CS 70 Fall 2020 — Discrete Mathematics and Probability Theory — DIS 11A

1 Linearity

$X \sim \mathcal{D}_1$, $Y \sim \mathcal{D}_2$, $c, d \in \mathbb{R}$, $\mathbb{E}[cX+dY] = c\mathbb{E}[X] + d\mathbb{E}[Y]$

$\mathbb{E}[\cdot]$

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive? $T = T_A + T_B$ $\quad \mathbb{E}[T] = \mathbb{E}[T_A] + \mathbb{E}[T_B] = 26$

$\mathbb{E}[T_{B_i}] = \frac{1}{5}\cdot 4 + \frac{4}{5}\cdot 0 = \frac{4}{5}$ $\quad \mathbb{E}[T_{A_i}] = \frac{1}{3}\cdot 3 + \frac{2}{3}\cdot 0 = 1$

$\mathbb{E}[T_B] = \sum_{i=1}^{20} \mathbb{E}[T_{B_i}] = 20\cdot\mathbb{E}[T_{B_1}] = 16$ $\quad \mathbb{E}[T_A] = \sum_{i=1}^{10} \mathbb{E}[T_{A_i}] = 10\cdot\mathbb{E}[T_{A_1}] = 10$

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

book ... bo | ok

can't start here

$1000000 - 3 = 999997$ starting spots

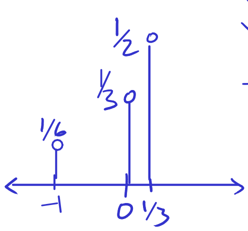
$$\mathbb{E}[X] = 999997\,\mathbb{E}[X_1] = 999997\left(\frac{1}{26^4}\cdot 1 + \left(1 - \frac{1}{26^4}\right)\cdot 0\right) \approx 2.17$$

$X_i \not\perp X_j$ $\qquad X_i = 1 \Rightarrow X_{i+1} = X_{i+2} = X_{i+3} = 0$

2 Joint Distributions

$X \sim \text{Bernoulli}(0.5)$ $\quad Y \sim \text{Bernoulli}(0.5)$ $\quad X=1 \Rightarrow Y=0$

(a) Give an example of discrete random variables X and Y with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of X and Y.



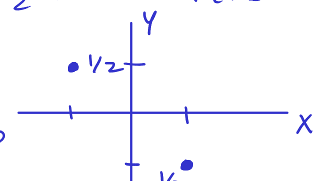
Y \ X	−1	1	
−1	0	1/2	1/2
1	1/2	0	1/2
	1/2	1/2	

Y \ X	−1	1	
−1	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

$\mathbb{E}[XY] = -1\cdot 1 + 1\cdot 0 = -1$

$\mathbb{E}[X] = \frac{1}{2}\cdot(-1) + \frac{1}{2}\cdot(1) = 0 = \mathbb{E}[Y]$

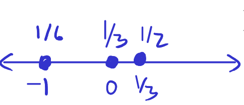
XY
−1 $\Pr[XY=-1] = 1$
1 $\Pr[XY=1] = 0$



(b) Give an example of discrete random variables X and Y that (i) are *not independent* and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of X and Y.

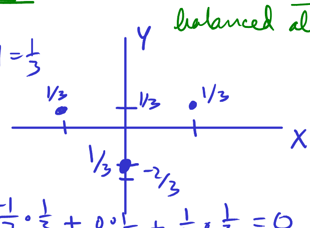
balanced about axes

Not indep $\Rightarrow$ Not



$P[X = 1 \cap Y = \frac{1}{3}] = \frac{1}{3}$

$P[X = \frac{1}{3}] = \frac{1}{2}$



$\mathbb{E}[X] = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot(-1) = 0$

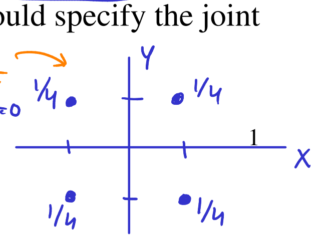
$P(Y = \frac{-2}{3}) = \frac{1}{3}$

$P[Y = \frac{-2}{3} \mid X = 0] = 1$

$X = 0 \Rightarrow Y = \frac{-2}{3}$

$\mathbb{E}[Y] =$

$\mathbb{E}[XY] = \frac{-1}{3}\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} = 0$



3 Ball in Bins

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

(a) What is $\mathbb{E}[X_i]$?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when two balls land in the same bin (if there are n balls in a bin, count that as $n-1$ collisions). What is the expected number of collisions?