10 Random Variables, Independence, Expectation

Computer Science Mentors 70

Nov 9 - Nov 13
Fall 2020

Prepared by: Aishani Sil, Austin Lei, Agnibho Roy, Debayan Bandyopadhyay, Abinav Routhu

I Random Variables

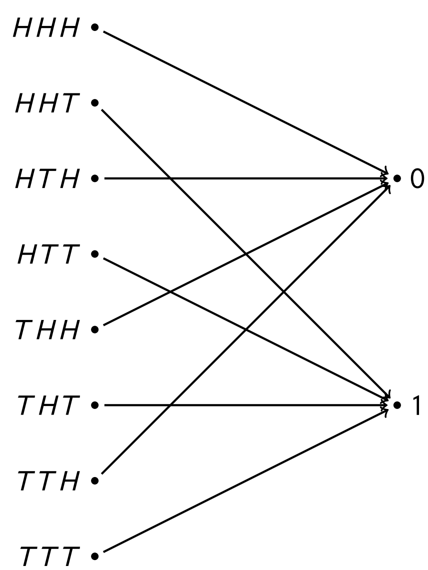
1. **Intro to Random Variables**

   Suppose that we are flipping 3 coins in a row. Let's try to define some different random variables relating to this process!

   (a) Define the random variable R_1 to represent whether the last coinflip was a head or a tail. Draw a mapping from the sample space to the value of R_1 that each event corresponds with.

   Solution:

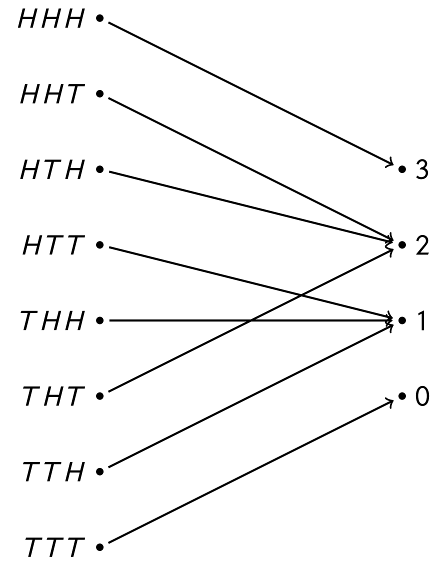
   We want to create a function that maps each event in the sample space to one value in $\mathbb{R}$. We do this in this case by assigning Heads to 0, and Tails to 1 (the reverse will also work). The following mapping has the sample space on the left and the corresponding value of R_1 on the right. This is an example of a Bernoulli random variable, which takes on a value of 0 or 1 depending on the success or failure of a single event.



   (b) Now, define the random variable R_2 to represent the number of heads seen in our series of flips. What are possible values that R_2 can take on? Draw a mapping from the sample space to the values of R_2 that each event corresponds with.

   Solution:

   R_2 can be either 0, 1, 2, or 3 since these are the possible number of heads we can see in 3 flips. Now we map each possible outcome to one of these possible values. The following mapping again has the sample space on the left and the corresponding value of R_2 on the right. This is an example of a binomial distribution!



2. **Dice Division** (practice.eecs.org Set 11 Problem 1)
Consider the following game: you roll two standard 6-sided dice, one after the other. If the number on the first dice divides the number on the second dice, you get 1 point. You get 1 additional point for each prime number you roll.

Define the random variable R_1 to be the result of the first roll, and define R_2 to be the result of the second roll. Define the random variable $X = R_1 + R_2$ to be the sum of the numbers that come up on both dice, define the random variable $Y = R_1 \cdot R_2$ to be the product of the numbers that come up on both dice, and define the random variable Z to be the number of points you win in the game.

(a) What values can the random variable Z take on (with nonzero probability)? Give examples of an event that would cause Z to take on that value.

Solution: 0 (e.g. (4, 6)),
1 (e.g. (6, 2)),
2 (e.g. (3, 5) or (2, 4)),
3 (e.g. (2,2)).

(b) Say that your first roll is a 3 and your second roll is a 6. What is the value of Z?

Solution: $Z = 2$. You get one point because 3 | 6 and a second point because 3 is prime.

(c) Say that your first roll is a 4 and your second roll is a 1. What is the value of $X^2 + Y + Z$?

Solution: Note that $X = R_1 + R_2 = 4 + 1 = 5$, $Y = R_1 \cdot R_2 = 4 \cdot 1 = 4$, and $Z = 0$ because neither rolls are prime and the first number does not divide the second number. Thus, $X^2 + Y + Z = 5^2 + 4 + 0 = 29$.

(d) Conditioned on the fact that your second roll is a 1, what is the probability that $Z = 1$?

Solution: For the score to be 1 when the second roll is 1 means the first roll is a prime number (2, 3, 5) or 1 (because $1|1$). Since there are 6 events with the second roll being 1 and 4 successes, $P[Z = 1|R_2 = 1] = 4/6 = 2/3$.

(e) Conditioned on the fact that your second roll is a 1, what is the probability that $Z = 2$?

Solution: With $R_2 = 1$, there is no possible roll that can garner 2 points. When $R_1 = 2, 4, 6$, $Z = 0$ and when $R_1 = 1, 3, 5$, $Z = 1$. This covers all the possible events, so $P[Z = 2|R_2 = 1] = 0/6 = 0$.

3. **Tired Of Flipping Coins Yet?**
Suppose you are flipping a fancy coin this time, where $P(\text{heads}) = 0.2$. Let us flip this coin 10 times.

(a) What is the probability that we get exactly 6 heads?

Solution: If we define random variable H to be the number of heads in these 10 trials, $H \sim$ Binom(10, 1/5). $P(H = 6) = \binom{10}{6}(\frac{1}{5})^6(\frac{4}{5})^4$.

(b) What is the probability that we get at least 3 heads but no more than 5 heads?

Solution: Sum up over the respective answers for 3, 4, and 5 successes. $\binom{10}{3}(\frac{1}{5})^3(\frac{4}{5})^7 + \binom{10}{4}(\frac{1}{5})^4(\frac{4}{5})^6 + \binom{10}{5}(\frac{1}{5})^5(\frac{4}{5})^5$.

(c) How many heads do you expect, and why?

Solution: $E[H] = 10 \times 0.2 = 2$. You can also get the same result by summing up (number of heads $\times$ probability of this number of heads) for all possible number of heads that can occur. This makes sense, since we have a very tail-biased coin!

4. **The Brown Family**
Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

Solution: The sample space is the set of all possible sequences of children that the Browns can have:
$\Omega = \{g, bg, bbg, bbb\}$. The probabilities of these sample points are:
$P(g) = \frac{1}{2}$
$P(bg) = \frac{1}{2} \times \frac{1}{2}$
$P(bbg) = (\frac{1}{2})^3$
$P(bbb) = (\frac{1}{2})^3$

(b) Compute the joint distribution of G and C. Fill in a table with each combination of possible values for G and C.

Solution:

	$C = 1$	$C = 2$	$C = 3$
$G = 0$	0	0	$P(bbb) = 1/8$
$G = 1$	$P(g) = 1/2$	$P(bg) = 1/4$	$P(bbg) = 1/8$

(c) Use the joint distribution to compute the marginal distributions of G and C.

Solution:
Marginal distribution of G:

$$P(G = 0) = 0 + 0 + \frac{1}{8} = \frac{1}{8}$$
$$P(G = 1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Marginal distribution of C:

$$P(C = 1) = 0 + \frac{1}{2} = \frac{1}{2}$$
$$P(C = 2) = 0 + \frac{1}{4} = \frac{1}{4}$$
$$P(C = 3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Note that the marginal distribution for each random variable should add up to 1.

(d) What is the expected number of girls that the Browns will have? Boys?

Solution: We can apply the definition of expectation directly for this problem, since we've computed the marginal distribution for both random variables.

$$E(G) = 0 \times P(G = 0) + 1 \times P(G = 1) = \frac{7}{8}.$$
$$E(\text{number of boys}) = 1 \times P(\text{1 boy}) + 2 \times P(\text{2 boys}) + 3 \times P(\text{3 boys}) = \frac{7}{8}.$$

We can also find the expected number of boys by subtracting $E(G)$ from $E(C)$. $E(C) = \frac{7}{4}$, so $E(\text{number of boys}) = \frac{7}{8}$.

II Expectation

1. **Introduction to Expectation**

   Imagine that we have two, 3-sided loaded (non-uniform probability) dice, which are represented by the random variables X and Y, respectively. X and Y are distributed as follows:

$$P(X=1)=\frac{1}{2} \qquad P(Y=1)=\frac{1}{6}$$
$$P(X=2)=\frac{1}{4} \qquad P(Y=2)=\frac{1}{6}$$
$$P(X=3)=\frac{1}{4} \qquad P(Y=3)=\frac{2}{3}$$

   (a) What is expected value of a roll of the first die, represented by random variable X?

   Solution:
$$E[X]=\frac{1}{2}(1)+\frac{1}{4}(2)+\frac{1}{4}(3)=\frac{7}{4}$$

   (b) What is expected value of a roll of the second die, represented by random variable Y?

   Solution:
$$E[Y]=\frac{1}{6}(1)+\frac{1}{6}(2)+\frac{2}{3}(3)=\frac{5}{2}$$

   (c) What is the expected value of the product of the two dice?

   Solution: We take the expectation of each possible product:
$$\begin{aligned}E[XY]=&\frac{1}{2}\cdot\frac{1}{6}(1)(1)+\frac{1}{2}\cdot\frac{1}{6}(1)(2)+\frac{1}{2}\cdot\frac{2}{3}(1)(3)+\frac{1}{4}\cdot\frac{1}{6}(2)(1)+\frac{1}{4}\cdot\frac{1}{6}(2)(2)\\&+\frac{1}{4}\cdot\frac{2}{3}(2)(3)+\frac{1}{4}\cdot\frac{1}{6}(3)(1)+\frac{1}{4}\cdot\frac{1}{6}(3)(2)+\frac{1}{4}\cdot\frac{2}{3}(3)(3)=4.375\end{aligned}$$

2. **Introduction to Indicators**

   Imagine drawing a poker hand of five cards from a deck of cards. What is the expected number of face cards that we get?

   Solution: Let X denote the number of face cards. We will denote the indicator X_i that takes on the value 1 when we have a face card, and 0 else. The probability that we have a face card is $\frac{3}{13}$ as there are three face cards out of the 13. (jack, queen, king). Thus, we have:
$$E[X]=E[\sum_{i=1}^{5}X_1]$$
   By the linearity of expectation:
$$E[X]=\sum_{i=1}^{5}E[X_i]=5E[X_1]=5*\frac{3}{13}=\frac{15}{13}$$

3. **Group Photo**

In this problem, we show that linearity of expectation can be thought of as "double counting"; we count the situation in two ways.

Leanne is arranging two Berkeley students and two Stanford students line up for a group photo. The four students randomly arrange themselves in a line. We wish to compute the expected number of times that a Berkeley and a Stanford student stand next to each other.

(a) Compute the expected value the naive way: list out all ways the students can be arranged, count the number of Berkeley-Stanford pairs for each arrangement, and compute the expected value. (Feel free to write out only a few cases just to get the feel for it, the table is very large!)

Solution: Let B_1, B_2 be the two Berkeley students, and let S_1, S_2 be the two Stanford students. We list out all possible arrangements of the students.

Arrangement	Pair at Pos. 1-2?	Pair at Pos. 2-3?	Pair at Pos. 3-4?	Total # of Pairs	# of Pairs · Prob. of Arr.
$B_1B_2S_1S_2$	0	1	0	1	$\frac{1}{24}$
$B_1B_2S_2S_1$	0	1	0	1	$\frac{1}{24}$
$B_1S_1B_2S_2$	1	1	1	3	$\frac{3}{24}$
$B_1S_1S_2B_2$	1	0	1	2	$\frac{2}{24}$
$B_1S_2S_1B_2$	1	0	1	2	$\frac{2}{24}$
$B_1S_2B_2S_1$	1	1	1	3	$\frac{3}{24}$
$B_2B_1S_1S_2$	0	1	0	1	$\frac{1}{24}$
$B_2B_1S_2S_1$	0	1	0	1	$\frac{1}{24}$
$B_2S_1B_1S_2$	1	1	1	3	$\frac{3}{24}$
$B_2S_1S_2B_1$	1	0	1	2	$\frac{2}{24}$
$B_2S_2S_1B_1$	1	0	1	2	$\frac{2}{24}$
$B_2S_2B_1S_1$	1	1	1	3	$\frac{3}{24}$
$S_1S_2B_1B_2$	0	1	0	1	$\frac{1}{24}$
$S_1S_2B_2B_1$	0	1	0	1	$\frac{1}{24}$
$S_1B_1S_2B_2$	1	1	1	3	$\frac{3}{24}$
$S_1B_1B_2S_2$	1	0	1	2	$\frac{2}{24}$
$S_1B_2B_1S_2$	1	0	1	2	$\frac{2}{24}$
$S_1B_2S_2B_1$	1	1	1	3	$\frac{3}{24}$
$S_2S_1B_1B_2$	0	1	0	1	$\frac{1}{24}$
$S_2S_1B_2B_1$	0	1	0	1	$\frac{1}{24}$
$S_2B_1S_1B_2$	1	1	1	3	$\frac{3}{24}$
$S_2B_1B_2S_1$	1	0	1	2	$\frac{2}{24}$
$S_2B_2B_1S_1$	1	0	1	2	$\frac{2}{24}$
$S_2B_2S_1B_1$	1	1	1	3	$\frac{3}{24}$
Total Sum	16	16	16	48	2
Expected Value	$\frac{16}{24} = \frac{2}{3}$	$\frac{16}{24} = \frac{2}{3}$	$\frac{16}{24} = \frac{2}{3}$	$\frac{48}{24} = 2$	

Each arrangement has $\frac{1}{24}$ of occurring, so we multiply the total sum by $\frac{1}{24}$ to get the expected number of pairs.

(b) Now, we wish to compute the expected value by using indicator variables. What indicator variables should we use?

Solution: Define the random variable I_i to be the indicator variable for the event that there is a Berkeley-Stanford pair at position i and $i+1$, for $i = 1, 2, 3$. We can express the number of pairs in the arrangement as $I_1 + I_2 + I_3$.

(c) Compute the expected value by using indicator variables. Do you see why this should give the same result as in part a)?

Solution: We now compute the probability that there is a Berkeley-Stanford pair at position i and $i+1$. We can choose anyone to be at position i, and there is a $\frac{2}{3}$ probability that the person at position $i+1$ goes to a different school than the person at position i (of the three other people, two of them are from a different school from the person at position i. Thus, $E[I_i] = \frac{2}{3}$.
By Linearity of Expectation, the expected number of pairs is $E[I_1 + I_2 + I_3] = E[I_1] + E[I_2] + E[I_3] = 2$. Thus, the expected number of pairs is 2, which matches what we got in part (a).
Note that the result in part (a) is gotten from summing the last column, and result in part (b) is gotten from summing the bottom row. Since these both count the total number of pairs multiplied by $\frac{1}{24}$, these values should be the same.

4. **Binomial Mean**
Show that mean of a Binomial random variable X with parameters n and p has mean np.

Solution: We can model X as the number of flips that show up heads if n independent coins are flipped, and each coin has probability p of being heads. Since a Bernoulli random variable with parameter p represents the whether or not a coin with probability p of being heads is heads, we can express X as the sum of n independent Bernoulli random variables $X_1 + X_2 + \cdots + X_n$, each with parameter p.
By Linearity of Expectation, we have that

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

Moreover, we know that the mean of each X_i is p. Thus, $E[X] = np$.

5. **Rectangles**
On the xy-plane, consider the 8×8 grid of lattice points (i, j), with $0 \le i, j \le 7$.

(a) Consider rectangles that are formed by taking 4 of the lattice points as vertices, such that the edges of the rectangles are parallel with the x and y axes. How many such rectangles are there?

Solution: To construct such a rectangle, we choose two numbers $0 \le a < b \le 7$ and two numbers $0 \le c < d \le 7$, then define the rectangle to have vertices (a, c), (a, d), (b, c), and (b, d). There are $\binom{8}{2}$ ways to choose a and b and $\binom{8}{2}$ ways to choose c and d, so in total there are $\binom{8}{2}^2 = 784$ rectangles.

(b) Each point on the grid is equally likely to be red or blue. What is the expected number of rectangles from part (a) that have all their vertices blue?

Solution: By Linearity of Expectation, we construct indicator variables for each of the rectangles, and the expected value of the indicator variable is the probability that the rectangle is blue. Each rectangle as a $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ probability of being blue, so the expected number of blue rectangles is $\frac{784}{16} = 49$.