Contents



## 1. UTMOST: Undergraduate Teaching in Mathematics with Open Software and Textbooks

Software for mathematical explorations, including computer algebra systems (CAS), have held great promise for education since their first appearance in the 1960s (Reduce, Macsyma (30; 40)) and the introduction of Maple and Mathematica in the 1980s (29; 59). Currently, CAS are widely found in the undergraduate classroom and a considerable amount of mathematics education research has focused on the use of CAS in learning undergraduate mathematics. Studies have documented the improvement in students' understanding (19; 20; 26; 34). For example, the use of technology allows students to avoid tedious or unreasonable computations, avoid long tables of integrals or Laplace transforms, visualize complicated 2d plots and 3d mathematical objects, and easily deal with unweildy matrices in linear algebra. There is strong agreement that most calculus students should learn the basic techniques of integration; however, CAS replaces the lengthy tables of integrals that were commonplace two decades ago. In addition, technology has allowed a qualitative approach to teaching differential equations. Most differential equations cannot be solved explicitly, but numerical and graphical software has enabled teachers to incorporate a qualitative approach in the courses that they teach (5). The ability to make large computations, quickly and without errors, with graphical output, can greatly aid a students' understanding of difficult ideas in mathematics and provides them with an incredible capacity for experimentation and conjecture.

Computer algebra systems are now common in the undergraduate classroom, yet we rarely see a seamless integration with the curriculum. To accommodate competing commercial systems that divide the market, textbooks typically offer supplements for several different CAS, or they are "technology-enhanced" with generic sidebars; otherwise the textbook can only be marketed to the subset of the audience that licenses a particular computer algebra system. Licensing restrictions for campus use, the expense of personal copies, and underpowered hardware often mean that students can only work with commercial software in campus labs. Some institutions are unable to afford the cost of building such labs for their students.

There have been efforts at seamless integration of technology and curriculum. An example of such an experiment is the Calculus & Mathematica project, which generated much excitement in the early 1990s and had demonstrable success in helping diverse students learn more effectively. However, today its use seems limited to the two institutions where it originated (32; 56). Likewise, the use of interactive Java applets to support teaching mathematics (such as (2; 7; 21)) does not seem to have been widely adopted. The undergraduate curriculum has not seen the broad transformative effect of these powerful tools for increasing the learning and understanding of mathematics. What is needed to increase the uptake of software for mathematical exploration across the spectrum of undergraduate education so we will see the benefits that isolated, successful pilot programs have predicted?

1.1. **An open approach.** Sage (4; 51) is free open source software designed to be an alternative to Magma, Maple, Mathematica and Matlab. Coinciding with the development of Sage, there is a general movement to freely-available open textbooks that includes many quality texts in mathematics ((9; 33), Appendix B). Our proposition is that freely-available open software, open textbooks, and other open curricular material can allow teachers everywhere to transform the undergraduate mathematics curriculum by tightly and seamlessly integrating mathematics software with more traditional curricular materials. We will test this hypothesis by integrating Sage into existing open textbooks and other curricular materials, placing the full computational power of Sage *directly into* a student's text, usable at all times and from anywhere simply via a web browser. For the institution and the instructor, the cost and time-consuming licensing inconveniences of commercial software are removed by open software, freely accessible with no license restrictions. Sage can also be used remotely from a standard web browser, eliminating the expenses and inconveniences of a dedicated local computer lab. Furthermore, the essential nature of open software means that curricular decisions and needs can drive the development of the software, with the classroom teacher actively advising, or actually doing, the software development. Likewise, open curricular materials can be adapted to course needs and teacher preferences, and can be distributed with the software freely in an integrated package. Leveraging these inherent advantages of an open approach to software and curricular materials, the promise of mathematical software in education can be fully realized by faculty and students. Our work will create and disseminate a model for this integration, addressing both the pedagogical and technical aspects, so that other faculty authors can realize the advantages in their own curricular materials and courses.

More specifically, UTMOST will build and test a model for easily integrating open mathematics software and open educational materials into the mathematics curriculum and classroom as follows.

- We plan to create a system that makes it easy for authors to convert open textbooks and other curricular materials to Sage worksheets interspersing runnable interactive demonstrations and exercises and live Sage code with publication-quality typeset mathematics.
- We plan to convert existing mature open textbooks to this format and create new curricular materials targeting this format, as demonstrations and tests of both the technical and pedagogical aspects of this new approach.
- We plan to partner with ten diverse institutions to test these materials in a wide variety of courses, while providing support for their use and assistance for the creation of new materials.
- We plan to evaluate the effectiveness of our model for making it easy to adopt open mathematics software and textbooks and making it easy to create integrated open curricular materials, and we plan to measure the resulting impact on teaching practices and the learning of mathematics with the expert assistance of professional evaluators.

In addition to workshops and other presentations, materials created or enhanced by UTMOST will be widely distributed with open licenses as much as possible. The materials will be available through the global Sage website (`sagemath.org`, (51)). As appropriate, they will also be available in Sage itself or through a new open textbook website created at the American Institute of Mathematics (1) as part of this grant.

## 2. Trends in Scientific Communication

The cost of academic research journals, especially in science, combined with dissemination restrictions imposed by copyright law and new dissemination possibilities afforded by technology, are collectively referred to by librarians as the "serials crisis" (6). Faculty have come a long way in their efforts to return scientific publication to a free exchange of ideas. Electronic journals and public repositories now publish articles with licenses that explicitly allow for sharing new results easily via the Internet without distribution fees. Government initiatives, such as the Public Access Policy of the National Institutes of Health (31) and the Policy Forum on Public Access of the White House Office of Science and Technology (58), are working to accelerate this trend.

Following on the sea change in research publication, the next wave is open textbooks. Faculty are all too familiar with the problems that plague commercial textbook publication, such as high prices, edition churn, and orphaned works. Open licenses are now being used to assert control of these critical resources for education, with faculty in mathematics, computer science and business as the leaders (13; 52). Government is poised to accelerate this open textbook trend at all levels, with Senator Durbin proposing legislation directing the Department of Education to award grants for the creation of open textbooks by faculty (12), Washington State's initiative to provide open textbooks for the eighty highest-enrollment courses in their community college system (57), and California's initiative to create free digital textbooks for its high schools (46).

With the emergence of viable and comprehensive open source software for mathematics, there is now a spectacular opportunity for mathematics teachers to use, extend, and create this important software for mathematics education and shape it to reach its full potential in the service of educating students. UTMOST will create an easy path for all faculty to make the initial transition to open software, open textbooks, and open curricular materials in their courses. This path will make it easy for diverse schools and faculty to employ mathematical software to transform the classroom into an interactive laboratory which takes the study of mathematics to a new level.

2.1. **Imagine this!** A student is learning about row-reducing matrices in a beginning linear algebra course. The electronic version of their textbook is an online Sage worksheet they can view from anywhere. Mathematical equations, with publication-quality typesetting, describe the procedure. An interactive demonstration, embedded in the worksheet where an example normally would be, allows the student to step through row-reducing a matrix that was generated on-the-fly (see Appendix A).

When ready to guide the procedure themselves, the student may choose the row operation to apply at each step of the reduction. The correct notation for each operation is displayed and the operation itself is highlighted with color-coded entries in the displayed matrix. Another example shows the student how to use a built-in Sage command to row-reduce a matrix. With a single click, the student creates an empty code cell under the example to experiment with the command.

In class, the instructor does one simple example on the board. Then the instructor opens a Sage worksheet, displaying the same demonstration that is in every student's electronic textbook. Students guide the choice of row operations at each step for several example matrices of increasing complexity. At the end of class, the instructor clicks a button to publish the work from that day's class session to the class Sage server. After class, students view the worksheet as they review their notes. They can also, with a click, copy the worksheet into their notes, as well as annotate or change it.

The book's exercises include an interactive problem generator that creates matrices of student-specified sizes and complexity for the student to row-reduce. There is a "Solution" button which generates a step-by-step solution. An advanced exercise guides a student through investigating numerical issues that arise in row reduction. By changing one parameter within a command, the matrices track numeric error bounds via interval arithmetic, and the student discovers that numerical errors can be a significant problem for some matrices.

A group exercise asks students to collaboratively write and test a short function to implement row-reduction using row operations. The student clicks a button to open a new Sage worksheet and types in a few lines using Sage matrices and commands. The student evaluates the code with a single click. Satisfied with their work, the student clicks another button, publishing the worksheet to the rest of the group for testing. Later that evening, the student checks the worksheet and sees that other students have tested the function, found an error and repaired it.

The next week, the instructor logs into the online Sage server with a web browser and looks over all of the published worksheets for the group projects. The instructor annotates each worksheet online, using an embedded word processor that supports mathematical typesetting (55; 10).

The instructor writes a quiz covering row-reduction techniques. Inside of the LaTeX file for the quiz, the instructor includes a Sage command to create a simple matrix in reduced-row echelon form, which will be the *answer* to a quiz problem. The instructor then uses Sage commands inside his LaTeX file to do several simple row operations using integers to transform the answer matrix to the matrix the students will row-reduce. The embedded Sage commands are executed automatically when the quiz is formatted with LaTeX, and the resulting question matrix is automatically incorporated in the printed quiz, while the answer matrix and a detailed solution are automatically incorporated in the answer key.

## 3. Why Sage?

Sage is a natural choice for software to realize the benefits of an open approach to the undergraduate mathematics curriculum. Sage is a comprehensive program with an open development process, a modular design philosophy, an easy-to-use interface utilizing standard web browsers, an industry-standard programming language, and tight integration with LaTeX. These are consequences of a steadfast commitment to open design principles and concrete examples of the benefits of this approach. With an open license, the ability to run on a remote server, and a platform-independent interface, Sage removes substantial financial and logistical barriers to classroom use of mathematics software. In this section, we describe the many features of Sage that make it a good choice for integrating mathematics software with open textbooks and other curricular materials.
[i] i

[i] I reordered the following sections, because it seemed like the comprehensive section naturally goes with the building the car section, which is naturally followed by the glue language section (Python). However, this pushes the notebook

**3.1. A comprehensive program.** Sage is becoming a comprehensive program for mathematics. Its modular design allows symbolic, exact, and numerical approaches to mathematics to coexist equally. Mathematical objects, such as functions, differential equations, rings, fields, modules, and vector spaces are "objects" that look and behave as their abstract mathematical definitions intend. Sage has been designed from the outset to incorporate many different computational strategies, and so does not rely solely on a single computational strategy, such as pattern-matching, which favors symbolic computation, or floating-point numbers, which favors numerical work.

Institutional, departmental, and course-by-course decisions about the use of a CAS require a commitment to a particular vendor's vision for the curriculum. With a modular design that allows for packages specializing in different approaches, Sage can seamlessly support many approaches to computationally exploring mathematical problems. Additions and extensions to Sage, driven by real curricular needs, can be easily and quickly incorporated. As individuals and institutions extend the software, the entire mathematics community benefits from a comprehensive program that students and faculty can employ in courses ranging from pre-calculus and introductory statistics through advanced courses like abstract algebra and number theory, as well as in graduate courses and professional careers.

**3.2. Building the car, not reinventing the wheel.** The genius of Sage is the way it unifies over one hundred mature, best-of-breed, open source packages. These packages range from focused libraries that excel at specific types of computations (e.g., Fast Library for Number Theory for the world's fastest implementation of integer polynomial arithmetic (14), the Integer Matrix Library for the fastest implementation of solutions to linear systems over the rationals (22), or M4RI for the fastest implementation of exact computations with binary matrices (28)) to complete applications or general libraries for broad areas of mathematics (e.g., R for statistics, GAP for group theory, and SciPy for numeric scientific computations (16; 39; 47)). Sage ties these packages into a single open source system with a consistent interface, making it easy for a teacher or student to smoothly explore vast areas of mathematics seamlessly, using the best tools under the hood for each computation.

As a student moves from course to course, the Sage notebook interface and commands remain consistent, even if the particular computations relevant to a new course may be performed by an entirely different package under the hood.

**3.3. A standard programming language.** The many components of Sage are held together with a significant library of new code, written in the industry-standard programming language Python (36). Through the power of Python, new open source packages written in a variety of programming languages are added easily to Sage, bringing users new functionality, or improving existing functionality[ii]. More and more packages are appearing for mathematics and science written in Python, often with open licenses (38; 47). $\boxed{\textbf{ii}}$

Students may use Sage through point-and-click interactive demonstrations written by others, or they may execute a sequence of single-line commands in the notebook. However, for more involved computations, they can use Python, since it also serves as the user language in Sage. This is in contrast to other comprehensive programs for mathematics that have chosen to create and develop their own programming languages, which are of no use outside of the CAS itself. A user with knowledge of Python is ready to be incredibly productive in Sage immediately, while a student using Sage that is new to programming receives a basic familiarity and education in Python—a skill that is readily transferable to mathematical and non-mathematical applications in a wide variety of fields, including art, business, science, engineering, and many other disciplines (37).

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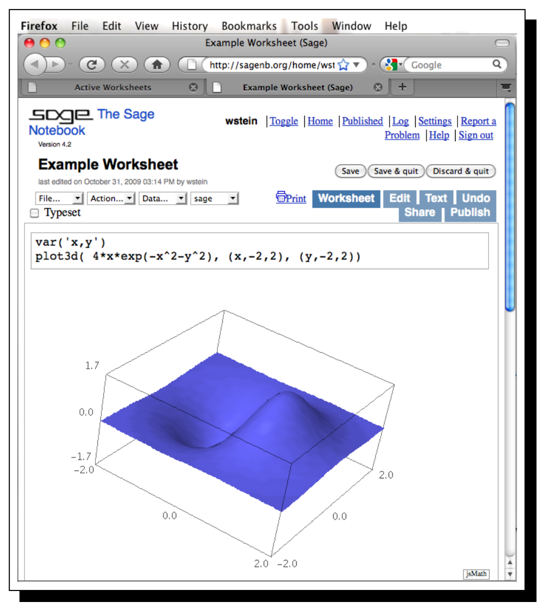
section and the open development section towards the bottom, which seems to downplay their importance. I'm not sure what the right order here should be.

[ii]Also, interfaces are easily written to non-open software. Also mention that these programs don't have to be written in python, as I can see some people assuming that they do have to be.

3.4. **Curriculum and open development.** The closed development process for commercial software creates a high barrier for teachers and students to alter or extend the software to meet their curricular needs. For example, while external extensions, such as libraries or packages, can extend proprietary systems, these must be purchased, distributed, and installed by every end-user. However, an open development process allows students and faculty to shape the core technology in a timely fashion to support the transformation of teaching and learning of mathematics on a broad scale. Software developed openly and collaboratively, such as Sage, allows the teaching and learning of mathematics to drive the technology, rather than the technology driving the teaching and learning. Two concrete examples of this important principle are given in Section 4.

3.5. **Sage anytime, anywhere.** Sage's notebook interface allows a student to communicate with a Sage server through a standard web browser—interactively running commands, viewing textual and graphic output, and annotating their computations with an online word processor that supports mathematical typesetting. The notebook interface removes many of the logistical barriers to using software in the study of mathematics. A student can employ the full power of Sage using a remote Sage server with only a web browser and a minimal network connection on a desktop computer, a small laptop or netbook, or even a cell phone (44)! This flexibility means that instead of an institution providing an expensive lab of machines with commercial software that can only be used on the dedicated computers, most any computer located anywhere can harness the full power of Sage.

[iii]The Sage notebook interface relies heavily on industry-standard Javascript and acts as a web application similar to Google Docs, GMail, or wiki software. Students have designed and implemented much of the current notebook. For example, Tim Dumol, a high school student in the Philippines, has been a very active developer in Fall 2009. **iii**




The Sage notebook interface running in Mac OS X on Firefox

Every copy of Sage includes the Sage server software. To establish a local server, colleges can download and use the ready-made Sage virtual computer image on a computer that supports their needs. There are also publicly-available servers that can be used to support students and classes. The flagship public Sage server is `sagenb.org`, a $100,000 rack of servers located at the University of Washington that was fully funded by an NSF SCREMS grant (DMS-0821725), and is currently home to over 21,000 accounts. Because it is publicly accessible, students from the poorest universities all over the world are running computations on the same version of Sage, on the same hardware cluster, with leading researchers in computational number theory. Constructed in January 2009, this powerful server should be a viable resource for many years to come. Publicly-accessible notebook servers continue to appear throughout the world (most recently in Hungary (43)), and many more run behind campus firewalls for dedicated use (45).

For occasions when there is no network access, Sage can be installed directly on a computer and used as a private server with exactly the same online notebook interface as the remote Sage servers use.

[iii]figure todo: make the window wider, so the buttons aren't squished, and maybe show it on Windows, so people know we can access it from Windows (to take care of complaints people may have heard about Sage not running on Windows).

The flexibility of freely-available online access is a key consequence of the open nature of Sage and its importance cannot be overemphasized in education. As student populations become increasingly mobile, having technology that can be used freely from anywhere, especially from underpowered mobile devices, supports natural trends in student life.

3.6. **Communicating mathematics.** TEX, with its add-on package LATEX, is another open source success story, and is the typesetting language of choice for mathematicians and other technical disciplines. Every mathematical object in Sage can be typeset in LATEX automatically. The Sage notebook uses jsMath (10) to typeset mathematics beautifully in a Sage worksheet (within a standard web browser). Furthermore, the Sage notebook interface allows a user to insert new text, including typeset mathematics. This is another example of how open standards and open software combine to make powerful tools, which is especially relevant for our plan to convert LATEX documents into Sage worksheets.

In addition to using LATEX in the notebook interface, Sage can be used from within any LATEX file to perform computations and create figures. Dan Drake, a Visiting Professor at Korea Advanced Institute of Science and Technology, authored a SageTEX (11), a LATEX package which allows Sage code in the file to easily be run as part of the process of formatting a LATEX document. Output (as typeset mathematics or images) is then automatically embedded in the resulting document.

## 4. Two Examples of Teachers Driving Open Development

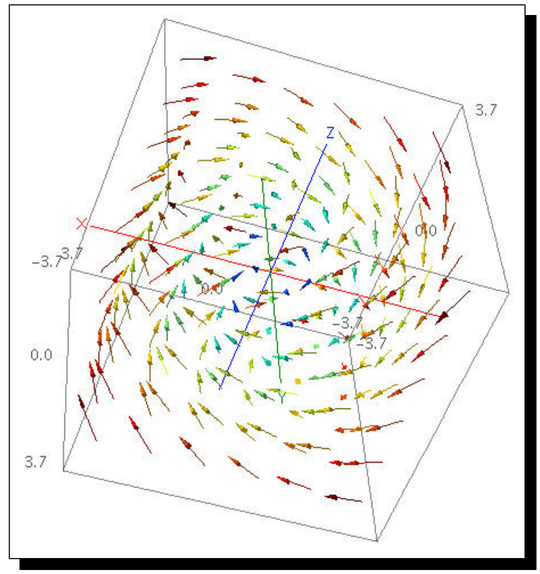
In this section, we provide two concrete examples of how teachers and classroom needs drive the devlopment of Sage to directly create better and more comprehensive tools for use in the teaching and learning of mathematics.

4.1. **3D vector fields.** Jason Grout, then a postdoctoral associate at Iowa State University, was asked one day by a meterology student if Sage had a 3D vector field plotting function. Grout quickly wrote a simple one. Open development communities encourage sharing even experimental code, so he posted his code online in Sage's publicly-accessible online database of enhancements and bugs. Robert Bradshaw, a graduate student at University of Washington who had written much of the 3D graphing code in Sage, suggested a small change that greatly increased the efficiency of the function. Several months later, another person authored a very similar function which had slightly improved options and published their code on the public Sage notebook server. In Fall 2008, Grout needed the 3D vector field plotting function for his multivariable calculus class. He made a few suggested changes to the function and posted it for his class to use. If he had been using a commercial CAS, this is where the story likely would have ended.

In the next few months, improvements were made to the internals of Sage 3D graphics (many by William Cauchois, a University of Washington freshman supported for a summer by an NSF VIGRE grant). Some of these improvements were directly the result of wanting to make the 3D vector field plotting function easier to write. Jason then taught multivariable calculus again. He incorporated the best ideas from the various sources and posted a documented, efficient version of the function, this time asking the community for a formal review (a prerequisite for his code to be added to the core Sage library). Marshall Hampton, an Assistant Professor at University of Minnesota, Duluth, who has been active in the Sage project, reviewed the function immediately, with the comment "Very nice, positive review. This is great timing since I am about to teach vector fields in a week or two." The function was incorporated into the next release of Sage, two weeks later, ready for immediate use by the entire Sage user community.

Now not only do the original students benefit from the invested effort, but the collaborative effort of at least five teachers and students has directly led to a function that will serve all who use Sage.

4.2. **Dicyclic groups.** When Rob Beezer teaches introductory group theory, as he did in Fall 2009 with Tom Judson's open source text, he explores classifying all the finite groups of order less than 16 as a way of exhibiting the power of the theorems developed through the term. Each of these groups is easily constructed and realized in Sage as a permutation group, except for one of the three non-abelian groups of order 12.




$\vec{f}(x, y, z) = y\vec{\imath} + z\vec{\jmath} + x\vec{k}$ plotted with Sage's `plot_vector_field3d` command

In Fall 2009, Beezer decided to add this group to the collection of constructions in Sage. First, on a Friday evening, he polled the Sage development email list about the best way to realize this group. John Palmieri, an Associate Professor at the University of Washington, who is an expert in algebraic topology, responded 90 minutes later with information about the dicyclic groups, a sequence of groups of order $4m$, including the group of order 12 in question, the quaternion group of order 8, and the generalized quaternions of order $2^n$. David Joyner, a Professor at the US Naval Academy who was the second Sage developer after Stein, and who implemented much of Sage's support for group theory, joined the conversation an hour later. Jaap Spies weighed in from Holland with a source for more information. With this help, Beezer was able to easily create an efficient permutation representation of these groups in the next few days, and posted code and documentation, including comprehensive information and references about the group's construction and properties, to Sage's enhancement database. Joyner positively reviewed the code and documentation several days later. Two weeks later, this new code and documentation was incorporated in a preliminary Sage release. Within a month, Mike Hansen, a volunteer working from Thailand, oversaw the release of the next version of Sage, which included this code.

In this example, the net result is new capabilities for Sage, relevant to the undergraduate curriculum, with associated mathematical explanations in the Sage documentation. Beezer's course was improved by this exercise, and the benefits are now available to all through Sage. The dicyclic groups are now a worthwhile future addition to Judson's text, completing a virtuous cycle.

These two examples illustrate many strengths of the open development process used by Sage, where teachers and curriculum drive the technology. Teachers, students, and individuals bring their talents as educators, mathematicians, and programmers together to make Sage an astonishing product of open development.

## 5. Implementation

Our goal is to have a broad, positive influence on the teaching and learning of mathematics. By using open curricular materials, integrated with powerful and comprehensive open software, we will realize a transformative effect. UTMOST will revolve around three coordinated efforts: (1) creating Sage-enhanced curricular materials, (2) working with, and providing classroom support for, teachers using these new materials in the classroom, and (3) providing the Sage infrastructure to support these materials. More specifically, UTMOST can be divided into five main activities.

(1) We plan to create a system to convert open mathematics textbooks to Sage worksheets that can incorporate live Sage code, as well as interactive demonstrations and exercises powered by Sage. Our work to make these textbooks Sage-enhanced will provide a model for other authors to create Sage-enhanced textbooks. The first courses we will address are linear algebra, abstract algebra, and number theory.

(2) We plan to further implement our model for authoring Sage-enhanced curricular materials by creating modules for standard undergraduate courses, especially courses not covered by the textbooks we are converting. These materials will include subject-specific guides to introduce faculty and students to Sage in the context of a particular topic. These materials will be created by members of the UTMOST team and by faculty at our ten test sites. Topics under consideration include single variable calculus, multivariable calculus, differential equations, and complex variables.
(3) We plan to improve the Sage library and its surrounding infrastructure (e.g., server design, notebook usability, and collaboration tools) where the improvements have a direct and obvious benefit for undergraduate education.
(4) We plan to identify and work closely with teachers at ten different institutions to test, refine, improve, and extend the work described above. We will select a broad range of institutions, including some with diverse student profiles. Dedicated Sage community workshops ("Sage Days") will provide opportunities for the UTMOST team and representatives from the test sites to all collaborate on the project's activities. These ten sites will provide extensive opportunities for our evaluation team to measure the success of our efforts and the impact on teaching and learning.
(5) We plan to disseminate our materials and the results of UTMOST through Sage Days workshops, MAA PREP workshops, and MAA minicourses (national and regional). We will use these opportunities to train faculty outside of the ten test sites in the use of these materials for their courses, and in the process for creating their own new materials. Our materials will all carry open licenses and be available through the global Sage website. As appropriate, they will also be available in Sage itself or through a new open textbook website created at the American Institute of Mathematics as part of this grant.

5.1. **Sage-enhanced open textbooks.** Our principal activity is the design and creation of Sage-enhanced textbooks.

5.1.1. *Infrastructure for textbooks.* UTMOST proposes creating and fully testing a system for integrating Sage and textbooks. Pilot projects have already demonstrated the feasibility of this process for short documents, and technical improvements have been identified to make integrating Sage and book-length documents easy for authors to do. The primary tool is the existing tex4ht translator (NSF IIS-0312487) (53) which can convert LaTeX into the jsMath format adopted by the Sage notebook for rendering publication-quality mathematics. The end result will be a folder of Sage worksheets, each a mixture of text, typeset mathematics, Sage input cells and Sage interactive demonstrations and exercises. Because LaTeX is the accepted standard for authoring mathematics, we expect this process to be applicable to a broad range of current and future mathematics textbooks.

5.1.2. *Targeted textbooks.* Appendix B lists many open textbooks that could be Sage-enhanced by our process. In addition to converting existing material to Sage worksheets, we plan to incorporate interactive demonstrations and exercises, live Sage code, and guidance on the Sage library itself in the texts. There is significant care and thought required to integrate these uses of mathematical software properly. Our work will provide a tested model other authors can consult when authoring new textbooks.

Listed below are our initial candidates for conversion to Sage-enhanced textbooks that bring the full power of mathematics software to the student *directly within their book*. The open textbook landscape is rapidly growing, and beyond the following list we will make final decisions based on the most promising open textbooks that will help us achieve the project's goals. We expect to have the three texts below available for use in the Fall 2011 term at our test sites. Each of these is authored by a project member, and so provides an excellent testbed for the enhancement process, since project members will then be able to directly refine the conversion process.

- Beezer's *A First Course in Linear Algebra* (3). This is a mature textbook designed from the start to be open source, and thus has a highly modular design. The first version of this book was released in December 2006, and has been used as the primary text in courses at 13 different universities. The author has taught from various versions of the book ten times. It will be an excellent choice as a full-scale test of the technical conversion process, the integration of Sage code, and classroom use.
- Judson's *Abstract Algebra: Theory and Applications* (23). This book was originally published by PWS-Kent in the early 1990s, but the author has received the copyright back and released it under an open source license. Beezer has taught from this text the past two years and created supplementary material utilizing Sage and contributed code to the Sage library to support the book. First released as open source in Spring 2009, eight universities adopted the book for the Fall 2009 term. As a year-long course, this is a larger project to tackle next, and will be a collaborative project between two team members.
- Stein's *Elementary Number Theory* (50). This text already has an extensive collection of Sage examples. Additionally, the author has taught from the book four times. It is currently being published by Springer-Verlag, but Stein has the rights to make it freely available starting May 2010, so will be able create a Sage-enhanced version for free distribution. While it will not carry a true open license that allows modifications by others, it will be a good test of the technical aspects of the conversion process and will be made freely available.

5.2. **Sage-enhanced curricular materials.** In addition to enhancing selected open textbooks, we will also enhance and develop smaller-scale curricular materials for other undergraduate courses.

5.2.1. *Subject-specific guides.* For selected courses where we do not provide a Sage-enhanced textbook, or no suitable open book currently exists, we will create subject-specific tutorials. These short guides will explain how to use Sage to investigate questions in a standard course, following a typical order for such a course. Existing examples of these smaller-scale materials include multivariate calculus worksheets by Jason Grout and Ben Woodruff [iv], John Perry's course notes for a Mathematical Computing course (35), and Stein and Mazur's Riemann Hypothesis book (41), which targets a curious undergraduate. **iv**

Stipends will be provided to faculty outside the project team, ideally from our test sites, to author such materials. These will be designed from the start for conversion to Sage worksheets, providing a test of our system for this conversion.[v] **v**

5.2.2. *Interactive demonstrations and exercises.* Sage has a powerful, but extremely simple-to-use framework for creating web-based interactive demonstrations. Input boxes, sliders, selectors, and other controls are simple to create. Output can use HTML, tables, typeset mathematics, and 2D or 3D graphics. Computations have the full Sage library at their disposal. The example in the screenshot in Appendix A is an example of such an interactive demonstration. Stein created the first implementation and continues to improve it with significant contributions from Grout and many other Sage developers. Current work by Mitesh Patel[vi], supported by an NSF FRG grant (DMS-0757627), will allow embedding these interactive demonstrations into standard web pages, usable without having to log into a Sage server. **vi**

For select courses we will create comprehensive collections of demonstrations for inclusion in our Sage-enhanced textbooks and into Sage itself. We will also create an easily-searchable repository of high quality, reviewed Sage interacts, which will be included in every copy of Sage.

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[iv] Jason: publish these as a set of worksheets

[v] Have we included this in the budget? Mention later in test-sites.

[vi] a student? If so, say so

5.3. **Sage into the classroom.** The heart of UTMOST is putting Sage-enhanced materials in the hands of teachers and students to assist them in teaching and learning mathematics. We will work with selected faculty at ten partner institutions by providing support for using Sage in their classrooms, offering technical support for Sage, assisting with the Sage-enhanced materials that we have created, and helping them write their own Sage-enhanced materials. Our teacher-authors and their students will receive access to Sage on a server dedicated to the UTMOST project, with dedicated technical support provided through this project.[vii] **vii**

We will recruit two groups of teacher-authors for UTMOST, with the first group beginning in Summer 2011 for a two-year commitment and the second group beginning Summer 2012 for a one-year commitment, for a total of ten test sites. Teacher-authors will receive a stipend for their participation in UTMOST and support to attend a workshop on using Sage in the classroom. In order to forge a strong relationship with the teacher-authors and provide the best possible support, each teacher-author will be paired with a member of the UTMOST team who will serve as a contact and adviser. This adviser will also make visits to the test site to provide support and gather evaluation data. We will expect the following from these participants.

- Teacher-authors will commit to using Sage in their classrooms during their time with UTMOST. More specifically, they will be expected to use Sage-enhanced materials extensively in at least one semester-long course during the academic year.
- Teacher-authors will write and test comprehensive Sage-enhanced curricular materials for the class that they are teaching.
- Teacher-authors will collect evaluation data from their students, and organize focus groups to gather feedback on the design and use of Sage-enhanced curricular materials. Focus groups will be led by the project member who is advising the teacher-author.
- Teacher-authors will offer training to others at their institution on how to use Sage and how to use Sage-enhanced materials in the classroom. Such training could be offered as departmental workshops or seminars.
- Teacher-authors from the first group will be selected to mentor the new teacher-authors in the second group.
- Teacher-authors will submit a report each year on their teaching experiences with Sage and new curricular materials.

5.3.1. *Recruitment plan.* Selection of the teacher-authors for UTMOST will be based on the following points.

- A narrative statement by the applicant regarding rationale for applying, the course or courses in which they plan to use UTMOST materials, and their familiarity with Sage.
- Limited or no familiarity with Sage, since we aim to make it easy for all faculty to integrate mathematics software into their courses, not just those who already have this experience.
- A letter of support from the department chair, indicating the flexibility to schedule faculty for the appropriate courses, a commitment to using Sage and new curricular materials in their department, and an awareness of evaluation activities.
- Consideration will be given to achieving a diverse mix of institutions and student populations, along with geographic proximity to a member of the UTMOST team.

5.4. **Sage infrastructure.** While the heart of Sage is its library of commands, the notebook interface and server configurations are key elements of a successful experience for students and faculty using Sage in their courses. A portion of our work will be to improve Sage itself, in those areas where the improvements *directly support* educational applications of Sage. Discussions and initial queries in the Sage-Edu email list indicate that faculty around the world are excited about setting up Sage servers for their students' use (45). We see three main areas for work on Sage.

[vii] In the budget?

5.4.1. *The Sage library.* Experience has shown us that teaching with Sage invariably suggests new capabilities or exposes needed commands. Our work enhancing textbooks, and our teacher-authors at test sites, will certainly suggest extensions and corrections to the Sage library. We will continue to refine and enhance Sage at an accelerated pace with support from this grant. All changes will undergo the existing code review process, and will then become permanent contributions to Sage, maintained and further extended by the worldwide Sage community. We have already identified specific improvements for linear algebra, abstract algebra, calculus, differential equations, number theory, and combinatorics, as well as broader areas like 2D and 3D graphics. Targeted fixes and improvements will greatly improve the educational experience for students. In many cases, undergraduate students can, and already have, contributed code and other suggestions; we plan to employ several students in these efforts.

5.4.2. *Notebook development.* The Sage notebook interface is a powerful tool for experimentation and collaboration. Stein worked fulltime during Fall 2009, supported by the University of Washington, on improving the robustness and scalability of the notebook. In Fall 2009, we have typically seen two thousand new accounts created on `sagenb.org` every month. The notebook interface is the face of Sage for students and can be improved in many ways[viii] **viii**

5.4.3. *Sage servers the easy way.* Sage servers allow a user with just a standard web browser to use Sage over the network. Initiating and maintaining a server should be as simple as possible, making it easy for faculty and system administrators everywhere to get their students started with Sage.

[ix]We will create a Virtual Box Sage notebook server appliance with a graphical interface, two virtual machines that provide a rock-solid secure Sage notebook server setup. Users will be able to very easily install this appliance on Windows, Linux, OS X, and Solaris (x86) servers. The management interface will provide clear and easy documentation about setting up this server, creating new notebook servers for specific classes or instructors, starting and stopping notebook servers, monitoring resource usage, adding users and authentication frameworks (such as LDAP), and upgrading Sage with minimal user downtime. **ix**

For users at our ten test sites, we will create a new mailing list for server administration issues, and in addition to the help we provide ourselves, we will hire a knowledgeable employee to answer support questions on a daily basis.

Several specific goals for notebook and server development include the following.

- We plan to improve the notebook so that it will robustly handle up to 250 simultaneous users viewing worksheets and doing typical computations for an undergraduate course when running on a single high-end server, as demonstrated by a robust automated test suite.
- We plan to implement management tools so administrators can manage the notebook load and better balance resources.
- We plan to create easy-to-use tools for educators to get automatic feedback about how their students use Sage.

In summary, we propose to provide high-quality enterprise-level software infrastructure and support to educators.

## 6. Evaluation

[x] Formative and summative evaluation will be conducted at different phases of the project to address questions about the effectiveness of the project in creating and implementing the UTMOST **x**

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[viii]William: like collaboration and....

[ix]Explain something about virtual images and how easy they are to use

[x]Check changes in this section. For example, lots of mentions of the "Sage tool and material" has been changed to "model and Sage-enhanced materials". Do the evaluators know we are evaluating a model, as well as specific curricular materials?

model and materials for undergraduate mathematics teaching and learning. Drs. Judson and Hassi[xi] will coordinate internal evaluation data-gathering, and will serve as the liaisons to evaluators from Ethnography & Evaluation Research at the University of Colorado at Boulder. The evaluation questions include: **xi**

(1) What aspects of the Sage-based tools and materials are beneficial to instructors, what challenges do they face, and what kinds of support do they need in using these tools?
(2) How do instructors use and apply the Sage-integrated curriculum materials, and how do these benefit (or fail to benefit) their teaching of mathematics?
(3) How do the tools and materials impact instructors' content knowledge, pedagogical content knowledge, classroom instructional practice, and their students' learning?

Information will be gathered on both the processes and outcomes of the project at different stages, and results will be organized as case studies of participating instructors and their students. Formative components will focus on monitoring the quality of project activities, enabling the project to make mid-course corrections and plan for future development. Summative components will focus on the impact of the project on instructors' instruction and student learning at the test sites. Instructors and classrooms will be sampled, taking into account instructor interest and local institutional cooperation. In addition to offering feedback on[xii] **xii**

6.1. **Study design.** The study design includes pre- and post-surveys, follow-up surveys included in yearly self-reports, and interviews of the[xiii] participating instructors. In addition, students will answer an online post-survey focusing on their experiences and gains in learning mathematics, including their classroom use of the Sage tool and materials. The design is informed by previous evaluation studies on professional development, education, and workshops in STEM fields (8; 15; 27; 48; 54) and on student outcomes of active instructional methods in undergraduate mathematics (17; 18). **xiii**

6.1.1. *Pre-survey.* While registering for the summer workshops, participants will complete a short pre-survey to gather demographic and contact data, and information about their classes, institution, current teaching practices, and pedagogical needs. This information will also help the project leaders to plan workshops and later support.

6.1.2. *Post-survey and feedback.* Participants will complete a survey on the summer workshops so that project leaders can make adjustments for future workshops and implementation support. The survey will ask about participants plans for using the model and Sage-enhanced materials, to help guide later components of the study. They will also be asked to sign a consent form to participate in a follow-up interview. Evaluators will observe the summer workshop and conduct a focus group with current and past participants.

6.1.3. *Follow-up survey and reports.* After using the model and Sage-enhanced materials for one year, the participants will file a report on their implementation, including some follow-up survey questions. They will report their use of the model and materials in their own classrooms and their future plans. Additional data on implementation will be gathered by the project team during site visits; they will conduct student focus groups using a protocol co-developed with the evaluators and share site visit notes that will be used as data sources[xiv]. **xiv**

[xi] added Hassi here
[xii] never got the end of this sentence
[xiii] removed number
[xiv] data sources for what?

6.1.4. *Student learning assessment.* Student learning will be evaluated after the first year of implementation with a post-survey based on the NSF-supported SALG instrument (DUE-0920801) (49), in which students report about their experiences of and learning gains from their mathematics course. This will provide information on student outcomes as well as formative feedback for the instructors using the model and Sage-enhanced materials in their classroom. A mathematics-specific version (SALG-M) has already been validated and used in a large evaluation study, and is sensitive to differences by student group and classroom practice (17; 25).

6.1.5. *Follow-up interviews.* Based on instructor and student responses on other measures, about half[xv] of the instructors will be interviewed to study factors that affect their success in implementing Sage-enhanced materials in their class. The interviews will explore classroom use of the model and Sage-enhanced materials, impacts on their instructional practices, and instructors' perception of students' response. **xv**

6.2. **Management and dissemination.** The evaluation will be led by Dr. Marja-Liisa Hassi from Ethnography & Evaluation Research (E&ER) at the University of Colorado at Boulder. E&ER has extensive experience evaluating large mathematics and science education projects (NSF DUE-0920126, DUE-0723600, DUE-0450088).

Drs. Judson and Hassi will collaborate with the project director, members of the project, and partner institutions to conduct the evaluation study. Dr. Judson and the project PIs will conduct site visits to partner institutions to observe and document the use of Sage-enhanced materials. Dr. Judson will have primary responsibility for extracting information from the site visits and workshop data to feed back to developers to improve the technology and classroom resources. Surveys and interview protocols will be prepared and conducted by Drs. Hassi and Judson. They will analyze all data and will prepare an annual report to document the evaluation activities and results. This will provide formative evaluation feedback to the project PIs and participating instructors to inform design decisions and mid-course corrections. A final report will gather results after the implementation of the tools and material by the second group of instructors in 2013. These findings will be shared also with the broader mathematics education community through a presentation and a coauthored article about the impacts of using CAS-integrated materials in teaching and learning undergraduate mathematics.

## 7. Dissemination

utmost is principally about dissemination—the focus issue is how we can enable the widespread implementation of approaches which have already been proven to work. In addition to working with our test sites, we will broadly disseminate the results of utmost through a variety of forums, several of which are already in place.

7.1. **Sage website.** The Sage website, `sagemath.org` (51), is an established venue for sharing materials related to Sage. This central location receives 90,000 visitors a month, with 42% of the visitors from the Americas (North, Central, and South) and 45% of the visitors from Europe. What we learn and create will be made available on the Sage website with open licenses. Further, some of our work will be incorporated into Sage itself, which already has an effective distribution system, designed to make Sage extremely easy to download and run on a personal machine or a server. Complete, integrated documentation ships with every copy of Sage (such as the 4,641 page reference manual), and it is possible that much of our material, *including complete textbooks*, could ship as a core component of Sage itself. A library of high-quality interactive demonstrations will also be included in Sage, where they will be able to be browsed topically or searched. These included demonstrations will benefit from Sage's automated test-suite that users routinely run on a wide variety of hardware.

[xv] still only half? Or are we now interviewing all 10 instructors?

7.2. **Sage Days workshops.** Sage Days, which are a series of conferences and workshops devoted to the development of Sage, are an ideal venue for dissemination. To date there have been nineteen Sage Days conferences, with nine more planned. These workshops now include sessions dedicated to using Sage in the undergraduate classroom, with the first such session taking place in December 2009 at the Clay Mathematics Institute on the final day of a week-long workshop on Sage and number theory. It attracted roughly thirty college faculty (and a few high school faculty) from around the Northeast, all eager to learn more about the use of Sage in educational settings.

A Sage Days devoted to training others to use our model and materials, and receiving and incorporating feedback and new ideas, will be entirely consistent to the way Sage Days have been used to drive Sage development. We plan to have a Sage Days event in Summer 2012, involving the UTMOST team and all ten test sites, as an opportunity to discuss the project at the midpoint of our experiences in the classroom. Another Sage Days event at the American Institute of Mathematics in Summer 2013 will be an opportunity to more generally review the experiences of the project and plan for the future of integrating open software and open textbooks in the undergraduate mathematics curriculum.

7.3. **National workshops.** We will apply to offer special sessions and workshops on the use of Sage in the classroom at the national Joint Mathematics Meetings and MathFest conferences, where we will be able to share the results of UTMOST with the undergraduate teaching community. We will also publish articles about the impact of using our materials on the teaching and learning of undergraduate mathematics. Drs. Beezer and Grout are part of a team delivering a workshop in Summer 2010 on how to use Sage in the classroom as part of the NSF-funded Mathematical Association of America Professional Enhancement Program (MAA PREP). We will apply to lead future PREP workshops that will incorporate the results of UTMOST. Notably, Stein has recently been invited to propose an American Mathematical Society Short Course on Sage for the 2011 Joint Meetings.

7.4. **AIM open textbook initiative.** [xvi]The American Institute of Mathematics is a respected NSF-funded institute that has supported both leading research mathematicians and mathematics education at all levels. As the sponsoring organization for UTMOST, AIM will conduct a pilot project to test the feasibility of sponsoring and supporting a series of open mathematics textbooks. A hindrance to the adoption of open textbooks is the lack of an acknowledged authority to vouch for the content. Usually this is a role played by a publisher, so a recognized organization within the mathematics community with a trusted reputation, such as AIM, will be another component to broadly disseminating our materials. This has the potential to positively impact open textbooks, in mathematics and more generally. **xvi**

## 8. Future Directions

The limited scope of this proposal suggests several broader new directions. In particular, the project team was intrigued by the possibilities for using Sage to generate and check homework through integration with a system like WebWork, but decided such an initiative was too ambitious and disjoint to be included in this proposal. [xvii] **xvii**

- Create new open textbooks with Sage enhancements designed in from the start.
- Create Sage-enhanced open textbooks for every standard course in the undergraduate mathematics curriculum.
- Integrate Sage with an open source homework delivery system such as WebWork.

[xvi]It feels like this AIM website is an afterthought. It feels like it should be fleshed out more clearly; what exactly will the criteria for endorsement be, how will it be done, what is the time frame, etc.
[xvii]Delete this, or clean it up.

- Integrate collaboration features of the Sage notebook with a course management system such as Moodle.
- More????

## 9. Qualifications and Preparation

The UTMOST team consists of five senior personnel plus an experienced evaluator of STEM education initiatives. The five senior personnel together have many years of experience teaching undergraduates at a wide range of institutions, four are active Sage developers (including its founder), three are authors of open textbooks, all have significant mathematics research experience, and one specializes in mathematics education research. As Sage developers, they are in a position to quickly and accurately shape changes in Sage based on the experience of working with faculty at other institutions that are new to Sage. Each is familiar with, and has extensive experience using, open source software and tools in their teaching and other professional activities. Besides the applicability of this experience to the project's central theme, they are also extremely adept at working collaboratively while still being separated geographically. Working together, they have the wide range of complementary experience and skills that will make UTMOST a success.

### 9.1. Personnel.

**Dr. William A. Stein**, Associate Professor of Mathematics at the University of Washington, is a leader in both computational and theoretical number theory, and the author of two published number theory books. He started the Sage project in 2005, and has passionately led its development ever since, including co-organizing over 25 workshops during the last 4 years and directing dozens of Sage development projects by students at Harvard, UC San Diego, and U of Washington. Dr. Stein will implement improvements to Sage, direct work by students, contribute curricular materials he has authored, supervise test sites, and co-organize workshops.

**Dr. Kiran S. Kedlaya**, Associate Professor of Mathematics at Massachusetts Institute of Technology and University of California, San Diego, is a leading researcher in arithmetic algebraic geometry. He has received an NSF CAREER grant, a PECASE award, and a Sloan Fellowship, and was selected to speak at the 2010 International Congress of Mathematicians. Dr. Kedlaya brings extensive experience in undergraduate and graduate education at all levels, from calculus to mentoring Ph.D. students. He is also deeply involved with mathematics competitions and other programs for exceptional students, such as the USA Math Olympiad, the Putnam competition, and the Art of Problem Solving Foundation. In addition, Dr. Kedlaya brings extensive knowledge of the Sage system, having been an active user and developer since 2005. Dr. Kedlaya will review improvements for Sage related to the project, supervise test sites, and provide additional overall guidance.

**Dr. Jason Grout**, Assistant Professor of Mathematics at Drake University, is an undergraduate teacher, an active researcher in combinatorial matrix theory and graph theory, and a Sage developer. Dr. Grout has contributed extensively to the Sage code base over the last three years in the linear algebra, graph theory, plotting, and notebook interface components of Sage. Dr. Grout has used Sage in research and in teaching a number of undergraduate courses. He has also given numerous presentations and tutorials on Sage, and in Summer 2010 will be co-directing an MAA PREP workshop with Dr. Beezer and Dr. Karl-Dieter Crisman to help participants develop new curricular materials in Sage. Dr. Grout will implement improvements to Sage, direct work by students, supervise test sites, help organize workshops, and contribute curricular materials.

**Dr. Robert Beezer**, Professor of Mathematics at the University of Puget Sound, is a undergraduate teacher with 31 years of experience, an active researcher in algebraic graph theory, one of the first open textbook authors, and a Sage developer. He began writing his open source linear algebra textbook in 2004 and has assisted Dr. Judson with the recent release of his very successful open source abstract algebra text. He began using Sage in 2007 and began contributing code in early 2009. He will lead the technical process of converting textbooks from LaTeX to Sage worksheets, producing a simple system for other authors to use. The pedagogical implications of this new capability will be explored as he incorporates Sage into existing textbooks on topics he teaches frequently, such as linear algebra, abstract algebra, combinatorics, calculus and cryptography. He will continue to contribute code to Sage where the new functionality enables a more complete experience for undergraduate students, and will suggest, review and test the project's improvements to interactive demonstrations and the notebook interface.

**Dr. Thomas Judson**, Associate Professor of Mathematics at Stephen F. Austin Sate University, is an active researcher in both mathematics and mathematics education, with 32 years of teaching experience. He is the author of an open source undergraduate abstract algebra textbook, and will work with Dr. Beezer to produce Sage-enhanced materials for abstract algebra. Dr. Judson has worked extensively with undergraduate mathematics teachers and has mentored graduate students in the teaching of mathematics. In addition to his teaching and research accomplishments, Dr. Judson brings added experience in working with diverse groups both in the US and abroad. He will work with Dr. Hassi to guide research and evaluation efforts for UTMOST. With experience in working with undergraduate faculty and graduate student mentoring, Dr. Judson will also provide insight and assistance in the implementation of Sage-enhanced materials in the undergraduate classroom. Project efforts connected to research in mathematics education will be under the direction of Dr. Judson.

**Dr. Marja-Liisa Hassi**, Ethnography & Evaluation Research, University of Colorado at Boulder, has a Master's Degree in Mathematics and a Ph.D. in Education. She has expertise in both theory and methodology of mathematics education research, as well as teaching experience in education, mathematics education, and research methods for undergraduate and graduate students. Her recent publications address theory and measurement of affect in undergraduate and adult mathematics learners and comparative student outcomes of inquiry-based and traditional undergraduate mathematics courses. She will be assisted by Dr. Sandra Laursen, co-director of E&ER and an experienced evaluator of STEM education initiatives in higher education.

### 9.2. Grant experience and support.

- As a sponsoring organization, the American Institute of Mathematics has a ten-year history of successfully administering, supporting and executing 38 NSF grants in mathematics, including a recent CCLI Phase 1 award.
- Stein has successfully administered many grants supporting Sage development from varied organizations such as NSF, UC San Diego, U of Washington, Google, Sun, Microsoft, and the US Department of Defense. National Science Foundation grants include awards from the SCREMS program for the `sagenb.org` computing cluster (DMS-0821725), the FRG program (sponsored by the American Institute for Mathematics, DMS-0757627) and the COMPMATH program to fund two postdoctoral positions (DMS-0713225).
- Judson works with middle and high school mathematics teachers from high-needs school districts to help them become teacher-leaders in their schools and districts, and preparing them to deliver pedagogical content and mathematical content to their colleagues. He also collaborates with the PIs on these NSF grants to help direct the mathematics education research component. (Texas Leadership Initiative: Mathematics Instruction Transformed, "Texas LIMIT," DUE-0934878; Texas Middle and Secondary Mathematics Project—Supplemental Funds, DUE-0227128)
- Beezer has been awarded a competitive year-long sabbatical leave from the University of Puget Sound for the 2010-11 academic year. This will allow him to begin converting

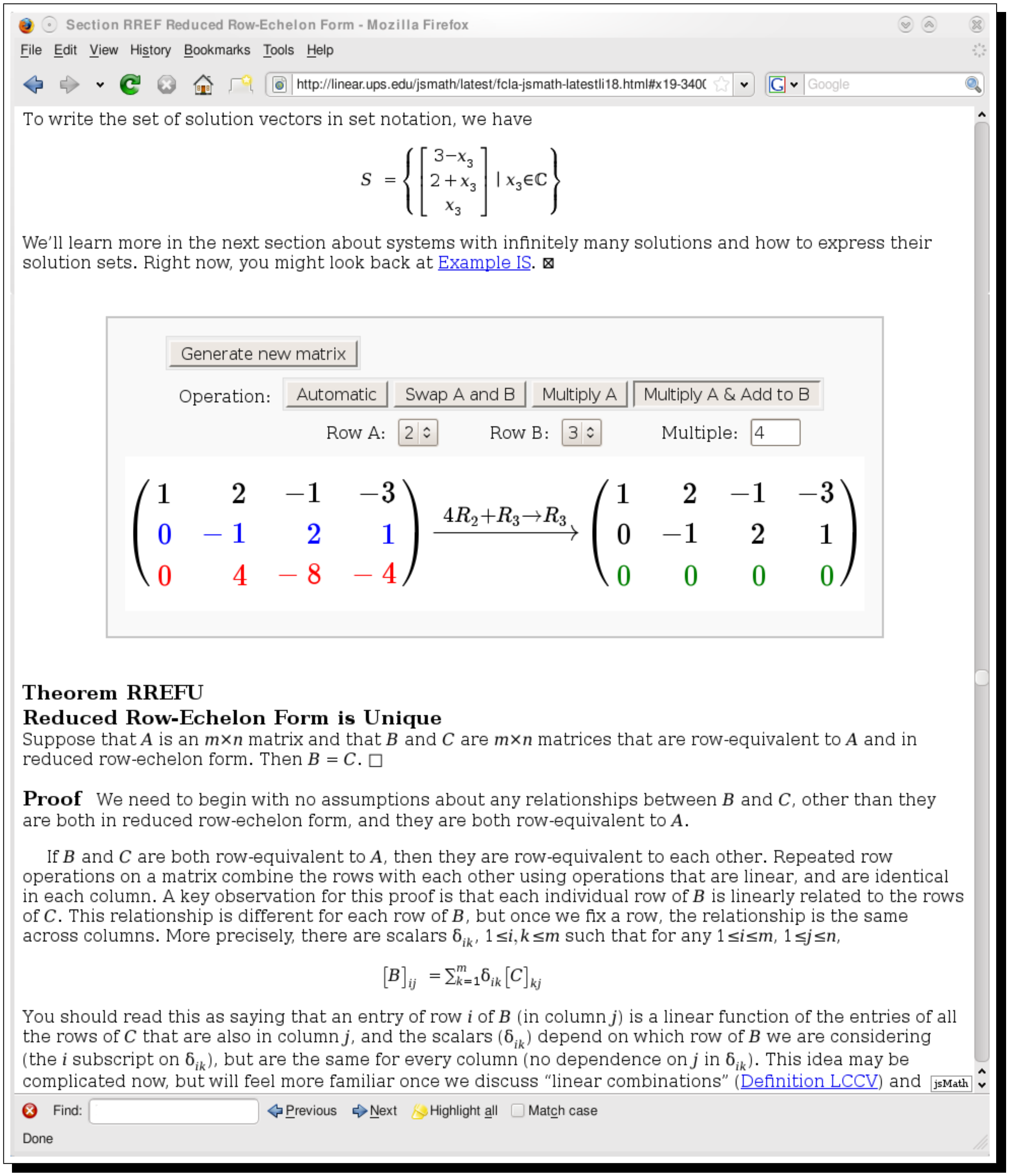
textbooks, increasing the possible textbooks available for test sites to use beginning in the Fall 2011 term.

- Some of Grout's recent work on Sage been partially supported by an NSF postdoctoral appointment in Summer 2008.[xviii] **xviii**
- Beezer and Grout will co-direct (with Karl-Dieter Crisman) an MAA PREP workshop on Sage in Summer 2009, as part of a program funded by the NSF (DUE-0817071).
- Dr. Hassi is an expert in the research and evaluation of mathematics education and is currently working with a large evaluation study focused on inquiry-based learning and teaching of undergraduate mathematics at four large research universities (18; 24). She is also currently working with an evaluation study of NSF DUE-funded workshops on inquiry-based learning for instructors[xix]. **xix**

[xviii] Jason: find award number

[xix] citation? Award number?

Appendix A. Prototype of a Sage-Enhanced Textbook




To write the set of solution vectors in set notation, we have

$$S = \left\{ \left[\begin{array}{c} 3-x_3 \\ 2+x_3 \\ x_3 \end{array}\right] \mid x_3 \in \mathbb{C} \right\}$$

We'll learn more in the next section about systems with infinitely many solutions and how to express their solution sets. Right now, you might look back at Example IS. ⊠

Generate new matrix

Operation: Automatic | Swap A and B | Multiply A | Multiply A & Add to B

Row A: 2 Row B: 3 Multiple: 4

$$\begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & -1 & 2 & 1 \\ 0 & 4 & -8 & -4 \end{pmatrix} \xrightarrow{4R_2+R_3 \to R_3} \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Theorem RREFU**
**Reduced Row-Echelon Form is Unique**
Suppose that $A$ is an $m \times n$ matrix and that $B$ and $C$ are $m \times n$ matrices that are row-equivalent to $A$ and in reduced row-echelon form. Then $B = C$. □

**Proof** We need to begin with no assumptions about any relationships between $B$ and $C$, other than they are both in reduced row-echelon form, and they are both row-equivalent to $A$.

If $B$ and $C$ are both row-equivalent to $A$, then they are row-equivalent to each other. Repeated row operations on a matrix combine the rows with each other using operations that are linear, and are identical in each column. A key observation for this proof is that each individual row of $B$ is linearly related to the rows of $C$. This relationship is different for each row of $B$, but once we fix a row, the relationship is the same across columns. More precisely, there are scalars $\delta_{ik}$, $1 \le i, k \le m$ such that for any $1 \le i \le m$, $1 \le j \le n$,

$$[B]_{ij} = \sum_{k=1}^{m} \delta_{ik} [C]_{kj}$$

You should read this as saying that an entry of row $i$ of $B$ (in column $j$) is a linear function of the entries of all the rows of $C$ that are also in column $j$, and the scalars ($\delta_{ik}$) depend on which row of $B$ we are considering (the $i$ subscript on $\delta_{ik}$), but are the same for every column (no dependence on $j$ in $\delta_{ik}$). This idea may be complicated now, but will feel more familiar once we discuss "linear combinations" (Definition LCCV) and



Prototype of a Sage-enhanced mathematics open textbook

xx

**xx**

[xx] TODO for Jason: Make sure the color is dark enough to look good on black and white. Highlight the rows by surrounding the row with a box. Color the $R_2$ and $R_3$ above the arrow to emphasize which row is which.

## Appendix B. Open Source Mathematics Textbooks

Beyond our initial selection of candidate textbooks, there is a wide range of books available that could be converted to cover portions of the standard undergraduate curriculum. Except where noted, the textbooks below are licensed with a GNU Free Documentation Licenses (GFDL) or a Creative Commons License that allows anyone to distribute modified versions, typically only requiring attribution of the original author's work and the use of the same license. New open source textbooks continue to be released and evidence suggests many authors are considering the option seriously.

- *A First Course in Linear Algebra*, Robert Beezer `http://linear.pugetsound.edu/`
  Second-year university level text, concentrating on proof techniques. Used at thirteen schools since Fall 2007. Contains some Sage commands.
- *Abstract Algebra: Theory and Applications*, Tom Judson `http://abstract.ups.edu/`
  Standard upper-division treatment of groups, rings, domains, fields, Galois Theory. Used by seven schools in its first year.
- *Combinatorics Through Guided Discovery*, Kenneth Bogart
  `http://www.math.dartmouth.edu/news-resources/electronic/kpbogart/`
  Problem book in combinatorics with roughly 400 problems designed to teach the subject. (NSF DUE-0087466)
- *Elementary Number Theory: Primes, Congruences, and Secrets*, William Stein
  `http://wstein.org/ent/`
  Classical elementary number theory and elliptic curves, with applications and extensive Sage examples. Published by Springer-Verlag, freely available (not open source) May 2010. (NSF DMS-0653968)
- *Vector Calculus*, Michael Corral `http://www.mecmath.net/`
  Standard treatment of multivariate calculus. Topics include vector calculus, partial derivatives, multiple integrals, theorems of Green and Stokes.
- *Trigonometry*, Michael Corral `http://mecmath.net/trig/`
  An in-depth, comprehensive and unified treatment of the standard topics.
- *Cryptography*, David Kohel
  `http://echidna.maths.usyd.edu.au/~kohel/tch/Crypto/crypto.pdf`
  Classical ciphers and their cryptanalysis, modern stream ciphers and public-key cryptography. Significant appendices on Sage.
- *What is Riemann's Hypothesis?*, Barry Mazur, William Stein `http://wstein.org/rh/`
  Richly illustrated elementary treatment of a central problem of modern mathematics, designed for an undergraduate audience. Backed up with extensive Sage worksheets already. (NSF DMS-0653968)
- *Numerical Analysis*, Steven Pav `http://scicomp.ucsd.edu/~spav/pub/numas.pdf`
  Comprehensive book-quality lecture notes. Extensive use of Octave, an open source clone of Matlab.
- *Calculus: Modeling and Application*, David A. Smith, Lawrence C. Moore
  `http://www.math.duke.edu/education/calculustext/`
  Calculus textbook with significant electronic components. Mathematical Association of America project with "all rights reserved" (i.e. does not currently have an open license). (NSF DUE-0231083)
- *Differential Calculus and Sage*, David Joyner, William Granville
  `http://sage.math.washington.edu/home/wdj/teaching/calc1-sage/`

A classic text, now in the public domain, significantly expanded by a Sage developer with Sage code and graphics.

## References


[1] American Institute of Mathematics, http://www.aimath.org/

[2] Thomas Banchoff, *Interactive Geometry and Multivariable Calculus on the Internet*, in Ki-hyong Ko, Deane Arganbright (Eds.), Enhancing university mathematics: Proceedings of the first KAIST international symposium on teaching, Conference Board of the Mathematical Sciences, Issues in Mathematics Education, **14** (2007) 17–31

[3] Robert A. Beezer, A First Course in Linear Algebra, http://linear.pugetsound.edu

[4] Robert A. Beezer, *Sage (Version 3.4)*, SIAM Review, **51** (2009) no. 4, 785–807

[5] Paul Blanchard, Robert L. Devaney, Glen R. Hall, *The Boston University Ordinary Differential Equations Project*, http://math.bu.edu/odes/

[6] Joseph J. Branin, Mary Case, *Reforming Scholarly Publishing in the Sciences: A Librarian Perspective*, Notices of the American Mathematical Society, **45** (1998) no. 4, 475–486, http://www.ams.org/notices/199804/branin.pdf

[7] Brown University, *Interactive Internet-Based Teaching and Learning in Mathematics*, NSF DUE-0428280

[8] K. A. Burke, T. J.Greenbowe, J. Gelder, *The Multi-Initiative Dissemination Project Workshops: Who attends and how effective are they?*, Journal of Chemical Education, **81** 2004 no. 6, 897–902

[9] George Cain, *Online Mathematics Textbooks*, http://people.math.gatech.edu/~cain/textbooks/onlinebooks.html

[10] Davide P. Cervone, jsMath: A Method of Including Mathematics in Web Pages, http://www.math.union.edu/~dpvc/jsMath/

[11] Dan Drake, SageTEX, http://tug.ctan.org/pkg/sagetex

[12] Senator Dick Durbin, News Release, October 6, 2009, http://durbin.senate.gov/showRelease.cfm?releaseId=318797

[13] Flat World Knowledge, http://www.flatworldknowledge.com/

[14] Fast Library for Number Theory, http://www.flintlib.org/

[15] L. Gafney, P. Varma-Nelson, *Peer-led team learning: Evaluation, dissemination and institutionalization of a college-level initiative*, Innovations in Science Education and Technology, **16** (2008) New York: Springer

[16] GAP—Groups, Algorithms, Programming—a System for Computational Discrete Algebra, http://www.gap-system.org/

[17] M.-L. Hassi, *Empowering undergraduate students through mathematical thinking and learning*, Safford-Ramus, K. (ed.), A Declaration of Numeracy: Empowering Adults through Mathematics Education, Proceedings of the 15th International Conference of Adults Learning Mathematics (ALM) (2009) 53–69. Lancaster, PA: DEStech Publications

[18] M.-L. Hassi, S. Laursen, *Studying undergraduate mathematics: Exploring students' beliefs, experiences and gains*, in S.L. Swars, D.W. Stinson, S. Lemons-Smith (Eds.), Proceedings of the 31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), (2009) 113–121. Atlanta, GA: Georgia State University.

[19] M. Kathleen Heid, *The technological revolution and the reform of school mathematics*, American Journal of Education, **106** (1997) no. 1, 5–61

[20] M. Kathleen Heid, Michael T. Edwards, *Computer algebra systems: revolution or retrofit for today's mathematics classrooms?*, Theory into Practice, **40** (2001) no. 2, 128–136

[21] Hobart and William Smith Colleges, Java Components for Mathematics, NSF DUE-9950473, http://math.hws.edu/javamath/

[22] Integer Matrix Library, http://www.cs.uwaterloo.ca/~astorjoh/iml.html

[23] Thomas W. Judson, *Abstract Algebra: Theory and Applications*, http://abstract.pugetsound.edu

[24] S. L., Laursen, M.-L. Hassi, *Inquiring about inquiry: Progress on research and evaluation studies of Inquiry-Based Learning in undergraduate mathematics at four campuses*, Presented at the Joint Mathematics Meetings, January 5–8, 2009, Washington, DC.

[25] S. Laursen, M.-L. Hassi, R. Crane, *First findings from evaluation studies of the IBL Mathematics Projects*, presented to the Legacy of R. L. Moore Conference, Austin, TX, July 16-18, 2009

[26] David Leigh-Lancaster, et al., *The 2007 Common Technology Free Examination for Victorian Certicate of Education (VCE) Mathematical Methods and Mathematical Methods Computer Algebra System (CAS)*, in M. Goos, R. Brown, K. Makar (Eds.), Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia, 2008.

[27] S. E. Lewis, J. E. Lewis, *Effectiveness of a workshop to encourage action: Evaluation from a post-workshop survey*, Journal of Chemical Education, **83**(2006) no. 2, 299–304, http://www.wcer.wisc.edu/publications/LEADcenter/nt-wkshop.pdf

[28] M4RI, Linear Algebra over $GF(2)$, http://m4ri.sagemath.org/performance.html

[29] Maplesoft, *Maple*, http://www.maplesoft.com/Products/Maple/

[30] Maxima Wiki, *The Macsyma Saga*, http://maxima-project.org/wiki/index.php?title=The_Macsyma_Saga

[31] National Institutes of Health, *Public Access Policy*, http://publicaccess.nih.gov/

[32] The Ohio State University, *Calculus & Mathematica*, http://socrates.math.ohio-state.edu/

[33] Open Knowledge Foundation, *Open Text Book*, http://www.opentextbook.org/category/maths/

[34] Jeanette R. Palmiter, *Effects of computer algebra systems on concept and skill acquisition in calculus*. Journal for Research in Mathematics Education, **22** (1991) no. 2, 151–156

[35] John Perry, *Mathematical Computing, MAT 305, University of Mississippi*, http://www.math.usm.edu/perry/mat305fa09/index.html

[36] Python Website, *Python*, http://www.python.org/.

[37] Python Website, *Python Success Stories*, http://www.python.org/about/success/

[38] Python Wiki, *Numeric and Scientific Packages*, http://wiki.python.org/moin/NumericAndScientific

[39] The R Project for Statistical Computing, http://www.r-project.org/

[40] REDUCE, A portable general-purpose computer algebra system. http://reduce-algebra.sourceforge.net/

[41] Barry Mazur, William Stein, *What is Riemann's Hypothesis?*, http://wstein.org/rh/

[42] Sage Notebook Server, http://sagenb.org

[43] Sage Server (public), Hungary, http://sage.math.u-szeged.hu/

[44] Sage Server, cell phone interface, Sungkyunkwan University, Korea, http://math1.skku.ac.kr/wap_html

[45] Sage Wiki, *Sage Notebook Servers*, http://wiki.sagemath.org/sagenb

[46] Governor Arnold Schwarzenegger, Press Release, Free Digital Textbook Initiative, http://gov.ca.gov/press-release/12225/

[47] SciPy, Scientific Tools for Python, http://www.scipy.org/

[48] G. R. Sell, *A review of research-based literature pertinent to an evaluation of workshop programs and related professional development activities for undergraduate faculty in the sciences, mathematics and engineering*, report commissioned by SRI International as part of an evaluation for the Undergraduate Faculty Enhancement (UFE) program for the National

Science Foundation, (1998) (Available from the author)

[49] Student Assessment of their Learning Gains (SALG), `www.salgsite.org`

[50] William Stein, *Elementary Number Theory: Primes, Congruences, and Secrets*, Undergraduate Texts in Mathematics, 2009, Springer-Verlag, `http://modular.math.washington.edu/ent/`

[51] William Stein et al., *Sage, Version 4.2*, `hhttp://www.sagemath.org`

[52] The Student PIRGs, *Open Textbook Catalog*, `http://www.studentpirgs.org/open-textbooks/catalog`

[53] TEX4ht: LaTeX and TeX for Hypertext, `http://www.cse.ohio-state.edu/~gurari/TeX4ht/`

[54] H. Thiry, S. L. Laursen, A.-B. Hunter, *Professional development needs and outcomes for education-engaged scientists: A research-based framework and its application*, Journal of Geoscience Education, **56** (2008) no. 3, 235–246.

[55] TinyMCE Javascript Editor, `http://tinymce.moxiecode.com/`

[56] University of Illinois at Urbana-Champaign, *Calculus & Mathematica*, `http://www-cm.math.uiuc.edu`

[57] Washington State Board for Community and Technical Colleges, Washington State Student Completion Initiative, `http://www.sbctc.ctc.edu/college/e_studentcompletioninitiative.aspx`

[58] White House Office of Science and Technology Policy, *Public Consultation on Public Access Policy*,
`http://www.whitehouse.gov/blog/2009/12/09/ostp-launch-public-forum-how-best-make-federally`

[59] Wolfram Research, *Mathematica*, `http://www.wolfram.com/products/mathematica/index.html`