$X \sim \mathcal{D}$ "support" of X is set of values x : $\Pr[X=x] > 0$.

$$\mathbb{E}[X] = \sum_{x \in X} x \cdot \Pr[X=x]$$

$X \sim \text{Bern}(0.5)$
support $(X) = \{0,1\}$

$X \sim \text{Binom}(n,p)$
support $(X) = \{0,1,\dots,n\}$

$X \sim \text{Normal}(0,1)$
support $(X) = \mathbb{R}$

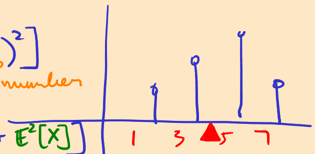
$X \sim \text{Exponential}(\lambda)$
support $(X) = \mathbb{R}_{\geq 0}$

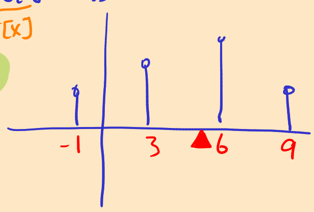
$$\text{Var}(X) = \mathbb{E}[(\underset{\uparrow \text{RV}}{X} - \underbrace{\mathbb{E}[X]}_{\text{just a number}})^2]$$

$$= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}^2[X]]$$

$$= \mathbb{E}[X^2] - \underbrace{2\mathbb{E}[X]\mathbb{E}[X]}_{-2\mathbb{E}^2[X]} + \mathbb{E}^2[X]$$

$$= \mathbb{E}[X^2] - \underbrace{(\mathbb{E}[X])^2}_{\mathbb{E}^2[X]}$$





Law of the Unconscious Statistician (LOTUS)

$$\mathbb{E}[g(X)] = \sum_{x \in X} g(x) \cdot \Pr[X=x]$$

$g(x) = x^2$

$$\mathbb{E}[X^2] = \sum_{x \in X} x^2 \cdot \Pr[X=x]$$

X	X^2	Pr
-1	1	1/4
0	0	1/3
1	1	1/4
2	4	1/6

$$\mathbb{E}[X] = \frac{-1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{1}{3}$$

$$\mathbb{E}[X^2] = \frac{1}{4} + \frac{1}{4} + \frac{2}{3} = \underline{\quad\quad}$$

$X \sim \mathcal{D}_1$, $Y \sim \mathcal{D}_2$

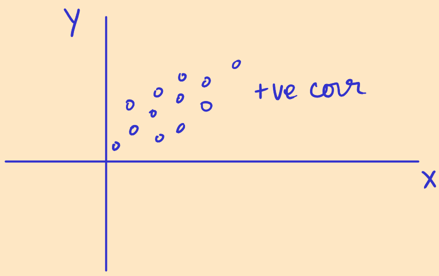
$$\text{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

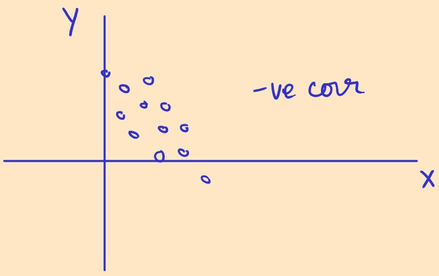
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

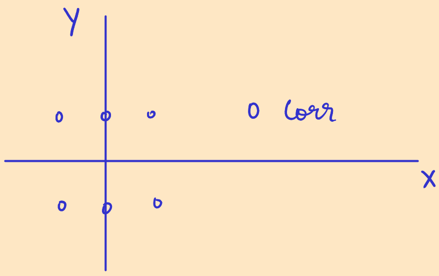
$\text{Cov}(X,X) = \text{Var}(X)$

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$-1 \leq \text{Corr}(X,Y) \leq 1$$







Bilinearity of Cov

$a, b \in \mathbb{R}$

$\text{Cov}(aX, bY) = ab\, \text{Cov}(X,Y)$

$\text{Cov}(X_1+X_2, Y_1+Y_2)$

$= \text{Cov}(X_1,Y_1) + \text{Cov}(X_2,Y_1) + \text{Cov}(X_1,Y_2) + \text{Cov}(X_2,Y_2)$

$\text{Var}(aX) = \text{Cov}(aX, aX) = a^2\, \text{Cov}(X,X) = a^2\, \text{Var}(X)$

$\text{Var}(X+b) = \text{Var}(X)$

$$\text{Var}(X+b) = \mathbb{E}[(X+b-\mathbb{E}[X+b])^2]$$

$$= \mathbb{E}[(X+\cancel{b}-\mathbb{E}[X]-\cancel{b})^2]$$

$$= \mathbb{E}[(X-\mathbb{E}[X])^2]$$

$$= \text{Var}(X)$$

$$= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + 2\sum_{1 \leq i < j \leq n} \mathbb{E}[X_i X_j]$$

$$= \sum_{i=1}^{n} \left(\mathbb{E}[X_i^2] - \underbrace{(\mathbb{E}[X_i])^2}_{0}\right) + 2\sum_{1 \leq i < j \leq n}\left(\mathbb{E}[X_i X_j] - \underbrace{\mathbb{E}[X_i]\mathbb{E}[X_j]}_{0}\right)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)$$

CS 70 Fall 2020 — Discrete Mathematics and Probability Theory — DIS 11B

1 Variance Proofs

(a) Let X be a random variable. Prove that:

$Var(X) = \mathbb{E}[\underbrace{(X-\mathbb{E}[X])^2}_{\geq 0}] = \sum_{x \in X} \underbrace{(x-\mathbb{E}[X])^2}_{\geq 0} \underbrace{Pr[X=x]}_{\geq 0} \geq 0$

$$\mathrm{Var}(X) \geq 0$$

(b) Let $X_1,\ldots,X_n$ be random variables. Prove that:

$(X_1+X_2+\cdots+X_n)^2 =$

$\mathbb{E}[(\sum_{i=1}^n X_i - \mathbb{E}[\sum_{i=1}^n X_i])^2]$

$\mathbb{E}[(X_1+X_2+\cdots+X_n)^2] = \mathbb{E}[\sum_{i=1}^n X_i^2 + 2\sum_{1\leq i<j\leq n} X_iX_j]$

$$\mathrm{Var}(X_1+\cdots+X_n) = \sum_{i=1}^n \mathrm{Var}(X_i) + 2\sum_{1\leq i<j\leq n} \mathrm{cov}(X_i,X_j)$$

$= \sum_{i=1}^n \mathbb{E}[X_i^2] + 2\sum_{1\leq i<j\leq n} \mathbb{E}[X_i X_j]$

$= \mathbb{E}[(\sum_{i=1}^n X_i - \sum_{i=1}^n \cancel{\mathbb{E}[X_i]}^{0})^2] = \mathbb{E}[(\sum_{i=1}^n X_i)^2]$

Hint: Without loss of generality we can assume that $\mathbb{E}[X_1] = \cdots = \mathbb{E}[X_n] = 0$. Why?

$\mathbb{E}[X] = 5$

$\sum X_i - \sum \mathbb{E}[X_i]$ $X_1 \leftarrow X_1 - \mathbb{E}[X_1],\ X_2 \leftarrow X_2 - \mathbb{E}[X_2], \ldots,$

$\mathbb{E}[X-5] = 0$

$\mathbb{E}[X-\mathbb{E}[X]] = 0$

(c) Let $a_1,\ldots,a_n \in \mathbb{R}$, and $X_1,\ldots,X_n$ be random variables. Prove that:

$a_i^2 Var(X_i) = Var(a_i X_i)$

$a_i a_j Cov(X_i, X_j) = Cov(a_i X_i, a_j X_j)$

$$\sum_{i=1}^n a_i^2 \cdot \mathrm{Var}(X_i) + 2\sum_{1\leq i<j\leq n} a_i \cdot a_j \cdot \mathrm{cov}(X_i,X_j) \geq 0$$

$a_1 = -10^7$ $2a_1a_2 = -10^{14}$

$a_1^2 = 10^{14}$ $a_2^2 = 10^{14}$

$\sum_{i=1}^n Var(a_i X_i) + 2\sum_{1\leq i<j\leq n} Cov(a_i X_i, a_j X_j) = Var(\sum_{i=1}^n a_i X_i) \geq 0$

2 Subset Card Game

Jonathan and Yiming are playing a card game. Jonathan has $k > 2$ cards, and each card has a real number written on it. Jonathan tells Yiming (truthfully), that the sum of the card values is 0, and that the sum of squares of the values on the cards is 1. Specifically, if the card values are $c_1,c_2,\ldots,c_k$, then we have $\sum_{i=1}^k c_i = 0$ and $\sum_{i=1}^k c_i^2 = 1$.

The cards are then going to be dealt randomly in the following fashion: for each card in the deck, a fair coin is flipped. If the coin lands heads, then the card goes to Yiming, and if the coin lands tails, the card goes to Jonathan. Note that it is possible for either player to end up with no cards/all the cards.

(a) Calculate $\mathrm{Var}(S)$, where S is the sum of value of cards in Yiming's hand. The answer should not include a summation.

(b) Now suppose that instead, Jonathan and Yiming take turns flipping a weighted coin, that lands heads with probability p. Whoever flips a heads keeps the card, and the first person to begin flipping the coin is determined by flipping a fair coin. Calculate $\mathrm{Var}(S)$ in terms of p.

3 Variance

A building has n upper floors numbered $1, 2, \ldots, n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each person gets off at one of the n upper floors uniformly at random and independently of everyone else. What is the *variance* of the number of floors the elevator *does not* stop at?