Solutions to Terence Tao Analysis I (Third Edition).

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Exercise 2.2.1. Prove Proposition 2.2.5. (Hint: fix two of the variables and induct on the third.) solution: Let's follow the hint. Induct on b.

Base case $b = 0$: $(a + 0) + c = a + c = a + (0 + c) = a + c$. Moreover, by the cancelation law we get that $a + c = a + c$, implies $0 = 0$, which is certainly true.

Exercise 2.2.2. Prove Lemma 2.2.10. (Hint: use induction.) solution: Suppose, there exists some other number $c \in N$, such that $c++ = a$. However then we get $c++ = b++$, which by the Peano axiom 4 implies that $c = b$.

Exercise 2.2.3. Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

solution: a) $a \geq a$. By the definition we have that $a = a + c$. for some natural number c. In particular taking $c = 0$, we get $a = a + 0 = a$.

b) $a = b + m, b = c + n$ for some $m, n \in N$. $a = c + n + m = c + d$, for some $d = m + n \in N$. Therefore, $a \geq c$ by the definition.

c) $a = b+m, b = a+n$ for some $m, n \in N$, then $b = b+m+n, b+0 = b+m+n$, which by the cancelation law gives $0 = m + n$. Moreover, we have that if $m, n \in N$ and $0 = m + n$, then $m = n = 0$, by the corollary 2.2.9. Therefore, $a = b + 0 = b$

d) $a = b + m$, for some $m \in N$, then $a + c = b + m + c = b + c + m$ by the commutativity of the natural numbers. Therefore, $a + c \geq b + c$ by definition.

e) The if part: $b = a + m$, for some $m \in N$ such that $m \neq 0$. Therefore, $m = n++$ for some $n \in N$, by the Lemma 2.2.10. Then by the commutativity of the natural numbers $b = a + m = a + (n++) = (a++) + n$. So, $b \geq a++$

The only if part: $b = (a++) + m$ for some $m \in N$, by the commutativity of the natural numbers we have $b = a + (m++) = a + n$ for $n = m++ \in N$. Moreover, $n \neq 0$ by the Peano axiom 2. Therefore, $b > a$ by definition.

f) The if part: $b = a+m$ for some $m \in N$, then by definition $m \neq 0$, because otherwise $b = a + 0 = a$ which will contradict to $a < b$.

The only if part: $b = a+d$ for some $d \in N$ such that $d \neq 0$, then by definition $a < b$.

Exercise 2.2.4. Justify the three statements marked (why?) in the proof of Proposition 2.2.13.

solution: When $a = 0$, we have $b \geq 0$, since any $b \in N$, by the construction of N is equal to 0 or some increment of 0, that is $b = 0+b$, so by definition $b \geq 0$. If

$a = b+m$ for some $m \neq 0 \in N$, then $a{+}{+} = a{+}1 = b{+}m{+}1 = b{+}m{+}{+} = b{+}n$, therefore, $a + + > b$. If $a = b, a + + > a$, then $a + + > b$.

Exercise 2.2.5.

solution:

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m + +)$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the principle of backwards induction. (Hint: apply induction to the variable n.)

solution: Base case $n = 0$: We assume that $P(0)$ is true, thus for all $m \leq 0$, we have that $P(m) = P(n) = P(0)$ is true by assumption. Inductive step: Suppose, $P(m)$ is true for all natural numbers $m \leq n$. Let's consider $n + +$. Since $P(n + +)$ is true, then we have that $P(n)$ is true by the given property. Therefore, $P(m) = P(n)$ is true. Moreover, by the induction hypothesis we have that $P(m)$ is true for all $m \leq n$, then $P(m)$ is true for all $m \leq n + +$.

Exercise 2.3.1. Prove Lemma 2.3.2.

solution: Show that: $m \times 0 = 0$. Induct on m. Base case: $m = 0$, then $0 \times 0 = 0$. Inductive step: Suppose, $m \times 0 = 0$, then $m + + \times 0 = m \times 0 + 0$ according to the definition. Since $m \times 0 = 0$ by the inductive hypothesis and $0 + 0 = 0$, then $m + + \times 0 = 0$.

Show that: $n \times m + + = n \times m + n$. Base case: $n = 0$, then $0 \times m + + = 0$. Inductive step: Assume $n \times m + + = n \times m + n$, show that $n + + \times m + + = n + + \times m + (n + +)$.

We have by definition that $n + + \times m + + = n \times (m + +) + m + + = (n \times m + n) + m + + = (n \times m + m) + (n + +) = (n + + \times m) + (n + +)$.

Now we can show that for all $m, n \in N$, $m \times n = n \times m$. Induct on n. Base case: $n = 0, m \times 0 = 0 \times n = 0$. Inductive step: Assume $m \times n = n \times m$, then show that $m \times n + + = n + + \times m$. We have that $m \times n + + = m \times n + m = n \times m + m = n + + \times m$.

Exercise 2.3.2. Prove Lemma 2.3.3.

solution: Suppose both are positive then by the Lemma 2.2.10, there exsist some b, c such that $b{+}{+} = m, c{+}{+} = n$, then $n{+}{+}{\times}m{+}{+} = n{\times}m{+}{+}{+}(m{+}{+})$, which is positive by the Lemma 2.2.9. Now for $n \times m = 0$, suppose, $n, m \neq 0$ then their product must be positive a contradiction.

Exercise 2.3.3. Prove Proposition 2.3.5.

solution: Induct on c. Base case: $c = 0$, $0 = 0$. Inductive step: Assume $(a \times b) \times c = a \times (b \times c)$, show that $(a \times b) \times c + + = a \times (b \times c + +)$.

$(a \times b) \times c + + = (a \times b) \times c + (a \times b) = [a \times (b \times c)] + (a \times b) = a[bc + b) = a \times (b \times c + +)$

Exercise 2.3.4. Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

solution: Using the distributive law and the exponentiation and the commutativity of natural numbers we get $(a{+}b)^2 = (a{+}b){*}(a{+}b) = a(a{+}b){+}b(a{+}b) = a * a + a * b + b * a + b * b = a^2 + 2ab + b^2$.

Exercise 3.1.1. Show that the definition of equality in Definition 3.1.4 is reflexive, symmetric, and transitive.

solution: Reflexivity: $A = A$. Suppose, it is not like that. Suppose, there exists some $x \in A$ such that $x \notin A$. Clearly, we get a contradiction. Therefore, $A = A$.

Transitivity: $A = B, B = C, A = C$. Take any $x \in A$. Since $A = B$, then by definition $x \in B$. Moreover, since $B = C$, then $x \in C$. The same argument works for any element of C too. Therefore, $A = C$.

Symmetry: $A = B, B = A$. Suppose, $A = B$, but $B \neq A$. Then there exists without loss of generality we can assume that there exists some $x \in B$ such that $x \not A$. However, according to the definition of the equality of the sets this contradicts to the fact that $A = B$. Therefore, $B = A$.

Exercise 3.1.2. Show that the definition of equality in Definition 3.1.4 is reflexive, symmetric, and transitive.

solution: $\emptyset \neq \{\emptyset\}$: Clearly by the axiom 3.3, $\emptyset \in \{\emptyset\}$, however, not the other way around because then any element of the $\{\emptyset\}$ must also be an element of $\emptyset$ and since $\emptyset$ has no elements by the axiom 3.2, then we get a contradiction. Hence $\emptyset \neq \{\emptyset\}$

$\emptyset \neq \{\{\emptyset\}\}$: If $x \in \{\{\emptyset\}\}$, then by the axiom 3.3, $x = \emptyset\}$ and we just showed that $x \notin \emptyset$. Moreover, $x \notin \{\emptyset\}$ too, since than we would have that there is some element in $\emptyset$, which is a contradiction by the axiom 3.2. Also, $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$, since $\emptyset \notin \{\{\emptyset\}\}$.

Exercise 3.1.3. Prove the remaining claims in Lemma 3.1.13.

solution: $A \cup B = B \cup A$. Let $x \in A \cup B$, then by the axiom 3.4, $x \in A$ or $x \in B$. If $x \in A$, then $x \in B \cup A$, by axiom 3.4. The same way if $x \in B$, then it is in $B \cup A$. The argument goes in reverse direction too. Therefore, union is a commutative operation.

If $x \in A \cup A$, then by the axiom 3.4, $x \in A$. Therefore $x \in A \cup \emptyset$ and $x \in \emptyset \cup A$. If $x \in A$, then.

Exercise 3.1.4. Prove the remaining claims in Lemma 3.1.18.

solution: If $A \subset B$, then any $x \in A$ is also in B. Moreover, if $B \subset A$, then any $y \in B$ is also in A. Therefore, according to the definition the sets are equal.

If $A \subset B$. then for any $x \in A$, $x \in B$, but there exists some $y \in B$ such that $y \notin A$. Since $B \subset C$, then any $y \in B$ is also $y \in C$, but there exists some $z \in C$ such that $y \notin B$. Combining these two we get that any $x \in A$ is also $x \in B$, therefore $x \in C$. However, since there is such $z \in C$, that $z \in B$, then $z \not A$, otherwise, we would have that $z \in B$ (by $A \subset B$), a contradiction. Therefore, $A \subset C$.

Exercise 3.1.5. Let A, B be sets. Show that the three statements $A \subseteq B, A \cup B = B, A \cap B = A$ are logically equivalent (any one of them implies the other two).

solution: Suppose, $A \subseteq B$, then any $x \in A$ is also $x \in B$. Now according to axiom 3.4, if $y \in A \cup B$, then $y \in A$ or $y \in B$. However due to $A \subseteq B$, and reflexivity of the union operator both $y \in A$ and $y \in B$ imply that $y \in B$. Also, from the axiom 3.4, it is clear that if $y \in B$, then $y \in A \cup A$. Hence, $A \cup B = B$.

Now let $x \in A \cap B$, then according to the definition $x \in A$ and $x \in B$, therefore, $A \cap B \subset A$. Moreover, if for any $y \in A$ we have that it is in $A \cap B$, since every $y \in A$ is also in B, according to $A \subseteq B$. Therefore, $A \cap B = A$.

Exercise 3.1.6. Prove Proposition 3.1.28. (Hint: one can use some of these claims to prove others. Some of the claims have also appeared previously in Lemma 3.1.13.)

solution: a) If $x \in A \cup \emptyset$, then $x \in A$ or $x \in \emptyset$. However, since there is no $x \in \emptyset$, then $x \in A$ and $A \cup \emptyset \subset A$. Moreover, by the axiom 3.4, if $x \in A$, then $x \in A \cup \emptyset$. Hence, $A \cup \emptyset = A$.

If $x \in A \cap \emptyset$, then by defintion $x \in A$ and $x \in \emptyset$. However, since there is no $x \in \emptyset$ by the axiom 3.2, then there is no such x exists. So, $A \cap \emptyset = \emptyset$.

b) We already showed this in the previous exercise,

c) If $x \in A \cap A$, then $x \in A$ and $x \in A$, by definition. Therefore, $A \cap A \subset A$. Moreover, if $x \in A$, then from the b, and the fact that $A \subset A$, we have $A \cap A = A$, the same works for $A \cup A = A$,

d) Commutativity of the union was already shown. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Therefore $x \in B \cap A$. The argument works in the reverse direction too. Therefore, the intersection is also a commutative operator.

e) The associativity of the union was already shown. The intersection is straightforward too.

f) Let's show only the first claim. Let $x \in A \cap (B \cup C)$, then by the definition $x \in A$ and $x \in (B \cup C)$. The former implies that $x \in B$ or $x \in C$. Now combining this we get that $x \in A$ and $x \in B$ or $x \in C$. If $x \in A$ and $x \in B$, then $x \in (A \cap B) \cup (A \cap C)$ by definition since $x \in (A \cap B)$. The same works if $x \in A$ and $x \in C$ since then $x \in (A \cap C)$. So, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Now let $x \in (A \cap B) \cup (A \cap C)$, then $x \in (A \cap B)$ or $x \in (A \cap C)$. So, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. Thus, if $x \in A$ and $x \in B$, then $x \in A \cap (B \cup C)$ by definition since $x \in A$ and $x \in (B \cup C)$ (by axiom 3.4). The similar argument works in case of $x \in A$ and $x \in C$. Therefore, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

g) Let $x \in A \cup (X \setminus A)$, then $x \in A$ or $x \in XA$. So, $x \in A$ or $x \in X$ and $x \notin X \cap A$.

Now suppose, $A \cup (X \setminus A) \not\subset X$. If $x \in A$, then $x \in X$, since $A \subset X$. If $x \in (XA)$ then $x \in X$ by definition, so $A \cup (X \setminus A) \subset X$.

Now for any $x \in X$, it is true that $x \in A$ or $x \notin A$. If $x \in A$, then $x \in A \cup (XA)$ by the axiom 3.4 so we would get a contradiction. If $x \notin A$ but $x \in X$, then by definition $x \in (XA)$, so we would get that $X \subset A \cup (X \setminus A)$.

h) $x \in XA \cup B$, then by definition $x \in X$ and $x \notin A \cup B$. ???

Exercise 3.1.8. Let A, B be sets. Prove the absorption laws $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$

solution: If $x \in A \cap (A \cup B)$, then by definition and by the axiom 3.4, $x \in A$ and $x \in A$ or $x \in B$, therefore $A \cap (A \cup B) \subset A$. If $x \in A$, then $x \in A \cap B$. From this and the fact that $x \in A$, we get that $x \in A \cap (A \cup B)$. Therefore, $A \cap (A \cup B) = A$.

Exercise 3.1.9. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Show that $A = XB$ and $B = X \setminus A$.

solution: Use De Morgan's law.

Exercise 3.1.10. Let A and B be sets. Show that the three sets AB, $A \cap B$, and BA are disjoint, and that their union is $A \cup B$.

solution: If any $x \in AB$, then $x \in A$ and $x \notin B$ by definition. Hence, there is no x such that $x \in AB$ and $x \in A \cap B$. Moreover, x is not in BA, by definition. The same argument is applicable to BA and $A \cap B$. Therefore, the given three sets are disjoint. Now for any x in their union it is true the fact that $x \in A$ is enough to state that $x \in AB$, provided such x exists of course (that is A is not an empty set). Therefore, $AB \cup A \cap B \cup BA \subset A \cup B$. Also, for any $x \in AcupB$, it is true that $x \in A$ or $x \in B$, therefore $x \in A \cap B \cup BA$. So, $A \cup B \subset AB \cup A \cap B \cup BA$.

Exercise 3.1.11. Show that the axiom of replacement implies the axiom of specification.

solution: Axiom of replacement states that for any object x of the set A assuming that we have a statement $P(x, y)$ pertaining to x, there is at most one y for which for which $P(x, y)$ is true. Then, $\{y : P(x, y)\text{is true}\}$ forms a set.

Now we can take the statement $P(x, y)$ to be $y = x$ and $P(x)$.

Leting for the axiom of specification the staement perteining to x be $P(x)$, then we can show that the sets constructed by these two axioms are equal. In fact, if $x \in \{y : P(x, y)\text{is true}\}$, then $y = x$ and $P(x)$ is true. Which implies that $x \in \{x : P(x)\text{is true}\}$ and if $x \in \{x : P(x)\text{is true}\}$, then $P(x, x)$ is true, so $x \in y : P(x, y)\text{is true}\}$.

Exercise 3.3.1. Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if $f, \hat{f} : X \to Y$ and $g, \hat{g} : Y \to Z$ are functions such that $f = \hat{f}$ and $g = \hat{g}$, then $gf = \hat{g}\hat{f}$.

solution: Reflexivity: $f(x) = f(x)$, since by the definition of the function to any x in the domain of f there is a corresponding unique $y = f(x)$ in the range of f.

Symmetry: $f = g, g = f$. Since $f = g$, then by definition $f(x) = g(x)$ for all x in the domain of f, g. Suppose, $g \neq f$, that is there is some x in the domain of g, f such that $g(x) \neq f(x)$. However, this contradicts to the statment that $f = g$.

Transitivity: Let $f, g, h : X \to Y$. $f = g, g = h, f = h$. Since $f = g$, the for any $x \in X$, $f(x) = g(x)$, also from $g = h$, we have that for any $y \in X$, $g(y) = h(y)$. Since the equalities hold for any $x, y \in X$, than in particular they hold for $x = y$. Therefore, for any $x \in X$, $f(x) = h(x)$ holds.

$f(x) = \hat{f}(x) = y$ for any $x \in X$. Also, we have that for any $y \in Y$ $g(y) = \hat{g}(y)$, so, $gof(x) = \hat{g}o\hat{f}(x)$ for any $x \in X$.

Exercise 3.3.2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

solution: If f, g are both injective then $f(x) = f(x')$ implies that $x = x'$, and $g(y) = g(y')$ implies that $y = y'$ by definition for all $x, x' \in X, y, y' \in Y$. Now, if $(g \circ f)(x) = (g \circ f)(x')$, then $g(f(x)) = g(f(x'))$, from the injectivity of g, we get $f(x) = f(x')$. From the injectivity of f, we get $x = x'$.

Exercise 3.3.3. When is the empty function injective? surjective? bijective?

solution: Let $f : X \to \emptyset$. Then, f is always injective since for any $x, x' \in X$

we have that $f(x) = f(x') = \emptyset$. However, f is not a surjective function since there is no $y \in \emptyset$ such that $f(x) = y$. Therefore, it is never bijective. https://math.stackexchange.com/questions/3584055/when-is-the-empty-function-injective-surjective-bijective

Exercise 3.3.4. In this section we give some cancellation laws for composition. Let $f : X \to Y, \hat{f} : X \to Y, g : Y \to Z$, and $\hat{g} : YZ$ be functions. Show that if $g \circ f = g \circ \hat{f}$ and g is injective, then $f = \hat{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \hat{g} \circ f$ and f is surjective, then $g = \hat{g}$. Is the same statement true if f is not surjective?

solution: If $gf = g\hat{f}$ and g is injective then $f(x) = \hat{f}(x)$ for all $x \in X$ which be definition implies that $f = \hat{f}$. If g is not injective then it can be that $f(x) = y, \hat{f}(x) = y'$, such that $y \neq y$, but $g(y) = g(y')$. If f is surjective then for any $y \in Y$, there is some $x \in X$ such that $f(x) = y$. Therefore, for any $y \in Y$, $g(y) = g(f(x)) = \hat{g}(y) = \hat{g}(f(x))$. Therefore, $g = \hat{g}$.

Exercise 3.3.5. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must also be surjective?

solution: If gof is injective. If $x \neq x'$, then $g(f(x)) \neq g(f(x'))$, which means that $f(x) \neq f(x')$. Therefore, f is injective by definition.

If gof is surjective, then for any $z \in Z$ there is some $x \in X$ such that $(gof)(x) = g(f(x)) = g(y) = z$. Therefore, for any $z \in Z$, there is some $y \in Y$ such that $g(y) = z$.

Exercise 3.3.6. Let $f : X \to Y$ be a bijective function, and let $f^1 : Y \to X$ be its inverse. Verify the cancellation laws $f^1(f(x)) = x$ for all$x \in X$ and $f(f^1(y)) = y$ for all$y \in Y$. Conclude that $f1$is also invertible, and has$f$ as its inverse (thus $(f^1)^1 = f$).

solution: By the definition the value of x such that $f(x) = y$ is defined as $f^{-1}(y)$. Therefore, we can write $f(x) = f(f^{-1}(y)) = y$ for any $y \in Y$ (because of the bijection). By the same argument $f^{-1}(y) = f^{-1}(f(x)) = x$ for all $x \in X$.

Exercise 3.3.7. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)1 = f1 \circ g^1$.

solution: Since gof is bijective than for any $z \in Z$, there is some $x \in X$ such that $g(f(x)) = z$. Where we denote x as $(gof)^{-1}(z)$. Moreover, we have that x is denoted as $f^{-1}(y)$ and y is denoted as $g^{-1}(z)$. Therefore, $f^{-1}(y) = f^{-1}g^{-1}(z) = x = (gof)^{-1}(z)$ since remember we denoted x as $(gof)^{-1}(z)$.

Exercise 3.3.8.

solution: a) By the given description(definition) of the inclusion map we have that $l_{X\to Y}(x) = x$, and $l_{Y\to Z}(x) = x$ for any $x \in X$, since $x \in Y$ too. Also, $l_{X\to Z}(x) = x$ for any $x \in X$. Therefore $l_{Y\to Z}ol_{X\to Y} = l_{X\to Z}$.

b) $f : A \to B$. $l_{A\to A}(a) = a$, $l_{B\to B}(b) = b$-by description(definition). So, $(fol_{A\to A})(a) = f(a) = b$ for some $b \in B$. Also, $l_{B\to B}f(a) = l_{B\to B}(b) = b$ for not just some, but... bear with me, the same b. Because according to the definition of the function for every $a \in A$, there is just a single corresponding $b \in B$ such that $f(a) = b$.

c) We already know from exercise 3.3.6 that $(fof^{-1})(b) = b$, also $l_{B\to B} = b$, is the identity map on B. Therefore, for any $b \in B$ we have that $(fof^{-1}) = l_{B\to B}$.

d) It is given that $(hol_{X\to X\cup Y})(x) = h(x) = f(x)$ and $(hol_{Y\to X\cup Y})(y) = h(y) = g(y)$.

Let there be another such function $h'(x) = f(x)$ for all $x \in X$ and $h'(y) = g(y)$ for all $y \in Y$. Then, by the transitivity $h'(x) = h(x), h'(y) = h(y)$. Which, is a contradiction.

Exercise 3.4.1. Let $f : X \to Y$ be a bijective function, and let $f^1 : Y \to X$ be its inverse. Let V be any subset of Y . Prove that the forward image of V under f^1 is the same set as the inverse image of V under f ; thus the fact that both sets are denoted by $f^1(V) willnotleadtoanyinconsistency.$

solution: The forward image of V under f^{-1} is the set $\{f^{-1}(y) = x \in X : f(x) = y \in V$, The inverse image of V under f, is $\{x \in X : f(x) \in V\}$.

Exercise 3.4.2. Let $f : XY$ be a function from one set X to another set Y , let $beasubsetof$X$, andlet$U$beasubsetof$Y$.What, ingeneral, canonesayabout$f$^1(f(S))and$S$?Whatabout$f$(f^1(U))a$

https://math.stackexchange.com/questions/3572048/what-in-general-can-one-say-about-f-1fs-and-s-what-about-ff-1

solution: $L = f(S) = \{f(x) : x \in S\}$, $f^{-1}(f(S)) = \{x \in X : f(x) \in L\}$

Therefore, if $x \in L$ that is $x \in S$, then $x \in f^{-1}(f(S))$. So, $S \subset f^{-1}(f(S))$.

$L' = f^{-1}(U) = \{x \in X : f(x) \in U\}$, $f(f^{-1}(U)) = \{f(x) : x \in L'\}$. Than, $x \in f(f^{-1}(U))$, implies that $x \in$

Exercise 3.4.3. Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that $f(AB) \subseteq f(A)capf(B), that$f (A)$\backslash f(B) \subseteq f(A \setminus B)$,f (A $\cup B) = f(A) \cup f(B).Forthefirsttwostatements, isittruethatthe \subseteq$ relation can be improved to =?

solution: $f(A\cap B) = \{f(x) : x \in A\cap B\}$, therefore, if $y \in f(A\cap B)$, then $y = f(x)$ and $x \in A$ and $x \in B$, so, $\{f(x) : x \in A\}\cap\{f(x) : x \in B\}$. The relation can not be improved to =. For example, take $A = \{-1, 0\}, B = \{0, 1\}, A \cap B = \{0\}$. Than, $f(A \cap B) = \{0\}, f(A) = \{0, 1\}, f(B) = \{0, 1\}$.

$f(A) = \{f(x) : x \in A\}$, $f(B) = \{f(x) : x\ inB\}$

If $y \in f(A)f(B)$, then $y = f(x)$ such that $x \in Ax \notin B$. Therefore, $f(A)f(B) \subset f(AB)$. Let $f(x) = x^2, A = \{2, 1\}, B = \{-1\}$, so $f(A) = \{4, 1\}, f(B) = \{1\}, AB = \{4, 1\}$.

$f(A\cup B) = \{f(x) : x \in A\cup B\}$. If $y \in f(A\cup B)$, then $y = f(x)$ such that $x \in A$ or $x \in B$. Therefore, $f(A \cup B) \subset f(A) \cup f(B)$. Moreover, if $y \in f(A) \cup f(B)$, then $y = f(x)$ such that $x \in A$ or $x \in B$. So, $f(A) \cup f(B) \subset f(A \cup B)$.

Exercise 3.4.4. Let $f : X \to Y$ be a function from one set X to another set Y , and let U, V be subsets of Y . Show that $f^1(U \cup V) = f^1(U) \cup f^1(V)$, that $f^1(U \cap V) = f^1(U) \cap f^1(V), andthat$f 1(U$\backslash$V) $= f^1(U)^1(V)$

solution: $f^{-1}(U) = \{x : f(x) \in U\}$ $f^{-1}(V) = \{x : f(x) \in V\}$ $f^{-1}(U \cup V) = \{x : f(x) \in U \cup V\}$

Exercise 3.4.5. Let $f : X \to Y$ be a function from one set X to another set Y . Show that $f(f^1(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective. Show that $f^1(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

solution: $f(f^{-1}(S)) = S$, where $L = f^{-1}(S) = \{x \in X : f(x) \in S\}$ $f(L) = \{f(x) : x \in L\}$ Suppose, $f(L) = \{f(x) \in S\}$ too. Than for any $y \in S$, there is some $x \in L$, i.e. $x \in X$ such that $f(x) = y$ (bacause the sets are equal). Therefore, f is surjective on S. Since the given relation holds for every $S \subset Y$, then f is surjective.

If f is surjective, then $s \in S$, there exists some $x \in X$ such that $f(x) = s$. Therefore, $f^{-1}(S)$ is the set of such $x \in X$, and $f(x)$ is equal to S, by definition.

$L' = f(S) = \{f(x) : x \in S\}$, $f^{-1}(L') = \{x \in X : f(x) \in L'\} = \{x \in S\}$. Let $f^{-1}f(S) = S$ for any $S \subset X$. If $x \in f^{-1}(f(S))$ then there is some $y \in f(S) = L'$ such that $y = f(x)$ and $x \in S$. Suppose, $x \neq x' \in S$, $f(x) \neq f(x')$, otherwise, if $f(x) = f(x')$, then $f^{-1}f(\{x\}) = \{x\} = \{x'\} = f^{-1}f(\{x'\})$, a contradiction. Therefore, the f is injective.

Exercise 3.4.6. Prove Lemma 3.4.9. (Hint: start with the set $\{0,1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^1(\{1\})$.)

solution: The set $\{0,1\}^X$ is the set of all functions from $X \to \{0,1\}$. Let $P(f, f^{-1}(\{1\}))$ be the property pertainig to f. Let it be true if $f(S) = \{1\}$ for some $S \subset X$. Since by the Power set axiom $\{0,1\}^X$ is the set of all the functions from $X \to \{0,1\}$, then for every subset S of X there is a function f such that $f(S) = \{1\}$, otherwise, the set will not be complete, i.e. there wouldn't exists function from some subset S of X to $\{1\}$. Therefore, by the usage of the replacement axiom we get the desired result.

Exercise 3.4.7.

solution: Let $f : X' \to Y'$, then $f : X' \to Y$. The collection of all functions $Y^{X'}$ is a set by the power set axiom. For any subset of X we can create a set of functions from it to Y, by taking $P(x, y = Y^x)$ to hold if $x \in 2^X$, and if $x \notin 2^X$, then $y = \emptyset$. Now, by the replacement theorem it is a set, applying the union axiom we would get $\cup\{Y^{X'} : X' \in 2^X\}$ a set.

Exercise 3.5.1.

solution: The property 3.5 states that $(x, y) = (x', y')$ iff $x = x', y = y'$. Now, let's check the property having $(x, y) = \{\{x\}, \{x, y\}\}$. Suppose, $(x, y) = (x', y')$, then we have that $A = \{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} = B$. Now suppose, $\{x\} \neq \{x'\}$, i.e. ($x' \neq x$ by the axiom 3.3). Then to have the equality of the given sets by the axiom 3.3 we should have that $\{x\} = \{x', y'\}(1)$ and $\{x, y\} = \{x'\}(2)$. The (1) is possible if $y' = x = x'$ and the (2) is possible if $y = x' = x$. In either way we have a contradiction, therefore $\{x\} = \{x'\}$. Moreover, from the axiom 3.3, we have that if $\{x\} = \{x'\}$, than surely $\{x, y\} = \{x', y'\}$. The converse is trivial (by the definition of the equality of the sets) since if $x = x', y = y'$, then $x \in (x', y')$ (in the set representation) and $x' \in (x, y)$. The same can be stated about $\{x, y\}$ and

$$\{x', y'\}$$

.

Exercise 3.5.1.

solution: Lets show that $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ than $x_i = y_i$ for all $i \in [1, n]$ by induction. Base case: $n = 1$ we have that $(x_i)_{1 \leq i \leq n} = (x_i)_{i=1} = x(1) = x_1$ and $(y_i)_{1 \leq i \leq n} = (y_i)_{i=1} = y(1) = y_1$ are equal, then $x_1 = y_1$. Inductive step:

Assume the given property holds for $1 \leq i \leq n-1$. Then $X = Y$, $x_n \in Y, y_n \in X$ $x_n = y_1 = x_1, y_n = x_1 = y_4$

Exercise 6.1.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number n. Prove that whenever n and m are natural numbers such that $m > n$, then we have $a_m > a_n$. solution: Let's do an induction on k where $k = m - n$. Base case: $k = 1$. Then $m = n + 1$ and we already have that $a_m = a_{n+1} > a_n$. Inductive step: Suppose the proposition is true for all k's up to $k - 1$. Show that $a_m > a_n$ if $m - n = k$. We know that for $m - n = k - 1, m = n + k - 1, a_m = a_{n+k-1} > a_n$ by inductive hypothesis. We also know that $a_{n+k} > a_{n+k-1}$ from it. Therefore, by the transitivity of the $>$ for real numbers we have $a_m = a_{n+k} > a_n$.

Exercise 6.1.2. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let L be a real number. Show that $(a_n)_{n=m}^{\infty}$ converges to L if and only if, given any real $\epsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \epsilon$ for all $n \geq N$. solution: By the definition 6.1.5, we have that the sequence $(a_n)_{n=m}^{\infty}$ convorges to L iff for every real $\epsilon > 0$ it is eventually epsilon close to L. A sequence $(a_n)_{n=m}^{\infty}$ is eventually epsilon close if $\exists N > m$ s.t. for all $n, |a_n - L| < \epsilon$, where $\epsilon > 0$ is a real number. Therefore, putting all together I get $(a_n)_{n=m}^{\infty}$ converges to L iff for every $\epsilon > 0$, there exists an $N > m$ s.t. for all $n, |a_n - L| < \epsilon$. Exactly what was asked to show.

Exercise 6.1.3. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{n=m'}^{\infty}$ converges to c. solution: * By the definition of convergence if $(a_n)_{n=m}^{\infty}$ converges to c, then for any $\epsilon > 0, \exists N > m$ s.t. $\forall n \geq N, |a_n - c| < \epsilon$. Since this is true $\forall n \geq N$ then it is eventually true for all $\forall n \geq m'$. Therefore, for every real $\epsilon > 0$ we can take $N_0 > max(N, m')$ s.t for all $n > N_0$ we have $|a_n - c| < \epsilon$.

* If $(a_n)_{n=m'}^{\infty}$ converges to c, then $\forall \epsilon > 0, \exists N > m' > m$ s.t. $|a_n - c| < \epsilon$. That is what we intended to show.

Exercise 6.1.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_{n+k})_{n=m}^{\infty}$ converges to c.

solution: * If $(a_n)_{n=m}^{\infty}$ converges to c, then for any $\epsilon > 0, \exists N > m$ s.t. $\forall n \geq N, |a_n - c| < \epsilon$. Since for $k > 0$ $n + k > n \geq N$ by the transitivity of natural numbers, then it is true that $|a_{n+k} - c| < \epsilon, \forall n + k \geq N$.

* Here the sequence $(a_{n+k})_{n=m}^{\infty}$ can be written as $a_{m+k}, a_{m+k+1}, a_{m+k+2}, \ldots$, therefore $(a_{n+k})_{n=m}^{\infty}$ can be rewritten as $(a_n)_{n=m+k}^{\infty}$. Now, if $(a_{n+k})_{n=m}^{\infty}$ converges to c, that is $(a_n)_{n=m+k}^{\infty}$ convorges to c , then $\forall \epsilon > 0, \exists N > m + k$ s.t. $\forall n \geq N |a_n - c| < \epsilon$. Since there exists $N > m+k$, then there exists $N > m$ such that $\forall n \geq N |a_n - c| < \epsilon$, which is the definition of $(a_n)_{n=m}^{\infty}$ being convergent to c.

Exercise 6.1.5. Prove Proposition 6.1.12. (Hint: use the triangle inequality, or Proposition 4.3.7.). (Convergent sequences are Cauchy). Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.

solution: The definition 6.1.3 states $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence iff it is eventually epsilon steady for every epsilon. $(a_n)_{n=m}^{\infty}$ is eventually epsilon steady for every epsilon iff $\exists N > m$ s.t $\forall i, j > N, |a_i - a_j| < \epsilon$. Hence, summing up the $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence iff $\forall \epsilon > 0, \exists N > m$ s.t. $\forall i, j > N, |a_i - a_j| < \epsilon$. We will also use the definition 6.1.5 of convergence.

Since $(a_n)_{n=m}^{\infty}$ is convergent to some c then $\forall \epsilon > 0, \exists N > m$ s.t. $\forall i > N, |a_i - c| < \epsilon$. Hence, for the same N's take $i, j > N$ and have a look at $|a_i - a_j| = |a_i - c + c - a_j| \geq |a_i - c| + |c - a_j|$ (was there a proposition for reals that states $|x - y| \leq |x - c| + |c - y|$? I remember one for rationals but can't recall for reals.) From $|a_i - a_j| = |a_i - c + c - a_j| \geq |a_i - c| + |c - a_j| < 2 * \epsilon/2 = \epsilon$, therefore is a Cauchy indeed.

Exercise 6.1.6. Prove Proposition 6.1.15 (Suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=m}^{\infty}$ converges to $LIM_{n\to}a_n$, i.e. $LIM_{n\to}a_n = lim_{n\to}a_n$.), using the following outline. Let $(a_n)_{n=m}^{\infty}$ be a Cauchy sequence of rationals, and write $L := LIM_n a_n$. We have to show that $(a_n)_{n=m}^{\infty}$ converges to L. Let $\epsilon > 0$. Assume for sake of contradiction that sequence an is not eventually ϵ-close to L. Use this, and the fact that $(a_n)_{n=m}^{\infty}$ is Cauchy, to show that there is an $N \geq m$ such that either $a_n > L + \epsilon/2$ for all $n \geq N$, or $a_n < L - \epsilon/2$ for all $n \geq N$. Then use Exercise 5.4.8.

solution: Following the hint. Suppose the $(a_n)_{n=m}^{\infty}$ is not ϵ close to L, i.e. $\exists \epsilon > 0$ s.t. $\forall N_0 > m, \exists n \geq N_0$ s.t. $|a_n - L| > \epsilon$. Now, use the fact that $(a_n)_{n=m}^{\infty}$ is Cauchy, i.e. real $\forall \epsilon > 0, \exists N_1 > m$ s.t. $\forall n, i > N_1, |a_n - a_i| < \epsilon_0$. We can also get that for $M > max(N_0, N_1)$'s, $\epsilon - |L - a_i| < |a_n - L| - |L - a_i| \leq |a_n - L + L - a_i| < \epsilon_0$. Let's watch at the possible two cases:

Take $\epsilon_0 = \epsilon/2$

if $L > a_i$ $\epsilon - (L - a_i) < \epsilon_0, \epsilon - L + a_i < \epsilon_0, a_i < L + \epsilon_0 - \epsilon = L - \epsilon/2$

if $L < a_i$ $\epsilon - (a_i - L) < \epsilon_0, \epsilon + L - a_i < \epsilon_0, a_i > L - \epsilon_0 + \epsilon = L + \epsilon/2$

Now by the exercise 5.4.8 (not sure about this part) $a_i < L - \epsilon$ for all $i > M$, then $LIMa_i < L - \epsilon$ and by the same logic $LIMa_i > L + \epsilon$, therefore $|a_i - L| < \epsilon$.

Exercise 6.1.7. Show that Definition 6.1.16 (A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is bounded by a real number M iff we have $|a_n| \leq M$ for all $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is bounded iff it is bounded by M for some real number $M > 0$) is consistent with Definition 5.1.12 (i.e., prove an analogue of Proposition 6.1.4 for bounded sequences instead of Cauchy sequences).

solution: Following the Hint: Let $M > 0$ be real then $|a_n| < M$. We know that there exists a rational number between any two reals in our case $M < q < M + 1$. Therefore $|a_n| < q$ for a $q > 0$.

Let $M > 0$ be rational. Then I guess, we can go by the same logic and say that there exists $L < M < L + 1$, so $|a_n| < L$. (Are there any other ways to solve the problem?)

Exercise 6.1.8. Prove Theorem 6.1.19.

solution: I am going to show only for (a) since I am not sure if I am doing this right. If you confirm the logic I may try to apply it to the other ones too.

a) $\forall \epsilon > 0, \exists N_0 > m$ s.t. $\forall n > N_0, |a_n - x| < \epsilon$ and $\forall \epsilon > 0, \exists N_1 > m$ s.t. $\forall n > N_1, |b_n - y| < \epsilon$. Therefore, for $\forall \epsilon > 0, \exists M > max(N_0, N_1)$ s.t. $\forall n > N, |a_n + b_n - (x + y)| = |a_n - x + b_n - y| < 2 * \epsilon/2 = \epsilon$.

Exercise 6.1.9. Explain why Theorem 6.1.19(f) fails when the limit of the denominator is 0.

solution: Because we never defined the division by 0.

Exercise 6.2.1. Prove Proposition 6.2.5. (Hint: you may need Proposition 5.4.7.)

solution: Let x, y, z be extended real numbers.

a) Reflexivity: $x \leq x$. We may have three cases:

a.1) $x = \infty$. Suppose, $x > \infty$, but then take some set E that contains both ∞ and x. We would have that the $sup(E) \neq \infty$, however, as we know from definition 6.2.6 case (b), $sup(E) = \infty$, so we got a contradiction.

a.2) $x = -\infty$. ?

a.3) x is not equal ∞ nor $-\infty$. Set $x = lim_{n\to\infty} a_n$, where $a_n \in Q$. Then we have that $x \leq x$ because, otherwise if $x > x$, we would have that $x - x > 0$, so $a_n - a_n$ must be positively bounded away from zero, which is not true since $a_n - a_n = 0$ for any $a_n \in Q$.

b) Exactly one of the statements $x < y, x = y$, or $x > y$ is true.

b.1) If x, y are not equal to ∞, nor are equal to $-\infty$, then the statement is true by the trichotomy of the non-extended reals.

b.2) If $x = \infty, y \neq \infty$, then by definition 6.2.6, $x > y$.

b.3) If $x = \infty, y = \infty$, then, $x = y$. (I guess, by default, since nowhere is stated that $\infty = \infty$. Moreover, no operations other than equality of rational number x to ∞ and the negation of ∞ have been defined.)

b.4) If $x = \infty, y = -\infty$, then $x > y$ by definition 6.2.6

b.5) If $x = -\infty, y = -\infty$, the same thoguhts as in (b.3)

c) Transitivity:If $x \leq y$ and $y \leq z$, then $x \leq z$

c.1) If none of them is equal to ∞ or $-\infty$ then the statement follows from the transitivity of non-extended reals.

c.2) If $z = \infty$, then $z \geq x$ for any $x \neq \infty$.

c.3) If $x = \infty$ then $y = \infty$ and $z = \infty$. (b.3)

c.4) If $x = -\infty$, $y \neq -\infty$, then by definition 6.2.6, for any real number other z than $-\infty$ $z \geq -\infty$. In particular this is true for the real number that has $z > y$ property.

c.5) $x = -\infty, y = -\infty, z = -\infty$, then by the (b.5) $x = y = z$.

d) Negation reverses order: If $x \leq y$, then $y \leq x$.

d.1) If none of them is equal to $-\infty, \infty$ then it is true by the negation for the non-extended reals.

d.2) If $x = \infty$ and $x \leq y$, then $y = \infty$, so $-x = -\infty, -y = -\infty$, therefore $-y = -x$ or we can write this also as $-y \leq -x$

d.3) If $x = -\infty$, then for any real number y, we have $y \geq x$, and $-x = -\infty$, which is bigger than the negation of any real number.

6.2.2 Prove Theorem 6.2.11

solution: Let E be a subset of extended reals.

a) For every $x \in E$ we have $x \leq sup(E)$ and $x \geq inf(E)$.

a.1) None of the elements of E is equal to ∞ or $-\infty$, therefore by definition 5.5.10 the statement is true.

a.2) There exists elements in E that are equal to ∞. Therefore, by definition 6.2.6 $sup(E) = \infty$. By the previous exercise $sup(E) \geq x$ for every real number. $inf(E) = -sup(-E)$. Take any $-E = -x : x \in E$, therefore $sup(-E)$ by definition 6.2.6 falls under case (c). Therefore $sup(-E) > -x$ for any $-x \in -E$. Using the reversed sign law from the previous exercise we get that $x < -sup(-E)$ for any $x \in E$.

a.3) There exists elements in E that are equal to $-\infty$. Then we can get the $sup(E)$ by the case (a) of definition 6.2.6. The $inf(E)$ by the definition will be equal to $-(sup(-E)) = -(\infty)$. Hence, any real number is bigger than or equal than the $inf(E)$ by the previous exercise.

Exercise 6.3.1. Verify the claim in Example 6.3.4.: Let $a_n := 1/n$; thus $(a_n)_{n=1}^{\infty}$ is the sequence $1, 1/2, 1/3, ...$ Then the set $a_n : n \geq 1$ is the countable set $1, 1/2, 1/3, 1/4, ...$ Thus $sup(a_n)_{n=1}^{\infty} = 1$ and $inf(a_n)_{n=1}^{\infty} = 0$.

solution: The $sup(a_n)^{\infty}n = 1 = 1$, because for any integer $n \in Z$, $1/n \leq 1$ and $1 \in$a$_n : n \geq 1$, therefore, for every $y < 1$, y can not be a supremum. The infimum is equal to 0, because it was shown in the book that the $lim_{n\to\infty}a_n = 0$. Suppose $inf(a_n) = 0 + \epsilon = \epsilon$ for some positive real $\epsilon > 0$. Then, there exists some $N \in Z$ such that $\epsilon > 1/N$ by the Archimedian property. Since $1/N \in a_n$ therefore, ϵ can not be the infimum of the sequence. Also, since the limit of the sequence is approaching to the 0 (was shown in the book), then all the elements of the $a_n \geq 0$. Therefore, the $inf(a_n) = 0$.

Exercise 6.3.2. Prove Proposition 6.3.6: Let $a_n^{\infty}n = m$ be a sequence of real numbers, and let x be the extended real number $x := sup(a_n)^{\infty}n = m$ Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in R$ is an upper bound for a_n (i.e.,$a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.

solution: Suppose there does not exists $n \geq m$ such that $y < a_n \leq x$. Then for all a_n such that $n \geq m$ we would have that $a_n \leq y$. This will mean that $y = sup(a_n)$, since y will be an upper bound by definition and $y < x$.

Exercise 6.3.3. Prove Proposition 6.3.8: Let $a_n^{\infty}n = m$ be a sequence of real numbers which has some finite upper bound $M \in R$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$).Then $a_n^{\infty}n = m$ is convergent, and in fact $\lim_{n\infty} a_n = sup(a_n)^{\infty}n = m \leq M$.

solution: We know that there exists for any $y < sup(a_n)$ at least one $N \geq m$ such that $y < a_N \leq sup(a_n)$. In particular take $y = sup(a_n) - \epsilon$ for any positive ϵ. Then we would have $sup(a_n) - \epsilon < a_N < sup(a_n) + \epsilon$. Since a_n is increasing sequence we have that the expression holds for every such $n \geq N$. Therefore, by definition of convergence the sequence converges to $sup(a_n)$.

Exercise 6.3.4. Explain why Proposition 6.3.10 fails when $x > 1$. In fact, show that the sequence $(x_n)_{n=1}^{\infty}$ diverges when $x > 1$.

solution: Following the hint: suppose it converges to some y. Also note that $0 < 1/x < 1$ for $x > 1$, therfore, $lim_{n=1}^{\infty}(1/x)^n = 0$. By the case (b) of theorem 6.1.19 we have that $lim_{n=1}^{\infty}(1/x)^n x^n = 0 = lim_{n=1}^{\infty}1 = 1$. A contradiction.

6.4.1 Let $a_n^{\infty}n = m$ be a sequence which converges to a real number c. Then c is a limit point of $a_n^{\infty}n = m$, and in fact it is the only limit point of $a_n^{\infty}n = m$.

solution: Since c is a limit of the sequence then by definition of convergence we have that for any positive real $\epsilon > 0$ there exists $N > m$ such that for all $n \geq N|a_n - c| < \epsilon$. Assume c is not a limit point, then there would exists a positive real $\epsilon > 0$ such that for all $N > m$, there would exists $n > N|a_n - c| \geq \epsilon$, but then we would have that c is not a limit too since the definition of the limit will not hold. Therefore, by contrapositive limit point of the sequnce is a limit too.

Exercise 6.4.2.

solution: For limit point:

Let $a_n^\infty n = m$ be a sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that c is a limit point of $a_n^\infty n = m$ if and only if c is a limit point of $a_n^\infty n = m'$.

If c is a limit point of $a_n^\infty n = m$, then by definition of limit point for every real positive $\epsilon > 0$ for every $N > m$ there exist $n > N$ such that $|a_n - c| < \epsilon$. Since such n exists for every $N > m$, then in particular it exists for $N > m'$, therefore c is also a limit point of $a_n^\infty n = m'$. Conversley, if c is a limit point of $a_n^\infty n = m'$, then for every real positive $\epsilon > 0$ for every $N > m'$ there exist $n > N$ such that $|a_n - c| < \epsilon$. Since $N > m'$ and $m' > m$, then $N > m$ by transitivity of real numbers, therefore, c is also a limit point for $a_n^\infty n = m$.

Let $a_n^\infty n = m$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that c is a limit point of $a_n^\infty n = m$ if and only if c is a limit point of $(a_{n+k})_{n=m}^\infty$.

$(a_{n+k})_{n=m}^\infty$ can be written as $(a_n)_{n=m+k}^\infty$. Then the arguments are similar to the previous case.

For limit superior:

Let $a_n^\infty n = m$ be a sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that c is a limit superior of $a_n^\infty n = m$ if and only if c is a limit superior of $a_n^\infty n = m'$.

Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers, let c be a real number, and let $m' \geq m$ be an integer. Show that c is a limit superior of $(a_n)_{n=m}^\infty$ if and only if c is a limit superior of $(a_n)_{n=m'}^\infty$.

If c is a limit superior of $(a_n)_{n=m}^\infty$, then $c = inf(sup(a_n)_{n=N}^\infty)$ by definition. Hence, c is the infimum of the supremums of all the elements of the sequence from $n \geq N$ onwards for some $N \geq m$. In particular the statemnt holds for all $n > N + m' - m$, where $m' - m > 0$, since $m' > m$. Then for that we can have introduce a new variable $M = N + m' - m > m + m' - m > m'$. Therefore, we would have that c is the infimum of the supremums of all the elements of the sequence from $n \geq M$ onwards for some $M \geq m'$. Conversley, if c is the limit superior for $(a_n)_{n=m'}^\infty$, then c is the infimum of the supremums of all the elements of the sequence from $n \geq N$ onwards for some $N \geq m' > m$. Therefore it is limit superior of $(a_n)_{n=m}^\infty$.

Let $a_n^\infty n = m$ be a sequence of real numbers, let c be a real number, and let $k \geq 0$ be a non-negative integer. Show that c is a limit superior of $a_n^\infty n = m$ if and only if c is a limit superior of $(a_{n+k})_{n=m}^\infty$.

Solve, analogusly by doing the same transformation for sequence as for limit point case and using the above result.

Exercise 6.4.3. (c) We have $inf(a_n)_{n=m}^{\infty} \leq L \leq L^+ \leq sup(a_n)_{n=m}^{\infty}$ solution: (c) By case (b), we have that if $inf(a_n)_{n=m}^{\infty} > L$, then for every $N > m$ there would exist an $n > N$ such that $a_n < inf(a_n)_{n=m}^{\infty}$, a contradiction. Therefore $inf(a_n)_{n=m}^{\infty} \leq L$. Using the same logic if $sup(a_n)_{n=m}^{\infty} < L^+$, then for every $N > m$, there would exist $n \geq N$ such that $a_n > sup(a_n)_{n=m}^{\infty}$, a contradiction. Therefore, $L^+ \leq sup(a_n)_{n=m}^{\infty}$. Since we know that $inf(a_n)_{n=m}^{\infty} \leq sup(a_n)_{n=m}^{\infty}$, all that remains is to show that $L \leq L^+$. However, don't know yet how to show!

d) Suppose, $c < L^-$, which means that $c < L^+$ by the previous case. By the (a) we have if $c < L^-$, then there exists an $N_0 \geq m$ such that $a_n > c$ for all $n \geq N$. However, by the definition of the limit point for every real positive $\epsilon > 0$, for all $N > m$, there exist an $n \geq N$ such that $|a_n - c| \leq \epsilon$. In particular for $N = N_0$, take $\epsilon = (a_n - c)/2$ and we would get $|a_n - c| \leq (a_n - c)/2$, which is false therefore a contradiction. Now suppose, $c > L^+$, then by (a) there exist an $N_0 \geq m$ such that $a_n < c$ for all $n \geq N$. Then using the same logic for $N = N_0$ take $\epsilon = (c - a_n)/2$, then $|a_n - c| = c - a_n < (c - a_n)/2$, which is a contradiction. Therefore, we would have that $L^- \leq c \leq L^+$.

e) If L^+ is finite than $L^+ < L^+ + \epsilon$ for some positive real ϵ. So by the case (a) we have that there exists $N_0 > m$ such that for all $n \geq N$ $a_n < L^+ + \epsilon$. Now by case (b) for every $N \geq m$ there exists $n \geq N$ such that $a_n > L^+ - \epsilon$. In particular for $N = N_0$, $L^+ - \epsilon < a_n < L^+ + \epsilon$. Therefore, $|a_n - L^+| < \epsilon$.

f)

Exercise 6.4.4. Suppose that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are two sequences of real numbers such that $a_n \leq b_n$ for all $n \geq m$. Then we have: solution: a) Suppose Suppose that $sup((a_n)_{n=m}^{\infty}) > sup((b_n)_{n=m}^{\infty})$. But we also know that $sup((b_n)_{n=m}^{\infty}) > b_n \geq a_n$ for all $n \geq m$. This will mean that $sup((a_n)_{n=m}^{\infty})$ in fact is not the least upper bound since $sup((b_n)_{n=m}^{\infty})$ is less than it and it is bigger than or equal than all the elements of the $(a_n)_{n=m}^{\infty}$. Therefore, a contradiction.

b) $inf(a_n)_{n=m}^{\infty} \leq inf(b_n)_{n=m}^{\infty}$. Suppose, $inf(b_n)_{n=m}^{\infty} < inf(a_n)_{n=m}^{\infty}$. Then for all $n \geq m$ we would have by the definition of infimum $inf(a_n)_{n=m}^{\infty} \leq a_n \leq b_n$. From which it follows, that $inf(b_n)_{n=m}^{\infty}$ is not the upper lower bound for b_n. Therfore, a contradiction.

c) $\lim sup_{n\to\infty} a_n \leq \lim sup_{n\to\infty} b_n$. We know that for any $n \geq m$, $sup(b_n) \geq sup(a_n)$ since $b_n \geq a_n$, therefore we can apply the case (b) on the sets of supremums from some $N \geq m$, and get the desired result.

d) $\lim inf_{n\to\infty} a_n \leq liminf_{n\to\infty} b_n$. We know that for any $n \geq m$, $inf(a_n) \leq inf(b_n)$ apply case a on the set of infimums and get the desired result.

Exercise 6.4.5. (Squeeze test). Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, and $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \geq m$. Suppose also that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ both converge to the same limit L. Then $(b_n)_{n=m}^{\infty}$ is also convergent to L.

solution: Since $(a_n)_{n=m}^{\infty}$ converge to L, then for all positive real $\epsilon > 0$, there exists an $N_0 > m$ such that $|a_n - L| < \epsilon$ for all $n > N$. Similarly, for $(c_n)_{n=m}^{\infty}$ for all positive real $\epsilon > 0$, there exists an $N_1 > m$ such that $|c_n - L| < \epsilon$ for all $n > N_1$. Take $M > max(N_0, N_1)$ then for all $n > M$ $L - \epsilon < c_n < b_n < a_n < L + \epsilon$. Which means that b_n converges to L.

6.4.6 Give an example of two bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $a_n < b_n$ for all $n \geq 1$, but that $sup(a_n)_{n=1}^{\infty} \nless sup(b_n)_{n=1}^{\infty}$. Explain why this does not contradict Lemma 6.4.13.

solution: $a_n = 0.00002, 0.00003, 0.00004, ...$ and $b_n = 0.00001, 0.00002, 0.00003, ...$, and they are bounded by 1. Then the supremums of both are equal to 1.

$b_n = 0.9, 0.99, 0.999, 0.9999$ and $a_n = 0.99, 0.999, 0.9999, ...$ supremum of both is 1. I guess, this doesn't contradict to the lemma because lema speaks about less than or equal. In this case the supremums are equal.

Exercise 6.4.7. Prove Corollary 6.4.17 (Let $(a_n)_{n=M}^{\infty}$ be a sequence of real numbers. Then the limit $\lim_{n\to\infty} a_n$ exists and is equal to zero if and only if the limit $\lim_{n\to\infty |a_n|}$ exists and is equal to zero.). Is the corollary still true if we replace zero in the statement of this Corollary by some other number?

solution: I don't know how to procceed with the existance of the limit since I can't say that limit exists if the sequence is convergent, because the limit can be equal to ∞ for example. Therefore, let's think about the convergence to 0. If the sequence converges to 0, that means that for all $\epsilon > 0$, there exists $N > M$ such that $|a_n| < \epsilon$ for all $n > N$. Since $|a_n| < \epsilon$, that means that $||a_n|| < |\epsilon| = \epsilon$ because $|\epsilon| > 0$ and $|a_n| > 0$. Therefore the statemnt is true. Conversley, if for all $\epsilon > 0$, there exists $N > M$ such that $||a_n|| < \epsilon$ for all $n > N$. But we also know that $|a_n| \leq ||a_n|| < \epsilon$. Therefore, the statement is true in this direction too. If we replace the 0 by some other number c. Then for example, take $a_n = -0.9, -0.99, -0.999, -0.9999, ...$ The limit of this sequence is equal to $\lim_{n\to\infty} a_n = -1$, but at the same time the sequence $|a_n| = 0.9, 0.999, 0.9999, ...$ and its limit is $\lim_{n\to\infty} |a_n| = 1$. Therefore, the statement does not hold.

Exercise 6.4.8. Let us say that a sequence $(a_n)_{n=M}^{\infty}$ of real numbers has $+\infty$ as a limit point iff it has no finite upper bound, and that it has $-\infty$ as a limit point iff it has no finite lower bound. With this definition, show that $\lim sup_{n\to\infty} a_n$ is a limit point of $(a_n)_{n=M}^{\infty}$, and furthermore that it is larger than all the other limit points of $(a_n)_{n=M}^{\infty}$; in other words, the limit superior is the largest limit point of a sequence. Similarly, show that the limit inferior is the smallest limit point of a sequence.

solution: Not sure what is asked in the exercise. From the Proposition 6.4.12 case (c) we know that for any limit point c, $L^- \leq c \leq L^+$, where $L^+ = \lim sup_{n\to\infty} a_n$ and $L^+ = \lim inf_{n\to\infty} a_n$

6.4.9 Using the definition in Exercise 6.4.8, construct a sequence $(a_n)_{n=1}^{\infty}$ which has exactly three limit points, at $-\infty, 0, and +\infty$.

solution: I am no good at examples. Look at: https://math.stackexchange.com/questions/2444176/a-sequence-that-has-three-limit-points

Exercise 6.4.10. Let $(a_n)_{n=N}^{\infty}$ be a sequence of real numbers, and let $(b_n)_{n=M}^{\infty}$ be another sequence of real numbers such that each b_m is a limit point of $(a_n)_{n=N}^{\infty}$. Let c be a limit point of $(b_n)_{n=M}^{\infty}$. Prove that c is also a limit point of $(a_n)_{n=N}^{\infty}$.

solution: b_m is a limit point of $(a_n)_{n=N}^{\infty}$, so for every positvive real $\epsilon > 0$, for every $K > N$, there exist $n > K$ such that $|a_n - b_m| \leq \epsilon$. c is a limit point of $(b_n)_{n=M}^{\infty}$, so for every positvive real $\epsilon > 0$, for every $K_1 > M$, there exist $n > K_1$ such that $|b_n - c| \leq \epsilon$

Take $T > max(K, K_1)$. Since each b_m is a limit point of $(a_n)_{n=N}^{\infty}$, then in particular b_n such that $|b_n - c| \leq \epsilon$ is a limit point of $(a_n)_{n=N}^{\infty}$. We get $|a_n - c| = |a_n - b_n + b_n - c| \leq |a_n - b_n| + |b_n - c| < 2 * \epsilon_0/2 = \epsilon_0$. Therefore, c is a limit point of $(a_n)_{n=N}^{\infty}$.

Exercise 6.5.1. Show that $\lim_{n\to\infty}(1/n)^q = 0$ for any rational $q > 0$. (Hint: use Corollary 6.5.1 and the limit laws, Theorem 6.1.19.) Conclude that the limit $\lim_{n\to\infty} n^q$ does not exist. (Hint: argue by contradiction using Theorem 6.1.19(e).)

solution: As we already know $\lim_{n\to\infty}(1/n) = 0$. Also we know that if $\lim_{n\to\infty}(1/n) = L$ then, $\lim_{n\to\infty}(1/n)^q = \lim_{n\to\infty}(1/n)^{q-1}(1/n) = L^q = L^{q-1}L = 0$ by theorem 6.1.19 case (b). Suppose $\lim_{n\to\infty} n^q = L$. We know that by theorem 6.1.19(e) and by Lemma 5.6.9(c) $\lim_{n\to\infty}(n^q)^{-1} = \lim_{n\to\infty} n^{(1/n)q} = 0$ according to the previous case.$\lim_{n\to\infty}(1/n)^q = 0$ according to the previous case.

Exercise 6.5.2. Let x be a real number. Then the limit $\lim_{n\to\infty} x^n$ exists and is equal to zero when $|x| < 1$, exists and is equal to 1 when $x = 1$, and diverges when $x = 1$ or when $|x| > 1$.

solution: If $|x| < 1$, than lets have a look at the $\lim_{n\to\infty} |x^n| = \lim_{n\to\infty} |x|^n$. The sequence is converging to 0 by the Proposition 6.3.10. By the Corollary 6.4.17 this means that $\lim_{n\to\infty} x^n = 0$. If $x = 1$, then $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$ by definition of the rationals as reals. If $x = -1$ (I don't remember if the book defines -1^n for even/odd powers, therefore, I am a bit reluctant to solve the problem in this case). If the sequence converges when $x = -1$, then it is a Cauchy sequence $|(-1)^n - (-1)^{n+1}| < \epsilon$. From which $|(-1)^n||1 - (-1)| = |(-1)^n| * 2 < \epsilon$, than $|(-1)^n| < 2 * \epsilon/2 = \epsilon$. Therefore, the sequence converges to 0. However, it is bounded away from 0, therefore, it can not converge to 0. So we got a contradiction. The sequence is divergent. If $|x| > 1$. If $x > 1$ $\lim_{n\to\infty} x^n$ then by exercise 6.3.4 the sequence diverges. If $x < -1$, then ... DON'T KNOW HOW TO PROCEED!!!

Exercise 6.6.1. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $((a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Furthermore, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

solution: $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$ because we can take $f(n) = n$, to be our function $f : N \to N$. Since $n \in N$ is increasing the function is increasing too. Therfore it is a subsequnece. $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, then there exist an increasing $f : N \to N$. Also since $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then there exist an increasing $h : N \to N$. We can take the composition of the f and h to get an increasing function $fh :\to N$ such that $c_n = a_{fh(n)}$. 6.6.2 Can you find two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ which are not the same sequence, but such that each is a subsequence of the other?

solution: So we must find a $f, h : N \to N$ such that $b_n = a_{f(n)}$ and $a_n = b_{h(n)}$ $a_n = 1, 1/2, 1/3, 1/4..., b_n = 1, 1/4, 1/6, ...f(n) = 2n$, $h(n) = n/2$, if $n > 1$, and $h(n) = 1$ if $n = 1$

Exercise 6.6.3. Let $(a_n)_{n=0}^{\infty}$ be a sequence which is not bounded. Show that there exists a subsequence $(b_n)_{n=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} 1/b_n$ exists and is equal to zero. (Hint: for each natural number j, recursively introduce the quantity $n_j := minn \in N : |a_n| j; n > n_j 1$ (omitting the condition $n > nj1$ when $j = 0$), first explaining why the set $n \in N : |a_n| j; n > n_j 1$ is non-empty. Then set $b_j := a_{nj}$.)

solution: $n \in N : |a_n| j; n > n_j 1$ is not empty because otherwise we would have that there exist an $j \in N$ such that $j > |a_n|$ for all n, i.e. the sequence will be bounded, which is a contradiction. Now set $b_n = a_{nj}$. If $\lim_{n\to\infty} 1/b_n$ converges to 0, then $\lim_{n\to\infty} |1/b_n|$ converges to 0. $0 < \lim_{n\to\infty} 1/|b_n| = 1/|a_{nj}| < \lim_{j\to\infty} 1/j = 0$, therefore indeed $\lim_{n\to\infty} 1/|b_n| = 0$.

Exercise 6.6.4.

solution: https://math.stackexchange.com/questions/3446393/the-sequence-a-n-infty-n-0-converges-to-l-then-every-subsequence-of-acomment$7084531_3 446401.(Note: the_3$ should be just an underscore and 3)

Exercise 6.6.5. Let $(a_n)_0^\infty$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent: (a) L is a limit point of $(a_n)_0^\infty$ (b) There exists a subsequence of $(a_n)_0^\infty$ which converges to L.

solution: If L is a limit point then for every real positive $\epsilon > 0$, for all $N \geq 0$ there exist an $n \geq N$ such that $|a_n - L| \leq \epsilon$ Suppose no subsequence converges to L. Then for any increasing $f : N \to N$, such that $b_n = a_{f(n)}$, there exist a real positive $\epsilon > 0$, and there exist a $N > 0$ such that for all $f(n) \geq n > N$, $|b_n - L| = |a_{f(n)} - L| > \epsilon$. However, it is a contradiction, therefore, there exists a subsequence of $(a_n)_0^\infty$ which converges to L. Conversley, if there exists a subsequence of $(a_n)_0^\infty$ which converges to L, then we know that the limit of the sequence is also a limit point of it by Proposition 6.4.5. If L is a limit point of the subsequence b_n then by definition for every real $\epsilon > 0$, for all $N > 0$, there exist $f(n) \geq n \geq N$ such that $|a_{f(n)} - L| \leq \epsilon$. But since $f(n) = k \in N$, then this means that for all $\epsilon > 0$, for all $N > 0$, there exist $k \geq N$ such that $|a_k - L| \leq \epsilon$, which means that L is a limit point of $(a_n)_0^\infty$.

Exercise 7.7.1. Prove Lemma 7.1.4.

solution: a) $\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i$

b) Let $m \leq n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}$$

Base case: $n - m = 0$ than $m = n$

$$\sum_{i=m}^{n} a_i = a_m = \sum_{j=m+k}^{m+k} a_{j-k} = a_{m+k-k} = a_m$$

Inductive step: assume the statement holds for $n - m = k$, show for $k + 1$

$$\sum_{i=m}^{m+k} a_i = \sum_{j=m+k}^{m+k+k} a_{j-k} = \sum_{j=m+k}^{m+2k} a_{j-k}$$

$$\sum_{i=m}^{m+k+1} a_i = \sum_{i=m}^{m+k} a_i + a_{m+k+1} = \sum_{j=m+k}^{m+2k} a_{j-k} + a_{m+k+1} = \sum_{j=m+k}^{m+2k+1} a_{j-k}$$

c) Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

Let's induct on n. Base case: $n - m = 0$, $n = m$

$$\sum_{i=m}^{n}(a_i + b_i) = a_n + b_n = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i = a_n + b_n$$

Inductive step: Assume that the staement holds for $n-m = k$, show for $n-m = k+1$. If it is true that

$$\sum_{i=m}^{m+k}(a_i + b_i) = \sum_{i=m}^{m+k} a_i + \sum_{i=m}^{m+k} b_i$$

$$\sum_{i=m}^{m+k+1}(a_i + b_i) = \sum_{i=m}^{m+k+1} a_i + \sum_{i=m}^{m+k+1} b_i$$

$$\sum_{i=m}^{m+k}(a_i+b_i)+(a_{m+k+1}+b_{m+k+1}) = \sum_{i=m}^{m+k} a_i+\sum_{i=m}^{m+k} b_i+(a_{m+k+1}+b_{m+k+1}) = \sum_{i=m}^{m+k+1} a_i+\sum_{i=m}^{m+k+1} b_i$$

d) Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n, andletcbeanotherrealnumber.Thenwehave$

$$\sum_{i=n}^{m}(ca_i) = c\sum_{i=n}^{m}(a_i)$$

Induct on $m - n = k$ Base case $k = 0$ then $n = m$,

$$\sum_{i=n}^{n}(ca_i) = ca_n = c\sum_{i=n}^{n}(a_i) = ca_n$$

Inductive step: Assume the statment holds for k, show for $k+1$

If

$$\sum_{i=n}^{n+k}(ca_i) = c\sum_{i=n}^{n+k}(a_i)$$

then,

$$\sum_{i=n}^{n+k+1}(ca_i) = \sum_{i=n}^{n+k}(ca_i) + ca_{n+k+1} = c\sum_{i=n}^{n+k}(a_i) + ca_{n+k+1} = c\sum_{i=n}^{n+k+1}(a_i)$$

e) (Triangle inequality for finite series) Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have

$$|\sum_{i=m}^{n} a_i| \leq \sum_{i=m}^{n} |a_i|$$

Let's induct on $n - m = k$. Base case $k = 0$ then $n = m$

$$|\sum_{i=m}^{m} a_i| = |a_m| \leq \sum_{i=m}^{m} |a_i| = |a_m|$$

Inductive step: Assume the statement holds for k, show for $k+1$,

If

$$|\sum_{i=m}^{m+k} a_i| \leq \sum_{i=m}^{m+k} |a_i|$$

then,

$$|\sum_{i=m}^{m+k+1} a_i| = |\sum_{i=m}^{m+k} a_i + a_{m+k+1}| \leq |\sum_{i=m}^{m+k} a_i| + |a_{m+k+1}| \leq \sum_{i=m}^{m+k} |a_i| + |a_{m+k+1}| = \sum_{i=m}^{m+k+1} |a_i|$$

f) (Comparison test for finite series) Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Suppose that $a_i \leq b_i$ for all $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i$$

Let's apply induction on $n - m = k$ Base case: $k = 0$ then $n = m$,

$$\sum_{i=m}^{m} a_i = a_m \leq \sum_{i=m}^{m} b_i = b_m$$

Inductive step: Assume the statement holds for k, show for $k+1$. If

$$\sum_{i=m}^{m+k} a_i \leq \sum_{i=m}^{m+k} b_i$$

then

$$\sum_{i=m}^{m+k+1} a_i = \sum_{i=m}^{m+k} a_i + a_{n+m+k} \leq \sum_{i=m}^{m+k} b_i + b_{n+m+k} = \sum_{i=m}^{m+k+1} b_i$$

Exercise 7.1.2. Prove Lemma 7.1.11.

solution: a) If X is empty, and $f : X \to R$ is a function (i.e., f is the empty function), we have

$$\sum_{x \in X} f(x) = 0$$

If X is an empty set than $n = |X| = 0$. Every function is a bijection from $i \in N : 1 \leq i \leq n$ to empty set since $i \in N : 1 \leq i \leq 0$ is also an empty set. Thus for any $g : \emptyset \to \emptyset$

$$\sum_{x \in X} f(x) = \sum_{i=1}^{n} f(g(i) = \sum_{i=1}^{0} f(g(i) = 0$$

b) If X consists of a single element, $X = x_0$, and $f : X \to R$ is a function, we have

$$\sum_{x \in X} f(x) = f(x_0)$$

If X has a single element then $n = |X| = 1$. Take $g : i \in N : 1 \leq i \leq 1 \to X$, such that $g(i) = x_0$. Then $g(i)$ is surjective since for all elements of X, there exists an $i \in i \in N : 1 \leq i \leq 1$ such that $g(i) = x_0$. Moreover, it is injective since the cardinalities of the two sets are 1. Then,

$$\sum_{x \in X} f(x) = \sum_{i=n}^{1} f(g(i)) = f(g(i)) = f(x_0)$$

c) (Substitution, part I) If X is a finite set, $f : X \to R$ is a function, and $g : Y \to X$ is a bijection, then prove that

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y))$$

If X is a finite set then there is a bijection from $h : i \in N : 1 \leq i \leq n \to X$. Since g is also a bijection to X, then we know that for some $j \in i \in N : 1 \leq i \leq n$, $h(j) = g(y)$

$$\sum_{x \in X} f(x) = \sum_{i=1}^{n} h(i) = \sum_{j=1}^{n} h(j) = \sum_{y=1}^{n} g(y) = \sum_{y=1} f(g(y))$$

Where the very last equation holds by definition.

d) (Substitution, part II) Let $n \leq m$ be integers, and let X be the set $X = i \in Z : n \leq i \leq m$. If a_i is a real number assigned to each integer $i \in X$, then we have:

$$\sum_{i=n}^{m} a_i = \sum_{i \in X} a_i$$

X must be finite. There is a bijection from N to X. Set $f : X \to R$ to be $f(i) = a_i$ and $g : i \in N : 1 \leq i \leq m - n + 1 \to X$ to be a bijection. Since g is a bijection then we know that for any $k \in X$, there is some j such that $g(j) = k$, from which we get $f(g(j)) = f(k)$. Then using Lemma 7.1.4 case (b) we get almost the desired result.

$$\sum_{i \in X} f(i) = \sum_{i \in X} a_i = \sum_{j=1}^{m-n+1} f(g(j)) = \sum_{k=1}^{m-n+1} f(k) = \sum_{k=1+n-1}^{m-n+n-1} f(k-(n-1)) = \sum_{i=n}^{m-1} f(i) = \sum_{i=n}^{m-1} a_i$$

e) Let X, Y be disjoint finite sets (so $XY = \emptyset$), and $f : XY \to R$ is a function. Then we have

$$\sum_{z \in XY} f(z) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$$

Say we have a bijection $g : i \in N : 1 \leq i \leq n \to XY$, then by definition

$$\sum_{z \in XY} f(z) = \sum_{i=1}^{n} f(g(i))$$

When $g(i) \in XY$, then $g(i) \in X$ or $g(i) \in Y$. Say $g(i) = h(i)$ if $g(i) \in X$ and $g(i) = l(i)$ if $g(i) \in Y$.

$$\sum_{z \in XY} f(z) = \sum_{i=1}^{k} f(h(i)) + \sum_{i=k+1}^{n} f(l(i))$$

By lemma 7.1.4 case (a). At the same time by case (b)

$$\sum_{z \in XY} f(z) = \sum_{i=1}^{k} f(h(i)) + \sum_{i=k+1}^{n} f(l(i)) = \sum_{i=1}^{k} f(h(i)) + \sum_{i=1}^{n-k} f(l(i+k)) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$$

by definition.

f) ¿ (Linearity, part I) Let X be a finite set, and let $f : X \to R$ and $g : X \to R$ be functions. Then using point (c) of Lemma 7.1.4

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x)$$

If there is a bijective function from $h :1, ..., n \to X$, then by definition

$$\sum_{x \in X} (f(x)+g(x)) = \sum_{i=1}^{n} (f(h(i))+g(h(i))) = \sum_{i=1}^{n} f(h(i)) + \sum_{i=1}^{n} g(h(i)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x)$$

g) Let X be a finite set, let $f : X \to R$ be a function, and let c be a real number. Then

$$\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x)$$

By definition and with the usage of case (d) Lemma 7.1.4

$$\sum_{x \in X} cf(x) = \sum_{i=1}^{n} cf(g(i)) = c \sum_{i=1}^{n} f(g(i)) = c \sum_{x \in X} f(x)$$

h) (Monotonicity) Let X be a finite set, and let $f : X \to R$ and $g : X \to R$ be functions such that $f(x) \leq g(x)$ for all $x \in X$. Then we have

$$\sum_{x\in X} f(x) \le \sum_{x\in X} g(x)$$

.

If there is a bijection $h : 1, ..., n \to X$, then $f(h(i)) \le g(h(i))$ by the given condition. Using this, the definition and case (f) of Lemma 7.1.4, we have

$$\sum_{x\in X} f(x) = \sum_{i=1}^{n} f(h(i)) \le \sum_{i=1}^{n} g(h(i)) = \sum_{x\in X} g(x)$$

.

i) (Triangle inequality) Let X be a finite set, and let $f : X \to R$ be a function, then using case (e) of Lemma 7.1.4

$$|\sum_{x\in X} f(x)| \le \sum_{x\in X} |f(x)|$$

If there is a bijection $g : 1, ..., n \to X$, then

$$|\sum_{x\in X} f(x)| = |\sum_{i=1}^{n} f(g(i)| \le \sum_{i=1}^{n} |f(g(i))| = \sum_{x\in X} |f(x)|$$

Exercise 7.1.3. Form a definition for the finite products $\sqcap_{i=1}^{n} a_i$ and $\sqcap_{x\in X} f(x)$. Which of the above results for finite series have analogues for finite products? (Note that it is dangerous to apply logarithms because some of the a_i or $f(x)$ could be zero or negative. Besides, we haven't defined logarithms yet.)

solution: The definition: Given a finite X with n elements, if there is a function $f : X \to R$, and we can find a bijection $g : 1, ..., n \to X$, then $\sqcap_{x\in X} f(x) = \sqcap_{i=1}^{n} f((g(i))$. Also $\sqcap_{i=1}^{n} a_i = \sqcap_{i=1}^{n-1} a_i * a_n$.

From Lemma 7.1.4: a) $\sqcap_{i=m}^{p} a_i = \sqcap_{i=m}^{n} a_i * \sqcap_{i=n+1}^{p} a_i$ holds. By induction on p. Base case: $p = n + 1$, $\sqcap_{i=m}^{n+1} a_i = \sqcap_{i=m}^{n} a_i * \sqcap_{i=n+1}^{n+1} a_i = \sqcap_{i=m}^{n} a_i * a_{n+1} = \sqcap_{i=m}^{n+1} a_i *$. Inductive step: Assume the statement holds for p, show for $p + 1$

$\sqcap_{i=m}^{p+1} a_i = \sqcap_{i=m}^{p} a_i * a_p = \sqcap_{i=m}^{n} a_i * \sqcap_{i=n+1}^{p} a_i * a_p = \sqcap_{i=m}^{n} a_i * \sqcap_{i=n+1}^{p+1} a_i$

b) $\sqcap_{i=m}^{n} a_i = \sqcap_{j=m+k}^{n+k} a_{j-k}$ holds by induction on $n - m = p$. Base case $p = 0$, $n = m$ $\sqcap_{i=m}^{m} a_i = a_m = \sqcap_{j=m+k}^{m+k} a_{j-k} = a_{m+k-k} = a_m$ Inductive step: Assume the statment holds for p, show for $p + 1$.

$\sqcap_{i=m}^{m+p+1} a_i = \sqcap_{i=m}^{m+p} a_i * a_{m+p+1} = \sqcap_{j=m+k}^{m+p+k} a_{j-k} * a_{m+p+1} = \sqcap_{j=m+k}^{m+p+k} a_{j-k} * a_{m+p+k+1-k} = \sqcap_{j=m+k}^{m+p+k+1} a_{j-k}$.

c) It doesn't hold. For example, $(4 + 5)(3 + 6) \ne (4 + 3) * (5 + 6)$

d) It doesn't hold. For example, let $c = 10$, $10 * 5 * 10 * 6 \ne 10(5 * 6)$

e) Holds. Prove is similar to that for the sum.

d) Doesn't hold. $-10 < 5, -10 < 6$, $-10 * (-10) > 5 * 6$

Exercise 7.1.4.

Define the factorial function $n!$ for natural numbers n by the recursive definition $0! := 1$ and $(n + 1)! := n!(n + 1)$. If x and y are real numbers, prove the binomial formula

$$(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$$

solution: Following the hint, let's induct on n. Base Case: $n = 0$. $(x+y)^0 = 1 = \frac{0!}{0!0!} x^0 y^0 = 1$ Inductive step: Assume the statment holds for n, show for $n+1$.

$$(x+y)^{n+1} * (x+y) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j} (x+y) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} + \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j} = x^{n+1} \sum_{j=0}^{n}$$

https://math.stackexchange.com/questions/1695270/binomial-theorem-proof-by-induction

7.1.5

7.2.1 Is the series $\sum_{n=1}^{\infty} (1)^n$ convergent or divergent? Justify your answer.

solution: According to the zero test if the series are convergent than $(a_n)_{n=1}^{\infty}$ must converge to 0. However, we can see that $(-1)^n$ is bounded away from 0, therefore it can not converge to 0. Thus, the sequence is divergent.

Exercise 7.2.2. Prove Proposition 7.2.5. (Hint: use Proposition 6.1.12 and Theorem 6.4.18.) Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if, for every real number > 0, there exists an integer Nm such that

$$|\sum_{n=p}^{q} a_n| \leq \epsilon$$

for all $p, q \geq N$

solution: First of all if $\sum_{n=m}^{\infty} a_n$ converges than by definition we have that for any $N \geq m$ the partial sum $S_N = \sum_{n=m}^{N} a_n$, $(S_N)_{N=m}^{\infty}$ converges to some limit L. In other words for any $\epsilon > 0$, there exist $N \geq m$ such that $|\sum_{n=m}^{t} a_n - L| < \epsilon$ for all $t > N$. Also we know that convergent sequences are Cauchy. So for any $\epsilon > 0$, there exist $N \geq m$ such that $|\sum_{n=m}^{q} a_n - \sum_{n=m}^{p-1} a_n| = |\sum_{n=p}^{q} a_n| \leq \epsilon$ for any $q, p-1 \geq N$, therefore for any $q, p \geq N$.

Conversely, if for every real $\epsilon > 0$, there exists an an integer Nm such that

$$|\sum_{n=p}^{q} a_n| = |\sum_{n=m}^{q} a_n - \sum_{n=m}^{p-1} a_n| \leq \epsilon$$

for all $p, q \geq N$. Then $\sum_{n=m}^{\infty} a_n$ is a Cacuhy sequence and since we know that every Cauchy is also convergent sequence then we have that $\sum_{n=m}^{\infty} a_n$ converges.

7.2.3 Use Proposition 7.2.5 to prove Corollary 7.2.6.

Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$. To put this another way, if $\lim_{n\to\infty} a_n = 0$ is non-zero or divergent, then the series$\sum_{n=m}^{\infty} a_n$ is divergent.

solution: Ok, since we have that the $\sum_{n=m}^{\infty} a_n$ is convergent than by the previous exercise we have that for every real number $\epsilon > 0$, there exists an integer Nm such that

$$|\sum_{n=p}^{q} a_n| \leq \epsilon$$

for all $p, q \geq N$

In particular, the statement is true if for all $q = p \geq N$, then $|a_p| < \epsilon$. Which by definition states that a_n converges to 0.

Exercise 7.2.4. Prove Proposition 7.2.9. (Hint: use Proposition 7.2.5 and Proposition 7.1.4(e).) Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case, we have the triangle inequality.

$$|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n|$$

.

solution: Suppose $|\sum_{n=m}^{\infty} a_n| > \sum_{n=m}^{\infty} |a_n|$. If $\sum_{n=m}^{\infty} a_n$ then by Proposition 7.2.5 its tail is convergent. So $|\sum_{n=p}^{q} a_n| < \epsilon$ for every $\epsilon > 0$, and some $N \geq m$ for all $p, q \geq N$. Now using the fact that $\sum_{n=m}^{\infty} |a_n|$ is also convergent and applying the same proposition $|\sum_{n=p}^{q} |a_n|| < \epsilon$ for every $\epsilon > 0$, and some $N_0 \geq m$ for all $p, q \geq N_0$. Now using Proposition 7.1.4 case (e) we would get that $|\sum_{n=p}^{q} a_n| \leq |\sum_{n=p}^{q} |a_n||$ for all $p, q \geq M$, where $M = max(N, N_0)$. Therefore, for $|\sum_{n=m}^{\infty} a_n| > \sum_{n=m}^{\infty} |a_n|$ to be true we should have that $|\sum_{n=m}^{t} a_n| > \sum_{n=m}^{t} |a_n|$ for the $t < M$, which is not true by proposition 7.1.4 case (e).

Exercise 7.2.5. Prove Proposition 7.2.14. (Hint: use Theorem 6.1.19.)

solution: a) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and $\sum_{n=m}^{\infty} b_n$ is a series of real numbers converging to y, then $\sum_{n=m}^{\infty}(a_n + b_n)$ is also a convergent series, and converges to $x + y$.

$\sum_{n=m}^{\infty}(a_n + b_n)$ is also a convergent series since $(a_n)_{n=m}^{\infty}$ converges to 0 and $(b_n)_{n=m}^{\infty}$ converges to 0, by zero test, and by Theorem 6.1.19 case (a) $(a_n + b_n)_{n=m}^{\infty}$ converges to 0 too.

If $\sum_{n=m}^{\infty} a_n$ converges to x, that means that $(S_N)_{N=m}^{\infty}$ converges to x, where $S_N = \sum_{n=m}^{N} a_n$. That is for every real $\epsilon > 0$, there exist $N_0 \geq m$, such that for all $n \geq N_0$, $|S_n - x| < \epsilon$. If $\sum_{n=m}^{\infty} b_n$ converges to y, that means that $(S'_N)_{N=m}^{\infty}$ converges to y, where $S'_N = \sum_{n=m}^{N} b_n$. That is for every real $\epsilon > 0$, there exist $N_1 \geq m$, such that for all $n \geq N_1$, $|S'_n - y| < \epsilon$. For $M \geq max(N_0, N_1)$, we would have $|S_n + S'_n - x - y| \leq |S_n - x| + |S'n - y| < 2 * \epsilon/2$, from which we get that $\sum_{n=m}^{\infty}(a_n + b_n)$ and converges to $x + y$.

b) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and c is a real number, then $\sum_{n=m}^{\infty} ca_n$ is also a convergent series, and converges to cx.

$\sum_{n=m}^{\infty} ca_n$ is convergent because ca_n converges to 0. For every real $\epsilon > 0$, there exist $N \geq m$, such that for all $n \geq N_0$, $|S_n - x| < \epsilon$ where $S_N = \sum_{n=m}^{N} a_n$. Now by Lemma 7.1.4 where $cS_N = \sum_{n=m}^{N} ca_n$ and by case (c) of Theorem 6.1.9 $(cS_N)_{n=m}^{\infty}$ converge to cx.

c) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k \geq 0$ be an integer. If one of the two series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m+k}^{\infty} a_n$ are convergent, then the other one is also, and we have the identity

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n$$

If one is convergent than $(a_n)_{n=m}^{\infty}$ converges which implies that $(a_n)_{n=m+k}^{\infty}$ converges by exercise 6.1.3 so the convergence is sattled down.

For every real $\epsilon > 0$, there exist $N \geq m$, such that for all $n \geq N_0$, $|S_n - x| = |\sum_{n=m}^{n} a_n - x| = |\sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{n} a_n - x| < \epsilon$ where $S_N = \sum_{n=m}^{N} a_n$. Which implies by definition that $\sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n$ also converges to x.

Exercise 7.2.5. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers which converge to 0, i.e., $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_{n=m}^{\infty}(a_n - a_{n+1})$ converges to a_0. (Hint: First work out what the partial sums $\sum_{n=0}^{N}(a_n - a_{n+1})$ should be, and prove your assertion using induction.) How does the proposition change if we assume that an does not converge to zero, but instead converges to some other real number L?

solution: Following the hint: by Lemma 7.1.4 cases (a), (b), (c) and (d) $\sum_{n=0}^{N}(a_n - a_{n+1}) = \sum_{n=0}^{N} a_n - \sum_{n=0}^{N} a_{n+1} = \sum_{n=0}^{N} a_n - \sum_{n=1}^{N+1} a_n = \sum_{n=0}^{N} a_n - \sum_{n=1}^{N} a_n - a_{N+1}$

If $N = 1$: $(a_0 - a_1) + (a_1 - a_2) = (a_0 + a_1) - a_1 - a_2 = a_0 - a_2$ so it holds.

Assume the statement holds for N, show for $N + 1$. $\sum_{n=0}^{N+1}(a_n - a_{n+1}) = \sum_{n=0}^{N}(a_n - a_{n+1}) + a_{N+1} - a_{N+2} = a_0 - a_{N+1} + a_{N+1} - a_{N+2} = a_0 - a_{N+2}$ holds.

Assuming $\sum_{n=m}^{\infty} a_n$ is convergent by proposition 7.2.14 using the cases a, b, c, d $\sum_{n=0}^{\infty}(a_n - a_{n+1}) = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} a_{n+1} = \sum_{n=0}^{0} a_n + \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n = a_0$.

If $\sum_{n=m}^{\infty} a_n$ is not convergent then just use the definition of the convergence of the series. In particular note that $S_N = a_0 - a_{N+1}$, and the limit of $(S_N)_{n=0}^{\infty}$ is a_0, since $(a_n)_{n=0}^{\infty}$ converges to 0. If $(a_n)_{n=0}^{\infty}$ doesn't converge to 0, then the statement doesn't hold.

Exercise 7.3.1. Use Proposition 7.3.1 to prove Corollary 7.3.2 Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact

$$|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n$$

solution: We know that $\sum_{n=m}^{\infty} |a_n|$ is a formal series of non-negative real numbers. It is convergent by the proposition 7.3.1 since it is bounded by the $\lim(S'_N)_{n=m}^{\infty}$ where $S'_N = \sum_{n=m}^{N} b_n$. Let $S_N = \sum_{n=m}^{N} |a_n|$. We know that $(S'_N)_{n=m}^{\infty}$ converges to some L. We also know that for all N $S_N \leq S'_N$ by

Lemma 7.1.4. Then I believe, we should have something like Corollary 5.4.10 proved for infinite sequence I just can't find it to directley reffer to it but anyway we should have that $lim(S_N)_{n=m}^{\infty} \leq \lim(S'_N)_{n=m}^{\infty}$. Which, by definition means that, $\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n$. Also by proposition 7.2.9

$$|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n$$

Exercise 7.3.2. Prove Lemma 7.3.3. (Geometric series). Let x be a real number. If $|x| \geq 1$,then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If however $|x| < 1$, then the series is absolutely convergent and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(Hint: for the first part, use the zero test. For the second part, first, use induction to establish the geometric series formula

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

and then apply Lemma 6.5.2).

solution: Following the hint: By the Lemma 6.5.2 we know that $(x^n)_{n=m}^{\infty}$ diverges when $|x| > 1$. Hence, by the zero test the series is divergent. If $|x| = 1$, then also by Lemma 6.5.2 the $(x^n)_{n=m}^{\infty}$ converges to 1, hence by the zero test the series is divergent. Let $S_N = \sum_{n=0}^{N} x^n$. Lets induct on N. Base case: $N = 0$ then $\sum_{n=0}^{0} x^n = x^0 = 1 = \frac{1-x}{1-x} = 1$ Inductive step: Assume the equation holds for N, show for $N+1$. $\sum_{n=0}^{N+1} x^n = \sum_{n=0}^{N} x^n + x^{N+1} = \frac{1-x^{N+1}}{1-x} + x^{N+1} = \frac{1-x^{N+1}+x^{N+1}-x^{N+2}}{1-x} = \frac{1-x^{N+2}}{1-x}$.

Then the $(S_N)_{n=0}^{\infty}$ converges to $\frac{1}{1-x}$ since by the Lemma 6.5.2 $(x^{N+1})_{n=0}^{\infty}$ as well as $(x^N)_{n=0}^{\infty}$ converges to 0, when $|x| < 1$, which is exactly our case.

Exercise 7.3.3. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers such that $\sum_{n=0}^{\infty} |a_n| = 0$. Show that $a_n = 0$ for every natural number n.

solution: We know that

$$0 \leq |\sum_{n=0}^{\infty} a_n| \leq \sum_{n=0}^{\infty} |a_n| = 0$$

Let $S_N = \sum_{n=0}^{N} |a_n|$. So we know that $(S_N)_{N=0}^{\infty}$ converges to 0. By induction we can show that if there exist an $a_t \neq 0$, then $S_N > 0$, for $N \geq t$. Which means that $(S_N)_{N=0}^{\infty}$ is bounded away from 0 for all $N \geq t$. Which means it can not converge to 0

Exercise 7.4.1. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers. Let $f : N \to N$ be an increasing function (i.e., $f(n + 1) > f(n)$ for all $n \in N$). Show that $\sum_{n=0}^{\infty} a_{f(n)}$ is also an absolutely convergent series. (Hint:

try to compare each partial sum of $\sum_{n=0}^{\infty} a_{f(n)}$ with a (slightly different) partial sum of $\sum_{n=0}^{\infty} a_n$.)

solution: Let $T_N = \sum_{n=0}^{N} |a_{f(n)}| = \sum_{n=0}^{f(N)} |a_n|$ be the partial sum of $\sum_{n=0}^{\infty} |a_{f(n)}|$. f is an increasing, injective function. Let $S_M = \sum_{n=0}^{M} |a_n|$ be the partial sum of $\sum_{n=0}^{\infty} |a_n|$ such that $M = max(f(n))$, then

$$T_N \leq S_M \leq L$$

Then don't know how to proceed? I think, that this can be proved by using 6.6.4 and its statement that The sequence converges to L then every subsequence of if converges to L. Here T_N is clearly a subspace of S_M and we know that $(S_M)_{n=0}^{\infty}$ converges to L.

Also I have found a link that may be useful: https://math.stackexchange.com/questions/1748486/show-that-sum-n-0-infty-a-fn-is-also-absolutely-convergent-series

Exercise 7.5.1. $\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf c_n^{1/n}$

solution: With analogy to the proof of lim supremum. Let $L = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$. If $L = -\infty$ then by definition $L \leq \liminf c_n^{1/n}$. Assume L a non-extended real number. Then by the proposition 6.4.1 for every there exists an $N \geq m$ such that $\frac{c_{n+1}}{c_n} \geq L - \epsilon$ for all $n \geq N$. Thus $c_{n+1} \geq c_n(L-\epsilon)$, $c_n \geq c_N(L-\epsilon)^{n-N}$ then denote $A = c_N(L-\epsilon)^{-N}, c_n = A(L-\epsilon)^n, c_n^{1/n} \geq A^{1/n}(L-\epsilon), \lim_{n\to\infty} A^{1/n}(L-\epsilon) = L-\epsilon$. That is $\liminf_{n\to\infty} c_n^{1/n} \geq L-\epsilon$, since this is true for all ϵ than $\liminf_{n\to\infty} c_n^{1/n} \geq L$

Exercise 7.5.2. $\frac{(n+1)^q x^{n+1}}{n^q x^n} = x(1+1/n)^q$ solution: Compute $\limsup(1+1/n)^q$ where $q \in R, n \in N$.

$L = 2^q, (1+1/2)^q, (1+1/3)^q, ...$ and since as $n \to \infty$ $1/n \to 0$, then $\inf(L) = 1$.

$\limsup(x(1+1/n)^q) = x \limsup(1+1/n)^q = x < 1$, therefore absouluteley convergent. At the same time $lim_{n\to\infty} n^q x^n = lim_{n\to\infty} n^q lim_{n\to\infty} x^n = 0$.

Exercise 8.1.1. Let X be a set. Show that X is infinite if and only if there exists a proper subset $Y \subset X$ of X which has the same cardinality as X.

solution: If there exists a proper subset $Y \subset X$ of X which has the same cardinality as X. Then X is infinite since there can be no bijection between X and its proper subset otherwise.

Let $Y = a_0, a_1, a_2, a_3, a_4, ...$ Let $Y_0 = a_0, a_2, a_4, ...$ and $Y_1 = a_1, a_3, a_5, ...$ We can construct a natural bijection from $f : Y \to Y_1$ with $f(a_n) = a_{2n+1}$. Also if we take $Y \cup (X \setminus Y) = X$ and $Y_1 \cup (X \setminus Y) = X \setminus Y_0$, we can extend our bijection just by assigning every a_n from $(X \setminus Y)$ to itself. This will mean that we constructed a bijection between X and one of its proper subsets. In other words, if X is infinite there exists a bijection between it and its proper subset.

Exercise 8.1.2. Proposition 8.1.4. (Well ordering principle). Let X be a non-empty subset of the natural numbers N. Then there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$. In other words, every non-empty set of natural numbers has a minimum element.

solution: Since X is a subset of natural number than it is bounded from below. By the Theorem 5.5.9 we have that there exists a single greatest lower bound n. n belongs to X, otherwise for $n+1 \in X, n+1 > n$ and every other element in X will be $m \geq n+1$, thus n would not be the greatest lower bound. Therefore, there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$.

Exercise 8.1.3. Fill in the gaps marked (?) in Proposition 8.1.5

solution:

- Also, since X is infinite, the set $x \in X : x \neq a_m$ for all $m < n$ is infinite(?), hence non-empty.

Because by the Well-Ordering-Principale every non-empty set of natural numbers has a minimum element. Now, by definition of $a_n$for every n there exist a minimum element in X. Since X is infinite then there are infinite amount of minimums, therefore, $x \in X : x \neq a_m$ for all $m < n$ is infinite.

Another trial is to construct a bijection from X to $x \in X : x \neq a_m$ for all $m < n$ just by taking $f(x) = x$ if $x \neq a_m$ for all $m < n$. It will be onto since every x from $x \in X : x \neq a_m$ for all $m < n$ will have its $x \in X$. Also it will be one-to-one since every xin $x \in X : x \neq a_m$ for all $m < n$ has exactly one x, such that $f(x) = x$ for all $m < n$.

- One can show(?) that an is an increasing sequence i.e.

$$a_0 < a_1 < a_2 < ...$$

and in particular that $a_n \neq a_m$ for all $n \neq m$.

Assume that for some $a_n > a_{n+1}$. But then $a_n \neq \min x \in X : x \neq a_m$ for all $m < n$ since there is clearly at least one element in the remaing elements, namely $\mathrm{a}_{n+1}, whichislessthan \mathrm{a}_n$. Therefore, a contradiction. Assume, $a_n = a_{n+1}$. By the definition of a_n it can not happen. Therefore, a contradiction. In particular, this means that the sequence is increasing and $a_n \neq a_m$ for all $n \neq m$

- Let $x \in X$. Suppose for sake of contradiction that $a_n \neq x$ for every natural number n. Then this implies(?) that x is an element of the set $x \in X : x \neq a_m$ for all $m < n$ for all n.

Assume it is not an element of the $x \in X : x \neq a_m$ for all $m < n$ set for all n. Then there exists some N such that for some $m < N$, $a_m = x$, but it is a contradiction.

- However, since a_n is an increasing sequence, we have $a_n \geq n$ (?), and hence $x \geq n$ for every natural number n.

Let's apply induction on n: Base case: $n = 0$. So min of all the elements of the set is a_0, since a_n is an increasing sequence. But then it is clear that $a_0 \geq 0$, since $a_0 \in N$. Inductive step: Assume $a_n \geq n$, show $a_{n+1} \geq n+1$. We already have that $a_{n+1} > a_n > n$. Then since $a_{n+1} \in N$ it is clear that $a_{n+1} \geq n+1$. In particular since $x \geq a_n$, then $x \geq n$.

Exercise 8.1.4. Prove Proposition 8.1.8: Let Y be a set, and let $f : N \to Y$ be a function. Then $f(N)$ is at most countable.

(Hint: the basic problem here is that f is not assumed to be one-to-one. Define A to be the set $A := n \in N : f(m) \neq f(n)$ for all $0 \leq m < n$; informally speaking, A is the set of natural numbers n for which $f(n)$ does not appear in the sequence $f(0), f(1), ...f(n1)$. Prove that when f is restricted to A, it becomes a bijection from A to $f(N)$. Then use Proposition 8.1.5).

solution: Let's restrict f to A. Then f is injective since if $f(m) \neq f(n)$, then $m < n$ by definition. Suppose there exist an $n \in N$ such that $f(n)$ exists but $n \notin A$. This will mean that $f(n) = f(m)$ for some $0 \leq m < n$. But since $f(n) = f(m)$ and f is injective $n = m$ which is a contradiction. Therefore, f is surjective. So we now want to say that A is at most countable. A is either finite or infinite. If it is finite (for instance, $f(n)$ might be the remainder when you divide n by 4, in which case $A = \{0, 1, 2, 3\}$), then it is at most countable by the definition of "at most countable". On the other hand, if A is infinite (for instance you might have $f(n) = \lfloor \frac{n}{2} \rfloor$, in which case A is the set of all even numbers), then it is at most countable by Proposition 8.1.5. If I know that A is an infinite subset of N then I have a proposition 8.1.5 (page 183, Terence Tao-Analysis 1), which says that A has the same cardinality as N, and therefore will be countable. From the existence of bijection from A to $f(N)$ it will follow that $f(N)$ is countable.

Exercise 8.1.5. Use Proposition 8.1.8 to prove Corollary 8.1.9. Let X be a countable set, and let $f : X \to Y$ be a function. Then $f(X)$ is at most countable.

solution: Since X is a countable set than it has equal cardinality with N by definition. Therefore, there exists a bijection between $g : N \to X$. Since g is a bijection $f(x) = f(g(n))$ for any $x \in X$ and $n \in N$. Which means, that $f(X) = f(g(N))$. Now let the composition of fg be a function from $fg : N \to Y$. Then by the proposition 8.1.8: if Y is a set and $h : N \to Y$, then Y is at most countable.

Exercise 8.1.6. Let A be a set. Show that A is at most countable if and only if there exists an injective map $f : A \to N$ from A to N.

solution: If A is at most countable than it is either finite or countable. If it is countable then there exists a bijection between it and N. If A is finite then we can construct an injection by taking for example $f(a_n) = n + 1$.

If there exists an injective map $f : A \to N$, then we can restrict N by $f(A)$ and get a bijection from A to $f(A)$ which is a subset of N. From Corollary 8.1.6 we know that all subsets of N are at most countable so, A is at most countable.

Exercise 8.1.7. Prove Proposition 8.1.10. Let X be a countable set, and let Y be a countable set. Then $X \cup Y$ is a countable set.

(Hint: by hypothesis, we have a bijection $f : N \to X$, and a bijection $g : N \to Y$. Now define $h : N \to X \cup Y$ by setting $h(2n) := f(n)$ and $h(2n + 1) := g(n)$ for every natural number n, and show that $h(N) = X \cup Y$. Then use Corollary 8.1.9, and show that $X \cup Y$ cannot possibly be finite.)

solution: Following the hint, define f, g, h as they were in the hint. If $t \in h(N)$ then $h(m) = t$ for some $m \in N$. If m is even than $t = f(n/2) \in X$, if m is odd then $t = g((m - 1)/2) \in Y$. Therefore, $t \in X \cup Y$. The other way: if $t \in X \cup Y$, then $t \in X$ or $t \in Y$. If $t \in X$, then $t = f(n) = h(2n)$, if $t \in Y$, then $t = g(n) = h(2n + 1)$ for some $n \in N$. So, $t \in h(N)$. Also, we can construct a bijection from $N \to X \cup Y$. If X is disjoint from Y than h is already a bijection. However, if they are not disjoint that is $h(2n) = h(2n + 1)$ than define a bijection between $X \setminus Y$ and N by $h(2n) = f(n)$ and between N and Y by $h(2n+1) = g(n)$, then $h(n)$ will be a bijection from $N \to (X \setminus Y) \cup Y = X \cup Y$. If $X \cup Y$ is finite then there exists some $M \in N$ such that $x \leq h(M)$ for every

$x \in X \cup Y$. But then, if M is even that will mean that $f(n) \leq M$ for all $n \in N$. Which is not true since X is not finite. The symmetric argument works for the case when M is odd. Therefore, by the proposition 8.1.8 in combination with our thoughts $X \cup Y$ is countable.

Exercise 8.1.8. Use Corollary 8.1.13 to prove Corollary 8.1.14: If X and Y are countable, then $X \times Y$ is countable.

solution: If X is countable then there exists a bijection $f : N \to X$. Similarly, there exists a bijection $g : N \to Y$. We can construct a bijection from $N \times N$ to $X \times Y$ by taking $h(n, m) = (f(n), g(m))$. Since by the Corollary 8.1.13 the $N \times N$ is countable then there exists a bijection $h : N \to N \times N$, from where because of the bijectivity being an equivalence relation we can get a bijection from $N \to X \times Y$. Therefore, it is countable.

Exercise 8.1.9. Suppose that I is an at most countable set, and for each $\alpha \in I$, let A_α be an at most countable set. Show that the set $\cup_{\alpha \in I} A_\alpha$ is also at most countable. In particular, countable unions of countable sets are countable. (This exercise requires the axiom of choice, see Section 8.4.)

solution: Here is the proof, however I am struggling to understand it. https://proofwiki.org/wiki/Countable$_Union_of_Countable_Sets_is_Countable$

Exercise 8.1.10. Find a bijection $f : N \to Q$ from the natural numbers to the rationals.

solution:

Exercise 8.2.1. Let X be an at most countable set, and let $f : XR$ be a function. Then the series $\sum_{x \in X} f(x)$ is absolutely convergent if and only if

$$sup(\sum_{x \in A} |f(x)| : A \subset X, Afinite) < \infty$$

solution: Since X is at most countable set than we have a bijection from $g : N \to X$. Since $\sum_{x \in X} f(x)$ is absolutely convergent then by the definition there exists some bijection $h : N \to X$ such that $\sum_{n=0}^{\infty} f(h(x))$ is absoluteley convergent and we define

$$\sum_{x \in X} f(x) = \sum_{n=0}^{\infty} |f(h(x))| = L$$

$$\sum_{x \in A} |f(x)| = \sum_{n \in g^{-1}(A)} |f(g(n))| \leq \sum_{n \in N} |f(g(n))| = L$$

. Hence our desired expression has been achieved.

We can take $h(x) = g(x)$, in the above definition. Let g :$1, .., n \to A$ and A is finite.

$$\sum_{x \in A} |f(x)| = \sum_{i=1}^{n} |f(g(i))| \leq L$$

for any $n \in N$. Which means by the proposition of monotone increasing bounded sequence the $\sum_{n \in N} |f(g(n))|$ is convergent. Therefore, $\sum_{x \in X} f(x)$ is absolutely convergent.

(https://math.stackexchange.com/questions/1812620/proof-of-lemma-8-2-3-in-terence-tao-analysis-1-book)
Exercise 8.2.2. Prove Lemma 8.2.5.

solution: https://math.stackexchange.com/questions/20661/the-sum-of-an-uncountable-number-of-positive-numbers
Exercise 8.2.3. Prove Proposition 8.2.6.

solution: https://math.stackexchange.com/questions/3475304/prove-that-he-series-sum-x

The series $\sum_{xX}(f(x)+g(x))$ is absolutely convergent, and

$$\sum_{xX}(f(x)+g(x)) = \sum_{xX} f(x) + \sum_{xX} g(x)$$

.

Well, if the set X is finite then we have the result by the Proposition 7.1.11 case (f). If it is countable, then $h : N \to X$ is a bijection and $\sum_{n=0}^{\infty}(f+g)(h(n))$ is absolutely convergent. Which means, that $\sum_{n=0}^{\infty} |f(h(n)+g(h(n))| = L$
Exercise 9.1.1. Let X be any subset of the real line, and let Y be a set such that $X \subset Y \subset X^-$. Show that $Y^- = X^-$.

solution: Let x be adherent point of X, i.e. $x \in \bar{X}$. Since it is an adherent point of X then by definition for every positive $epsilon > 0$ there exists some $u \in X$ such that $|x-u| < \epsilon$. However, since $u \in X$, then $u \in Y$ too according to the given conditions. Therefore we have $x \in \bar{Y}$ and $\bar{X} \subset \bar{Y}$. Now take into attention that $Y \subset \bar{X}$. Let $y \in \bar{Y}$. So, by definition for every $epsilon_1 > 0$, there exists some $u \in Y$ such that $|y-u| < \epsilon_1$. However, we have by the underlined condition that $u \in \bar{X}$ too, i.e. for every $\epsilon_2 > 0$ there exists some $x \in X$ such that $|u-x| < \epsilon_2$. Now to show that $\bar{Y} \subset \bar{X}$ we should show that for every $epsilon_0 > 0$ there exists some $x \in X$ such that $|y-x| < \epsilon_0$. We know that

$$|y-x| = |y-u+u-x| \leq |y-u| + |u-x| < \epsilon_0$$

. So, $\bar{Y} = \bar{X}$.
Exercise 9.1.2. Prove Lemma 9.1.11: Let X and Y be arbitrary subsets of R. Then $X \in \bar{X}$, $X \bar{\cup} Y = \bar{X} \cup \bar{Y}$ and $X \bar{\cap} Y \subset \bar{X} \cap \bar{Y}$. If $X \subset Y$, then $\bar{X} \subset \bar{Y}$.

solution: The last part was shown in the previous exercise. $X \in \bar{X}$: if $x \in X$, then for any $\epsilon > 0$ there exists $y \in X$ such that $|x-y| < \epsilon$, namely $y = x$. $X \bar{\cup} Y = \bar{X} \cup \bar{Y}$: if $x \in X \bar{\cup} Y$ then for every $\epsilon > 0$ there is some $y \in X \cup Y$ such that $|x-y| < \epsilon$. Since $y \in X \cup Y$, then either $y \in X$ and $|x-y| < \epsilon$ or $y \in Y$ and $|x-y| < \epsilon$, so

$$X \bar{\cup} Y \subset \bar{X} \cup \bar{Y}$$

, the converse is analogous. $X \bar{\cap} Y \subset \bar{X} \cap \bar{Y}$: if $x \in X \bar{\cap} Y$ then for any $\epsilon > 0$ there exists some $y \in X \cap Y$ such that $|x-y| < \epsilon$. Since $y \in X \cap Y$, then $y \in X$ such that $|x-y| < \epsilon$ and $y \in Y$ such that $|x-y| < \epsilon$. So,

$$X \bar{\cap} Y \subset \bar{X} \cap \bar{Y}$$

. However, if $x \in \bar{X} \cap \bar{Y}$, then for every ϵ_1 there is some $u \in X$ such that $|x-u| < \epsilon_1$ and there is some $w \in Y$ such that $|x-w| < \epsilon_1$ but it is not necessary for $u \in Y$ too, or for $w \in X$ too.

Exercise 9.1.3. Prove Lemma 9.1.13. (Hint: for computing the closure of Q, you will need Proposition 5.4.14.)

solution: The closure of N is N. If $n \in N$ then

Exercise 9.1.5. Let X be a subset of R, and let $x \in R$. Then x is an adherent point of X if and only if there exists a sequence sequence $(a_n)_{n=1}^{\infty}$ consisting entirely of elements in X, which converges to x.

solution: If x is an adherent point then the set $U = \{y \in X : |x - y| < \epsilon$, for some $\epsilon = \epsilon_0 > 0$ is not empty. So, by the axiom of choice that there exist a sequence which converges to x for any X.

Exercise 9.1.6. Let X be a subset of R. Show that $\bar{X}$ is closed (i.e., $\bar{\bar{X}} = \bar{X}$). Furthermore, show that if Y is any closed set that contains X, then Y also contains $\bar{X}$. Thus the Closure $\bar{X}$ of X is the smallest closed set which contains X.

solution: If $x \in \bar{\bar{X}}$, then for any $\epsilon_0 > 0$ there exists some $y \in \bar{X}$ such that $|x - y| < \epsilon_0$. Since $y \in \bar{X}$, then for any $\epsilon_1 > 0$ there exists some $u \in X$ such that $|y - u| < \epsilon_1$. Now, $|x - u| = |x - y + y - u| \leq |x - y| + |y - u| < \epsilon$. Which means that for any $\epsilon > 0$ there exists some $u \in X$ such that $|x - u| < \epsilon$ which by definition means that $x \in \bar{X}$.

Exercise 9.1.7. Let $n \geq 1$ be a positive integer, and let $X_1, ..., X_n$ be closed subsets of R. Show that $X_1 \cup X_2 \cup ... \cup X_n$ is also closed.

solution: By induction if $n = 1$, then X_1 is closed. Suppose that if $X_1, ..., X_{n-1}$ are closed then $X_1 \cup X_2 \cup ... \cup X_{n-1}$ are closed.

$X \bar{\cup} Y = \bar{X} \cup \bar{Y}$. $\bar{\bar{}}X \cup Y = \bar{X} \bar{\cup} \bar{Y} = \bar{\bar{X}} \cup \bar{\bar{Y}} = \bar{X} \cup \bar{Y} = X\ \bar{c}upY$. Where $X = X_1 \cup X_2 \cup ... \cup X_{n-1}$ and $Y = X_n$

Exercise 9.1.8. Let I be a set, and for each $\alpha \in I$ let X_α be a closed subset of R. Show that the intersection $\cap_{\alpha \in I} X_\alpha$ is also closed.

solution: The argument should be similar to the argument of the previous exercise. However, I am not so sure how to handle the infinite case.

Exercise 9.1.9. Let X be a subset of the real line, and x be a real number. Show that every adherent point of X is either a limit point or an isolated point of X, but cannot be both. Conversely, show that every limit point and every isolated point of X is an adherent point of X.

solution: We know that x is an adherent point iff there exists a sequence consisting of elements in X, which converge to x.

If x is an adherent point then for every $\epsilon > 0$ there exists $y \in X$ such that $|x - y| < \epsilon$. Suppose, $y \neq x$, then for every $\epsilon > 0$ there exists $y \in X \setminus \{x\}$ such that $|x - y| < \epsilon$. Hence, by definition x is a limit point. Now, suppose that $y = x$ is the only point such that for every $\epsilon > 0$ there exists $y \in X$ such that $|x - y| < \epsilon$. Therefore, there exists some $\epsilon > 0$ such that $|x - y| > \epsilon$ for all $y \in X \setminus \{x\}$. If x is a limit point then it is an adherent point of $X \setminus \{x\}$ and hence an adherent point of X. If x is an isolated point of X then it is an adherent point of X because for every $\epsilon > 0$ there exists some $y \in X$, namely $y = x$ such that $|x - y| < \epsilon$.

Exercise 9.1.10. If X is a non-empty subset of R, show that X is bounded if and only if $inf(X)$ and $sup(X)$ are finite.

solution: Let X be bounded. Then by the definition we know that $X \subset [-M, M]$, for some real number $M > 0$. Which means that the $sup(X)$ i.e. the least upper bound of X is less than or equal to M and $inf(X)$ i.e. the biggest lower bound is bigger than or equal to $-M$. Therefore, they are both finite. Now suppose that $inf(X)$ and $sup(X)$ are finite. Let $M = max\{inf(X), sup(X)\}$. Now by definition we know that $|sup(X)| \leq M$, so we found an upper bound. Moreover, $|inf(X)| \leq M$, If $inf(X) < 0$, then $-inf(X) \leq M$ or $inf(X) \geq -M$ so we found a lower bound too. However, if $inf(X) \geq 0$, then $inf(X) \leq M$ or $-inf(X) \geq -M$ and since $inf(X) \geq -inf(X) \geq -M$, then $-M$ is the lower bound, so X is bounded.

Exercise 9.1.11. Show that if X is a bounded subset of R, then the closure $\bar{X}$ is also bounded.

solution: Let $X \in [-M, M]$. We aslo know that $X \subset \bar{X}$. Let's proceed with contradiction. Suppose $\bar{X}$ is not bounded. Then let's take $x = sup(\bar{X})$. As we know by the preceeding exercise x is not finite. Therefore there for any $\epsilon > 0$ there does not exsit some $y \in X$ such that $|x - y| < \epsilon$. Or since $\bar{X}$ is not bounded then there exists an element $x > M$. However, for such an element for any $\epsilon > 0$ there doesn't exist some $y \in X$ such that $|x - y| < \epsilon$.

Exercise 9.1.12. Show that the union of any finite collection of bounded subsets of R is still a bounded set. Is this conclusion still true if one takes an infinite collection of bounded subsets of R?

solution: In case of finite collection we can take $M = max(M_1, M_2, ..., M_n)$ and then the union will be bounded. However, I am not so sure about the infinite case.

Exercise 9.1.13. Let X be a subset of R. Then the following two statements are equivalent: (a) X is closed and bounded. (b) Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X (i.e., $a_n \in X$ for all n), there exists a subsequence $(a_nj)_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.

solution: Following the hints: $a \to b$: Since X is closed there exists a sequence of elements in X that are converging to an element that also lies in X. Moreover, since the X is bounded, i.e. the sequence is bounded then by Bolzano-Weierstrass there is at least one subsequence of the original sequence which converges. $b \to a$: Suppose X is not closed, then there doesn't exsist a sequence of elements of X that converge to an element that lies in X. Hence, we have got a contradiction. X is closed.

Exercise 9.1.15. Let E be a bounded subset of R, and let $S := \sup(E) betheleastupperboundof$E.$(Notefro$

Exercise solution:. Suppose S is not an adherent point of E. Then for every $\epsilon > 0$ there does not exists some $y \in E$ such that $|S - y| < \epsilon$, or in other words there exists $\epsilon_0 > 0$ such that for all $y \in E$, $|S - y| \geq \epsilon_0$. However, saying that $|S - \epsilon_0 - y| < \epsilon$ for any $\epsilon > 0$, so S is not the supremum.

Exercise 9.2.1. Let $f : R \to R$, $g : R \to R$, $h : R \to R$. Which of the following identities are true, and which ones are false? In the former case, give a proof; in the latter case, give a counterexample.

solution: Since it was mentioned that

$$(f+g)ohx = (f+g)(h(x)) = f(h(x)) + g(h(x)) = (foh)(x) + (goh)(x)$$

$$fo(g+h)x = f((g+h)(x)) = f(g(x)+h(x)) \neq (fog)(x) + (foh)(x)$$

For example, $x^2o(x+\sqrt{x})(4) = ((x+\sqrt{x})(4))^2 = (4+2)^2 = 36 \neq 4^2+2^2 = 16+4 = 20$

$$(f+g)h(x) = (f+g)(x)h(x) = (f(x)+g(x))h(x) = f(x)h(x) + g(x)h(x)$$

$$f(g+h)(x) = f(x)(g+h)(x) = f(x)(g(x)+h(x)) = f(x)g(x)+f(x)h(x) = fg(x)+fh(x)$$

Exercise 9.3.1. Prove proposition 9.3.9.

solution: f converges to L at x_0 in E, means by definition 9.3.6 that for every $\epsilon > 0$ there exists $\sigma > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in E$ such that $|x - x_0| < \sigma$. Let be any sequnece $(a_n)_{n=0}$ converging to x_0 i.e by definition 6.1.5 for every $\sigma > 0$ there exists $N \geq 0$ such that for every $n \geq N$ $|a_n - x_0| < \sigma$. Since $|f(x) - L| < \epsilon$ for all $x \in E$, such that $|x - x_0| < \sigma$ then in particular it is true that $|f(a_n) - L| < \epsilon$ since $a_n \in E$ for all n by constraction and $|a_n - x| = |a_n - x_0 + x_0 - x| \leq |a_n - x_0| + |x_0 - x| < \sigma$ Conversley, suppose for every $\sigma > 0$ there exists some $N \geq 0$ such that for all $n \geq N$ $|a_n - x| < \sigma$. Then for $\epsilon > 0$ there exists $N_1 \geq 0$ such that for all $m \geq N_1$ then $|f(a_n) - L| < \epsilon$. Suppose, that $\lim_{x \to x_0, x \in E} f(x) \not\to L$. Then there exists some $\epsilon > 0$ such that for all $\sigma > 0$ there exists some x such that $|x - x_0| < \sigma$ then $|f(x) - L| \geq \epsilon$. However, https://math.stackexchange.com/questions/3356983/how-to-prove-terence-tao-proposition-9-3-9 http://yaoyao.codes/math/2018/07/26/digest-of-terence-tao-analysis

Exercise 9.3.2. Prove proposition 9.3.14.

solution: Since x_0 is an adherent point of E, then there exists some sequence $(a_n)_{n=0}^{\infty}$ of elements E such that it converges to x_0. Since f has a limit L at $x_0 \in E$ we thus see $(f(a_n)_{n=0}^{\infty}$ converges to L. Similarly $(g(a_n)_{n=0}^{\infty}$ converges to M. Therefore by the limit laws of the sequence $(f(a_n)_{n=0}^{\infty} - (g(a_n)_{n=0}^{\infty}$ has limit $L - M$.

Exercise 9.3.3. Prove proposition 9.3.18.

solution: For every $\epsilon > 0$ there exists $\sigma > 0$ such that $|x - x_0| < \epsilon$ for $x \in E$ such that $|f(x) - L| < \epsilon$. Since $|x - x_0| < \epsilon$ and $x \in E$ then $x \in E \cup (x_0 - \sigma, x_0 + \sigma)$. Conversley, if

Exercise 9.3.5. Continuous version of squeeze test

solution: There exists a sequence of the elements of E such that $(a_n)_{n=0}^{\infty}$ it converges to x_0. Moreover, for every such sequence we know that the sequence $(f(a_n))_{n=0}^{\infty}$ and sequence $(g(a_n))_{n=0}^{\infty}$ are converging to L. Therefore, since $h(x)$

is between f and g for all $x \in E$, by the squize test of sequences we know it converges to L too.

Exercise 9.4.1. Prove the Proposition 9.4.7

solution: $a \to b$ Since f is continuous we know that the limit of f is $f(x_0)$ as $x \in X$ converges to x_0. Or according to the Proposition 9.3.9 for every sequence of the elements of X as $(a_n)_{n=0}^{\infty}$ converges to x_0, we know that $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$. $b \to c$ Since $(a_n)_{n=0}^{\infty}$ converges to x_0, we know that for any $\sigma > 0$ there exists some $N \geq 0$ such that $|a_N - x_0| < \sigma$. Also, we know that for the same sequence of elements $(a_n)_{n=0}^{\infty}$ and the same $N \geq 0$, for every $\epsilon > 0$, $|f(a_n) - f(x_0)| < \epsilon$. Then we can constrauct a contradiction and get the desired result. $c \to d$ is trivial. $d \to a$?????

Exercise 9.4.2. Let X be a subset of R, and let $c \in R$. Show that the constant function $f : X \to R$ defined by $f(x) := c$ is continuous, and show that the identity function $g : XR$ defined by $g(x) := x$ is also continuous.

solution: For every $\epsilon > 0$, there exists $\sigma > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all X such that $|x - x_0| < \sigma$, in particular since $f(x) = c$ this definition holds for all $|x - x_0| < \sigma$ because both $f(x) = f(x_0) = c$. If $f(x) = x$, then we have that For every $\epsilon > 0$, there exists $\sigma > 0$ such that $|f(x) - f(x_0)| = |x - x_0| < \epsilon$ for all X such that $|x - x_0| < \sigma$, in particular take $\sigma = \epsilon$.

Exercise 9.4.3. Prove Proposition 9.4.10. (Hint: you can use Lemma 6.5.3, combined with the squeeze test (Corollary 6.4.14) and Proposition 6.7.3.)

Let $a > 0$ be a positive real number. Then the function $f : R \to R$ defined by $f(x) = a^x$ is continuous

solution: So, we got to show that for every $\epsilon > 0$ there exists $\sigma > 0$ such that $|a^x - a^{x_0}| < \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$.

We can notice that $|a^x - a^{x_0}| = |a^x||a^{x-x_0} - 1|$, therefore it is sufficient to prove that a^x converges to 1 as x converges to 0. Let's do this by contradiction, suppose that there exists $\epsilon > 0$ such that for all $\sigma > 0$ such that $|a^x - a^{x_0}| > \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$.

Then since $|x - x_0| < \sigma < 1/n$, so we know that $a^{-1/n} < a^{x-x_0} < a^{1/n}$. Then by the Lemma 6.5.3, combined with the squeeze test (Corollary 6.4.14) we have that $\lim_{n\to\infty} a^{x-x_0} = 1$ or in other words for all $\epsilon > 0$ there exists $N \geq 0$ such that $|a_N^{x-x_0} - 1| = |a^{x_N - x_0} - 1| = |a^{x_N}/a^{x_0} - 1| = |a^{x_N} - a^{x_0}| < \epsilon$ by the Proposition 6.7.3. Therefore we got a contradiction.

Exercise 9.4.4. Prove Proposition 9.4.11. (Hint: from limit laws (Proposition 9.3.14) one can show that $lim_{x\to1} x^n = 1$ for all integers n. From this and the squeeze test (Corollary 6.4.14) deduce that $lim_{x\to1} x^p = 1$ for all real numbers p. Finally, apply Proposition 6.7.3)

solution: By the Proposition 9.3.14 we can use the multiplication part. We already know that $lim_{x\to1} x = 1$. And we know that $lim_{x\to1} x^2 = (lim_{x\to1} x) * (lim_{x\to1} x) = 1$, and by induction it can be shown that this relationship holds for all $n > 0$. The same way using the limit of the division of the functions we can have the $lim_{x\to1} x^n = 1$ for all $n < 0$. If $n = 0$, then $x = 1$ and by the continuity of constant. $lim_{x\to1} x^p = 1$. We know that $x^{-n} < x^p < x^n$, so that x converges to 1 by the squize test. We want to show that $lim_{x\to a} x^p = a^p$, but

we just showed that $lim_{x \to a}(x/a)^p = 1$, then using the limit laws we know that $lim_{x \to a}x^p = a^p$ holds indeed.
Exercise 9.4.5. Let X and Y be subsets of R, and let $f : X \to Y$ and $g : Y \to R$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $gf : X \to R$ is continuous at x_0.

solution: We want to show that for every $\epsilon > 0$ there exists $\sigma > 0$ such that $|g(f(x)) - g(f(x_0))| < \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$. We know that $\epsilon_0 > 0$ there exists $\sigma_0 > 0$ such that $|f(x) - f(x_0)| < \epsilon_0$ for all $x \in X$ such that $|x - x_0| < \sigma_0$. Also, $\epsilon_1 > 0$ there exists $\sigma_1 > 0$ such that $|g(f(x)) - g(f(x_0))| < \epsilon_1$ for all $f(x) \in X$ such that $|f(x) - f(x_0)| < \sigma_1$. But then combining these two statements we have that $\epsilon > 0$ there exists $\sigma > 0$ such that $|g(f(x)) - g(f(x_0))| < \epsilon$ for all $f(x) \in X$ such that $|f(x) - f(x_0)| < \sigma$. Since for every such $|f(x) - f(x_0)| < \sigma$ there is a σ_0 such that $|x - x_0| < \sigma_0$.
Exercise 9.4.6. Let X be a subset of R, and let $f : X \to R$ be a continuous function. If Y is a subset of X, show that the restriction $f|Y : YR$ of f to Y is also a continuous function. (Hint: this is a simple result, but it requires you to follow the definitions carefully.)

solution: We know that for every $\epsilon > 0$ there exists $\sigma > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$. Since this is true for all $x \in X$, then in particular it is true for $x \in Y$, therefore, the restriction function is continuous.
Exercise 9.4.7. Show that P is continuous.

solution: We already know that x^p is continuous and using the limit laws we can deduce that the polynomial is continuous too.
Exercise 9.5.1. Let E be a subset of R, let $f : E \to R$ be a function, and let x_0 be an adherent point of E. Write down a definition of what it would mean for the limit $lim_{x \to x_0; x \in E} f(x)$ to exist and equal $+\infty$ or $-\infty$. If $f : R\{0\} \to R$ is the function $f(x) := 1/x$, use your definition to conclude $f(0+) = +$ and $f(0) = \infty$. Also, state and prove some analogue of Proposition 9.3.9 when $L = +\infty or L = \infty$.

solution: $+\infty$: For every $\epsilon > 0$ there exists a $\sigma > 0$ such that $f(x) > \epsilon$ for all $x \in E$ as $|x - x_0| < \sigma$. $-\infty$: For every $\epsilon > 0$ there exists a $\sigma > 0$ such that $f(x) > -\epsilon$ for all $x \in E$ as $|x - x_0| < \sigma$.

Now we should show for every $\epsilon > 0$ there exists a $\sigma > 0$ such that $f(x) > \epsilon$ for all $x \in E \cap (0, \infty)$ as $|x| < \sigma$. Let's proceed by contradiction. For all $\sigma > 0$ there exists $\epsilon > 0$ $f(x) < \epsilon$ for some $x \in E \cap (0, \infty)$ as $|x| < \sigma$. Since $|x| < \sigma$ and $x \in E \cap (0, \infty)$, then $1/x > \sigma$ for all $\sigma > 0$, so a contradiction. For the left limit. For all $\sigma > 0$ there exists $\epsilon > 0$ $f(x) < -\epsilon$ for some $x \in E \cap (0, \infty)$ as $|x| < \sigma$. Since $|x| < \sigma$ and $x \in E \cap (-\infty, 0)$, then $-x < \sigma, x > -\sigma, 1/x < -1/\sigma$ for all $\sigma > 0$, so a contradiction.

That means that as $a_n \in E$ and $a_n > x$ as $(a_n)_{n=0}^{\infty}$ converges to x_0, then $(f(a_n))_{n=0}^{\infty}$ diverges. In case of $-\infty$ we have that as $a_n < x$, $(a_n)_{n=0}^{\infty}$ converges to x_0, then $(f(a_n))_{n=0}^{\infty}$ diverges.
Exercise 9.6.1.

solution: a) The interval is open and the definition speaks about the closed interval. b) The interval is half open. c) There is no word about continouty. d)

The function can be discontious at the endpoints of the interval.

Exercise 9.7.1. Prove Corollary 9.7.4 Let $a < b$, and let $f : [a, b] \to R$ be a continuous function on $[a, b]$. Let $M := sup_{x[a,b]} f(x)$ be the maximum value of f, and let $m := inf_{x[a,b]} f(x)$ be the minimum value. Let y be a real number between m and M (i.e., myM). Then there exists a $c \in [a, b]$ such that $f(c) = y$. Furthermore, we have $f([a, b]) = [m, M]$.

solution: We know that by the intermediate value theorem if we let $a < b$, and let $f : [a, b] \to R$ be a continuous function on $[a, b]$ then for every y between $f(a), f(b)$ there exists some $c \in [a, b]$ such that $f(c) = y$. Moreover, we know that such f attains its maximum and minimum values at some $x_max, x_min \in [a, b]$. Therefore, f is continuous at x_max, x_min and we can use the intermediate value theorem to conclude that for every y between m, M there exists some $c \in [a, b]$ such that $f(c) = y$

Exercise 9.8.1. Explain why the maximum principle remains true if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone.

solution: Let the function $f : [a, b] \to R$ be a monotone function and let $a < b$. Then we should prove that f attains at some $x_max \in [a, b]$ its maximum value and at some $x_min \in [a, b]$ its minimum value.

Suppose f is a monotone increasing function then we have that for all $x, y \in [a, b]$ if $x < y$ then $f(x) < f(y)$. Since $[a, b]$ is a bounded set then the sequence $a \leq b \leq a_1, ..., \leq a_n \in [a, b]$ is bounded and monotone increasing. Therefore, we know that it is convergent to the $sup[a, b] = a_n$, and since $a_n > x$ for all $x \in [a, b]$ and $a_n \in [a, b]$ then $f(a_n) > f(x)$ for all $x \in [a, b]$. Therefore, it is the maximum of the f.

Exercise 9.8.2. Give an example to show that the intermediate value theorem becomes false if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone.

solution: Take the example from the book of the picewise function of f.

Exercise 9.8.3. Let $a < b$ be real numbers, and let $f : [a, b] \to R$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (Hint: divide into the three cases $f(a) < f(b)$, $f(a) = f(b)$, $f(a) > f(b)$. The second case leads directly to a contradiction. In the first case, use contradiction and the intermediate value theorem to show that f is strictly monotone increasing; in the third case, argue similarly to show f is strictly monotone decreasing.

solution: Let $f(a) = f(b)$ since f is continuous and one-to-one we know that $lim_{x \to a} f(x) = f(a) = f(b) = lim_{x \to b}$, then to preserve one-to-one property we should have $a = b$ and f in a single point interval is monotone, otherwise there would exists some $x \in [a, b]$ such that $x < b$ but $f(x) > f(b)$, which is not the case. Let $f(a) < f(b)$. Show that for all $c \in (a, b)$, $f(a) < f(c) < f(b)$. Suppose not, then either $f(c) < f(a) < f(b)$ or $f(b) < f(a) < f(c)$. Suppose, the first is the case than there by the IVT there is some $x \in [c, b]$ such that $f(x) = f(a)$ but f is also one-to-one, so there can not be such x, a contradiction.

Exercise 9.8.4. Let $a < b$ be real numbers, and let $f : [a, b] \to R$ be a function which is both continuous and strictly monotone increasing. Then f

is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse $f^1 : [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing. (Hint: to show that f$^1 iscontinuous, itiseasiesttousethe\backslash epsilon - delta" definitionofcontinuity$).

solution: Since f is continuous by the intermediate value theorem we know that it is subjective and also by the strict monotone increasing nature we know that it is invective, otherwise, there would be some $f(x) = f(y)$ such that $x \neq y$ that is $x < y$ or $y < x$. In either case, we would get a contradiction that f is strictly monotone increasing. Now, let's use the given hint. If $f^{-1} : [f(a), f(b)] \to [a, b]$ is continuous then for every $\epsilon > 0$ there exists some $\sigma > 0$ such that $|f^{-1}f(x) - f^{-1}f(x_0)| = |x - x_0| < \epsilon$ for all $f(x) \in [f(a), f(b)]$ with $|f(x) - f(x_0)| < \sigma$. Suppose, it is not the case then there exists $\epsilon > 0$ such that for all $\sigma > 0$, $|f^{-1}f(x) - f^{-1}f(x_0)| = |x - x_0| > \epsilon$ for some $f(x) \in [f(a), f(b)]$ with $|f(x) - f(x_0)| < \sigma$. However, this will contradict to the fact that f is continuous, therefore, it is not possible and f^{-1} is continous.

Exercise 9.9.1. Prove Lemma 9.9.7. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers (not necessarily bounded or convergent). Then $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent if and only if $lim_{n\to\infty}(a_n b_n) = 0$.

solution: Suppose, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent, then by the definition for every $\epsilon > 0$ there exists a $N \geq 1$ such that $|a_n - b_n| < \epsilon$, which is exactly the definition of $lim_{n\to\infty}(a_n b_n) = 0$.

Exercise 9.9.2. Prove Proposition 9.9.8. (Hint: you should avoid Lemma 9.9.7, and instead go back to the definition of equivalent sequences in Definition 9.9.5.)

solution: $(a \to b)$ If $f : X \to R$ is uniformly continuous, then for every $\epsilon > 0$, there exists a $\sigma > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$. Since f is uniformly continuous we can take $\sigma > 0$ such that $|x - y| < \sigma$ and $|f(x) - f(y)| < \epsilon$ for every $\epsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are equivalent, i.e. for every $\epsilon_0 > 0$ there exists some $N \geq 0$ such that for all $n \geq N$ $|x_n - y_n| < \epsilon_0$, then in particular this holds for $\epsilon_0 = \sigma$. So, $(f(x_n))_{n=1}^{\infty}$ and $(f(y_n))_{n=1}^{\infty}$ are equivalent. $(b \to a)$ Since for all $\sigma > 0$ there exists some $N \geq 0$ such that for all $n \geq N$ $|x_n - y_n| < \sigma$, then for all $\epsilon > 0$ there exists some $M \geq 0$ such that for all $m \geq M$ $|f(x_m) - f(y_m)| < \epsilon$, then in particular for some fixed $\sigma > 0$ there exists $K = max\{N, M\}$ such that for all for all $k \geq K$ and for all $\epsilon > 0$ $|f(x_k) - f(y_k)| < \epsilon$ $m \geq M$, and $|x_k - y_k| < \sigma$. Therefore, f is uniformly continuous.

Exercise 9.9.3. Prove Proposition 9.9.12. (Hint: use Definition 9.9.2 directly.) Let X be a subset of R, and let $f : X \to R$ be a uniformly continuous function. Let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence consisting entirely of elements in X. Then $(f(x_n))_{n=0}^{\infty}$ is also a Cauchy sequence.

solution: Suppose that $(f(x_n))_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists some $\sigma > 0$ such that for all $N > 0$ and $i, j \geq N$, $|x_i - x_j| < \sigma$ yet for all $\epsilon > 0$, for all $M > 0$ and $k, l > M$, $|f(x_k) - f(x_l)| \geq \epsilon$. In particular we can take $i, j > R$, where $R = max\{N, M\}$, then we would get a contradiction, since it will contradict to the fact that f is a uniformly continuous function.

Exercise 9.9.4. Use Proposition 9.9.12 to prove Corollary 9.9.14. Use this corollary to give an alternate demonstration of the results in Example 9.9.10.

solution: Since x_0 is an adherent point then there exists a sequence of elements of X i.e $(x_n)_{n=0}^{\infty}$ a Cauchy, convergent sequence. Then f is uniform continuous, so it is $(f(a_n))$ is Cauchy too.

Exercise 9.9.5. Prove Proposition 9.9.15. (Hint: mimic the proof of Lemma 9.6.3. At some point you will need either Proposition 9.9.12 or Corollary 9.9.14.) Let X be a subset of R, and let $f : X \to R$ be a uniformly continuous function. Suppose that E is a bounded subset of X. Then $f(E)$ is also bounded.

solution: Let $E = (a, b)$, and $a < b$. Suppose, $f(E)$ is not bounded, that is for every real number M there exists an element x from E, such that $|f(x)| \geq M$. In particular, for every $n \in N$, the set $\{x \in E : |f(x)| \geq n\}$ is non-empty. Therefore, we can choose a sequence $(x_{n_j})^{j=0\infty}$ which converges to some limit L, since the sequence is convergent it is a Cauchy sequence, and therefore, $(f(x_{n_j}))^{j=0\infty}$ is also a Cauchy seuqence, i.e. a convergent one. Thus, it is bounded. However, we had that by the construction $|f(x_{nj})| \geq n_j \geq j$, and hence is not bounded, a contradiction.

Exercise 9.10.1. . Let (an) n=0 be a sequence of real numbers, then an can also be thought of as a function from N to R, which takes each natural number n to a real number an. Show that lim n+;nN an = limn an where the left-hand limit is defined by Definition 9.10.3 and the right-hand limit is defined by Definition 6.1.8. More precisely, show that if one of the above two limits exists then so does the other, and then they both have the same value. Thus the two notions of limit here are compatible.

solution: Suppose, that the left-handed limit exists. Then according to the definition for every $\epsilon > 0$, there exsists an N such that $|a_n - L| \leq \epsilon$ for all $n \in N$ such that $n > N$. However, this is also the exact definition of the right-handed limit, so it exists if the left-handed exists and vice versa. Suppose for all $\epsilon > 0$ $|a_n - M| \leq \epsilon$ for all $n \geq K$, then $|L - M| \leq |a_n - L| + |M - a_n| < \epsilon$, so the limits are equal too.

Exercise 10.1.1. Suppose that X is a subset of R, x_0 is a limit point of X, and $f : X \to R$ is a function which is differentiable at x_0. Let $Y \subset X$ be such that $x_0 \in Y$, and x_0 is also a limit point of Y . Prove that the restricted function $f|Y : Y \to R$ is also differentiable at x_0, and has the same derivative as f at x_0. Explain why this does not contradict the discussion in Remark 10.1.2

solution: We have that x_0 is a limit point for both X and its subset Y. Since the function is differentiable then for every $\epsilon > 0$ there exists some $\sigma > 0$ such that $|\frac{f(x)-f(x_0)}{x-x_0} - L| < \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$. Since the definition is true for all $x \in X$, then it holds in particular for all $x \in Y$. Thus, we have that $f(x)$ is differentiable in Y too. Moreover, it converges to the same point L, therefore the derivative doesn't change.

Exercise 10.1.2. Prove Proposition 10.1.7. (Hint: the cases $x = x_0$ and $x \neq x_0$ have to be treated separately.)

Let X be a subset of R, let $x_0 \in X$ be a limit point of X, let $f : X \to R$ be a function, and let L be a real number. Then the following statements are logically equivalent:

(a) f is differentiable at x_0 on X with derivative L.

(b) For every $\epsilon > 0$, there exists a $\sigma > 0$ such that $f(x)$ is $\epsilon|xx_0|$- close to $f(x_0) + L(xx_0)$ whenever $x \in X$ is σ-close to x_0, i.e., we have $|f(x)(f(x_0) + L(xx_0))| \leq \epsilon|xx_0|$ whenever $x \in X$ and $|xx_0| \leq \sigma$.

solution: $a \to b$ we know that for every $\epsilon > 0$, there exists $\sigma > 0$ such that $|\frac{f(x)-f(x_0)}{x-x_0} - L| < \epsilon$ for all $x \in X$ such that $|x - x_0| < \sigma$. In particular if $x \neq x_0$ and $|x - x_0| < \sigma$, then $|\frac{f(x)-f(x_0)}{x-x_0} - L| = |f(x)(f(x_0) + L(xx_0))| < \epsilon$, so the theroem is true. If $x = x_0$, then for every $\epsilon > 0$ there is a σ such that $|f(x) - f(x) - L(x - x)| = |0| < \epsilon$ for $|x - x_0| = |x - x| = |0| < \sigma$

$b \to a$ for $x \neq x_0$ it is trivial. For $x = x_0$, then the limit is undefined as the definition says that x should be a limit point of X, i.e. the adherent point of Xx_0.

Exercise 10.1.3. Prove Proposition 10.1.10. (Hint: either use the limit laws (Proposition 9.3.14), or use Proposition 10.1.7) Let X be a subset of R, let $x_0 \in X$ be a limit point of X, and let $f : X \to R$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.

solution: If f is differentiable at x_0, then by the Newton-law for every $\epsilon > 0$ there exists $\sigma > 0$ such that $|f(x)(f(x_0) + L(xx_0))| \leq \epsilon|xx_0|$ whenever $x \in X$ and $|xx_0| \leq \sigma$. So, for every $\epsilon > 0$ and the given $\sigma > 0$ such that $|x-x_0| < \sigma$ we have that $|f(x) - f(x_0)| \leq |f(x)(f(x_0) + L(xx_0))| \leq \epsilon|xx_0| \leq \epsilon\sigma$. Now taking $\epsilon_0 = \epsilon\sigma$, we would get that for every $\epsilon_0 = \epsilon\sigma > 0$ there exists a $\sigma = \epsilon/\epsilon_0 > 0$ such that $|f(x) - f(x_0)| \leq \epsilon_0$ for all $x \in X$ such that $|x - x_0| \leq \sigma$.

Exercise 10.1.4. Prove Theorem 10.1.13. (Hint: use the limit laws in Proposition 9.3.14. Use earlier parts of this theorem to prove the latter. For the product rule, use the identity $f(x)g(x)f(x_0)g(x_0) = f(x)g(x)f(x)g(x_0)+f(x)g(x_0)f(x_0)g(x_0) = f(x)(g(x)g(x_0)) + (f(x)f(x_0))g(x_0)$; this trick of adding and subtracting an intermediate term is sometimes known as the "middle-man trick" and is very useful in analysis.)

solution: a) If f is a constant function, i.e., there exists a real number c such that $f(x) = c$ for all $x \in X$, then $\lim 0/(x - x_0) = 0$.

b) If $f(x) = x$ is, then $\lim x - x_0/(x - x_0) = lim1 = 1$

c) If $\lim((f + g)(x) - (f + g)(x_0))/(x - x_0) = \lim f(x) - f(x_0)/(x - x_0) + \lim g(x) - g(x_0)/(x - x_0) = f'(x_0) + g'(x_0)$

d) $\lim(fg(x)-fg(x_0))/(x-x_0) = \lim(f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)f(x_0)g(x_0))/(x-x_0) = \lim f(x)(g(x)g(x_0))/(x-x_0)+\lim(f(x)f(x_0))g(x_0)/(x-x_0) = f(x_0)g'(x_0)+g(x_0)f'(x_0)$

f) $\lim((f-g)x-(f-g)x_0)/(x-x_0) = \lim(f(x)-g(x)-f(x_0)+g(x_0))/(x-x_0) = \lim(f(x) - f(x_0))/(x - x_0) + \lim(g(x) - g(x_0))/(x - x_0) = f'(x_0) + g'(x_0)$?????

g) $\lim \frac{1/g(x)-1/g(x_0)}{x-x_0} = \lim \frac{(g(x_0)-g(x))/g(x)g(x_0)}{x-x_0} = \lim \frac{(g(x_0)-g(x))}{g(x)g(x_0)(x-x_0)} = -\frac{1}{g(x_0)} \frac{g(x)-g(x_0)}{g(x)(x-x_0)} = -\frac{g'(x_0)}{g(x_0)^2}$

Exercise 10.1.5. Let n be a natural number, and let $f : R \to R$ be the function $f(x) := x^n$. Show that f is differentiable on R and $f'(x) = nx^{n1}$ for all $x \in R$. (Hint: use Theorem 10.1.13 and induction.)

solution: Induct on n. Base case: $n = 1$, then according to the Theorem

10.1.13 point b $f'(x) = 1 = 1 * x^0$. Inductive step: Let $f(x) = x^{n-1}$ and $f'(x) = n - 1x^{n-2}$. Now, take $f(x) = x^n = x * x^{n-1}$, $f'(x) = (x * x^{n-1})' = x^{n-1} + n - 1x^{n-2} * x = n * x^{n-1}$ according to the Theorem 10.1.13 point d.

Exercise 10.1.6. Let n be a negative integer, and let $f : R0 \to R$ be the function $f(x) := x^n$. Show that f is differentiable on R and $f'(x) = n * x^{n1}$ for all $x \in R\{0\}$.

solution: If $n \geq 0$ then by the previous exercise the statement holds. If $n < 0$, i.e. $f(x) = x^n$, Let $m = -n$ then $f(x) = x^n = x^{m-2m}$

Exercise 10.1.7. Prove Theorem 10.1.15. (Hint: one way to do this is via Newton's approximation, Proposition 10.1.7. Another way is to use Proposition 9.3.9 and Proposition 10.1.10 to convert this problem into one involving limits of sequences, however with the latter strategy one has to treat the case $f'(x_0) = 0$ separately, as some division-by-zero subtleties can occur in that case.)

solution:

Exercise 10.2.1.

solution: Let f attain a local maximum at x_0. Since f is differentiable then for any $\epsilon > 0$ such that $|\frac{f(x)-f(x_0)}{x-x_0} - L| < \epsilon$ there exists some $\sigma > 0$ such that $|x - x_0| < \sigma$. Now lets take $x \in (x_0 - \sigma, x_0)$, then $\frac{f(x)-f(x_0)}{x-x_0} > 0$ since $f(x) < f(x_0)$. Now let $x \in (x_0, x_0 + \sigma)$ then $\frac{f(x)-f(x_0)}{x-x_0} < 0$.

Exercise 10.2.2. Give an example of a function $f : (1, 1) \to R$ which is continuous and attains a global maximum at 0, but which is not differentiable at 0. Explain why this does not contradict Proposition 10.2.6

solution: Take $f(x) = -|x|$. It does not contradict the Proposition 10.2.6 because the proposition speaks about the differentiable functions and $f(x)$ is not differentiable at 0.

Exercise 10.2.4. Prove Theorem 10.2.7.

solution: Since f is continuous at $[a, b]$ according to the Proposition 9.6.7 it attains its maximum/minimum at some points $x_max, x_min \in [a, b]$. Suppose, the maximum/minimum are attained at the endpoints. Since $f(a) = f(b)$ and the function is contiuous then f is just a constant function. Therefore, $f'(a) = f'(b) = 0$. Suppose that they are not in endpoints, then according to the Proposition 10.2.6, at the maximum/minimum the derivative of the function is equal to 0.

Exercise 10.2.5. Use Theorem 10.2.7 to prove Corollary 10.2.9.

solution: Lets try to apply the Roll's theorem. Lets take $g(x) = f(x) - cx$ for some c. The goal is to take c such that the $g(a) = g(b)$ so that we can use the Roll's theorem. $g(a) = f(a) - ca = f(b) - cb$, $cb - ca = c(b - a) = f(b) - f(a)$ so $c = \frac{f(b)-f(a)}{b-a}$. Now we according to the Roll's theorem we know that $g(x) = f(x) - \frac{f(b)-f(a)}{b-a} * x$ has some point in $(a.b)$ such that $g'(x) = 0$, therefore $f'(x) = \frac{f(b)-f(a)}{b-a}$

Exercise 10.2.6. Let $M > 0$, and let $f : [a, b]R$ be a function which is continuous on $[a, b]anddifferentiableon(a, b), andsuchthat$—f (x)— $\leq M$ for all $x \in (a, b)(i.e., thederivativeoffisbounded).Showthatforany$x, y $\in [a, b]wehavetheinequality$—f (x) f (y)— $\leq M|xy|$. (Hint: apply the mean value theorem (Corollary 10.2.9) to a suitable restriction of f .) Functions which obey the bound $|f(x)f(y)| \leq M|xy|$ are known as Lipschitz continuous functions with Lipschitz constant M; thus this exercise shows that functions with bounded derivative are Lipschitz continuous.

solution: ToDo: Show why?

Since the f is contiouns on $[a, b]$, then it is contious in any subset $X \subset [a, b]$, in particual $X = [x, y]$ for any $x, y \in [a, b]$ such that $x < y$. Since f is differentiable on (a, b) then it is differentiable on every subset of (a, b). Then we can apply the MVT, to get $|f'(x)| = |\frac{f(x)-f(y)}{x-y}| \leq M$ since for any $[x, y] \subset [a, b]$ the MVT conditions hold and we can find some x_0 such that $f'(x_0) = \frac{f(x)-f(y)}{x-y}$.

Exercise 10.3.1. Let X be a subset of R, let $x_0 \in X$ be a limit point of X, and let $f : X \to R$ be a function. If f is monotone increasing and f is differentiable at x_0, then $f'(x_0) \geq 0$. If f is monotone decreasing and f is differentiable at x_0, then $f'(x_0) \leq 0$.

solution: By definition since f is differentiable at x_0 then $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = L$ exsists. Moreover, since f is monotone increasing then for any $x > x_0$ we have that $f(x) \geq f(x_0)$.

Exercise 10.3.2. Give an example of a function $f : (1, 1) \to R$ which is continuous and monotone increasing, but which is not differentiable at 0. Explain why this does not contradict Proposition 10.3.1.

solution: $f(x) = |x| + 4x$. This does not contradict the previous exercise because f is not differentiable at 0.

Exercise 10.3.3. Give an example of a function $f : R \to R$ which is strictly monotone increasing and differentiable, but whose derivative at 0 is zero. Explain why this does not contradict Proposition 10.3.1 or Proposition 10.3.3. (Hint: look at Exercise 10.2.3.)

solution: Take $f(x) = x^3$, since $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} x^3/x = \lim_{x \to 0} x^2 = 0$.

Exercise 10.3.4. Let $a < b$, and let $f : [a, b] \to R$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a, b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant function.

solution: $f'(x) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} > 0$ for all $x, x_0 \in [a, b]$, hence for all $x > x_0$, that $f(x) > f(x_0)$, therefore, f is strictly monotone increasing. Or via MVT $f'(x) = \frac{f(b)-f(a)}{b-a}$ for some $x \in [a, b]$. Suppose that f is not strictly monotone increasing that is for some $x_0, x_1 \in [a, b]$ such that $x_1 > x_0$, $f(x_1) \leq f(x_0)$, but then by the MVT there is some $x \in [x_0, x_1]$ such that $f'(x) = \frac{f(x_0)-f(x_1)}{x_0-x_1} \leq 0$, a contradiction. The same way for the $f'(x) < 0$ case.

Exercise 10.3.5. Give an example of a subset $X \subset R$ and a function $f : X \to R$ which is differentiable on X, is such that $f'(x) > 0$ for all $x \in X$, but f is not strictly monotone increasing.

solution: Let $X = [0, \infty)$, $f(x) = |x|$, then $f'(x) > 0$ for all $x \in X$, however, f is not strictly monotone increasing function in general.

Exercise 10.4.1. Let $n \geq 1$ be a natural number, and let $g : (0, \infty)(0, \infty)$ be the function $g(x) := x^{1/n}$

solution: a) Show that $\lim_{x\to x_0} g(x) = g(x_0)$. Following the hint, we can show from the Proposition 9.4.9 that $f(x) = x^n$ is continous by induction. Moreover, for $n \geq 1$, we have that if $x > x_0$, then $f(x) = x^n > x_0^n = f(x_0)$ is a monotone increasing function.

from the Proposition 9.8.3, we can say that if f is both monotone and continous on $(0, \infty)$ then g is too monotone and continuous on $(0, \infty)$.

b) From the continuouty of g and the inverse function theorem we get the desired result.

Exercise 10.4.2. Let q be a rational number, and let $f : (0, \infty) \to R$ be the function $f(x) = x^q$.

solution: Let $q = m/n$ for $m, n \in N$, and $m, n \geq 1$ then $f(x) = x^q = (x^m)^{1/n}$,

From the chain rule we get $f'(x) = 1/n(x^m)^{1/n-1}mx^{m-1} = m/nx^{m/n-m+m-1} = m/nx^{m/n-1} = qx^{q-1}$

b) $f'(1) = \lim_{x\to 1} \frac{f(x)-f(x_0)}{x-x_0} = q1^{q-1} = q$ from the definition of the limit and the a.

Exercise 10.5.1. Prove Proposition 10.5.1. (Hint: to show that $g(x) \neq 0$ near x_0, you may wish to use Newton's approximation (Proposition 10.1.7). For the rest of the proposition, use limit laws, Proposition 9.3.14.)

solution: Suppose, that for all $\sigma > 0$, we have that $g(x) = 0$ for all $x \in X \cap (x_0 - \sigma, x_0 + \sigma)\{x\}$. Now, according to the Newton's approximation we have that $g(x) = g(x_0) + g'(x_0)(x - x_0) = g'(x_0)(x - x_0)$ since $g(x_0) = 0$. Moreover, since $x \neq x_0$, and $g'(x_) \neq 0$ then for at least one $x \in X \cap (x_0 - \sigma, x_0 + \sigma)\{x\}$, $g(x) \neq 0$. Then, $\lim_{x\to x_0; x\in X\cap(x_0-\sigma,x_0+\sigma)\{x\}} \frac{f(x)}{x-x_0} / \frac{g(x)}{x-x_0} = \lim_{x\to x_0; x\in X\cap(x_0-\sigma,x_0+\sigma)\{x\}} \frac{f'(x)}{g'(x)}$ by the Proposition 9.3.14.

Exercise 10.5.2. Explain why Exercise 1.2.12 does not contradict either of the propositions in this section.

solution: As for the Proposition 10.5.1, we have that $x_0 = 0$, and $g(x_0) = 1$, therefore, it does not contradict to the Proposition 10.5.1. For the Proposition 10.5.2, there is not point $x \in R$, such that $f(x) = g(x)$, because $x = 1 + x$, gives $0 = 1$, a contradiction.