-x}{x}}} dx\,dy$$

$$= 2 \times \int_0^{2a} \sqrt{\frac{2a-x}{x}}\,dx,$$

Put $x = 2at$, $dx = 2a\,dt$

Limits :

x	0	2a
t	0	1

$$\therefore \quad \text{Area} = 2\int_0^1 (2a - 2at)^{1/2}(2at)^{-1/2}\,2a\,dt$$

$$= 8a^2 \int_0^1 t^{-1/2}(1-t)^{1/2}\,dt$$

$$= 8a^2 \beta\left(\frac{1}{2}, \frac{3}{2}\right) = 8a^2 \frac{\Gamma\frac{1}{2}\ \Gamma\frac{3}{2}}{\Gamma 2}$$

$$\boxed{\text{Area} = 4a^2\pi} \quad \text{... Ans.}$$

Q.7. : Show that the area of the curve $a^2x^2 = y^3(2a - y)$ is πa^2.

[SPPU : Dec.-13, Marks 6]

Ans. : Given that

$$a^2x^2 = y^3(2a-y)$$

- It is symmetric about y-axis.

- Passes through origin.
- It intersects y-axis at (0, 0) (0, 2a).
- Y-axis is the tangent to the curve at (0, 0).

The tangent at (0, 2a) is parallel to X-axis.

The required area is shown shaded i.e. OABCA.

The required area is symmetric about Y-axis

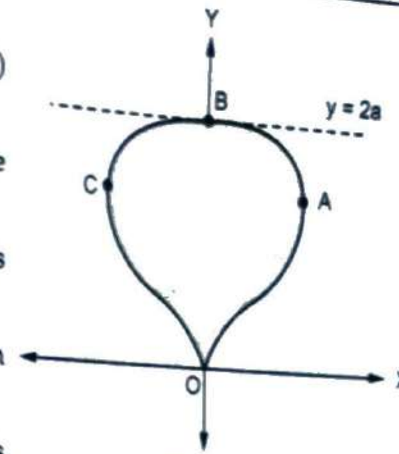


Fig. Q.7.1

$$\therefore \quad \text{Area} = 2 \text{ Area about Y-axis}$$

$$= 2\int_0^{2a} x\,dy = 2\int_0^{2a} \frac{y^{3/2}(2a-y)^{1/2}}{a}\,dy$$

$$= \frac{2}{a}\int_0^{2a} y^{3/2}(2a-y)^{1/2}\,dy$$

Put, $y = 2at \Rightarrow dy = 2a\,dt$

y	0	2a
t	0	1

$$\therefore \quad \text{Area} = \frac{2}{a}\int_0^1 (2at)^{3/2}(2a-2at)^{1/2}(2a)\,dt$$

$$= \frac{2}{a}(2a)^3 \int_0^1 t^{3/2}(1-t)^{1/2}\,dt$$

$$= 16a^2 \beta\left(\frac{5}{2}, \frac{3}{2}\right) = 16a^2 \frac{\frac{3}{2}\left|\frac{3}{2}\right.\ \frac{1}{2}\left|\frac{1}{2}\right.}{\left|4\right.}$$

$$= 16a^2 \frac{\frac{3}{2}\,\frac{1}{2}\,\Gamma\!\left(\frac{1}{2}\right)\,\frac{1}{2}\,\Gamma\!\left(\frac{1}{2}\right)}{3!} = 16a^2\,\frac{3}{8\times 6}\sqrt{\pi}\sqrt{\pi}$$

$$\text{Area} = \pi a^2$$

Type II : Area by Polar form

Q.8 : Find area inside the cardiode $r = 2a\,(1 + \cos\theta)$ and outside the parabola $r = \dfrac{2a}{1 + \cos\theta}$.

Ans. : $\text{Area} = 2\iint\limits_A r\, d\theta\, dr$

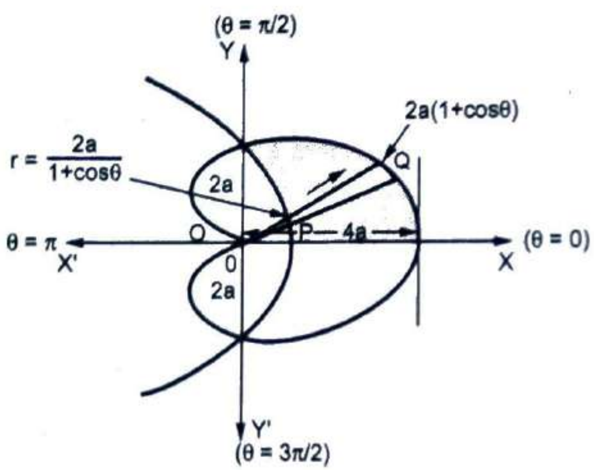


Fig. Q.8.1

$= 2 \times$ Area of the upper half.

$$= 2\int_{\theta=0}^{\pi/2}\int_{\frac{2a}{1+\cos\theta}}^{2a(1+\cos\theta)} r\, d\theta\, dr$$

$$= 2\int_{\theta}^{\pi/2}\left(\frac{r^2}{2}\right)_{\frac{2a}{1+\cos\theta}}^{2a(1+\cos\theta)} d\theta\; dr$$

$$= \int_{\theta}^{\pi/2}\left[4a^2(1+\cos\theta)^2 - \frac{4a^2}{(1+\cos\theta)^2}\right] d\theta$$

$$= 4a^2\int_0^{\pi/2}\left[1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right] d\theta$$

$$- 4a^2\int_0^{\pi/2}\frac{d\theta}{\left(2\cos\frac{\theta}{2}\right)^2}$$

$$= 4a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$- \frac{4a^2}{4}\int_0^{\pi/2}\sec^4\frac{\theta}{2}\, d\theta$$

$$= 4a^2\left[\frac{3\pi}{4} + 2\right] - a^2\int_0^{\pi/2}\left(1 + \tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2}\, d\theta$$

Put $\tan\frac{\theta}{2} = t$ $\quad\because \frac{1}{2}\sec^2\frac{\theta}{2}\, d\theta = dt$

Limits :

0	0	$\pi/2$
t	1	1

$$= 4a^2\left[\frac{3\pi}{2} + 2\right] - a^2\int_0^1\left(1+t^2\right) 2\, dt$$

$$= 4a^2\left[\frac{3\pi}{2} + 2\right] - \frac{8a^2}{3}$$

$$= 4a^2\left[\frac{3\pi}{2}+2-\frac{2}{3}\right]$$

$$\boxed{\text{Area } = a^2\left(4\pi+\frac{16}{3}\right)} \quad \text{...Ans.}$$

Q.9 : Find area of the cardiode $r = a(1 + \cos\theta)$.

Ans. : $\text{Area } = \iint_A dx\,dy = \iint_A r\,d\theta\,dr \quad \text{... (1)}$

Where A is area of the cardiode as shown in the Fig. Q.9.1 from equation (1),

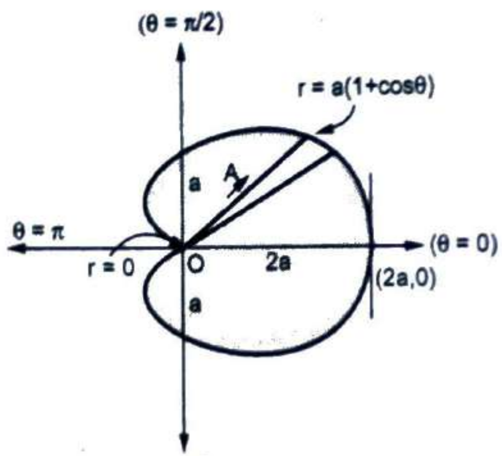


Fig. Q.9.1

$$\text{Area } = 2 \times \text{Area of the upper half}$$

$$= 2\iint_A r\,d\theta\,dr$$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,d\theta\,dr$$

$$= 2\int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right]d\theta$$

$$= a^2\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$\boxed{\text{Area } = a^2\left[\frac{3\pi}{2}+0\right] = \frac{3\pi a^2}{2}} \quad \text{... Ans.}$$

Q.10 : Find the total area included between the two cardioides $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$.

[SPPU : Dec.-14, 18, May-17, Marks 6]

Ans. : Given that $r = a(1+\cos\theta)$, $r = a(1-\cos\theta)$

The required area is shown in the Fig. Q.10.1.

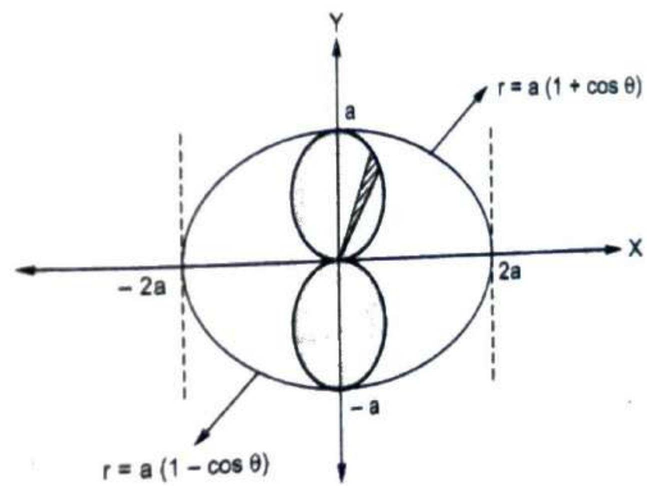


Fig. Q.10.1

The region is symmetric about X and Y axes.

$\therefore$ A = The required area

= $4\times$ Area in the first quadrant

$$A = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta = 4\int_{0}^{\pi/2}\left[\frac{x^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2} a^2(1-\cos\theta)^2\,d\theta = 2a^2\int_0^{\pi/2}[1-2\cos\theta+\cos^2\theta]\,d\theta$$

$$= 2a^2\left\{\frac{\pi}{2}-2(1)+\frac{1}{2}\frac{\pi}{2}\right\} = 2a^2\left(\frac{3\pi}{4}-2\right)$$

Q.11 : Find area inside the circle $r = a\sin\theta$ and outside the cardiod $r = a(1-\cos\theta)$. **[SPPU : Dec.-17, Marks 6]**

Ans. : The area $= \iint_A r\,d\theta\,dr$ ... (1)

Where A is Area outside $r = a(1-\cos\theta)$ and inside

$$r = a\sin\theta$$

or

$$r^2 = ar\sin\theta$$

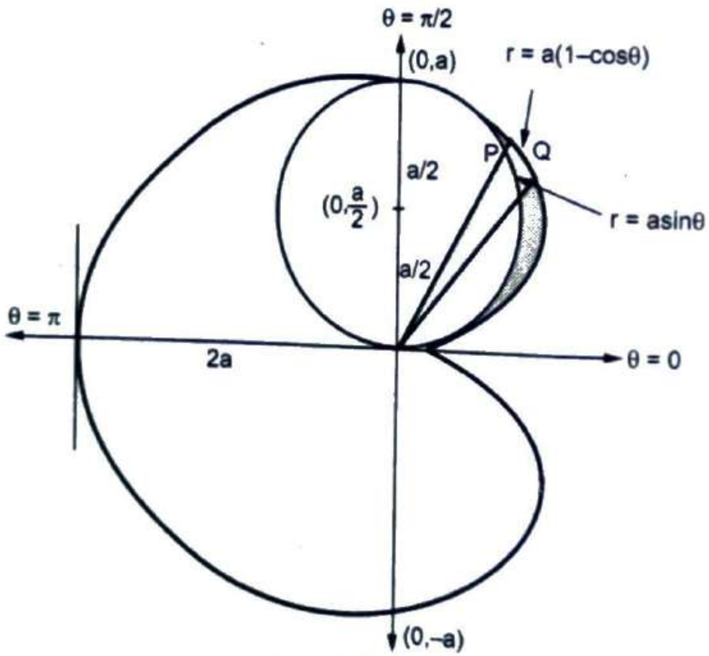


Fig. Q.11.1

$$x^2 + y^2 = ay$$

$$\Rightarrow \quad x^2 + \left(y - \frac{a^2}{2}\right) = \left(\frac{a}{2}\right)^2 \text{ is a circle.}$$

with centre $= \left(0, \frac{a}{2}\right)$ and Radius $= \frac{a}{2}$

From equation (1)

$$\text{Area} = \iint r\,d\theta\,dr = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,d\theta\,dr$$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{\pi}{2} + 2(1) - \frac{1}{2}\frac{\pi}{2}\right]$$

$$\boxed{\text{Area} = a^2\left(1-\frac{\pi}{4}\right)} \quad \text{... Ans.}$$

Type III : Area by Parametric form

Q.12 : Find area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ **[SPPU : Dec.-16, Marks 6]**

Ans. : The parametric equation for astroid are

$$x = a\cos^3\theta,\ y = a\sin^3\theta$$

The curve is symmetrical about X and Y axis as shown in the Fig. Q.12.1.

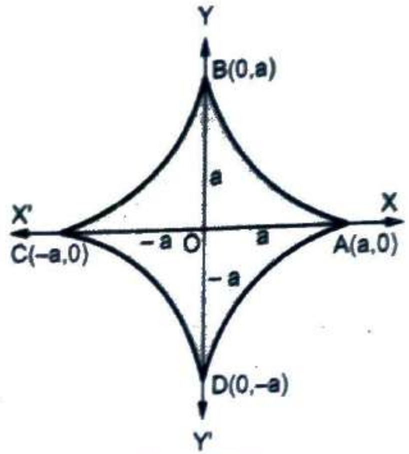


Fig. Q.12.1

$$\therefore \quad \text{Total area} = 4 \times \text{Area in first quadrant.}$$

$$= 4 \iint dx\,dy$$

$$= 4 \int x dy$$

$$= 4 \int_0^{\pi/2} x \frac{dy}{d\theta} d\theta \qquad \dots (1)$$

But $\quad x = a \cos^3\theta, \; y = a \sin^3\theta$

$$\therefore \quad \frac{dy}{d\theta} = 3a \sin^2\theta \cos\theta \; d\theta \qquad \text{from (1)}$$

$$\text{Total area} = 4 \int_0^{\pi/2} a \cos^3\theta \; 3a \sin^2\theta \cos\theta \; d\theta$$

$$= 12a^2 \int_0^{\pi/2} \sin^2\theta \cos^4\theta \; d\theta$$

$$= 12a^2 \frac{(2-1)\left[(4-1)(4-3)\right]}{6(6-2)(6-4)} \frac{\pi}{2}$$

$$\boxed{\text{Area} = \frac{3\pi a^2}{8}} \qquad \dots \textbf{Ans.}$$

Q.13 : Find the mass of a lamina in the form of the cardiode $r = a(1 + \cos\theta)$ whose density at any point varies as the square of it's distance from the initial line.

Ans. :

$$\text{Mass} = \iint_R F(r,\theta)\, r \, d\theta \, dr \qquad \dots (1)$$

Where R is cardiode $r = a(1 + \cos\theta)$ as shown in the Fig. Q.13.1.

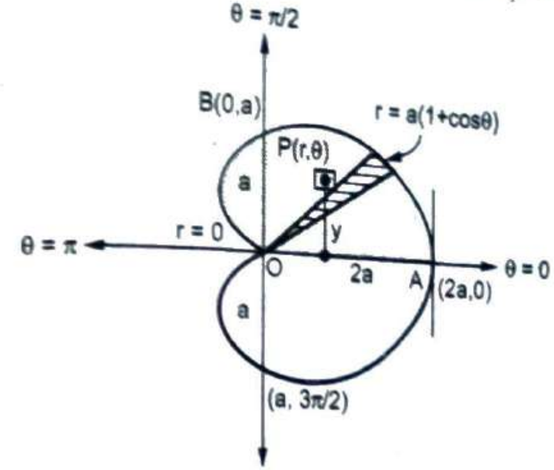


Fig. Q.13.1

The distance of any point P (r, θ) from the initial line is $r \sin\theta$.

$$\therefore \quad \text{Density} = \rho = k(r \sin\theta)^2 = kr^2 \sin^2\theta$$

By symmetry,

$$\text{Mass} = 2 \int_{\theta=0}^{\pi} \int_0^{a(1+\cos\theta)} \rho\, r \, d\theta \, dr$$

$$= 2 \int_0^{\pi} \int_0^{a(1+\cos\theta)} k\, r^2 \sin^2\theta \; r \, d\theta \, dr$$

$$= 2k \int_0^{\pi} \left(\int_0^{a(1+\cos\theta)} r^3 dr \right) \sin^2\theta \; d\theta$$

$$= 2k \int_0^{\pi} \left(\frac{r^4}{4} \right)_0^{a(1+\cos\theta)} \sin^2\theta \; d\theta$$

$$= \frac{k}{2}\int_0^{\pi} a^4 (1+\cos\theta)^4 \sin^2\theta \, d\theta$$

$$= \frac{ka^4}{2}\int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^4 \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 d\theta$$

Put, $\frac{\theta}{2} = u$

$$= 64\,k\,a^4 \int_0^{\pi/2} \sin^2 u \cos^{10} u (2du) \quad \therefore d\theta = 2\,du$$

$$= 64\,k\,a^4 \frac{(1)(9\cdot7\cdot5\cdot3\cdot1)}{12\cdot10\cdot8\cdot6\cdot4\cdot2}\frac{\pi}{2}$$

$$\boxed{\text{Mass} = \frac{21\,k\,\pi\,a^4}{32}} \quad \text{... Ans.}$$

14.3 : Volume of Solid Region by Double Integrals

We know that when a surface $z = f(x, y)$ is continuous on given region R and

$f(x, y) \geq 0 \; \forall \; (x, y) \in R$, then

$\iint_R f(x, y)\, dA$ is the volume of the solid between the surface

$z = f(x, y)$ and region R in xy-plane.

Moreover, when the solid region lies between the two surfaces $z_1 = f_1(x, y)$ and $z_2 = f_2(x, y)$ where $z_1 = f_1(x, y)$ is the lower surface and $z_2 = f_2(x, y)$ is the upper surface. Then

$$V = \iint_R [f_2(x, y) - f_1(x, y)]\, dA$$

In polar form, volume can be obtained by the integral

$$V = \iint_R z\, r\, dr\, d\theta.$$

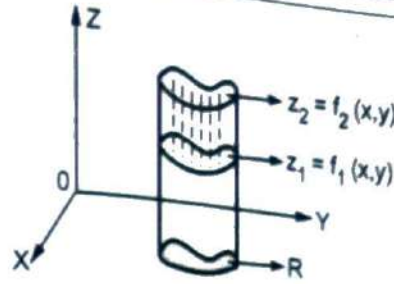


Fig. 14.2

Volume of Solids by Triple Integrals

1) The volume of solid by triple integration is given by

$$\text{Volume} = V = \iiint_V dv = \iiint_V dx\, dy\, dz \quad \text{... (4)}$$

2) In spherical polar co-ordinates

$$V = \iiint_V r^2 \sin\theta\, dr\, d\theta\, d\phi \quad \text{... (5)}$$

3) In cylindrical polar co-ordinates

$$V = \iiint_V \rho\, d\rho\, d\phi\, dz \quad \text{... (6)}$$

Q.14 : Find the volume bounded by the cylinder $x^2 + y^2 = 4$ between $y + z = 3$ and $z = 0$.

Ans. : Given that $x^2 + y^2 = 4$ and $y + z = 3$ and $z = 0$. The projection of solid region V is a circle on $z = 0$ plane.

Here $z = 3 - y$, $x^2 + y^2 = 4$

$\Rightarrow r = 2.$

Limits are in polar form

$0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

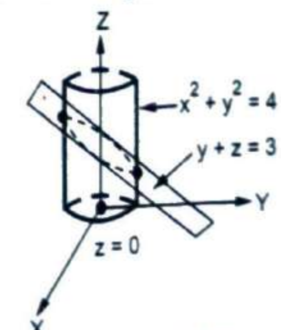


Fig. Q.14.1 (a)

Thus the required volume = V.

$$V = \iint z\, r\, dr\, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (3 - r \sin\theta)\, r dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^2}{2} - \frac{r^3}{3} \sin\theta \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left[6 - \frac{8}{3} \sin\theta \right] d\theta$$

$$= \left[6\theta + \frac{8}{3} \cos\theta \right]_0^{2\pi}$$

$$= 12\pi + \frac{8}{3} - \frac{8}{3} = 12\pi$$

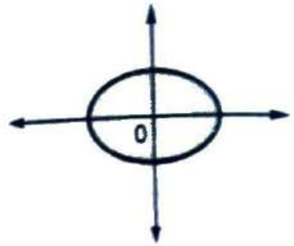


Fig. Q.14.1 (b) Projection on XY plane

Q.15 : Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$ in the XY-plane.

Ans. : Given that $z = x^2 + y^2$, $y = x$, $x = 0$ and $x + y = 2$.

The required volume $V = \iint_R z\, dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x}^{2-x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx$$

$$= \int_0^1 \left[x^2 (2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \left[\frac{-8x^3}{3} + 4x^2 - 4x + \frac{8}{3} \right] dx$$

$$= \left[-\frac{8}{3} \frac{x^4}{4} + \frac{4x^3}{3} - \frac{4x^2}{2} + \frac{8}{3} x \right]_0^1$$

$$= -\frac{8}{3} + \frac{4}{3} - \frac{4}{2} + \frac{8}{3} = \frac{4}{3}$$

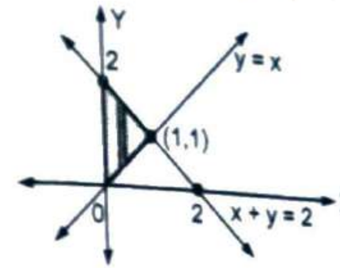


Fig. Q.15.1

Q.16 : Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. [SPPU : Dec.-18, May-18, Marks 6]

Ans. : From Fig. Q.16.1, the upper surface is the surface of $x^2 + z^2 = a^2$. Volume is same in all octants

∴ We find volume only in first quadrant.

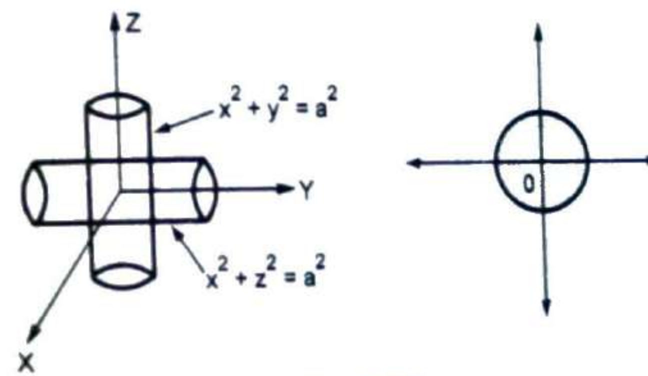


Fig. Q.16.1

∴ The limits are : $0 \le y \le \sqrt{a^2 - x^2}$

$0 \le x \le a.$

∴ $$V = 8 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2}\, dy\, dx$$

$$= 8\int_0^a [\sqrt{a^2 - x^2}]\,[y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= 8\int_0^a (a^2 - x^2)\,dx = 8\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$V = 8\left[a^3 - \frac{a^3}{3}\right] = 8\left(\frac{2a^3}{3}\right) = \frac{16\,a^3}{3}$$

Q.17 : Find volume of the tetrahedron bounded by the co-ordinates planes and the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$. **[SPPU : May-14,18, Dec.-14, Marks 6]**

Ans. : Volume $= \iiint dx\,dy\,dz$ ... (1)

Put $\frac{x}{2} = u,\ \frac{y}{3} = v,\ \frac{z}{4} = w$

$\Rightarrow$ $x = 2u,\ y = 3v,\ z = 4w$ from equation (1)

$$V = \iiint 24\,du\,dv\,dw$$

$$= 24\iiint du\,dv\,dw$$

$$= 24\iiint u^{1-1}v^{1-1}w^{1-1}\,du\,dv\,dw,\ (u + v + w = 1)$$

$$= 24\,\frac{\lceil 1 \cdot \lceil 1 \cdot \lceil 1}{\lceil 1+1+1+1}$$

$$= \frac{24}{4} = \frac{24}{3!}$$

$$\boxed{\text{Volume} = \frac{24}{6} = 4}$$ ... Ans.

Q.18 : Find the volume enclosed by the cone $x^2 + y^2 = z^2$ and the paraboloid $x^2 + y^2 = z$. **[SPPU : Dec.-17, Marks 6]**

Ans. :
$$V = \iint \int_{z = x^2+y^2}^{\sqrt{x^2+y^2}} dx\,dy\,dz$$

$$= \iint\left(\int_{z = x^2+y^2}^{\sqrt{x^2+y^2}} dz\right) dx\,dy$$

$$V = \iint\left[\sqrt{x^2+y^2} - (x^2+y^2)\right] dx\,dy \quad \text{... (1)}$$

The intersection of the cone and the paraboloid is

$$\sqrt{x^2+y^2} = x^2 + y^2 \text{ i.e.}$$

$$x^2 + y^2 = 1$$

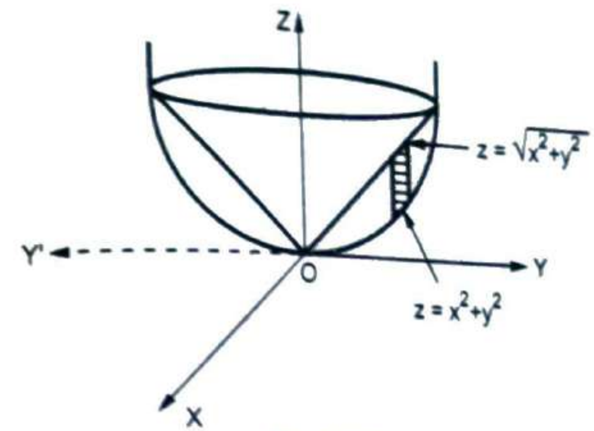


Fig. Q.18.1

Changing equation (1) into polar co-ordinates by

Putting

$$x = r\cos\theta$$

$$y = r\sin\theta,$$

$$dx\,dy = r\,d\theta\,dr$$

$$V = 4\int_0^{\pi/2}\int_0^1 (r - r^2)\,r\,d\theta\,dr$$

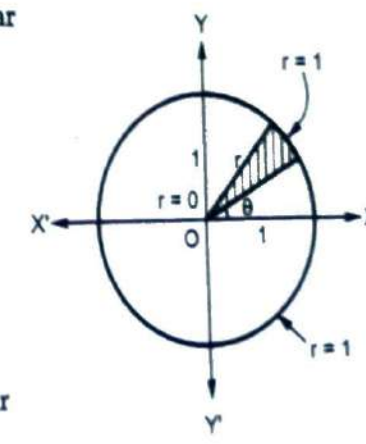


Fig. Q.18.2

$$= 4\int_0^{\pi/2}\left[\int_0^1 \left(r^2 - r^3\right) dr\right] d\theta = 4\int_0^{\pi/2}\left(\frac{r^3}{3} - \frac{r^4}{4}\right)_0^1 d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{1}{3} - \frac{1}{4}\right) d\theta = 4\int_0^{\pi/2}\left(\frac{1}{12}\right) d\theta = 4\cdot\frac{1}{12}\cdot\frac{\pi}{2}$$

$$\boxed{\text{Volume} = \frac{\pi}{6}}$$ ... **Ans.**

Q.19 : Find volume of the region bounded by paraboloid $x^2 + y^2 = 2z$ and the cylinder $x^2 + y^2 = 4$. [SPPU : Dec.-13,18, Marks 6]

Ans. : By using cylindrical polar co-ordinates

$$x = \rho \cos\theta, \; y = \rho \sin\theta, \; z = z$$

and $x^2 + y^2 = \rho^2$

$$dx\, dy\, dz = \rho\, d\rho\, d\phi\, dz$$

$\therefore$ $$V = 4\iiint dx\, dy\, dz = 4\iiint \rho\, d\rho\, d\phi\, dz$$

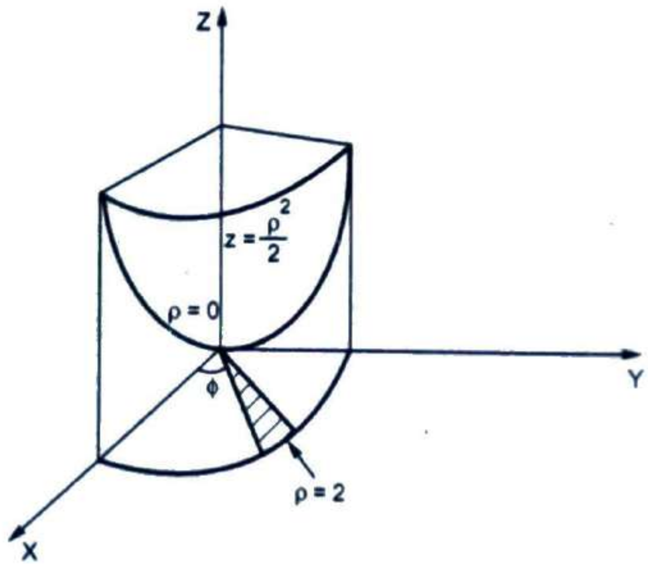


Fig. Q.19.1

$$= 4\int_0^{\pi/2} d\phi \int_0^2 \rho\left(\int_{z=0}^{\rho^2/2} dz\right) d\rho$$

$$= 4 \times \frac{\pi}{2} \times \int_0^2 \frac{\rho^3}{2}\, d\rho$$

$$= 4\cdot\frac{\pi}{2}\cdot\frac{1}{2}\left(\frac{e^4}{4}\right)_0^2 = \pi\,\frac{(2)^4}{4}$$

$$\boxed{\text{Volume} = 4\pi}$$

... **Ans.**

Q.20 : Find the volume common to the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$. [SPPU : May-16, Dec.-16, Marks 5]

Ans. : For given cylinders,

$$x^2 + y^2 = a^2, \; x^2 + z^2 = a^2.$$

z varies from

$$z = -\sqrt{a^2 - x^2} \text{ to } z = \sqrt{a^2 - x^2}$$

y varies from

$$y = -\sqrt{a^2 - x^2} \text{ to } y = \sqrt{a^2 - x^2}$$

x varies from $x = -a$ to $x = a$

By symmetry,

Required volume = 8 times the volume in the first octant

$$= 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2}} dx\, dy\, dz$$

$$= 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} \left(\sqrt{a^2 - x^2}\right) dx\, dy$$

$$= 8\int_0^a \sqrt{a^2 - x^2}\cdot(y)_0^{\sqrt{a^2 - x^2}}\, dx$$

$$= 8\int_0^a \left(a^2 - x^2\right) dx = 8\left(a^2 x - \frac{x^3}{3}\right)_0^a$$

$$= 8\left(a^3 - \frac{a^3}{3}\right)$$

$$\boxed{\text{Volume} = 16\,\frac{a^3}{3}}$$

... Ans.

Q.21 : Find volume of the cylinder $x^2 + y^2 = 2ax$, intercepted between paraboloid $x^2 + y^2 = 2az$ and the XY - plane.

[SPPU : May-13, Marks 6]

Ans. :

$$\text{Volume} = \iint \int_0^{\frac{x^2+y^2}{2a}} dx\,dy\,dz$$

$$= \iint_R \frac{x^2 + y^2}{2a}\,dx\,dy$$

Where R is $x^2 + y^2 = 2ax$ as shown in the Fig. Q.21.1.

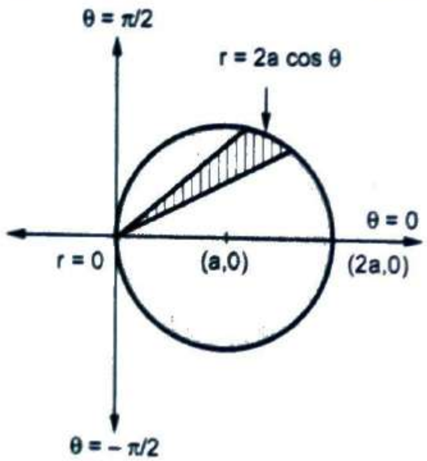


Fig. Q.21.1

$$\text{Volume} = \frac{1}{2a}\,2\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\, r\, d\theta\, dr$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta = \frac{1}{4a}\int_0^{\pi/2}(2a\cos\theta)^4\, d\theta$$

$$\boxed{\text{Volume} = \frac{3\pi a^3}{4}}$$

... Ans.

Q.22 : Find the volume cut off from the paraboloid $x^2 + \frac{y^2}{4} + z = 1$ by the plane $z = 0$.

[SPPU : May-17, Marks 6]

Ans. :

$$V = \text{Required volume} = \iint \int_0^{1-x^2-\frac{y^2}{4}} dz\,dx\,dy$$

$$V = \iint\left(1 - x^2 - \frac{y^2}{4}\right)dx\,dy$$

The projection of the paraboloid on XY plane is $x^2 + \frac{y^2}{4} = 1$ which is an ellipse.

$\therefore$ Put $x = a\,r\cos\theta = r\cos\theta$,

$$y = b\,r\sin\theta = 2\,r\sin\theta$$

$$dxdy = abr\,dr\,d\theta = 2\,r\,dr\,d\theta,$$

$$x^2 + \frac{y^2}{4} = r^2$$

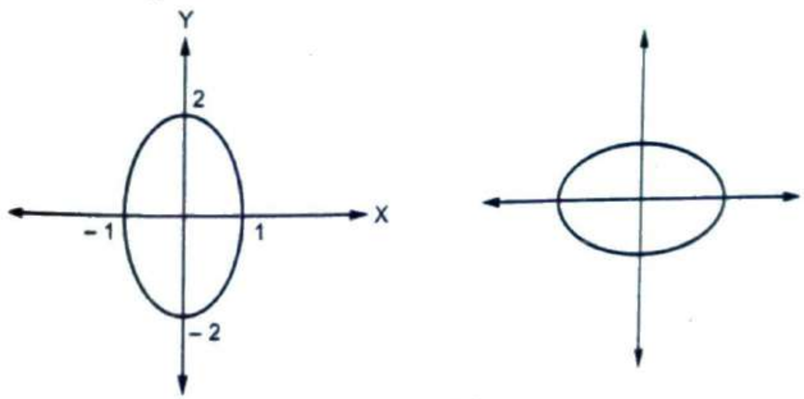


Fig. Q.22.1

It is symmetric about x and y axes. It is transformed to the unit circle.

$\therefore$ Limits are $r = 0$ to $r = 1$, $\theta = 0$, to $\theta = \pi/2$.

$$\therefore \quad V = 4\iint_R \left(1-x^2-\frac{y^2}{4}\right) dx\,dy$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} [1-r^2]2r\, dr\, d\theta$$

$$= 8\int_0^{\pi/2}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta$$

$$= 8\int_0^{\pi/2}\left[\frac{1}{2}-\frac{1}{4}\right] d\theta = 8\left(\frac{1}{4}\right)(\theta)_0^{\pi/2}$$

$$V = 2\left(\frac{\pi}{2}\right) = \pi$$

Q.23 : Find the volume of the paraboloid $x^2+y^2 = 4z$ cut-off by the plane $z = 4$. **[SPPU : May-19, Marks 6]**

Ans. :

$$\text{Required volume} = \iiint dz\, dx\, dy$$

$$V = \iint \int_{z=\frac{x^2+y^2}{4}}^{4} dz\, dx\, dy$$

$$= \iint\left(4-\frac{x^2+y^2}{4}\right) dx\,dy$$

This is transformed to

$$x^2+y^2 = 4\ (4)$$

$$x^2+y^2 = 16$$

$\therefore$ Put $\quad x = r\cos\theta$

$$y = r\sin\theta$$

$$x^2+y^2 = r^2$$

and $0 \le r \le 4$ and $0 \le \theta \le \pi/2$.

$$V = 4\int_0^{\pi/2}\int_0^4\left(4-\frac{r^2}{4}\right) r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left[4\frac{r^2}{2}-\frac{r^4}{4.4}\right]_0^4 d\theta$$

$$= 4\int_0^{\pi/2}[32-16]\, d\theta$$

$$V = 64\,[\theta]_0^{\pi/2} = 64\left(\frac{\pi}{2}\right) = 32\,\pi$$

Q.24 : Find the volume bounded by the cylinder $x^2+y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Ans. : For cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$

$\therefore y = \pm\sqrt{4-x^2}$ and $z = 4 - y$, Given $z = 0$

z varies from $\quad z = 0$ to $z = 4 - y$

y varies from $\quad y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$

x varies from $\quad x = -2$ to $x = 2$

Required volume $= V = \iiint dx\, dy\, dz$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-y} dx\, dy\, dz$$

$$= \int_{-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dx\, dy$$

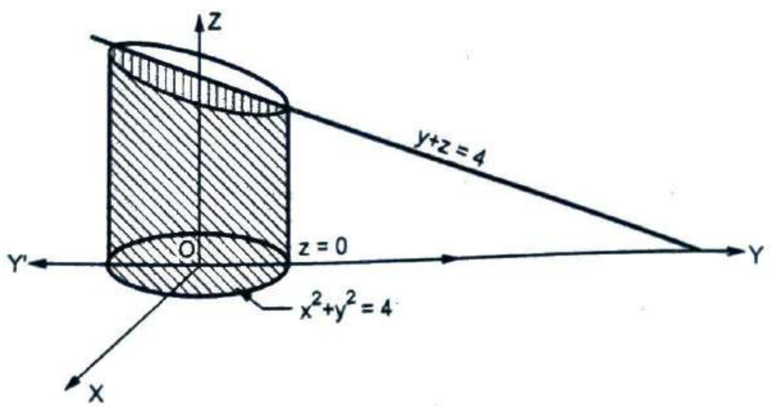


Fig. Q.24.1

$$= \int_{-2}^{2} \left(4y - \frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 8 \cdot \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$$

$$\boxed{\text{Volume} = 16\pi}$$ ... **Ans.**

14.4 : Centre of Gravity

The centre of gravity (CG) of an object is the point at which weight is evenly dispersed and all sides are in balance.

A human's centre of gravity can change as he takes different positions, but in many other objects, it's a fixed location.

Consider the following experiment.

Attempt to balance the pencil on the edge you have selected.

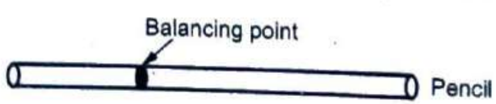


Fig. 14.3

Once the pencil is balanced, mark the location of the balancing point. Now measure the distance between the ends of the pencil and the balancing point. Are two lengths equal ?

For the above pencil two lengths are not equal. The balancing point is the centre of the gravity of this pencil. If we cut the pencil at the balancing point, the lengths of these parts are not equal but their weight are equal.

The centre of gravity is an important concept in determining the stability of a structure.

Centre of Gravity for Different Curves

If $m_1, m_2 \ldots m_n$ are the point masses situated at the points $(x_1, y_1, z_1), (x_2, y_2, z_2) \ldots\ldots (x_n, y_n, z_n)$ respectively and $(\bar{x}, \bar{y}, \bar{z})$ are the co-ordinates of centre of gravity of the system, then

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}; \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}; \bar{z} = \frac{\sum_{i=1}^{n} m_1 z_i}{\sum_{i=1}^{n} m} \quad \ldots (7)$$

Where $m_1 + m_2 + \ldots + m_n = \sum_{i=1}^{n} m_i$ and

$m_1 x_1 + m_2 x_2 + \ldots . m_n x_n = \sum_{i=1}^{n} m_i x_i \ldots$ etc.

Instead of discrete masses, if the mass distribution is contineous (i.e. rigid body) then from equation (7)

$$\bar{x} = \frac{\int x\, dm}{\int dm}; \bar{y} = \frac{\int y\, dm}{\int dm}; \bar{z} = \frac{\int z\, dm}{\int dm} \quad \ldots (8)$$

Where 'dm' is an element of the distributed mass of the body. $(\bar{x}, \bar{y}, \bar{z})$ may be considered as C.G. of mass distribution.

A) Centre of Gravity of an Arc :

Let the mass be distributed in the form of curve $y = f(x)$, 'ds' be an elementary arc at the point P(x, y).

If ρ is density at the point P(x, y) then mass of this element is

$$dm = \rho\, ds \qquad \text{... (9)}$$

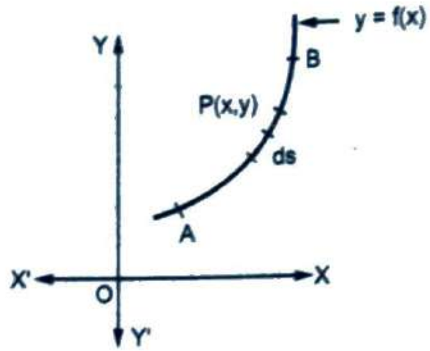


Fig. 14.4 (a)

If $(\bar{x}, \bar{y})$ be centre of gravity of arc AB, then

$$\bar{x} = \frac{\int x\, dm}{\int dm}, \quad \bar{y} = \frac{\int y\, dm}{\int dm}$$

Or

$$\boxed{\bar{x} = \frac{\int x\, \rho ds}{\int \rho\, ds}, \quad \bar{y} = \frac{\int y\, \rho ds}{\int \rho ds}} \qquad (\because \text{from (9)})$$

If ρ is constant then

$$\boxed{\bar{x} = \frac{\int x\, ds}{\int ds}, \quad \bar{y} = \frac{\int y\, ds}{\int ds}} \qquad \text{.... (10)}$$

Note :

1) If $y = f(x)$ then $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$

2) If $x = f(y)$ then $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$

3) If $x = f_1(t)$, $y = f_2(t)$ then

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

4) If $r = f(\theta)$ then $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$

5) If $\theta = f(r)$ then $ds = \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} \cdot dr$

B) Centre of Geravity of Plane Lamina :

Let $(\bar{x}, \bar{y})$ be the co-ordinates of C.G. of plane lamina bounded by the curve C and 'ρ' is density at the point P(x, y), then

$$dm = \rho\, dA \qquad \text{... (11)}$$

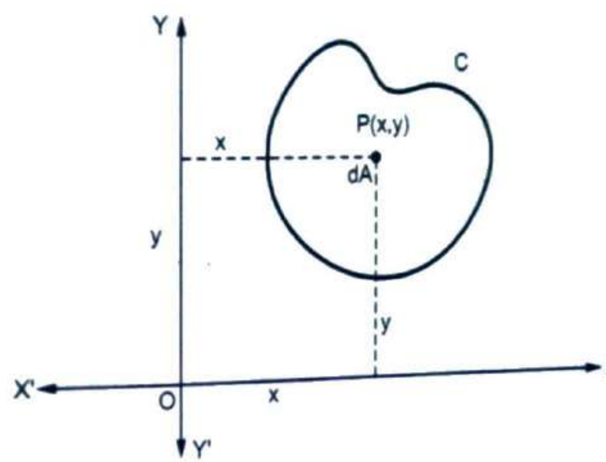


Fig. 14.4 (b)

and $\bar{x} = \dfrac{\int x\ dm}{\int dm}$, $\bar{y} = \dfrac{\int y\ dm}{\int dm}$, $(dA = dx\ dy)$

or
$$\boxed{\begin{aligned}\bar{x} &= \frac{\iint_R x\rho\ dx\ dy}{\iint_R \rho\ dx\ dy},\\ \bar{y} &= \frac{\iint_R y\rho\ dx\ dy}{\iint_R \rho\ dx\ dy}\end{aligned}} \quad \text{... (from equation (11))}$$

If ρ is constant then

$$\bar{x} = \frac{\iint_R x\ dx\ dy}{\iint_R dx\ dy} \quad \text{... (12)}$$

$$\bar{y} = \frac{\iint_R y\ dx\ dy}{\iint_R dx\ dy} \quad \text{... (13)}$$

Where R is region bounded by the curve C or lamina.

Note :

For the polar curve,

Put $dA = r\ d\theta\ dr$, $x = r \cos\theta$, $y = r \sin\theta$ in equation (12) and (13), we get,

$$\bar{x} = \frac{\iint_R r \cos\theta\ \rho\ r\ d\theta\ dr}{\iint_R \rho\ r\ d\theta\ dr};$$

$$\bar{y} = \frac{\iint_R r \sin\theta\ \rho\ r\ d\theta\ dr}{\iint_R \rho\ r\ d\theta\ dr}$$

If ρ is constant, then

$$\bar{x} = \frac{\iint_R r \cos\theta\ r\ d\theta\ dr}{\iint_R r\ d\theta\ dr};$$

$$\bar{y} = \frac{\iint_R r \sin\theta\ r\ d\theta\ dr}{\iint_R r\ d\theta\ dr}$$

C) Centre of Gravity of a solid :

If the $(\bar{x}, \bar{y}, \bar{z})$ be co-ordinates of C.G. of the solid which encloses volume V.

ρ is density at the point P(x, y, z) then

$$dm = \rho\ dv = \rho\ dx\ dy\ dz \quad \text{... (14)}$$

$$\therefore \quad \bar{x} = \frac{\iiint_V x\ \rho\ dx\ dy\ dz}{\iiint_V \rho\ dx\ dy\ dz},$$

$$\bar{y} = \frac{\iiint_V y\ \rho\ dx\ dy\ dz}{\iiint_V \rho\ dx\ dy\ dz} \quad \text{... (15)}$$

$$\text{and} \quad \bar{z} = \frac{\iiint_V z\rho\ dx\ dy\ dz}{\iiint_V \rho\ dx\ dy\ dz} \quad \text{... (16)}$$

Note :

For transforming into spherical polar and cylindrical polar co-ordinates.

Put $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$

and $x = u \cos\phi$, $y = u \sin\phi$, $z = z$ respectively in equation (15) and (16)

Type I : Examples on Centre of Gravity of an Arc

Q.25 : Find the C.G. of the arc of the cardiode $r = a(1 + \cos\theta)$ lying above the initial line $\theta = 0$. [SPPU : Dec.-11,17, May-17, Marks 6]

Ans. : Let $(\bar{x}, \bar{y})$ be co-ordinates of C.G.

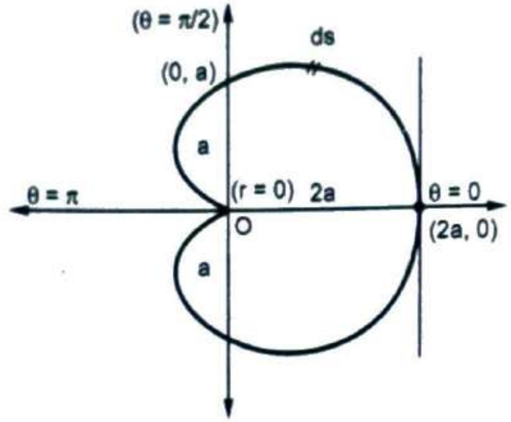


Fig. Q.25.1

Where $$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds}$$

Now $$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$r = a(1 + \cos\theta), \frac{dr}{d\theta} = -a\sin\theta$$

∴ $$ds = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= 2a\cos\frac{\theta}{2}\,d\theta$$

and $$x = r\cos\theta = a(1+\cos\theta)\cos\theta$$

∴ $$\int x\,ds = \int_0^{\pi} a(1+\cos\theta)\cos\theta\; 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 2a^2\int_0^{\pi} 2\cos^2\frac{\theta}{2}\cos\theta\cos\frac{\theta}{2}\,d\theta$$

$$= 4a^2\int_0^{\pi}\cos^3\frac{\theta}{2}\left[2\cos^2\frac{\theta}{2} - 1\right]d\theta,$$

Put $\frac{\theta}{2} = t$

∴ $$\int x\,ds = 4a^2\left[4\int_0^{\pi/2}\cos^5 t\,dt - 2\int_0^{\pi/2}\cos^3 t\,dt\right]$$

$$\int x\,ds = 4a^2\left[4\,\frac{4\cdot 2\cdot 1}{5\cdot 3} - 2\cdot\frac{2}{3}\right] = \frac{16a^2}{5}$$

and $$\int ds = \int_0^{\pi} 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 2a\cdot 2\left(\sin\frac{\theta}{2}\right)_0^{\pi}$$

$$= 4a$$

∴ $$\bar{x} = \frac{16a^2}{5}\cdot\frac{1}{4a} = \frac{4a}{5}$$

Now, $$\int y\,ds = \int_0^{\pi} r\sin\theta\,ds$$

$$= \int_0^{\pi} a(1+\cos\theta)\sin\theta\,ds$$

$$= \int_0^{\pi} a(1+\cos\theta)\sin\theta\cdot 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 2a^2\int_0^{\pi} 2\cos^2\frac{\theta}{2}\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\,d\theta$$

$$= 8a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

$$= 8a^2 \left\{-\frac{1}{5}\cos^5\frac{\theta}{2}\ 2\right\}_0^{\pi}$$

$$= \frac{16}{5}a^2$$

$\therefore$
$$\bar{y} = \frac{16a^2}{5}\cdot\frac{1}{4a} = \frac{4a}{5}$$

$\therefore$ Hence C.G. is $\boxed{(\bar{x}, \bar{y}) = \left(\frac{4a}{5}, \frac{4a}{5}\right)}$ .... **Ans.**

Q.26 : Find the C.G. of the arc of a uniform sector of a circle of radius 'a' angle at centre being 2α. Deduce the same to semicircle. **[SPPU : Dec.-08, Marks 4]**

Ans. : Let the equation of circle be

$$x^2 + y^2 = a^2,$$

Parametric equations :

$$x = a\cos\theta,\ y = a\sin\theta$$

As shown in Fig. Q.26.1.

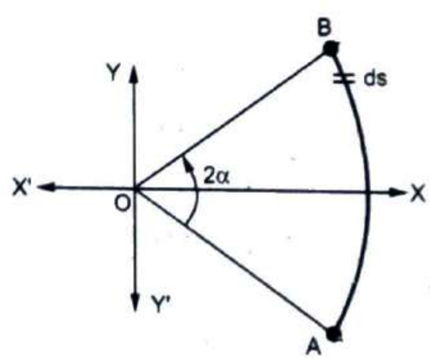


Fig. Q.26.1

X - axis bisecting the central angle of sector.

By symmetry C.G. of arc AB lies on X - axis

$\therefore$
$$\bar{y} = 0$$

and
$$\bar{x} = \frac{\int x\rho\, ds}{\int \rho\, ds} = \frac{\int x\, ds}{\int ds}\ (\rho \text{ is constant}) \quad \text{... (1)}$$

$$s = a\theta$$

$\therefore$
$$ds = a\, d\theta$$

$\therefore$
$$\int x\,ds = 2\int_0^{\alpha} a\cos\theta \cdot d\theta$$

$$= 2a^2\sin\alpha$$

$$\int ds = 2\int_0^{\alpha} a\, d\theta = 2a\alpha$$

$\therefore$
$$\bar{x} = \frac{2a^2\sin\alpha}{2a\alpha} = \frac{a\sin\alpha}{\alpha}$$

As for semicircle $\alpha = \frac{\pi}{2}$

$$\boxed{\bar{x} = \frac{2a}{\pi},\ \bar{y} = 0}$$ .... **Ans.**

Type II : Examples on Centre of Gravity of the Area

Q.27 : Find the C.G. of the area bounded by $y^2 = x$ and $x + y = 2$. **[SPPU : May-14, Marks 6]**

Ans. : Let $(\bar{x}, \bar{y})$ be co-ordinates of C.G. of area bounded by $y^2 = x$ and $x + y = 2$ as shown in Fig. Q.27.1.

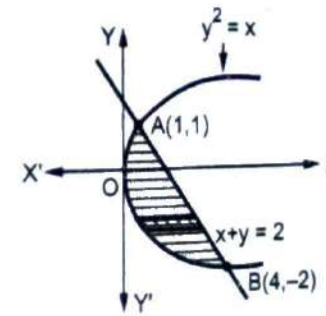


Fig. Q.27.1

$$\bar{x} = \frac{\iint x\, dx\, dy}{\iint dx\, dy} = \frac{N}{D} \quad \text{... (1)}$$

where
$$N = \int_{-2}^{1}\int_{y^2}^{2-y} x\, dx\, dy$$

$$= \int_{-2}^{1}\left(\frac{x^2}{2}\right)_{y^2}^{2-y} dy$$

$$= \frac{1}{2}\int_{-2}^{1} [(2-y)^2 - y^4]\, dy$$

$$= \frac{1}{2}\left[\frac{(2-y)^3}{-3} - \frac{y^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{13}{3} - \frac{11}{5}\right) - \left(\frac{-56}{3} + \frac{32}{5}\right)\right]$$

$$N = \frac{36}{5}$$

and
$$D = \iint dx\, dy = \int_{-2}^{1}\int_{y^2}^{2-y} dx\, dy$$

$$= \int_{-2}^{1}\left(2 - y - y^2\right) dy$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{a-2}^{1} = 9/2$$

$$\therefore \quad \bar{x} = \frac{36/5}{9/2} = \frac{8}{5}$$

Now,
$$\bar{y} = \frac{\iint y\, dx\, dy}{\iint dx\, dy}$$

$$= \frac{\int_{-2}^{1}\int_{y^2}^{2-y} y\, dx\, dy}{9/2}$$

$$= \frac{2}{9}\int_{-2}^{1} y\left[(2-y) - y^2\right] dy$$

$$= \frac{2}{9}\int_{-2}^{1}\left(2y - y^2 - y^3\right) dy$$

$$= \frac{2}{9}\left(y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right)_{-2}^{1}$$

$$= \frac{2}{9}\left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(4 + \frac{8}{3} - \frac{16}{4}\right)\right]$$

$$= \frac{2}{9}\left(-\frac{27}{12}\right) = \frac{2}{9}\left(-\frac{9}{4}\right)$$

$$\bar{y} = -\frac{1}{2}$$

$$\therefore \quad \boxed{\bar{x} = \frac{8}{5}, \quad \bar{y} = -\frac{1}{2}} \quad \text{.... Ans.}$$

Q.28 : Find the C.G. (Centre of Gravity) of the area enclosed by the curves $y^2 = 4ax$, $y = 2x$. [SPPU : May-10]

Ans. : Given that area is enclosed by the curves $y^2 = 4ax$ and $y = 2x$.

$$\therefore \quad 4x^2 = 4ax$$

$$\Rightarrow \quad x^2 - ax = 0$$

$$x(x-a) = 0$$

$$\Rightarrow \quad x = 0$$

and $\quad x = a$

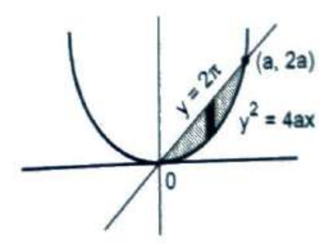


Fig. Q.28.1 (a)

Now
$$\iint dx\,dy = \int_0^a \left\{\int_{\sqrt{4ax}}^{2x} dy\right\} dx = \int_0^a [2x - \sqrt{4a}\sqrt{x}]\,dx$$
$$= \left(x^2 - \sqrt{4a}\,\frac{x^{3/2}}{3/2}\right)_0^a$$
$$= a^2 - \frac{4}{3}a^2 = \frac{-a^2}{3}$$
$$\iint x\,dy\,dx = \int_0^a \left\{\int_{\sqrt{4ax}}^{2x} x\,dy\right\} dx$$
$$= \int_0^a x\,dx\left\{2x - \sqrt{4a}\sqrt{x}\right\}$$
$$= \int_0^a \left[2x^2 - \sqrt{4a}\,x^{3/2}\right] dx$$
$$= \left(\frac{2x^3}{2} - \frac{\sqrt{4}\,a x^{5/2}}{5/2}\right)_0^a$$
$$= \left(\frac{2}{3}a^3 - \frac{4}{5}a^3\right) = -\frac{2}{15}a^3$$
$$\iint y\,dy\,dx = \int_0^a \left(\int_{\sqrt{4ax}}^{2x} y\,dy\right) dx = \int_0^a \left[\frac{y^2}{2}\right]_{\sqrt{4ax}}^{2x} dx$$
$$= \int_0^a \frac{1}{2}[4x^2 - 4a\,x]\,dx = \int_0^a (2x^2 - 2ax)\,dx$$
$$= \left(\frac{2x^3}{3} - \frac{2ax^2}{2}\right)_0^a = \frac{2a^3}{3} - a^3 = \frac{-a^3}{3}$$

We have
$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} = \left(\frac{-2}{15}a^3\right)\left(-\frac{3}{a^2}\right) = \frac{2}{5}a$$

$$\bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy} = \left(-\frac{a^3}{3}\right)\left(-\frac{3}{a^2}\right) = a$$

Thus centre of gravity $(\bar{x}, \bar{y}) = \left(\frac{2}{5}a,\ a\right)$

Q.29 : Find the centroid of the loop of the Laminscate $r^2 = a^2 \cos 2\theta$.

[SPPU : May-15, 19, Dec.-13, Marks 6]

Ans. : Since the loop is symmetrical about x - axis.

$\therefore$ $\bar{y} = 0$

and
$$\bar{x} = \frac{\iint x\ dx\ dy}{\iint dx\ dy}$$
$$= \frac{\iint r\cos\theta\ r\ d\theta\ dr}{\iint r\ d\theta\ dr}$$
$$= \frac{N}{D} \qquad \text{... (1)}$$

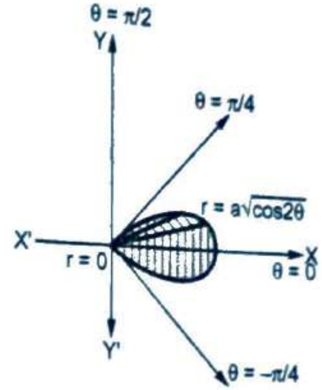


Fig. Q.29.1

For the loop, 'r' varies from **r = 0 to r = a** $\sqrt{\cos 2\theta}$ **, 'θ' varies from** $\theta = \frac{-\pi}{4}$ **to** $\theta = \frac{\pi}{4}$

$\therefore$
$$N = \int_{\theta=-\frac{\pi}{4}}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r^2\cos\theta\ d\theta\ dr$$
$$= \int_{\theta=-\frac{\pi}{4}}^{\pi/4} \cos\theta \left(\frac{r^3}{3}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$
$$= \frac{2a^3}{3}\int_0^{\pi/4} \cos\theta\left(\sqrt{\cos 2\theta}\right)^3 d\theta$$
$$= \frac{2a^3}{3}\int_0^{\pi/4} \left(1 - 2\sin^2\theta\right)^{3/2}\cos\theta\ d\theta,$$

Put $\sqrt{2}\sin\theta = \sin t,\ \sqrt{2}\cos\theta\, d\theta = \cos t\, dt$

Limits :

θ	0	π/4
t	0	π/2

$$= \frac{2a^3}{3}\frac{1}{\sqrt{2}}\int_0^{\pi/2} \cos^3 t \cos t\, dt$$

$$= \frac{2a^3}{3\sqrt{2}}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi a^3}{8\sqrt{2}} \quad \text{... (2)}$$

$$D = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\, d\theta\, dr$$

$$= \int_{-\pi/4}^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}}$$

$$D = a^2\int_0^{\pi/4}\cos 2\theta\, d\theta$$

$$= a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$= \frac{a^2}{2}[1-0] = \frac{a^2}{2} \quad \text{... (3)}$$

$$\therefore \quad \bar{x} = \frac{\frac{\pi a^3}{8\sqrt{2}}}{\frac{a^2}{2}} = \frac{\pi a}{4\sqrt{2}}$$

$$\therefore \quad \boxed{\text{Centroid} = (\bar{x}, \bar{y}) = \left(\frac{\pi a}{4\sqrt{2}}, 0\right)} \quad \text{... Ans.}$$

Q.30 : Find the centroid of the area bounded by $y^2(2a - x) = x^3$ and it's asymtote.

[SPPU : Dec.-01,09,10, May-03,11]

Ans. : The curve is CISSOID as shown in the Fig. Q.30.1. $x = 2a$ is asymotote. The curve is symmetrical about X - axis. $\therefore\ \bar{y} = 0$.

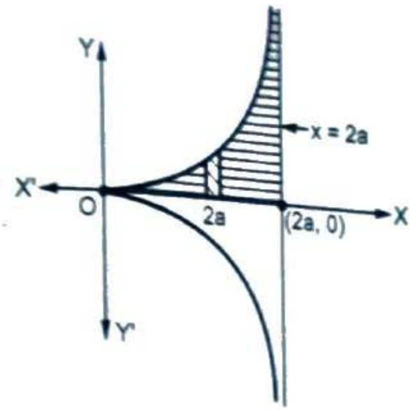


Fig. Q.30.1

and
$$\bar{x} = \frac{\iint x\, dx\, dy}{\iint dx\, dy} = \frac{N}{D} \quad \text{... (1)}$$

$$N = \int_0^{2a}\int_0^{y} x\, dx\, dy = \int_0^{2a} x\,(y)_0^y\, dx$$

$$= \int_0^{2a} xy\, dx = \int_0^{2a} x\,\frac{x^{3/2}}{\sqrt{2a-x}}\, dx$$

$$N = \int_0^{2a} x\,\frac{x^{3/2}}{\sqrt{2a-x}}\, dx \ \text{Put } x = 2a\sin^2\theta,$$

$$dx = 4a\sin\theta\cos\theta\, d\theta$$

Limits :

x	0	2a
θ	0	π/2

$$\therefore \qquad N = \int_0^{\pi/2} \frac{2a\sin^2\theta \cdot \left(2a\sin^2\theta\right)^{3/2}}{\sqrt{2a - 2a\sin^2\theta}}\, 4a\sin\theta\cos\theta\, d\theta$$

$$= \frac{(2a)^{5/2}}{(2a)^{1/2}} \int_0^{\pi/2} \frac{\sin^5\theta\ 4a\sin\theta\cos\theta}{\cos\theta}\, d\theta$$

$$= (2a)^2 (4a) \int_0^{\pi/2} \sin^6\theta\, d\theta$$

$$= 16a^3\, \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{5\pi a^3}{2} \qquad \ldots (2)$$

and
$$D = \int_0^{2a}\int_0^{y} dx\, dy = \int_0^{2a} y\, dx$$

$$= \int_0^{2a} \frac{x^{3/2}}{\sqrt{2a - x}}\, dx \qquad \ldots \text{Put } x = 2a\sin^2\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4\theta\, d\theta = 8a^2\, \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi a^2}{2} \qquad \ldots (3)$$

From equations (1), (2) and (3),

$$\bar{x} = \frac{5\pi a^3}{2} \Big/ \frac{3\pi a^2}{2}$$

$$= \frac{5}{3}\, a$$

$\therefore$ C.G. is $\boxed{(\bar{x}, \bar{y}) = \left(\frac{5a}{3}, 0\right)}$ .... Ans.

Q.31 : Find the C.G. of one loop of $r = a \sin 2\theta$.

[SPPU : May-13, Dec.-14, Marks 6]

Ans. : The curve $r = a\sin 2\theta$ is four leaved rose lies within the circle $r = a$,

Consider a loop lies between $\theta = 0$ to $\theta = \pi/2$ and is symmetrical about the line $\theta = \pi/4$ as shown in the Fig. Q.31.1.

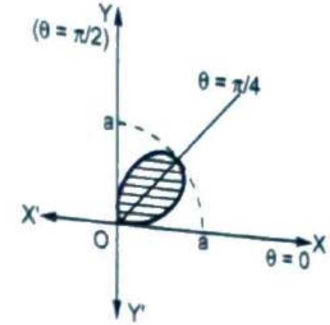


Fig. Q.31.1

$$\therefore \qquad \bar{x} = \bar{y},$$

$$\bar{x} = \frac{\iint x\, dx\, dy}{\iint dx\, dy}$$

$$= \frac{N}{D} \qquad \ldots (1)$$

Where
$$N = \iint x\, dx\, dy$$

$$= \int_0^{\pi/2}\int_0^{a\sin 2\theta} r\cos\theta\, r\, d\theta\, dr$$

$$= \int_0^{\pi/2} \cos\theta\, \frac{a^3\sin^3 2\theta}{3}\, d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \sin^3\theta\cos^4\theta\, d\theta$$

$$= \frac{8a^3}{3}\, \frac{[(3-1)]\cdot[(4-1)(4-3)]}{(7)(7-2)(7-4)(7-6)}\cdot 1$$

$$= \frac{16a^3}{105} \qquad \ldots (2)$$

and
$$D = \iint dx\, dy$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin 2\theta} r\, d\theta\, dr$$

$$= \int_{\theta=0}^{\pi/2} \frac{a^2 \sin^2 2\theta}{2} \, d\theta$$

$$= 2a^2 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta = 2a^2 \left(\frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}\right)$$

$$= \frac{\pi a^2}{8} \qquad \text{... (5)}$$

From equations (1), (2) and (3)

$$\bar{x} = \frac{\frac{16a^3}{105}}{\frac{\pi a^2}{8}}$$

$$\boxed{\bar{x} = \frac{128\,a}{105\,\pi}, \text{ Also } \bar{y} = \frac{128\,a}{105\,\pi}} \qquad \text{.... Ans.}$$

Q.32 : Find the centre of gravity of the area bounded by r = a sin θ and r = 2a sin θ. **[SPPU : May-19, Marks 4]**

Ans. : The circles as shown in the Fig. Q.32.1.

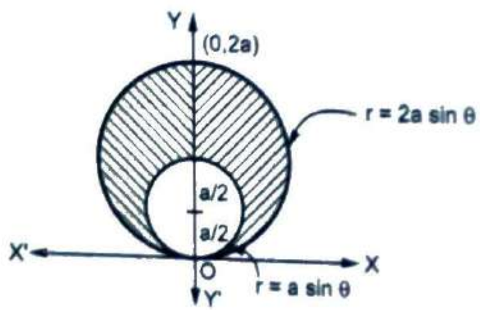


Fig. Q.32.1

i) $r = a \sin\theta \Rightarrow r^2 = a\, r \sin\theta \Rightarrow x^2 + y^2 = ay$

$$\Rightarrow x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

Centre = $\left(0, \frac{a}{2}\right)$ **and Radius =** $\frac{a}{2}$

ii) $r = 2a \sin\theta \Rightarrow r^2 = 2a\, r \sin\theta \Rightarrow x^2 + y^2 = 2ay$

$$\Rightarrow x^2 + (y - a)^2 = a^2$$

Centre = (0, a) and Radius = a

The C.G. lies on Y - axis $\therefore \bar{x} = 0$

and
$$\bar{y} = \frac{\iint y \, dx \, dy}{\iint dx \, dy} = \frac{N}{D} \qquad \text{... (1)}$$

Where
$$N = \iint y \, dx \, dy = \int_0^{\pi} \int_{a \sin\theta}^{2a \sin\theta} r \sin\theta \; r \, d\theta \, dr$$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^3}{3}\right)_{a \sin\theta}^{2a \sin\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi} \sin\theta \, [(2a \sin\theta)^3 - (a \sin\theta)^3] \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} 7 \sin^4\theta \, d\theta = \frac{14a^2}{3} \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= \frac{14a^3}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{7\pi a^3}{8} \qquad \text{... (2)}$$

and
$$D = \iint dx \, dy = \int_0^{\pi} \int_{a \sin\theta}^{2a \sin\theta} r \, d\theta \, dr$$

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_{a \sin\theta}^{2a \sin\theta} d\theta$$

$$= \frac{3a^2}{2}\int_0^{\pi} \sin^2\theta \, d\theta = \frac{3a^2}{2}\, 2\int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= 3a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \textbf{OR}$$

$$\text{Area} = \pi a^2 - \frac{\pi a^2}{4} = \frac{3\pi a^2}{4}$$

$$= \frac{3\pi a^2}{4} \qquad \text{... (3)}$$

From equations (1), (2) and (3)

$$\bar{y} = \frac{7\pi a^3}{8} \cdot \frac{4}{3\pi a^2} = \frac{7a}{6}$$

$\therefore$ The C.G. is $\boxed{\left(0, \frac{7a}{6}\right)}$ .... **Ans.**

Type III : Example on Centre of Gravity of Solid

Q.33 : Find the centroid of the region in the first octant bounded by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ($a > 0$, $b > 0$, $c > 0$).

Ans. : Let the centroid be $(\bar{x}, \bar{y}, \bar{z})$

Where, $$\bar{x} = \frac{\iiint x \, dx\, dy\, dz}{\iiint dx\, dy\, dz},$$

$$\bar{y} = \frac{\iiint y \, dx\, dy\, dz}{\iiint dx\, dy\, dz}$$

and $$\bar{z} = \frac{\iiint z \, dx\, dy\, dz}{\iiint dx\, dy\, dz} \qquad \text{... (1)}$$

Put $x = au$, $y = bv$, $z = cw$, ρ = constant

$\therefore$ $dx\, dy\, dz = abc\, du\, dv\, dw$ and $u+v+w=1$

$\therefore$ $\iiint x \, dx\, dy\, dz = \iiint au \, du\, dv\, dw$

$$= a^2bc \iiint u^{2-1} v^{1-1} w^{1-1} du\, dv\, dw$$

$$= a^2bc \frac{\Gamma 2\, \Gamma 1\, \Gamma 1}{\Gamma(1+2+1+1)} = \frac{a^2bc}{4!}$$

$$= \frac{a^2bc}{24} \qquad \text{(Using Dirichlet's theorem)}$$

Similarly $\iiint y \, dx\, dy\, dz = \frac{ab^2c}{24}$

And $\iiint z \, dx\, dy\, dz = \frac{abc^2}{24}$

Also $\iiint dx\, dy\, dz$ = Volume of the tetrahydron $= \frac{abc}{6}$

$$\bar{x} = \frac{\frac{a^2bc}{24}}{\frac{abc}{6}} = \frac{a}{4}$$

Similarly $\bar{y} = \frac{b}{4}$ and $\bar{z} = \frac{c}{4}$

$\therefore$ $\boxed{\bar{x} = \frac{a}{4}, \bar{y} = \frac{b}{4}, \bar{z} = \frac{c}{4}}$ ... **Ans.**

14.5 : Moment of Inertia

The moment of inertia is a physical quantity which describes how easily a body can be rotated about a given axis. It is a rotational analogue of mass which describes an objects resistance to translational motion.

Inertia is the property of matter which resists change in it's state for motion. The larger the inertia, the greater force that is required to bring some change in it's velocity, in a given amount of time.

Moment of inertia is that property where matter resists change in it's state of rotatory motion. The larger the moment of inertia the greater the amount of torque that will be required to bring the same change in it's angular velocity in a given amount of time.

1) Moment of Inertia of a plane lamina :

Consider a plane lamina R bounded by the curve C.

If ρ is density at the the point P(x, y) then $dm = \rho\, dx\, dy$.

i) If p is the distance of this elementary mass from the axis, the M.I. about this axis is

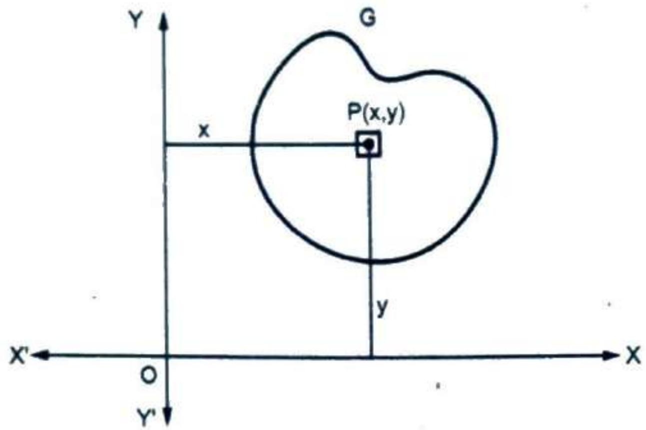


Fig. 14.4

$$\text{M.I.} = \iint_R \rho\, p^2\, dx\, dy$$

ii) The moment of Inertia of the Lamina about X-axis is

$$\boxed{\text{M.I.} = \iint_R \rho\, y^2\, dx\, dy} \qquad (\because p = y)$$

iii) The moment of Inertia of the lamina about Y-axis is

$$\boxed{\text{M.I.} = \iint_R \rho\, x^2\, dx\, dy} \qquad (\because p = x)$$

iv) The M.I. in polar co-ordinates is

$$\boxed{\text{M.I.} = \iint_R \rho\, p^2\, r\, d\theta\, dr}$$

2) Moment of Inertia of solid :

Consider a solid of volume V and ρ is at a density at the point p (x, y, z) then

$$dm = \rho\, dx\, dy\, dz$$

The moment of inertia of solid which is at a distance p from the axis is

$$\boxed{\text{M.I.} = \iiint_V \rho\, p^2\, dx\, dy\, dz}$$

i) The moment of inertia about X-axis is

$$\boxed{\text{M.I.} = \iiint_V \rho\left(y^2 + z^2\right) dx\, dy\, dz} \qquad (\because p = \sqrt{y^2 + z^2})$$

ii) The moment of inertia about Y-axis is

$$\boxed{\text{M.I.} = \iiint_V \rho\, (x^2 + z^2)\, dx\, dy\, dz} \qquad (\because p = \sqrt{x^2 + z^2})$$

iii) The moment of inertia about Z-axis is

$$\boxed{\text{M.I.} = \iiint_V \rho\left(x^2 + y^2\right) dx\, dy\, dz} \qquad (\because p = \sqrt{x^2 + y^2})$$

3) Moment of Inertia of an arc :

$$\boxed{\begin{aligned}\text{M.I.} &= \int p^2 \rho\, ds \\ &= \int p^2 \rho \frac{ds}{dx}\, dx\end{aligned}} \qquad (\text{where } dm = \rho\, ds)$$

or

$$\boxed{\text{M.I.} = \int p^2 \rho \frac{ds}{dy}\, dy}$$

or

$$\boxed{\text{M.I.} = \int p^2 \rho \frac{ds}{dt}\, dt}$$

4) Theorem of perpendicular axes :

If I_x, I_y are the moments of Inertia of a plane Lamina about two perpendicular axes OX and OY respectively then the moment if Inertia I_z about the axis perpendicular to the plane of the lamina through 'O' is

$$\boxed{I_z = I_x + I_y}$$

5) Theorem of parallel axes (Steiner's Theorem) :

If ' I_g ' is the moment of inertia of a mass m about an axis through it's Centre of Gravity (C.G.) then it's moment of Inertia I_p about a line parallel to the above axis at a distance 'd' is given by,

$$\boxed{I_p = I_g + Md^2}$$

6) Radius of gyration :

If the moment of Inertia of a body of mass 'M' is Mk^2, then 'k' is called radius of Gyration and it is given by,

$$\boxed{k = \sqrt{\frac{\text{M.I.}}{\text{M}}} = \sqrt{\frac{\text{M.I.}}{\text{Mass}}}}$$

Q.34 : Find the moment of inertia of the portion of the parabola $y^2 = 4ax$ bounded by X-axis and latus rectum, about X-axis, if the density at each point varies as the cube of the abscissa.

[SPPU : May-18, Dec.-17, Marks 6]

Ans. : Consider a small element dx dy at P(x, y). The density at P varies as the cube of the abscissa i.e. $\rho \propto x^3$.

$\therefore$ $$\rho = \lambda x^3$$

$$\text{M.I.} = \iint \rho y^2 \, dx\, dy$$

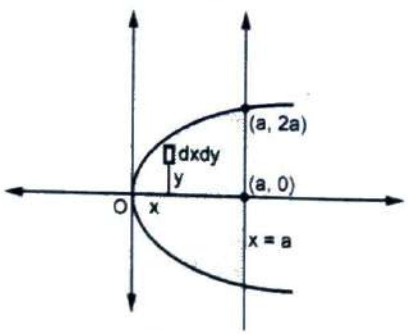


Fig. Q.34.1

$$= \int_0^a \int_0^{2\sqrt{ax}} \lambda x^3 y^2 \, dx\, dy$$

$$= \lambda \int_0^a x^3 \left[\frac{y^3}{3}\right]_0^{2\sqrt{ax}} dx$$

$$= \frac{\lambda}{3} \int_0^a x^3 [8a^{3/2}\, x^{3/2}] dx$$

$$= \frac{8\lambda}{3} \int_0^a a^{3/2} x^{9/2}\, dx = \frac{8\lambda a^{3/2}}{3} \left[\frac{x^{11/2}}{11/2}\right]_0^a$$

$$= \frac{16}{33} \lambda a^7$$

$$\text{Now, Mass} = M = \iint \rho \, dx\, dy = \int_0^a \int_0^{2\sqrt{ax}} \lambda x^3 \, dx\, dy$$

$$M = \lambda \int_0^a x^3 [y]_0^{2\sqrt{ax}} dx = \lambda \int_0^a x^3 \sqrt{a}\, \sqrt{x}\, dx$$

$$M = 2\lambda\sqrt{a} \int_0^a x^{7/2} dx = 2\lambda\sqrt{a} \left[\frac{x^{9/2}}{9/2}\right]_0^a = \frac{4}{9} \lambda a^5$$

$$\therefore \quad \boxed{\lambda = \frac{9}{4} \frac{M}{a^5}}$$

$$\therefore \quad \text{M.I.} = \frac{16}{33} \lambda a^7 = \frac{16}{33} \cdot \frac{9}{4} \frac{M}{a^5} a^7 = \frac{12}{11} Ma^2$$

$$\therefore \quad \boxed{\text{M.I.} = \frac{12}{11} Ma^2}$$

Q.35 : Prove that the moment of inertia of the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ about X-axis is $\frac{144}{35} Ma^2$, where M is the mass of the area included between the curves.

[SPPU : May-05,15, Marks 6]

Ans. : $$\text{M.I.} = \iint_A \rho\, y^2\, dx\, dy \qquad (\because p = y)$$

Where *A is* area included between two curves as shown in the Fig. Q.35.1.

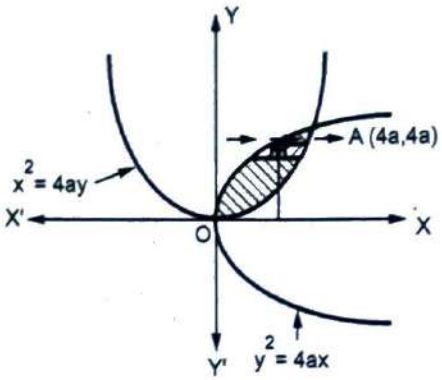


Fig. Q.35.1

$$\therefore \quad \text{M.I.} = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{\sqrt{4ay}} \rho\, y^2\, dx\, dy$$

$$= \int_0^{4a} \rho\, y^2\, (x)_{\frac{y^2}{4a}}^{\sqrt{4ay}}\, dy$$

$$= \int_0^{4a} \rho\, y^2 \left[\sqrt{4ay} - \frac{y^2}{4a}\right] dy$$

$$= \rho \int_0^{4a} \left(2\sqrt{a}\, y^{5/2} - \frac{y^4}{4a}\right) dy$$

$$= \rho \left[2\sqrt{a} \cdot \frac{2}{7}\, y^{7/2} - \frac{1}{4a}\, \frac{y^5}{5}\right]_0^{4a}$$

$$= \rho (4a)^4 \left(\frac{2}{7} - \frac{1}{5}\right) = \frac{3}{35}(4a)^4 \rho$$

Now mass of the area included between the curves is

$$M = \int_0^{4a} \int_{\frac{y^2}{4a}}^{\sqrt{4ay}} \rho\, dx\, dy$$

$$= \rho \int_0^{4a} (x)_{\frac{y^2}{4a}}^{\sqrt{4ay}}\, dy$$

$$= \rho \int_0^{4a} \left[\sqrt{4ay} - \frac{y^2}{4a}\right] dy$$

$$= \rho \left[\sqrt{4a}\, \frac{y^{3/2}}{\frac{3}{2}} - \frac{1}{4a}\, \frac{y^3}{3}\right]_0^{4a}$$

$$= \rho \left[\frac{2}{3}(4a)^2 - \frac{(4a)^2}{3}\right]$$

$$M = \rho\, \frac{(4a)^2}{3}$$

$$\therefore \quad \rho = \frac{3M}{16a^2}$$

But $$\text{M.I.} = \frac{3}{35}(4a)^4 \rho$$

$$= \frac{3}{35}(4a)^4 \cdot \frac{3M}{16a^2}$$

or $$\boxed{\text{M.I.} = \frac{144}{35}\, a^2 M} \quad \text{... Ans.}$$

Q.36 : Show that M.I. if a rectangle of sides a, b about it's diagonal is $\frac{M}{6}\left(\frac{a^2 b^2}{a^2 + b^2}\right)$, where M is the mass of the rectangle. **[SPPU : May-11]**

Ans. : $$\text{M.I.} = \iint_A p^2 \rho\, dx\, dy \qquad \text{... (1)}$$

Where A is area of the rectangle as shown in the Fig. Q.36.1.

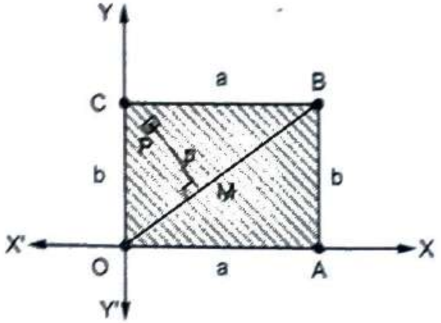


Fig. Q.36.1

(Bounded by x = 0, x = b, y = 0, y = a)

p is the length of perpendicular from point P to the diagonal OB.

The equation of the line OB is (slope $\frac{b}{a}$)

$y = \frac{b}{a}x \Rightarrow bx - ay = 0$

Now, perpendicular distance = $p = \left|\frac{bx - ay}{\sqrt{a^2 + b^2}}\right|$

$$\text{M.I.} = \int_{x=0}^{a}\int_{y=0}^{b} \frac{(bx - ay)^2}{a^2 + b^2}\rho\, dx\, dy$$

$$= \frac{\rho}{a^2 + b^2}\int_0^a \left[\frac{(bx - ay)^3}{-3a}\right]_0^b dx$$

$$= \frac{\rho}{3\left(a^2 + b^2\right)a}\int_0^a [(bx - ab)^3 - b^3x^3]\, dx$$

$$= -\frac{\rho b^3}{3a\left(a^2 + b^2\right)}\left[\frac{(x - a)^4}{4} - \frac{x^4}{4}\right]_0^a$$

$$= -\frac{\rho b^3}{3a\left(a^2 + b^2\right)}\left(\frac{-a^4 - a^4}{4}\right) = \frac{\rho a^3 b^3}{6\left(a^2 + b^2\right)}$$

But mass of the rectangle, M = ρ ab

$$\therefore \quad \rho = \frac{M}{ab}$$

$$\therefore \quad \text{M.I.} = \frac{M}{ab}\frac{a^3b^3}{6\left(a^2 + b^2\right)}$$

$$\boxed{\text{M.I.} = \frac{M\, a^2b^2}{6\left(a^2 + b^2\right)}} \qquad \text{... Ans.}$$

Q.37 : Find the moment of inertia about the line $\theta = \frac{\pi}{2}$ of the area enclosed by $r = a(1 + \cos\theta)$. **[SPPU : May-17, 14, Dec.-15, Marks 6]**

Ans. : Here $\rho = x = r\cos\theta$

$$\text{M.I.} = \iint \rho^2 dA = \iint r^2\cos^2\theta\ r\, dr\, d$$

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\cos^2\theta\, dr\, d\theta$$

$$= 2\int_0^{\pi}\cos^2\theta\left(\frac{r^4}{4}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi}\cos^2\theta\,(1+\cos\theta)^4\, d\theta$$

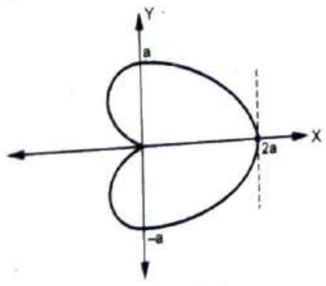


Fig. Q.37.1

$$= \frac{a^4}{2}\int_0^{\pi} \cos^2\theta\,(1+4\cos\theta+6\cos^2\theta+4\cos^3\theta+\cos^4\theta)d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi} [\cos^2\theta+4\cos^3\theta+6\cos^4\theta+4\cos^5\theta+\cos^6\theta)d\theta$$

$$= \frac{a^4}{2}2\int_0^{\pi/2} [\cos^2\theta+4\cos^3\theta+6\cos^4\theta+4\cos^5\theta+\cos^6\theta]\,d\theta$$

$$\text{M.I.} = a^4\left\{\frac{1}{2}\cdot\frac{\pi}{2}+6\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\} = \frac{49\pi a^4}{32}$$

Q.38 : The density at any point (x, y) on a square lamina of side 'a' units, varies as the square of its distance from one of the diagonals. show that the moment of inertia (M.I.) about the diagonal is $\frac{Ma^2}{5}$, where M is the mass of the lamina.

[SPPU : Dec.-09]

Ans. : We have the equation on line OB is y = x, i.e. y − x = 0 and the distance between line OB and A(x, y) is,

$$P = \left|\frac{y-x}{\sqrt{2}}\right| \text{ and } \rho = \frac{(y-x)^2}{2}$$

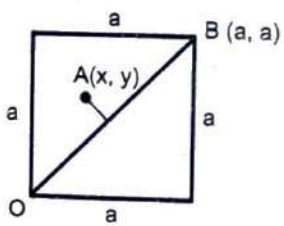


Fig. Q.38.1

$$\therefore \quad \text{M.I.} = \int_0^a\int_0^a \frac{(y-x)^4}{4}\,dx\,dy$$

$$= \frac{1}{4}\int_0^a \left[\frac{(y-x)^5}{5}\right]_0^a dx$$

$$= \frac{1}{20}\int_0^a [(a-x)^5+x^5]dx$$

$$= \frac{1}{20}\left[\frac{a^6}{6}+\frac{a^6}{6}\right]$$

$$\text{M.I.} = \frac{1}{60}a^6$$

$$\text{Mass} = M = \frac{1}{2}\int_0^a\int_0^a (y-x)^2\,dx\,dy$$

$$= \frac{1}{2}\int_0^a \left[\frac{(y-x)^3}{3}\right]_0^a dx = \frac{1}{6}\int_0^a [(a-x)^3+x^3]dx$$

$$M = \frac{1}{6}\left[\frac{a^4}{4}+\frac{a^4}{4}\right] = \frac{a^4}{12}$$

$$\therefore \quad \text{M.I.} = \frac{1}{60}a^6 = \frac{a^4}{12}\frac{a^2}{5} = \frac{Ma^2}{5}$$

Q.39 : Find the moment of inertia of one loop of the Laminscate $r^2 = a^2\cos 2\theta$ about the initial line θ = 0.

[SPPU : Dec.-15, May-17, Marks 6]

Ans. :

$$\text{M.I.} = \iint_A p^2\rho\ r\ d\theta\ dr$$

$$= \iint_A r^2\sin^2\theta\ r\rho\ d\theta\ dr \qquad (\because p = y = r\sin\theta) \quad \dots (1)$$

Where A is area of the loop as shown in the Fig. Q.39.1.

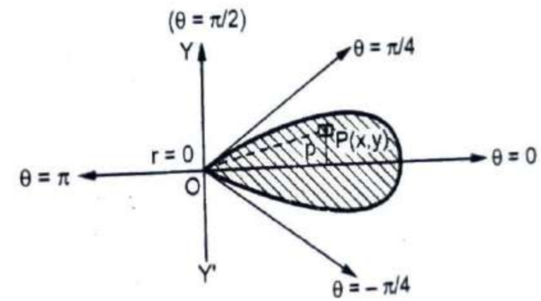


Fig. Q.39.1

From equation (1)

$$\text{M.I.} = \rho\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r^3\sin^2\theta\ d\theta\ dr$$

$$= 2\rho \int_0^{\pi/4} \sin^2\theta \left(\int_0^{a\sqrt{\cos 2\theta}} r^3 \, dr \right) d\theta$$

$$= 2\rho \int_0^{\pi/4} \sin^2\theta \left(\frac{r^4}{4} \right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\rho \frac{a^4}{4} \int_0^{\pi/4} \sin^2\theta \cos^2 2\theta \, d\theta$$

$$= \frac{\rho a^4}{2} \int_0^{\pi/4} \cos^2 2\theta \frac{(1 - \cos 2\theta)}{2} d\theta \qquad \because (\text{Put } 2\theta = t)$$

$$= \frac{\rho a^4}{4} \int_0^{\pi/2} \cos^2 t \, (1 - \cos t) \frac{dt}{2} \qquad \because \left(d\theta = \frac{dt}{2} \right)$$

$$= \frac{\rho a^4}{4} \left[\frac{1}{2} \frac{\pi}{2} - \frac{2}{3} \right] = \frac{\rho a^4}{96} (3\pi - 8)$$

Now, mass of the loop is

$$M = \iint \rho \, r \, d\theta \, dr = \rho \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r \, d\theta \, dr$$

$$= 2\rho \int_0^{\pi/4} \left(\int_0^{a\sqrt{\cos 2\theta}} r \, dr \right) d\theta$$

$$= 2\rho \int_0^{\pi/4} \left(\frac{r^2}{2} \right)_0^{a\sqrt{\cos 2\theta}} d\theta = \rho \int_0^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= \rho a^2 \left(\frac{\sin 2\theta}{2} \right)_0^{\pi/4} = \frac{\rho a^2}{2} (1 - 0)$$

$$M = \frac{\rho a^2}{2}$$

$$\therefore \quad \rho = \frac{2M}{a^2}$$

But $\quad \text{M.I.} = \rho \frac{a^4}{96} (3\pi - 8) = \frac{2M}{a^2} \frac{a^4}{96} (3\pi - 8)$

$$\boxed{\text{M.I.} = \frac{(3\pi - 8)}{48} Ma^2}$$

... Ans.

Q.40 : Show that the M.I. of a loop of $r^2 = a^2 \cos 2\theta$ about a line through the pole perpendicular to it's plane is $Ma^2\pi/8$, when M is the mass of the loop. [SPPU : May-10]

Ans. :

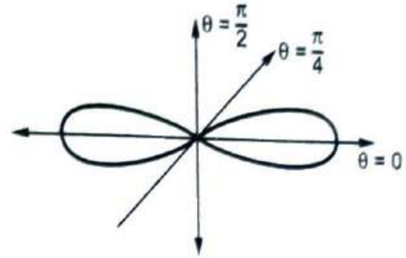


Fig. Q.40.1

We have $\quad P = \sqrt{x^2 + y^2} = r$

$$M_1 = \rho \iint_R P^2 dx dy = \rho \int_0^{\pi/4} \int_0^{\sqrt{a^2 \cos 2\theta}} r^3 dr \, d\theta$$

$$= \rho \frac{a^4}{4} \int_0^{\pi/4} \cos^2 2\theta \, d\theta = \rho \frac{a^4}{4} \int_0^{\pi/4} \left(\frac{1 + \cos 4\theta}{2} \right) d\theta$$

$$M_1 = \rho \frac{a^4 \pi}{32}$$

$$M_2 = \iint \rho \, dx \, dy = \rho \int_0^{\pi/4} \int_0^{\sqrt{a^2 \cos 2\theta}} r \, dr \, d\theta$$

$$= \rho \int_0^{\pi/4} \frac{a^2 \cos 2\theta}{2} d\theta = \frac{\rho}{2} a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$M_2 = \frac{\rho a^2}{4} (1 - 0) = \frac{\rho a^2}{4}$$

$$\text{Moment of inertia} = \frac{M_1}{M_2} = \frac{\rho a^4 \pi}{32} \cdot \frac{4}{\rho a^2} = \frac{\rho a^2 \pi}{8} \quad (\rho = M)$$

$$\text{M.I.} = \frac{Ma^2 \pi}{8}$$

Q.41 : Find the M.I. about the X-axis of the area enclosed by the lines $x = 0, \frac{x}{a} + \frac{y}{b} = 1$. **[SPPU : Dec.-14, Marks 6]**

Ans. :

$$\text{M.I.} = \iint_A \rho P^2 \, dx\, dy$$

$$\text{M.I.} = \iint_A \rho \, y^2 \, dx\, dy \qquad (\because P = y) \quad \ldots (1)$$

where A is area as shown in the Fig. Q.41.1.

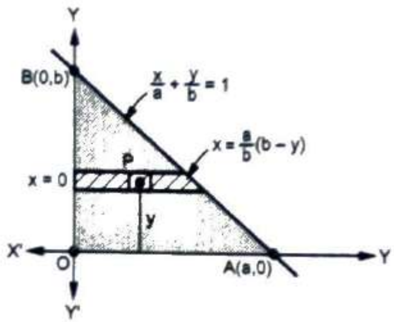


Fig. Q.41.1

Consider small area dxdy at a distance y from the X-axis.

From equation (1),

$$\text{M.I.} = \rho \int_{y=0}^{b} \int_{0}^{\frac{a}{b}(b-y)} y^2 \, dx\, dy$$

$$= \rho \int_0^b y^2 (x)_0^{\frac{a}{b}(b-y)} dy = \rho \int_0^b y^2 \frac{a(b-y)}{b} dy$$

$$= \frac{\rho a}{b} \int_0^b \left(by^2 - y^3\right) dy = \frac{\rho a}{b} \left(b\frac{y^3}{3} - \frac{y^4}{4}\right)_0^b$$

$$\text{M.I.} = \frac{\rho \, ab^3}{12}$$

Now, mass M of the area is

$$M = \rho \times \text{area of the } \Delta \, OAB$$

$$M = \rho \frac{ab}{2}$$

$$\therefore \quad \rho = \frac{2M}{ab}$$

$$\therefore \quad \text{M.I.} = \frac{ab^3}{12} \times \frac{2M}{ab}$$

$$\boxed{\text{M.I.} = \frac{b^2 M}{6}} \qquad \ldots \text{Ans.}$$

Q.42 : Find the moment of inertia of a sphere about a diameter. **[SPPU : May-03, 07]**

Ans. : Let the equation of the sphere be $x^2 + y^2 + z^2 = a^2$. Let Z-axis be diameter about which M.I. is to be obtained.

$$\therefore \quad \text{M.I.} = 8 \iiint \left(x^2 + y^2\right) \rho \, dx\, dy\, dz \qquad \ldots (1)$$

Transform integral (1) into spherical polar form by putting

$x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$

and $dx\, dy\, dz = r^2 \sin\theta \, d\theta \, d\phi \, dr$

$$\therefore \quad x^2 + y^2 = r^2 \sin^2\theta$$

$$\therefore \quad \text{M.I.} = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho \, r^2 \sin\theta \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 8\rho \int_0^{\pi/2} \sin^3\theta \, d\theta \int_0^{\pi/2} d\phi \cdot \int_0^a r^4 dr$$

$$= 8\rho \cdot \frac{2}{3} \cdot \frac{\pi}{2} \cdot \frac{a^5}{5} = \rho \cdot \frac{8\pi a^5}{15}$$

Now, the mass of the sphere M is

$$M = \rho\ V = \rho \cdot \frac{4}{3}\pi a^3$$

$$\therefore \qquad \rho = \frac{3M}{4\pi a^3}$$

Hence

$$M.I. = \frac{3M}{4\pi a^3}\,\frac{8\pi a^5}{15}$$

$$\boxed{M.I. = \frac{2}{5}\pi a^2} \qquad \text{... Ans.}$$

Memory Map

1) $\iint f(x,y)\,dx\,dy = \iint_S f[g(u,v), h(u,v)]\ |J|\ du\,dv$

where $dxdy = |J|\ du\ dv = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dx\ dy$

2) $\text{Area} = \iint_R dx\,dy = \iint r\,dr\,d\theta = \int y\,dx$

3) $\text{volume} = \iiint_V dx\,dy\,dz = \iint f(x,y)\,dA$

$$= \iiint_V r^2 \sin\theta\,dr\,d\theta\,d\phi = \iiint_V \rho\,d\rho\,d\phi\ dz$$

4) Centre of gravity

i) $\bar{x} = \frac{\int x\,ds}{\int ds},\ \bar{y} = \frac{\int y\,ds}{\int ds}$ where $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

ii) $\bar{x} = \frac{\iint x\rho\,dx\,dy}{\iint \rho\,dx\,dy},\ \bar{y} = \frac{\iint y\rho\,dx\,dy}{\iint \rho\,dx\,dy},$

$$\bar{z} = \frac{\iint z\rho\,dx\,dy}{\iint \rho\,dx\,dy}$$

5) Moment of Inertia

a) About X-axis : $M.I. = \iint \rho y^2\ dx\ dy$

b) About Y-axis : $M.I. = \iint \rho x^2\ dx\ dy$

c) In polar form : $M.I. = \iint_R \rho\,p^2\ r\ dr\ d\theta$

d) $M.I. = \iiint_V \rho\,p^2\ dx\ dy\ dz$

END...