Injection / Injective Function $\quad f: S \to T$

one-to-one

$$f(s_1) = f(s_2) \Rightarrow s_1 = s_2$$
$$s_1 \neq s_2 \Rightarrow f(s_1) \neq f(s_2)$$

$e^x \quad \mathbb{R} \to \mathbb{R}$

Surjection / Surjective Function $\quad f: S \to T$

onto

$$\forall t \in T, \exists s \in S : t = f(s)$$

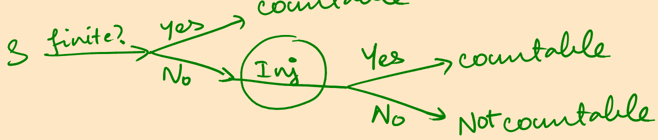
$\log x \quad \mathbb{R}_{>0} \to \mathbb{R}$

Bijection = inj & surj

S, T

① Is S countable? $\qquad \{1, 2, 3\}$

Bij betⁿ S & $\mathbb{N}$



$\exists$ inj $S \to \mathbb{N}$
$\Rightarrow |S| \leq |\mathbb{N}|$

② $|S| = |T|$?

If $\exists$ some bij between S & T, then $|S| = |T|$, else $|S| \neq |T|$.

$S = \{1, 2\}$
$T = \{3, 4\}$ $\qquad f(x) = x + 2, \; x \in S$

$f = \{(1,3), (2,4)\}$

$g = \{(1,3), (2,3)\}$

$h = \{(2,4)\}$

$S = \{1, 2\}$
$T = \{3, 4, 5\}$

i. Is there a bij betⁿ S & T? No!
ii. " " inj $S \to T$? $\{(1,3), (2,4)\}$
iii. " " surj $S \to T$? No!
iv. " " inj $T \to S$? No!
v. " " surj $T \to S$? $\{(3,1), (4,2)\}$
$\{(1,3), (2,4)\}$

$\{(1,3), (2,3)\}$
$\{(1,3), (1,4)\}$

$|S| < |T|$

$\exists$ inj $S \to T \Rightarrow |S| \leq |T|$

$\exists$ surj $T \to S \Rightarrow |T| \geq |S|$

$(\exists \text{ inj } S \to T) \wedge (\exists \text{ surj } S \to T)$
$\Rightarrow (|S| \leq |T|) \wedge (|S| \geq |T|)$
$\Rightarrow |S| = |T|$

7 Countability

Computer Science Mentors 70

Oct 12 - Oct 16
Fall 2020

Roast us here: *https://tinyurl.com/csm70-feedback20*

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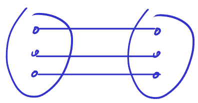
1. Show that for any positive integer n, an injective (one-to-one) function $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ must be a bijection.



Since f is an inj, the size of the image of f must be n (same size as dom).
But, the size of the range is n, so f must be a surj. ⊛ Assume not surj. ⇒ not inj ⇓ contra.
∴ f is a bij.

2. Find a bijection between $\mathbb{N}$ and the set of all integers congruent to 1 mod $\cancel{n}$ 6, for a fixed n.

$g: \mathbb{Z} \to S$, $g(x) = 1 + kx$

$\mathbb{N}$ $\{\ldots, -11, -5, 1, 7, 13, 19, 25, \ldots\}$

$f: \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} n/2, & n \text{ is even} \\ -(n+1)/2, & n \text{ is odd} \end{cases}$

$S = \{1 + kn : k \in \mathbb{Z}\} = \{\ldots, -2n+1, -n+1, 1, n+1, 2n+1, 3n+1, \ldots\}$

$g \circ f$ is a bij from $\mathbb{N} \to S$. $(g \circ f)(x) = g(f(x))$

$\mathbb{N} \to \boxed{f} \to \mathbb{Z} \to \boxed{g} \to S$ ($g \circ f$)

3. Are these sets countably infinite/uncountably infinite/finite? If finite, what is the order of the set? Reminder: A bit string is a sequence of digits where each digit corresponds to either a 1 (on) or a 0 (off).

   (a) Finite bit strings of length n.

   (b) All finite bit strings of length 1 to n.

   (c) All finite bit strings

   (d) All infinite bit strings

   (e) All finite or infinite bit strings.

   (f) Suppose that we try to show that infinite bit strings are countable by induction. We show that for any positive integer n, a bit string of length n is countable. Why does this not work for infinite strings?

4. Is the power set $\mathcal{P}(S)$, where S is countably infinite, finite, countably infinite, or uncountably infinite? Provide a proof for your answer. Reminder: the power set of a set is the set of all possible subsets of that set. Ex: $S = \{A, B\}$, $\mathcal{P}(S) = \{\{\}, \{A\}, \{B\}, \{A, B\}\}$