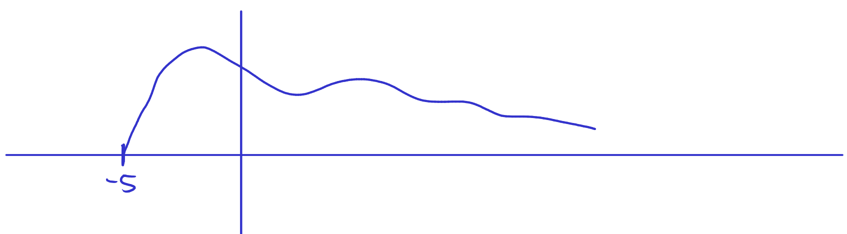
1 Inequality Practice

(a) X is a random variable such that $X > -5$ and $\mathbb{E}[X] = -3$. Find an upper bound for the probability of X being greater than or equal to -1.

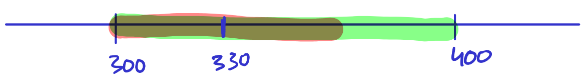
$X' = X + 5$, $X' > 5$

$E[X'] = -3 + 5 = 2$



$\Pr[X \geq -1] = \Pr[X+5 \geq -1+5]$
$= \Pr[X' \geq 4]$
$\leq \frac{2}{4} = \boxed{\frac{1}{2}}$

(b) Y is a random variable such that $Y < 10$ and $\mathbb{E}[Y] = 1$. Find an upper bound for the probability of Y being less than or equal to -1.



$\mathbb{E}[Z] = 330$

$\Pr[Z \notin [300, 400]] \geq \Pr[Z \notin [270, 400]]$
$\leq \Pr[Z \notin [300, 360]]$
$\leq \frac{Var(Z)}{30^2}$

(c) You roll a die 100 times. Let Z be the sum of the numbers that appear on the die throughout the 100 rolls. Compute Var(Z). Then use Chebyshev's inequality to bound the probability of the sum Z being greater than 400 or less than 300. 350−50 350+50

Let $Z = Z_1 + Z_2 + \cdots + Z_{100}$ Z_i is the # on i^{th} roll.

$\mathbb{E}[Z] = \mathbb{E}[\sum_{i=1}^{100} Z_i] = \sum_{i=1}^{100} \mathbb{E}[Z_i] = 100 \cdot \left(\frac{7}{2}\right) = 350$

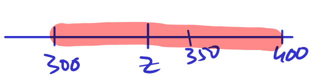
$Var(Z) = Var\left(\sum_{i=1}^{100} Z_i\right) = \sum_{i=1}^{100} Var(Z_i) = 100 \cdot Var(Z_1) = \frac{3500}{12}$

b/y indep

$E[Z^2] - (350)^2$

$E\left[\left(\sum_{i=1}^{100} Z_i\right)^2\right] = E\left[\sum_{i=1}^{100} Z_i^2 + 2\sum_{1 \leq i < j \leq n} Z_i Z_j\right] = \sum_{i=1}^{100} E[Z_i^2] + 2\sum_{1 \leq i < j \leq n} E[Z_i Z_j]$

$Var(Z_i) = \mathbb{E}[Z_i^2] - \cancel{\mathbb{E}^2[Z_i]}^{\frac{49}{4}} = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$



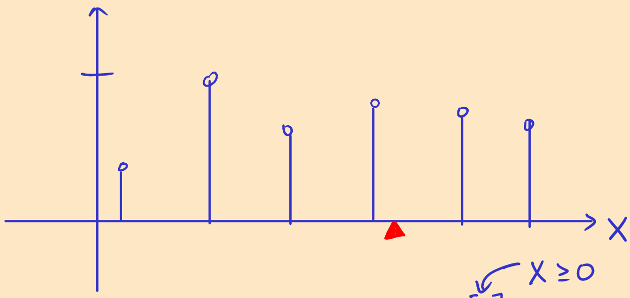
$\frac{1}{6}\sum_{k=1}^{6} k^2 = \frac{91}{6}$

$Cov(Z_i, Z_j) = E[Z_i Z_j] - E[Z_i]E[Z_j]$

b/y indep → $0 = E[Z_i Z_j] - E[Z_i]E[Z_j]$

$E[Z_i Z_j] = E[Z_i]E[Z_j] = \frac{49}{4}$

$\Pr[Z < 300 \cup Z > 400]$
$= \Pr[|Z - 350| \geq 50] \leq \frac{3500/12}{50^2} = \boxed{\frac{7}{60}}$



$X \geq 0$

$$\Pr[X > a] \leq \frac{\mathbb{E}[X]}{a}$$

$\mathbb{E}[X]$

X

for any X

$$\Pr\left[\,|X - \mathbb{E}[X]| > c\,\right] \leq \frac{Var(X)}{c^2}$$

$\mathbb{E}[X]$, $Var(X)$

2 Vegas

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip his or her coin. Assume that each person is independently likely to carry a fair or a trick coin.

1. Given the results of your experiment, how should you estimate p? Find an RV $\hat{p}$ such that $E[\hat{p}] = p$.
   (*Hint:* Construct an (unbiased) estimator for p such that $E[\hat{p}] = p$.)

   $\boxed{\hat{p} = 2X - 1}$

   Let X be the # of H. $X = \sum_{i=1}^{n} X_i$ p

   $E[\hat{p}] = 2E[X] - 1 = p$

   $$E[X] = E\left[\sum X_i\right] = \sum_{i=1}^{n} E[X_i] = \underbrace{p \cdot 1}_{\text{i is a cheater}} + \underbrace{\frac{(1-p)}{2} \cdot 1}_{\text{i is honest AND H}} + \underbrace{\frac{(1-p)}{2} \cdot 0}_{\text{i is honest AND T}} = p + \frac{1}{2} - \frac{p}{2} = \frac{p}{2} + \frac{1}{2}$$

   $\Rightarrow 2E[X] - 1 = p$

2. How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

3 Working with the Law of Large Numbers

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.