

1 First Exponential to Die

Let X and Y be $\text{Exponential}(\lambda_1)$ and $\text{Exponential}(\lambda_2)$ respectively, independent. What is

$$\mathbb{P}(\min(X, Y) = X),$$

the probability that the first of the two to die is X ?

2 Chebyshev's Inequality vs. Central Limit Theorem

Let n be a positive integer. Let X_1, X_2, \dots, X_n be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \quad \mathbb{P}[X_i = 1] = \frac{9}{12}; \quad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1 , $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \geq 2]$.

- (c) Can you use b to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?
- (d) As $n \rightarrow \infty$, what is the distribution of Z_n ?
- (e) We know that if $Z \sim \mathcal{N}(0, 1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \rightarrow \infty$, can you provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

3 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2 . Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. One approach is to start with the cumulative distribution function of Y and use it to derive the probability density function of Y .

[1. You can use without proof that the pdf for any gaussian with mean and sd is given by the formula $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean value for X and σ^2 is the variance. 2. The derivative of CDF gives PDF.]