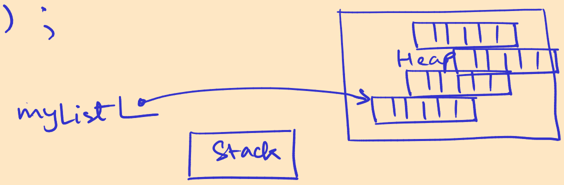
```
void foo (int x) {
    List* l = new List( ) ;
    while (true) {
        List* l = new List( ) ;
    }
    l.~List();
}
```



TestDestruct (P, x)

```
def TestHalt(P, x):
  def P'(x):
    List* list = new List();
    P(x), clean the heap
    list.~List();
  return TestDestruct(P', x)
```

```
def foo(x):
    myList = [ . . . ]
    ___
    ___
    ___
    return None
```

list

Proof of the Halting Problem

```
def Turing(P, x):
    if TestHalt(P, x):
        loop
    else:
        return   //halt
```

P \ x	x_1	x_2	x_3	. . .
P_1	H	L	L	
P_2	H	L	H	. . .
P_3	L	L	H	
⋮		⋮		⋱

$P_i(x_i)$ halts

$$((TH \Rightarrow Turing) \wedge \neg Turing) \Rightarrow \neg TH$$

```
def Turing(P):
    if TestHalt(P, P):
        loop
    else:
        return   //halt
```

Turing (Turing)

"Reduce TestHalt to TestFix"

```
def TestHalt:
    /*
     Do behavior tested by
     given func
    */
    Test ( . . . )
```

8 Computability and Introduction to Probability

Computer Science Mentors 70

Oct 26 - Oct 30
Fall 2020

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Computability

1. Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program. Hint: we can use the fact that TestHalt can't exist to prove other statements.

   Proof of Halting Problem

   (a) You want to determine whether a program P on input x prints "Hello World!" Is there a computer program that can perform this task? Justify your answer.

   Undecidable

   ```
   def TestHalt(P, x):
       def P'(x):
           P(x) suppressing printing
           print("Hello World")
       return TestHello(P', x)
   ```

   TestHello(P, x)

   P(x) halts $\Updownarrow$ P'(x) prints HW

   (b) You want to determine whether a program P prints "Hello World!" before running the kth line of the program.

   Undecidable

   ```
   def TestHalt(P, x):
       def P'():
   1 →     P(x)
   2 →     print("Hello World")
   3 →     print("blah")
       return TestHelloK(P', 3)
   ```

   TestHelloK(P, k)

   P(x) halts $\Updownarrow$ P' prints helloworld by line ___

   (c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task?

   TestHelloK(P, k)

   Run P(x) for k steps.
   If P(x) prints HW, return True.
   O/w return False.

2. Say that we have a program M that decides whether any input program halts as long as it prints out the string "ABC" as the first operation that it carries out. Can such a program exist? Prove your answer.

Introduction to Probability

1. **Rolling Die**

   Leanne rolls two fair six-sided die. For parts (c) through (e), imagine that Leanne wishes to compute the probability that the sum of the two ~~die~~ dice is 5.

   (a) How could you represent the sample space?

   $$\Omega = \{(x,y) : 1 \le x \le 6,\ 1 \le y \le 6\}$$

   (b) What is the size of the sample space?

   $$|\Omega| = 36$$

   $$(\underset{6}{\_},\ \underset{6}{\_})$$

   (c) How could we represent the event we are looking for?

   $$E = \{(x,y) \in \Omega : x+y = 5\}$$

   $$E \subseteq \Omega$$

   (d) What is the size of the event?

   $$|E| = 4$$

1,1	2 1	3 1	4 1	5 1	6 1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

   (e) What is the probability that Leanne rolls a 5?

   $$P(E) = \frac{|E|}{|\Omega|} = \frac{1}{9}$$

2. **Probably Poker**

   (a) What is the probability of drawing a hand with four of a kind (four cards of the same rank)?

   1–12, A

   $$P[E] = \frac{|E|}{|\Omega|} = \frac{\binom{13}{1} \times 1 \times 1 \times 1 \times 48}{\binom{52}{5}}$$

   3 ♤ 3 ♢ 3 ♡ 3 ♧ 2 ♢

   $$|\Omega| = \binom{52}{5}$$

   Ω?

   (b) What is the probability of drawing a straight (five cards in numerical order)?

   $$\frac{10 \cdot 4^5\ |E|}{\binom{52}{5}\ |\Omega|}$$

   $$\binom{8}{1} 4^4$$

   1–5, 2–6, 3–7, 4–8, 5–9, 6–10, 7–J, 8–Q, 9–K, 10–A

   1 2 3 4 5

   ♤♢♡♧ ♤♢♡♧ ♤♢♡♧ ♤♢♡♧ ♤♢♡♧

   $$\frac{52!}{5!\,47!}$$

   (c) What is the probability of drawing a flush (five cards all of the same suit)?

   $$|E| = \binom{13}{5} \cdot 4$$

   $$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}$$

   $$P(E) = \frac{|E|}{|\Omega|} = \frac{\binom{13}{5} \cdot 4}{\binom{52}{5}}$$

   $$|\Omega| = \binom{52}{5}$$

   $$\binom{13}{1} \cdot 48 \cdot 5! /$$

(d) What is the probability of drawing a straight flush (five cards in numerical order, all of the same suit)?

$|E| = 10 \cdot 4$

$$\frac{40}{\binom{52}{5}} \approx \frac{40}{2.6\times 10^6} = \frac{40}{260\times 10^4} = \frac{1}{65}\times 10^{-4} \approx 10^{-6}$$

(e) What is the probability of drawing a hand with exactly one pair (two cards of matching rank)?

$|E| = \binom{52}{1}\cdot 3 \cdot \binom{48}{3}$ $\qquad 13\cdot\binom{4}{2}\cdot 4^3$ $\qquad 13\cdot 12\cdot 11\cdot 12\cdot 4$

ranks suits

$13\cdot\binom{4}{2}\cdot 12\binom{4}{1}\cdot 11\binom{4}{1}\cdot 10\binom{4}{1}$

$$P(E) = 13\cdot\binom{4}{2}\cdot\left(\binom{12}{3}\cdot 4^3 + 12\cdot\binom{4}{3}\right)\Big/\binom{52}{5}$$

3 diff ranks $\qquad$ 1 rank $\qquad$ ranks $\qquad$ suits

3. Oski is rolling 10 six-sided dice; what is the probability that he gets exactly two 4s?

$\underbrace{4\,4}\,1\,1\,2\,3\,5\,6\,6\,6$

$|E| = \binom{10}{2}\cdot 1^2 \cdot 5^8$

$|\Omega| = 6^{10}$

$$\frac{|E|}{|\Omega|} = \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$$

# of ways $\qquad$ P(get 2 4's) $\qquad$ P(get 8 not 4's)

4. Eric is looking at the weather forecast for the next few days. On Monday, there is a 30% chance it will rain, and a 70% chance it will be sunny. On Tuesday, there is a 20% chance it will rain, and a 80% chance it will be sunny. The weather on Monday does not affect the weather on Tuesday. What is the probability that there will be rain on at least one day?

A = rain on Monday $\qquad$ B = Rain on Tuesday

$$P(A\cup B) = 1 - P(\bar{A}\cap\bar{B}) = 1 - P(\bar{A})P(\bar{B}) = 1 - 0.7\times 0.8 = 0.44$$

$\|$

$$P(A) + P(B) - P(A\cap B) = 0.3 + 0.2 - (0.3\times 0.2) = 0.5 - 0.06 = 0.44$$

$$P(B|A) = P(B)$$

$$P(A\cup B) = P(A) + P(B) \qquad P(A\cap B) = 0 \qquad P(B|A) = 0 = \frac{P(A\cap B)}{P(A)}$$

5. For each permutation σ of 1 through n, let $\sigma(i)$ denote the value at position i. For example, if the permutation is 2, 4, 1, 3 we have $\sigma(1) = 2$ and $\sigma(2) = 4$ (Source: CS 70 SP19 MT2)

(a) For a fixed $1 \leq k \leq n$, what is the probability that a permutation σ of 1 through n satisfies the property that $\forall\, i < k, \sigma(i) < \sigma(k)$? Express your answers in terms of n and k.

$k=3$ $\qquad$ 1 2 3 4 5 6 $\qquad$ $n=6,\ k=4$ $\qquad$ $\sigma(4)=6$

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

$\sigma(k)$ is the largest of the first k elems

$\sigma(k)$ is the 2nd largest of the first k elems

$\sigma(k)$ is the k^{th} largest of the first k elems

$\sigma(k)$ is the 17^{th} largest of the first k elems

$\boxed{\frac{1}{k}}$

1 2 3 6 5 4
4 3 5 6 1 2
1 3 2 6 5 4
3 1 2 5 4 6
2 3 1 4 6 5
3 1 4 2 5 6

$\sigma(i)$ 1 2 3 6 5 4

(b) What is the probability that a permutation from 1 through n satisfies the property that for each i, $\sigma(\sigma(i)) = i$ and

$\sigma(i) \neq i$? (For example, the permutation 3, 4, 1, 2 is such a permutation, since for example $\sigma(\sigma(1)) = \sigma(3) = 1$. You may assume n is even.

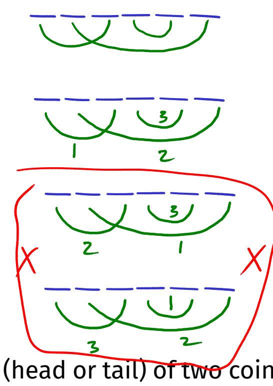
3 6 1 5 4 2

# of pairings $= \dfrac{n!}{2^{n/2}\left(\frac{n}{2}\right)!}$

$$\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}$$

$$\frac{n!}{2!\,\cancel{(n-2)!}} \quad \frac{\cancel{(n-2)!}}{2!\,(n-4)!}$$

$$\prod_{i=0}^{\frac{n}{2}-1}\binom{n-2i}{2} = \frac{n!}{2^{n/2}}$$



$$\Pr[E] = \frac{n!}{2^{n/2}\left(\frac{n}{2}\right)!}\cdot\frac{1}{n!}$$

$$\frac{\left(\prod_{i=0}^{\frac{n}{2}-1}\binom{n-2i}{2}\right)/\left(\frac{n}{2}\right)!}{n!} = \boxed{\frac{1}{2^{n/2}\left(\frac{n}{2}\right)!}}$$

6. Eve is playing a game with Steve where she guesses the value (head or tail) of two coins that Steve has just flipped. She is told that one coin is facing heads. Given this information, what is the probability that both coins are facing heads?

— —

{HH, HT, TH}

$\frac{1}{3}$

$\frac{1}{2}$

$$P(2H \mid 1H) = \frac{P(1H \mid 2H)\,P(2H)}{P(1H)}$$

$$= \frac{1\cdot\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$$\frac{n!}{2!\,\cancel{(n-2)!}}\cdot\frac{\cancel{(n-2)!}}{2!\,\cancel{(n-4)!}}\cdot\frac{\cancel{(n-4)!}}{2!\,\cancel{(n-6)!}}\cdots\frac{\cancel{2!}}{2!\,2!} = \frac{n!}{2^{n/2}}$$