Ω

$\overset{3}{HHH}$ $\overset{2}{HHT}$

$\overset{2}{HTH}$ $\overset{1}{HTT}$ $\overset{2}{THH}$ $\overset{1}{THT}$

^{1}TTH TTT^{0}

$$P(X=2) = \frac{3}{8}$$

$X : \Omega \to \mathbb{R}$

X is # of H.

$X(HHH) = 3$

$X(THH) = 2$

$X(TTT) = 0$

$X(TTH) = 1$

$X(THT) = 1$

X	Pr
0	1/8
1	3/8
2	3/8
3	1/8

Binom$(3, 1/2)$

Bern$(1/2)$

$$\mathbb{E}[X] = \frac{1}{8}\cdot 0 + \frac{3}{8}\cdot 1 + \frac{3}{8}\cdot 2 + \frac{1}{8}\cdot 3$$

$= 0.375 + 0.75 + 0.375$

$= 1.5$

10 Random Variables, Independence, Expectation

Computer Science Mentors 70

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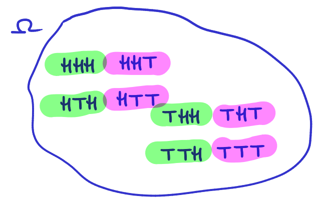
I Random Variables

1. **Intro to Random Variables**
   Suppose that we are flipping 3 coins in a row. Let's try to define some different random variables relating to this process!

   (a) Define the random variable R_1 to represent whether the last coinflip was a head or a tail. Draw a mapping from the sample space to the value of R_1 that each event corresponds with.

   $R_1 = \begin{cases} 1 & \text{if last flip is H} \\ 0 & \text{o/w} \end{cases}$

   (b) Now, define the random variable R_2 to represent the number of heads seen in our series of flips. What are possible values that R_2 can take on? Draw a mapping from the sample space to the values of R_2 that each event corresponds with.

2. **Dice Division** (practice.eecs.org Set 11 Problem 1)
   Consider the following game: you roll two standard 6-sided dice, one after the other. If the number on the first die divides the number on the second die, you get 1 point. You get 1 additional point for each prime number you roll.

   Define the random variable R_1 to be the result of the first roll, and define R_2 to be the result of the second roll. Define the random variable $X = R_1 + R_2$ to be the sum of the numbers that come up on both dice, define the random variable $Y = R_1 \cdot R_2$ to be the product of the numbers that come up on both dice, and define the random variable Z to be the number of points you win in the game.

   (a) What values can the random variable Z take on (with nonzero probability)? Give examples of an event that would cause Z to take on that value.

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[[(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1)],
 [(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)],
 [(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3)],
 [(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)],
 [(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5)],
 [(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)]]
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   0 R_1, R_2 not prime, $R_1 \nmid R_2$ $(4,6)$
   1 R_1 or R_2 is prime or $R_1 | R_2$ $(6,2)$
   2 R_1 & R_2 prime or R_1 prime & $R_1 | R_2$ $(2,4), (3,5)$
   3 R_1 & R_2 prime & $R_1 | R_2$ $(2,2), (3,3), (5,5)$

   (b) Say that your first roll is a 3 and your second roll is a 6. What is the value of Z?

   $R_1 = 3, R_2 = 6$

   $Z = 2$ $3|6$

   (c) Say that your first roll is a 4 and your second roll is a 1. What is the value of $X^2 + Y + Z$?

   $R_1 = 4, \quad R_2 = 1$

   $X = 4+1 = 5$
   $Y = 4 \cdot 1 = 4$
   $Z = 0$

   $X^2 + Y + Z$
   $= 25 + 4 + 0 = \boxed{29}$

(d) Conditioned on the fact that your second roll is a 1, what is the probability that $Z = 1$?

$Pr[Z=1 | R_2 = 1] = \frac{4}{6} = \boxed{\frac{2}{3}}$

$R_1 = \{2,3,5,1\}$

(e) Conditioned on the fact that your second roll is a 1, what is the probability that $Z = 2$?

$Pr[Z=2 | R_2 = 1] = 0$

3. **Tired Of Flipping Coins Yet?**

Suppose you are flipping a fancy coin this time, where $P(\text{heads}) = 0.2$. Let us flip this coin 10 times.

(a) What is the probability that we get exactly 6 heads?

X is # of H. $X \sim \text{Binom}(10, 0.2)$

$Pr[X=6] = \binom{10}{6}(0.2)^6(0.8)^{10-6}$

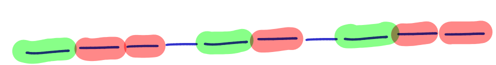
$Pr[X \geq 6] = \sum_{i=6}^{10} \binom{10}{i} 0.2^i \, 0.8^{10-i} < \binom{10}{6} 0.2^6$

$\binom{10}{6}0.8^4 + \binom{10}{7} 0.2 \times 0.8^3 + \binom{10}{8} 0.2^2 \times 0.8^2 + \binom{10}{9} \quad + \binom{10}{10}$

(b) What is the probability that we get at least 3 heads but no more than 5 heads?

$Pr[3 \leq X \leq 5] = \sum_{i=3}^{5} \binom{10}{i} 0.2^i \, 0.8^{10-i} < \binom{10}{3} 0.2^3 \binom{7}{5} 0.8^5$

NOTE: This bound is vacuous since it is >1.



(c) How many heads do you expect, and why?

$\mathbb{E}[X] = 10 \times 0.2 = 2$

np

4. **The Brown Family**
Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C. Fill in a table with each combination of possible values for G and C.

(c) Use the joint distribution to compute the marginal distributions of G and C.

(d) What is the expected number of girls that the Browns will have? Boys?

II Expectation

1. **Introduction to Expectation**
   Imagine that we have two, 3-sided loaded (non-uniform probability) dice, which are represented by the random variables X and Y, respectively. X and Y are distributed as follows:

$$P(X = 1) = \frac{1}{2} \qquad P(Y = 1) = \frac{1}{6}$$
$$P(X = 2) = \frac{1}{4} \qquad P(Y = 2) = \frac{1}{6}$$
$$P(X = 3) = \frac{1}{4} \qquad P(Y = 3) = \frac{2}{3}$$

   (a) What is expected value of a roll of the first die, represented by random variable X?

   (b) What is expected value of a roll of the second die, represented by random variable Y?

   (c) What is the expected value of the product of the two dice?

2. **Introduction to Indicators**
   Imagine drawing a poker hand of five cards from a deck of cards. What is the expected number of face cards that we get?

3. **Group Photo**

In this problem, we show that linearity of expectation can be thought of as "double counting"; we count the situation in two ways.

Leanne is arranging two Berkeley students and two Stanford students line up for a group photo. The four students randomly arrange themselves in a line. We wish to compute the expected number of times that a Berkeley and a Stanford student stand next to each other.

(a) Compute the expected value the naive way: list out all ways the students can be arranged, count the number of Berkeley-Stanford pairs for each arrangement, and compute the expected value. (Feel free to write out only a few cases just to get the feel for it, the table is very large!)

(b) Now, we wish to compute the expected value by using indicator variables. What indicator variables should we use?

(c) Compute the expected value by using indicator variables. Do you see why this should give the same result as in part a)?

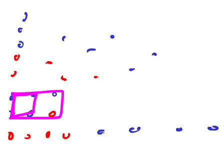
4. **Binomial Mean**

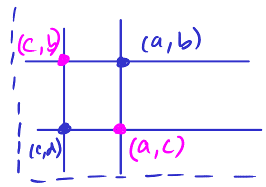
Show that mean of a Binomial random variable X with parameters n and p has mean np.

5. **Rectangles**

On the xy-plane, consider the 8×8 grid of lattice points (i, j), with $0 \le i, j \le 7$.

(a) Consider rectangles that are formed by taking 4 of the lattice points as vertices, such that the edges of the rectangles are parallel with the x and y axes. How many such rectangles are there?



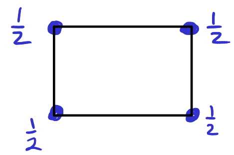


$$\binom{8}{2} \cdot \binom{8}{2} = 28^2 = 784$$

(a

$a > c \quad b > d$

(b) Each point on the grid is equally likely to be red or blue. What is the expected number of rectangles from part (a) that have all their vertices blue?



$$X_i = \begin{cases} 1 & \text{if all corners blue} \\ 0 & \text{o/w} \end{cases}$$

$$E[X_i] = \frac{1}{16} \cdot 1 + \frac{15}{16} \cdot 1 = \frac{1}{16}$$

X = total # of rects w/ all corners blue

$$X = \sum_{i=1}^{784} X_i$$

$$E[X] = E\left[\sum_{i=1}^{784} X_i\right] = \sum_{i=1}^{784} E[X_i] = 784 \cdot E[X_1]$$

$$= 784 \cdot \frac{1}{16} = \boxed{49}$$