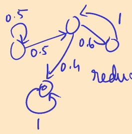
$\mathcal{X} = \{ \quad \}$



reducible!

X_i — state at time i

$\pi_i \quad [\pi_i(1) \;\; \pi_i(2) \; \cdots \; \pi_i(k)]$

P row k, col j: $\Pr[X_{i+1} = j \mid X_i = k]$

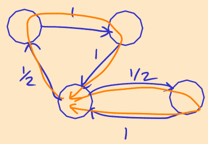
Irreducibility

Go to any state from any other

Periodicity

$d(i) = \gcd\{\text{lengths of all cycles}\}$

$d(i) = 1 \quad \gcd\{1, \ldots\}$



(invariant)

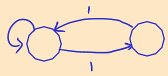
Stationary Distribution (π)

MC is irred & aperiodic

then a unique π

$\pi P = \pi$

Thm 21.4



$X_0 = H$

$\pi_0 = [1 \quad 0]$

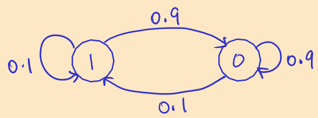
$$\begin{array}{c|cc} \text{from} \backslash \text{to} & H & T \\ \hline H & 1/2 & 1/2 \\ T & 1/2 & 1/2 \end{array}$$

Metropolis-Hastings

$\pi_1 = \pi_0 P = [1/2 \quad 1/2]$

$\pi_2 = \pi_1 P = [1/2 \quad 1/2]$

$\pi_3 = \pi_2 P = [1/2 \quad 1/2]$



$t_0^* =$

$t_1 = 1 + p t_2 + (1-p) t_0$

$x = 1 + \frac{1}{2} y = 2$

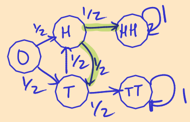
$y = 1 + \frac{1}{2} x = 1 + \frac{1}{2} + \frac{1}{4} y = \frac{3}{2} + \frac{1}{4} y$

$\Rightarrow \frac{3}{4} y = \frac{3}{2} \Rightarrow y = 2$

LOTUS $\quad \mathbb{E}[g(X)] = \sum_{x \in X} g(x) \Pr[X = x]$

Hitting Times (starting from H) $\rightarrow \beta(H)$

I'm at H, how long until $\{HH, TT\}$?



$\beta(HH) = 0$

$\beta(H) = 1 + \frac{1}{2}\cancel{\beta(HH)}^{0} + \frac{1}{2}\beta(T)$

$\beta(T) = 1 + \frac{1}{2}\beta(H) + \frac{1}{2}\cancel{\beta(TT)}^{0}$

$\beta(TT) = 0$

$\beta(H) = 1 + \frac{1}{2}\beta(T) = 2$

$\beta(T) = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\beta(T)\right)$

$= 1 + \frac{1}{2} + \frac{1}{4}\beta(T)$

$\Rightarrow \frac{3}{4}\beta(T) = \frac{3}{2}, \Rightarrow \beta(T) = 2$

Prob of hitting $\mathcal{A}$ before $\mathcal{B}$ (starting from $\overset{H}{0}$) $\alpha(H)$ $\alpha(0)$

$\mathcal{A} = \{HH\}, \mathcal{B} = \{TT\}$

$\alpha(HH) = 1$

$\alpha(H) = \frac{1}{2}\cancel{\alpha(HH)}^{1} + \frac{1}{2}\alpha(T) = \frac{1}{2} + \frac{1}{2}\alpha(T)$

$\alpha(T) = \frac{1}{2}\cancel{\alpha(TT)}^{0} + \frac{1}{2}\alpha(H) = \frac{1}{4} + \frac{1}{4}\alpha(T)$

$\alpha(TT) = 0$

$\alpha(0) = \frac{1}{2}\alpha(H) + \frac{1}{2}\alpha(T) = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$

$\frac{3}{4}\alpha(T) = \frac{1}{4} \Rightarrow \alpha(T) = \frac{1}{3}$

$\alpha(H) = \frac{2}{3}$

THH before HHT

transitive games

James Grime

Prepared by: Aishani Sil, Austin Lei, Agnibho Roy, Debayan Bandyopadhyay, Abinav Routhu
Roast us here: *https://tinyurl.com/csm70-feedback20*

I Pre-Midterm

1. **Vertex Colorings**
Show by induction that for $n \geq 3$, K_n can be vertex colored by n colors.

2. **CRT, FLT, and Friends**
    (a) Find $7^{17} \pmod{15}$.

    (b) Find $7^{60} \pmod{77}$.

3. **Breaking the Encryption**
Austin, having found one large prime p but struggling to find another for RSA, gives up and decides to make $N = p^2$, with e and d, such that $ed \equiv 1 \pmod{p(p-1)}$. (Recall that $x^{p(p-1)} \equiv 1 \pmod{p^2}$ when $p \nmid x$.) Aishani knows that $N = p^2$ and knows the value of e, but her computer is unable to do square roots or division quickly. However, her computer is very good at addition, subtraction, and multiplication, including taking powers. How can Aishani quickly determine the value of p?

II Counting and Introductory Probability

1. **Fun with Polya's Urn**
Consider an urn of balls, that initially contains r red balls and g green balls. The scheme is that on every turn, if we draw a

ball of some color, then we put the ball back, and then put another ball of the same color back in the urn. For this question, consider $r = g = 1$.

(a) What is the probability that we have more red balls than green balls after n turns? Assume n to be odd.

(b) Still considering $r = g = 1$, show that the probability that we have the same number of red balls as green balls after n turns is $\frac{1}{n+1}$.

(c) Still considering $r = g$, what is the probability that we have more red balls than green balls after n turns? Assume n to be even this time.

(d) Let the event R_1 denote the first time that we draw a red ball from this bag. Show that $E[R_1]$ is infinite. (*hint*: think about how you can use the tail sum definition of expectation)

2. **Guess and Check**

Thomas did not attend any of his CSM sections, and has resorted to guessing answers on the CS70 final. The questions are structured in a mysterious way, and the number of minutes it takes him to get a question right by guessing is distributed $\sim Geom(\frac{1}{10})$ (he also gets some indication that his answer is correct). To save time, Thomas has decided that he will give up on a question if it takes him more than 5 minutes, and he will also randomly skip $\frac{1}{4}$ of the questions. What is the expected amount of time it will take Thomas to finish a 20 question exam?

III Distributions

1. **Basketball**

   Jordan and James are both playing basketball. Jordan makes shots with probability 0.6, and James makes shots with probability 0.7, and the probabilities of making shots are independent between different shots. Starting at time $t = 1$, Jordan and James take a shot simultaneously, and continue to do so at every integer time t.

   (a) Suppose that exactly one person makes their first shot. What is the probability that it was Jordan?

   (b) What is the expected amount of time until someone makes a shot?

   (c) What is the probability that James makes a shot before Jordan? (Note that this does not count the situation where they both make their first shot at the same time.)

2. **Buggy Code**

Leanne is debugging her partner's project code. She knows the number of mistakes that her partner makes is Poisson(λ) distributed, where λ is an integer chosen uniformly at random from the interval $[1, 10]$, inclusive, but she does not know the value of λ. Let X be the random variable representing the number of mistakes that Leanne's partner makes.

(a) What is the expected value of X?

(b) Frustrated with her partner, Leanne decides to write her own code. However, the number of mistakes she makes is Poisson(λ) distributed, where λ is chosen from a Binomial$(10, 0.2)$ distribution, but the value of λ is unknown. Let Y be the random variable representing the number of mistakes that Leanne makes. Compute the expected value of Y.

3. **Balls in Bins™**

You are playing a game where you are randomly throwing balls into 15 lined-up bins, and each bin has an Oski sticker that you get to keep if at least one ball lands in that bin.

(a) If you're given 10 balls to throw, what is the expected number of stickers you win?

(b) What is the variance of the above?

IV Continuous RVs and Applications

1. **Lognormal Distribution**

   In this question, we will explore an interesting distribution called the lognormal distribution. The distribution is described as $X = e^{\mu+\sigma Z}$, where $Z \sim \mathcal{N}(0, 1)$.

   (a) First, derive the CDF of X. You may represent your answer in terms of ϕ, the CDF of the standard normal distribution.

   (b) Now, find the PDF of the lognormal distribution (remember the chain rule here when taking the derivative).

   (c) How would you find the expectation of the lognormal distribution? Leave your answer as an integral in terms of the the PDF of the normal distribution, $f_Z(z)$.

2. **Bounding the Irwin-Hall Distribution**

   There is a very interesting distribution, which is known as the Irwin-Hall Distribution. It is defined as the sum of i.i.d continuous uniform distributions $U_i \sim U[a, b]$. Thus, I can be defined as:

$$I = \sum_{i=1}^{n} U_i$$

   The PDF of this distribution is quite ugly, so we will attempt to bound some probabilities using the inequalities learned from this class. For all parts, assume $a \geq 0$. Let $c = \frac{3n(b-a)}{4}$

   (a) Using Markov's inequality, bound the probability that the Irwin-Hall distribution exceeds c.

(b) Using Chebyshev's inequality, bound the probability that the Irwin-Hall distribution exceeds c.

3. **Tired of Balls and Urns Yet?**
An urn contains six balls, of which three are red and three are green. In each step, two balls are selected at random. If one of them is red, and the other is green, then we discard them and replace them by two blue balls, and if both of the balls are blue, then we replace those blue balls with an equal amount of red and green balls. Otherwise, we do not do anything. Find the probability that if we start with an equal number of balls of every color, what is the probability that we reach 6 blue balls before 0 blue balls in the bag?