$X \sim \text{Bernoulli}(p)$ $\xrightarrow[\text{add}]{n \text{ iid}}$ $X \sim \text{Binom}(n, p)$

is this a success?

fix # of trials

$X \sim \text{Binom}(n, p)$

I tried n times.
How many succ?

$X \sim \text{Poisson}(\lambda)$

I wait for 1 unit of time.
How many arrivals?

rate

λ $\dfrac{\text{arrivals}}{\text{time}}$

$\lambda = 6$ customers/hr
$= 1/10$ customers/min
$\vdots$

$Pr[H = 0.1]$

TTTTH

$X = 5$

$X \sim \text{Geom}(p)$

How many trials
until 1^{st} succ

fix # of succ

$X \sim \text{Exponential}(\lambda)$

time

How long to wait until
1^{st} arrival

Bern(p)

Normal

$$\lim_{n \to \infty} \frac{\frac{1}{n}\sum_{i=1}^{n} X_i \cancel{- p}}{\cancel{\sqrt{np(1-p)}}} \sim \mathcal{N}(0, 1)$$

$\mathcal{N}(p, np(1-p))$

CS 70 Fall 2020 — Discrete Mathematics and Probability Theory — DIS 14B

1 Markov Chains: Prove/Disprove

Prove or disprove the following statements, using the definitions from the previous question.

(a) There exists an irreducible, finite Markov chain for which there exist initial distributions that converge to different distributions.

(b) There exists an irreducible, aperiodic, finite Markov chain for which $\mathbb{P}(X_{n+1} = j \mid X_n = i) = 1$ or 0 for all i, j.

(c) There exists an irreducible, non-aperiodic Markov chain for which $\mathbb{P}(X_{n+1} = j \mid X_n = i) \neq 1$ for all i, j.

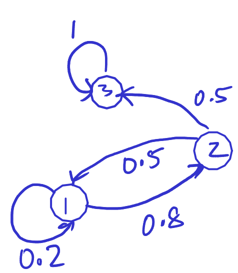
(d) For an irreducible, non-aperiodic Markov chain, any initial distribution not equal to the invariant distribution does not converge to any distribution.

2 Can it be a Markov Chain?

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that hte fly starts between positions 1 and m, model this process as a Markov Chain.

(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain?

$\mathcal{X} = \{1, 2, 3\}$

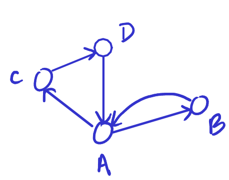


$$\begin{array}{c|ccc} \text{from} \backslash \text{to} & 1 & 2 & 3 \\ \hline 1 & 0.2 & 0.8 & 0 \\ 2 & 0.5 & 0 & 0.5 \\ 3 & 0 & 0 & 1 \end{array}$$

$[0.5 \;\; 0.4 \;\; 0.1]$

3 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his



$\gcd(2,3) = 1$

umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p. Model the # of umbrellas

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

1. tired
2. aper

$\mathcal{X} = \{H, S\}$

$$\begin{array}{c|cc} & H & S \\ \hline H & 1-p & p \\ S & p & 1-p \end{array}$$

$\mathcal{X} = \{0, 1, 2\}$

0 = (Soda, 0) or (Home, 0)
1 = (" , 1) or (" , 1)
2 = (" , 2) or (" , 2)

Soda Home

$$P = \begin{array}{c} \text{from} \backslash \text{to} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} \end{array}$$

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges yes to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

P^2, P^n

$\pi = [\pi(0) \quad \pi(1) \quad \pi(2)]$

$\pi P = \pi$

$$[\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} = [\pi(0) \quad \pi(1) \quad \pi(2)]$$

$(1-p)\pi(2) = \pi(0)$

$(1-p)\pi(1) + p\,\pi(2) = \pi(1)$

$\pi(0) + p\,\pi(1) = \pi(2)$

$\pi(0) + \pi(1) + \pi(2) = 1$

$$\pi = \left[\frac{1-p}{3-p} \quad \frac{1}{3-p} \quad \frac{1}{3-p}\right]$$

$n \to \infty$

$\pi_0 P^n \to \pi$

$$P = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$$

$$p\pi(0) = p\frac{1-p}{3-p}$$

$$P^2 \overset{?}{=} \cancel{\begin{bmatrix} 0.01 & 0.81 \\ 0.81 & 0.01 \end{bmatrix}}$$

$P^2 = PP =$ ______

$P^n = \underbrace{PPP\cdots P}_{n \text{ times}} =$ ______

4 Three Tails

You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: How is this different than the number of *coins* flipped until getting TTT?