The Visible Hand: Price Discrimination under Heterogeneous Precision*

Juan Sagredo†

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Abstract

Technological innovations have allowed some sellers to collect detailed information about buyers. We study these changes in a standard search-theoretic model of imperfect competition, featuring buyers with heterogeneous private valuations for quality, and introduce sellers who observe valuation signals of heterogeneous precision. Signals induce third-degree price discrimination, and their precision largely dictates whether they are used to increase trade or increase markups - impacting aggregate surplus and its distribution. When buyers' valuations are more heterogeneous, imprecisely informed sellers prioritize high-markups despite limiting trade, and precision relaxes this tension, not only allowing them to pursue high-markups when it is least obstructive but also primarily incentivizing low-markup offers that increase trade upon signals indicative of low valuation - increasing aggregate surplus and benefiting (hurting) buyers with a low (high) valuation. However, when valuations are more homogeneous, imprecisely informed sellers prioritize trade, and precision can primarily incentivize high-markup offers that limit trade upon signals indicative of high valuation, hurting all buyers and even decreasing aggregate surplus. In either case, precision makes sellers more profitable, but its effect on competitors can be positive or negative. Generally, competitors suffer (benefit) when laggards (leaders) gain precision.

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†Economics Department, University of Pennsylvania, 133 South 36th Street. Can be reached at sagju@sas.upenn.edu.

1 Introduction

Matt Murray (Wall Street Journal, Editor in Chief): *The perception of a lot of people is that you've morphed...In a lot of ways, you're thought of as a data company more than a retail company.*
Jeff Wilke (CEO, Amazon Worldwide Consumer): *There was a corner pharmacy where I grew up. The pharmacist had been there forever. When you walked in, he knew what you liked to buy...That's the same thing we're doing. Our main purpose in storing your purchases is so that we can recommend something that you might want to buy the next time.*

-"Amazon's Defense of Private Brands" (*WSJ*, 10/24/19).

Amazon's well-documented harvesting and leveraging of consumer data exemplifies a sweeping transformation in how contemporary firms operate. A surge in the availability of data and in the power of analytical methods that uncover its insights[1] now helps firms better discern consumer preferences and tailor offers[2]. These methods have spread across industries, but diffusion within each has been heterogeneous. However, despite the importance of this pipeline, its impact on markets is not well understood.

There are several natural questions about this technological change. First, whether it benefits us in aggregate and as consumers is of immediate importance. In this vein, policymakers have raised concerns about potentially detrimental effects on competition (FTC (2012, 2013, 2014), CEA (2015), UK Competition and Markets Authority (2021)) and proactively moved towards containing them with wide-ranging measures, such as the EU's General Data Protection Regulation (European Commission (2012)). Second, how do firms benefit from investments in predictive technologies? The growing analytics gap[3] between Amazons of the world and more traditional businesses suggests that investment is profitable, but in the right context. So, what is that context? And, how does this type of investment affect competitors? To address these questions, we leverage a standard search-theoretic framework of imperfect competition, featuring buyers with heterogeneous private valuations for quality, and introduce sellers with buyer valuation signals of heterogeneous precision. In particular, sellers are differentiated by an ex-ante characteristic, their predictive skill, which is reflected in the precision of their signals. We characterize equilibria analytically, linking properties of information with properties of offers, and study the comparative statics of both precision and competition, documenting their effects on social surplus, in aggregate terms, as well as its distribution between and within buyers and sellers.

Precision fundamentally determines the impact of information on trade and the distribution of its gains. It shapes sellers' trade-off between profiting through higher sales versus higher markups, so sellers with different precisions have different offer strategies. On the one hand, sellers with more precise information are more willing to extend low-markup offers that increase sales upon signals indicative of low buyer valuations, and high-markup offers that decrease sales upon signals indicative of high buyer valuations - with opposite effects on trade efficiency. On the other hand, precision always improves the efficiency of any offer strategy because it allows sellers to extend high-markup

[1] Since 2018, the McKinsey Global Institute has conducted a yearly survey on the "State of AI", "representing the full range of regions, industries, company sizes, functional specialties, and tenures". Approximately half of all firms consistently report the adoption of AI in at least one business function, while the average number of functions has doubled since the first survey.

[2] Mikians et al. (2012, 2013), Hannak et al. (2014), Chen et al. (2015) document the pervasiveness of these methods among sellers, both large and small, while industry surveys by Deloitte (2018) and McKinsey (2023) echo these points, respectively finding widespread use of AI for personalization and that "Marketing and Sales" along with "Product/Service Development" are the most common business functions for AI applications.

[3] In McKinsey's surveys, high-performing organizations are more than three times as likely to report that their data and analytics contributed at least 20% of earnings before interest and taxes.

offers when these are least obstructive to trade (when buyers have a high willingness to pay) and low-markup offers when these are most needed to trade (when buyers have a low willingness to pay). The net of these effects is such that precision can increase or decrease the efficiency of trade. When buyers' valuations are more heterogeneous, imprecisely informed sellers pursue inefficient high-markup strategies, and precision primarily relaxes this motive; however, when buyers' valuations are more homogeneous, imprecisely informed sellers pursue efficient high-sale strategies, and precision can weaken this motive. Beyond its aggregate impact, precision is inherently redistributive, shifting not only the share of surplus between buyers and sellers, but also its distribution within each side of the market. On the demand side, high-valuation buyers - the principal targets of high markups - broadly suffer from precision, whereas low-valuation buyers - the principal victims of rationed trade - broadly benefit from it. On the supply side, additional precision improves the profitability of sellers, but its impact on competing sellers can be positive or negative. Precision relaxes competition for high-valuation buyers but intensifies competition for low-valuation ones, and either effect can dominate. Generally, we find that precision growth among laggards has negative profit externalities, whereas precision growth among leaders has positive profit externalities -benefiting all sellers.

We confirm the traditional effects of competition, but also find a new - concerning - one. Characteristically, competition increases trade efficiency and buyer surplus. However, competition and imprecision both decrease the sensitivity of sellers' offers to their signals, even becoming signal insensitive. This complementarity results in low-precision sellers being particularly prone to forego using their predictive skill, and implies that competition can exacerbate the documented disparity in firm usage of these technologies.

Before proceeding with more detailed results, it is helpful to describe the components of our model. On one side of the market, there is a unit mass of buyers with low or high private valuations for the quality of a good. On the other side, there is a unit mass of sellers who produce the good with a common production technology. We introduce imperfect competition by matching buyers and sellers in the style of Burdett and Judd (1983): each buyer matches with either one or two sellers who make simultaneous take-it-or-leave-it offers, without any information about the number of competitors in the match, beyond the commonly known matching protocol. In this reduced-form formulation, we capture the full range of competitive intensity, from monopoly to perfect competition, by varying the probability of matching with only one seller. Our innovation is the (a) introduction of additional seller information about buyers' valuations with flexible precision, and (b) allowing this ex-ante characteristic to be heterogeneous among sellers. We model this by assuming that the population distribution of buyer valuations is common knowledge but that each seller observes an informative signal about the matched buyer's valuation. Signals are essentially pairwise independent conditionally on the buyer's valuation, so they are only informative about the type of competing sellers through their information about the buyer's valuation. For our purposes, a discretization with two relative levels of precision suffices: less precisely informed *amateurs* and more precisely informed *sharks*. This allows us to capture the full range of seller valuation information, from perfect[4] to fully imperfect (only prior information), by varying the absolute level of amateur and shark precision[5].

We introduce most of the core ideas in a simpler environment, where sellers have homogeneous precision and offer an identical good of exogenously specified quality (a commodity), so that predictive skill only orients pricing, and then proceed to the general environment, where each sellers have heterogeneous precision and can also choose the quality of the goods that they offer, so that

[4]Where third-degree price discrimination becomes first-degree price discrimination.

[5]We could also study the effects of precision by varying the proportion of amateur and shark sellers, but the analysis would be analogous and less direct.

predictive skill orients pricing and production. We find that information imbues equilibrium offers with several ordering properties across both environments. Recalling the timing of this incomplete information game, sellers match with buyers, observe a valuation-relevant signal, and update their valuation beliefs. At this interim stage, a seller's type is summarized by its posterior probability that the matched buyer is of low-valuation. In equilibrium, we find that high-markup offers are more profitable in matches with high-valuation buyers, whereas high-sale offers, which share more of the gains from trade with buyers, are more profitable in matches with low-valuation buyers. As such, sellers of larger type, who place more weight on being in a low-valuation match, extend offers that are more attractive to buyers and more efficient. Precision enters the picture through its close connection with sellers' types. Precision (a) makes it more likely that a seller observes a low (high) signal in a low- (high-) valuation match, and is of relatively high (low) type, and (b) makes posterior valuation beliefs more extreme, because more precise signals trigger larger updates of sellers' valuation beliefs. Since sellers' prioritization of sales versus markups is dictated by their type, sharks who observe high signals make the least attractive/highest markup/most inefficient offers, while sharks who observe low signals make the most attractive/lowest markup/most efficient offers - amateur offers are intermediate in these aspects. Sharks are more profitable because (a) they are more likely to observe the signal that orients toward the right profitability lever (sales versus markups), and (b) they exercise each lever more aggressively.

When signals can inform production, there are important differences. Equilibria change in form as sellers augment signal valuation information with that which can be extracted through screening menus, made up of the price-quality pairs for each type of buyer; in other words, they perform second- and third- degree price discrimination concurrently. The enhanced ability of sellers to extract buyer information rents changes how reliant they are on predictive skill. In particular, we find that under sufficient competition, amateur sellers ignore their signals, simply choosing to extend maximally efficient offers to all buyers, all the time. Efficiency is also measured differently. When demand information only informs pricing, efficiency of trade is measured at the extensive margin (whether a buyer finds an acceptable offer), but when demand information also informs production, trade efficiency is measured at intensive margin (how much quality a buyer obtains), since buyers always accept some seller's offer. Lastly, we learn that trends in precision provide an alternative explanation for trends in markups (De Loecker 2020), as precision leaders can have high markups (from trade with high-valuation buyers), high sales (from sales to low-valuation buyers), and high costs (from trading more quality with low-valuation buyers).

Literature Review - Our approach extends a tradition that leverages the canonical search-theoretic framework of Burdett and Judd (1983), which teaches us that imperfect information about the level of competition can generate price dispersion, as sellers choose mixed strategies. Garrett[6] et al. (2019) introduce buyers with heterogeneous valuations for a vertically differentiated good to this setting, which induces sellers to mix, but over screening menus. We advance this literature by enhancing sellers' information: our sellers (a) observe informative signals about matched buyers' valuations, and (b) the signal precision is heterogeneous. By introducing additional information, we can flexibly characterize the impact of precision on prices, products, and welfare; whereas, by accounting for its heterogeneity, we can characterize the impact of trends at different points in the precision distribution.

Generally, we contribute to the extensive literature on price discrimination, and particularly to the branches that study the welfare ramifications of second- and third-degree price discrimination. The latter's tradition extends back to Pigou (1920), while the former has been an active field since the

[6]Contemporaneously, Lester et al. (2019) study the flipside of Garret et al.'s setting - a lemons problem with an analogous matching mechanism where privately informed sellers obtain bids from uninformed buyers.

seminal work of Mussa and Rosen (1978). A salient theme has been the double-edged nature of price discrimination: potentially increasing efficiency but also redistributing surplus. Our concern for the principal factors that determine the sign of these effects aligns with research that has analyzed the influence of market structure[7] and demand characteristics[8]. Our contribution to this branch is two-fold. First, we bridge second- and third- degree price discrimination, as sellers practice both concurrently when quality is endogenous, which allows us to obtain the insight that less precisely informed sellers can choose to exclusively practice second-degree price discrimination. And second, we link the nature of price discrimination to novel factors. In detail, by having our sellers learn structurally, about individual buyers' preferences (rather than about aggregate reduced-form parameters), we are able to connect preference heterogeneity to the effects of price discrimination, whereas by modeling sellers' information in a rich fashion, we are able to do the same for the level and distribution of precision.

Our work also forms part of a growing literature that investigates the microfoundations and macroeconomic implications of the analytics pipeline. At the data collection stage, research[9] has focused on the incentives of buyers to disclose information about their preferences - trading off the desirability of product offers with their price or privacy costs. At the insight extraction stage, the role of intermediaries and their use of information design (repackaging data) to achieve various objectives, including aggregate and distributional welfare objectives[10], has been a main concern. These works are complementary to ours, since they investigate the process of generating predictive precision, meanwhile, we study its implications in diverse settings. In this vein, the link between data and market structure has been studied by a multidisciplinary literature that incorporates frameworks from macroeconomics, industrial organization, and finance[11]. We share some of the pro-welfare and pro-consumer effects of information and competition, but explicitly modeling the documented heterogeneity on both sides of the market allows us to parse their composition and identify the factors that reverse them, while our concern for the use of information to price discriminate allows us to derive new relations between precision, offer dispersion/order, and the profits of sales to each type of buyer that can inform empirical work.

The paper proceeds as follows: in Section 2, we introduce the environment, in Section 3, we study economies where sellers are homogeneous and quality is exogenous, and in Section 4, we study economies where sellers are heterogeneous and quality is endogenous. Primarily technical aspects are developed in the Appendix.

2 Environment

We study a two-sided market that features heterogeneity on both the buyer and seller sides. A unit mass of buyers have single-unit demands and heterogeneous tastes for the quality of a traded good.

Assumption 2.1 (Additively Separable Utility). *A buyer with a marginal value for quality θ_i obtains utility $u(q, x; \theta_i) = \theta_i q - x$ from consuming a good of quality q at a price of x.*

The utility of not trading is normalized to zero.

Sellers can produce a good of quality q using a common cost function for quality $\phi(q)$. Upon a successful transaction with a buyer for a contract (q, x) - a good of quality q and price x - sellers

[7]See Holmes (1989), Armstrong and Vickers (2001), Rochet and Stole (1997, 2002), Stole (2007), Vives (2011), and Rhodes and Zhou (2022).

[8]See Robinson (1933), Schmalensee (1981), Varian (1985), and Aguirre et al. (2010)

[9]See Braghieri (2019), Ichihashi (2020), Bonatti and Cisternas (2020), Hidir and Vellodi (2021), and Ali et al. (2023).

[10]See Bergemann et al. (2015, 2018), Elliot et al. (2021), Guo et al. (2022), Yang (2022), Haghpanah and Siegel (2023), Galperti et al. (2023), and Ichihashi and Smolin (2023).

[11]See Begenau at al. (2018), Agrawal et al. (2018, 2019), Kehoe et al. (2020), Farboodi and Veldkamp (2022), and Eeckhout and Veldkamp (2022)

receive profits $x-\phi(q)$. The market between buyers and sellers follows an exogenously defined matching process. Following the style of Burdett and Judd (1983), a buyer matches with one or two randomly and independently drawn sellers with the probability of a single match given by $\tilde{\rho} \in (0,1)$. Because we assume the number of matches is only privately known by the buyer, sellers face uncertainty as to whether they are competing against a peer for a buyer's purchase within a given match. As such, matched sellers assign not competing a probability,

$$\rho = P(\text{single match}|\text{matched}) = \frac{\tilde{\rho}}{\tilde{\rho} + 2(1-\tilde{\rho})} \tag{2.1}$$

where ρ governs the level of competition, ranging from the extremes of perfect competition ($\rho = 0$) to monopoly ($\rho = 1$).

While the ex-ante distribution of buyer valuations is common knowledge, sellers also observe a private signal about the valuation of each buyer with whom they match. Our principal innovation is to introduce heterogeneity amongst sellers in the precision of these signals[12]. We consider the simplest discretization of precision, where the sellers are either less precisely informed *amateurs* or more precisely informed *sharks*. Their respective signal precisions are denoted

$$\alpha_e = P^e(\text{signal} = i|\text{matched buyer's valuation} = \theta_i) \quad i \in \{l, h\} \tag{2.2}$$

$$0.5 \leq \alpha_a < \alpha_s < 1 \tag{2.3}$$

In this sense, the two relative precision categories of amateur and shark will remain fixed throughout the paper, even if the absolute level of precision that they represent differs. The mass of amateurs (sharks) is $\mu(e) = \gamma$ ($\mu(s) = 1-\gamma$).

Agents do not make choices before matching, so our setting is that of an incomplete information game that is solved at the interim stage, after matches have formed and sellers have observed their signals. A seller's type is then its posterior belief of being in a low-valuation match, $p_L^{e,j} = P^e(\theta = \theta_L|j)$,

$$p_L^{e,j} = \frac{P^e(j|\theta_L)p(\theta_L)}{P^e(j|\theta_L)p(\theta_L) + P^e(j|\theta_H)p(\theta_H)} = \frac{\alpha_e p(\theta_L)}{\alpha_e p(\theta_L) + (1-\alpha_e)p(\theta_H)} \tag{2.4}$$

Having introduced the environment, we will first analyze the role of information in a simple setting where sellers produce a good of exogenously fixed quality, such as a commodity, so optimal offers take the form of a single quoted price per unit x. Subsequently, we will endogenize quality, allowing sellers to choose the quality of the goods that they offer. This will allow us to analyze the connection between precision and fit of the products available to buyers, but result in a more complex set of optimal offers that take the form of screening menus composed of quality-price contracts $((q_L, x_L), (q_H, x_H))$.

3 Exogenous Quality Setting

For the remainder of this section, we assume that all sellers produce a homogeneous good of exogenously fixed quality for free.

Assumption 3.1 (Exogenous Quality Assumptions)**.** *The traded good has quality $q = 1$ and a cost of production, $\phi(1) = 0$.*

[12] In practice, precision can be improved by acquiring more/better data or by improving the methods with which data are analyzed. Heterogeneity in precision is, therefore, natural when we consider the variability in firms' data and analytical resources.

This means that there are gains from trade with both types of buyers, and that each buyer simply chooses the lowest price offer it obtains, provided it is below her valuation θ_i. The exclusion of product design (quality) choices will allow us to isolate the role of the pricing channel, but still characterize some of the main relationships between precision, the properties of offers, and welfare that extend to settings where quality is endogenous. To obtain these insights in the most straightforward fashion, we will also assume that sellers are homogeneous.

Assumption 3.2 (Homogeneous Sellers)**.** *All sellers have signal precision α.*

There will only be two types of seller: sellers of type p_L^h, who observed an h signal, and sellers of type p_L^l, who observed an l signal.

Sellers who are more convinced that they are matched with high-valuation buyers - those who observe high signals - offer higher prices, often beyond the willingness to pay of low-valuation buyers, and thus forego trade in some matches. Inversely, sellers who are more convinced that they are matched with low-valuation buyers - those who observe low signals - offer lower prices, often substantially below the willingness to pay of high-valuation buyers. Both types of mispricing lower profitability, but pricing out low-valuation buyers also decreases the efficiency of trade. More precise valuation information makes any given (offer) strategy more profitable and efficient because it reduces these instances of mispricing; however, it can also induce sellers to adjust their strategies. Precision gives sellers the confidence to offer larger and more frequent discounts upon a signal indicative of low-valuation, but also to offer larger and more frequent price hikes upon a signal indicative of high-valuation. While the former increases trade, the latter limits it. Naturally, any adjustment in strategy is towards greater profitability, and, in equilibrium, additional precision is always individually beneficial for sellers, but its externalities on other agents (buyers or competitors) can be positive or negative, as these are always the net of the direct effect on any given seller strategy and the equilibrium effect that induces sellers to choose different strategies. As a result, both buyers, sellers, and efficiency can suffer from additional precision.

Additional precision improves the efficiency of trade when valuations are sufficiently heterogeneous, but this can go in the opposite direction when valuations are sufficiently homogeneous. Accounting for the winning and losing sides, precision growth will generally make the sellers who obtain it more profitable and reduce high-valuation buyers' surplus, whereas low-valuation buyer surplus responds in the same direction as efficiency. Before characterizing the determinants of these distributional and aggregate effects more comprehensively, we will formally introduce the seller's problem and the structure of equilibria.

3.1 Seller Problem and Equilibrium Concept

The expected profits of a type p_L^j seller from a price offer x are the product of its profits per sale, x, times its expected sales, $P(\text{sale at price } x|j)$,

$$\Pi(x) = P(\text{sale at price } x|j)x \tag{3.1}$$

As in private value auctions, sellers know the value of having an offer accepted at the time of submission, the profits per sale x, but must infer the probability of its acceptance. We call the *sales* associated with an offer exactly this probability, which reflects the uncertainty a seller has about four variables: (1) the buyer's valuation, (2) whether the buyer is matched with another seller, (3) the type of such a competitor, and (4) the offer that this type of competitor would make. A type p_L^j seller's optimal prices are therefore,

$$x^j(P(\cdot|j)) = \underset{x\in\mathbb{R}}{\operatorname{argmax}}\,\Pi(x)P(\text{sale at price } x|j)x \tag{3.2}$$

and its strategy is a cumulative distribution function $F^j(x)$ over prices. This type's expected sales can then be decomposed as,

$$\begin{aligned} P(\text{sale at price } x|j) &= \rho P(\text{sale at price } x|\text{monopoly match}, j) \\ &+ (1-\rho)\Bigg(P(\theta_L|\text{competitive match}, j)P(\text{sale at price } x|\theta_L, \text{competitive match}, j) \\ &+ P(\theta_H|\text{competitive match}, j)P(\text{sale at price } x\theta_H, \text{competitive match}, j)\Bigg) \\ &= \rho(p_L^j \mathbb{1}(x \le \theta_L) + p_H^j) + (1-\rho)\Bigg(p_L^j(1-F(x|\theta_L))\mathbb{1}(x\le\theta_L) + p_H^j(1-F(x|\theta_H))\Bigg) \end{aligned} \tag{3.3}$$

where $F(x|\theta_i)$ is the seller's expected distribution of prices offered by competitors in matches with θ_i valuation buyers. This decomposition highlights and quantifies the distinct sources of uncertainty facing sellers - the buyer's valuation (p_i^j terms), the existence of competitors (ρ terms), and their offers ($F(x|\theta)$ terms). This last term encapsulates uncertainty about the type of competitors and their chosen price, emerging naturally as an average of each type of seller strategy,

$$\begin{aligned} F(x|\theta) &= \sum_{j\in\{l,h\}} P(\text{seller observes } j|\theta)P(\text{seller offer price below } x|\text{signal } j) \\ &= \sum_{j\in\{l,h\}} P(j|\theta)F^j(x) \end{aligned} \tag{3.4}$$

where the weights originate from the commonly known matching protocol and signal structure: $P(j|\theta)$ is the probability that a competitor observes a j value signal in a match with a θ valuation buyer and $F^j(x)$ is the cumulative distribution of prices they would offer upon observing j.

Definition 3.1 (Bayes-Nash Equilibrium)**.** *A Bayes-Nash equilibrium of an economy with exogenous quality is a vector of seller strategies, $\{F^j\}_{j\in\{l,h\}}$, for each type of seller, satisfying* $\operatorname{supp}(F^j) \subseteq x^j(P(\cdot|j))$ *for all $j \in \{l,h\}$, when their expected sales are given by* (3.3)-(3.4).

These equilibria are quite tractable. In particular, strategies and offer distributions are continuous, almost everywhere differentiable, and strictly increasing on at most two convex interval(s). Furthermore, equilibria are essentially unique in the sense that profitability, buyer surplus, and trade efficiency are invariant across them. Their qualitative properties are intuitive and revealing, with the principal point being that higher prices are more profitable in high-valuation matches and are offered by sellers whose posteriors place more weight on being matched with a high-valuation buyer - crisply illustrating the role of information.

3.2 Equilibrium

3.2.1 Corner Cases: Monopoly and Perfect Competition

We first analyze the two corner cases of monopoly ($\rho = 1$) and perfect competition ($\rho = 0$) to fix ideas. These easily yield key insights that extend to economies where competition is imperfect ($\rho \in (0,1)$) while also motivating their study, as the limitations of the analysis in the corners will be apparent.

Monopoly - A monopolist's pricing choice involves the traditional trade-off between sales and profits per sale. Matched buyers have no alternative offers; they either accept the seller's offer to buy the good for a price of x and obtain utility $\theta_i - x$, or they reject it and obtain zero utility. Optimal take-it-or-leave-it offers must, therefore, give zero surplus to the lowest (valuation) buyer who would accept it, meaning that a seller's optimal offer must be equal to some buyer's valuation. A monopolist sets a sales maximizing price of $x = \theta_L$, and trades with every buyer, or a profit-per-sale maximizing price of $x = \theta_H$, and only trades with high-valuation buyers depending on which has the highest expected revenues (recall that costs are zero),

$$\theta_L \lesseqgtr (1 - p_L^j)\theta_H \iff p_L^j \theta_L \lesseqgtr (1 - p_L^j)(\theta_H - \theta_L)$$

Which of the two prices the monopolist prefers is determined by her valuation beliefs, summarized by p_L^j. If she is sufficiently convinced that the buyer's valuation is low (high p_L^j), the low price $x = \theta_L$ is optimal, since the expected value of its additional sales, $p_L^j \theta_L$, is greater than the expected cost of its discount, $(1 - p_L^j)(\theta_H - \theta_L)$; otherwise, if she is sufficiently convinced that the buyer's valuation is high, she expects extracting their information rents to be the most profitable action and chooses a price of $x = \theta_H$. The threshold probability p_L^* dictating this choice is given by,

Proposition 3.1 (Monopoly). *If $\rho = 1$, a seller of type p_L^j offers a price*[13]*,*

$$x^*(p_L^j) = \begin{cases} \theta_H & \text{if } p_L^j < p_L^* \\ \theta_L & \text{otherwise} \end{cases} \tag{3.5}$$

for the threshold probability,

$$p_L^* = \frac{\theta_H - \theta_L}{\theta_H} \tag{3.6}$$

Our first general takeaway is that precision increases price dispersion, as the posteriors of sellers with more precise information (larger α_e) are more dispersed ($p_L^l - p_L^h$ increases), and posterior dispersion increases the likelihood that upon a low versus high signal the seller is above versus below the threshold p_L^*, respectively. Said differently, precision dictates whether sellers choose to practice third-degree price discrimination, a salient point that will reappear when we endogenize quality choices. A monopolist who does not price discriminate sets a uniformly low or high price, so a precision-induced switch to price discrimination impacts trade efficiency and buyer welfare very differently, depending on which of the two signal-invariant prices would have been chosen with less precise information.

When buyers' valuations are sufficiently heterogeneous, imprecisely informed sellers face severe adverse selection, so they set a uniformly high price (θ_H) and only trade with high-valuation buyers. They only have an incentive to offer a low price (θ_L) upon a low signal, when signals are precise enough, so precision is crucial for efficient trade in these settings. The switch to price discrimination not only increases trade with low-valuation buyers - the gains of which are entirely captured by sellers - but it also allows high-valuation buyers to obtain some information rents from instances of mispricing (with probability $1 - \alpha$ in each match). In this sense, precision growth can make all agents weakly better off, when buyers are different enough.

However, when buyers' valuations are relatively homogeneous, imprecisely informed sellers face little adverse selection, so they set a uniformly low price (θ_L) and trade with both types of buyer. Analogously, they only have an incentive to offer a high price (θ_H) upon a high signal, when signals are precise enough, and the switch to price discrimination also introduces mispricing. Low price offers to high-valuation buyers simply redistribute - transfering seller surplus to high-valuation buyers - but

[13] We break profitability ties in favor of trade, but that is immaterial.

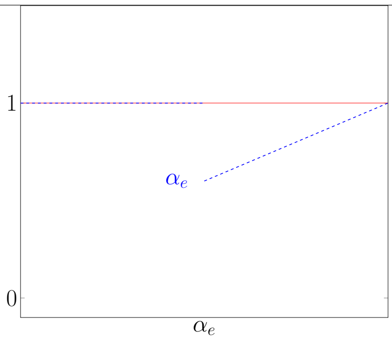
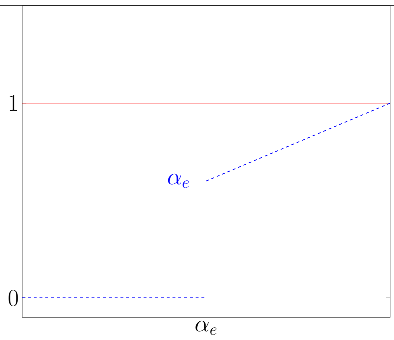
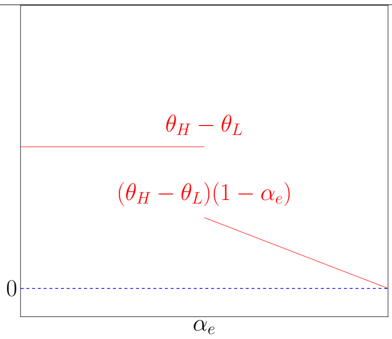
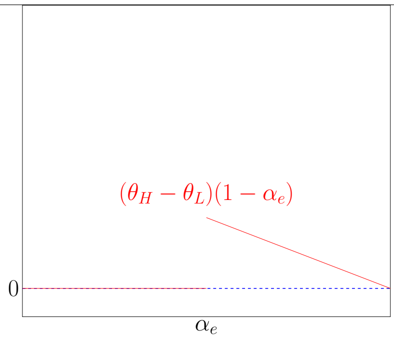
Valuation Heterogeneity $\theta_H - \theta_L$ / Y-axis Variable	**Low**	**High**
Probability of Trade: $\mathbb{P}(x \leq \theta_i \mid \theta_i)$		
Buyer Surplus: $\mathbb{E}[\,u_i\,]$		

Table 1: Consider a setting buyers always match with a single seller ($\tilde{\rho} = \rho = 1$). Low (dashed) versus High valuation buyer outcomes.

high price offers to low-valuation buyers are inefficient, decreasing aggregate surplus. In this sense, precision growth can exclusively benefit sellers and decrease trade efficiency, when buyers are similar enough. This effect of preference heterogeneity in the relation between precision and efficiency, as well as the distribution of surplus is robust: also present when competition is imperfect and when quality is endogenously chosen.

Monopoly settings constrain the analysis of precision in two important respects. When it comes to buyers, low-valuation buyer surplus is trivial (zero), whereas with some competition, they obtain some of the gains from trade. When it comes to sellers, monopoly shuts down profit externalities. Imperfect levels of competition will allow us to conduct a richer analysis of the relationship between precision and both low-value buyer surplus and competitor profitability.

Perfect Competition - At the other end of perfect competition ($\rho = 0$), where sellers engage in Bertrand price competition and the unique equilibrium is one where they offer the good at a price equal to its cost and buyers capture all gains from trade.

Proposition 3.2 (Perfect Competition)**.** *If $\rho = 0$, the unique Bayes-Nash equilibrium is for almost every seller to price the good at cost, $x = 0$, with probability 1.*

Perfect competition constrains the analysis of precision in an even more decisive fashion, mainly by making it irrelevant. Nevertheless, perfect competition highlights the classic pro-efficiency and pro-consumer surplus effects of competition that extend to economies where this is imperfect.

3.2.2 Structure

We will now discuss the structure of equilibria when competition is imperfect[14]. It is helpful to first focus on the distributions of prices offered in matches with each type of buyer $\{F(x|\theta)\}_{\theta\in\{\theta_L,\theta_H\}}$. These distributions are averages of each type's strategy, so their supports are identical. They start at the top, with the highest equilibrium price, which only allows sellers to trade when they are in a monopoly match, so it is offered by sellers who expect the most sales from high prices in these, mainly sellers who observe high signals, who pick a price of θ_H or θ_L when bidding as such. Analogously, if any seller offers prices that low-valuation buyers find acceptable, then the highest such one must be θ_L, for it only beats prices that are unacceptable ($x > \theta_L$).

Proposition 3.3 (Highest Prices). *The highest overall equilibrium price,*

$$\overline{x} = \begin{cases} \theta_H & \textit{if } p_L^h \leq p_L^* \\ \theta_L & \textit{otherwise} \end{cases} \tag{3.7}$$

for p_L^* from (3.6). *Whereas, if any prices acceptable to low-valuation buyers are offered, then the highest one is θ_L.*

It turns out that when high signal sellers would always offer prices that allow trade, there exists a type-invariant equilibrium where both types of sellers have the same strategy and, due to the fact that expected sales are then identical in either type of match, signals play no role. We select this equilibrium in these settings and derive it in the Appendix, but proceed with the analysis of economies where precision is certain to have equilibrium effects.

Assumption 3.3 (Some Separating Offers). *Seller precision is such that $p_L^h \leq p_L^*$.*

In these economies, prices are supported on, at most, two convex intervals: one of prices that are only acceptable to high-valuation buyers, $[\hat{x}, \theta_H]$ with $\hat{x} > \theta_L$, and another of prices that are acceptable to all buyers.

Proposition 3.4 (Continuous Distributions, Support Convexity). *Given Assumption 3.3, distributions of offers $\{F(x|\theta)\}_{\theta\in\{\theta_L,\theta_H\}}$ are atomless and supported on at most two convex interval.*

$$\text{supp}(F(x|\theta)) = [\hat{x}, \theta_H] \quad \textit{or} \quad [\underline{x}, \theta_L] \cup [\hat{x}, \theta_H] \tag{3.8}$$

These properties are necessary to rule out deviations: atomlessness, rules out discrete sales increases in competitive matches from infinitesimal discounts, and convexity, rules out price increases that raise profits-per-sale without sacrificing sales.

It is conceptually important to highlight that offers above low-valuation buyers' valuation are separating: they are only ever accepted by high-valuation buyers and always rejected by low ones. As such, they are chosen to be optimal in high-valuation matches and, to not be strictly dominated, are the most profitable among all offers in high-valuation matches. Separating prices, therefore, maximize (minimize) high- (low-) valuation match profitability.

Proposition 3.5 (Offer Distribution over Separating Prices). *The distribution of offers over separating prices is uniquely determined by,*

$$\rho\theta_H = \Pi(\theta_H|\theta_H) = \Pi(x|\theta_H) = (\rho + (1-\rho)(1 - F^h(x|\theta_H)))x \quad \textit{for } x > \theta_L$$

[14]We present a summarized version of the logic behind results here, but a detailed exposition of these is found in the Appendix

All sellers, therefore, agree on the optimal separating prices, because they choose these conditioning on being in a high-valuation match. The difference is that sellers who observe low signals can expect separating offers to sacrifice too many sales, so despite agreeing on the optimal ones, they prefer to offer lower pooling offers that allow them to trade with both types of buyers.

Proposition 3.6 (Threshold Posterior)**.** *Given any equilibrium, there exists a threshold posterior,*

$$\hat{p}_L = \sup_{x' \leq \theta_L} \frac{\Pi(x = \theta_H|\theta_H) - \Pi(x = x'|\theta_L)}{\Pi(x = \theta_H|\theta_H) + \Pi(x = x'|\theta_L) - \Pi(x = x'|\theta_H)} \tag{3.9}$$

such that,

- ***Separating Types:*** *Sellers of lower type, $p^j_L < \hat{p}_L$, only offer separating prices.*
- ***Dual Type:*** *Sellers of equal type, $p^j_L = \hat{p}_L$, offer separating and pooling prices.*
- ***Pooling Types:*** *Sellers of larger type, $\hat{p}_L < p^j_L$, only offer pooling prices.*

This coarse grouping of prices is a manifestation of an ordering property, linking the indirect utility that sellers offer buyers with their posteriors, which is characteristic of economies where sellers have predictive skill - heterogeneous or not. Both in settings where quality is exogenous, and, as we will see, in settings where quality is endogenous, sellers who place more posterior weight on low-valuation matches extend offers that are more attractive on average to buyers of either valuation.

Moving onto the offer distributions over pooling prices, there are two cases to consider: economies where only low signal sellers make pooling offers and economies where both types of sellers make pooling offers. In the first case, low and high signal sellers' strategies have disjoint supports, and offer distributions maintain low signal sellers indifferent by making lower pooling prices more (less) profitable in low- (high-) valuation matches. This last point follows because there are more low signal sellers in low-valuation matches, so a discount from θ_L to $x < \theta_L$ forfeits the same amount of revenue in a monopoly match, but yields a strictly larger sales gain in low-valuation matches.

Proposition 3.7 (Only Low Signal Sellers Pool)**.** *If only low signal sellers extend pooling offers, offer distributions are uniquely determined by the profitability,*

$$\begin{aligned} \Pi(\theta_L; p^l_L) &= \left[p^l_L(\rho + (1-\rho)(1 - F(\hat{x}|\theta_H))) + p^l_H(\rho + (1-\rho)(1 - F(\hat{x}|\theta_L)))\right]\theta_L \\ = \Pi(x; p^l_L) &= \left[p^l_L(\rho + (1-\rho)(1 - F(x|\theta_L))) + p^l_H(\rho + (1-\rho)(1 - F(x|\theta_H)))\right] x \end{aligned}$$

and aggregation conditions, $F(x|\theta_L) = \alpha F^l(x), F(x|\theta_H) = (1-\alpha)F^l(x)$. Further, given two prices $x < x' \leq \theta_L$,

$$\Pi(x'|\theta_H) > \Pi(x|\theta_H) \quad \Pi(x'|\theta_L) < \Pi(x|\theta_L)$$

If high signal sellers offer pooling prices, however, the support of strategies necessarily[15] overlaps over every pooling price offered by a high signal seller, $[\underline{x}^h, \theta_L]$, and offer distributions make both types of sellers indifferent by maintaining the conditional profitability, $\Pi(x|\theta)$, of prices. This overlapping region is followed below by a contiguous interval of prices that are only offered by low signal sellers, and over which conditional profitability trends are as per the logic of the previous case.

Proposition 3.8 (Some High Signal Sellers Pool)**.** *If high signal sellers offer pooling prices and $\underline{x}^h$ is their lowest pooling offer, then* $\text{supp}(F^l) \cap \text{supp}(F^h) = [\underline{x}^h, \theta_L]$, *over which the offer distributions*

[15]See the Appendix, but the logic is similar to that which sets the profitability trend among pooling prices that are exclusively offered by low signal sellers.

are uniquely determined by the profit invariance conditions,

$$\Pi(\theta_L|\theta) = \Pi(x|\theta) = (\rho + (1-\rho)(1-F(x|\theta)))x \quad \text{for } x \leq \theta_L,\ \theta \in \{\theta_L, \theta_H\}$$

And, $(\text{supp}(F^l) \cap \text{supp}(F^h)^c) \cap [0, \theta_L] = [\underline{x}^l, \underline{x}^h]$ *with a uniquely determined distribution of offers over these prices,*

$$\Pi(\underline{x}^l; p_L^l) = \Pi(x; p_L^l) = \left[p_L^l(\rho + (1-\rho)(1-F(x|\theta_L))) + p_H^l(\rho + (1-\rho)(1-F(x|\theta_H)))\right] x$$

where $F(x|\theta_L) = \alpha F^l(x), F(x|\theta_H) = (1-\alpha)F^l(x) = \Pi(x; p_L^l)$. *Further, given two prices* $x < x' \leq \underline{x}^h$,

$$\Pi(x'|\theta_H) > \Pi(x|\theta_H) \quad \Pi(x'|\theta_L) < \Pi(x|\theta_L) \quad x < x', x, x''$$

Jointly, Proposition 3.5, Proposition 3.7, and Proposition 3.8 imply that distributions of offers are unique.

Theorem 3.1 (Unique Distributions of Offers)**.** *The equilibrium distribution of offers in each type of match,* $\{F(x|\theta)\}_{\theta \in \{\theta_L, \theta_H]}$, *are unique.*

Offer distributions determine the probability that low-valuation buyers find an acceptable offer, and so efficiency, as well as the average price that each type of buyer trades at, and so consumer surplus, so both are uniquely determined. While the multiplicity of equilibria comes from possible shifts in mass across seller strategies among separating or overlapping prices, but only up to the point of preserving the unique offer distributions that make all these prices equally profitable, whereby seller surplus is also uniquely determined.

Corollary 3.1 (Efficiency and Welfare Outcome Uniqueness)**.** *In any equilibrium, the probability of trade of low-valuation buyers, the surplus of low- and high-valuation buyers, and the profits of each type of seller in each type of match are identical.*

This allows us to conduct unambiguous comparative static analysis and to construct - in the Appendix - equilibrium strategies through a simple algorithm that weakly orders prices by the seller's type, in the sense that the highest separating prices are only offered by high signal sellers, below which are separating prices offered by low signal sellers or the set of pooling prices offered by both types of sellers, and below which there can be pooling prices that are only offered by low signal sellers.

3.3 Over- and Under-bidding

In these settings, there is no winner's curse: sellers' value of winning is identical conditionally or unconditionally on winning - mainly their profit margin. However, sellers do over- and under-bid, in the sense that the optimal bid ex-post is often different from the one they made. Precision's profit externalities will come, in part, from exacerbating the size of these mistakes, so we will analyze them before proceeding to precision's comparative statics.

Sellers' offers are optimal conditionally on their information, however, ex-post optimality requires conditioning on all information about the match - the buyer's valuation, the existence of competing sellers, and their type - much of which sellers only have imperfect information about. Consider the optimal offers under three levels of nested information, where the seller knows (1) the buyer's valuation, the presence of a competing seller, and the competitor's type, (2) the buyer's valuation and the presence of a competing seller, and (3) only the buyer's valuation.

Corollary 3.2 (Optimal Prices Conditionally on Match Type)**.** *In the ordered equilibrium, conditionally on the buyer's valuation, the number of competitors, the type of the competitor,*

- *If the buyer's valuation is high, the optimal competitive price is* $X^*(\theta_H, \textit{competitive}, p_L^j \textit{ type competitor}) = \underline{x}^j$.

- *If the buyer's valuation is low, the optimal competitive price is* $X^*(\theta_L, \textit{competitive}, p_L^j \textit{ type competitor}) = \min(\underline{x}^j, \theta_L)$.

whereas, conditionally on only the buyer's valuation and the number of competitors,

- *If the buyer's valuation is high, the optimal price is* $X^*(\theta_H, \textit{competitive}) = \hat{x}$.

- *If the buyer's valuation is low, the optimal price is* $X^*(\theta_L, \textit{competitive}) = \underline{x}^l$.

and, conditionally on only the buyer's valuation,

- *If the buyer's valuation is high, the set of optimal price offers is* $X^*(\theta_H) = [\hat{x}, \theta_H]$.

- *If the buyer's valuation is low, the optimal price is* $X^*(\theta_L) = \underline{x}^l$.

Whereas, if the seller knows that it has no competitors and the buyer's valuation, then its optimal price offer is the valuation.

The claims about optimality conditionally on only the buyer's valuation are immediate from the profitability trends that we said offer distributions generate[16], whereas the about optimality conditionally on also the level and type of competition are immediate from the point that lower prices in each type of sellers' support, which are less profitable in monopoly matches, are incentivized by making them more profitable in a competitive matches.

Since equilibrium distributions of prices are atomless, almost every offer that sellers make is suboptimal under the perfect information posterior: if they sell, their offer in that particular match was too low (over-bid buyer surplus), and inversely if they do not (under-bid buyer surplus). However, much of this regret is unavoidable, as sellers do not have information about the number of competitors, and even knowing a competitor's type, there would remain uncertainty about the price it would offer. A part of regret is linked to the buyer's valuation, as winning (losing) makes having been in a high (low) match more likely, and it is here that precision helps minimize regret.

Precision shapes a seller's distribution of posteriors, hence types, in two ways. First, it changes how often the seller's signal correctly classifies the buyer; in other words, how often the seller's type is relatively low or high in matches with a buyer of each valuation. Second, it changes how extreme its posterior valuation beliefs are upon each signal; in other words, how large its type is upon each signal. Together, these bring the expected sales of sellers in each type of match closer to the perfect information forecast $P(\text{sale at price } x|\theta)$, which allows them to improve their profitability by trading-off sales with mark-ups more precisely.

3.4 Comparative Statics

With a clear understanding of the equilibrium structure, we proceed to study the effect of competition and precision on prices, trade efficiency, and the level as well as the distribution of aggregate surplus between and within buyers and sellers.

[16]Maintaining profits in each type of match over separating or overlapping pooling prices, and increasing (decreasing) profits in low- (high-) valuation matches of lower prices that only low signal sellers offer.

In the case of buyer welfare, we examine the average utility that buyers of each valuation obtain,

$$\mathscr{W}_b(\theta) = \rho E[u(1, \min(\theta, x_i); \theta)] + (1-\rho)E[\max_{i \in \{1,2\}} u(1, \min(\theta, x_i); \theta)] \tag{3.10}$$

where x_i are iid draws from the conditional equilibrium marginals $F_i(x_i|\theta_i)$. Seller profits from low- and high-valuation buyer matches depend on (1) the average number of matches they are in, (2) their sales per match, and (3) their profits per sale. Since sellers are homogeneous, we track profits by their ex-ante average across sellers,

$$\mathscr{M}(\tilde{\rho}) = \tilde{\rho} + 2(1-\tilde{\rho}) \tag{3.11}$$

$$\Pi = \Big[\underbrace{\mathscr{M}(\tilde{\rho})}_{\text{number of matches}} \quad \underbrace{P(\text{sale at price } x, j)}_{\text{probability of selling}} \quad \underbrace{x}_{\text{profits-per-sale}} \Big] \tag{3.12}$$

where we average over the probabilities that the seller observes each signal $j \in \{l, h\}$ in a match with a buyer of each valuation $\theta_i \in \{\theta_L, \theta_H\}$ and the prices $x \sim F^j$ offered upon each signal. Finally, because inefficient trade occurs solely from low-valuation buyers being priced out, we track efficiency by the probability low-valuation buyers trade,

$$\mathscr{Q} = \quad \rho F(x \leq \theta_L|\theta_L) + (1-\rho)P_{x_1,x_2 \sim F(\cdot|\theta_L)}(\min(x_1, x_2) \leq \theta_L|\theta_L) \tag{3.13}$$

We find that competition promotes greater efficiency and buyer surplus. Sellers, anticipating "more" competitors in the average match, offer more attractive terms to buyers to sustain optimal sales. Since sellers decrease prices across the board, low-valuation buyers have a higher probability of receiving an acceptable offer, thus boosting trade efficiency and the aggregate surplus, all of which is captured by buyers.

The effects of variation in precision are more nuanced and depend mainly on the level of adverse selection. Low-valuation typically benefit from additional precision, while high-valuation buyers are harmed by it. This difference is rooted in the point that buyers obtain more attractive offers when sellers observe low signals, and precision reduces incidents of misclassification - benefiting low-valuation buyers, but hurting the high. However, these general trends can be reversed. When buyers' valuations are relatively homogeneous, imprecisely informed sellers prefer to obtain large sales, and precision can increase the share of separating offers upon a high signal much more than the share of pooling offers upon a low signal - decreasing trade efficiency and competition for low-valuation buyers, hence their surplus. When buyers' valuations are relatively heterogeneous, however, imprecisely informed sellers prefer to obtain large markups, and precision increases the share of pooling offers upon a low signal much more than the share of separating offers upon a high signal. For sellers, as we discussed, precision is always individually beneficial, but it changes the nature of competition, so it has externalities on competitors. This is why when we change the precision of all sellers' signals, profitability will sometimes decrease. The settings most prone to this large negative profit externality are exactly the ones where precision is most beneficial for buyers: those where preferences are sufficiently heterogeneous and the level of competition is high.

3.4.1 Competition

When buyers obtain more offers, lower ρ, sellers expect them to have more alternatives, particularly more alternatives below any price. Given a fixed set of strategies, this reduces the sales that sellers expect at any price. They respond by lowering prices to restore sales, and since their equilibrium strategies are mixed, this is done by shifting mass towards lower prices. As such, the general equilibrium effect on sellers' strategies reinforces the partial equilibrium effect on the number of offers, both improving buyers' offers.

We take a closer look at this dynamic in the more relevant set of equilibria - those in which separating prices are offered, so that predictive skill has a role. The highest overall price $\overline{x}$ does not depend on the level of competition - only on the posterior of sellers who observe a high signal (by (A.4)). However, the probability of prices below is closely related to the level of competition. Inspecting the distributions of offers in each type of match,

$$F(x|\theta;\rho_2) < F(x|\theta;\rho_1) \quad ,\rho_1 < \rho_2$$

In other words, when there is more competition ($1-\rho$ increases), sellers place more mass on lower prices in both types of matches (first-order stochastic dominance relation). Therefore, both types of buyers obtain larger surplus, and the additional mass on pooling prices also increases trade efficiency. However, sellers' profits drop in both types of matches, and the reasoning is straightforward. Recall that the sales a matched seller expects from a price x are,

$$P(\text{sale at a price } x|j) = \rho(p_L^j \mathbb{1}(x \leq \theta_L) + p_H^j) + (1-\rho)\left(p_L^j(1-F(x|\theta_L))\mathbb{1}(x \leq \theta_L) + p_H^j(1-F(x|\theta_H))\right)$$

and these decrease both because competitive matches are more likely (ρ and $1-\rho$ terms) and because the distribution of competitor bids shifts mass towards lower prices ($F(x|\theta)$ terms), making any offer is less profitable in any match. It is true that there exists a compensating effect, as competition allows sellers to match with more buyers (larger $M(\tilde{\rho}) = \tilde{\rho} + 2(1-\tilde{\rho})$), but this is insufficient. Mechanically, the average number of matches also appears in the denominator of seller's conditional probabilities of being in a monopoly or competitive match, $\rho = \frac{\tilde{\rho}}{M(\tilde{\rho})}$ and $1-\rho = \frac{2(1-\tilde{\rho})}{M(\tilde{\rho})}$, so it cancels in the ex-ante profit equation,

$$\begin{aligned}&\mathscr{M}(\rho)\Pi(p_L^j,\rho)\\&= E_{x\sim F^j}\left[\tilde{\rho}(p_L^j\mathbb{1}(x\leq\theta_L)+p_H^j) + 2*(1-\tilde{\rho})\left(p_L^j\mathbb{1}(x\leq\theta_L)(1-F(x|\theta_L)) + p_H^j(1-F(x|\theta_H))\right)\right]\end{aligned}$$

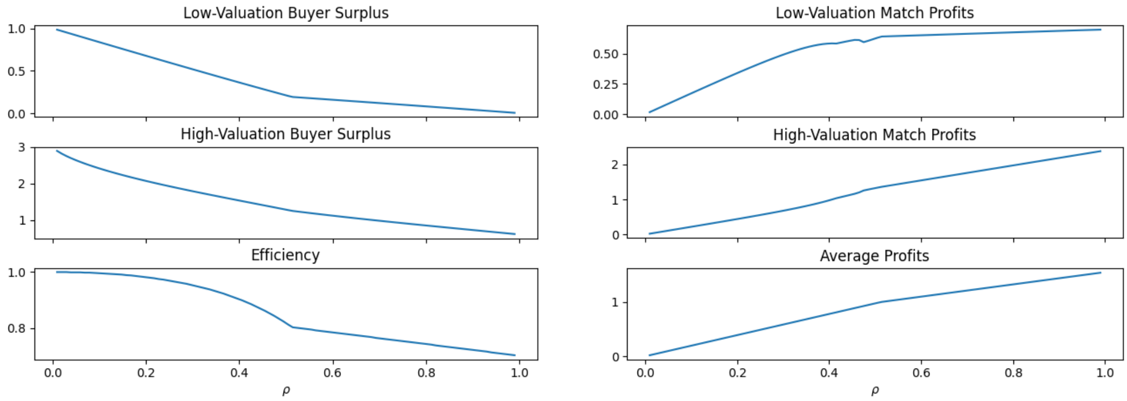
meaning that ex-ante profits also decrease with competition[17]. The additional sales that some types achieve are not insufficient compensation for the lower prices it takes to get them, so the profits of every type of seller decrease.

Theorem 3.2 (ρ Comparative Statics)**.** *Consider two economies that differ only in the level of competition $\rho_1 < \rho_2$. Then,*

- ***Distributions Strictly Increasing:*** $F(x|\theta;\rho_2) < F(x|\theta;\rho_1)$.
- ***Buyer Surplus Strictly Increasing:*** $\mathscr{W}_b(\theta;\rho_2) < \mathscr{W}_b(\theta;\rho_1)\forall\theta$.
- ***Trade Efficiency Increasing:*** $\mathscr{Q}(\rho_2) \leq \mathscr{Q}(\rho_1)$.
- ***Profits Strictly Decreasing:*** $\Pi(\rho_1) < \Pi(\rho_2)$.

We illustrate these effects in Figure 1.

[17] $\tilde{\rho}$ decreases if and only if ρ does, so $F^j(x;\rho)$ increases if and only if $\tilde{\rho}$ decreases. As such, the terms inside the expectation, once we cancel the average number of matches, are pointwise (in x) increasing in ρ.



(a) Buyer Surplus and Efficiency **(b)** Seller Surplus

Figure 1: Welfare and Efficiency Effects of Competition The common parameters are $[\theta_L, \theta_H, p(\theta_L)] = [1, 3, 0.5]$ for buyers and $\alpha = 0.7$ for sellers' precision.

3.4.2 Precision

Competition is traditionally dictated only by the number of sellers in a market, or search-theoretic models in the style of Burdett and Judd, by the number of offers that a buyer obtains. However, when sellers have information about preferences, its precision also affects the distribution of prices that buyers obtain, and so the degree of competition that sellers expect. This is why precision has profit externalities when competition is imperfect.

As we know from the equilibrium structure, the more confident a seller is about being in a high-valuation match, the more sales it expects at any price - including separating prices. Precision, therefore, changes the strategy of each type of seller: sellers who observe high (low) signals extend more (fewer) separating offers upon a high (low) signal. Said differently, precision makes the offers of sellers who observe high (low) signals less (more) competitive. The net effect of precision, whether we consider a discrete or marginal change, is a combination of its impact on the strategy of each type of seller (low and high signal *type effect*), and its impact on the distribution of seller-types in matches with buyer of each valuation (*classification effect*).

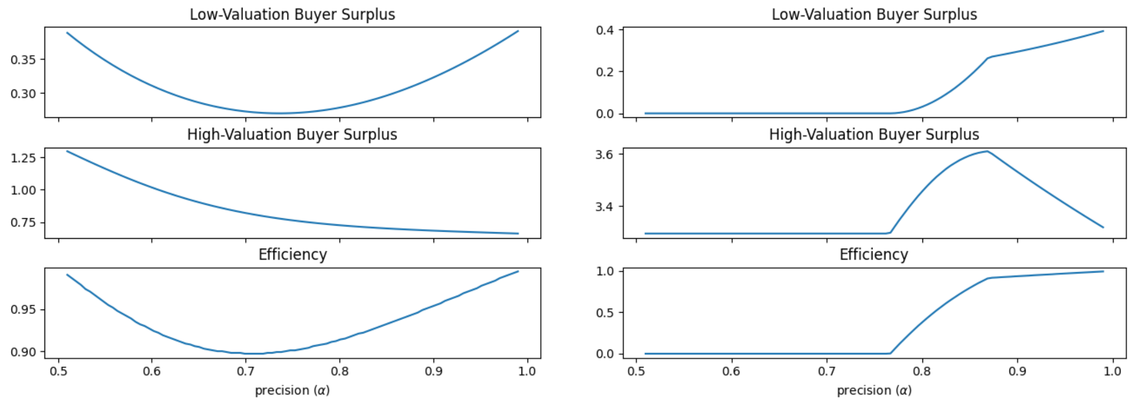
Buyer Surplus and Efficiency Pooling prices benefit high-valuation buyers, who obtain information rents, and low-valuation buyers, as competition for their purchases is stronger when sellers expect them to have more acceptable offers. Therefore, precision is beneficial for buyers to the degree that it increases the number of pooling offers that they receive. In this sense, the classification effect hurts high-valuation buyers and benefits low-valuation buyers, so their surplus generally responds accordingly to changes in precision. However, the type effects upon each signal can dominate and reverse these points. The heterogeneity of the buyers' valuations is the main variable that determines the sign of the net effect of precision.

In economies where valuations are sufficiently heterogeneous, poorly informed sellers face high adverse selection, so they prioritize high markups. As such, precision leads to a small increase in the share of separating offers upon a high signal (weak high signal type effect), but large increases in the share of pooling offers upon a low signal (strong low signal type effect). It allows low-valuation buyers to trade more often, improving efficiency, and to obtain more attractive offers, as the classification effect (more sellers observe low signals in low-valuation matches) and strong low signal type effects dominate the weak high signal type effect; indeed, even high-valuation buyers can benefit from precision, due to misclassification, when the low signal type effect is strong enough.

In economies where valuations are relatively homogeneous, however, poorly informed sellers face low adverse selection, so they prioritize sales. As such, precision leads to a small increase in the share of pooling offers upon a low signal (weak low signal type effect), but large increase in the share of pooling offers upon a high signal (strong high signal type effect). Increases in precision can then make all buyers worse off and decrease the efficiency of trade, unless they are large enough for sellers to rarely misclassify buyers. A simple application of the product rule shows why,

$$\begin{aligned}
&F(\theta_L|\theta_L;\alpha) = \alpha F^l(\theta_L;\alpha) + (1-\alpha)F^h(\theta_L;\alpha) \\
&\frac{\partial F}{\partial \alpha}(\theta_L|\theta_L;\alpha) = \underbrace{F^l(\theta_L;\alpha) - F^h(\theta_L;\alpha)}_{\geq 0} + \alpha \underbrace{\frac{\partial F^l}{\partial \alpha}(\theta_L;\alpha)}_{>0} + (1-\alpha)\underbrace{\frac{\partial F^h}{\partial \alpha}(\theta_L;\alpha))}_{<0}
\end{aligned}$$

Therefore, when precision is still too low (large $1-\alpha$), the increase in separating offers upon a high signal is dominant (high signal type effect), whereas if precision is higher, the decrease in misclassification (classification effect) and increase in pooling offers upon a high signal (low signal type effect) dominate.



(a) Similar Valuations: $[\theta_L, \theta_H] = [1, 2]$

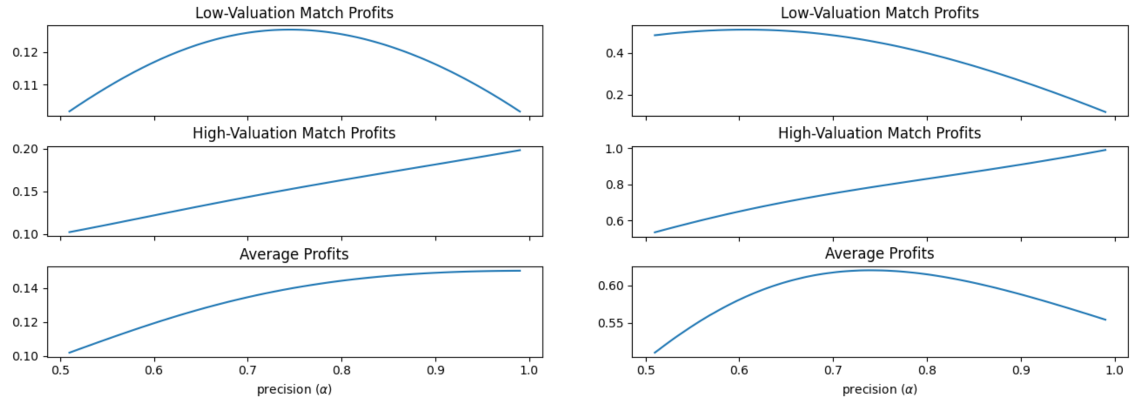
(b) Dissimilar Valuations: $[\theta_L, \theta_H] = [1, 10]$

Figure 2: Buyer Surplus and Efficiency Effects of Precision The common parameters are $p(\theta_L) = 0.5$ for the mass of low-valuation buyers and $\rho = 0.6$ for the level of competition.

Profits Sellers individually benefit from precision, but since we are considering changes in the precision of all sellers in settings with imperfect competition, the profit externalities generated by other sellers' precision also affect them. When the level of competition is low (high ρ), the individual benefit of precision dominates profit externalities - making sellers more profitable at the margin. However, when the level of competition is high enough, profit externalities are dominant.

As we have mentioned, precision makes the offers of sellers who observe a low (high) signal more (less) competitive - the type effects - and the low (high) signal type effect is strong (weak) in economies where buyers' valuations are dissimilar. In these economies, the profit externalities of additional precision are negative, when precision is large enough for sellers to be very exposed to peers' more competitive offers in low-valuation matches, and large, when there is sufficient competition for sellers' profits to be largely dictated by their conditional profitability in competitive matches - making sellers less profitable at the margin.

In summary, more precise information generally makes sellers more profitable; however, when markets are competitive and buyers are heterogeneous, there is a point beyond which it makes sellers less profitable.



(a) Similar Valuations: $[\theta_L, \theta_H] = [1, 2]$ **(b)** Dissimilar Valuations: $[\theta_L, \theta_H] = [1, 10]$

Figure 3: Seller Surplus Effects of Precision The common parameters are $p(\theta_L) = 0.5$ for the mass of low-valuation buyers and $\rho = 0.1$ for the level of competition.

4 Endogenous Quality Setting

Although pricing is one of the main use cases of demand information, another that is at least as important and particularly prevalent in sellers' recent data analytics applications is production - the problem of choosing what to offer to each buyer. Abstracting away from this problem has been convenient analytically and even practically negligible in situations where buyers perceive goods as highly substitutable or sellers' production is inflexible. However, to study settings where product differentiation and design are significant aspects, we will incorporate product choice by allowing sellers to pick the quality of the goods that they offer buyers. Although many of the core insights from exogenous product choice economies will generalize, some in exact form and others with close analogs, the significant differences lend nuance to the effects of precision on welfare and efficiency.

4.1 Seller Problem and Equilibrium Concept

The interesting cases of endogenous quality are those where the efficient quality of trade with low- and high-valuation buyers is different. This requires some cost convexity. We will work with a piecewise-linear cost function that kinks at the efficient qualities of trade with low- and high-valuation buyers.

Assumption 4.1 (Piecewise Linear Costs)**.** *The cost function $\phi(\cdot)$ is piecewise linear, convex, strictly increasing with*

$$\phi(q) = \begin{cases} \kappa_L q & q \le q_L^* \\ \kappa_L q_L^* + \kappa_m q & q_L^* < q \le q_H^* \\ \kappa_L q_L^* + \kappa_m (q_H^* - q_L^*) + \kappa_H q & q_H^* < q \end{cases}$$

where $0 < \kappa_L < \theta_L$, $\theta_L < \kappa_m < \theta_H$, and $\theta_H < \kappa_H$ and the efficient qualities of trade with a buyer of each valuation are given by,

$$q_i^* = \operatorname*{argmax}_q \theta_i q - \phi(q) \tag{4.1}$$

Piecewise linearity makes the marginal cost of quality revisions locally constant, lending considerable tractability[18]. Since sellers can choose quality and its valuation among buyers is heterogeneous,

[18] Our main results are not dependent on this restriction.

they have the ability to improve their screening of buyers by going beyond single quality-price offers and instead offering each buyer a menu $((q_L, x_L), (q_H, x_H))$ composed of a pair of quality-price contracts, where the contract (q_i, x_i) is intended for a buyer of valuation θ_i. When both contracts are identical, the menu is equivalent to a single contract offer, but we will see that such a menu is never optimal for sellers, the first sign that the exogenous product choice assumption imposes serious economic limitations.

There are three components to the profits that a matched seller expects from a menu: (1) the profits per sale from each contract, (2) the probability that a buyer of a given valuation chooses each contract, and (3) the probabilities that the buyer has each valuation. Profits per sale from a contract, $\pi(q_i, x_i) = x_i - \phi(q_i)$, are simply the difference between the lump sum price x_i and the seller's cost of producing the quality q_i. The probability that a buyer of a given valuation chooses a contract is the probability that it has no better offers. By the Revelation Principle, it is sufficient to restrict attention to individually rational and incentive-compatible menus, so the probability that a buyer of valuation θ_i chooses the contract (q_j, x_j) is zero if $i \neq j$, and otherwise equal to the sum of the probability that the seller's offer is the only one[19] (ρ) plus the probability that it has an inferior offer from another seller. To compute the probability that the seller's offer beats that of a competitor, we define the marginal distribution $F_i(u_i) = F(u_i \times [0, \infty]|\theta_i)$ over indirect utilities offered to θ_i valuation buyers by sellers of each type via the joint distribution of utilities in such matches,

$$F(u_L, u_H|\theta_i) = \sum_{\substack{e\in\{a,s\}\\ j\in\{l,h\}}} \mu(e) P^e(j|\theta_i) F^{e,j}(u_L, u_H) \tag{4.2}$$

where $\mu(e)$ is the mass of sellers with precision α_e, $P^e(j|\theta_i)$ is the proportion of them that would observe j signals when matched with a θ_i valuation buyer (and so that would be of type $p_L^{e,j}$), and $F^{e,j}(u_L, u_H)$ is the joint distribution over indirect utility offers implied the menus that sellers of type $p_L^{e,j}$ mix over. Therefore, the probability that a θ_i valuation buyer chooses a contract (q_i, x_i) contract is $\Psi_i(u_i) = \rho + (1-\rho)F_i(u_i)$, where $u_i = u(q_i, x_i; \theta_i)$ is the indirect utility it offers θ_i valuation buyers. Lastly, a matched seller's probability that the buyer has low and high-valuation is given by its type $p_L^{e,j}$ and the respective complementary probability $1 - p_L^{e,j}$. Since each contract (q_i, x_i) determines the seller's expected profits in a match with each type of buyer, the expected profits from the menu are therefore given by the average of these,

$$\begin{aligned}\Pi^{e,j}(q_L, x_L, q_H, x_H) &= p_L^{e,j}\Psi_L(u(q_L, x_L; \theta_L))\pi(q_L, x_L) + p_H^{e,j}\Psi_H(u(q_H, x_H; \theta_H))\pi(q_H, x_H) \\ &= \sum_{i=l,h} p_i^{e,j}\Psi_i(u(q_i, x_i; \theta_i))\pi(q_i, x_i)\end{aligned} \tag{4.3}$$

weighted by the seller's valuation beliefs.

4.2 Seller Strategies

We so far have implicitly allowed for both pooling and separating offers; however, as our notation suggests, only separating menus are offered in equilibrium.

Corollary 4.1 (No Pooling in Equilibrium). *Equilibrium menus separate buyers of each valuation.*

Cost convexity is at the core of this result. It is the reason why a pooling contract can always be improved through a separating revision that adds quality to the high contract, at a price (above

[19] The probability that the seller is a monopolist given that it is matched.

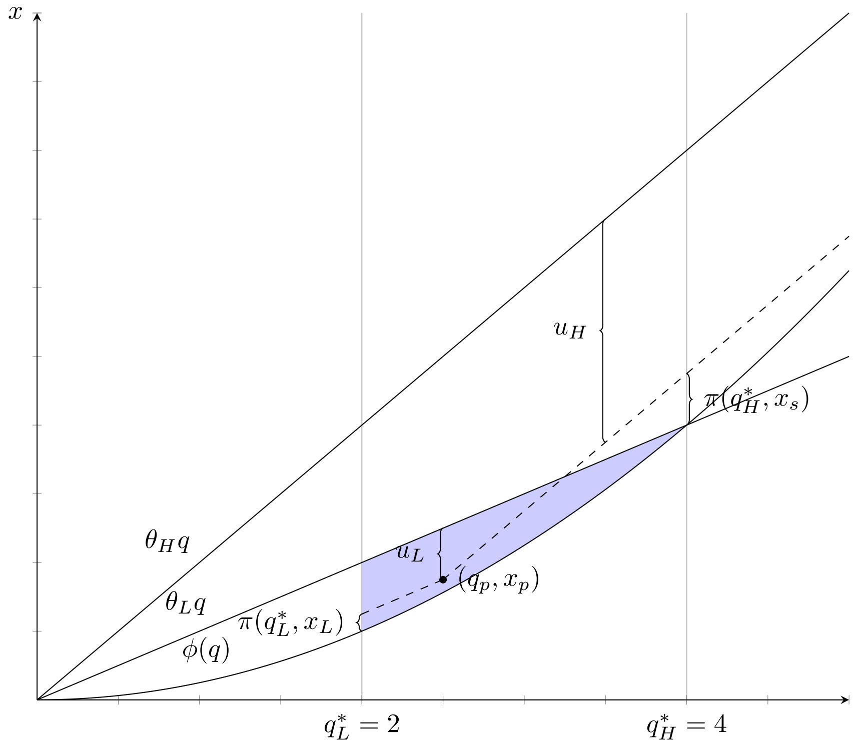


Figure 4: Example with $\theta_H = 4$, $\theta_L = 2$, $\phi(q) = \frac{1}{2}x^2$. Contracts are depicted as points in the (q, x) space. Implied utilities are given by the vertical distance of a contract's y-axis coordinate to the zero utility indifference curves of the respective θ_i valuation buyers, while profits per sale are given by the vertical distance to the seller's cost function $\phi(q)$. The blue region represents the set of pooling contracts that are individually rational for buyers of either valuation, imply non-negative profits, and are not dominated by another pooling contract.

A candidate pooling offer (q_p, x_p) lies on dashed iso-utility lines for buyers with low and high-valuation; we follow these to the right and left, respectively, until reaching the efficient qualities of trade with each type of buyer at prices $x_i = \theta_j q_i^* - u(q_p, x_p; \theta_i)$. This alternative separating bid $((q_L = q_L^*, x_L = x_p - \theta_L(q_p - q_L^*), (q_H = q_H^*, x_H = x_p + \theta(q_H^* - q_p))$ remains incentive compatible and strictly dominates the pooling offer: the same utility to buyers of either valuation (hence the same probability of winning) but strictly higher profits per sale.

marginal costs) only high-valuation buyers are willing to pay, and reduces the quality of the low, in exchange for a discount only low-valuation buyers are interested in. We provide a simple graphical representation of this procedure in Figure 4. Beyond its economic relevance, this result is analytically convenient because it allows us to avoid both the classical threat to existence of equilibrium (a la Rothschild Stiglitz (1976)) and a more complicated diversity of offers (as in Lester et al. (2019)).

Beyond separating buyers, equilibrium menus have quite a bit of additional structure. In particular, the quality-price terms of each contract featured in a menu are closely linked to the utility they allow their targeted buyer to obtain. The forward direction is obvious since incentive constraints and individual rationality of sellers menus imply buyers will always choose their intended contract from $((q_L, x_L), (q_H, x_H))$,

$$
\begin{aligned}
(IC_i) &: \quad u(q_i, x_i; \theta_i) \geq u(q_{\neg i}, x_{\neg i}; \theta_i) \ \forall i \in \{l, h\} \\
(IR_i) &: \quad u(q_i, x_i; \theta_i) \geq 0
\end{aligned}
$$

so the indirect utility offered to buyers of each respective valuation by a menu is simply that offered by their respective contracts,

$$u_L = \theta_L q_L - x_L$$
$$u_H = \theta_H q_H - x_H$$

The backward direction follows from conditions that profit maximality imposes on optimal offers, and they imply the existence of a bijection between the indirect utilities (u_L, u_H) and quality-price terms $((q_L, x_L), (q_H, x_H))$ of contracts in any *equilibrium* menu.

Theorem 4.1 (Converting to Indirect Utilities). *Consider an equilibrium menu $((q_L, x_L), (q_H, x_H))$ with associated indirect utilities (u_L, u_H). Qualities are then given by,*

$$q_L(u_L, u_H) = \begin{cases} \frac{u_H - u_L}{\Delta\theta} & u_H - u_L < q_L^* \Delta\theta \\ q_L^* & u_H - u_L \geq q_L^* \Delta\theta \end{cases} \qquad q_H(u_L, u_H) = \begin{cases} \frac{u_H - u_L}{\Delta\theta} & u_H - u_L > q_H^* \Delta\theta \\ q_H^* & u_H - u_L \leq q_H^* \Delta\theta \end{cases} \tag{4.4}$$

and prices by $x_i = \theta_i q_i - u_i$.

Therefore, when an incentive constraint binds at an optimal menu, the difference in utilities $u_H - u_L$ uniquely pins down the qualities q_L, q_H offered in each contract, while the level of these utilities then uniquely determines their respective prices. When both incentive constraints are slack, however, it is optimal to offer efficient qualities to each type of buyer, which is why we will refer to these menus as *dually efficient* from now on, and the prices follow uniquely in the same fashion.

This result connecting the quality-price form of menus to their associated indirect utilities is standard in mechanism design - referred to as the *parametric-utility approach* (Rochet and Stole (2006)) - and originates from the fact that seller surplus (profits per sale) equals the gains from trade $S_i(q) = \theta_i q - \phi(q)$ net of buyer surplus,

$$\pi(q_i, x_H) = x_i - \phi(q_i) = (x_i - \theta_i q_i) + (\theta_i q_i - \phi(q_i)) = S_i(q_i) - u(q_i, x_i; \theta_i)$$

Profits then increase both from minimizing buyer surplus *and* maximizing trade efficiency. As such, consider the optimal way to offer a pair of utilities (u_L, u_H). If it is possible to do so with a dually efficient menu that respects incentive constraints, then it is optimal to do so and unlock a larger social surplus. If it is not possible and a buyer's incentive constraint would be violated by such an offer, then a problematic IC_H (IC_L) constraint is corrected most profitably by under-providing (over-providing) quality in the low (high) contract. In these cases, a valuation θ_i buyer - whose incentive constraint binds at the optimal menu that offers (u_L, u_H) - is indifferent between their contract and the one intended for the opposite $\theta_{\neg i}$ valuation buyer, $u_H - u_L = q_{\neg i}\Delta\theta$.

Since optimal offers can be expressed in terms of indirect utilities, it is natural to recast the strategy of each type of seller $p_L^{e,j}$ as a distribution $F^{e,j}(u_L, u_H)$ over pairs of utility offers (u_L, u_H). In this notation, the profits that a type $p_L^{e,j}$ seller expects from an offer (u_L, u_H) are,

$$\Pi_i(u_L, u_H) = \Psi_i(u_i)\pi_i(u_L, u_H)$$
$$\Pi^{e,j}(u_L, u_H) = \sum_{i=l,h} p_i^{e,j} \Pi_i(u_L, u_H)$$

Therefore, the *level* of indirect utilities (u_L, u_H) determines both the probability of winning the contested offers $1 - F_i(u_i)$ and the part of the buyer surplus of the profits per sale $\pi(u_L, u_H) = S(u_L, u_H) - u_i$, while the *difference* in generosity towards a high and low-valuation buyer, $u_H - u_L$, determines the part of the social surplus of the profits per sale, through the efficiency of trade with them. Given the sample space $\Omega = [0, S_L^*] \times [0, S_H^*]$, Borel σ-algebra, and set of countably additive probability measures $\mathscr{P}$ over it, this concise specification of the seller's problem allows us to introduce the equilibrium concept.

Definition 4.1 (Bayes-Nash Equilibrium). *An equilibrium is a vector of strategies* $\{F^{e,j}\}_{\substack{e\in\{a,s\}\\ j\in\{l,h\}}}$, *for each type of seller, such that,*

$$\text{supp}(F^{e,j}) \subseteq \underset{(u_L,u_H)}{\text{argmax}}\, \Pi^{e,j}(u_L, u_H) \quad \forall\ (e,j) \in \{a,s\} \times \{l,h\}$$

4.2.1 Corner Cases: Monopoly and Perfect Competition

We start building intuition by analyzing the corner cases of competition, as economies with exogenous quality economies. The screening problem of a monopolist is well understood from Mussa Rosen (1978). The only trade-off for this seller is between the efficiency of trade with low-valuation buyers and the share of efficient social surplus S_H^* that it captures in trade with the high-valuation buyer. A type $p_L^{e,j}$ monopolist thus solves

$$\max_{u_L,u_H\geq 0} p_L^{e,j}(S_L(u_L,u_H) - u_L) + p_H^{e,j}(S_H(u_L,u_H) - u_H)$$

where incentive compatibility is implicit in the functional form of social surplus terms. We can ignore menus with (1) $u_H - u_L > q_L^*\Delta\theta$ or (2) $u_L > 0$, as they would not entail greater efficiency of trade and strictly lower profits per sale from buyers of at least one valuation than the offer $(u_L, u_H) = (0, q_L^*\Delta\theta)$. The incentive constraint of high-valuation buyers, therefore, binds at an optimal menu, and the optimal quality in the low contract. Our sellers have piecewise linear costs that give rise to a bang-bang monopolist outcome, with a fully efficient or inefficient offer, depending on where the seller's type $p_L^{e,j}$ is beyond a threshold belief p_L^* determined by the marginal benefit of increasing the indirect utility that is offered to high-valuation buyers,

$$p_L^* \underbrace{\frac{\theta_L - \kappa_L}{\Delta\theta}}_{\text{efficiency gain}} - \underbrace{p_H^*}_{\text{rent loss}} = 0 \tag{4.5}$$

The left-hand term is the marginal gain in efficiency made possible by increasing the indirect utility offered to high-valuation buyers. This relaxes their incentive constraint and allows the seller to provide a more efficient quality to low-valuation buyers, increasing the social surplus from trade with them and thus increasing the profitability of each low-valuation sale. The right-hand term represents the rents surrendered by making a more generous offer to high-valuation buyers - featuring the same quality at a strictly lower price.

Proposition 4.1 (Monopoly). *If $\rho = 1$, a seller of type $p_L^{e,j}$ offer a menu,*

$$(u_L, u_H) = \begin{cases} (0,0) & \textit{if } p_L^{e,j} < p_L^* \\ (0, q_L^*\Delta\theta) & \textit{if } p_L^{e,j} \geq p_L^* \end{cases} \tag{4.6}$$

with p_L^* as per (4.5).

We obtain an ordering property that is familiar from exogenous quality economies, mainly that the efficiency and generosity of offers is monotonically ranked by the seller's type, the nonmonotonicity of trade efficiency and buyer surplus, along with its dependence on preference heterogeneity $\Delta\theta$, which explicitly determines the size of the efficiency gain $\frac{\theta_L - \kappa_L}{\Delta\theta}$ from relaxing high-valuation buyers' incentive constraint. In detail, when buyers' valuations are sufficiently homogeneous, imprecisely informed monopolists always trade efficiently ($p_L^{e,j} > p_L^*$ for both signals $j \in \{l, h\}$) and additional precision can decrease the efficiency of trade ($p_L^{e,h} < p_L^*$ for α_e large) and hurt high-valuation buyers (low-valuation buyers always get zero surplus). However, when buyers' valuation are sufficiently heterogeneous, imprecisely informed monopolists ration low-valuation buyers upon the high and even low signal, so precision increases efficiency of trade and can allow high-valuation buyers to obtain some information rents from misclassification. These ordering, efficiency, and welfare relations extend when competition is imperfect, with the usual caveat that competition allows low-valuation buyers to obtain some of the gains from trade and thus benefit from precision, so long as it is not too efficiency reducing.

The other corner where sellers are guaranteed to bid against another seller in every match ($\rho = 0$) is also well understood, as each match again becomes a setting of Bertrand price competition. Therefore, sellers make dually efficient offers, and buyers capture all gains from trade.

Proposition 4.2 (Perfect Competition)**.** *If $\rho = 0$, the unique Bayes-Nash equilibrium involves almost every seller offering $((q_L, x_L), (q_H, x_H)) = ((q_L^*, \phi(q_L^*)), (q_H^*, \phi(q_H^*)))$ with probability 1.*

Second-degree price discrimination does not change the point that predictive skill is of little use under perfect competition. Competition also promotes efficiency and consumer surplus when it is at interior ($\rho \in (0, 1)$) levels, but it has different effects on how much sharks and amateurs use their predictive skill, because all of the latter's offers are dually efficient and thus invariant to their signal in more settings. Lastly, we note that the combination of imperfect competition and heterogeneous seller precision allows us to isolate its profit externalities, by fixing the precision of one group of sellers (for example, amateurs) and varying the precision of the other (for example, sharks). We find that sellers become more profitable through their own predictive skill, and sometimes even that of competitors.

4.3 Equilibrium Structure

In this section, we discuss some general properties satisfied by the menus of a candidate equilibrium. Each has intuitive appeal, either from an economic standpoint or because of the mathematical tractability that they impart. By leveraging these in conjunction with the optimality conditions of sellers' problems, we can solve for this equilibrium analytically and obtain a complete characterization. In the Appendix, we show that these properties hold in any equilibrium where at least some offer rations low-valuation buyers.

4.3.1 Claims about Equilibrium Structure

There are three main areas that benefit from additional structure: (1) the distributions over indirect utilities offered by sellers in low- and high-valuation matches, F_i, (2) the relationship of incentive compatibility constraints to the generosity (indirect utility) of menus, and (3) the relationship between generosity of a menu with the type $p_L^{e,j}$ of the seller who offers it.

The equilibrium distributions of indirect utilities F_i offered in each match are weighted averages of each seller type's mixed strategy, and it inherits their properties.

Claim 4.1. *The equilibrium marginal distributions over indirect utilities F_i for $i \in \{l, h\}$,*

1. *are atomless.*

2. *have a connected support of low utility offerings* $\Upsilon_L = [\underline{u}_L, \overline{u}_L]$.

3. *have a support of high utility offerings* $\Upsilon_H = \cup_{\substack{e\in\{a,s\},\\ j\in\{l,h\}}} [\underline{u}_H^{e,j}, \overline{u}_H^{e,j}]$, *made up of the contiguous bids by type* $p_L^{e,j}$ *sellers.*

4. *are continuously differentiable with densities* f_i *in the interior and one-sided derivatives at the boundaries.*

By avoiding atoms and gaps in the supports of each type of seller's mixture, erratic behavior is curtailed. It is never optimal to bunch up and compete at a single point in the space of utility offers, as one might see in the corner cases of monopoly and Bertrand, nor do we see discontinuous jumps in generosity among sellers of the same type. While continuous densities allow us to consider marginal incentives, which helps convey economic intuition and solve for the equilibrium as the solution of a standard differential system.

As in the closely related work of Lester et al. (2019) and Garret et al. (2019), a property called *ordering*, which refers to an equilibrium where ranking menus by the utility offered to low- or high-valuation buyers is identical, lends a great degree of tractability. This property turns out to be necessary in any equilibrium where the incentive constraint of low-valuation buyers is slack in every menu that is offered, and a sufficient condition for this is the assumption that sellers cannot profitably trade the efficient high quality with low-valuation buyers.

Assumption 4.2 (No IC_L Cost Condition)**.** *The piecewise linear cost function* $\phi(\cdot)$ *is such that,*

$$\theta_L q_H^* \leq \kappa_m(q_H^* - q_L^*) + \kappa_L q_L^* \tag{4.7}$$

Claim 4.2. *Low-valuation buyers' incentive constraint never binds in equilibrium menus.*

Eliminating the possibility of a binding low-valuation buyer constraint means that high-valuation buyers always obtain their efficient quality (q_H^*), but low-valuation buyers are rationed ($q_L < q_L^*$) whenever the offer intended for them is part of a menu in which high-valuation buyers' incentive constraint binds, as in the monopoly setting.

We will focus on equilibria in which offers are (a) ordered by their generosity, but also (b) monotone in the seller's type, meaning that generosity weakly increases with the seller's belief of being matched with a low-valuation buyer $p_L^{e,j}$, as in the ordered equilibria of economies where quality is exogenous.

Claim 4.3. *Given two equilibrium menus* (u_L, u_H) *and* $(\tilde{u}_L, \tilde{u}_H)$ *offered by sellers of respective types* p_L *and* $\tilde{p}_L$ *with* $u_i > \tilde{u}_i$ *for some* $i \in \{l, h\}$,

1. *It is also true that* $u_{\neg i} \geq \tilde{u}_{\neg i}$ *and strictly so if both menus are offered by sellers of the same type.*

2. *The gap between low and high utilities* $u_H - u_L$ *increases strictly with generosity.*

3. *Each support* $\Upsilon_i = [\underline{u}_i, \overline{u}_i]$ *is such that* $\overline{u}_i \leq S_i^*$. *Further, there exists a* $u_i^{de} \in [\underline{u}_i, \overline{u}_i]$ *such that all* $u_i < u_i^{de}$ *are in* IC_H *binding menus and all* $u_i \geq u_i^{de}$ *are dually efficient.*

Due to Theorem 4.1, the efficiency of any menu is directly pinned down by the difference in its respective utility offers $u_H - u_L$, and since low-valuation buyers' incentive constraint does not bind in the menus of these equilibria, a larger difference in utility offers only improves efficiency - weakly raising the quality of low trade towards the optimal, q_L^*. Orderedness in generosity then follows from the complementarity between relaxing the incentive compatibility constraint of a high-valuation buyer (through a more generous u_H term) - which permits more profitable sales to low-valuation buyers - and increasing low sales (through a more generous u_L term). In equilibrium, the first effect dominates - sellers face stronger incentives to increase the generosity of their high- versus low-valuation offer - so that the difference in utilities $u_H - u_L$ and, consequently, efficiency is weakly increasing in generosity. This upward efficiency progression then also generates a natural grouping of menus, with the least generous menus also being the least efficient (constrained by high-valuation buyers' incentive compatibility), but above a generosity threshold, becoming dually efficient (unconstrained by either buyer's incentive compatibility).

Equilibrium profits conditionally on matching with a low (high) buyer provide the incentives that order the offers of sellers with different interim valuation beliefs (types). These give rise to the second ordering relation that links a seller's type to the generosity of its offers. In particular, the profitability of menus in low- (high-) valuation matches increases (decrease) with their generosity and lead sellers who place more posterior weight on low-valuation matches to offer more generous contracts.

Claim 4.4. *Given two equilibrium menus (u_L, u_H) and $(\tilde{u}_L, \tilde{u}_H)$ offered by sellers of respective types p_L and $\tilde{p}_L$ with $u_i > \tilde{u}_i$ for some $i \in \{l, h\}$,*

1. *Profitability conditionally on matching with a buyer of low- (high-) valuation is increasing (decreasing) in generosity,*

$$\Pi_L(u_L, u_H) \geq \Pi_L(\tilde{u}_L, \tilde{u}_H) \text{ and } \Pi_H(u_L, u_H) \leq \Pi_H(\tilde{u}_L, \tilde{u}_H)$$

with both inequalities being strict if high-valuation buyers' incentive constraint binds at (u_L, u_H)

2. *Generosity is increasing in the seller's conviction of facing a low-valuation buyer, $p_L > \tilde{p}_L$, and strictly so if high-valuation buyers' incentive constraint binds at (u_L, u_H).*

The origin of this conditional profit monotonicity is clear when we consider the first-order condition satisfied by the bids of sellers offering menus at which high-valuation buyers' incentive constraint binds,

$$0 = p_H^{e,j} \frac{\partial \Pi_H}{\partial u_H}(u_H, u_L) + p_L^{e,j} \underbrace{\frac{\partial \Pi_L}{\partial u_H}(u_H, u_L)}_{\geq 0}$$

$$0 = p_L^{e,j} \frac{\partial \Pi_L}{\partial u_L}(u_H, u_L)$$

When high-valuation buyers' incentive constraint binds at a menu, greater generosity toward them (larger u_H) relaxes the constraint and makes low trade more profitable, so the term $\frac{\partial \Pi_L}{\partial u_H}(u_H, u_L)$ is strictly positive. As such, the profits from high-valuation sales of constrained menus (Π_H) are locally decreasing in generosity towards high-valuation buyers (u_H) but invariant in that towards the low (u_L), while their profits from low-valuation sales (Π_L) are increasing generosity towards high-valuation buyers and at a local maximum with respect to low-valuation buyer generosity. When we consider the set of constrained menus, the least generous are offered by the sellers who are comparatively more concerned about the profitability in high-valuation matches, mainly those sellers of the lowest type $p_L^{e,j}$. The efficiency-generosity relation places these constrained menus below any dually efficient ones, so sellers who are sufficiently concerned about low-valuation match profits offer the latter (dually efficient) menus. However, when menus are dually efficient, incentive constraints are slack, and changing the generosity of the contract offered to a buyer of either type does not impact the profitability of the paired contract, offered to a buyer of the alternate type (by Theorem 4.1). The term $\frac{\partial \Pi_L}{\partial u_H}(u_H, u_L)$ is, therefore, equal to zero at dually efficient offers, so these satisfy the pair of equations,

$$0 = \frac{\partial \Pi_H}{\partial u_H}(u_H, u_L)$$
$$0 = \frac{\partial \Pi_L}{\partial u_L}(u_H, u_L)$$

implying that dually efficient offers are equally profitable and that all sellers - irrespective of their type - are indifferent between them. The most that we can necessarily say about the relation between a seller's type and dually efficient menus is that the type must be high enough for the seller to offer one. Any additional relation between sellers' type and the generosity of dually efficient menus is not just unnecessary but unappealing because it would allow precision to shape the share of surplus that buyers of each valuation obtain, purely due to the equilibrium selection rule. Instead, we consider the unique equilibrium where any seller has the same distribution of dually efficient offers conditionally on making any.

Ordering in economies with endogenous quality, therefore, differs from that with exogenous quality in an important sense. In the latter, the mass that each type of seller places on inefficient offers (price larger than θ_L) can be arbitrarily shifted while preserving the aggregate offer distribution, whereas among efficient offers (price lower than θ_L) the lowest ones must be offered by sellers of the largest type (unless all offers are efficient). In economies with endogenous quality, the opposite is true, as inefficient offers must be strictly ranked by the type of the seller who offers them, whereas efficient offers are not.

We close by connecting these points about profitability, generosity, and a seller's type to the benefits of precision. In our setting, with two information structures (levels of precision), each of which features a random variable (signal) that can take two possible values (low or high), there are four types of sellers at the interim stage, corresponding to each precision and signal combination. And, sellers with the highest precision have the highest conviction about their signals and the most extreme beliefs upon each signal,

$$p_L^{s,h} < p_L^{a,h} < p_L^{a,l} < p_L^{s,l} \tag{4.8}$$

As such, in an equilibrium where at least some inefficient menus are offered, the least generous menus, towards a buyer of either valuation, are offered by sharks who observe high signals, while the most generous are offered by sharks who observe low signals. Amateur offers have intermediate generosity and only overlap with those of sharks if they are dually efficient. As such, in low- (high-) valuation matches, a greater share of sharks (than amateurs) observe signals that orient their offers to the top (bottom) of the generosity distribution, where the most profitable low- (high-) valuation match menus are found, and makes them more profitable than amateurs - the benefit of predictive skill.

4.4 Comparative Statics

We will perform an analogous comparative static analysis to Section 3.4, highlighting points of commonality and departure. The aggregate statistics for welfare and trade efficiency are similar after adjusting for the fact that we express strategies in the space of indirect utility offers and that sellers are now heterogeneous. Recycling notation, the welfare of each type of buyer follows as,

$$\mathscr{W}_b(\theta) = \rho E[u_i] + (1-\rho)E[\max(u_i, \tilde{u}_i)] \tag{4.9}$$

where u_i are iid draws from the conditional equilibrium marginals $F_i(u_i|\theta_i)$. Seller profits are given by,

$$\Pi^e = E\Big[\underbrace{\mathscr{M}}_{\text{number of matches}} \quad \underbrace{\Psi_i(u_i)}_{\text{probability of selling}} \quad \underbrace{S_i(u_L, u_H) - u_i}_{\text{profits-per-sale}}\Big| e\Big] \tag{4.10}$$

where we also average over the probabilities that sellers with precision α_e observe each signal $j \in \{l, h\}$ in a match with a buyer of each valuation $\theta_i \in \{\theta_L, \theta_H\}$, and that they mix over menus as per $F^{e,j}$ upon each signal. Lastly, in the case of efficiency, high-valuation buyers always receive efficient offers, but the low may get rationed, so we measure the efficiency of trade by the average quality offered to the latter,

$$\rho E[q_L|\theta_L] + (1-\rho)E[\max(q_L, q'_L)|\theta_L] \tag{4.11}$$

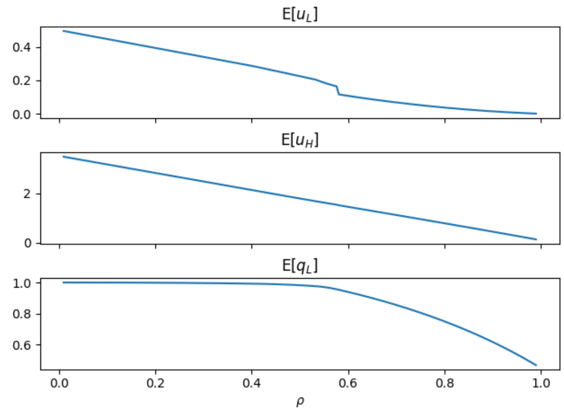
where we rely on the equilibrium property that generosity and efficiency are positively correlated, so buyers always select an offer with the greatest efficiency available.

The qualitative effects of competition and precision on efficiency and buyer surplus do not change when we endogenize quality - directionally similar, under similar environments. However, this changes where we track efficiency. Whereas pricing uniquely impacts trade efficiency at the extensive margin - determining whether a buyer trades or not by finding any acceptable prices - when quality is exogenous, removing this barrier allows information to shape the level of trade at the intensive margin - determining the quality of trade a buyer obtains. Stronger still, the optimality of screening menus allows almost every buyer to trade in *some* amount, so that *all* action is at the intensive margin - a theoretical insight that informs empirical analysis.

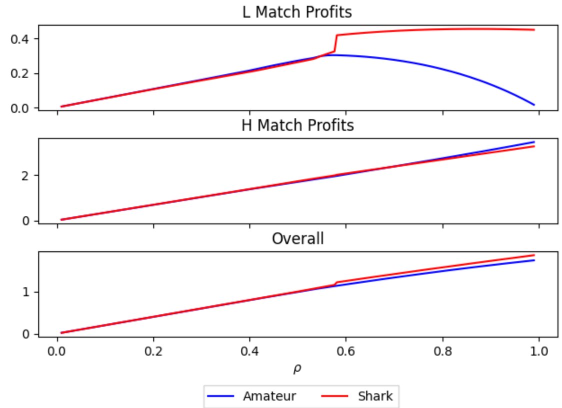
Seller heterogeneity introduces more significant differences. For one, it allows us to cleanly analyze precision's profit externalities and find that precision growth at the frontier (shark precision) is broadly beneficial for *all* sellers, even laggards, whereas precision growth among laggards generally only benefits them and hurts more advanced sellers. Second, it gives rise to an ordering of interim types (4.8) that connects with the ordering of their offers, and that, in turn, implies that in sufficiently competitive economies amateurs do not use their predictive skill, as their offers upon either signal are dually efficient and equally distributed. In other words, competition neutralizes the predictive skill of amateurs, when quality is endogenous.

4.4.1 Competition

The aggregate effects of competition are similar when we allow information to also orient production - increasing trade efficiency, buyer surplus, and decreasing seller surplus - but the mechanism giving rise to them is different. In particular, competition increases the sales gains from generosity, so sellers offer more generous menus, and complementarity between the utility of offers to each type of buyer then links both their joint progression and the quality of the good that is offered to low-valuation buyers.



(a) Buyer Surplus and Efficiency

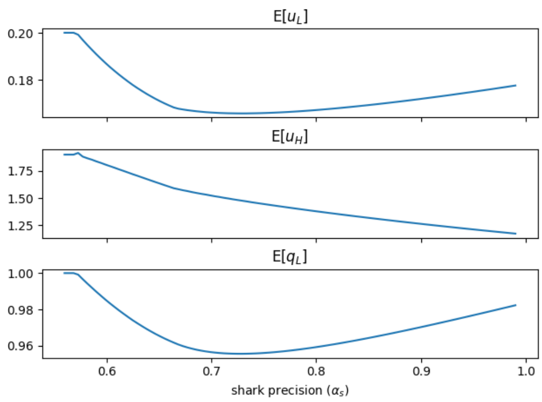


(b) Seller Surplus

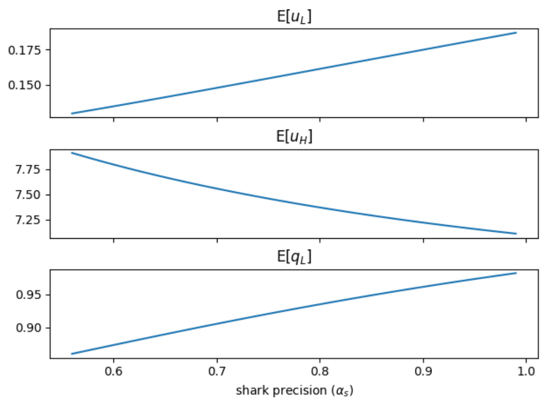
Figure 5: Buyer Surplus and Efficiency Effects of Competition The common parameters are $[\theta_L, \theta_H, P(\theta_L)] = [1, 3, 0.5]$ for buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*] = [0.5, 2, 1, 2]$ for sellers' cost functions, $[\alpha_a, \alpha_s] = [0.55, 0.95]$ for sellers' precision, and $P(a) = 0.5$ for proportion of amateurs.

4.4.2 Precision

The qualitative effects of precision on the efficiency of trade and buyer surplus are similar as well, with valuation heterogeneity largely determining precision's effects: in economies where buyers' preferences are similar, precision can decrease trade efficiency and hurt all buyers, whereas in economies where buyers' preferences are dissimilar, precision increases trade efficiency and low-valuation buyer surplus, but generally[20] hurts high-valuation buyers. We illustrate these points in Figure 6 by tracking the response of buyer surplus and trade efficiency to changes in shark precision.



(a) Similar Valuations: $[\theta_L, \theta_H] = [1, 2.5]$



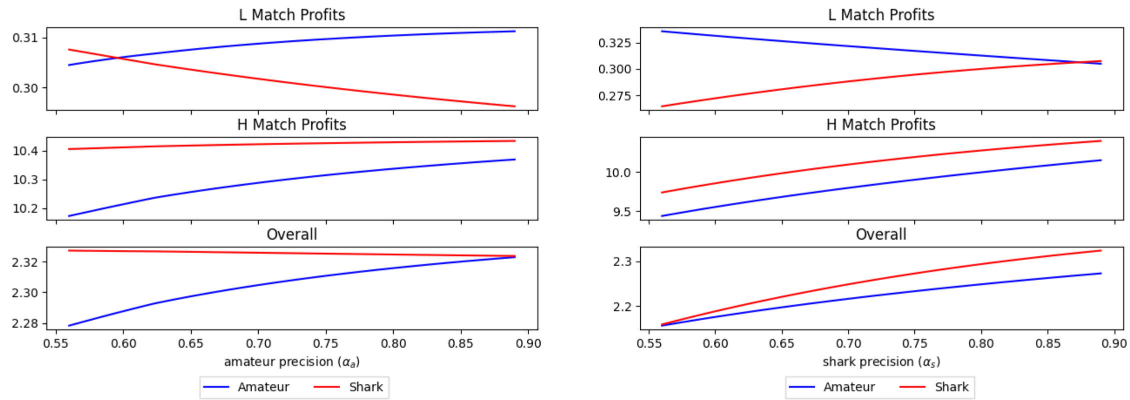
(b) Dissimilar Valuations: $[\theta_L, \theta_H] = [1, 10]$

Figure 6: Buyer Surplus and Efficiency Effects of Precision Half of the sellers are sharks and we increase their precision α_s. The common parameters are $P(\theta_L) = 0.8$ for the mass of low-valuation buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*] = [0.5, 2, 1, 2]$ for sellers' cost functions, $\alpha_a = 0.55$ for amateur precision, and $\rho = 0.6$ for the level of competition.

When it comes to sellers, the ordering of offers is fundamental. Inefficient offers from amateurs only beat sharks that observe high signals, and the profitability of dually efficient offers is only determined by the generosity of inefficient offers below, so shark precision sets off a chain of downward generosity

[20]There are still cases where additional precision benefits high-valuation buyers, by encouraging generosity upon a low signal (low signal type effect).

revisions that benefit them and amateurs: the inefficient offers of high signal sharks become less competitive (generous) when their information is more precise (lower $p_L^{s,h}$ type under larger α_s), which makes any of the following inefficient offers of amateurs above less competitive/generous, which makes any of the following inefficient offers of low signal sharks above less competitive/generous, which ultimately makes any of the following dually efficient offers above less competitive generous.



(a) Amateur Precision Change, Shark Precision $\alpha_s = 0.9$

(b) Shark Precision Change, Amateur Precision $\alpha_a = 0.55$

Figure 7: Seller Surplus Effects of Precision The common parameters are $[\theta_L, \theta_H, P(\theta_L)] = [1, 10, 0.8]$ for buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*, p(a)] = [0.5, 2, 1, 2, 0.5]$ for sellers, and $\rho = 0.6$ for the level of competition.

On the other hand, sharks generally suffer from amateur expertise. Sharks who observe a high signal always lose against amateurs, so their profitability is invariant to amateur precision, but sharks who observe a low signal always beat amateurs, so they are exposed to the effect of precision on amateurs who observe high signals, who become less competitive, and low signals, who become more competitive. High signal amateurs roll back generosity under greater precision, in particular that to high-valuation buyers, and this chains up the distribution, so sharks' sales to high-valuation buyers generally become more profitable, but low signal amateurs become willing to compete more aggressively for any amount of low-valuation trade, so they strongly increase the generosity of their low-valuation offers, decreasing the profitability of sharks' sales to low-valuation buyers. It is the latter effect that generally dominates; the exceptions, where all sellers benefit from amateur precision at the margin, are found in settings where buyers have very similar preferences and amateur precision is relatively low. We illustrate the general profitability trends in Figure 7 and the more exceptional one, where amateur precision benefits all sellers, in Figure 8.

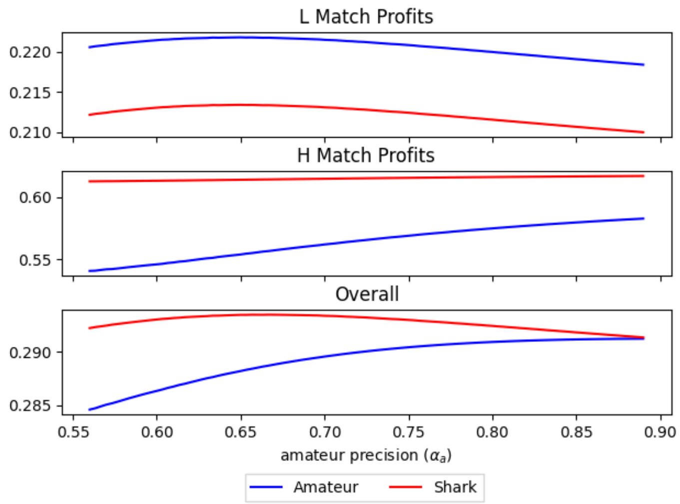


Figure 8: All Sellers Benefit from Amateur Precision The parameters are $[\theta_L, \theta_H, P(\theta_L)] = [1, 2.1, 0.8]$ for buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*, p(a)] = [0.5, 2, 1, 2, 0.5]$ for sellers, and $\rho = 0.4$ for the level of competition.

5 Conclusion

A race in analytics is currently taking place, where firms' performance is intrinsically linked to their ability to source and analyze data. Although it may be too early to tell, an emerging literature[21] argues that many of the technologies that are being deployed - machine learning, big data, data mining, natural language processing, etc. - are indeed general-purpose technologies (GPT) i.e. "widely used, capable of ongoing technical improvement, and enabling innovation in application sectors" (Bresnahan et al. (1995), Bresnahan (2010)). Since the consequences are likely to be deep, pervasive, and persistent, research on this analytics pipeline is fundamental and urgent.

Motivated by the size of the retail sector and its intense reliance on data to optimize product and pricing decisions, we answer a "crucial" question posed by Bergemann and Bonatti (2019) in their review of markets for information: "what are the implications of acquiring an advantage in a downstream market by means of better data (e.g. improvements in the predictive power of an algorithm)?" The problem is addressed within a model that features sellers with predictive skill that has strategic value, as they face adverse selection from buyers and imperfect competition from other sellers. Aligning with empirical documentation of broad endemic differences in firms' use of predictive technologies, as well as forward-looking policy concerns about these disparities, we allow the precision of firms' predictions to be heterogeneous. We find predictive precision to be generically efficiency-enhancing but redistributive. On the demand side, it tends to benefit (hurt) low- (high-) valuation buyers. On the supply side, unsurprisingly, firms use more precise forecasts to obtain more profitable sales, but, more interestingly, their predictive skill can also benefit competitors. This suggests the need for nuanced analysis of distributional impacts, in contrast to fears of endemic harm that have tinged public discussion.

Several modeling compromises have left open avenues for future research with connections to concurrent work. For one, predictive skill is exogenous in our model, despite the indications that there are potentially very interesting forces at work in its initial acquisition. Second, once an initial level of predictive skill has been acquired, our reduced-form model of precision cannot consider the

[21] Brynjolfsson, Rock and Syverson (2019), Cockburn et al. (2019), Trajtenberg (2019), Goldfarb (2019)

important feedback loop between data and the strategic aspects that it endows sellers with. These choices constrain our model to be static in its most natural interpretation, inhibiting us from exploring essential dynamics, such as learning within and across each side of the market, market structure, and prices as well as production. A formal characterization of these would be very helpful in understanding the drivers of observed asymmetries. Lastly, the naivete of buyers with respect to sellers' information could also benefit from relaxation, given its importance to outcomes, even within the present setting, as well as its role in a host of results emerging from the literature on consumer privacy. Much like these agents, we are using a model to understand a complicated problem; however, structural concerns have led us to sacrifice additional complexity for the sake of intelligibility. Our hope is that this work contributes to an ongoing discussion that advances the precision of our understanding.

Appendix A Exogenous Quality Setting: Equilibrium Structure

We will first establish the properties of the distribution of prices offered by the average seller to low- and high-valuation buyers, $\{F(x|\theta\}_{\theta\in\{\theta_L,\theta_H\}}$. Since these distributions are unique, buyer surplus and trade efficiency will be invariant across equilibria, which differ in the allocation of mass among the strategies of each type of seller, but not in the distribution of offers that buyers obtain. These distributions will also give rise to monotone relations in the profitability of prices in each type of match: higher prices are weakly more profitable in high-valuation matches, whereas separating prices ($x > \theta_H$) and any pooling prices ($x \leq \theta_L$) in both type of sellers' supports maintain profits in either type of match. Since the multiplicity of equilibria is only from shift in mass among sellers' strategies over prices covered by the latter point - where profits in either type of match are constant - it is also true that across equilibria seller surplus is invariant.

Then, we will derive sellers' strategies in the unique equilibrium that is simplest to derive. This is an equilibrium where seller's strategies are weakly ordered and which has the following structure: the highest prices are offered by high signal sellers, below which there can be a set of prices that is offered by both types of sellers, and below which there is a region of prices offered only by low signal sellers.

A.1 Distribution of Offers

By (3.4), equilibrium distributions of offers in a low or high-valuation match - $F(x|\theta_L)$ and $F(x|\theta_H)$, respectively - are averages of each type of seller's strategy, weighted by the mass of sellers with the respective type in matches with buyers of the respective valuation. Since buyers of either valuation have a strictly positive probability of matching with any type of seller, these distributions share identical supports - their differences lie in how mass is allocated within these.

Proposition A.1 (Identical Supports)**.** *Equilibrium price distributions $(F(x|\theta_L), F(x|\theta_H))$ of competitor price offers in low- and high-valuation matches have identical supports,*

$$\text{supp}(F(x|\theta_L)) = \text{supp}(F(x|\theta_H)) \tag{A.1}$$

Further, so long as some buyers obtain one offer but others obtain two, the equilibrium distributions - both aggregate price distributions and, by extension, seller strategies - are atomless.

Proposition A.2 (Continuous Distributions)**.** *Equilibrium distributions $F(\cdot\,|\,\theta_i)$ and $F^{e,j}(\cdot)$ are atomless.*

Sellers always have a strictly positive probability of competing against a peer of identical type, so a price greater than its unit cost (0) cannot be an atom, because deviating with an infinitesimal discounts would sacrifice negligible profits per sale in exchange for a discrete increase in sales. Additionally, any price offer equal to the unit cost is strictly dominated by some more expensive one because the latter would generate some profits in each sale and at least generate sales in monopoly matches, ruling out an atom at the unit cost and implying that equilibrium price distributions are continuous.

Since a loss of sales is the only deterrent to a price increase, and sales are only lost by becoming more expensive than another seller's offer or a buyer's valuation, offer distributions must be locally increasing at any price in their support that is not equal to some buyer's valuation.

Proposition A.3 (Strictly Increasing Distributions)**.** *Given prices $x, x' \in \text{supp}(F(x|\theta))$ such that $x < x' < \theta_L$ or $\theta_L < x < x' < \theta_H$, $F(x|\theta)$ increases strictly in the interval $[x, x']$ for $\theta \in \{\theta_L, \theta_H\}$.*

An immediate implication is that the prices offered to any buyer are distributed in at most two disjoint intervals: one formed by prices weakly below the low-valuation, $x \leq \theta_L$, and another formed by prices strictly above it, $\theta_L < x$. As such, we only need to find the endpoints of these intervals to fully characterize the support of prices.

We start with the suprema of low-trade-permitting and overall prices. Equilibrium price distributions are atomless (Proposition A.2), so sellers who offer the highest overall price only expect to obtain sales with it in monopoly matches and bid accordingly, as per Equation (3.5). Comparing the choices of the two types of sellers, the one most willing to offer a high price is the type who expects the greatest sales from it: sellers who observe a high signal. As such, the highest price offered in equilibrium is that chosen by a monopolist who observes a high signal.

Proposition A.4 (Highest Price)**.** *The highest overall equilibrium price,*

$$\overline{x} = \begin{cases} \theta_H & \textit{if } p_L^h \leq p_L^* \\ \theta_L & \textit{otherwise} \end{cases} \tag{A.2}$$

for p_L^* from (3.6)

However, when all equilibrium offers allow trade ($x \leq \theta_L$), sellers' preference information is superfluous, in a sense, as there exists an equilibrium where every type of seller has the same mixed strategy, so sellers' expected sales in either type of match are identical. We solve for this invariant equilibrium in the next section, but proceed to explain the results of economies where Assumption 3.3 holds and some separating prices are offered.

Continuing with the most expensive offer that allows trade with both types of buyers, a seller who offers the highest low-trade-permitting price only expects to sell to a low-valuation buyer without better offers from competitors, so it chooses the highest price that these buyers would accept, mainly $x = \theta_L$.

Proposition A.5 (Form of Supports)**.** *For $\theta_i \in \{\theta_L, \theta_H\}$,*

$$\text{supp}(F(x|\theta_i)) = \begin{cases} [\underline{x}, \theta_L] \cup [\hat{x}, \theta_H] & \textit{if } p_H^h \theta_H > \theta_L \\ [\underline{x}, \theta_L] & \textit{otherwise} \end{cases} \tag{A.3}$$

where $\theta_L < \hat{x}$ is the lowest equilibrium price offer that only allows trade with high-valuation buyers.

The equation that determines the unique aggregate distribution of prices $x > \theta_L$ is that of profit equality in high-valuation matches. If any such prices are offered, we have argued that θ_H is the highest one, which only allows sellers to trade in monopoly matches, so other offers that only allow trade with high-valuation buyers should be as profitable as this one,

$$\rho\theta_H = (\rho + (1-\rho)(1 - F(x|\theta_H)))\, x \tag{A.4}$$

where we implicitly apply the convexity of the separating support (Proposition A.5). An offer of θ_H maximizes (minimizes) high (low) match profitability,

$$\Pi(x = \theta_H|\theta = \theta_L) = 0$$
$$\Pi(x = \theta_H|\theta = \theta_H) = \theta_H$$

and separating prices preserve these conditional profits, so they are offered by sellers who place sufficient belief on high-valuation matches.

Switching to sellers who offer pooling prices. Consider the type of sellers who offer the highest equilibrium pooling price $x = \theta_L$. If high signal sellers only offer separating prices, then it is without loss to assume that prices are strictly ordered by the seller's type: (a) the distribution of separating prices is unique (**??**), so total welfare and its distribution are invariant in shifts of mass among seller strategies that preserve it, and (b) only low signal sellers offer pooling prices. However, when high signal sellers offer pooling prices, low signal sellers' support must overlap with these. To understand why, consider the highest pooling price offered by low signal sellers, $\overline{x}^l$, and suppose that it was strictly below θ_L. There are more high signal sellers in high-valuation matches than in low-valuation matches, so if they were the only ones who bid in $[\overline{x}^l, \theta_l]$, then an offer of $\overline{x}^l$ would entail an identical profit decrease in monopoly matches, but a strictly larger profit increase in competitive matches when the buyer's valuation was high,

$$\begin{aligned}\Pi(\theta_L|\theta_L) - \Pi(\overline{x}^l|\theta_L) = & -(\rho + (1-\rho)(1-\alpha)(1-F^h(\theta_l))(\theta_L - \overline{x}^l) \\ & + (1-\rho)(1-\alpha)(F^h(\theta_L)\theta_L - F^h(\theta_L)\overline{x}^l) \\ < \Pi(\theta_L|\theta_H) - \Pi(\overline{x}^l|\theta_H) = & -(\rho + (1-\rho)\alpha(1-F^h(\theta_l))(\theta_L - \overline{x}^h) \\ & + (1-\rho)\alpha(F^h(\theta_L)\theta_L - F^h(\theta_L)\overline{x}^l)\end{aligned} \tag{A.5}$$

so indifference of high signal sellers, which requires $\overline{x}^l$ to be less profitable in some type of match, could only be maintained if $\overline{x}^l$ was less profitable in low-valuation matches, meaning that low signal sellers would then strictly prefer to bid above $\overline{x}^l$ - a contradiction. Extending this reasoning, we rule out any interval of pooling prices that is exclusively offered by high signal sellers. And, inversely, since there are more low signal sellers in low-valuation matches, high signal sellers cannot make offers below intervals of pooling offers exclusively offered by low signal buyers[22].

Proposition A.6 (Support Overlap). *If only low signal sellers offer pooling prices, it is without loss to consider the unique ordered equilibrium where strategies have disjoint support* $\text{supp}(F^l) < \text{supp}(F^h)$. *Whereas, if high signal sellers offer pooling prices, then given their lowest pooling price offer* $\underline{x}^h$, $\text{supp}(F^l) \cap \text{supp}(F^h) \cap [0, \theta_L] = [\underline{x}^h, \theta_L]$.

The only way two sellers with different weights on low-valuation match and high-valuation match profitability can be indifferent over the same set of pooling prices, however, is if these maintain profits conditionally on the buyer's valuation,

$$\Pi(\theta_L|\theta) = \Pi(x|\theta) \quad \forall x \in [\underline{x}^h, \theta_L] \text{ and } \theta \in \{\theta_L, \theta_H\} \tag{A.6}$$

Note that since the aggregate distributions of separating prices $F(x|\theta)$ for $x > \theta_L$ are unique, this pair of equations also uniquely pins down the aggregate distributions of prices in the overlapping support, as $\Pi(x|\theta) = (\rho + (1-\rho)(1 - F(x|\theta)))x$.

Since low and high signal sellers expect different profits from an offer of θ_L, the support of low signal sellers' strategy must extend below that of high signal sellers' when their supports overlap over pooling prices,

$$\Pi(\overline{x}^h; p_L^h) = \Pi(\theta_L; p_L^h) > \Pi(\overline{x}^h; p_L^h) = \Pi(\theta_L; p_L^l)$$

Proposition A.7 (Lowest Pooling Offers). *If some separating and pooling offers are made in equilibrium, the lowest are made by low signal sellers* $\text{supp}(F^l) \cup \text{supp}(F^h)^c \cap [0, \theta_L] = [\underline{x}^l, \underline{x}^h]$.

[22] Same monopoly match revenue loss, strictly higher compensating profit gain in competitive matches with low-valuation buyers, so the price at the top of the interval would be more profitable in high-valuation matches (and less in low-valuation matches), and be strictly preferred by high signal sellers than the bottom one.

The fact that the aggregate distribution of offers $F(x|\theta)$ is uniquely determined over separating and overlapping pooling prices, then also implies that these are uniquely determined over the pooling prices that only by low signal sellers offer,

$$(\rho + (1-\rho)(1 - F(\underline{x}^h|\theta)))\underline{x}^h = (\rho + (1-\rho)(1 - F(x|\theta)))x \quad x \in [\underline{x}^l, \underline{x}^h] \text{ and } \theta \in \{\theta_L, \theta_H\} \tag{A.7}$$

We close by noting that the profitability of prices in each type of match also has an ordering property. For one, both separating prices and any pooling prices in an overlapping region maintain profits in either type of match, whereas in the case of pooling prices $x \in [\underline{x}^l, \underline{x}^h]$ that are only offered by low signal sellers, we can apply familiar logic: discounts involve identical revenue losses in monopoly matches but strictly larger compensating profit gains in competitive matches with low-valuation buyers, so the only way these can keep low signal sellers indifferent is if lower prices are more profitable in low-valuation matches and less profitable in high-valuation matches.

Proposition A.8 (Profit Ordering)**.** *Given two prices $x < x'$,*

$$\Pi(x|\theta_L) \geq \Pi(x'|\theta_L) \ , \ \Pi(x|\theta_H) \leq \Pi(x'|\theta_H) \tag{A.8}$$

with equality if both are separating or in $[\overline{x}^h, \theta_L]$ and with strict inequalities otherwise.

A.2 Seller Strategies

Starting with the strategy of seller types that offer the (weakly) highest prices (sharks who observe high signals), the supremum of their distribution's support is as specified by Proposition A.4. If their highest price is θ_L, we focus on the symmetric equilibrium where all sellers have the same mixture to avoid introducing a purely coordination role for predictive skill. When every type of seller has the same strategy, however, the distribution of competitor offers in both low and high-valuation matches is precisely equal to it, so a single equation pins down this distribution $F(x)$ at every price offer; mainly, that which guarantees that every offer yields the same profits as the highest one,

$$\rho\theta = (\rho + (1-\rho)(1 - F(x)))x \tag{A.9}$$

Proposition A.9 (All Pooling Type-Invariant Equilibrium)**.** *If the highest equilibrium is pooling, then the distribution of offers in low- and high-valuation matches is,*

$$F(x) = 1 - \frac{\rho}{1-\rho}(\theta - x) \tag{A.10}$$

and it is also the strategy of both types of seller.

However, when the highest equilibrium price is separating, we study the unique ordered equilibrium. When only low signal sellers offer pooling prices, prices are strictly ordered by the seller's type, as low signal sellers offer strictly lower prices than high signal sellers. When some pooling prices are offered by high signal sellers, however, prices are weakly ordered by the seller's type, as the support of high signal sellers' strategy intersects to top of low signal sellers' on a continuous overlapping region, but then there exists another region below made up of only bids from low signal sellers.

Strictly Ordered Equilibrium - We will first derive strategies for the strictly ordered case. Since seller strategies are ordered, we derive them sequentially, starting with that of sellers who observe high signals. The lower bound on high signal sellers' offers is,

$$\rho\theta_H = (\rho + (1-\rho)\alpha)\,\underline{x}^h \implies \underline{x}^h = \frac{\rho}{\rho + (1-\rho)\alpha}\theta_H \tag{A.11}$$

where α is the mass of sellers who observe h that a seller expects to face in competitive matches if the buyer is of high-valuation - it would always lose against sellers who observe low signals, as they offer lower prices in the ordered equilibrium. The allocation of mass between the upper and lower bound is then uniquely determined by (A.4),

$$F^h(x) = \begin{cases} 1 & \forall\ x \in (\theta_H, \infty) \\ 1 - \frac{\rho}{(1-\rho)\alpha}\frac{\theta_H - x}{x} & \forall\ x \in [\underline{x}^h, \theta_H] \\ 0 & \forall\ x \in (-\infty, \underline{x}^h) \end{cases} \tag{A.12}$$

By induction, when low signal sellers offer any separating prices, their strategy is,

$$F^l(x) = \begin{cases} 1 & \forall\ x \in (\underline{x}^h, \infty) \\ 1 - \frac{P(\text{sale at price } \underline{x}^h|\theta_H)}{(1-\rho)(1-\alpha)}\frac{\underline{x}^h - x}{x} & \forall\ x \in [\hat{x}, \underline{x}^h] \\ 1 - \frac{P(\text{sale at price } \theta_L; p_L^l)}{(1-\rho)P(p_L^l|l)}\frac{\theta_L - x}{x} & \forall\ x \in [\underline{x}^l, \theta_L] \\ 0 & \forall\ x \in (-\infty, \underline{x}^l) \end{cases} \tag{A.13}$$

$$P(\text{sale at price } x^h|\theta_H) = \rho + (1-\rho)\alpha \tag{A.14}$$

$$P(\text{sale at price } \theta_L; p_L^l) = \rho + (1-\rho)((1 - F(\hat{x}|\theta_H))p_H^l + (1 - F(\hat{x}|\theta_L))p_L^l) \tag{A.15}$$

where $\hat{x}$ is the lowest separating price offered by low signal sellers. This price, also in the case where sharks offer pooling prices, follows from the equation,

$$P(\text{sale at price } \hat{x}|\theta_h)p_H^j\hat{x} = P(\text{sale at price } \theta_L; p_L^j)\theta_L$$

$$P(\text{sale at price } \theta_L; p_L^j) = \rho + (1-\rho)((1 - F(\hat{x}|\theta_H))p_H^j + (1 - F(\hat{x}|\theta_L))p_L^l p_L^j)$$

where p_l^j is the type of seller who offers the price.

And, when low signal sellers only offer pooling prices, the equation for their strategy is instead,

$$F^l(x) = \begin{cases} 1 & \forall\ x \in (\theta_L, \infty) \\ 1 - \frac{P(\text{sale at price } \theta_L; p_L^l)}{(1-\rho)P(p_L^l|l)}\frac{\theta_L - x}{x} & \forall\ x \in [\underline{x}^l, \theta_L] \\ 0 & \forall\ x \in (-\infty, \underline{x}^l) \end{cases} \tag{A.16}$$

where the lowest price offered by a low signal seller is the lowest one in equilibrium, so it beats every other price, yields profits $\underline{x}^l$, and must be as profitable as the highest price offered by this seller,

$$\underline{x}^l = P(\text{sale at price } \overline{x}^h; p_L^l)\overline{x}^h$$

Weakly Ordered Equilibrium - In settings where some high signal seller offers are pooling, their strategy over pooling prices and the lowest separating price that they offer, $\hat{x}$, is given by the formulas that we've already derived, but in the overlapping pooling region $[\underline{x}^h, \theta_L]$ the strategies of high and low signal sellers at each price is pinned down by condition that profits conditionally matching with a buyer of either valuation are preserved $\Pi(\theta_L|\theta) = \Pi(x|\theta)$ for $\theta \in \{\theta_L, \theta_H\}$,

$$\begin{aligned} |\theta_H \text{ condition:} &\quad (\rho + (1-\rho)\alpha)\theta_L = (\rho + (1-\rho)(\alpha(1-F^h(x)) + (1-\alpha)(1-F^l(x)))x \\ |\theta_L \text{ condition:} &\quad (\rho + (1-\rho)(1-\alpha))\theta_L = (\rho + (1-\rho)((1-\alpha)(1-F^h(x)) + \alpha(1-F^l(x)))x \end{aligned} \tag{A.17}$$

and then the strategy of low signal sellers over the remaining pooling prices (below the overlapping region) that only they offer follows through the same formula in the strictly ordered equilibrium.

Appendix B Endogenous Quality Setting: Equilibrium Properties

In this section, we will discuss the technical aspects of the candidate equilibrium introduced in Section 4.3 and explain why many of its distinguishing properties hold in all equilibria. An even stronger result will follow that the ordered symmetric equilibrium is unique, which allows us to affirm the genericity of the comparative static analysis in Section 4.4.

B.1 Equilibrium Distributions and Orderedness

We begin by recalling the concept of an ordered equilibrium, which connects the level of indirect utility offered to low and high-valuation buyers in a menu. Put simply, in an ordered equilibrium, sellers who offer more indirect utility in the contract intended for a high-valuation buyer must do the same in the contract intended for a low-valuation one.

Definition B.1 (Orderedness)**.** *An equilibrium is said to be weakly-ordered if, for any two equilibrium menus* (u_L, u_H) *and* (u'_L, u'_H)*,*

$$(u_H - u'_H)(u_L - u'_L) \geq 0$$

When the inequality holds strictly in almost every[23] *comparison, we refer to the equilibrium as ordered.*

These two properties have also been referred to as rank preserving and strictly rank preserving in related work (including Lester et al. (2019)). Like them, we find that orderedness is necessary holds whenever a buyer's incentive compatibility constraint binds at one of the menus that is being compared.

Lemma B.1 (Ordered Equilibrium)**.** *Almost every equilibrium menu* (u_L, u_H) *featuring a binding incentive compatibility constraint is ordered when compared to another equilibrium menu* $(u_L{}', u_H{}')$*. That is,*

$$(u_H - u_H{}')(u_L - u_L{}') > 0$$

The economic rationale underlying this complementarity in indirect utilities is familiar from Garrett et al. (2019) and intuitively explained in Section 4.3. Consider a seller who increases the indirect utility it offers to high-valuation buyers u_H. If their incentive constraint binds, this relaxes it and allows for an increase in the quality provided to buyers of low-valuation in the paired contract (q_L, x_L), as per $q_L = \frac{u_H - u_L}{\Delta\theta}$. The seller can then offer low-valuation buyers the same utility u_L, while obtaining strictly larger profits in each sale to them; however, when low sales are more profitable, the seller also bids more aggressively for them, and this is done by making them more appealing through a low utility increase ($\uparrow u_L$). The channel that creates this complementarity is, therefore, the connection

[23] Up to a measure zero set of menus.

between the efficiency of the contracts and the utility they offer to both buyers. Since this link is missing among dually efficient offers, at which incentive compatibility constraints are slack, they do not need to be ordered.

Orderedness simplifies the equilibrium structure substantially, and heterogeneity in sellers' interim beliefs does not alter the fundamental complementarity between providing additional utility to low and high-valuation buyers, but rather how interested sellers are in forfeiting high sale profitability for low one. Therefore, the heterogeneity of the posteriors *moderates* the joint progression of u_L and u_H, but does not change the correlation between these.

Theorem B.1 (Type Monotonicity). *Let (u_L, u_H) and (u'_L, u'_H) be two equilibrium menus sharing a common binding incentive compatibility constraint with $u_i < u_i{}'$. Then,*

1. *High sale profits per-match are decreasing in generosity, $\Pi_H(u_L, u_H) > \Pi_H(u'_L, u'_H)$, while low ones increase, $(u_L, u_H) < \Pi_L(u'_L, u^j_H{}')$.*

2. *If the menus are offered by sellers of respective types $p_L \neq p'_L$, then $p_L < p'_L$*

Consider the menu $(u_L(p), u_H(p))$ occupying the p^{th} generosity-percentile in the equilibrium distribution. Then, $u_H(p)$ increases enough for profits from high sales $\Psi_H(p)(S^*_H - u_H(p))$ to decrease, but $u_L(p)$ grows passively enough to not undo the additional profitability of profits from low sales. The relationship between low/high trade profits per-match and generosity creates an equilibrium structure where seller types comparatively more interested in profits from high sales make offers that are less generous towards buyers with either valuation than those made by seller types comparatively more interested in profits from low sales. In particular, sellers relatively more convinced that they face high-valuation buyers will aim to depress bids as much as possible so as to extract these buyers' (information) rents, whereas sellers who are relatively more convinced that they face low-valuation buyers give greater consideration to capturing profitable trade with them and cede additional rents to both low and high-valuation buyers to do so.

Efficiency gains thus far have been described as taking place within low trade, implicitly treating the incentive constraint of high-valuation buyers as the only relevant one. In fact, this is a necessary property of equilibria in any economy where sellers' costs satisfy Assumption 4.2. Furthermore, in these, offers are grouped in two sets of menus. The most rationed menus, at which high-valuation buyers' incentive constraints binds, are also the least generous, and then any that are dually efficient are also more generous towards low and high-valuation buyers.

Lemma B.2 (Stacking). *The menus offered in an equilibrium where firms' costs satisfy Assumption 4.2 are partitioned into separate incentive compatibility regions such that*

1. *low-valuation buyers' incentive constraint does not bind in at any menu.*

2. *If some dually efficient menus are offered, there exists a u_i^{de} such that all utilities $u_i < u_i^{de}$ are offered in menus at which high-valuation buyers' incentive constraint binds, whereas all utilities $u_i \geq u_i^{de}$ are offered in dually efficient menus.*

Consider the logic that drives this, from the least generous bid to the most generous. The least generous menu is offered by a seller who only expects to sell if it is in a lone match, so it offers exactly the menu it'd choose if it was a monopolist with the same assessment of the buyer's probable valuation, and a monopolist would never offer a menu at which low-valuation buyers' incentive constraint binds - featuring $u_H - u_L > q^*_H\Delta\theta$ - since they could strictly increase profits from high sales by offering fewer rents to high-valuation buyers. Competition seller types who make more generous offers drives the efficiency of these alongside their generosity (through $u_H - u_L$ growth). When there is sufficient upward pressure on generosity/efficiency, such that the utility gap reaches $u_H - u_L = q_L^* \Delta\theta$, menus become dual efficient. Without an efficiency benefit to high rent concession ($\uparrow u_H$) or efficiency loss to low rent extraction ($\downarrow u_L$) among dually efficient offers, the growth of $u_H - u_L$ slows so that low-valuation buyers' incentive constraint never binds.

The results we have covered so far do not rely on the differentiability in any way, but it is a convenient feature to convey intuition and maintain tractability. It is even better to be able to work with continuously differentiable conditional distributions $F_i(u_i|\theta_i)$ over the utilities offered by the average seller to each type of buyer. Fortunately, equilibria also have these properties.

Lemma B.3 (Equilibrium Distributions)**.** *Equilibrium distributions $F_i(u_i|\theta_i)$ for $i \in \{l, h\}$.*

1. *Do not have atoms in their supports Υ_i.*

2. *Have a convex, connected low support $\Upsilon_L = [\underline{u}_L, \overline{u}_L]$. The high support Υ_H is the union of at most two convex sets disjoint sets, composed of the high utilities offered in menus where high-valuation buyers' incentive constraint bids and is slack, respectively. Furthermore, suprema over utilities always satisfy $\overline{u}_i \leq S_i^*$.*

3. *Are continuously differentiable on the interior of their supports with one-sided derivatives at the boundaries.*

Atoms make it possible for sellers to obtain discrete increases in sales in exchange for infinitesimal discounts in profits per sale, so it is clear that these cannot exist. Given that low-valuation buyers' incentive constraint does not bind in equilibrium, gaps in the low support would allow the sellers offering a menu with implied utility at the top of the gap to increase their profitability in low-valuation matches by decreasing the rents that the menu offers to low-valuation buyers, which would achieve identical low sales but strictly higher profits in each one. The logic for the convexity claim among high offers depends on whether the point u_H that we are considering is such that menus that offer it are dually efficient or constrained. In the former case, high-valuation buyers' incentive constraint is slack, so a gap below any such utility would allow a seller that offers it to become more profitable by lowering these high rents, which would preserve high sales, increase high profits per sale, and not affect the efficinecy/profitability of its low-valuation buyer sales. In the latter case, a constrained menu is ordered when compared with any other, so a gap on the high side is either accompanied by one on the low side (which we have ruled out), an atom at its low utility offer (which we have ruled out), or a situation there are two constrained menus (with the same low utility term, but one has the high utility term at the top of the gap and the other the one at the bottom) and low sales vary locally in low generosity in such a way that it is not preferrable to alter the low offer whether the menu is more or less profitable in each low sale, which is not possible. Lastly, the differentiability claims follow because sellers' indifference must be maintained by the probability of winning in combination with profits per sale, and additively separable utilities allowed us to rewrite profits per sale as $S_i(u_L, u_H) - u_i$, which is smooth in marginal changes to either utility, so differentiability of equilibrium distributions becomes necessary to rule out infinitesimal deviations.

B.2 Equilibrium System of Equations Derivation

We now briefly derive the system of equations that allow us to obtain the candidate equilibrium's analytical closed form. Recall that the problem of a type $p_L^{e,j}$ seller is to offer a menu (u_L, u_H) that maximizes her expected profits,

$$\Pi^{e,j}(u_L, u_H) = \sum_{i=l,h} p_i^{e,j} \Psi_i(u_i)(S_i(u_L, u_H) - u_i)$$

subject to the constraint $u_H \geq u_L \geq 0$. The first-order conditions of this problem highlight the interdependence between the optimal amount of utility extended to each type of buyer, as well as the role of posteriors in determining the relative importance of various trade-offs. Based on the fact that only high-valuation buyers' incentive constraint can bind in the candidate equilibrium, the seller's optimality conditions are

$$\frac{\partial}{\partial u_L} : \quad \underbrace{p_L^{e,j}(1-\rho) f_L(u_L|\theta_L)(S_L(u_L, u_H) - u_L)}_{\text{sales gains}} - \underbrace{p_L^{e,j}\Psi_L(u_L)}_{\text{rent losses}} + \underbrace{p_L^{e,j}\Psi_L(u_L)\frac{\partial S_L}{\partial u_L}(u_L, u_H)}_{\text{efficiency losses}} = 0 \quad \text{(B.1)}$$

$$\frac{\partial}{\partial u_H} : \quad \underbrace{p_H^{e,j}(1-\rho) f_H(u_{h|\theta_H})(S_H^* - u_H)}_{\text{sales gains}} - \underbrace{p_H^{e,j}\Psi_H(u_H)}_{\text{rent losses}} + \underbrace{p_L^{e,j}\Psi_L(u_L)\frac{\partial S_L}{\partial u_H}(u_L, u_H)}_{\text{efficiency gains}} = 0 \quad \text{(B.2)}$$

Similar terms appear in both equations. The first two capture a typical trade-off between expected sales versus rents per-sale. By increasing indirect utility u_i, a seller makes her offer more attractive to θ_i valuation buyers, thus increasing the probability of selling to them by the mass of equilibrium menus that it would be preferred over in contested matches, mainly $p_i^{e,j}(1-\rho) f_i(u_i|\theta_i)$, whereas the cost of surrendering said rents is directly proportional to the likelihood of trading $p_i^{e,j}\Psi_i(u_i)$ with buyers of this valuation. The third term determines the efficiency effect of an increase in generosity towards θ_i valuation buyers (u_i), and it stems from the point that univariate changes in generosity u_i alter the difference in offered utilities $u_H - u_L$, which drives efficiency. When high-valuation buyers' incentive constraint binds at a menu, generosity towards low-valuation buyers (u_L increases) requires further rationing (q_L decrease), thereby reducing the gains of trade with them $S_L(u_L, u_H)$ and, by extension, the profitability of their purchases; the opposite holding for generosity towards high-valuation buyers. In other words, generosity in the low (high) offer has an efficiency cost (benefit), when high-valuation buyers' incentive constraint is locally binding. Whereas if low-valuation buyer's incentive constraint is slack, the efficiency term disappears and the only consideration for the seller is the aforementioned trade-off between the from sales and rents given to buyers of the same valuation.

The implicit objects of immediate interest are the marginal conditional utility distributions $F_i(u_i|\theta_i)$, which shape the nature of competition. Marginals have densities that measure the mass at points on the supports of some seller types' mixed strategy. These supports are atomless, convex, and monotone in the seller's type (overlapping only among any dually efficient bids). Locally, each conditional marginal density $f_i(u_i|\theta_i)$, therefore, corresponds to a weighted density of each seller type's conditional marginal density. In particular, if the utility is offer in an inefficient menu, then these is a unique seller type $p_i^{e,j}$ that offers it and the conditional marginal density is given by,

$$f_i(u_i|\theta_i) = \mu(e) P^e(j|\theta_i) f_i^{e,j}(u_i|\theta_i)$$

For utilities that are offered in dually efficient menus, the conditional marginal density still takes a weighted average form but there is a much simpler way to solve for the utility offers, so we will not use that relation.

To obtain the distribution over utilities in inefficient menus, we will further rewrite the first-order conditions of sellers offering these by applying additional equilibrium properties. In particular, we recall that menus are ordered, so the particular ones of seller types $p_L^{e,j}$ who make inefficient offers are as well, and can be written as functions $(u_L(Q), u_H(Q))$ of the menu's generosity quantile Q in seller type $p_L^{e,j}$'s mixed strategy. This allows us to apply the inverse function theorem and link the conditional marginal densities $f_i(u_i|\theta_i)$ to the progression of utilities,

$$f_i(u_i|\theta_i) = \frac{\mu(e)P^e(j|\theta_i)}{\dot{u}_i^{e,j}(Q)} \tag{B.3}$$

so the conditional marginal distributions take the form,

$$F_i(u_i|\theta_i) = \mu(e)P^e(j|\theta_i)u_i^{e,j,(-1)}(u_i) + \sum_{\substack{e',j' \\ \text{s.t. } p^{e',j'} < p^{e,j}}} \mu(e')P^{e'}(j'|\theta_i) \tag{B.4}$$

where $u_i^{e,j,(-1)}(\cdot)$ is understood to be the inverse of the strictly monotone functions $u_i^{e,j}(Q)$. And, the Q^{th} quantile menu from a $p_L^{e,j}$ type obtains average sales per match,

$$\Psi_i^{e,j}(Q) = \rho + (1-\rho)F_i(u_i^{e,j}(0)|\theta_i) + (1-\rho)\mu(e)P^e(j|h)Q$$

in matches with θ_i valuation buyers.

Substituting (B.3) and (B.4) into the first-order conditions produces a standard system of ordinary differential equations that pins down the indirect utilities offered in constrained equilibrium menus. Piecewise linear costs make marginal efficiency effects locally constant, which decouples these equations and allows us to obtain analytical solutions: the equation that drives high utility $u_H^{e,j}(Q)$ is independent, under piecewise linear costs, and we can then substitute its solution into the equation driving the progression of low utility $u_L^{e,j}(Q)$. Specifically, note the marginal efficiency term becomes,

$$\frac{\partial S_L}{\partial u_H}(u_L, u_H) = (\theta_L - \kappa_L)\frac{\partial q_L}{\partial u_H} = \frac{\theta_L - \kappa_L}{\Delta\theta}$$

So, the differential system governing the progression of utilities in inefficient menus offered by a type $p_L^{e,j}$ seller is,

$$\dot{u}_L^{e,j}(Q)\left[-\frac{\theta_L - \kappa_L}{\Delta\theta} - 1\right]\Psi_L^{e,j}(Q) + (1-\rho)\mu(e)P^e(j|l)(S_L(u_L^{e,j}(Q), u_H^{e,j}(Q)) - u_L^{e,j}(Q)) = 0 \tag{B.5}$$

$$\dot{u}_H^{e,j}(Q)\left[\frac{p_L^{e,j}}{p_H^{e,j}}\frac{\theta_L - \kappa_L}{\Delta\theta}\Psi_L^{e,j}(Q) - \Psi_H^{e,j}(Q)\right] + (1-\rho)\mu(e)P^e(j|h)(S_H^* - u_H^{e,j}(Q)) = 0 \tag{B.6}$$

If offers become dually efficient, either among the offer seller type or because this seller type prefers to jump right to making dually efficient offers when it transitions from those of the adjacent seller type below, we will solve for the remaining utility offers with a different equation. As for the exact form of the utilities that solve the system (B.5)-(B.6), we define $\Xi^{e,j} = \frac{p_H^{e,j}P(j|h)}{p_H^{e,j}P(j|h) - p_L^{e,j}P(j|l)\frac{\theta_L - \kappa_L}{\Delta\theta}}$ to tighten the expressions and write the high utility term as,

$$u_H^{e,j}(Q) = S_H^* - C_H^{e,j}\left(p_H^{e,j}\Psi_H^{e,j}(Q) - p_L^{e,j}\Psi_L^{e,j}(Q)\frac{\theta_L - \kappa_L}{\Delta\theta}\right)^{-\Xi^{e,j}} \tag{B.7}$$

with,

$$C_H^{e,j} = \frac{S_H^* - u_H^{e,j}(0)}{\left(p_H^{e,j}\Psi_H^{e,j}(0) - p_L^{e,j}\Psi_L^{e,j}(0)\frac{\theta_L - \kappa_L}{\Delta\theta}\right)^{-\Xi^{e,j}}} \tag{B.8}$$

which simplifies to,

$$u_H^{e,j}(Q) = S_H^* - (S_H^* - u_H^{e,j}(0)) \left(\frac{p_H^{e,j}\Psi_H^{e,j}(0) - p_L^{e,j}\Psi_L^{e,j}(0)\frac{\theta_L-\kappa_L}{\Delta\theta}}{p_H^{e,j}\Psi_H^{e,j}(Q) - p_L^{e,j}\Psi_L^{e,j}(Q)\frac{\theta_L-\kappa_L}{\Delta\theta}} \right)^{\Xi^{e,j}} \tag{B.9}$$

Substituting this explicit form of high utility into the differential equation for low utility, we then obtain its functional form,

$$\begin{aligned} u_L^{e,j}(Q) =& \frac{\Psi_L^{e,j}(0)}{\Psi_L^{e,j}(Q)} \left(C_L^{e,j} + \frac{(1-\rho)\mu(e)P^e(j|l)}{\Psi_L^{e,j}(0)} \frac{\theta_L - \kappa_L}{\theta_H - \kappa_L} \left(\frac{C_H^{e,j}}{-p_H^{e,j}P^e(j|h)} \right.\right. & \text{(B.10)} \\ & * \left.\left. \left(\left(p_H^{e,j}\Psi_H^{e,j}(Q) - p_L^{e,j}\Psi_L^{e,j}(Q)\frac{\theta_L-\kappa_L}{\Delta\theta} \right)^{1-\Xi^{e,j}} - \left(p_H^{e,j}\Psi_H^{e,j}(0) - p_L^{e,j}\Psi_L^{e,j}(0)\frac{\theta_L-\kappa_L}{\Delta\theta} \right)^{1-\Xi^{e,j}} \right) \right) \right) & \text{(B.11)} \end{aligned}$$

with $C_L^{e,j} = u_L^{e,j}(0)$. This heavy expression is not very informative, but we can derive an implicit form of the low utility term - as a function of both its generosity quantile Q and the high utility it is paired with $u_H^{e,j}(\cdot)$ - that is quite helpful. To do this, we simplify (B.6) and rewrite it as,

$$\begin{aligned} \Psi_L^{e,j}(Q)\dot{u}_L^{e,j}(Q) + \psi_L^{e,j}(Q)u_L^{e,j}(Q) &= \psi_L^{e,j}(Q)\frac{\theta_L-\kappa_L}{\theta_H-\kappa_L}u_H^{e,j}(Q) \\ \implies \frac{\mathrm{d}}{\mathrm{d}Q}\left(\Psi_L^{e,j}(Q)u_L^{e,j}(Q)\right) &= \psi_L^{e,j}(Q)\frac{\theta_L-\kappa_L}{\theta_H-\kappa_L}u_H^{e,j}(Q) \end{aligned}$$

so that,

$$u_L^{e,j}(Q) = u_L^{e,j}(0)\frac{\Psi_L^{e,j}(0)}{\Psi_L^{e,j}(Q)} + \frac{\theta_L-\kappa_L}{\theta_H-\kappa_L}\int_0^Q \frac{\psi_L^{e,j}(x)}{\Psi_L^{e,j}(Q)}u_H^{e,j}(x)\,\mathrm{d}x \tag{B.12}$$

This expression highlights the fact that the low offer is a conditional expectation of the high offers made in less generous menus. This characterization provides an immediate proof for the point that the difference in utilities, $u_H^{e,j}(Q) - u_L^{e,j}(Q)$, and hence efficiency, increases with respect to generosity, and allows us to more easily think about the response of buyer welfare to parameter perturbations, by focusing on the response of high-valuation buyer surplus and then averaging these to get that of low-valuation buyers.

The initial condition of adjacent types $p_L^{e,j} < p_L^{e',j'}$ depends on whether (a) high-valuation buyers' incentive constraint binds at most generous menu $(u_L^{e,j}(1), u_H^{e,j}(1))$ of the lower type $p_L^{e,j}$ and (b) when that happens, whether the next type of seller would prefer for their least generous bid to also inefficient or dually efficient. This decision is determined by the efficiency gain from additional high rents on the profitability of the least generous low offer,

$$\begin{aligned} &- p_H^{e',j'}\Psi_H^{e',j'}(0) + p_L^{e',j'}\Psi_L^{e',j'}(0)(\theta_L - \kappa_L)\frac{\partial q_L}{\partial u_H} \\ &= -p_H^{e',j'}\Psi_H^{e',j'}(0) + p_L^{e',j'}\Psi_L^{e',j'}(0)\frac{\theta_L-\kappa_L}{\Delta\theta} \end{aligned}$$

When this is > 0, sellers of the higher type $p_L^{e',j'}$ prefer for their lowest bid to be dually efficient, and there is a discontinuous jump in generosity between $u_H^{e,j}(1) < u_H^{e',j'}(0)$. When the condition is < 0, instead, sellers of the higher type prefer for their least generous offers to also be inefficient, and there is no discountinuity between $u_H^{e,j}(1) < u_H^{e',j'}(0)$. In either case, the progression of the low utility terms is continuous and the higher seller type's least generous low utility offer is exactly the most generous one of the lower seller type, $u_L^{e,j}(1) = u_L^{e',j'}(0)$.

We will prove that the difference in utilities $u_H^{e,j}(Q) - u_L^{e,j}(Q)$ increases in the seller's quantile and hence the efficiency of the offer, so if these reach the point $u_H^{e,j}(Q) - u_L^{e,j}(Q) = q_L^* \Delta\theta$ such that the menu becomes dually efficient, then we solve for the dually efficient menus with a simpler equation. In particular, given the utility pair $(u_L^{de}(0), u_H^{de}(0))$ at which menus become dually efficient and the mass of sellers that makes constrained offers in θ_i matches $F_i(u_i^{de})$, then the equations that low and high offers must satisfy for dually efficient bids to have the necessary property of being equally profitable in low (high) matches is,

$$(\rho + (1-\rho)F_i(u_i^{de}(0)|\theta_i))(S_i^* - u_i^{de}(0))) = (\rho + (1-\rho)(1 - F_i(u_i^{de}(0)|\theta_i))Q)(S_i^* - \underline{u}_i^{de}(Q)) \quad \text{for } i \in \{l, h\} \tag{B.13}$$

where Q is the quantile among dually efficient menus of the utility $u^{de}(Q)$, and $1 - F_i(u_i^{de}(0)|\theta_i)$ is the probability that a seller makes a dually efficient offer in a θ_i match. By construction, these indirect utility functions have all the properties stipulated earlier: strictly increasing in generosity, monotone in seller type, identically ranked by their low and high utility offerings, and jointly forming a bottom pair of intervals comprised of utilities offered in menus where high-valuation buyers' incentive constraint binds, potentially followed above by another pair of intervals comprised of utilities offered in dually efficient menus.

Appendix C Endogenous Quality: Equilibrium Property Proofs

Proof of Theorem 4.1. Per sale profits from θ_i valuation buyer intended contracts take the form:

$$\pi(q_i, x_i) = x_i - \phi(q_i) = (\theta_i q_i - \phi(q_i)) + (x_i - \theta_i q_i) = S_i(q_i) - u_i \tag{C.1}$$

and $S_i(q_i)$ is concave, reaching a maximum at q_i^*. Due to concavity, any incentive-compatible menu featuring a low quality $q_L > q_L^*$ is strictly dominated by one with a revised low contract of $(q_L^*, u_L - \theta_L(q_L - q_L^*))$, while any incentive-compatible menu featuring a high quality $q_H < q_H^*$ is strictly dominated by one with a revised high contract of $(q_H^*, u_H + \theta_H(q_H^* - q_H))$. In other words, the optimal incentive-compatible menus feature $q_L \le q_L^*$ and $q_H \ge q_H^*$.

Furthermore, IC_i must bind[24] if the $\theta_{\neg i}$ intended contract is not efficient; else, the revised contract $(\tilde{q}_{\neg i}, \tilde{x}_{\neg i})$ featuring[25] $\tilde{q}_{\neg i} = \frac{u_H - u_L}{\Delta\theta}$ and $\tilde{x}_{\neg i} = \theta_{\neg i}\tilde{q}_{\neg i} - u_{\neg i}$ could be paired with the former θ_i contract for a strictly dominant menu - same expected $\theta_{\neg i}$ sales (by preserving the utility that a $\theta_{\neg i}$ valuation buyer obtains), strictly higher profits-per-sale in $\theta_{\neg i}$ matches (by preserving $u_{\neg i}$ and increasing the gains from trade (see (C.1))), and maintaining the incentive compatibility of θ_i buyers ($\theta_i \tilde{q}_{\neg i} - \tilde{x}_{\neg i} = u_i$). Inversely, since only one constraint IC can bind in a given menu, profit maximality implies that any menu featuring an inefficient $q_{\neg i}$ offer, also features efficient $q_i = q_i^*$ quality provision.

Incentive compatibility bounds the qualities offered in each contract by the suggested ratio: $q_L \le \frac{u_H - u_L}{\Delta\theta}$ (to satisfy IC_H) and $q_H \ge \frac{u_H - u_L}{\Delta\theta}$ (to satisfy IC_L). We have shown that IC_i binds when $q_{\neg i} \neq q_i^*$ though, so equality of the respective bound yields the form of the inefficient quality $q_{\neg i} = \frac{u_H - u_L}{\Delta\theta}$. Lastly, any menu featuring efficient quality provision in both contracts must feature utility offerings satisfying $q_L^* \Delta\theta \le u_H - u_L \le q_H^* \Delta\theta$.

We conclude that: (1) menus featuring inefficient low-valuation buyer provision are IC_H binding, feature $q_L = \frac{u_H - u_L}{\Delta\theta} < q_L^*$, and are paired with efficient high-valuation buyer contracts, (2) menus featuring inefficient high-valuation buyer provision are IC_L binding, feature $\frac{u_H - u_L}{\Delta\theta} = q_H < q_H^*$, and are paired with efficient low-valuation buyer contracts, and (3) doubly efficient menus correspond to those for which $q_L^* \Delta\theta \le u_H - u_L \le q_H^* \Delta\theta$ almost all of which have locally slack IC constraints with

[24] It is elementary to check that only one IC constraint can bind in a menu.

[25] Where $\tilde{q}_{\neg i} < q_{\neg i}$ if $\neg i = l$ and $>$ if $\neg i = h$.

the exception of boundary ones satisfying $q_i^* \Delta\theta = \frac{u_H - u_L}{\Delta\theta}$ at which $IC_{\neg i}$ binds. $\square$

Proof of Proposition 4.2. Recall that seller types with degenerate beliefs have measure zero, so the following arguments apply to almost every bid offered in a match, in particular those that would be offered by types with nondegenerate conditional beliefs.

Consider the essential infimum and supremum, $\underline{u}_i$ and $\overline{u}_i$, respectively, on the equilibrium indirect utility offered to θ_i valuation buyers. By Theorem 4.1, equilibrium menus are separating and these must correspond to the bounds on utilities extended in θ_i intended contracts, with profits $S_i(u_L, u_H) - u_i$. If $\overline{u}_i > S_i^*$, then a discrete mass of menus would feature contracts with a non-zero probability of being accepted and entail negative profits per sale. As such, $\overline{u}_i \leq S_i^*$ and we will argue that the lower bound $\underline{u}_i$ is exactly S_i^*.

Suppose that $\underline{u}_i < S_i^*$. By Theorem 4.1, optimal contracts with $u_i < S_i^*$ entail strictly positive[26] profits per sale if $q_i > 0$ and zero profits per sale if $q_i = 0$. Further, there are only two possibilities in a neighborhood of $\underline{u}_i$: either $\underline{u}_i$ is an atom, or half-open sets $[\underline{u}_i, \underline{u}_i + \delta)$ are assigned arbitrarily small mass, as $\delta \searrow 0$, by the equilibrium distribution of indirect utility offerings by competitors in θ_i matches $F_i(u_i|\theta_i)$.

Recall that every contract makes nonnegative profits per sale, since not trading (offering zero quality) is always an option. As such, if there were an atom at $\underline{u}_i$, the menu implied by pairing a slightly more generous $\theta_{\neg j}$ offering $u_{\neg j} + 2\varepsilon$ and $\underline{u}_i + \varepsilon$ (for $\varepsilon > 0$ small) would be strictly dominant, entailing (at worst) an arbitrarily small decrease in $\theta_{\neg j}$ per sale profits, a discrete increase in θ_i expected sales, and thus a discrete increase in θ_i profits per match. Whereas if there were no atom at the lower bound, contracts offering u_i arbitrarily close have θ_i match profits arbitrarily close to zero (they almost surely compete against a more generous seller); so a discrete mass of these is strictly dominated by a menu of the form given in the atom case.

We conclude that almost every seller selects a pair of contracts that extend implied utilities (S_L^*, S_H^*). By Theorem 4.1, the unique optimal menu that satisfies these conditions is $((q_L^*, \phi(q_L^*)), (q_H^*, \phi(q_H^*)))$. $\square$

No Atoms. Toward a contradiction, suppose that F_H had an atom at u_H and let (u_L, u_H) be an equilibrium menu featuring this high bid generosity.

We begin by showing that $S_H(u_L, u_H) - u_H > 0$ in any equilibrium offer (u_L, u_H). Suppose not. Then we must have $S_L(u_L, u_H) - u_L \leq 0$, otherwise offering a pooling menu of only the low-valuation buyer's contract would strictly increase the seller's expected profits - strictly positive per-sale profits from θ_H sales and strictly positive probability of being accepted ($\rho > 0$), even by low-valuation buyers. But then expected profits $\Pi \leq 0$, which contradicts seller optimization: the seller can always offer the menu $((0,0), (q_H^*, \theta_H q_H^*))$ and obtain strictly positive expected profits. With this fact, we can rule out an atom at any u_H in $\text{supp}(\Psi_H)$.

[26]If $q_i = q_i^*$, the claim follows. If $q_i \neq q_i^*$ and $u_i = S_i(q_i)$, then lowering u_i by a small $\varepsilon < u_i - \underline{u}_i$ would allow the seller to still win matches with some probability u_i and make strictly positive profits in these, as the revision would both increase the efficiency and lower the generosity of these sales.

In particular, note that at any u_H with discrete mass and for any type of seller p_L,

$$
\begin{aligned}
&\lim_{\varepsilon \searrow 0} \Pi(u_L+\varepsilon, u_H+\varepsilon) - \Pi(u_L, u_H) \\
&= \lim_{\varepsilon \searrow 0} \left\{ \sum_{k=l,h} p_k \Psi_k(u_k+\varepsilon)(S_k(u_L+\varepsilon, u_H+\varepsilon) - u_k - \varepsilon) - \sum_{k=l,h} p_k \Psi_k(u_k)(S_k(u_L, u_H) - u_k) \right\} \\
&= \lim_{\varepsilon \searrow 0} \big((1-\rho)(F_H(u_H+\varepsilon) - F_H(u_H))\big)(S_H(u_L, u_H) - u_H) \\
&> 0
\end{aligned}
\tag{C.2}
$$

and so (u_L, u_H) would be strictly dominated by $(u_L+\varepsilon, u_H+\varepsilon)$, for some $\varepsilon > 0$.

Furthermore, $S_L(u_L, u_H) - u_L \geq 0$ in equilibrium; otherwise, this menu would be strictly dominated by one with the paired offers $(q_L, x_L) = (0, 0)$ and $(q_H, x_H) = (q_H^*, \theta_H q_H^* - u_H)$, which maintain expected high-valuation match profits and strictly those of low ones (these buyers either select a no-loss contract, $(0, 0)$, or one with strictly positive profits per sale, (q_H, x_H)).

We close by ruling out atoms among low bids. Suppose that supp(Ψ_L) had an atom at u_L and let (u_L, u_H) be an equilibrium menu featuring this low bid generosity. Inequalities (C.2) rule out $S_L(u_L, u_H) - u_L > 0$, so the only possible menu with such a low offering must be one that makes zero profits in low-valuation matches $S_L(u_L, u_H) - u_L = 0$. Suppose that $u_L > 0$. If IC_L is slack, then menu with slightly less low generosity $(u_L - \varepsilon, u_H)$ is strictly dominant (low efficiency nondecreasing hence positive profits per low sale, some low sales since $\rho > 0$, high trade profitability not affected), whereas if IC_L binds, low-valuation buyers obtain utility $u_L > 0$ from the high contract, which has strictly positive profits per sale, so the seller could just pool all buyers on this contract and makes strictly positive profits in low-valuation matches as well. Lastly, if $u_L = 0$ and $S_L(u_L, u_H) - u_L = 0$, it follows (by Theorem 4.1) that $u_H = 0$, so a strictly positive mass of such (u_L, u_H) menus would give rise to an atom in supp(Ψ_H). □

Weak-Orderedness. Consider two equilibrium menus (u_L, u_H) and $(\tilde{u}_L, \tilde{u}_H)$, offered by sellers of (possibly equal) respective types p_L and $\tilde{p}_L$, which violate weak-orderedness; without loss, suppose that this takes place via $\tilde{u}_H > u_H$ and $u_L > \tilde{u}_L$. We proceed case-by-case, depending on the incentive compatibility constraint that binds at (u_L, u_H).

Suppose that IC_H binds at (u_L, u_H). Then,

$$
\begin{aligned}
\Psi_L(u_L) &> \Psi_L(\tilde{u}_L) \\
S_L(u_L, \tilde{u}_H) - S_L(u_L, u_H) &\geq S_L(\tilde{u}_L, \tilde{u}_H) - S_L(\tilde{u}_L, u_H) \geq 0 \\
S_L(u_L, \tilde{u}_H) - S_L(u_L, u_H) &> 0
\end{aligned}
$$

where the second set of inequalities follows by the convexity of costs, Theorem 4.1,

$$
\begin{aligned}
S_L(u_L, u_H) &= \theta_L q_L(u_L, u_H) - \phi\left(q_L(u_L, u_H)\right) \\
q_L(u_L, u_H) &= \begin{cases} \frac{u_H - u_L}{\Delta\theta} & \text{if } u_H - u_L < q_L^* \Delta\theta \\ q_L^* & \text{otherwise} \end{cases}
\end{aligned}
$$

while the third is due to the fact that IC_H binds at (u_L, u_H) (by assumption), so the efficiency of low trade achieved by $(u_L, \tilde{u}_H)$ must be strictly greater. Thus,

$$
\Psi_L(u_L)\left(S_L(u_L, \tilde{u}_H) - S_L(u_L, u_H)\right) - \Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L, \tilde{u}_H) - S_L(\tilde{u}_L, u_H)\right) > 0
$$

and,

$$\Psi_L(u_L)\left(S_L(u_L,\tilde{u}_H)-u_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_L\right)>\Psi_L(u_L)\left(S_L(u_L,u_H)-u_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,u_H)-\tilde{u}_L\right) \tag{C.3}$$

When facing high-valuation buyers, Theorem 4.1 indicates that,

$$S_H(u_L,u_H)=\theta_H q_H(u_L,u_H)-\phi\left(q_H(u_L,u_H)\right)$$
$$q_H(u_L,u_H)=\begin{cases}\frac{u_H-u_L}{\Delta\theta} & \text{if } u_H-u_L>q_H^*\Delta\theta\\ q_H^* & \text{otherwise}\end{cases}$$

so high contracts are decreasing in efficiency with respect to the difference in utility that a menu offers to low- versus high-valuation buyers. But the weak-orderedness violation implies $\tilde{u}_H-u_L<\tilde{u}_H-\tilde{u}_L$ and $u_H-u_L<u_H-\tilde{u}_L$, so that cost convexity also implies,

$$S_H(u_L,\tilde{u}_H)-S_H(\tilde{u}_L,\tilde{u}_H)\geq S_H(u_L,u_H)-S_H(\tilde{u}_L,u_H)\geq 0$$

which is equivalent to,

$$\Psi_H(\tilde{u}_H)\left((S_H(u_L,\tilde{u}_H)-\tilde{u}_H)-(S_H(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_H)\right)\geq\Psi_H(u_H)\left((S_H(u_L,u_H)-u_H)-(S_H(\tilde{u}_L,u_H)-u_H)\right)\geq 0 \tag{C.4}$$

Jointly (C.3) and (C.4) yield,

$$\begin{aligned}
&p_L\left[\Psi_L(u_L)\left(S_L(u_L,\tilde{u}_H)-u_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_L\right)\right]\\
&+p_H\left[\Psi_H(\tilde{u}_H)\left(S_H(u_L,\tilde{u}_H)-\tilde{u}_H\right)-\Psi_H(\tilde{u}_H)\left(S_H(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_H\right)\right]\\
&>p_L\left[\Psi_L(u_L)\left(S_L(u_L,u_H)-u_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,u_H)-\tilde{u}_L\right)\right]\\
&+p_H\left[\Psi_H(u_H)\left(S_H(u_L,u_H)-u_H\right)-\Psi_H(u_H)\left(S_H(\tilde{u}_L,u_H)-u_H\right)\right]\\
&\geq 0
\end{aligned}$$

where the last inequality comes from the optimality of (u_L,u_H) for a p_L type seller. Necessarily then, at least one of the terms in brackets at the topmost expression must be >0 and we know, by (C.4), that the second term is ≥ 0. If the first was >0 or if it were $=0$ (and so the second term in brackets was >0), then any seller - including a $\tilde{p}_L$ type - would strictly prefer $(u_L,\tilde{u}_H)$ over $(\tilde{u}_L,\tilde{u}_H)$. The only alternative then is that this first bracket term is <0 and so that the second bracket term is >0. Given this fact, we can take advantage of (C.3), to note that the first term in brackets of the bottom expression is also <0 and the second >0. The latter strict inequality can only hold when IC_L binds at both $(\tilde{u}_L,u_H)$ and, since $\tilde{u}_H>u_H$, also at $(\tilde{u}_L,\tilde{u}_H)$.

However, for (u_L,u_H) to be preferred by a type p_L seller over $(u_L,\tilde{u}_H)$, which is strictly more profitable in low-valuation matches (since (u_L,u_H) is IC_H binding), the chosen menu must be strictly more profitable in the high. But then,

$$\begin{aligned}
&\Psi_H(u_H)\left(S_H(\tilde{u}_L,u_H)-u_H\right)-\Psi_H(\tilde{u}_H)\left(S_H(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_H\right)\\
&\geq\Psi_H(u_H)\left(S_H(u_L,u_H)-u_H\right)-\Psi_H(\tilde{u}_H)\left(S_H(u_L,\tilde{u}_H)-\tilde{u}_H\right)\\
&>0
\end{aligned} \tag{C.5}$$

and $(\tilde{u}_L,u_H)$ is strictly better than $(\tilde{u}_L,\tilde{u}_H)$ in high-valuation matches and identical in those with buyers of low-valuation (same sales, same profits per-sale) - making it strictly preferable for a $\tilde{p}_L$ type.

These arguments are sufficient to establish that[27] for a given optimal menu (u_L, u_H) at which IC_H binds, then any other equilibrium menu $(\tilde{u}_L, \tilde{u}_H)$ with $\tilde{u}_i > u_i$ - offered by any type of seller - must have $\tilde{u}_{\neg i} \geq u_{\neg i}$.

The only comparisons left to consider are those between two IC_L binding menus and those between one at which IC_L binds one with another that is dually efficient; any comparison where IC_H binds in a menu is covered by the previous reasoning. But, if IC_L is to bind in either of the potentially non-weakly-ordered menus, giving rise to $\tilde{u}_H > u_H$ and $u_L > \tilde{u}_L$, then it must bind at the menu with the largest utility difference, $(\tilde{u}_L, \tilde{u}_H)$. Suppose that this is so and consider the alternative bid $(\tilde{u}_L, u_H)$.

By the fact that (u_L, u_H) is either dually efficient or IC_L binding, the same must be true for $(\tilde{u}_L, u_H)$ and $(u_L, \tilde{u}_H)$, which have strictly larger differences in offered utilities, so $S_L^* = S_L(u_L, u_H) = S_L(\tilde{u}_L, u_H) = S_L(\tilde{u}_L, \tilde{u}_H) = S_L(u_L, \tilde{u}_H)$ and,

$$\begin{aligned} &\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L, u_H) - \tilde{u}_L\right) - \Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L\right) \\ &= \Psi_L(u_L)\left(S_L(u_L, u_H) - u_L\right) - \Psi_L(u_L)\left(S_L(u_L, \tilde{u}_H) - u_L\right) \\ &= 0 \end{aligned} \tag{C.6}$$

Given that IC_L binds at $(\tilde{u}_L, \tilde{u}_H)$ however, $S_H(u_L, \tilde{u}_H) - S_H(\tilde{u}_L, \tilde{u}_H) > S_H(u_L, u_H) - S_H(\tilde{u}_L, u_H)$ and,

$$\begin{aligned} &\Psi_H(u_H)\left(S_H(\tilde{u}_L, u_H) - u_H\right) - \Psi_H(\tilde{u}_H)\left(S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H\right) \\ &> \Psi_H(u_H)\left(S_H(u_L, u_H) - u_H\right) - \Psi_H(\tilde{u}_H)\left(S_H(u_L, \tilde{u}_H) - \tilde{u}_H\right) \\ &\geq 0 \end{aligned} \tag{C.7}$$

where the weak inequality is due to the choice of (u_L, u_H) over $(u_L, \tilde{u}_H)$ by the p_L type seller, while the strict inequality follows from the IC_L constraint binding in all four of menus under consideration. Jointly (C.6) and (C.7) then imply that $(\tilde{u}_L, \tilde{u}_H)$ is strictly dominated by $(\tilde{u}_L, u_H)$ for any seller type however, since the latter is equally profitable in low-valuation matches and strictly better in the high. $\square$

Ordered, Support Convexity, and Profit Ranking. We will now strengthen the claim that equilibrium menus are weakly ordered to one of (strict) orderedness. The proof is broken up into 8 steps which will show the monotonicity of profits per low- and high-valuation match with respect to generosity as well as convex supports for the probability of winning distributions Ψ_j.

Step 1 (Conditional Expected Sales): *Every type of seller expects the same sales, $\Psi_i(u_i)$ from an indirect utility offer of u_i.*

The buyer's type determines the distribution of signals observed by sellers with each precision, α_e, and so the distribution of seller types,

$$P(p_L = p|\theta_i) = \sum_{e,\, j} \mathbb{1}\left(p = P^e(\theta_L|j)\right) P^e(j|\theta_i)\mu(e)$$

[27]Note that the case of IC_H binding at (u_L, u_H) and $u_H > \tilde{u}_H$ but $\tilde{u}_L > u_L$ is subsumed in one we've established, because IC_H must also bind at $(\tilde{u}_L, \tilde{u}_H)$ in these other inequalities.

A seller's mixing distribution, $P((u_L, u_H)|p_L)$, is determined by her type. As such, every seller's expected distribution of competitor bids,

$$F(\tilde{u}_i \leq u_i|\theta_i) = \sum_{p_L} P(\tilde{u}_i \leq u_i|p)P(p_L|\theta_i)$$

is identical when the buyer's type is θ_i.$\square$

Step 2 (No IC_L): *Under Assumption 4.2, IC_L binding menus are not offered in equilibrium.*

The profits per sale high sale of any menu (u_L, u_H) are given by,

$$S_H(u_L, u_H) - u_H \leq S_H^* - u_H$$

so $u_H \leq S_H^*$, as losses are strictly increasing in u_H for larger values. The gap between utilities of any menu is therefore bounded by,

$$\begin{aligned} u_H - u_L \leq u_H \leq S_H^* &= (\theta_H - \kappa_m)(q_H^* - q_L^*) + (\theta_H - \kappa_L)q_L^* \\ &\leq (\theta_H - \theta_L)(q_H^* - q_L^*) + (\theta_H - \theta_L)q_L^* \\ &= \Delta\theta q_H^* \end{aligned}$$

Step 3 (Dually Efficient Characterization): *If $\Delta\theta q_L^* < \Delta u < \Delta\theta q_H^*$, then sufficiently small changes u_H or u_L leave expected profits per low- and high-valuation match unchanged. Further, the existence of a menu $(\overline{u}_L^{de}, \overline{u}_H^{de})$ with $\Delta\theta q_L^* < \overline{u}_H^{de} - \overline{u}_L^{de} \leq \Delta\theta q_H^*$ implies the existence of another menu $(\underline{u}_L^{de}, \underline{u}_H^{de})$ with $\underline{u}_H^{de} - \underline{u}_L^{de} = \Delta\theta q_L^*$ and convex regions $[\underline{u}_i^{de}, \overline{u}_i^{de}]$ for $i \in \{L, H\}$ made up of offerings from dually efficient menus that satisfy the strict inequalities. Lastly, if $\Delta\overline{u}^{de} = \Delta\theta q_H^*$, then there exists a $\delta > 0$, such that a bid of $(u_L - \delta, u_H - \delta)$ has identical profits in low- and high-valuation matches.*

Since lowering either offering does not affect the efficiency of bids when the inequalities are strict, a gap below either u_i would allow a strict increase in profitability from undercutting the original utility by some $\epsilon > 0$. Given that there aren't gaps under either coordinate then, there can't be IC_H binding menus immediately below - by weak-orderedness and these menus' strictly smaller Δu respectively - so, all coordinates $u_i' \in [u_i - \delta, u_i]$ for $\delta > 0$ small and $j \in \{l, h\}$ belong to dually efficient menus, and consequently, $\Psi_i(u_i')(S_i^* - u_i') = \Psi_i(u_i)(S_i^* - u_i)$ (otherwise, some seller would be able to deviate to a bid with higher expected profits per-match without affecting the efficiency of the paired contract). This establishes the first sentence's claim.

The preceding logic implies that dually efficient menus satisfying the pair of strict inequalities have other dually efficient menus immediately below them in generosity and that the profitability of offering these menus, conditionally on matching with either type of buyer, is identical. Consider the infimum u_i coordinates of the contiguous intervals made up by these dually efficient menus and let us refer to it as $\underline{u}_i^{de}$ for $i \in \{l, h\}$. Note that $\Delta\underline{u}^{de} = \Delta\theta q_L^*$, since otherwise necessarily there'd be a gap below one of the $\underline{u}_i^{de}$ coordinates (no sufficiently close dually efficient menus with coordinates below, whereas coordinates from IC_H binding menus would create gaps), which would allow an analogous deviation[28]

[28]Profits conditionally on buyer type are continuous with respect to generosity due to the lack of atoms in the bid distribution, while the continuity of profits per sale follows from the efficiency formulation $S_i(u_L, u_H) - u_i$.

as in the previous paragraph for those seller types offering dually-efficient menus with u_i's close to $\underline{u}_i^{de}$. This establishes the second sentence.

For the case that $\Delta\overline{u}^{de} = \Delta\theta q_H^*$, it is sufficient to observe that all entries immediately below the menu $(\overline{u}_L^{de}, \overline{u}_H^{de})$ and sufficiently close must also be dually efficient to avoid creating a gap. By continuing down distribution in either coordinate one must eventually reach a dually efficient menu with $\Delta u < \Delta\theta q_H^*$; else, there'd either be a gap (allowing the familiar deviation) if one encountered an IC_H menu before reaching a dually efficient one that satisfied the pair of strict inequalities, or it'd hold that $\underline{u}_L^{de} = 0$ with a paired high utility of $\tilde{u}_H = \underline{u}_L^{de} + \Delta\theta q_H^*$. In this last case, the lack of atoms, weak-orderedness (ruling out a non-zero mass of IC_H binding ones having entries below $\tilde{u}_H$), and lack of dually efficient bids with $\Delta u < \Delta\theta q_H^*$ would imply that there is a gap below $\tilde{u}_H$ and so that this menu's profitability could be strictly improved by lowering its high offering (more profits per high sale without decreasing the efficiency of low sales).

To prove the statement in the last sentence, consider the supremum over dually efficient bids u_i with $u_i < \overline{u}_j^{de}$ that belong to a menu with $\Delta u < \Delta\theta q_H^*$, and refer to it as u_j'. If $u_j' = \overline{u}_j^{de}$, we are done by the previous claims. If $u_j' < \overline{u}_j^{de}$, then all menus with bids in $[u_j', \overline{u}_j^{de}]$ for either $j \in \{l, h\}$ must have $\Delta u = \Delta\theta q_H^*$ and it is sufficient to establish the invariance of profits in this region, as the previous arguments take over for dually efficient menus with bids below. Note that the existence of a high offering u_H that allows greater expected profits per high-valuation match than neighboring ones above would allow these sellers above to strictly increase their expected profits per high-valuation match (while leaving low-valuation match ones unperturbed), by instead choosing u_H as the high pairing; by extension of this local argument, expected high-valuation match profits are maintained by menus satisfying $\Delta\theta q_H^*$ in the interval $[u_H', \overline{u}_H^{de}]$. The invariance of expected low-valuation match profits for menus with bids in $[u_L', \overline{u}_L^{de}]$ follows by a similar logic: the existence of a point u_L in this interval allowing larger expected low-valuation match profits would allow sellers above to shift their high and low offering by the same amount so as to obtain the superior expected profits per low-valuation match while preserving the same expected profits per high-valuation match.

Step 4 (Convex Ψ_L Support): *There are no gaps in* supp(Ψ_L).

Since IC_L never binds in an equilibrium bid by Step 2, gaps in supp Ψ_L would imply the existence of a deviation for sellers bidding menus featuring a u_L offer close to the top of said gap. Such a seller could decrease its low offer to some value in the gap, so as to obtain a discrete increase in profits per low sale, while sacrificing an arbitrarily small number of low sales (sales continuous in generosity) and maintaining expected profits per high-valuation match, thereby strictly increasing expected profits. The only non-obvious case is if $\Delta u = \Delta\theta q_H^*$, but then we know from Step 3 that all the u_H' immediately below u_H preserve expected profits per high-valuation match, so a revision of u_L and u_H by the same δ downward would do what is described.

Step 5 (Ordereness Violations): *The only possible violation of strict-orderedness between two equilibrium menus (u_L, u_H) and (u_L', u_H'), where IC_H binds in at least one of them, is if $u_L = u_L'$, $u_H' > u_H$, and $p_L' > p_L$ for the respective seller types that offer these.*

Given two equilibrium menus (u_L, u_H) and (u_L', u_H'), such that IC_H binds in at least one, these must be weakly ordered, so the violation of strict ordering must involve either $u_L = u'_L$ and $u_H > u'_H$ or $u_H = u'_H$ and $u_L > u'_L$. However, the second case cannot be.

To see this point, note that if $u_H = u'_H$ and $u_L > u'_L$, IC_H must bind at the menu with the smaller generosity difference, (u_L, u_H). As such, any other bid $(\tilde{u}_L, \tilde{u}_H)$ with $\tilde{u}_L \in [u'_L, u_L]$ is weakly ordered with respect to this menu, so $\tilde{u}_H \leq u_H$. And if IC_H bound at $(\tilde{u}_L, \tilde{u}_H)$, then it too would be weakly ordered with respect to (u'_L, u'_H), so that $u'_H \leq \tilde{u}_H$. Jointly, these statements imply that $u_H = \tilde{u}_H = u'_H$ for any IC_H binding bid with a low offering in interval $[u'_L, u_L]$ and thus that a nonzero mass of such menus gives rise to an atom at u_H (contradicting the lack of atoms). We only have the mass of dually efficient menus left to fill the intervals below u_L. Weak ordering requires these to satisfy $\tilde{u}_H \leq u_H$ and dual efficiency $\Delta\theta q^*_L \leq \Delta\tilde{u}$, so that,

$$\tilde{u}_L \leq \tilde{u}_H - \Delta\theta q^*_L \leq u_H - \Delta\theta q^*_L < u_L$$

Implying the existence of a $\delta > 0$, for which $\Psi_L([u_L - \delta, u_L]) = 0$ (contradicting Step 4).

If $u'_L = u_L$ and $u'_H > u_H$ instead, IC_H must bind at (u_L, u_H); therefore, (u'_L, u'_H) is strictly more profitable in low-valuation matches (more efficient, same low generosity, same sales). The menu (u_L, u_H) is offered in equilibrium though, so it cannot be strictly dominated and must be superior to (u'_L, u'_H) in high-valuation matches. This difference in profits conditionally on a buyer type can only be optimal for two sellers if they differ in seller type, $p'_L \neq p_L$, with the one who places more weight on low-valuation matches $p'_L > p_L$ offering the menu (u'_L, u'_H) that is more profitable in them.

Step 6 (Profit Ranking): *Given two menus (u_L, u_H) and (u'_L, u'_H) with $u_L < u'_L$, it must be that the expected profits per high-valuation match satisfy $\Psi_H(u'_H)(S^*_H - u'_H) \leq \Psi_H(u_H)(S^*_H - u_H)$ and inversely for low-valuation match profits, $\Psi_L(u'_L)(S_L(u'_L, u'_H) - u'_L) \geq \Psi_L(u_L)(S_L(u_L, u_H) - u_L)$, with both inequalities either strict or equal. In particular, if the inequalities are strict, the type p'_L of the seller who bids (u'_L, u'_H) must be strictly greater than that of the seller who bids (u_L, u_H).*

Consider menus (u_L, u_H) and (u'_L, u'_H) with $u_L < u'_L$ close. If $\Delta\theta q^*_L \leq \Delta u' \leq \Delta\theta q^*_H$, any (u_L, u_H) with u_L sufficiently close must be dually efficient and maintain expected profits per high and low-valuation match by Step 3. Whereas if $\Delta u' < q^*_L\Delta\theta$, for u_L close: (a) $\Pi_H(u'_L, u'_H) = \Pi_H(u_L, u'_H)$, (b) to rule out a deviation toward the more efficient bid (u_L, u'_H) by the type bidding (u_L, u_H), necessarily $\Pi_H(u_L, u'_H) < \Pi_H(u_L, u_H)$, and (c) to rule out a deviation, by the type offering (u'_L, u'_H), towards the more high-valuation match profitable bid (u_L, u_H), necessarily $\Pi_L(u'_H, u'_L) > \Pi_L(u_H, u_L)$. The menu with the more generous low offer is therefore more profitable in low-valuation matches and must be offered by the seller of type p_L, which places more weight on low-valuation matches. This local reasoning around any point $\text{supp}(\Psi_L)$ yields the global claim.

□

Continuous Differentiable Distributions. We will show that the functions $\Psi_H(\cdot)$ and $\Psi_L(\cdot)$ are continuously differentiable. Since these functions are given by the composition of F_H and F_L with a continuous mononote function, this will prove the continuous differentiability of the equilibrium offer distributions.

We present the case of Ψ_H (Ψ_L's is analogous). Let u_H be a utility level offered in the interior of the support of F_H and u_L be its paired low utility offering such that the menu (u_L, u_H) is optimal for some seller of type p_L. We proceed to bound the difference $\Psi_H(u_H+\varepsilon)-\Psi_H(u_H)$ from above and below.

For any $\varepsilon \in \mathbb{R}$, the expected profit to this type of seller from the deviation menu $(u_L, u_H+\varepsilon)$ can be decomposed as:

$$\begin{aligned}
&p_L\Psi_L(u_L)(S_L(u_L,u_H+\varepsilon)-u_L)+p_H\Psi_H(u_H+\varepsilon)(S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon)\\
&=p_L\Psi_L(u_L)(S_L(u_L,u_H)-u_L)+p_H\Psi_H(u_H)(S_H(u_L,u_H)-u_H)\\
&+p_L\Psi_L(u_L)(S_L(u_L,u_H+\varepsilon)-S_L(u_L,u_H))+p_H\Psi_H(u_H)(S_H(u_L,u_H+\varepsilon)-\varepsilon-S_H(u_L,u_H))\\
&+p_H(\Psi_H(u_H+\varepsilon)-\Psi_H(u_H))(S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon)
\end{aligned}$$

and by optimality, we must have $\Pi(u_L,u_H;p_L)\geq \Pi(u_L,u_H+\varepsilon;p_L)$, which implies the following inequality,

$$\begin{aligned}
&p_H(\Psi_H(u_H+\varepsilon)-\Psi_H(u_H))(S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon) \qquad \text{(C.8)}\\
&\leq p_L\Psi_L(u_L)(S_L(u_L,u_H)-S_L(u_L,u_H+\varepsilon))+p_H\Psi_H(u_H)(S_H(u_L,u_H)-S_H(u_L,u_H+\varepsilon)+\varepsilon)
\end{aligned}$$

Similarly, for any $\varepsilon \in \mathbb{R}$ such that $u_H+\varepsilon$ is in the interior of the support of F_H, there exists $u_{l,\varepsilon}$ for which $(u_{l,\varepsilon}, u_H+\varepsilon)$ is the optimal bid of some seller with type $\tilde{p}_L$. And, decomposing the profits from this bid to said seller as above:

$$\begin{aligned}
&\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H+\varepsilon)-u_{l,\varepsilon})+\tilde{p}_H\Psi_H(u_H+\varepsilon)(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon)\\
&=\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H)-u_{l,\varepsilon})+\tilde{p}_H\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H)-u_H)\\
&+\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H+\varepsilon)-S_L(u_{l,\varepsilon},u_H))+\tilde{p}_H\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H+\varepsilon)-\varepsilon-S_H(u_{l,\varepsilon},u_H))\\
&+\tilde{p}_H(\Psi_H(u_H+\varepsilon)-\Psi_H(u_H))(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon)
\end{aligned}$$

Again, since here $(u_{l,\varepsilon}, u_H+\varepsilon)$ is optimal for a seller of type $\tilde{p}_L$, we must have $\Pi(u_{l,\varepsilon},u_H+\varepsilon;\tilde{p}_L)\geq \Pi(u_{l,\varepsilon},u_H;\tilde{p}_L)$, which implies the following inequality,

$$\begin{aligned}
&\tilde{p}_H(\Psi_H(u_H+\varepsilon)-\Psi_H(u_H))(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon) \qquad \text{(C.9)}\\
&\geq \tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H)-S_L(u_{l,\varepsilon},u_H+\varepsilon))+\tilde{p}_H\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H)-S_H(u_{l,\varepsilon},u_H+\varepsilon)+\varepsilon)
\end{aligned}$$

So, by (C.8) and (C.9), we can form a squeezed inequality for the derivative of $\Psi_H(\cdot)$,

$$\begin{aligned}
&\frac{\frac{\tilde{p}_L}{\tilde{p}_H}\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H)-S_L(u_{l,\varepsilon},u_H+\varepsilon))+\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H)-S_H(u_{l,\varepsilon},u_H+\varepsilon)+\varepsilon)}{\varepsilon(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon))}\\
&\leq \frac{\Psi_H(u_H+\varepsilon)-\Psi_H(u_H)}{\varepsilon}\leq\\
&\frac{\frac{p_L}{p_H}\Psi_L(u_L)(S_L(u_L,u_H)-S_L(u_L,u_H+\varepsilon))+\Psi_H(u_H)(S_H(u_L,u_H)-S_H(u_L,u_H+\varepsilon)+\varepsilon)}{\varepsilon(S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon)}
\end{aligned}$$

Considering the right-hand derivative, for $\varepsilon \searrow 0$ small $S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon>0$, since (u_L,u_H) is an equilibrium menu and (as we've argued in Section C in the proof ruling out atoms) $S_H(u_L,u_H)-u_H>0$ for all such menus. Once the nonzero denominator is established, we can recall that $q_i(u_L,u_H)$ (specified in Theorem 4.1) is everywhere left as well as right differentiable in each variable and that $S_i(u_L,u_H)=\theta_i q_i(u_L,u_H)-\phi(q_i(u_L,u_H))$, so taking the limit of the right-hand expression, we obtain:

$$\frac{-\frac{p_L}{p_H}\Psi_L(u_L)\frac{\partial S_L}{\partial_+ u_H}(u_L,u_H)+\Psi_H(u_H)\left(1-\frac{\partial S_H}{\partial_+ u_H}(u_L,u_H)\right)}{S_H(u_L,u_H)-u_H}$$

The left-hand side expression of the inequality is similar to the right, with the exception that $u_{l,\varepsilon}$ is substituted in for u_L and the weighting probabilities are $(\tilde{p}_L,\tilde{p}_H)$ instead of (p_L,p_H), thus requiring a different argument. If $u_H-u_L>q_L^*\Delta\theta$, we establish in the orderedness proof that offerings locally around both u_L and u_H are those of dually efficient bids and that they preserve expected profits from both low- and high-valuation matches - $\Pi_i(u_{l,\varepsilon},u_H+\varepsilon)=\Pi_i(u_{l,\varepsilon},u_H)$ for $i\in\{l,h\}$. So, if a seller of type $\tilde{p}_L$ finds $(u_{l,\varepsilon},u_H+\varepsilon)$ optimal, then so would a seller of type p_L, yielding a limit of the left-hand side expression as above (replacing $\tilde{p}_L$ with p_L for ε small). On the other hand, if $u_H-u_L<q_L^*\Delta\theta$, IC_H binds locally for all menus with $\varepsilon>0$ small, so the strict ordering of profits by type requires $\tilde{p}_L\searrow p_L$. Lastly, if $u_H-u_L=q_L^*\Delta\theta$, then we must be able to select a subsequence with $u_H-u_{l,\varepsilon_{\tilde{n}}}>u_H-u_L$ or $<u_H-u_L$. In the former (latter) case, IC_H is slack (binds) at the menus of the subsequence, so the argument from the $u_H-u_L>q_L^*\Delta\theta$ ($u_H-u_L<q_L^*\Delta\theta$) case applies, and this is sufficient to obtain a convergent left-hand inequality.

Therefore, the limit as $\varepsilon\to 0$ of the left-hand and right-hand inequalities is identical, thus establishing the right-hand derivative claim:

$$\frac{\mathrm{d}\Psi_H}{\mathrm{d}_+u_H}(u_H)=\frac{-\frac{p_L}{p_H}\Psi_L(u_L)\frac{\partial S_L(u_L,u_H)}{\partial_+ u_H}+\Psi_H(u_H)(1-\frac{\partial S_H(u_L,u_H)}{\partial_+ u_H})}{S_H(u_L,u_H)-u_H}$$

The left-hand derivative argument is analogous except that we consider the menu $(u_L,u_H-\varepsilon)$.

As stated at the start, the case of $\Psi_L(\cdot)$ is analogous, ultimately yielding,

$$\frac{\mathrm{d}\Psi_L}{\mathrm{d}u_L}(u_L)=\frac{-\frac{p_H}{p_L}\Psi_H(u_H)\frac{\partial S_H(u_L,u_H)}{\partial u_L}+\Psi_L(u_L)(1-\frac{\partial S_L(u_L,u_H)}{\partial u_L})}{S_L(u_L,u_H)-u_L}$$

Under strictly convex costs, at the sole[29] points of possible non-differentiability where $u_H-u_L=\Delta\theta q_i^*$,

$$\frac{\partial S_i(u_L,u_H)}{\partial_+ u_i}(u_L,u_H)=\frac{\partial S_i(u_L,u_H)}{\partial_- u_i}(u_L,u_H)=\frac{\theta_i-\phi'(q_i^*)}{\Delta\theta}=0$$

hence the stronger differentiability claim for Ψ_i. If costs instead take a piecewise form, unless $u_H-u_L=\Delta\theta q_i^*$, then $S_i(u_L,u_H)$ is locally linear (or constant) in each variable, so that the left and right derivatives are also equal. Since $S_i(u_L,u_H)-u_i$ are continuous in (u_L,u_H) and $\Psi_i(\cdot)$ are continuous by the lack of atoms, the distributions Ψ_i are also continuously differentiable on the interior of the supports with the exception of points u_i corresponding to the (no more than two) bids (u_L,u_H) at which $u_H-u_L=\Delta\theta q_i^*$. $\square$

[29]For a given equilibrium distribution.

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