CS 70 Fall 2020 — Discrete Mathematics and Probability Theory — DIS 01A

1 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then n^2+1 is even. (Recall that n is odd if $n = 2k+1$ for some natural number k.)

$(2k+1)2l = 2(2k+1)l \quad (2k+1)l \in \mathbb{Z}$
$2l+2k = 2(l+k)\ ,\ (l+k) \in \mathbb{Z}$
NICE BUT NOT NEEDED

$$(2k+1)^2 + 1 = 4k^2+4k+1+1 = 2(\underbrace{2k^2+2k+1}_{\in \mathbb{Z}})$$

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$. (Recall, that the definition of absolute value for a real number z, is

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

$x<y$: $\dfrac{x+y-|x-y|}{2}$ <0

$= \dfrac{x+y-(-x+y)}{2}$

$= x$

$x \geq y$: $\dfrac{x+y-|x-y|}{2}$ >0

$= \dfrac{x+y-(x-y)}{2}$

$= y$

$(A \subseteq B) \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

(c) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. (Recall that $A' \in \mathscr{P}(A)$ if and only if $A' \subseteq A$.)

$A = \{1,2\} \quad \mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$

$(\forall x \in A')\ x \in B$
$\Rightarrow A' \subseteq B$

$(A' \in \mathcal{P}(A) \Rightarrow A' \in \mathcal{P}(B)) \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

$A' \subseteq A$

Consider $x \in A'$. Then, $x \in A$. Since $A \subseteq B$, $x \in B$.

$(\forall A' \in \mathcal{P}(A))\ A' \subseteq B$
$\Rightarrow (\forall A' \in \mathcal{P}(A))\ A' \in \mathcal{P}(B)$
$\Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

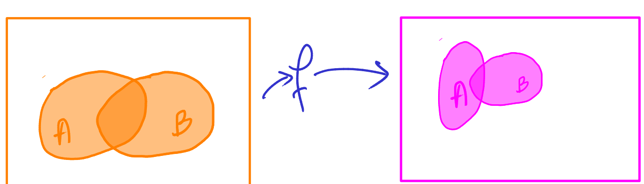
For all $A' \in \mathcal{P}(A)$, $A' \subseteq B$. Then, $A' \in \mathcal{P}(B)$. Hence, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

2 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. (such that) Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

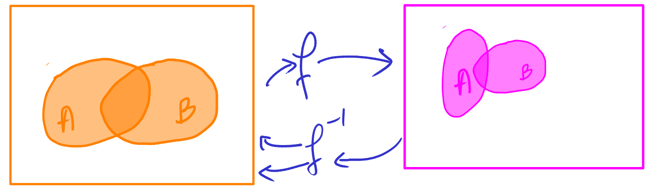
Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.



$x \in f^{-1}(A \cup B) \Rightarrow x \in (f^{-1}(A) \cup f^{-1}(B))$

(b) $f(A \cup B) = f(A) \cup f(B)$.



(f^{-1} might map one element to many, e.g., consider $f(x) = x^2$)

3 Fermat's Contradiction

Prove that $2^{1/n}$ is not rational for any integer $n \geq 3$. (*Hint*: Use Fermat's Last Theorem. It states that there exists no positive integers a, b, c s.t. $a^n + b^n = c^n$ for $n \geq 3$.)

4 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.