The Bombay Salesian Society's

Don Bosco Institute of Technology, Mumbai

(An Autonomous Institute Affiliated to the University of Mumbai)

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Prepared By:

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The Bombay Salesian Society's

DON BOSCO INSTITUTE OF TECHNOLOGY

An Autonomous Institute Affiliated to the University of Mumbai

SUBJECT ______________________ BRANCH/DIV.__________

SEMESTER __________ ROLL NO.__________

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CERTIFICATE

This is to certify that Mr./Ms. ______________________ Roll No. __________ has completed the specified termwork in the subject of ______________________ in a satisfactory manner in the college during Academic Year 202 - 202 .

Date: __________ ______________________

. Lecturer in charge

Module 1 : Complex Numbers and its Functions

- Cartesian form: $z = x + iy$
- Polar form: $z = r[\cos\theta + i\sin\theta]$
- Exponential form: $z = re^{i\theta}$
- $r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(\frac{y}{x}),\ e^{i\theta} = \cos\theta + i\sin\theta$
  Use the following rules for determining θ :

z in	θ
I Quadrant	$\theta = \tan^{-1}(\frac{y}{x})$
II Quadrant	$\theta = \pi - \tan^{-1}(\frac{y}{x})$
III Quadrant	$\theta = -\pi + \tan^{-1}(\frac{y}{x})$
IV Quadrant	$\theta = -\tan^{-1}(\frac{y}{x})$

- De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
- If $Z^p = r(\cos\theta + i\sin\theta)$ then $Z = r^{1/p}[cos(2n\pi + \theta) + isin(2n\pi + \theta)]^{1/p}$
  That is, $Z = r^{1/p}[cos(\frac{2n\pi + \theta}{p}) + isin(\frac{2n\pi + \theta}{p})]$
- $-1 = \cos\pi + i\sin\pi, \quad 1 = \cos 0 + i\sin 0$
  $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \quad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$
- $1 + x + x^2 + ... + x^{n-1} = \frac{x^n - 1}{x - 1}$ $(if\ |x| > 1)$.
  . $= \frac{1 - x^n}{1 - x}(if\ |x| < 1)$
- $1 + x + x^2 + ... = \frac{1}{1 - x}(if\ |x| < 1)$

(I)**Expansion of $\sin n\theta, \cos n\theta$ in powers of $\sin\theta, \cos\theta$**

- By the De Moivre's Theorem, $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

- Use Binomial Theorem on the R.H.S
  i.e. $(a+b)^n = a^n + nC_1a^{n-1}b + nC_2a^{n-2}b^2 + \cdots + nC_{n-1}ab^{n-1} + b^n$
  Here $a = \cos\theta, \quad b = i\sin\theta$

- By equating real and imaginary parts, we get expansion of $\sin n\theta, \cos n\theta$ in powers of $\sin\theta, \cos\theta$

(II) **Expansions of $\sin^n\theta, \cos^n\theta$ in terms of sines and cosines of multiples of θ :** We know that,

- If $x = \cos\theta + i\sin\theta \Rightarrow \dfrac{1}{x} = \cos\theta - i\sin\theta$

- $x^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

- $x^n + \dfrac{1}{x^n} = 2\cos n\theta$ and $x^n - \frac{1}{x^n} = 2i\sin n\theta$

(III)Procedure to find the n th roots of a given complex number z. That is we need to find $z^{\frac{1}{n}}$. Then

- Step 1: Find the polar form of the given complex number $z = r(\cos\theta + i\sin\theta)$

- Step 2: Write the general value of the argument θ as $\theta + 2k\pi$ where k is an integer

- Step 3: Raise the number obtained in Step 2 to the power $\dfrac{1}{n}$ by using De Moivre 's theorem

- Step 4: For $k = 0, 1, 2 \cdots, n-1$ obtain **distinct** roots

Note: Suppose we need to find $z^{\frac{m}{n}}$. Then

- Step 1 : Find the polar form of the given complex number $z = r(\cos\theta + i\sin\theta)$

- Step 2 A: Find $z^m = r^m(cosm\theta + i\sin m\theta)$ by using De Moivre's Theorem

- Step 2 B: Write the general value of the argument θ as $\theta + 2k\pi$ where k is an integer

- Step 3 : Raise the number obtained in Step 2 to the power $\dfrac{1}{n}$ by using De Moivre 's theorem

- Step 4 : For $k = 0, 1, 2 \cdots, n-1$ obtain **distinct** roots

Complex Functions: Hyperbolic Functions and Logarithm of Complex Numbers

1. $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \dfrac{e^{ix}-e^{-ix}}{2i}, \quad \tan x = \dfrac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}$

2. $\cosh x = \dfrac{e^{x}+e^{-x}}{2}, \quad \sinh x = \dfrac{e^{x}-e^{-x}}{2}, \quad \tanh x = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

3. $sin(ix) = i\sinh x, \quad \cos(ix) = \cosh x, \quad \tan(ix) = i\tanh x$

4. $\cosh^2 x - sinh^2 x = 1$
   $sech^2 x + tanh^2 x = 1$
   $coth^2 x + cosech^2 x = 1$

5. $sinh^{-1}x = log(x+\sqrt{x^2+1}), \qquad \cosh^{-1}x = log(x+\sqrt{x^2-1})$
   $tanh^{-1}x = \dfrac{1}{2}log(\dfrac{1+x}{1-x}), \qquad sech^{-1}x = log(\dfrac{1+\sqrt{x^2-1}}{x})$
   $coth^{-1}x = \dfrac{1}{2}log(\dfrac{1-x}{1+x}), \qquad cosech^{-1}x = log(\dfrac{1+\sqrt{1+x^2}}{x})$

6. $log(x+iy) = \log r + i\theta$
   $Log(x+iy) = \log(\sqrt{x^2+y^2}) + i(2\pi n + tan^{-1}\dfrac{y}{x})$

7. $\sin\ (x \pm iy) = \sin x.\cosh y \pm i\cos x.\sinh y$
   $\cos\ (x \pm iy) = \cos x.\cosh y \pm i\sin x.\sinh y$
   $tan\ (x \pm iy) = \dfrac{\sin(x\pm iy)}{\cos(x\pm iy)} = \dfrac{\sin 2x}{\cos 2x + \cosh 2y} + i\dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$

8. $\dfrac{d}{dx}(\sinh x) = \cosh x, \qquad \dfrac{d}{dx}(\cosh x) = \sinh x$
   $\dfrac{d}{dx}(\tanh x) = \ sech^2 x, \qquad \dfrac{d}{dx}(\ cosecx) = -\ cosechx.\coth x$
   $\dfrac{d}{dx}(\ sechx) = -\ sechx.\tanh x, \qquad \dfrac{d}{dx}(\coth x) = -\ cosech^2 x$

9. $\sinh(-x) = -\sinh x, \quad \cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x$

10. $\cosh(A+B) = \cosh A.\cosh B + \sinh A.\sinh B$

11. $\sinh(u) = \dfrac{2\tanh(\frac{u}{2})}{1-\tanh^2(\frac{u}{2})}, \quad \cosh(u) = \dfrac{1+\tanh^2(\frac{u}{2})}{1-\tanh^2(\frac{u}{2})}, \quad \tanh(u) = \dfrac{2\tanh(\frac{u}{2})}{1+\tanh^2(\frac{u}{2})}$

12. $\tanh(A\pm B) = \dfrac{\tanh A \pm \tanh B}{1 \pm \tanh A.\tanh B}, \quad \tanh(2\theta) = \dfrac{2\tanh\theta}{1+\tanh^2\theta},$

13. $\sinh(2x) = 2\sinh x.\cosh x, \quad \cosh(2x) = \cosh^2 x + \sinh^2 x$
14. $\sinh(3x) = 3\sinh x + 4\sinh^3 x, \quad \cosh(3x) = 4\cosh^3 x - 3\cosh x$
15. $x^m + \dfrac{1}{x^m} = 2\cos m\theta, \quad x^m - \dfrac{1}{x^m} = 2i\sin m\theta$, where $x = \cos\theta + i\sin\theta$

<u>Circular Functions Formulae</u>	<u>Corresponding Hyperbolic Functions Formulae</u>
a) $\cos^2 x + \sin^2 x = 1$	a) $\cosh^2 x - \sinh^2 x = 1$
b) $\sec^2 x - \tan^2 x = 1$	b) $sech^2 x + \tanh^2 x = 1$
c)$cosec^2 x - \cot^2 x = 1$	c) $-cosech^2 x + \coth^2 x = 1$
d)$\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$	d) $\sinh 2x = \dfrac{2\tanh x}{1-\tanh^2 x}$
e) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$	e) $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$
f) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$	f) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
g) $\sin 3x = 3\sin x - 4\sin^3 x$	g) $\sinh 3x = 3\sinh x + 4\sinh^3 x$
h) $\cos 3x = 4\cos^3 x - 3\cos x$	h) $\cosh 3x = 4\cosh^3 x - 3\cosh x$
i) $\cos 2x = \cos^2 x - \sin^2 x$	i) $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cos^2 x - 1$	$= 2\cosh^2 x - 1$
$= 1 - 2\sin^2 x$	$= 1 + 2\sinh^2 x$

Tutorial Questions

1. Evaluate (i) $\dfrac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9}$ (ii) $\dfrac{(1+i\sqrt{3})^9(1-i)^4}{(i+1)^4(i+\sqrt{3})^{12}}$

2. Show that $\dfrac{1+cos\alpha+i\sin\alpha}{1-\cos\alpha+i\sin\alpha}=\cot(\alpha/2)e^{i(\alpha-\pi/2)}$

3. If α, β are the roots of the equation $x^2-2x+4=0$,
   Prove that $\alpha^n+\beta^n=2^{n+1}\cos(\dfrac{n\pi}{3})$. Hence evaluate $\alpha^{15}+\beta^{15}$.

4. Prove that $\alpha^n+\beta^n = 2\cos n\theta.\operatorname{cosec}^n\theta$, if α, β are the roots of the equation $z^2\sin^2\theta - z\sin 2\theta+1=0$.

5. If n is a positive integer show that $(1+i)^n+(1-i)^n=2.2^{n/2}\cos(\dfrac{n\pi}{4})$.
   Hence evaluate $(1+i)^{10}+(1-i)^{10}$.

6. If $z=-\dfrac{1+i\sqrt{3}}{2}$, then show that $(z^{3k}+\overline{z}^{3k})=2$

7. If $\sin 5\theta = a\sin\theta + b\sin^3\theta + c\sin^5\theta$ then find the value of a+b+c.

8. Use De Movire's theorem to show that
   $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$
   Hence deduce that $5\tan^4(\pi/10)-10\tan^2(\pi/10)+1=0$

9. Using De Moivre's theorem show that
   (i) $\dfrac{\sin 7\theta}{\sin\theta}=7-56\sin^2\theta+112\sin^4\theta-64\sin^6\theta$;
   (ii) $\dfrac{\sin 6\theta}{\sin 2\theta}=a\cos^4\theta+b\cos^2\theta+c$ find the values of a,b,c.

10. Show that $\cos^8\theta+\sin^8\theta=\dfrac{1}{64}(\cos 8\theta+28\cos 4\theta+35)$

11. If $\cos^6\theta-\sin^6\theta=\alpha\cos 6\theta+\beta\cos 2\theta$ then find the value of α, β.

12. Express $\sin^5\theta$ in a series of sines of multiple of θ.

13. Show that $cos^5\theta.\ sin^3\theta=\dfrac{-1}{2^7}[\sin 8\theta+2\sin 6\theta-2\sin 4\theta-6\sin 2\theta]$

14. Express $\cos^7\theta$ in a series of cosines of multiples of θ.

15. If $\cos^6\theta+\sin^6\theta=\alpha\cos 4\theta+\beta$ then prove that $\alpha+\beta=1$

16. Solve (i)$x^4 - x^3 + x^2 - x + 1 = 0$ (ii)$x^5 - x^4 + x^3 - x^2 + x - 1 = 0$

17. Find the continued product of roots of the equation
(i)$(1+i)^{\frac{2}{3}}$ (ii)$(\frac{1}{2} - i\frac{\sqrt{3}}{2})^{\frac{3}{4}}$

18. If $z^3 = (z+1)^3$ then show that $z = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}$.

19. Find the roots common to
    (a) $x^4 + 1 = 0$ *and* $x^6 - i = 0$
    (b) $x^{12} - 1 = 0$ *and* $x^4 - x^2 + 1 = 0$

20. If $\cosh x - 5\sinh x - 5 = 0$ then find $\tanh x$.

21. Solve $17\cosh x + 18\sinh x - 1 = 0$, for real values of x.

22. If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$, Prove that $\cos 2\theta \cdot \cosh 2\phi = 3$

23. If $\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$, Prove that $\phi = \frac{1}{2}\log\left(\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right)$

24. If $\text{cosec}(\frac{\pi}{4} + ix) = u + iv$, prove that $(u^2+v^2)^2 = 2(u^2 - v^2)$

25. If $\cos\alpha.\cosh\beta = \frac{x}{2}$ and $\sin\alpha.\sinh\beta = \frac{y}{2}$, prove that
(i) $\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{4x}{x^2+y^2}$
(ii) $\sec(\alpha - i\beta) - \sec(\alpha + i\beta) = \frac{-4iy}{x^2+y^2}$

26. If $\tan(x+iy) = \alpha + i\beta$. Prove that $\frac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2} = \frac{\cos 2x}{\cosh 2y}$.

27. If $\tan(x+iy) = a + ib$. Prove that $\tanh(2y) = \frac{2b}{1+a^2+b^2}$

28. Separate into real and imaginary parts
(i) $\sin^{-1}\left(e^{i\theta}\right)$ (ii)$\cos^{-1}\left(e^{i\theta}\right)$ (iii)$(\sqrt{3}+i)^{(\sqrt{3}-i)}$ (iii)$(1+\sqrt{3}i)^{(1+\sqrt{3}i)}$

29. If $\sqrt{i}^{\sqrt{i}^{\sqrt{i}^{\cdots}}} = \alpha + i\beta$. Prove that $\alpha^2 + \beta^2 = e^{\frac{-\pi\beta}{2}}$ and $\tan^{-1}(\frac{\beta}{\alpha}) = \frac{\pi\alpha}{4}$

30. Prove that $\sec h^{-1}(\sin\theta) = \log(\cot(\theta/2))$

31. $\tanh(\alpha + i\beta) = x + iy$ then prove that
    (i)$x^2 + y^2 - 2x\coth(2\alpha) = -1$ (ii)$x^2 + y^2 + 2y\cot(2\beta) = 1$

32. Considering the principal value only prove that $\log_2(-3) = \dfrac{\log 3 + i\pi}{\log 2}$

33. Prove that $Log_i(i) = \dfrac{4n+1}{4m+1}$ where m, n are integers

34. If $\log(x + iy) = e^{p+iq}$ Find the value of y

Module 2 : Successive Differentation and Expansion of Functions

n^{th} order derivative of some standard functions

Standard function	n^{th} derivative
e^{ax}	$a^n e^{ax}$
a^{mx}	$m^n a^{mx}(loga)^n$
$(ax+b)^m$	$\frac{a^n m!(ax+b)^{m-n}}{(m-n)!}$, if $m > n$ $n!a^n$, if $m = n$ 0, if $m < n$
$(ax+b)^{-m}$, m is a positive integer	$(-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{a^n}{(ax+b)^{m+n}}$ $\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$,when m=1
$\log(ax+b)$	$\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
$\sin(ax+b)$	$a^n \sin(\frac{n\pi}{2} + ax + b)$
$\cos(ax+b)$	$a^n \cos(\frac{n\pi}{2} + ax + b)$
$e^{ax}\cos(bx+c)$	$r^n e^{ax}\cos(bx+c+n\theta)$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$
$e^{ax}\sin(bx+c)$	$r^n e^{ax}\sin(bx+c+n\theta)$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$

Leibnitz theorem: Suppose u and v are both functions of x and $y = uv$. Then the nth order derivative of y is given by
$y_n = (uv)_n$
$= nC_0 u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \cdots + nC_r u_{n-r} v_r + \cdots + nC_n u v_n$
$\Rightarrow y_n = (uv)_n = u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \cdots + nC_r u_{n-r} v_r + \cdots + u v_n$
that is,$y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2} u_{n-2} v_2 + \cdots + \frac{n(n-1)\cdots(n-(r-1)}{r!} u_{n-r} v_r + \cdots + u v_n$

Taylor's Theorem :

If $f(x)$ and its first $(n-1)$ derivative be continuous in $[a, a+h]$, and $f^n(x)$ exists for every value of x in $(a, a+h)$, then there is at least one number $\theta \ (0 < \theta < 1)$, such that

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n$$

where $R_n = \dfrac{h^n}{n!}f^n(a + \theta h)$

If $f(x)$ is infinitely differentiable at a and $R_n \to 0$ as $n \to \infty$. Then we say that the Taylor Series converges to the function $f(x)$ at the point x. Hence

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + ...$$

Let $a + h = x \implies h = x - a$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + ...$$

gives the expansion of $f(x)$ as a power series(**Taylor's series**) about point a.

Maclaurin's Series: In Taylor's series if we put $a = 0$, we get Maclaurin's Series

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

Tutorial Questions

1. if $y = (2x+5)^{25}$, find y_{20}, y_{28}, y_{25}
2. Find the nth derivatives of

   (a) $\dfrac{1}{(x-1)(x-2)(x-3)}$ (b) $\dfrac{1}{(3x-2)(x-3)^2}$ (c) $y = sin^5 x.cos^3 x$

   (d) $\dfrac{x^2}{(x+2)(2x+3)}$ (e) $\dfrac{x^2+4x+1}{x^3+2x^2-x-2}$ (f) $e^{2x}\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)\sin 3x$

3. If $y = \dfrac{1}{x^2+a^2}$ prove that $y_n = \dfrac{(-1)^n n! \sin(n+1)\theta(\sin\theta)^{n+1}}{a^{n+2}}$ where $\theta = \tan^{-1}(a/x)$
4. If $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ prove that $y_n = (-1)^{n-1}(n-1)!(\sin\theta)^n \sin n\theta$
5. If $y = (x^2-1)^n$ prove that $(x^2-1)y_{n+2} + 2nxy_{n+1} - n(n+1)y_n = 0$
6. If $x = \cos[logy^{\frac{1}{m}}]$ show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. Hence, evaluate $y_n(0)$
7. If $y = \log\left(x+\sqrt{1+x^2}\right)^2$ show that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2)y_n = 0$
8. Expand using Taylor Series $f(x) = x^6 + 2x^4 - x^3 + x^2 - x - 2$ in powers of $(x-1)$. Hence find $f\left(\dfrac{3}{10}\right)$.
9. Expand $log(1+x+x^2+x^3)$ upto x^8.
10. Expand $(1+x)^x$ upto x^4.
11. Expand $f(x) = x^3 - x^2 + x - 1$ in powers of $(x+1)$.
12. Using Taylor's theorem to find $\sqrt{35.25}$
13. Using Taylor's theorem to find $\sqrt{15.25}$

Module 3 : Partial Differentiation and its Application

Partial differentiation of composite function

Let $u = f(x, y, z)$ and $x = x(s, t), y = y(s, t), z = z(s, t)$.

The function f is considered as a function of s and t via the intermediate variables x, y, z.
That is,

$$f \to (x, y, z) \to (s, t)$$

Now the partial derivative of f w.r.t. s keeping t constant is:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

In the similar way, we get,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Chain rule

If $u = f(x, y, z)$ and $x = x(t), y = y(t), z = z(t)$
That is,

$$f \to (x, y, z) \to t$$

then the total derivative of f is,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} + \frac{\partial f}{\partial z}.\frac{dz}{dt}$$

Definition A function $f(x, y)$ in two variables x and y is said to be a **homogeneous function** of degree n, if for any positive number λ,

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Euler's theorem on Homogeneous functions

Statement: If f is a homogeneous function in x and y of degree n, then,
$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

Corollary: If f is a homogeneous function of degree n, then,
$x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f$

Corollary: If z is a homogeneous function of degree n in x and y and $z = f(u)$, then,

$$\begin{aligned}
\textbf{(i)} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= n\frac{f(u)}{f'(u)} \\
\textbf{(ii)} \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= G(u)[G'(u) - 1] \\
\text{where } G(u) &= n\frac{f(u)}{f'(u)}
\end{aligned}$$

Method of Finding Extrema of f(x,y)

1. Solving $f_x = 0$ & $f_y = 0$ yields critical or stationary point $P(a, b)$ of f.
2. Calculate $f_{xx}(a, b) = r, f_{xy}(a, b) = t, f_{xy}(a, b) = s$ at the critical point P.
3. (a) Maximum: if $rt - s^2 > 0,\ r < 0$ then f has a maximum at P.
   (b) Minimum: if $rt - s^2 > 0,\ r > 0$ then f has a minimum at P.
   (c) Saddle point: if $rt - s^2 < 0$ then f has neither maximum nor minimum.
   (d) No Conclusion: if $rt - s^2 = 0$, further investigation needed.

Tutorial Questions

1. $If\ u = x^y,\ Show\ that\ \dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$

2. $Verify\ that\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}, for(i)\ u = x^y + y^x (ii) u = x^3 y + e^{xy^2}$

3. $If\ u = tan^{-1}(\dfrac{x}{y}),\ where\ x = 2t,\ y = 1 - t^2,\ find\ \dfrac{du}{dt}$

4. $If\ z = sin^{-1}(x - y),\ x = 3t,\ y = 4t^3,\ Prove\ \dfrac{dz}{dt} = \dfrac{3}{\sqrt{1-t^2}}$

5. $If\ u = log(x^3 + y^3 - x^2 y - xy^2),\ Prove\ that$
$\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{-4}{(x+y)^2}$

6. $If\ u = log(x^3 + y^3 + z^3 - 3xyz),\ Prove\ that$
$\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$

7. $If\ f = \phi(u, v)\ and\ u = x^2 - y^2,\ v = 2xy,$
$Prove\ that\ (\dfrac{\partial f}{\partial x})^2 + (\dfrac{\partial f}{\partial y})^2 = 4(u^2 + v^2)^{\frac{1}{2}}[(\dfrac{\partial f}{\partial u})^2 + (\dfrac{\partial f}{\partial v})^2]$

8. $If\ z = f(u, v), u = lx + my,\ v = ly - mx,$
$Prove\ that\ \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = (l^2 + m^2)(\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2})$

9. $If\ x = \sqrt{vw},\ y = \sqrt{uw},\ z = \sqrt{uv},\ Prove\ that$
$x\dfrac{\partial \phi}{\partial x} + y\dfrac{\partial \phi}{\partial y} + z\dfrac{\partial \phi}{\partial z} = u\dfrac{\partial \phi}{\partial u} + v\dfrac{\partial \phi}{\partial v} + w\dfrac{\partial \phi}{\partial w}\ where\ \phi\ is\ a\ function\ of\ x, y, z$

10. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\dfrac{1}{x}\dfrac{\partial u}{\partial x} + \dfrac{1}{y}\dfrac{\partial u}{\partial y} + \dfrac{1}{z}\dfrac{\partial u}{\partial z} = 0$

11. If $u = f(\dfrac{y-x}{xy}, \dfrac{z-x}{xz})$, prove that $x^2\dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} + z^2\dfrac{\partial u}{\partial z} = 0$

12. If $u = \dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{\log x - \log y}{x^2 + y^2}$, prove that $xu_x + yu_y + 2u = 0$

1. If $u = x^y$, Show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Starting off with the LHS first $\frac{\partial^3 u}{\partial x \partial x \partial y}$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right) = x^y \qquad \frac{\partial u}{\partial y} = \underbrace{x^y}_{u}\,\underbrace{\log x}_{v} \qquad uv' + vu'$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^y \frac{1}{x} + y x^{y-1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^{y-1} + y(\underbrace{x^{y-1}}_{u}\,\underbrace{\log x}_{v})$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right) = y-1 x^{y-2} + y\left(x^{y-1}\frac{1}{x} + \log x\,(y-1)\,x^{y-2}\right.$$

$$\boxed{= y-1x^{y-2} + yx^{y-2} + (y-1)\log x\, x^{y-2}} \quad \text{LHS}$$

now for the RHS $\frac{\partial^3 u}{\partial x \partial y \partial x}$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right) = \frac{\partial u}{\partial x}\, x^y$$

$$\boxed{\frac{\partial u}{\partial x} = y x^{y-1}}$$

$$\boxed{\frac{\partial u}{\partial x} = y x^{y-1}} \qquad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \underset{u}{y}\ \underset{v}{x^{y-1}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = y\left(\underset{u}{x^{y-1}}\ \underset{v}{\log x}\right) + (1)\ x^{y-1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right) = y\left(x^{y-1}\ \frac{1}{x} + \log x\ (y-1)\, x^{y-2}\right)$$

$$+\ y-1 \cdot x^{y-2}$$

$$\boxed{= y\, x^{y-2} + \log x\,(y-1)\, x^{y-2} + x^{y-2}(y-1)}$$

Since LHS = RHS hence Proved

2. *Verify that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, *for* (i) $u = x^y + y^x$ (ii) $u = x^3 y + e^{xy^2}$

(i) $u = x^y + y^x$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \text{To Prove this one}$$

LHS

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial y} = \quad u = x^y + y^x$$

$$\frac{\partial u}{\partial y} = \underbrace{x^y}_{u}\,\underbrace{\log x}_{v} + \underbrace{x}_{u}\,\underbrace{y^{x-1}}_{v}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^y \frac{1}{x} + \log x \; y\, x^{y-1} + x\,(x-1)\, y^{x-2} + y^{x-1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \underline{x^{y-1}} + \underline{\log x \; y \; x^{y-1}} + \boxed{x(x-1)\, y^{x-2}} + y^{x-1}$$

now RHS $= \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$$\frac{\partial u}{\partial x} = \underbrace{y}_{u}\cdot\underbrace{x^{y-1}}_{v} + \underbrace{y^x}_{u}\,\underbrace{\log y}_{v}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) = \underline{y\, x^{y-1} \log x} + \underline{x^{y-1}} + y^x \frac{1}{y} + \boxed{x\, y^{x-1} \log y}$$

Since both LHS = RHS $\underline{\underline{H.P}}$

3. *If* $u = tan^{-1}(\frac{x}{y})$, *where* $x = 2t,\ y = 1 - t^2,$ *find* $\frac{du}{dt}$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{tikzcd} & u \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & t & \end{tikzcd}$$

$$\frac{\partial x}{\partial t} = 2 \qquad \frac{\partial y}{\partial t} = -2t$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = 2 \qquad \frac{\partial y}{\partial t} = -2t$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}(2) + \frac{\partial u}{\partial y}(-2t)$$

$$\boxed{\frac{\partial u}{\partial t} = 2\frac{\partial u}{\partial x} - 2t\left(\frac{\partial u}{\partial y}\right)}$$

$$\frac{du}{dx} = \tan^{-1}\left(\frac{x}{y}\right) \Rightarrow \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\frac{y^2+x^2}{x^2}}\left(\frac{1}{y}\right) = \frac{x^2}{y^2+x^2}\left(\frac{1}{y}\right)$$

$$\frac{du}{dy} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) \Rightarrow \frac{\cancel{y^2}}{y^2+x^2}\left(-\frac{x}{\cancel{y^2}}\right)$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y^2+x^2}$$

$$\frac{\partial u}{\partial t} = 2\left(\frac{1}{y}\right)\left(\frac{x^2}{y^2+x^2}\right) + 2t\left(\frac{x}{y^2+x^2}\right)$$

$$= \frac{2x^2}{y^3+yx^2} + \frac{2tx}{y^2+x^2}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{2x^2 + 2txy}{y(y^2+x^2)}}$$

4 If $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, Prove $\dfrac{dz}{dt} = \dfrac{3}{\sqrt{1-t^2}}$

(A) okay now you stupid ass nigga yo fucked up by substituting x & y into $\sin^{-1}(x-y)$'s place

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & t & \end{tikzcd}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$= \frac{1}{\sqrt{1-(x^2-2xy+y^2)}} = \frac{1}{\sqrt{1-x^2+2xy-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(-1) \Rightarrow -\frac{1}{\sqrt{1-x^2+2xy+y^2}}$$

$$\frac{\partial x}{\partial t} = 3 \quad \frac{\partial y}{\partial t} = 12t^2$$

$$\frac{\partial z}{\partial t} = \frac{1}{\sqrt{1-(x-y)^2}}(3) - \frac{12t^2}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial z}{\partial t} = \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

5. *If* $u = \log(x^3 + y^3 - x^2y - xy^2)$, *Prove that*

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$$

} Extremely lengthy Question

Starting off with LHS

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$\left(\frac{\partial u}{\partial x}\right) = \frac{1}{x^3+y^3-x^2y-xy^2} \cdot (3x^2 - 2xy - y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{3x^2 - 2xy - y^2}{x^3 + y^3 - x^2y - xy^2} \qquad \boxed{\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}}$$

$$= \frac{(x^3+y^3-x^2y-xy^2)\cdot(6x-2y) - (3x^2-2xy-y^2)\cdot(3x^2-2xy-y^2)}{(x^3-y^3-x^2y-xy^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^3+y^3-x^2y-xy^2)(6x-2y) - (3x^2-2xy-y^2)(3x^2-2xy-y^2)}{(x^3-y^3-x^2y-xy^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{(x^3+y^3-x^2y-xy^2)} \cdot 3y^2 - x^2 - 2xy$$

$$\left(\frac{\partial u}{\partial y}\right) = \frac{3y^2 - x^2 - 2xy}{x^3+y^3-x^2y-xy^2}$$

$$\left(\frac{\partial u}{\partial y}\right) = \frac{3y^2 - x^2 - 2xy}{x^3 + y^3 - x^2y - xy^2} \qquad \boxed{\frac{u}{v} = \frac{vu' - uv'}{v^2}}$$

$$2\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right) = 2\left(\frac{(x^3+y^3-x^2y-xy^2\,(-2x-2y) - 3y^2-x^2-2xy\,(3x^2-2xy-y^2))}{(x^3+y^3-x^2y-xy^2)^2}\right)$$

$$\frac{2(x^3+y^3-x^2y-xy^2\,(-2x-2y) - 3y^2-x^2-2xy\,(3x^2-2xy-y^2)}{(x^3+y^3-x^2y-xy^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3-x^2y-xy^2}\cdot(3y^2-x^2-2xy) \quad \frac{u}{v}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) =$$

$$\frac{(x^3+y^3-x^2y-xy^2)(6y^2-2x) - (3y^2-x^2-2xy)(3y^2-x^2-2xy)}{(x^3+y^3-x^2y-xy^2)^2}$$

on doing everything imaginable we finally get

$$\frac{-y}{(x+y)^2}$$ hence LHS = RHS

6. *If* $u = log(x^3 + y^3 + z^3 - 3xyz)$, *Prove that*

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

now on the LHS side on Expanding it

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} + \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} + \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}\right)$$

Simplifying this stuff by using

$$x^3+y^3+z^3-3xyz = (x^2+y^2+z^2-xz-yz-xy)(x+y+z)$$

it boils down to $3\left(\frac{1}{x+y+z}\right)$

$$\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= 3\left(\frac{-1}{(x+y+z)^2} - \frac{3(1)}{(x+y+z)^2} - \frac{3(1)}{(x+y+z)^2}\right)$$

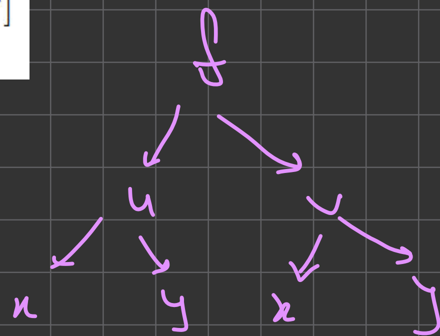
$$\boxed{= \frac{-9}{(x+y+z)^2}}$$

fuck u nigger

7. *If* $f = \phi(u, v)$ *and* $u = x^2 - y^2$, $v = 2xy$,

Prove that $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 = 4(u^2+v^2)^{\frac{1}{2}}[(\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2]$

Starting off with LHS



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(2y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}(-2y) + \frac{\partial f}{\partial v}(2x)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(2y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}(-2y) + \frac{\partial f}{\partial v}(2x)$$

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

$$\Rightarrow \left(\frac{\partial f}{\partial u}(2x) + 2y\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial f}{\partial u}(-2y) + \frac{\partial f}{\partial v}(2x)\right)^2$$

$$\left(\frac{\partial f}{\partial u}\right)^2 4x^2 + \cancel{8xy\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}} + 4y^2\left(\frac{\partial f}{\partial v}\right)^2 +$$

$$4y^2\left(\frac{\partial f}{\partial u}\right)^2 - \cancel{8xy\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}} + 4x^2\left(\frac{\partial f}{\partial v}\right)^2$$

$$4x^2+4y^2\left(\frac{\partial f}{\partial u}\right)^2 + 4x + 4y^2\left(\frac{\partial f}{\partial v}\right)^2$$

$$\Rightarrow 4x^2+4y^2\left(\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right)$$

13. If $u = \left(\frac{x}{y}\right)^{\left(\frac{y}{x}\right)}$ then evaluate $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

14. $If\ u = log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right),\ \ Prove\ that\ x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$

15. If $u = tan^{-1}(\frac{x^3+y^3}{x-y})$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} =$ $2\cos 3u \sin u$.

16. $u = \ log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right) - \frac{1}{3}\log(\frac{x^3+y^3}{x^2+y^2})$ then evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

17. $If\ u = sin^{-1}\sqrt{\frac{x^2+y^2}{x+y}} + sinh^{-1}(\frac{x^3+y^3}{x^2+y^2}),$
$Show\ that\ x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{1}{4}tanu.(tan^2u-1) - \tanh^3 u$

18. Show that the minimum value of $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$

19. Discuss maximum and minimum values of the function $u = \sin x. \sin y. \sin{(x+y)}$

20. Find the extreme values of the following functions:
$(a)\ x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ $(b)\ y^2 + 4xy + 3x^2 + x^3$
$(c)\ x^4 + y^4 - 2x^2 + 4xy - 2y^2$ $(d)\ x^4 + y^4 - 2(x-y)^2$
$(e)\ x^3 + xy^2 + 21x - 12x^2 - 2y^2$ $(e) x^3 + y^3 - 63(x+y) + 12xy$

Module 4 : Matrices

1. A square matrix A is called an **Orthogonal matrix**, if $A.A^t = A^t.A = I$. i.e., If A is Orthogonal then $A^{-1} = A^t$

2. A square matrix A is called an **Unitary matrix**, if $A.A^\theta = A^\theta.A = I$. i.e., If A is unitary then $A^{-1} = A^\theta$

3. **Row Echelon Form:** Let A be a matrix of order m x n, then the Row echelon form of A is a matrix in which:

   (a) The first non-zero entry in a non-zero row is a 1

   (b) In consecutive non-zero rows, the first entry 1 in the lower row appears to the right of the 1 in the higher row.

   (c) Rows consisting of all zeros are at the bottom of the matrix.

4. **Rank of a Matrix:** Rank of matrix A is number of non zero rows in Row Echelon form of A.

5. **Normal Form/ Canonical Form:** By performing elementary transformation (Row as well as column), any non-zero matrix A can be reduced to one of the following four forms, called the Normal form/ Canonical form of A :

   (i) I_r (ii) $[I_r \quad 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

   The number r so obtained is called the rank of A and we write $\rho(A) = r$.

   The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called **first canonical form** of A.

6. **PAQ Form:** If A is a matrix of rank r, then there exists non-singular matrices P and Q such that PAQ is in the normal form

   i.e, $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

   Steps to find P & Q:

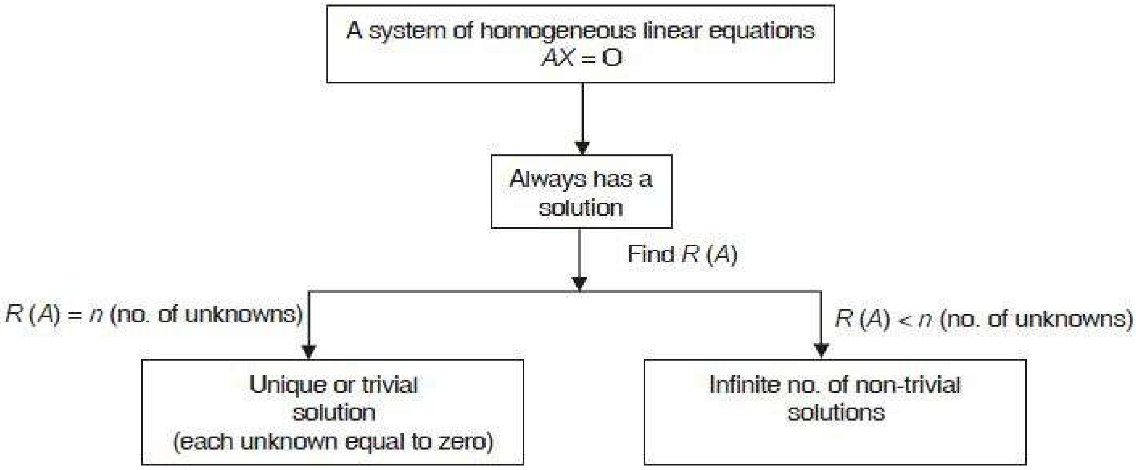
   (a) If A is a matrix of order m x n, then write $A = I_m A I_n$ where I_m & I_n are identity matrices of order m x m & n x n respectively.

   (b) Whatever row operations we will use to convert A into normal form same row operations must be used on Prefactor $(I_m)$of R.H.S.

   (c) Whatever column operations we will use to convert A into normal form same column operations must be used on Postfactor $(I_n)$of R.H.S.

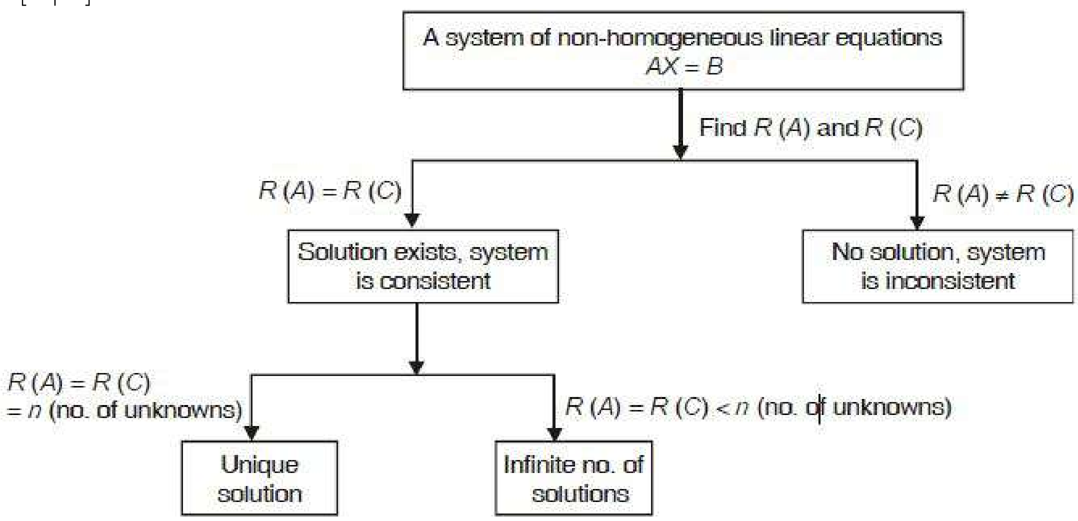
(d) Finally when A will be converted to normal form I_m & I_n will be converted to P and Q respectively.

Condition for Consistency of Homogeneous Equations



Condition for Consistency of Non-Homogeneous Equations

Let $[A|B] = C$



Gauss Elimination Method

The Gauss Elimination method is a direct method for solving a system of linear equations because, within a finite number of determined steps, we can solve the given system of linear equations.

The steps of the Gauss elimination method are

(1) Write the given system of linear equations in matrix form AX = B, where A is the coefficient matrix, X is a column matrix of unknowns and B is the column matrix of the constants.

(2) Reduce the augmented matrix [A : B] by elementary row operations to

get [A' : B'].
(3) We get A' as an upper triangular matrix.
(4) By the backward substitution in A'X = B', we get the solution of the given system of linear equations.

Gauss Jordan Method
1) Write the augmented matrix. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
2) Use a row operation to get a 1 as the entry in the first row and first column.
3) Use row operations to make all other entries as zeros in column one.
4) Interchange rows if necessary to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.
5) Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero. The final matrix is called the reduced row-echelon form.

Tutorial Questions

1. If A is orthogonal, find a, b, c where $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$

2. Show that $\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal and find its inverse

3. Prove that $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$ is unitary.

4. Show that the matrix $\begin{bmatrix} a+ib & -c+id \\ c+id & a-ib \end{bmatrix}$ is unitary if $a^2+b^2+c^2+d^2 = 1$

5. Find the value of p for which the matrix $A = \begin{bmatrix} p & p & 2 \\ 2 & p & p \\ p & 2 & p \end{bmatrix}$ will have

   (i) rank 1 (ii) rank 2 (iii) rank 3

6. Find the value of k for which the following matrix A will have

   (i) rank 1 (ii) rank 2 (iii) rank 3: where $A = \begin{bmatrix} 1 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$

7. Reduce the following matrices to normal form and find their rank:

   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 6 & 10 \\ 4 & 6 & 8 \\ 15 & 27 & 39 \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{3}{4} & \frac{9}{5} & -\frac{1}{2} \\ \frac{30}{2} & -18 & 5 \\ \frac{57}{4} & -\frac{81}{5} & \frac{9}{2} \end{bmatrix}$

8. Find non singular matrices P and Q such that PAQ is in the normal form. Find Inverse of A, if exists:

   (a) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 1 \\ -2 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

9. Solve the following systems of equations using Gauss Elimination Method:
(a) $x + 2y - z = 1;\ x + y + 2z = 9;\ x + y - z = 2$
(b)$x_1 + 2x_2 - x_3 = 1;\ 3x_1 - 2x_2 + 2x_3 = 2;\ 7x_1 - 2x_2 + 3x_3 = 5$
(c)$3x + y - z = 3;\ 2x - 8y + z = -5;\ x - 2y + 9z = 8$

10. Solve the following systems of equations using Gauss Jacobi Method:
(a) $x + y + z = 3;\ x + 2y + 3z = 4;\ x + 4y + 9z = 6$
(b)$5x_1 + 2x_2 + x_3 = 12;\ -x_1 + 4x_2 + 2x_3 = 2;\ 2x_1 - 3x_2 + 10x_3 = -45$
(c)$3x + 4y - z = 8;\ -2x + y + z = 3;\ x + 2y - z = 2$

11. Find for what values of λ and μ the equations
$x + 2y + 3z = 4;\ x + 3y + 4z = 5;\ x + 3y + \lambda z = \mu$ have
(1) no solution (2) unique solution (3) infinite no. of solutions

12. For what value of λ the following system of equations possesses a non-trivial solution, obtain solution for real values of λ
$3x_1 + x_2 - \lambda x_3 = 0;\ 4x_1 - 2x_2 - 3x_3 = 0;\ 2\lambda x_1 + 4x_2 - \lambda x_3 = 0$

13. Show that the only real value of 'λ' for which the following equations have non zero solution is 6.
$x + 2y + 3z = \lambda x;\ 3x + y + 2z = \lambda y;\ 2x + 3y + z = \lambda z$

14. Find for what value of λ and μ the system of equations
$2x + 3y + 5z = 9;\ 7x + 3y - 2z = 8;\ 2x + 3y + \lambda z = \mu$
has (1) no solution (2) unique solution (3) an infinite no. of solutions

15. Show that the system of equations
$2x - 2y + z = \lambda x;\ 2x - 3y + 2z = \lambda y;\ -x + 2y = \lambda z$
can possess a non-trivial solution if $\lambda = 1,\ \lambda = -3$ obtain the general solution in each case.

Module 5: Introduction to Vector Calculus

- **Gradient:**
  Let $\phi(x, y, z)$ be a real valued (scalar valued) function of variables x, y, z. Derivative of $\phi(x, y, z)$ is given by,
  . $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$
  $\therefore d\phi = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right).(dxi + dyj + dzk)$
  Let $d\bar{r} = dxi + dyj + dzk$ and $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$
  $\therefore d\phi = \nabla\phi.d\bar{r}$, where $d\bar{r}$ is unit vector.

- **Remark:**
  1. Directional derivative of ϕ in the direction $'a'$ is given by $\dfrac{\nabla\phi.\bar{a}}{|\bar{a}|}$
  2. Directional derivative is maximum when $'a'$ is in direction of $\nabla\phi$
     Hence, the maximum value of directional derivative is,
  3. If the function $\phi(x, y, z)$ is constant then derivative $(\dfrac{\nabla\phi.\bar{a}}{|\bar{a}|})$ is zero.
     hence, if $\phi(x, y, z) = c$ then $\nabla\phi = 0$
     i.e., $\nabla\phi$ is perpendicular to surface $\phi(x, y, z) = c$

- **The Divergence & Curl of a vector field**
  Suppose $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector field.
  Then the **divergence of** F is defined by

  $$div\ \bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}, \quad div\ \bar{F} \text{ is a scalar quantity}$$

  And **curl of** F is defined by, $curl\ \bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

  $= \hat{i}(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}) - \hat{j}(\dfrac{\partial F_3}{\partial x} - \dfrac{\partial F_1}{\partial z}) + \hat{k}(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y})$
  $curl\ \bar{F}$ is a vector quantity.

- **Solenoidal, Irrotaional vector fields**
  A vector field $\bar{F}$ is said to be **Solenoidal** if its divergence is zero.
  That is, vector field $\bar{F}$ is **Solenoidal** if

$\nabla \cdot \bar{F} = 0$

A vector field $\bar{F}$ is said to be **Irrotational** or **Conservative** if its curl is zero.
That is, vector field $\bar{F}$ is **Irrotational** if
$\nabla \times \bar{F} = \bar{0}$

- **Conservative Fields, Scalar Potential and Work Done**
  If vector field $\bar{F}$ is irrotational (Conservative), then $\bar{F}$ can be written as a gradient of a scalar function ϕ
  That is, if $\nabla \times \bar{F} = \bar{0}$ then $\bar{F} = \nabla\phi$
  This scalar filed ϕ is called as the **Scalar Potential** of $\bar{F}$

Tutorial Questions

1. Find directional derivative of the function

   (a) $f(x, y, z) = xy^2 + yz^3$ at the point (2,-1,1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$

   (b) $\phi = x^2yz + 4xz^2$ at the point (1,-2,-1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$

   (c) $\phi = 4xz^2 - 3x^2yz^2$ at (2,-1,2) along z - axis.

   (d) $\phi = xy^2 + yz^3$ at the point P(2,-1,1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at (-1, 2, 1).

   (e) $\phi = \dfrac{y}{x^2 + y^2}$ at the point (0,1) making an angle 30° with the positive X-axis.

2. In what direction from (3,1,-2) is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude?

3. What is the greatest rate of increase of $u = x^2 + yz^2$ at the point (1,-1,3)?

4. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1,1,2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

5. Find a and b such that $\vec{F} = (axy + z^3)i + x^2j + bz^2xk$ is irrotational. **(Dec 2022)**

6. Show that $\bar{F} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2z)\hat{k}$ is both solenoidal and irrotational.**(Dec 2024)**

7. Find the value of constant $'a'$ such that $\bar{A} = (ax + 4y^2z)\hat{i} + (x^3 \sin z - 3y)\hat{j} - (e^x + 4\cos x^2y)\hat{k}$ is solenoidal.

8. If $\bar{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational then find the value of a, b, c. **(May 2023)**

9. Show that $\bar{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is solenoidal and irrotational.**(Dec 2023)**

10. Show that the vector field $\vec{V} = (\sin y + z)\hat{i} + (x \cos y - z)\hat{j} + (x - y)\hat{k}$ is irrotational. Hence find its scalar potential.

11. Show that the vector field $\vec{A}$, where $\vec{A} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is irrotational and find its scalar potential. Also find the work done by $\bar{F}$ in moving a particle from A(1,1,1) to B(1,2,3) along the straight line AB.

12. Given $\bar{F} = (2xy + z)i + (x^2 + 2yz^3)j + (3y^2z^2 + x)k$,

    (a) Prove that F is conservative.

    (b) Find scalar potential ϕ such that $\bar{F} = \nabla\phi$

    (c) Find the work done by $\bar{F}$ in moving a particle from A(1,2,0) to B(2,2,1) along the straight line AB.

Module 6 : Numerical methods

Numerical Solutions of Transcendental Equations:

(I)Bisection Method:

1. Initial Interval:Start with an interval [a, b] where a continuous function $f(x)$ has opposite signs at the endpoints, i.e., $f(a) * f(b) < 0$.
2. Find the Midpoint: Calculate the midpoint of the interval: $c = \frac{(a+b)}{2}$.
3. Check the Midpoint: If f(c) = 0, then c is the root, and the process ends.
   If f(c) has the same sign as f(a) ($i.e., \quad f(c) * f(a) > 0$), the root lies in the interval [c, b]. The new interval becomes [c, b].
   If f(c) has the same sign as f(b) ($i.e., \quad f(c) * f(b) > 0$), the root lies in the interval [a, c]. The new interval becomes [a, c].
4. Repeat: Repeat steps 2 and 3 with the new, smaller interval [a, b] until the interval width is smaller than a desired tolerance or you find a value of f(c) that is sufficiently close to zero.

(II)Newton Raphson Method:

1. Find the interval where the function $f(x)$ have the opposite signs.
2. Take the initial approximation x_0 in the interval.
3. Find $f(x_0)$ and $f'(x_0)$
4. Compute the approximation's using
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \qquad n = 0, 1, 2, \ldots$$
5. Continue this process till, the last two successive approximations are almost same.
   i.e $|x_{n+1} - x_n|$ <error, $n = 0, 1, 2, \ldots$

(III)Regula- Falsi Method:

1. Choose two points a and b such that $f(a)$ and $f(b)$ are of opposite signs.

2. Compute the approximation's using

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \dots\dots (I)$$

3. If $f(x_1)$ and $f(a)$ are of opposite signs, then the root lies between a and x_1 and we replace b by x_1 in (I) and obtain the next approximation x_2. Otherwise, we replace a by x_1 and generate the next approximation.

4. The procedure is repeated till the root is obtained to the desired accuracy.

Numerical Solutions of System of Linear Equations :

(I)Gauss - Jacobi method:

Consider the system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} -\text{(I)}$$

let us assume,

$$|a_1| > |b_1| + |c_1| \ , \qquad |b_2| > |a_2| + |c_2| \ , \qquad |c_3| > |a_3| + |b_3|$$

Now given system can be written as,

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} -\text{(II)}$$

If $x^{(0)}, \ \ y^{(0)}, \ \ z^{(0)}$ are then initial values of $x, \ \ y, \ \ z$ respectively, then

$$\left.\begin{aligned} x^{(1)} &= \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\ y^{(1)} &= \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}) \\ z^{(1)} &= \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)}) \end{aligned}\right\} -\text{(III)}$$

Again using these values $x^{(1)}, \ \ y^{(1)}, \ \ z^{(1)}$ in (II) , we get

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$
$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(1)})$$
$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

Proceed in same way till the convergence is assured (correct to required decimals).
Note:- In the absence of the initial values of $x,\ y,\ z$ we take usually $(0,\ 0,\ 0)$ as the initial value.

(II)Gauss - Seidel method

This method is only a refinement of Gauss - Jacobi method. As we have seen.

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \tag{1}$$
$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \tag{2}$$
$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \tag{3}$$

We start with initial values $y^{(0)}, z^{(0)}$ for 'y' and 'z' and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

while using the second equation , we use $z^{(0)}$ for 'z' and $x^{(1)}$ for 'x' instead of $x^{(0)}$ as in the Jacobi's method , we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ and $y^{(1)}$, use $x^{(1)}$ for 'x' and $y^{(1)}$ for 'y' in (3) eqn. we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

This is the completion of First iteration. Now for Second iteration :-

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)}) \tag{1}$$
$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(2)} - c_2 z^{(1)}) \tag{2}$$
$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(2)} - b_3 y^{(2)}) \tag{3}$$

continue until the convergence is assured.
Note:- As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns , convergence in Gauss - Seidel method is very fast (roughly twice) as compared to Gauss - Jacobi method.

Solve everything upto 3 iteration

Tutorial Questions

1. Solve the following equations by Bisection method:

   (a) $3x + \sin x - e^x = 0$

   (b) $x^3 - 4x - 9 = 0$

   what to take the interval but in this sum

   (c) $x^4 - x - 10 = 0$ (between 1 and 2)

   (d) $x^x + 5x = 1000$ (Hint: $x = (1000 - 5x)^{\frac{1}{x}}$)

2. Solve the following equations by Regula-Falsi method:

   (a) $x^3 = 3x - 4$

   (b) $x^3 - x - 1 = 0$

   (c) $xe^x = \sin x$

   (d) $3x = \cos x + 1$

3. Solve the following equations by Newton Raphson Method:

   (a) $e^{-x} = \sin x$

   (b) $x^4 = 5x + 5$

   (c) $x^3 + 2x - 5 = 0$

   (d) $\sqrt[3]{18}$

4. Solve the following systems of equations using Gauss Jacobi method.

   (a) $10x - 5y - 2z = 3;\ 4x - 10y + 3z = -3;\ x + 6y + 10z = -3$

   (b) $8x - 3y + 2z = 20;\ 4x + 11y - z = 33;\ 6x + 3y + 12z = 72$

   (c) $x + y + 54z = 110;\ 27x + 6y - z = 85;\ 6x + 15y + 2z = 72$

   (d) $28x + 4y - z = 32;\ x + 3y + 10z = 24;\ 2x + 17y + 4z = 35$

5. Solve the following system of equations by Gauss Seidel method

   (a) $5x - y + z = 10,\quad x + 2y = 6,\quad x + y + 5z = -1$

   (b) $5x + 2y + z = 12,\quad x + 4y + 2z = 15,\quad x + 2y + 5z = 20$

   (c) $2x + y + 6z = 9,\quad 8x + 3y + 2z = 13,\quad x + 5y + z = 7$

   (d) $10x + y + z = 12,\quad 2x + 10y + z = 13,\quad 2x + 2y + 10z = 14$

Regula falsi method

(a) $e^{-x} = \sin x$

$e^{-x} - \sin x = 0$

$f(x) = e^{-x} - \sin x = 0$

$f(0\ 5) = 0\ 1271 > 0$

$f(-1) = -0\ 473 < 0$

. the root lies in between $(\overset{a}{-1}, \overset{b}{0\ 5})$

by using the formula $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

$$= \frac{(-1)(0\ 1271) - 0\ 5(-0\ 473)}{0\ 1271 - (-0\ 473)}$$

$x_1 = 0\ 15230 > 0$

$f(x_1) = e^{-(0\ 15230)} - \sin(0\ 15230) = 0\ 7070$

the root lies in between $(\overset{a}{-1}, \overset{b}{0\ 15230})$

the root lies in between $(\overset{a}{-1}, \overset{b}{0\ 15230})$

$$f(a) = e^{-(1)} - \sin(-) = 1\ 209 > 0$$

$$f(b) = f(0\ 15230) = e^{0\ 15230} - \sin(0\ 15230) = 1\ 0127$$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(-)(1\ 0127) - 0\ 15230(1\ 209)}{1\ 0127 - 1.209}$$

$$x_2 = 6\ 0969 > 0$$

$$f(x_2) = e^{(6\ 0964)} - \sin(6\ 0964) = 444\ 66 > 0$$

(c) $xe^x = \sin x$ (doing upto 3rd iteration)

$xe^x - \sin x = 0$

$f(x) = xe^x - \sin x = 0$

The root lies between $(\overset{a}{-3}, \overset{b}{-2.5})$

$f(-3) = -8 \times 10^{-3}$

$f(-2.5) = 0.3932$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{-3(0.3932) - (-2.5)(-0.008)}{0.3932 - (-0.008)}$$

$x_1 = 3.2770$

$f(x_1) = 3.27 \times e^{(3.27)} - \sin(3.27) = 86.16670$

$\therefore$ The root lies in bwt $(\overset{a}{-3}, \overset{b}{3.27})$

$f(a) = f(-3) = -0.008$, $f(b) = f(3.27) = 86.166$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(-3)(86.166) - (3.27)(-0.008)}{86.166 - (-0.008)}$$

$x_2 = -2.9994$

$f(x_2) = f(-2.9994)$

$\Rightarrow xe^x - \sin x \Rightarrow -2.9994\, e^{-2.9994} - \sin(2.9994)$

$f(x_2) = -0.291134$

(a) $3x + \sin x - e^x = 0$

1 (a) $3x + \sin x - e^x = 0$ (Solve till 3 iteration)

let $\underbrace{f(x) = 3x + \sin x - e^x}$

Substitute this thing into calculator in f(x) & Set the table ranges from start = -5, end = 5 & step = 0.5

the root lies between (0.5, 1)

$f(0.5) = 3(0.5) + \sin(0.5) - e^{0.5} = -0.134 < 0$

$f(1) = 3(1) + \sin(1) - e^{(1)} = 0.2991 > 0$

now $x_0 = \left(\frac{0.5+1}{2}\right) = 0.75$

$f(x_0) = f(0.75) = 3(0.75) + \sin(0.75) - e^{0.75} = 0.146020$

now the root lies in between (0.14, 0.5)

$x_1 = \left(\frac{0.14+0.5}{2}\right) = 0.32$

$f(0.32) = 3(0.32) + \sin(0.32) - e^{0.32} = \underline{-0.41120}$

root lies between (1, 0 32)

$$x_2 = \left(\frac{1+0\,32}{2}\right) = 0\,66$$

$$f(0\,66) = 3(0\,66) + \sin(0\,66) - e^{0\,66} = 0\,056 > 0$$

. the root lies between (0 32, 0.66)

$$x_3 = \left(\frac{0\,32 + 0\,66}{2}\right) = \underline{\underline{0\,49}}$$

$$f(0\,49) = 3(0\,49) + \sin(0\,49) - e^{0\,49} = -0\,153 < 0$$

. The root lies between (0 49, 0 66)

(b) $x^3 - 4x - 9 = 0$ (Solving till 3 iteration)

The root lies between 2 5 & 3 (2 5, 3)

. $f(2\,5) = -3\,375 < 0$

$f(3) = 6 > 0$

$$x_0 = \frac{(2\,5+3)}{2} = 2\,75 > 0$$

$$f(x_0) = f(2\,75) = 0\,7968 > 0$$

the root lies in between (2 5, 2 75)

$$x_1 = \left(\frac{2\,5 + 2\,75}{2}\right) = 2\,625 > 0$$

$f(x_1) = (2\ 625)^3 - 4(2\ 625) - 9 = -1\ 412 < 0$

$x_1 = 0\ 265$

the root lies in between $(0\ 265, 2\ 75)$

$$x_2 = \left(\frac{0\ 265 + 2\ 75}{2}\right) = 1\ 5075 > 0$$

$f(x_2) = (1\ 5075)^3 - 4(1\ 5075) - 9 = -11\ 604 < 0$

. The roots are $(1\ 5075, 2\ 75)$

$$f(x_3) = \left(\frac{1\ 5075 + 2\ 75}{2}\right) = 2\ 12$$

$f(2\ 12) = (2\ 12)^3 - 4(2\ 12) - (9) = -7\ 95 < 0$

$\boxed{\text{The roots are } (2\ 12, 0\ 75)}$ ⊛

(c) $x^4 - x - 10 = 0$ (between 1 and 2)

Since we know that the root lies in b/w $(1, 2)$

$f(1) = -10 < 0$

$f(2) = 4 > 0$

$$x_0 = \left(\frac{1+2}{2}\right) = 1\ 5 > 0$$

$f(x_0) = (1\ 5)^4 - (1\ 5) - 10 = -6\ 43 < 0$

root lies between $(1\ 5, 2]$

$$x_1 = \left(\frac{1\ 5 + 2}{2}\right) = 1\ 75$$

Regular falsi method

2. Solve the following equations by Regula-Falsi method:

(a) $x^3 = 3x - 4$

(b) $x^3 - x - 1 = 0$

(c) $xe^x = \sin x$

(d) $3x = \cos x + 1$

2) a) $x^3 = 3x - 4$

$f(x) = x^3 - 3x + 4$

The roots of the equation are $(-2\ 5, -2)$
$a \quad , \quad b$

$f(a) = f(2\ 5) = (-2\ 5)^3 - 3(-2\ 5) + 4 = -4\ 125$

$f(b) = f(-2) = \underline{\underline{2}}$

The main formula $\boxed{x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}}$

$$x_1 = \frac{(-2\ 5)(2) - (-2)(-4\ 125)}{2 - (-4\ 125)} = \underline{\underline{-2\ 163}}$$

now to Calculate $f(x_1)$

$$(-2.163)^3 - 3(-2.163) + 4 = 0.369$$

$f(x_1) f(a) < 0 \Rightarrow$ new interval (a, x_1) or update b

$f(x_1) f(b) < 0 \Rightarrow$ new interval (x_1, b) or update a

$$(0.369)(-4.125) = -1.522 < 0$$

$\therefore$ the new interval would be $\underbrace{(\underbrace{-4.125}_{a}, \underbrace{0.369}_{b})}$

This is the new (a, b)

now the value of x_2

$$\boxed{x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

$f(a) = (-4.125)^3 - 3(-4.125) + 4 = -53.81 < 0$

$f(b) = (0.369)^3 - 3(0.369) + 4 = 2.943 > 0$

$$x_2 = \frac{(-4.125)(2.943) - 0.369(-53.81)}{2.943 - (-53.81)}$$

$$x_2 = 0.135 > 0$$

$$f(x_2) = (0.135)^3 - 3(0.135) + 4 = 3.5974$$

$f(x_2)\, f(a) = 3.5974\,(-53.81) = -193.576 < 0$

new interval $\Rightarrow (-4.125, 0.135)$

The Principal Formulae

Trigonometry

1. $sin^2\theta + cos^2\theta = 1,\ sec^2\theta = 1 + tan^2\theta,\ cosec^2\theta = 1 + cot^2\theta$

2. $sin0^o = 0,\ cos0^o = 1,\ sin45^o = cos45^o = \frac{1}{\sqrt{2}}, sin60^o = \frac{\sqrt{3}}{2},$
   $cos60^o = \frac{1}{2},\ sin90^o = 1,\ cos90^o = 0$

3. $sin(-\theta) = -sin\theta, \quad cos(-\theta) = cos\theta$
   $sin(90^o - \theta) = cos\theta, \quad cos(90^o - \theta) = sin\theta$
   $sin(90^o + \theta) = cos\theta, \quad cos(90^o + \theta) = -sin\theta$
   $sin(180^o - \theta) = sin\theta, \quad cos(180^o - \theta) = -cos\theta$
   $sin(180 + \theta) = -sin\theta, \quad cos(180 + \theta) = -cos\theta$

4. $sin(A \pm B) = sinA.cosB \pm cosA.sinB$
   $cos(A \pm B) = cosA.cosB \mp sinA.sinB$
   $$sinC + sinD = 2sin\frac{C+D}{2}.cos\frac{C-D}{2}, \quad sinC - sinD = 2cos\frac{C+D}{2}.sin\frac{C-D}{2}$$
   $$cosC + cosD = 2cos\frac{C+D}{2}.cos\frac{C-D}{2}, \quad cosC - cosD = -2sin\frac{C+D}{2}.sin\frac{C-D}{2}$$

5. $2sinA.cosB = sin(A+B) + sin(A-B)$
   $2cosA.sinB = sin(A+B) - sin(A-B)$
   $2cosA.cosB = cos(A+B) + cos(A-B)$
   $2sinA.sinB = cos(A-B) - cos(A+B)$
   $$tan(A \pm B) = \frac{(tanA \pm tanB)}{(1 \mp tanA.tanB)}$$
   $$sin2A = 2sinA.cosA = \frac{2tanA}{1 + tan^2A}$$
   $$cos2A = cos^2A - sin^2A = 2cos^2A - 1 = 1 - 2sin^2A = \frac{1 - tan^2A}{1 + tan^2A}$$
   $$sin3A = 3sinA - 4sin^3A \qquad cos3A = 4cos^3A - 3cosA$$
   $$tan2A = \frac{2tanA}{1 - tan^2A} \qquad tan3A = \frac{3tanA - tan^3A}{1 - 3tan^2A}$$

6. $sinA = \sqrt{\dfrac{1 - cosA}{2}}, \qquad cosA = \sqrt{\dfrac{1 + cosA}{2}}$

7. $log_a(mn) = log_a m + log_a n$ $\qquad log_a(\frac{m}{n}) = log_a m - log_a n$

   $log_a m^n = n log_a m$ $\qquad log_a m = \dfrac{logm}{loga}$

8. $Binomial Expansion:$
   $(1+x)^n = 1 +^n C_1 x^1 +^n C_2 x^2 + ... + x^n$
   $= 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + x^n$

Inverse Circular Functions

1. $sin^{-1}(x) = cosec^{-1}(\frac{1}{x}),$ $\qquad cosec^{-1}(x) = sin^{-1}(\frac{1}{x})$
   $cos^{-1}(x) = sec^{-1}(\frac{1}{x}),$ $\qquad sec^{-1}(x) = cos^{-1}(\frac{1}{x})$
   $tan^{-1}(x) = cot^{-1}(\frac{1}{x}),$ $\qquad cot^{-1}(x) = tan^{-1}(\frac{1}{x})$

2. $\theta = sin^{-1}(sin\theta) = cos^{-1}(cos\theta) = tan^{-1}(tan\theta), \quad etc$
   $x = sin(sin^{-1}x) = cos(cos^{-1}x) = tan(tan^{-1}x), \quad etc$

3. $sin^{-1}(-x) = -sin^{-1}(x), \quad tan^{-1}(-x) = -tan^{-1}(x),$
   $cos^{-1}(-x) = \pi - cos^{-1}(x), \quad etc$

4. $sin^{-1}(x) = cos^{-1}(\sqrt{1-x^2}) = tan^{-1}(\dfrac{x}{\sqrt{1-x^2}}) = sec^{-1}(\dfrac{x}{\sqrt{1-x^2}}), \quad etc$

   $cos^{-1}(x) = sin^{-1}(\sqrt{1-x^2}) = tan^{-1}(\dfrac{\sqrt{1-x^2}}{x}) = cot^{-1}(\dfrac{x}{\sqrt{1-x^2}}), \quad etc$

   $tan^{-1}(x) = sin^{-1}(\dfrac{x}{\sqrt{1+x^2}}) = cos^{-1}(\dfrac{1}{\sqrt{1+x^2}}) = sec^{-1}(\sqrt{1+x^2}), \quad etc$

5. $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$, $tan^{-1}x + cot^{1}x = \frac{\pi}{2}$, $sec^{-1}x + cosec^{-1}x = \frac{\pi}{2}$

6. $tan^{-1}x - tan^{-1}y = tan^{-1}(\dfrac{x-y}{1+xy})$
   $sin^{-1}x + sin^{-1}y = sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1+x^2})$
   $sin^{-1}x - sin^{-1}y = sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$
   $cos^{-1}x - cos^{-1}y = cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$

7. $2tan^{-1}x = tan^{-1}(\dfrac{2x}{1-x^2}) = sin^{-1}(\dfrac{2x}{1+x^2}) = cos^{-1}(\dfrac{1-x^2}{1+x^2})$

8. $3sin^{-1}x = sin^{-1}(3x - 4x^3), \quad 3cos^{-1}x = cos^{-1}(4x^3 - 3x),$
   $3tan^{-1}x = tan^{-1}(\dfrac{3x - x^3}{1 - 3x^2})$

Differentiation Formulae

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(logx) = \frac{1}{x}$
3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(a^x) = a^x loga$
5. $\frac{d}{dx}(sinx) = cosx$
6. $\frac{d}{dx}(cosx) = -sinx$
7. $\frac{d}{dx}(tanx) = sec^2 x$
8. $\frac{d}{dx}(cotx) = -cosec^2 x$
9. $\frac{d}{dx}(cosecx) = -cosecx.cotx$
10. $\frac{d}{dx}(secx) = secx.tanx$
11. $\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
12. $\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$
14. $\frac{d}{dx}(cot^{-1}x) = -\frac{1}{1+x^2}$
15. $\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
16. $\frac{d}{dx}(cosec^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$
17. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
18. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Standard Limits

1. $\lim_{n\to\infty} \frac{\log n}{n} = 0$
2. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$
3. $\lim_{n\to\infty} (n)^{1/n} = 1$
4. $\lim_{n\to\infty} (n!)^{1/n} = \infty$
5. $\lim_{n\to\infty} \left(\frac{n!}{n}\right)^{1/n} = \frac{1}{e}$
6. $\lim_{n\to\infty} x^n = 0$, if $x < 1$
7. $\lim_{n\to\infty} x^n = \infty$, if $x > 1$
8. $\lim_{n\to\infty} nx^n = 0$, if $x < 1$
9. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x
10. $\lim_{n\to 0} \left(\frac{a^n - 1}{n}\right) = \log a$
11. $\lim_{n\to\infty} \left(\frac{a^{1/n} - 1}{1/n}\right) = \log a$