Supplemental Proofs, Algorithms, and Experiments

A.1 PRM Pseudocode

Karaman and Frazzoli's [16] implementation of the PRM algorithm uses two separate subroutines: a pre-processing subroutine that grows a graph to a specified number of samples (Alg. 3), and a query routine that connects the starts and goals to the graph and solves a shortest-path search (using Dijkstra's algorithm, etc.; Alg. 4). For both algorithms, we assume several subroutines have already been provided:

- SampleFree($\cdot$), which samples a collision-free point from the configuration space.
- CollisionFree($\cdot$), which checks if a linear path between the query vertices is collision-free.

Standard implementations of the Near subroutine include a range search that identifies the vertices within a radius of the query vertex and a K-nearest neighbor search.

Algorithm 3 sPrmConstruction

Require: $n > 0$ number of new samples, X c-space, $G = (V, E)$ prev. PRM graph

$V \leftarrow V \cup \{\text{SampleFree}(X)\}_{i=|V|,\dots,n}$
for $v \in V$ **do**
  $U \leftarrow \text{Near}(v, V \backslash \{v\})$
  for $u \in U$ **do**
    if CollisionFree(v, u, X) **then**
      $E \leftarrow E \cup \{(u, v)\}$
return (V, E)

Algorithm 4 sPrmQuery

Require: x_s start, x_g goal, X c-space, $G = (V, E)$ PRM graph

$U_s, U_g \leftarrow \text{Near}(s, V), \text{Near}(g, V)$
for $u \in U_s$ **do**
  if CollisionFree(x_s, u, X) **then**
    $E \leftarrow E \cup \{(x_s, u)\}$
for $u \in U_g$ **do**
  if CollisionFree(x_g, u, X) **then**
    $E \leftarrow E \cup \{(x_g, u)\}$
return ShortestPath(G, x_s, x_g)

A.2 Proof of Lemmas 1 and 2 in Section 3

We begin with our proof of Lemma 1, which proceeds along similar reasoning to Tsao et al. [31] (though simpler since we do not discuss path optimality).

Proof (Lemma 1). Let $x_s, x_g \in X_{free}$ be a start and goal pose, respectively. Suppose there exists a continuous 2α-clear path $\sigma : [0, 1] \to X_{free}$ such that $\sigma(0) = x_s$ and $\sigma(1) = x_g$.

Let L denote the length of σ. Let $\{B^d_{2\alpha}(p_i)\}_{i\in[\lceil L/2\alpha\rceil]}$ denote a set of balls on $\sigma([0, 1])$, where $B^d_{2\alpha,1}(p_1)$ is centered on $x_s = p_1$ and $B^d_{2\alpha}(p_{\lceil L/2\alpha\rceil})$ is centered on $x_g = p_{\lceil L/2\alpha\rceil}$. The remaining balls (for $i = 2, \dots, \lceil L/2\alpha\rceil - 1$) are centered along $\sigma([0, 1])$, where the p_i is spaced $2\alpha(i - 1)$ away from x_s along the arclength of σ.

Next, we construct another set of balls, $\{B^d_{\alpha}(q_j)\}_{j\in[\lceil L/2\alpha\rceil-1]}$, where q_j is placed $\alpha + 2\alpha(j - 1)$ along the arclength of σ from $x_s = p_1$. We observe that q_j is placed exactly between p_j and p_{j+1} along σ's arclength. Since these points are in a Euclidean space, straight lines represent shortest paths, so

$$\|p_j - q_j\|, \|p_{j+1} - q_j\| \leq 2\alpha,$$

for all $j \in [\lceil L/2\alpha \rceil - 1]$. Thus, by construction, we know that

$$B^d_\alpha(q_j) \subset B^d_{2\alpha}(p_j), B^d_{2\alpha}(p_{j+1}).$$

Since N forms an α-net, we know that there must exist a point $x_j \in N \in B^d_\alpha(q_j)$ for all j (x_j must be at most α from ball center q_j). Furthermore, we observe that the linear path between x_j and x_{j+1} is collision-free for all j, since $x_j, x_{j+1} \in B^d_{2\alpha}(p_{j+1})$ and balls are convex.

The maximum distance between x_j, x_{j+1} is 4α (opposite sides of the 2α-ball, and so setting the connection radius to 4α ensures all $x_s, x_1, \ldots, x_{\lceil L/2\alpha \rceil - 1}, x_g$ will be connected to form a path to be returned by the radius PRM (the endpoints trivially included as the centers of the first and last 2α balls). ■

We now describe the our proof for Lemma 2. Our arguments are centered around covering point configurations and the probability they arise from random sampling processes, a topic well-studied at the intersection of statistics, combinatorics, and discrete geometry[2] [2, 18, 24, 26]. Our intuition behind the definitions is grounded in trying to 'hit' a class of subsets of some space X_{free} with a random finite set of points.

Definition 4. *Y be a (potentially infinite) set and let $\mathcal{F} \subset 2^Y$. We call the tuple $(Y, \mathcal{F})$ a range space.*[3]

We will take the base space to be $X_{free} \subset \mathbb{R}^n$, the obstacle-free configuration space of a robot. $\mathcal{F}$ will be the set of n-dimensional balls that are a proper subset of X_{free}. Often, hitting *all* the sets in $\mathcal{F}$ with a finite number of points in Y is too tall an order to ask. Only hitting the 'voluminous' sets in $\mathcal{F}$ is enough for our purposes.

Definition 5. *Let $(Y, \mathcal{F})$ be a range space. Suppose that μ is a measure on Y such that every $S \in \mathcal{F}$ is measurable with respect to μ.*

A finite subset $N \subset Y$ is an ϵ-transversal if for all $S \in \mathcal{F}$ such that $\mu(S) \geq \epsilon \cdot \mu(Y)$, then $N \cap S \neq \emptyset$.

Intuitively, ϵ is a *threshold* on the minimum volume of $S \in \mathcal{F}$ to be hit by a point in N. There could be many different ϵ-transversals on $(Y, \mathcal{F})$, and we prefer to find smaller ϵ-transversals if we can. Blumer et al. [2], generalizing Welzl and Haussler's work [11] from discrete to continuous range spaces showed that random sampling can find ϵ-transversals with high probability. We can derive the following statement by combining Theorem A2.1, Proposition A2.1, and Lemma A2.4 by Blumer et al. [2]:

Theorem 3. *Let $(Y, \mathcal{F})$ be a range space with VC-dimension $\mathrm{VC}(\mathcal{F})$ and $Y \subset \mathbb{R}^n$. Let P be a probability measure on Y such that all $S \in \mathcal{F}$ is measurable with respect to P.*

Let $N \subset Y$ be a set of m independent random samples of Y with respect to P, where

$$m \geq \max\left\{\frac{4}{\epsilon}\log_2\frac{2}{\gamma}, \frac{8d}{\epsilon}\log_2\frac{13}{\epsilon}\right\},$$

then N is an ϵ-transversal of Y with probability at least $1-\gamma$.

[2] We will be using nomenclature from both computational/combinatorial geometry and statistics to avoid conflicting terminology.

[3] For those who are familiar with learning theory: another name for $\mathcal{F}$ can be the *hypothesis space* of a set of binary classifiers or set of indicators associated with all elements of $\mathcal{F}$.

Intuitively, the VC-dimension quantifies the *complexity*, or intricacy, of how the sets in $\mathcal{F}$ intersect with Y and each other. We now move to the technical definition of the VC-dimension:

Definition 6. *Let $(Y, \mathcal{F})$ be a range space. We say that a subset $A \subset Y$ is shattered by $\mathcal{F}$ if each of the subsets of A can be obtained as the intersection of some $S \in \mathcal{F}$ with A. We define the VC-dimension of $\mathcal{F}$ as the supremum of the sizes of all finite shattered subsets of Y. If arbitrarily large subsets can be shattered, the VC-dimension is ∞.*

Our argument only requires that the VC-dimension of a set of d-dimensional (Euclidean) balls in $\mathbb{R}^d$ is $d+1$ [24, 26]. When uniformly sampling from a space X_{free}, we are in a favorable scenario where the definitions of ϵ-transversal (hitting set of balls in X_{free}) ϵ-net (covering X_{free} with balls) defined in Def. 2 coincide (but for different ϵ's).

Lemma 4. *Let $Y \subseteq \mathbb{R}^d$, and P be a uniform probability measure. Let $N \subset Y$ be a finite subset.*

N is an ϵ-net of Y if and only if N is an $\epsilon^d\,\mathrm{P}(B_1^d)$-transversal on the range space $(Y, \mathcal{F})$ where $\mathcal{F}$ is the set of d-dimensional balls in $\mathbb{R}^d$ that have non-empty intersection with Y.

Proof. The essence of this proof is a radius-volume conversion, which is only possible since we are working with range spaces with spheres.

We start with the forward direction. Suppose N is an ϵ-net. Let $S \in \mathcal{F}$ such that $\mathrm{P}(S) \geq \epsilon^d\,\mathrm{P}(B_1^d) \cdot \mathrm{P}(Y) = \epsilon^d\,\mathrm{P}(B_1^d)$ (i.e., S's volume is larger than that of a d-dimensional ball with radius ϵ). Since S is a ball, its radius must be larger than ϵ. Let x_S be its center. By the definition of geometric ϵ-net, there exists $x_n \in N$ such that $||x_c - x_n|| \leq \epsilon$, so $x_n \in S$.

The backward direction is even shorter. Let N be a $\epsilon^d\,\mathrm{P}(B_1^d)$-transversal. Let $x \in Y$. Let S be the d-dimensional ball centered at x. Then there must exist $x_n \in N$ such that $x_n \in S$, so $||x - x_n|| \leq \epsilon$. ■

We now have all the ingredients we require for Lemma 2.

Proof (Lemma 2). By Lemma 4, $\epsilon^d\,\mathrm{P}(B^d(1))$-transversals is an ϵ-net. We apply Lemma 3, and the result follows. ■

A.3 Proofs of Lemma 3 and Prop. 1

The proof of Lemma 3 is a simple proof by contradiction. The intuition for the proof is pictorially displayed in Fig. 3.

Proof (Lemma 3). We start with the forward direction. Let $r \in \mathbb{R}$ such that $0 < r < \min_{v \in V} \|v - \mathrm{NN}_{k+1}(v)\|$. Suppose, for the sake of contradiction, that there exists $u, v \in V$ such that $\textsc{CollisionFree}(u, v)$ returns $\textsc{True}$ but $(u, v) \notin E$. Then v must not be a Kth nearest neighbor of u, or vice versa. Without loss of generality, assume the former, and so v must be at least a $(K+1)$th nearest neighbor of u. But then $r < \min_{v \in V} \|v - \mathrm{NN}_{k+1}(v)\| < \|u - v\|$, so we have a contradiction.

The proof of reverse direction nearly follows the definition of an effective connection radius. Suppose $r \geq \min_{v \in V} ||v - \mathrm{NN}_{k+1}(v)||$. Let v^* be the minimizing vertex in the expression above. Then $(v^*, \mathrm{NN}_{k+1}(v))$ is a non-edge and $\|v^* - \mathrm{NN}_{k+1}(v^*)\| \leq r$. ■

Our proof of Prop. 1 is loosely inspired by the proof for Lemma 58 presented by Karaman and Frazzoli [16].

Proof (Proposition 1). We prove our bound by controlling the number of samples that land within a ball of radius r of another sample using a Chernoff bound.

Let $r = \min_{v \in V} \|v - \mathrm{NN}_{k+1}(v)\|$. Let $I_{v,w}$ be a Bernoulli random variable where vertex w falls within radius r for vertex v. We aim to find a lower bound to the probability the event:

$$\mathrm{P}\left(\forall v \in V, \sum_{w \in V \setminus \{v\}} I_{v,w}(r) < K + 1\right) \tag{12}$$

It is far easier to reason about an upper bound of the complement of this event and take a union bound over all vertices:

$$\begin{aligned}
&\mathrm{P}\left(\exists v \in V, \sum_{w \in V \setminus \{v\}} I_{v,w}(r) \geq K + 1\right) \\
\leq &\sum_{v \in V} \mathrm{P}\left(\sum_{w \in V \setminus \{v\}} I_{v,w}(r) \geq K + 1\right) \\
\leq &N \cdot \mathrm{P}\left(v \in V \text{ s.t. } B_r^d(v) \subset X, \sum_{w \in V \setminus \{v\}} I_{v,w}(r) \geq K + 1\right)
\end{aligned}$$

The second line takes the union bound over all vertices. The third line applies the bound $\mathrm{P}(B_r^d(v) \cap X) \leq \mathrm{P}(B_r^d(v))$. To obtain a tail estimate on the binomial random variable above, we use a Chernoff bound stated by Boucheron et al. [3]:

$$\begin{aligned}
&\mathrm{P}\left(v \in V \text{ s.t. } B_r^d(v) \subset X, \sum_{w \in V \setminus \{v\}} I_{v,w}(r) \geq K + 1\right) \\
\leq &\exp\left\{-D_{KL}(p + t || p)(N - 1)\right\}
\end{aligned}$$

where D_{KL} is the KL-divergence between two Bernoulli variables, and $p = \mathrm{P}(B_r^d)$ and $t = K/(n-1) - p$ (our bound on r will avoid a degenerate $t \leq 0$). Using the well-known inequality $D_{KL}(p_1 || p_2) \geq (p_1 - p_2)^2 / 2p_1$ when $p_1 > p_2$:

$$\begin{aligned}
\exp\left\{-D_{KL}(p + t || p)(N - 1)\right\} &\leq \exp\left\{\frac{-(N-1)t^2}{2(p+t)}\right\} \\
&\leq \exp\left\{-(N-1)\frac{\left(\frac{K}{N-1} - p\right)^2}{2\left(\frac{K}{N-1}\right)}\right\} \\
&= \exp\left\{-\frac{(K - p(N-1))^2}{2K}\right\}
\end{aligned}$$

Substituting the bound above back into our union bound, plugging in $p = \mathrm{P}(B_r^d)$, imposing that the last expression be less than $\gamma > 0$, and solving for r yields:.

$$\mathrm{P}(B_r^d) = p \leq \frac{K - \sqrt{2K(\log N - \log \gamma)}}{N - 1}$$

$$r \leq \left[\frac{K - \sqrt{2K(\log N - \log \gamma)}}{(N-1)\,\mathrm{P}(B_1^d)} \right]^{1/d} \tag{13}$$

What we have shown is that if we choose a ball radius r as expressed in Equation 13, then each ball of radius r centered at each vertex in the PRM contains less than $K+1$ other vertices with probability at least $1-\gamma$. By the construction of the event in Eq. 12, we know that:

$$\left[\frac{K - \sqrt{2K(\log N - \log \gamma)}}{(N-1)\,\mathrm{P}(B_1^d)} \right]^{1/d} < \min_{v \in V} \|v - \mathrm{NN}_{k+1}(v)\|,$$

with probability $1-\gamma$. ■

The conservative $\log N$ factor inside the radical may be removed with a tighter analysis that does not rely on a union bound over all vertices.

A.4 Numerical Algorithm Correctness Proof and Additional Results

Analysis of Correctness of Algorithm 2 We begin by stating the following theorem that formalizes the correctness of Algorithm 2 and then sketch out the proof:

Theorem 4. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ denote the function represents the Eq. 6.*

$$f(m) = \left(\sum_{i=1}^{d'} \binom{2m}{i} \right) \cdot 2^{-cm}$$

for $c > 0$, and $d' \in \mathbb{Z}^+$, which are computed by the input to Alg. 2.

Let the input failure probability be $\gamma \in (0,1)$. Then Alg. 2 will return m^* such that $f(m^*-1) > \gamma > f(m^*)$.

Proof. By the definition of the binomial coefficient operation, we know that the left-hand term is a d'-degree polynomial that is monotonically increasing in m with all positive coefficients:

$$\sum_{i=1}^{d'} \binom{2m}{i} = \sum_{i=0}^{d'} a_i m^i, \;\; a_i \geq 0 \forall i = 0, \ldots, d'$$

We then compute the derivative of f. Using the product rule, we see that:

$$f'(m) = \left(\sum_{i=1}^{d'} i a_{i-1} m^{i-1} \right) \cdot 2^{-cm} - cm \left(\sum_{i=0}^{d'} a_i m^i \right) \cdot (2^{-cm} \log 2)$$

We observe from the form of the expression (lower order polynomial and exponential on the left term, higher order polynomial and linear factor on the right term) and conclude two possibilities:

1. $f'(m) < 0$ for all $m \geq 1$ (f is monotonically decreasing), or

2. $f'(m) \geq 0$ for some $k > m$, and then $f(m) < 0$ for all $m > k$ (f is monotonically increasing until k, and then starts monotonically decreasing).

If we are in the first case, we write $k = 0$ for the convenience of the rest of the analysis. In either case, the doubling search of the numerical algorithm will double N_u until $N_u > k$ and $f(N_u) \leq \gamma$. Thus, we observe that the binary search will preserve two invariants:

1. $N_l \leq m^* \leq N_u$. The left inequality is true because the search increases N_l if either $f(\lfloor (N_l + N_u)/2 \rfloor) > \gamma$ or $N_l \leq k$ (by the check that verifies f is decreasing via the left inequality on Line 6). The right inequality is true via the right inequality on Line 6.
2. The intervals close by half each time.

Thus, the algorithm will return the correct answer since we are searching over a discrete set of integers while the invariants hold.

Numerical Algorithm for Proposition 1 and Tightness Experiments As mentioned in Section 4, we write an analogous algorithm (Alg. 5) to compute the *largest* effective connection radius given a failure of probability γ (and other technical details of the configuration space). The algorithm performs a binary search up to a prespecified additive approximation error. The correctness of the algorithm follows from a similar argument provided for Alg. 2 above.

Algorithm 5 NumericalRadiusBound

Require: $1 > \gamma > 0$, $\mathrm{P}(B_1^d)$ volume of unit ball by uniform measure on c-space X, K number of neighbors, n number of samples, r_u maximum possible radius in X, and maximum $\epsilon > 0$ additive error of solution.

$r_l \leftarrow \epsilon$
$p \leftarrow K/(n-1)$
while $r_u - r_l > \epsilon$ **do**
    $r_{mid} \leftarrow (r_l + r_u)/2$
    $q_{mid} \leftarrow r_{mid}^d \, \mathrm{P}(B_1^d)$
    $q_{mid-} \leftarrow (r_{mid} - \epsilon/2)^d \, \mathrm{P}(B_1^d)$ ▷ Ensure probability bound is decreasing.
    if $e^{-D_{KL}(p||q_{mid-})} \geq \exp^{-D_{KL}(p||q_{mid})}$ or $(n-1)\exp^{-D_{KL}(p||q_{mid})} > \gamma$ **then**
        $r_u \leftarrow r_{mid}$ ▷ Binary search to ensure radius corresponding to γ is in $[r_l, r_u]$.
    else
        $r_l \leftarrow r_{mid}$
return r_u

To verify the tightness of Prop. 1 and its numerical algorithm 5, we sample uniformly from a unit cube, run an approximate K-nearest neighbors algorithm, and compute the effective radius upper bound for varying numbers of neighbors, samples, and dimensions. The experiment is repeated for a variety of dimensions and nearest-neighbor counts. We see that the bound is overly conservative with small sample sets and then progressively tightens as samples increase (with a maximum number of samples of $n = 10^6$). For $d = 2, 3$ the numerical bound tightens quickly but does not fully converge for $d = 20$ within 10^6 samples. Eventually, as $n \to \infty$, we expect the bound to become loose again because of the conservative logarithmic factor.

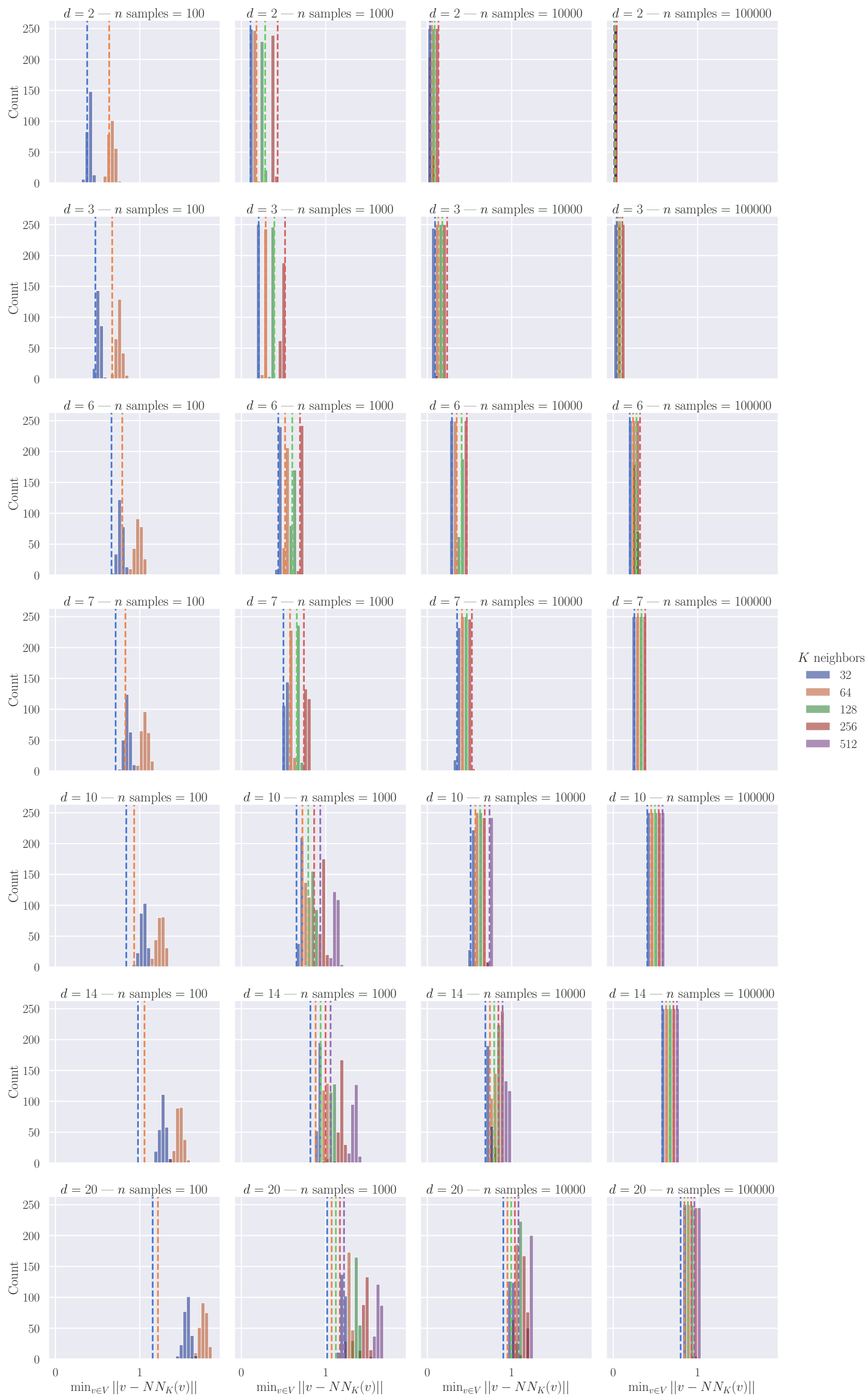

Fig. 5: The largest connection radius computed by the numerical bound by setting the probability of failure $\gamma = 0.01$ is represented by corresponding dotted vertical bars. Their right-to-left ordering is the same as the ordering of the histograms.

A.5 Code Implementation

The code that implements all the algorithms and experiments described in this paper is available in a public repository: `https://github.com/robustrobotics/nonasymptotic-mp`.