Fall 15 Midterm 1

6. Simple proofs.

(a) Prove or disprove that for integers a, b, if $a+b \geq 1016$ that either a is at least 508 or b is at least 508.

$a < 508$ and $b < 508 \Rightarrow a+b < 1016$

$a+b < 508+508 = 1016$

by contraposition.

(b) Prove or disprove that $\sqrt{8}$ is irrational.

Assume $\sqrt{2}$ is rational,

i.e., $\sqrt{2} = \frac{a}{b}$, where $\gcd(a,b)=1$, $a,b \in \mathbb{Z}$

$2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow 2 | a^2 \Rightarrow 2|a$

$\Rightarrow a = 2k,\ k \in \mathbb{Z}$

$a^2 = 4k^2 = 2b^2$

$\Rightarrow 2k^2 = b^2 \Rightarrow 2|b^2 \Rightarrow 2|b$

$\gcd(a,b) \geq 2$ Contradiction.

Assume $\sqrt{8} = \frac{a}{b} \Rightarrow 2\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2} = \frac{a}{2b}$. Contradiction.

(c) Let $R_0 = 0; R_1 = 2; R_n = 4R_{n-1} - 3R_{n-2}$ for $n \geq 2$.
Prove that $R_n = 3^n - 1$ for all $n \geq 0$.

Base case: $0 = 3^0 - 1 = 1 - 1$ ✓, $2 = 3^1 - 1$ ✓

IH: For all $k \geq n \geq 2$ $R_n = 4R_{n-1} - 3R_{n-2} = 3^n - 1$

IS: $R_{k+1} \overset{\text{show}}{=} 3^{k+1} - 1$

$= 4R_k - 3R_{k-1} = 4(3^k - 1) - 3(3^{k-1} - 1)$

$= 4 \cdot 3^k - 4 - 3^k + 3$

$= 3 \cdot 3^k - 1 = 3^{k+1} - 1$ ■

Spring 14 Midterm 1

3. Its Own Inverse (5pts)

For $p > 1$, prove that $p - 1$ is always its own multiplicative inverse in mod p arithmetic.

$p-1 \equiv -1 \pmod{p}$

$\Rightarrow (p-1)^2 \equiv (-1)^2 \equiv 1 \pmod{p}$

$\Rightarrow (p-1)(p-1) \equiv 1 \pmod{p}$

$(p-1)(p-1)$

$\equiv p^2 - 2p + 1 \pmod{p}$

$\equiv p(p-2) + 1 \pmod{p}$

$\equiv 0 + 1 \pmod{p}$

$\equiv 1 \pmod{p}$ ∎

6. Prove it by induction (10pts)

The j-th harmonic number is defined as

$$H_j = \sum_{i=1}^{j} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

So $H_1 = 1, H_2 = 1.5, \ldots$.

Use induction to prove that for any positive integer n,

$$\sum_{j=1}^{n} H_j = H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

$$H_j = \sum_{i=1}^{j} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

$$= \underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j} + \frac{1}{j+1}} - \frac{1}{j+1}$$

$$= \sum_{i=1}^{j+1} \frac{1}{i} - \frac{1}{j+1}$$

$$= H_{j+1} - \frac{1}{j+1}$$

Base case: $H_1 = 1$ $\quad \sum_{j=1}^{1} H_j = 1 = (1+1)1 - 1$

IH: $\sum_{j=1}^{k} H_j = (k+1)H_k - k$

IS:

$$\sum_{j=1}^{k+1} H_j = (k+2)H_{k+1} - (k+1)$$

$$= H_1 + H_2 + \cdots + H_k + H_{k+1}$$

$$= \sum_{j=1}^{k} H_j + H_{k+1}$$

$$= (k+1)H_k - k + H_{k+1}$$

$$= (k+1)\left(H_{k+1} - \frac{1}{k+1}\right) - k + H_{k+1}$$

$$= (k+2)H_{k+1} - 1 - k$$

$$= (k+2)H_{k+1} - (k+1)$$ ∎

Fall 09 Midterm 1

2. [Proofs.] [20 pts]

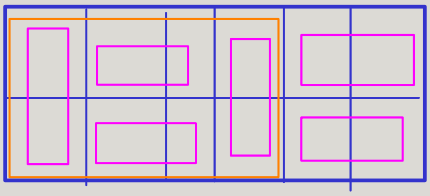
A. (10 pts) Let D_n be the number of ways to tile a $2 \times n$ checkerboard with dominos, where a domino is a 1×2 piece. Prove that $D_n \leq 2^n$ for all positive integers n. (Find a recurrence relation for D_n. No need to give a proof. Then inductively prove the upper bound on D_n.)

Note that dominos can only be placed exactly aligned with checkerboard squares, and cannot be placed diagonally.

$D_n = D_{n-1} + D_{n-2}$

Show $D_n \leq 2^n$

2



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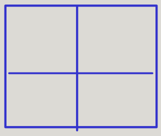

Base: $D_1 = 1 < 2^1$

$D_2 = 2 < 2^2$

IH: $\forall n \leq k,\ D_n \leq 2^n$

IS: $D_{k+1} \leq 2^{k+1}$

$= D_k + D_{k-1}$

$\leq 2^k + 2^{k-1}$

$\leq 2^k + 2^k = 2^{k+1}$ $\blacksquare$

B. (10 pts) Show that $\forall$ odd $a \in N, a^2 = 1 \mod 8$.