Module 3 : Partial Differentiation

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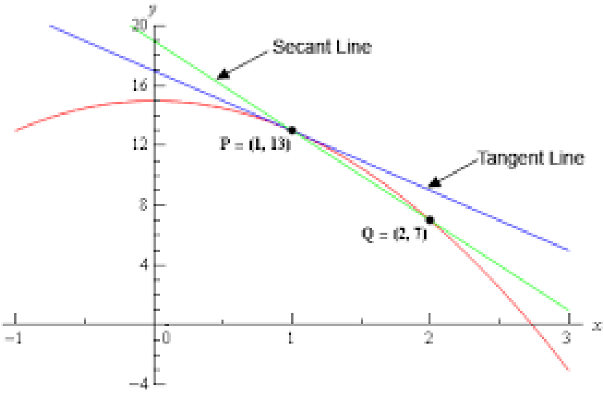
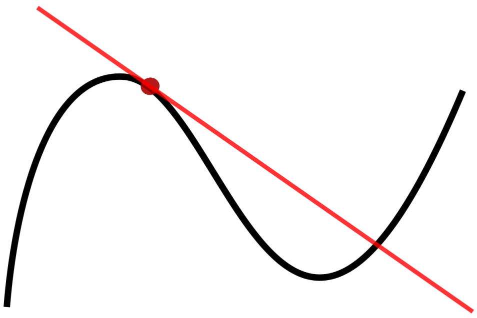
3.3. Maxima and Minima of a function of two independent variables

Self learning topics: Applications of partial differentiation in weather modeling, wave equation, sensitivity analysis.

Prerequisite: Differentiation

A **tangent line** to a function at a point P is the line that best approximates the function at that point better than any other line. This tangent line touches the given graph of the function at exactly this point P if we consider any neighbourhood of P





The slope of the function at a given point is the slope of the tangent line to the function at that point.
The derivative of f at $x = a$ is the slope, m, of the function f at the point $x = a$ (if m exists), denoted by $f'(a) = m$.
Some other notations for the derivative:
$y', \ \frac{dy}{dx}, \ \frac{d}{dx}f(x), \ D_x f(x), \ D_x(y).$

The function $f(x)$ is **differentiable** at a point x_0 if $f'(x_0)$ exists.
That is, if the following limit exits:
$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdots (1)$
The above limit can also be written as
$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdots (2)$ where $h = x - x_0$

If a function is differentiable at all points in its domain (i.e. $f'(x)$ is defined for all x in the domain), then we consider $f'(x)$ as a function and call it the **derivative of** $f(x)$. The derivative of f that we have been talking about is called the first derivative.

Derivatives of some standard Functions

1. $\frac{d}{dx}(c) = 0$, where c is a constant
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(\log x) = \frac{1}{x}$
4. $\frac{d}{dx}(e^x) = e^x$
5. $\frac{d}{dx}(a^x) = a^x \log a$
6. $\frac{d}{dx}(\sin x) = \cos x$
7. $\frac{d}{dx}(\cos x) = -\sin x$
8. $\frac{d}{dx}(\tan x) = \sec^2 x$
9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
10. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
17. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$
18. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
19. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Similarly we can define the second order derivative of a function to be the derivative of $f'(x)$, denoted by $f''(x)$ or $\frac{d}{dx}(\frac{df(x)}{dx}) = \frac{d^2 f}{dx^2}$

Partial Differentiation

Motivation : Why should we study Partial Differentiation?:

- Calculating errors:

  It is used to estimate errors in calculated quantities that depend on more than one uncertain experimental.

- Thermodynamics:

  Thermodynamic energy functions (enthalpy, Gibb's free energy, Hellholtz free energy) are function of two or more variables. Most thermodynamic quantities (temperature, entrophy, heat capacity) can be expressed as derivatives of these functions.

- Financial engineering:

  Financial engineers use partial derivatives to assess a portfolio's sensitivity to changes in market conditions (interest rates, volatility). Then can hedge against risk by designing portfolio's respect to market values.

- Partial differential equations:

  Many laws of nature are best expressed as relations between the partial derivatives of one or more quantities.

  $$ih\frac{\partial(\psi)}{\partial t} = -\frac{h^2}{2m}\nabla^2\psi + \nabla\psi$$

  and the Navier-Stokes equation describes all fluid motion.

- Important properties of functions that we encounter in engineering are checked by using partial derivatives
  For instance, the Laplace's equation
  $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
  is used to check if the function $u = u(x, y)$ is **harmonic**
  Harmonic functions have nice properties:
  Theorem (Maximum principle) Suppose $u = u(x, y)$ is harmonic on a open region Z. Then
  Suppose z_0 is in Z. If u has a relative maximum or minimum at z_0 then u is constant on a disk centered at z_0.
  Harmonic functions are shown to have several more useful properties which make them well suited for robotics applications
  (Ref: Applications of Harmonic Functions to Robotics Christopher I. Connolly and Roderic A. Grupen Laboratory for Perceptual Robotics, Computer and Information Science Department,J. Field Robotics,(1993))

Getting started with partial derivatives
Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as x changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable.
What do we do if we only want one of the variables to change, or if we want more than one of them to change?

Suppose we are designing a vehicle and we want to find out the effect of **temperature** on **mileage.**
We know that mileage depends on factors other than temperature - the pressure, velocity with which the vehicle is running, the terrain etc.
Then how can we go about to find out what we want?
What we do is, we keep the other factors (other than temperature, here) a *constant*
How do we do it?
If we know for what type of usage, the terrain etc. that we are designing the vehicle - for example, it can be a mountainous terrain with an average velocity of $50km/hr$ - then we can **suppress** the other factors and observe how changes in temperature affects the mileage
This is exactly what we are going to do here
We will consider a function of several variables and find out the rate at which the function changes when exactly one of the variables is changing and the remaining variables are kept constant

In fact, if we are going to allow more than one of the variables to change there are then going to be an infinite amount of ways for them to change. For instance, one variable could be changing faster than the other variable(s) in the function. Notice as well that it will be completely possible for the function to be changing differently depending on how we allow one or more of the variables to change.
If we consider an example from medicine:
An individual's health depends on various factors like blood pressure, glucose level, age, type of profession etc.
That is, $h = h(x, y, z, w)$
where h denotes health which is a function of
$x \to$ blood pressure, $y \to$ glucose level, $z \to$ age and $z \to$ profession.
Therefore, if a researcher wants to check if a drug is effective in lowering blood pressure, then she will be able to interpret the results properly only if she keeps the remaining factors such as glucose level, age and profession as constant for the induviduals she is testing.
This is because, the changes that one sees in the blood sugar levels can be attributed to (younger) age or absence of diabletes etc.
Hence, if she observes the changes in blood pressure for a sample with similar glucose level, age and profession, then the effect of the drug can be correctly interpreted for blood pressure.

Partial Differentiation
A partial derivative of a function of several variables is the ordinary derivative w.r.t. one of the variables are held constant. Partial differentiation is the process of finding partial derivatives. All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) w.r.t. one variable, all the other variables are treated (temporarily) as constants.

Consider a function u of three independent variables x,y,z,

$$u = f(x, y, z)$$

Keeping y,z constant and varying only x, the partial derivative of u w.r.t. x is denoted by $\frac{\partial u}{\partial x}$ and is defined as the limit

$$\lim_{\Delta x \to 0} \frac{\partial u}{\partial x} = \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

Partial derivatives of u w.r.t. u and z can defined similarly and are denoted by $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

<u>Notation:</u> The partial derivative $\frac{\partial u}{\partial x}$ or f_x or $f_x(x, y, z)$ or $D_x f$.
Thus we can have
$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = f_y = D_y f$ etc.
The value at a partial derivative at a point (a,b,c) is denoted by,

$$\frac{\partial u}{\partial x}|_{x=a,y=b,z=c} = \frac{\partial u}{\partial x}|_{(a,b,c)} = f_x(a, b, c)$$

Geometrical interpretation The partial derivative of a function of two variables $z = f(x, y)$ represents the equation of a surface in xyz co-ordinate system.
Let APB be the curve, which is cut by a plane through any point P on the surface parallel to the xz-plane.
As the point P moves along this curve APB, its co-ordinates z and x vary while y remains constant.
The slope of the tangent line at P to APB represents the rate at which z changes w.r.t. to x.

Thus $\dfrac{\partial z}{\partial x} = tan(\alpha) =$ slope of the curve of APB at the point P.

Similarly, $\dfrac{\partial z}{\partial y} = tan(\beta) =$ slope of the curve of CPD at the point P.

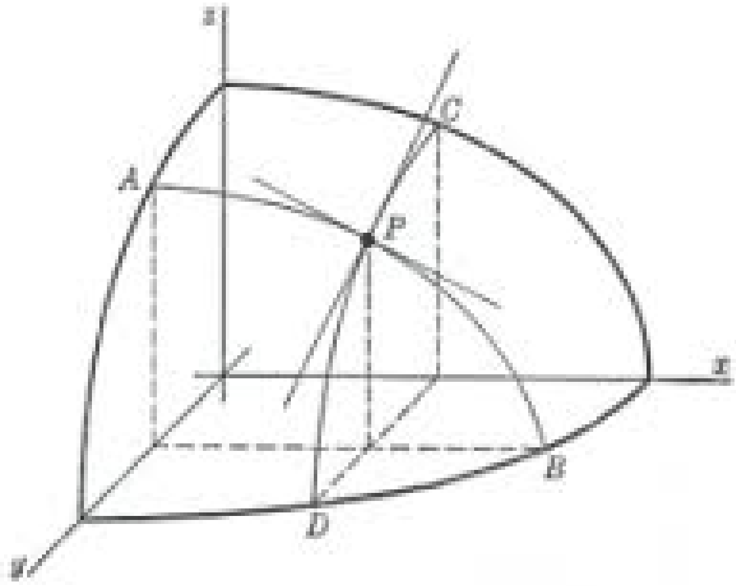


Fig. 1

Higher Order Partial Derivatives Partial derivatives of higher order, of a function f(x,y,z) are calculated by successive differentiation. Thus, if $u = f(x, y, z)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = f_{xx}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = f_{yy}$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = f_{xy}$$
$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = f_{yx}$$
$$\frac{\partial^3 u}{\partial z^2 \partial y} = \frac{\partial}{\partial z}(\frac{\partial^2 f}{\partial z \partial y}) = \frac{\partial}{\partial z}(\frac{\partial}{\partial z}(\frac{\partial f}{\partial y})) = f_{yzz}$$
$$\frac{\partial^4 u}{\partial x \partial y \partial z^2} = \frac{\partial}{\partial x}(\frac{\partial^3 f}{\partial y \partial z^2}) = \frac{\partial}{\partial z}(\frac{\partial}{\partial z}(\frac{\partial f}{\partial y})) = f_{yzz}$$

The partial derivative $\dfrac{\partial f}{\partial x}$ obtained by differentiating once is known as first order partial derivative, while $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ which are obtained by differentiating twice are known as second-order derivatives. 3^{rd} order, 4^{th} order derivatives involve 3, 4 times differentiation respectively.

Note 1: The crossed or mixed partial derivatives $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are equal that is,:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

if the first order partial derivatives involved are continuous. i.e, the order of differentiation is immaterial if the derivatives
$\dfrac{\partial f}{\partial x}\ and\ \dfrac{\partial f}{\partial y}$
are continuous.

Note 2: In the subscript notation, the subscripts are written in the same order in which differentiation is carries out; while in the '∂' notation the order is opposite, for example

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = f_{xy}$$

Note 3: A function of two variables has two first order derivatives, four second order derivatives and 2^n of the n^{th} order derivatives. A function of m independent variables will have m^n derivatives of order n

Let us evaluate some partial derivatives.

1. Find u_x, u_y if $u = e^{2x}\cos 2y + e^{2y}\sin 2x$
   Solution: We have

$$\begin{aligned} u &= e^{2x}\cos 2y + e^{2y}\sin 2x \\ \Rightarrow u_x = \frac{\partial u}{\partial x} &= 2(e^{2x}\cos 2y) + e^{2y}(2\cos 2x) \\ That\ is\ u_x &= 2(e^{2x}\cos 2y + e^{2y}\cos 2x) \\ Also\ u_y = \frac{\partial u}{\partial y} &= e^{2x}2(-\sin 2y) + 2(e^{2y})\sin 2x \\ \Rightarrow u_y &= 2(-e^{2x}\sin 2y + e^{2y}\sin 2x) \end{aligned}$$

2. Find f_x, f_y if $f = \dfrac{1}{\sqrt{x^2+y^2}}$
   Solution: We have

$$\begin{aligned} f(x,y) = \frac{1}{\sqrt{x^2+y^2}} &= (x^2+y^2)^{-1/2} \\ \Rightarrow f_x = \frac{\partial f}{\partial x} &= \frac{-1}{2}(x^2+y^2)^{-3/2}(2x) \\ That\ is\ f_x &= -x(x^2+y^2)^{-3/2} \\ \Rightarrow f_x &= \frac{-x}{(x^2+y^2)^{3/2}} \\ Similarly\ f_y &= \frac{-1}{2}(x^2+y^2)^{-3/2}(2y) \\ \Rightarrow f_y &= \frac{-y}{(x^2+y^2)^{3/2}} \end{aligned}$$

3. Find u_x, u_y if $u = \log xy + \tan^{-1}(\frac{y}{x})$
   Solution: We have

$$\begin{aligned} u &= \log xy + \tan^{-1}(\frac{y}{x}) \\ \Rightarrow u_x = \frac{\partial u}{\partial x} &= \frac{1}{xy}(y) + \frac{1}{1+(y^2/x^2)}(\frac{-y}{x^2}) \\ That\ is\ u_x &= \frac{1}{x} + \frac{-y}{x^2+y^2} \\ Also\ u_y = \frac{\partial u}{\partial y} &= \frac{1}{xy}(x) + \frac{1}{1+(y^2/x^2)}(\frac{1}{x}) \\ \Rightarrow u_y &= \frac{1}{y} + \frac{x}{x^2+y^2} \end{aligned}$$

Composite Functions and partial differentiation and total differentials

Refresher Quiz

1. If $f = f(x, y)$ is a function of two variables x and y then

(A) $\dfrac{\partial f}{\partial y}$ is found by keeping f as constant

(B) $\dfrac{\partial f}{\partial x}$ is found by keeping x as constant

(C) $\dfrac{\partial f}{\partial y}$ is found by keeping y as constant

(D) * $\dfrac{\partial f}{\partial x}$ is found by keeping y as constant

2. If $(x, y) = 3x^2y - \sin 2x \cos y$ then

(A) $\dfrac{\partial f}{\partial y} = 6xy + \sin 2x \sin y$

(B) * $\dfrac{\partial f}{\partial x} = 6xy - 2\cos 2x \cos y$

(C) $\dfrac{\partial f}{\partial y} = 3x^2 - 2\cos 2x \cos y$

(D) $\dfrac{\partial f}{\partial x} = 3x^2 - 2\cos 2x \cos y$

Example: If $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$, Show that $u_x + u_y + u_z = 2u$

Solution: Given: $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$

taking log on both sides,

$\log u = \log \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$

That is,

$\log u = \log(e^{x+y+z}) - \log\left(e^x + e^y + e^z\right)$

(Since $\log(\dfrac{A}{B}) = \log A - \log B$)

That is,

$\log u = x + y + z - \log\left(e^x + e^y + e^z\right) \cdots\cdots (*)$

(Since $\log(e^A) = A$)

Differentiating (*) partially w.r.t. x, we get,

$$\frac{1}{u}.\frac{\partial u}{\partial x} = 1 - \frac{e^x}{e^x + e^y + e^z} \cdots\cdots (1)$$

Similarly Differentiating (*) partially w.r.t. y and z we get respectively,

$$\begin{aligned} \frac{1}{u}.\frac{\partial u}{\partial y} &= 1 - \frac{e^x}{e^x + e^y + e^z} \cdots\cdots (2) \\ \frac{1}{u}.\frac{\partial u}{\partial z} &= 1 - \frac{e^x}{e^x + e^y + e^z} \cdots\cdots (3) \end{aligned}$$

$$\begin{aligned} &Adding (1),\ (2)\ and\ (3)\ we\ get \\ \frac{1}{u}.(u_x + u_y + u_z) &= 3 - \frac{e^x + e^y + e^z}{e^x + e^y + e^z} \\ That\ is,\ u_x + u_y + u_z &= 2u \end{aligned}$$

Composite Functions

- Suppose I want to paint my wall
  The cost of painting will **depend** on the cost of paint, which in turn will depend on the size of the wall
  Therefore, if we denote by f, the cost of painting, then f is a function of x, the cost of paint, which in turn is a function of l and b the length and breadth of the wall.
- Suppose I decide to buy a shirt from the mall because there is a discount of 40%. If the GST (Goods and Services Tax) ranges from 12% to 18% of the cost, write a composite function for the possible price I will pay at the register.
  Solution: Let u be the price of the shirt
  After reduction, the price is $0.6u$
  I have to calculate the GST for this amount, that is, $0.6u$
  That is, GST is $= 0.18(0.6u)$
  Let us denote the GST by x
  Then GST is $x = 0.18(0.6u) = 0.108u$
  Therefore, if I denote the amount I pay at the register as f, then
  $f = f(x, u) = 0.6u + x$
  Here, $f = f(x, u)$ and $x = x(u)$
  That is, f is a function of x and u and x is a function of u

Here we shall study about differentiating both the above types of functions

Partial differentiation of composite function

- Let u be a function of x, y, z. That is, $u = f(x, y, z)$ and let x, y, z be functions of two independent variables s and t; that is, $x = x(s,t), y = y(s,t), z = z(s,t)$.

  The function f is considered as a function of s and t via the intermediate variables x, y, z.
  That is,

  $$f \to (x, y, z) \to (s, t)$$

  Now the partial derivative of f w.r.t. s keeping t constant is:

  $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

  In the similar way, we get,

  $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

  For Example
  We have $z = x + iy$ where $x = r\cos\theta,\ y = r\sin\theta$

  $$z \to (x, y) \to (r, \theta)$$

Here,

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\
\Rightarrow \frac{\partial z}{\partial r} &= 1 \cdot \cos\theta + i \cdot \sin\theta \\
\text{that} \quad \text{is} \quad \Rightarrow \frac{\partial z}{\partial r} &= e^{i\theta}
\end{aligned}$$

(which is true when we directly obtain partial derivatives using $z = re^{i\theta}$)

- Suppose f is a function of x, y and x is a function of s and y is a function of t That is,

$$f \to (x, y);\ x \to s;\ y \to t$$

Then

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{dx}{ds} \\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial y}\frac{dy}{dt}
\end{aligned}$$

- Suppose f is a function of x, y and x, y are functions of t ; That is,

$$f \to (x, y);\ x, y \to t$$

Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The above equations are known as **chain rules** for partial differentiation.

Partial differentiation and total differentials of Composite Functions
Refresher Quiz

1. If $r^2 = x^2 + y^2 + z^2$ then $\dfrac{\partial r}{\partial x}$ equals

(A) $2x$ (B)* $\dfrac{x}{r}$ (C) $\dfrac{2x}{r}$ (D) $\dfrac{1}{r}$

2. If $z = x + y, \quad x = e^t,\ y = e^{-t}$ then

(A) $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} = 1 \cdot e^t$ (B) $\dfrac{\partial x}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = x \cdot e^t - y \cdot e^{-t}$

(C)* $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = 1 \cdot e^t - 1 \cdot e^{-t}$ (D) $\dfrac{\partial y}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = 1 \cdot e^t - 1 \cdot e^{-t}$

Solved Examples - Composite Functions

1. If $z = f(x, y)$ where $x = e^u \cos v; y = e^u \sin v$

   Show that $y\dfrac{\partial z}{\partial u} + x\dfrac{\partial z}{\partial v} = e^{2u}\dfrac{\partial z}{\partial y}$

   Solution: We have
   $z \to (x, y) \to (u, v)$
   We have
   $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}.\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}.\dfrac{\partial y}{\partial u}$
   (By the chain rule)
   Now, $x = e^u cosv \Rightarrow \dfrac{\partial x}{\partial u} = e^u \cos v; \quad y = e^u \sin v \Rightarrow \dfrac{\partial y}{\partial u} = e^u \sin v$

$$\begin{aligned}
\therefore \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} \\
\text{that is } \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}.e^u \cos v + \frac{\partial z}{\partial y}.e^u \sin v \\
\text{that is } \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}.x + \frac{\partial z}{\partial y}.y \\
\Rightarrow y\frac{\partial z}{\partial u} &= xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \cdots\cdots (1)
\end{aligned}$$

   Again, by chin rule, we have $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}.\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}.\dfrac{\partial y}{\partial v}$
   $\therefore x = e^u \cos v \Rightarrow \dfrac{\partial x}{\partial v} = -e^u \sin v; \quad y = e^u \sin v \Rightarrow \dfrac{\partial y}{\partial v} = e^u \cos v$

$$\begin{aligned}
\therefore \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} \\
\text{that is } \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}. - e^u \sin v + \frac{\partial z}{\partial y}.e^u \cos v \\
\text{that is } \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}.(-y) + \frac{\partial z}{\partial y}.x \\
\Rightarrow x\frac{\partial z}{\partial v} &= -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \cdots\cdots (2)
\end{aligned}$$

(1) + (2) gives

$$\begin{aligned}
y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} &= xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \\
&= \frac{\partial z}{\partial y}.(x^2 + y^2) \\
&= \frac{\partial z}{\partial y}.((e^u \cos v)^2 + (e^u \sin v)^2) \\
&= \frac{\partial z}{\partial y}.(e^{2u}(\cos^2 v + \sin^2 v)) \\
\text{that is } \; y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} &= e^{2u}.\frac{\partial z}{\partial y}
\end{aligned}$$

2. If $z = f(x, y);\; x = e^u + e^{-v};\; y = e^{-u} - e^v$
   Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$
   Solution: We have $z \to (x, y) \to (u, v)$
   We have

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} \\
&= \frac{\partial z}{\partial x}.e^u + \frac{\partial z}{\partial y}.(-e^{-u}) \\
\text{similarly}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} \\
&= \frac{\partial z}{\partial x}.(-e^{-v}) + \frac{\partial z}{\partial y}. - e^v \\
\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} &= (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y} \\
\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} &= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}
\end{aligned}$$

3. If $z = f(x, y)$; $x = \log u$; $y = \log v$
   Prove that $\dfrac{\partial^2 z}{\partial x \partial y} = uv \dfrac{\partial^2 z}{\partial u \partial v}$

   Solution: We have $z \to (x, y)$; $x \to u$, $y \to v$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}.\frac{dx}{dv} + \frac{\partial z}{\partial y}.\frac{dy}{dv} \\
&= \frac{\partial z}{\partial x}.0 + \frac{\partial z}{\partial y}.\frac{1}{v} \\
That\ is\ \frac{\partial z}{\partial v} &= \frac{1}{v}.\frac{\partial z}{\partial y}
\end{aligned}$$

   (Note that $\dfrac{\partial z}{\partial y}$ is also a function of x, y)
   Let $\dfrac{\partial z}{\partial y} = w$
   Now,

$$\begin{aligned}
\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\frac{\partial z}{\partial v} &= \frac{\partial}{\partial u}(\frac{1}{v}.w) \\
&= \frac{1}{v}(\frac{\partial w}{\partial u}) \\
&= \frac{1}{v}(\frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u}) \\
&= \frac{1}{v}(\frac{\partial w}{\partial x}.\frac{1}{u} + \frac{\partial w}{\partial y}.0) \\
&= \frac{1}{uv}(\frac{\partial w}{\partial x}) \\
&= \frac{1}{uv}(\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})) \\
&= \frac{1}{uv}(\frac{\partial^2 z}{\partial x \partial y}) \\
\Rightarrow \frac{\partial^2 z}{\partial x \partial y} &= uv\frac{\partial^2 z}{\partial u \partial v}
\end{aligned}$$

Partial differentiation and total differentials of Composite Functions - More examples
Refresher Quiz

1. If $u = log(\tan x + \tan y)$ then $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y}$ equals
(A) 1
(B) $2u$
(C) u
(D) * 2

2. If $u = \dfrac{e^{x+y}}{e^x + e^y}$ then $u_x + u_y$ equals
(A) 0
(B) 1
(C) * u
(D) $2u$

Recap:

Chain rule
If $u = f(x, y, z)$ and $x = x(t), y = y(t), z = z(t)$
That is,

$$f \to (x, y, z) \to t$$

then the total derivative of f is,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} + \frac{\partial f}{\partial z}.\frac{dz}{dt}$$

Suppose $u = f(x, y, z)$ and suppose $y\&z$ are function of x.
Then f is a function of the one independent variable x.
That is,

$$f \to (x, y, z) \quad \& \quad (y, z) \to x$$

$\dfrac{df}{dx} = \dfrac{\partial f}{\partial x}.\dfrac{dx}{dx} + \dfrac{\partial f}{\partial y}.\dfrac{dy}{dx} + \dfrac{\partial f}{\partial z}.\dfrac{dz}{dx}$
$\Rightarrow \dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}.\dfrac{dy}{dx} + \dfrac{\partial f}{\partial z}.\dfrac{dz}{dx}$

Solved Examples

1. If $u = f(x-y, y-z, z-x)$ prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

   Solution: Let $q = x-y,\ s = y-z,\ t = z-x$
   Then $u \to (q,s,t)\ \to (x,y,z)$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial q}.\frac{\partial q}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}.\frac{\partial t}{\partial x} \\
&= \frac{\partial u}{\partial q}.(1) + \frac{\partial u}{\partial s}.(0) + \frac{\partial u}{\partial t}.(-1) \\
\text{That is } \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial q} - \frac{\partial u}{\partial t} \\
Similarly\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial q}.\frac{\partial q}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}.\frac{\partial t}{\partial y} \\
\Rightarrow \frac{\partial u}{\partial y} &= -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial s} \\
Also\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial q}.\frac{\partial q}{\partial z} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}.\frac{\partial t}{\partial z} \\
\Rightarrow \frac{\partial u}{\partial z} &= -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \\
\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= 0
\end{aligned}
$$

2. If $z = f(x,y),\ u = x^2 - y^2;\ v = 2xy$
   Prove that $(\dfrac{\partial z}{\partial x})^2 + (\dfrac{\partial z}{\partial y})^2 = 4(u^2+v^2)^{1/2}.[(\dfrac{\partial z}{\partial u})^2 + (\dfrac{\partial z}{\partial v})^2]$

   Solution: We have

$$
\begin{aligned}
u = x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} &= 2x \\
\text{and } \frac{\partial u}{\partial y} &= -2y \\
v = 2xy \Rightarrow \frac{\partial v}{\partial x} &= 2y \\
\text{and } \frac{\partial v}{\partial y} &= 2x
\end{aligned}
$$

   Also $z \to (x,y) \to (u,v)$

$$
\begin{aligned}
\Rightarrow \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} \\
\Rightarrow \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}.2x + \frac{\partial z}{\partial v}.2y \quad \cdots\cdots(1) \\
\text{We have } \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} \\
\Rightarrow \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}.(-2y) + \frac{\partial z}{\partial v}.2x \quad \cdots\cdots(2)
\end{aligned}
$$

Squaring and adding (1) and (2), we get,

$$
\begin{aligned}
(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 &= (2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v})^2 + (-2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v})^2 \\
&= 4x^2[(\frac{\partial z}{\partial u})^2 + (\frac{\partial z}{\partial v})^2] + 4y^2[(\frac{\partial z}{\partial u})^2 + (\frac{\partial z}{\partial v})^2] \\
&= 4(x^2+y^2).[(\frac{\partial z}{\partial u})^2 + (\frac{\partial z}{\partial v})^2] \\
&= 4(u^2+v^2)^{1/2}.[(\frac{\partial z}{\partial u})^2 + (\frac{\partial z}{\partial v})^2]\ (\because u^2+v^2 = (x^2+y^2)^2)
\end{aligned}
$$

3. $If z = e^{ax+by}.f(ax-by), \quad Prove\ that\ b\frac{\partial z}{\partial x}\ + a\frac{\partial z}{\partial y} =\ 2abz$

Solution: Let $u = ax+by,\ v = ax-by$
Then $z = e^u f(v)$
$\Rightarrow z \to u, v \to x, y$
Therefore

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\
&= e^u f(v)\cdot a + e^u f'(v)\cdot a \\
\text{similarly } \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\
&= e^u f(v)\cdot b + e^u f'(v)\cdot(-b) \\
\therefore b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} &= 2abe^u f(v) \\
&= 2abz
\end{aligned}
$$

4. Suppose $z = \tan(y + ax) + (y - ax)^{3/2}$. Show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial z^2}$

Solution: Let $y + ax = u$ and $y - ax = v$
Then we have

$$\begin{aligned}
z &= \tan(y+ax) + (y-ax)^{3/2} \\
\Rightarrow z &= \tan u + v^{3/2} \\
\text{Now } \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\
\text{That is } \frac{\partial z}{\partial x} &= \sec^2 u(a) + \frac{3}{2}v^{1/2}(-a) \\
\Rightarrow \frac{\partial z}{\partial x} &= a(\sec^2 u - \frac{3}{2}v^{1/2}) \\
\text{Now } \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) \\
\Rightarrow \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial u}[a(\sec^2 u - \frac{3}{2}v^{1/2})]\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}[a(\sec^2 u - \frac{3}{2}v^{1/2})]\frac{\partial v}{\partial x} \\
\Rightarrow \frac{\partial^2 z}{\partial x^2} &= a(2\sec^2 u \tan u)(a) + a(-\frac{3}{2}\frac{1}{2}v^{-1/2})(-a) \\
\text{that is } \frac{\partial^2 z}{\partial x^2} &= a^2(2\sec^2 u \tan u - \frac{3}{4}v^{-1/2}) \quad \cdots (1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\
\text{That is } \frac{\partial z}{\partial y} &= \sec^2 u(1) + \frac{3}{2}v^{1/2}(1) \\
\Rightarrow \frac{\partial z}{\partial x} &= \sec^2 u - \frac{3}{2}v^{1/2}
\end{aligned}$$

$$\begin{aligned}
\text{Now } \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) \\
\Rightarrow \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial u}[\sec^2 u - \frac{3}{2}v^{1/2}]\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}[\sec^2 u - \frac{3}{2}v^{1/2}]\frac{\partial v}{\partial y} \\
\Rightarrow \frac{\partial^2 z}{\partial y^2} &= 2\sec^2 u \tan u(1) - \frac{3}{2}\frac{1}{2}v^{-1/2})(1) \\
\text{that is } \frac{\partial^2 z}{\partial y^2} &= 2\sec^2 u \tan u - \frac{3}{4}v^{-1/2}) \quad \cdots (2)
\end{aligned}$$

From (1) and (2) we get
$\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$

5. If $u = x^y$, Show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$

Solution: We have

$$\begin{aligned}
u &= x^y \\
\Rightarrow \log u &= y \log x \\
\Rightarrow \frac{1}{u}\frac{\partial u}{\partial x} &= \frac{y}{x} \\
\text{that is } \frac{\partial u}{\partial x} &= u\frac{y}{x} \\
\text{that is } \frac{\partial u}{\partial x} &= yx^{y-1} (\because u = x^y) \\
\Rightarrow \frac{\partial^2 u}{\partial y \partial x} &= x^{y-1} + yx^{y-1} \log x \\
\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial x} &= (y-1)x^{y-2} + y(y-1)x^{y-2} \log x + yx^{y-2} \quad \cdots\cdots (1) \\
Similarly & \\
u &= x^y \\
\Rightarrow \log u &= y \log x \\
\Rightarrow \frac{1}{u}\frac{\partial u}{\partial y} &= \log x \\
\text{that is } \frac{\partial u}{\partial y} &= x^y \log x \\
\Rightarrow \frac{\partial^2 u}{\partial x \partial y} &= yx^{y-1} \log x + x^{y-1} \\
\Rightarrow \frac{\partial^3 u}{\partial x^2 \partial y} &= y(y-1)x^{y-2} \log x + yx^{y-2} + (y-1)x^{y-2} \quad \cdots\cdots (2)
\end{aligned}$$

From (1) and (2) we get
$\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$

6. If $u = log(x^3 + y^3 + z^3 - 3xyz)$, Prove that $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u = \frac{-9}{(x+y+z)^2}$

Solution: We have

$$\begin{aligned}
(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u &= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})u \\
&= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}) \\
\text{now } \frac{\partial u}{\partial x} &= \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3x^2 - 3yz) \\
\text{that is } \frac{\partial u}{\partial x} &= \frac{3(x^2 - yz)}{(x^3 + y^3 + z^3 - 3xyz)} \\
\text{Similarly } \frac{\partial u}{\partial y} &= \frac{3(y^2 - xz)}{(x^3 + y^3 + z^3 - 3xyz)} \\
\text{and } \frac{\partial u}{\partial z} &= \frac{3(z^2 - xy)}{(x^3 + y^3 + z^3 - 3xyz)} \\
\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3(x^2 - yz)}{(x^3 + y^3 + z^3 - 3xyz)} + \frac{3(y^2 - xz)}{(x^3 + y^3 + z^3 - 3xyz)} \\
&\quad + \frac{3(z^2 - xy)}{(x^3 + y^3 + z^3 - 3xyz)} \\
&= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x^3 + y^3 + z^3 - 3xyz)} \\
&= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)} \\
(\because (x^3 + y^3 + z^3 - 3xyz) &= (x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)) \\
\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3}{(x + y + z)} \\
\therefore (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u &= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})\frac{3}{(x + y + z)} \\
&= 3[\frac{-1}{(x + y + z)^2} + \frac{-1}{(x + y + z)^2} + \frac{-1}{(x + y + z)^2}] \\
\Rightarrow (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u &= \frac{-9}{(x + y + z)^2}
\end{aligned}$$

Refresher Quiz

1. If $u = x^2 + y^2$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ equals

(A) 0
(B) 1
(C) u
(D) $2u$

3.2 Homogeneous functions and Euler's theorem
Homogeneous Functions
Introduction A polynomial in x and y is said to be **homogeneous** if all its terms are of the same degree.
For example, $x^3 + 2x^2y + y^3$ is homogeneous since all its terms are of degree 3, while $x^3 + 2x^2y + y^3 + 7$ is not homogeneous.
Definition A function $f(x, y)$ in two variables x and y is said to be a **homogeneous function** of degree n, if for any positive number λ,

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

For example,

1. $3x^2 - 2xy + \frac{15}{2}y^2$ is homogeneous in x and y of degree 2.
2. $\frac{\sqrt{x}+\sqrt{y}}{x+y}$ is homogeneous in x and y of degree -1/2.
3. $sin(\frac{y}{x}) + tan^{-1}(\frac{x}{y})$ is homogeneous in x and y of degree 0.
4. $\frac{x+y}{xy} + x^{2/3}.e^{x/y}$ is not homogeneous.
5. $x^{1/3}.y^{-2/3} + x^{2/3}.y^{-1/3}$ is not homogeneous.

Note: Here n need not be an integer, n could be positive, negative or zero.
A useful representation of a homogeneous function
Now suppose that $f(x, y)$ is homogeneous in x and y of degree n. Then $f(x, y)$ can be expressed as

$$f(x, y) = x^n\phi(\frac{y}{x}) \ or \ f(x, y) = y^n\psi(\frac{x}{y})$$

For example, consider $3x^2 - 2xy + \frac{15}{2}y^2$ which is homogeneous in x and y of degree 2. Then

$$\begin{aligned} 3x^2 - 2xy + \frac{15}{2}y^2 &= x^2(3 - \frac{2y}{x} + \frac{15}{2}(\frac{y}{x})^2) \\ &= x^2(\phi(\frac{y}{x})) \\ Similarly,\ 3x^2 - 2xy + \frac{15}{2}y^2 &= y^2(3(\frac{x}{y})^2 - \frac{2x}{y} + \frac{15}{2}) \\ &= y^2(\psi(\frac{x}{y})) \end{aligned}$$

Exercise: Show that if $f(x, y)$ is homogeneous in x and y of degree n. Then

$$f(x, y) = x^n\phi(\frac{y}{x}) \ or \ f(x, y) = y^n\psi(\frac{x}{y})$$

Euler's theorem on Homogeneous functions

Statement: If f is a homogeneous function in x and y of degree n, then,

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

Proof: Since f is a homogeneous function of degree n, f can be written in the form,

$$f(x,y) = x^n \phi(\frac{y}{x}) \tag{1}$$

Differentiating (1) partially w.r.t. x and y we get, respectively

$$\frac{\partial f}{\partial x} = nx^{n-1} \cdot \phi(\frac{y}{x}) + x^n \cdot \phi'(\frac{y}{x})\frac{-y}{x^2} \tag{2}$$

$$and\ \frac{\partial f}{\partial y} = x^n \cdot \phi'(\frac{y}{x})\frac{1}{x} \tag{3}$$

Multiplying (2) by x and (3) by y and adding we have,

$$\begin{aligned}
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x \cdot (nx^{n-1}\phi(\frac{y}{x}) + x^n\phi'(\frac{y}{x})\frac{-y}{x^2}) + y(x^n\phi'(\frac{y}{x})\frac{1}{x}) \\
&= nx^n\phi(\frac{y}{x}) - yx^{n-1}\phi'(\frac{y}{x}) + yx^{n-1}\phi'(\frac{y}{x}) \\
\text{that is } x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= n \cdot x^n\phi(\frac{y}{x}) \\
\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= nf \\
\text{that is } xf_x + yf_y &= nf
\end{aligned}$$

Note: Thus the differential operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ operating on a given homogeneous function f of degree n is equivalent to multiplying f by n.

Corollary: If f is a homogeneous function of degree n, then,

$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

Proof: We have by Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

Differentiating the above equation w.r.t. to x and y we get, respectively

$$\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x} \tag{4}$$

$$and\ x\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y} \tag{5}$$

Multiplying (2) by x and (3) by y and adding we have,

$$\begin{aligned}
x\frac{\partial f}{\partial x}+x^2\frac{\partial^2 f}{\partial x^2}+xy\frac{\partial^2 f}{\partial x\partial y}+xy\frac{\partial^2 f}{\partial y\partial x}+y\frac{\partial f}{\partial y}+y^2\frac{\partial^2 f}{\partial y^2} &= nx\frac{\partial f}{\partial x}+ny\frac{\partial f}{\partial y}\\
\Rightarrow x^2\frac{\partial^2 f}{\partial x^2}+y^2\frac{\partial^2 f}{\partial y^2}+2xy\frac{\partial^2 f}{\partial x\partial y} &= (n-1)(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y})\\
(\because \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial^2 f}{\partial y\partial x})\\
\text{that is } x^2\frac{\partial^2 f}{\partial x^2}+y^2\frac{\partial^2 f}{\partial y^2}+2xy\frac{\partial^2 f}{\partial x\partial y} &= (n-1)nf\\
(\text{ By Euler's theorem})\\
\text{that is } \ x^2f_{xx}+2xyf_{xy+}y^2f_{yy} &= n(n-1)f
\end{aligned}$$

Corollary: If z is a homogeneous function of degree n in x and y and $z = f(u)$, then,

$$\begin{aligned}
x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} &= n\frac{f(u)}{f'(u)}\\
and\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} &= G(u)[G'(u)-1]\\
\text{where } G(u) &= n\frac{f(u)}{f'(u)}
\end{aligned}$$

Euler's theorem for three variables: Statement: If f is a homogeneous function of three independent variables x, y, z of order n, then,
$xf_x + yf_y + zf_z \ = \ nf$
Note: If $u = f(x, y, z)$ is a homogeneous function in x, y and z of degree n, then
$f(\lambda x, \lambda y, \lambda y) = \lambda^n f(x, y, z)$

Examples

1. If $u = sin^{-1}\frac{x}{y} + tan^{-1}\frac{y}{x}$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

   Solution: We have

$$\begin{aligned}
u = f(x,y) &= sin^{-1}\frac{x}{y}+tan^{-1}\frac{y}{x}\\
\Rightarrow f(\lambda x,\lambda y) &= sin^{-1}\frac{\lambda x}{\lambda y}+tan^{-1}\frac{\lambda y}{\lambda x}\\
f(\lambda x,\lambda y) &= [sin^{-1}\frac{x}{y}+tan^{-1}\frac{y}{x}]\\
\text{that is } f(\lambda x,\lambda y) &= \lambda^0 f(x,y)
\end{aligned}$$

   $\Rightarrow u$ is a homogeneous function in x and y of degree 0
   Hence by Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.u = 0$

2. If $u = x^2 tan^{-1}\frac{y}{x} + y^2 sin^{-1}\frac{x}{y}$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$
Solution: We have

$$\begin{aligned} u = f(x,y) &= x^2 tan^{-1}\frac{y}{x} + y^2 sin^{-1}\frac{x}{y} \\ \Rightarrow f(\lambda x, \lambda y) &= \lambda^2 x^2 tan^{-1}\frac{\lambda y}{\lambda x} + \lambda^2 y^2 sin^{-1}\frac{\lambda x}{\lambda y} \\ f(\lambda x, \lambda y) &= \lambda^2 (x^2 tan^{-1}\frac{y}{x} + y^2 sin^{-1}\frac{x}{y}) \\ \text{that is } f(\lambda x, \lambda y) &= \lambda^2 f(x,y) \end{aligned}$$

$\Rightarrow u$ is a homogeneous function in x and y of degree 2
Hence by (corollary to) Euler's theorem,

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= 2(2-1)u \\ \text{that is } \ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= 2u \end{aligned}$$

3. If $y = x\cos u$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$
Solution: We have

$$\begin{aligned} y &= x\cos u \\ \Rightarrow u &= \cos^{-1}\frac{y}{x} \\ \text{that is } u = f(x,y) &= \cos^{-1}\frac{y}{x} \\ \Rightarrow f(\lambda x, \lambda y) &= \cos^{-1}\frac{\lambda y}{\lambda x} \\ &= \cos^{-1}\frac{y}{x} \\ \text{that is } f(\lambda x, \lambda y) &= \lambda^0 f(x,y) \end{aligned}$$

$\Rightarrow u$ is a homogeneous function in x and y of degree 0
Hence by (corollary to) Euler's theorem,

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= 0(0-1)u \\ \text{that is } \ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= 0 \end{aligned}$$

4. If $u = \dfrac{x^2+y^2}{\sqrt{x+y}}$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2}u$

Solution: We have

$$\begin{aligned} u = f(x,y) &= \frac{x^2+y^2}{\sqrt{x+y}} \\ \Rightarrow f(\lambda x, \lambda y) &= \frac{(\lambda x)^2+(\lambda y)^2}{\sqrt{\lambda x+\lambda y}} \\ f(\lambda x, \lambda y) &= \frac{\lambda^2}{\sqrt{\lambda}}(\frac{x^2+y^2}{\sqrt{x+y}}) \\ \text{that is } f(\lambda x, \lambda y) &= \lambda^{\frac{3}{2}} f(x,y) \end{aligned}$$

$\Rightarrow u = f(x,y)$ is a homogeneous function in x and y of degree $\dfrac{3}{2}$

Hence by Euler's theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3}{2} \cdot u$

5. If $u = \tan^{-1}(\dfrac{x^3+y^3}{x+y})$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

Solution: Now

$$\begin{aligned} u = f(x,y) &= tan^{-1}(\frac{x^3+y^3}{x+y}) \\ \Rightarrow f(\lambda x, \lambda y) &= tan^{-1}\frac{(\lambda x)^3+(\lambda y)^3}{\lambda x+\lambda y} \\ \text{that is } f(\lambda x, \lambda y) &= tan^{-1}\frac{\lambda^2 x^2+\lambda^2 y^2}{x+y} \\ \Rightarrow f(\lambda x, \lambda y) &\neq \lambda^k f(x,y) \text{ for any } k \end{aligned}$$

$\Rightarrow u$ is NOT a homogeneous function

But if we consider

$$\begin{aligned} z &= \tan u \\ &= \frac{x^3+y^3}{x+y} \\ \text{Then } z = F(x,y) &= \frac{x^3+y^3}{x+y} \\ \Rightarrow F(\lambda x, \lambda y) &= \frac{(\lambda x)^3+(\lambda y)^3}{\lambda x+\lambda y} \\ \text{that is } F(\lambda x, \lambda y) &= \lambda^2(\frac{x^3+y^3}{x+y}) \\ \text{that is } F(\lambda x, \lambda y) &= \lambda^2 F(x,y) \end{aligned}$$

$\Rightarrow z = \tan u$ is a homogeneous function in x and y of degree 2.

Hence by (corollary to) Euler's theorem,

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= n\frac{f(u)}{f'(u)} \\ &= 2\frac{\tan u}{\sec^2 u} = 2\sin u \cdot \cos u = \sin 2u \end{aligned}$$

6. If $z = \log(x^2+y^2) + \dfrac{x^2+y^2}{x+y} - 2\log(x+y)$ prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{x^2+y^2}{x+y}$

Solution: Let

$$\begin{aligned} u = u(x,y) &= \log(x^2+y^2) - 2\log(x+y) \\ \text{that is } u(x,y) &= \log(\frac{x^2+y^2}{(x+y)^2}) \\ v = v(x,y) &= \frac{x^2+y^2}{x+y} \\ \text{then } z &= u + v \quad \cdots(*) \\ \text{now} \quad u(\lambda x, \lambda y) &= \log(\frac{(\lambda x)^2 + (\lambda y)^2)}{(\lambda x + \lambda y)^2} = \frac{\lambda^2(\log(x^2+y^2))}{\lambda^2(x+y)} \\ \Rightarrow u(\lambda x, \lambda y) &= \lambda^0 u(x,y) \quad \cdots(A) \\ \text{Also} \quad v(\lambda x, \lambda y) &= \frac{(\lambda x)^2 + (\lambda y)^2}{\lambda x + \lambda y} = \lambda\frac{x^2+y^2}{x+y} \\ \Rightarrow v(\lambda x, \lambda y) &= \lambda v(x,y) \quad \cdots(B) \end{aligned}$$

$\Rightarrow u$ is a homogeneous function in x and y of degree 0 and v is a homogeneous function in x and y of degree 1.
Now

$$\begin{aligned} x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x\frac{\partial(u+v)}{\partial x} + y\frac{\partial(u+v)}{\partial y} \\ &= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \\ &= 0\cdot u + 1\cdot v \text{ (By Euler's theorem)} \\ \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= \frac{x^2+y^2}{x+y} \end{aligned}$$

7. *If* $u = \operatorname{cosec}^{-1}[\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}]$,

Prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \dfrac{\tan u}{144}(13+\tan^2 u)$

Solution: Now

$$\begin{aligned} u &= \operatorname{cosec}^{-1}[\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}] \\ \Rightarrow u(\lambda x,\lambda y) &= \operatorname{cosec}^{-1}[\sqrt{\frac{\lambda^{1/2}(x^{1/2}+y^{1/2})}{\lambda^{1/3}(x^{1/3}+y^{1/3})}}] \\ \Rightarrow u(\lambda x,\lambda y) &\neq \lambda^k f(x,y) \text{ for any } k \end{aligned}$$

$\Rightarrow u$ is NOT a homogeneous function

But if we consider

$$\begin{aligned} z &= f(u) = \operatorname{cosec} u \\ &= [\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}] \\ \text{Then } z(\lambda x,\lambda y) &= [\sqrt{\frac{\lambda^{1/2}(x^{1/2}+y^{1/2})}{\lambda^{1/3}(x^{1/3}+y^{1/3})}}] \\ \text{that is } z(\lambda x,\lambda y) &= \sqrt{\lambda^{1/2-1/3}}[\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}] \\ \text{that is } z(\lambda x,\lambda y) &= \lambda^{1/12} z \end{aligned}$$

$\Rightarrow z = \operatorname{cosec} u$ is a homogeneous function in x and y of degree $\dfrac{1}{12}$.

Hence by (corollary to) Euler's theorem,

$$\begin{aligned}
x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} &= G(u)(G'(u) - 1) \\
\text{where } G(u) &= n\frac{f(u)}{f'(u)} = \frac{1}{12}\left(\frac{\operatorname{cosec} u}{-\cot u \operatorname{cosec} u}\right) \\
\text{that is } \; G(u) &= \frac{-1}{12}\tan u \\
\Rightarrow G'(u) &= \frac{-1}{12}\sec^2 u \\
\textit{and } G'(u) - 1 &= \frac{-1}{12}\sec^2 u - 1 \\
&= \frac{-1}{12}(1 + \tan^2 u) - 1 \\
&= \frac{-1}{12}\tan^2 u - \frac{13}{12} \\
\text{that is } \; G'(u) - 1 &= \frac{-1}{12}(\tan^2 u + 13) \\
x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} &= G(u)(G'(u) - 1) \\
&= \left(\frac{-1}{12}\tan u\right) \cdot \left(\frac{-1}{12}(\tan^2 u + 13)\right) \\
\Rightarrow x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} &= \frac{\tan u}{144}(13 + \tan^2 u)
\end{aligned}$$

Euler's theorem for three variables: Statement: If f is a homogeneous function of three independent variables x, y, z of order n, then,
$xf_x + yf_y + zf_z \; = \; nf$
Example: Show that $xu_x + yu_y + zu_z = 2u$, where $u = 3y^2 + 5x^2 - 4z^2$
Solution: Given $u(x, y) = 3y^2 + 5x^2 - 4z^2$

$$\begin{aligned}
u(\lambda x, \lambda y) =& 3\lambda^2 y^2 + 5\lambda^2 x^2 - 4\lambda^2 z^2 \\
=& \lambda^2(3y^2 + 5x^2 - 4z^2) \\
=& \lambda^2 u(x, y)
\end{aligned}$$

Hence u is homogeneous of degree 2.
By Euler's Theorem (for 3 variables)
$xu_x + yu_y + zu_z = nu = 2u$

Practice problems

1. $If \; u \; = \; (\frac{x}{y} \; + \; \frac{y}{z} \; + \; \frac{z}{x})^n, \;\; Show \; that \; xu_x \; + \; yu_y \; + \; zu_z \; = \; 0$

2. $If \; z \; = \; x^n f(\frac{y}{x}) \; + \; y^{-n} f(\frac{x}{y}), \;\; Prove \; that$
   $x^2 z_{xx} \; + \; 2xyz_{xy} \; + \; y^2 z_{yy} \; + \; xz_x \; + \; yz_y \; = \; n^2 z$

3. $If \; u \; = \; x^3 sin^{-1}(\frac{y}{x}) \; + \; x^4 tan^{-1}(\frac{y}{x}), \;\; Find \; the \; value \; of$
   $x^2 u_{xx} \; + \; 2xyu_{xy} \; + \; y^2 u_{yy} \; + \; xu_x \; + \; yu_y$
   $at \; x = 1, y = 1$

3.3: Maxima and Minima of a function of two independent variables

Maxima and Minima

In this sub-unit, we will discuss about obtaining the points of maximum or minimum of a given function of two variables.
Obtaining the maximum or minimum points can together be called as **optimization.**
To optimize something means to maximise or minimise something.
Well, what would anyone want to optimize?
In real life and engineering we may want to find the **fastest or cheapest** way of doing a particular thing - for instance, reaching college from home or designing the most fuel efficient car. The most efficient algorithm, the bandwidth with the least error during transmission etc. are all optimization problems.
I recommend you to refer to the following article:
Kelley, T. R. (2010). Optimization, an important stage of engineering design. The Technology Teacher, 69(5), 18-23.

Some Real life examples
Maxima and minima can be found whenever we are interested in the highest and/or lowest value of a given system

Examples

- A meteorologist creates a model that predicts temperature variance with respect to time. The absolute maximum and minimum of this function over any 24-hour period are the forecasted high and low temperatures, as later reported on The Weather Channel or the evening news.

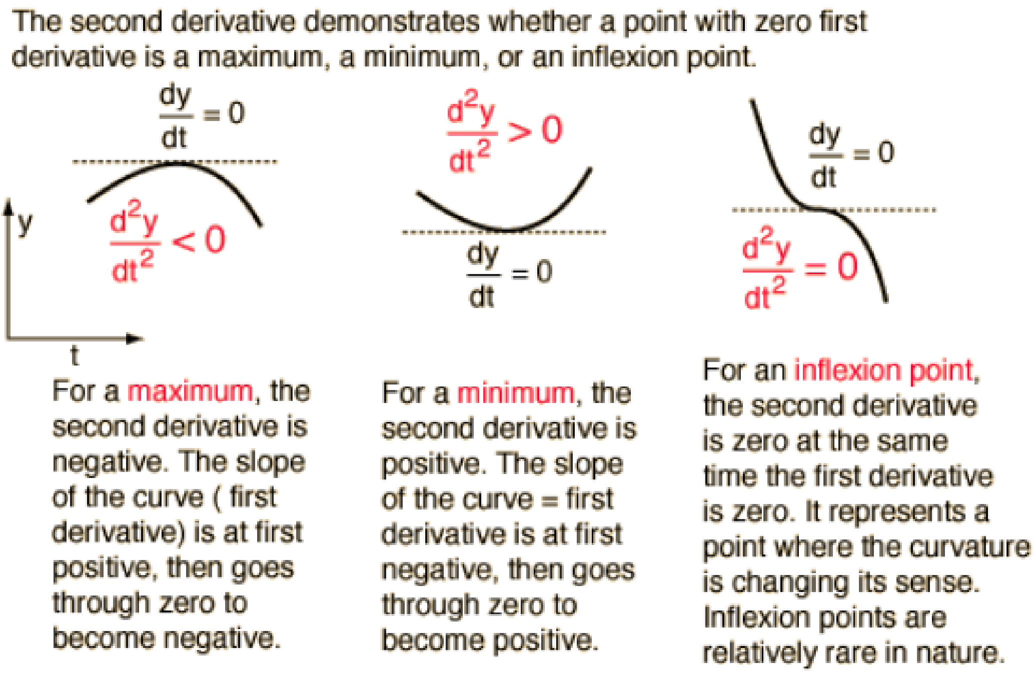
- The director of a theme park works with a model of total revenue as a function of admission price. The location of the absolute maximum of this function represents the ideal admission price (i.e., the one that will generate the most revenue).

- An actuary works with functions that represent the probability of various negative events occuring. The local minima of these functions correspond to lucrative markets for his/her insurance company – low-risk, high-reward ventures.

- A NASA engineer working on the next generation space shuttle studies a function that computes the pressure acting on the shuttle at a given altitude. The absolute maximum of this function represents the pressure that the shuttle must be designed to sustain.

- A conscientious consumer meticulously collects data on his/her cell phone usage over a certain interval (say, a month) and develops a function to represent cell phone usage over time. The local maxima and minima of this function give the consumer a better idea of his/her usage patterns, empowering him/her to choose the most appropriate cell phone plan.

(Reference: http://www.science.ubc.ca/ csp/life/StudentSamples /Website1/theory.html)



One of the great powers of calculus is in the determination of the maximum or minimum value of a function. Take f(x) to be a function of x. Then the value of x for which the derivative of f(x) with respect to x is equal to zero corresponds to a maximum, a minimum or an inflexion point of the function f(x).

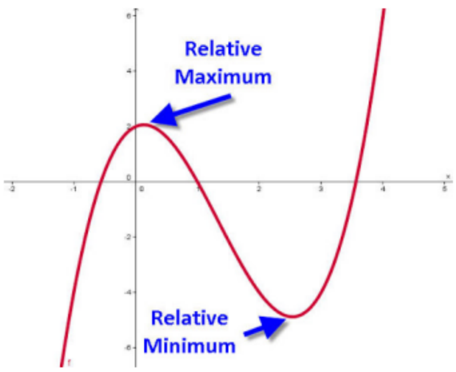
In this chapter, we will define **methods** to obtain the maximum or minimum of functions with and without some conditions imposed - remember in real life, we very often operate with some constraints like what is the fastest way to reach a place if I can spend a maximum of ₹200 or I may have to design a fuel efficient car with a capacity to seat exactly 4 people.

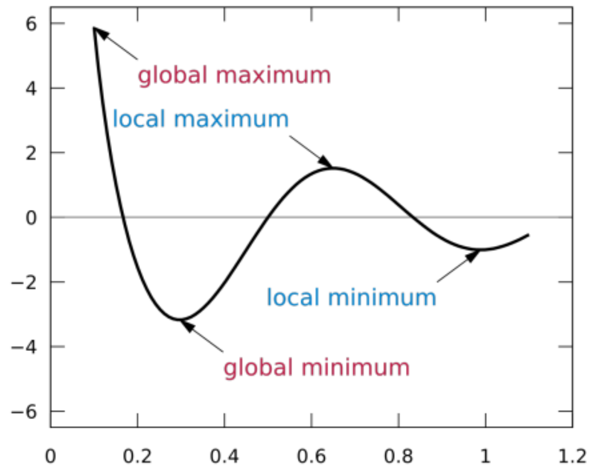
Lagrange multipliers method developed by Lagrange in 1755 is a powerful method for finding extreme values of constrained functions in economics, in designing multi-stage rockets, in engineering, in geometry etc.

Maxima and Minima of functions of two variables
Let $z = f(x, y)$ be a function of two independent variables x and y.
Relative minimum
$f(x, y)$ is said to have a **relative maximum** at a point (a, b) if

$$f(a, b) > f(a + h, b + k)$$

for small positive or negative values of $h,\ k$
i.e. $f(a, b)$ [the value of the function f at (a, b)] is **greater** than the value of the function f at all points in some small neighbourhood of (a, b).





Relative minimum
f(x,y) is said to have a relative minimum at a point (a,b) if

$$f(a, b) < f(a + h, b + k)$$

for small positive or negative values of $h,\ k$
i.e. $f(a, b)$ [the value of the function f at (a, b)] is **less** than the value of the function f at all points in some small neighbourhood of (a, b).

Extremum Point which is either a maximum or minimum.
The value of the function f at an extremum is either a maximum or minimum value of the function f.

Geometrically, $z = f(x, y)$ represents a surface.
The maximum is a point on the surface (hill top) from which the surface descends (comes down) in every direction towards the xy -plane.
The minimum is the bottom of depression from which the surface ascends (climbs up) in every direction.
In either case, the tangent planes to the surface at a maximum or minimum point is horizontal (parallel to xy-plane) and perpendicular to z-axis.
Saddle point Saddle point or minimax is a point where function f is neither a maximum nor a minimum.
At such a point f is maximum in one direction while minimum in another direction.

Geometrically, such a surface(looks like the leather seat on back of a horse) forms a ridge rising in one direction and falling in another direction.

Necessary and Sufficient conditions
Consider a function $f = f(x, y)$.
We would like to investigate if f has a maxima or minima at a point (a, b).
Denote $f_{xx}(a, b) = r, f_{xy}(a, b) = t, f_{xy}(a, b) = s$.
The sufficient (Lagrange's) conditions for extrema are:

1. f attains (has) a maximum at (a,b) if $rt - s^2 > 0,\ r < 0$.
2. f attains (has) a minimum at (a,b) if $rt - s^2 > 0,\ r > 0$.
3. Saddle Point: If $rt - s^2 < 0,\ then\ \Delta > 0\ or\ r < 0$ depending on h & k. Therefore f has a saddle point (minimax) at (a,b) if $rt - s^2 < 0$.
4. No Conclusion: If $rt - s^2 = 0$, further investigation is needed to determine the nature of function f.

Method of Finding Extrema of f(x,y)

1. Solving $f_x = 0\ and\ f_y = 0$ yields critical or stationary point P of f.
2. Calculate $f_{xx}(a, b) = r, f_{xy}(a, b) = t, f_{xy}(a, b) = s$ at the critical point P.
3. (a) Maximum: if $rt - s^2 > 0,\ r < 0$ then f has a maximum at P.
   (b) Minimum: if $rt - s^2 > 0,\ r > 0$ then f has a minimum at P.
   (c) Saddle point: if $rt - s^2 < 0$ then f has neither maximum nor minimum.
   (d) No Conclusion: if $rt - s^2 = 0$, further investigation needed.

Note: Extrema occur only at stationary points. However stationary points need not be extrema.

Remark: The conditions are obtained by using Taylor's theorem:

$$\begin{aligned} f(a+h,b+k) &= f(a,b)+[hf_x(a,b)+kf_{y(a,b)}] \\ &\quad +\tfrac{1}{2!}[h^2 f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)]+\ldots \quad \_\_\_\_(1) \end{aligned}$$

Neglecting higher order terms of h^2, hk, k^2, etc since, h,k are small, the above expression reduces to

$$\begin{aligned} \Delta &= [f(a+h,b+k)-f(a,b)] \\ &= hf_x(a,b)+kf_{y(a,b)} \quad \_\_\_\_(2) \end{aligned}$$

The necessary condition that Δ has the same positive or same negative sign is when $f_x(a,b)$ and $f_y(a,b)$ (even though h & k can take both positive and negative values).

With $f_x(a,b)$ and $f_y(a,b)$, expansion(1), neglecting higher order terms h^3, h^2k, k^3, reduces to

$$\Delta = \tfrac{1}{2!}[h^2 f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2 f_{yy}(a,b)] \quad \_\_\_\_(3)$$

From (3), we observe that the nature of sign of $h^2r+2hks+k^2t$. Rewriting

$$\begin{aligned} sign\ of\ \Delta &= sign\ of(h^2r+2hks+k^2t) \\ &= sign\ of(\frac{h^2r+2hks+k^2t}{r}) \\ &= sign\ of\{\frac{(hr+ks)^2+k^2(rt-s^2)}{r}\} \quad \_\_\_\_(4) \end{aligned}$$

If $rt-s^2>0$ then the numerator in RHS of (4) is positive. In that case of Δ =sign of r. Thus $\Delta<0\ if\ rt-s^2>0,\ r<0$ and $\Delta>0\ if\ rt-s^2>0,\ r>0$

(The above explanation is for reference only)

Solved Examples - Maxmima & Minima

1. Determine the points where the function $x^3 + y^3 - 3axy$ has a maximum or minimum.

   Solution: Here $f(x, y) = x^3 + y^3 - 3axy$

   The points of maxima and minima are given by,

   $\dfrac{\partial f}{\partial x} = 0 \Longrightarrow 3x^2 - 3ay = 0$ ——(1)

   $\dfrac{\partial f}{\partial y} = 0 \Longrightarrow 3y^2 - 3ax = 0$ ——(1)

   We now solve the equations (1) and (2) as simultaneous equation. From (1) $y = \dfrac{x^2}{a}$ substituting in (2), we have $x^4 - a^3x = 0$,

   $$x(x - a)(x^2 + ax + a^2) = 0$$

   Hence $x = 0$ or $x = a$ (discarding the imaginary roots).
   Corresponding y values are $y = 0$ or $y = a$.

   Therefore the functions is stationary at (0,0) and (a,a).

   Now we have $r = \dfrac{\partial^2 f}{\partial x^2} = 6x,\ s = \dfrac{\partial^2 f}{\partial x \partial y} = -3a,\ t = \dfrac{\partial^2 f}{\partial y^2} = 6y$

   At (0,0) $rt - s^2 = 0.0 - (-3a)^2 = -9a^2 < 0$

   Hence at the point (0,0), function$f(x, y)$ is neither maximum nor minimum and it is a saddle point.
   At (a,a) $rt - s^2 = 6a.6a - (-3a)^2 = 27a^2 > 0$

   Hence at the point (a,a), function f(x,y) has maximum or minimum. Since $r = 6a$ it is a maximum if a is negative (i.e. $r < 0$) and minimum if a is positive (i.e. $r > 0$)

2. Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

   Solution: Here $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

   The stationary points are given by,

   $\dfrac{\partial f}{\partial x} = 0 \Longrightarrow 4x^3 - 4x + 4y = 0$ ——(1)

$\dfrac{\partial f}{\partial y} = 0 \Longrightarrow 4y^3 + 4x - 4y = 0$ ______(1)

Solve equations (1) and (2) as simultaneous equations. Adding (1) and (2), we get, $4x^3 + 4y^3 = 0$ *or* $x = -y$ putting $y = -x$ in (1) we obtain $x^3 - 2x = 0$ which gives $x = 0, \sqrt{2}, -\sqrt{2}$ and corresponding values of y are $y = 0, \sqrt{2}, -\sqrt{2}$.

Therefore $(0,0), (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$ are the stationary points of the given function.

Now we have $r = \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4,\ s = \dfrac{\partial^2 f}{\partial x \partial y} = 4,\ t = \dfrac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

At (0,0) $rt - s^2 = (-4)(-4) - (-4)^2 = 4$

At (0,0) $rt - s^2 = 20.20 - 4^2 > 0$ *and* $r = 2$

At (0,0) $rt - s^2 = 20.20 - 4^2 > 0$ *and* $r = 2$

We note that at $(\pm\sqrt{2}, \pm\sqrt{2}),\ rt - s^2 > 0$ is satisfied and also $r > 0$. Hence at these two points, the functions has minima. The minimum value of f(x,y) is -8.

3. Find the extreme values of $xy(a - x - y)$.

Solution: Let $f(x,y) = xy(a - x - y) = axy - x^2y - xy^2$
Extreme values of f(x,y) are given by,

$\dfrac{\partial f}{\partial x} = 0 \Longrightarrow ay - 2xy - y^2 = 0$ ______(1)

$\dfrac{\partial f}{\partial y} = 0 \Longrightarrow ax - x^2 - 2xy = 0$ ______(2)

Solving (1) and (2), we get the points (0,0), (0,a), (a,0), (a/3,a/3)

Now we have $r = \dfrac{\partial^2 f}{\partial x^2} = -2y,\ s = \dfrac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y,\ t = \dfrac{\partial^2 f}{\partial y^2} = -2x$

At (0,0) $rt - s^2 = 0.0 - (a)^2 = -a^2 < 0$
Hence f(0,0) is not an extreme value of f(x,y)

At (0,a) $rt - s^2 = (-2a).0 - (-a)^2 < 0$
Hence f(0,0) is not an extreme value of f(x,y)

At (a,0) $rt - s^2 = 0.(-2a) - (-a)^2 < 0$
Hence f(0,0) is not an extreme value of f(x,y)

At (a/3,a/3) $rt - s^2 = (-2a/3)(-2a/3) - (a - 2a/3 - 2a/3)^2 = 4\frac{a^2}{9} - (-\frac{a}{3})^2$

$\therefore rt - s^2 = \frac{a^2}{3} > 0 \ and \ r = -2a/3$

Hence f(a/3,a/3) is an extreme value and the extreme value is $\frac{a^3}{27}$

Practice Problems

(a) Find the extreme values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
(Ans: $(6,0)(Min), (4,0)(Max), (5,1), (5,-1)(neither)$)

(b) Find the stationary values of $sinx \sin y \sin(x + y)$
(Ans: $(0,0)(Cannot\ decide), (\frac{\pi}{3}, \frac{\pi}{3})(Max)$)