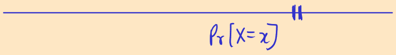
$Pr[X=x]$ X is discrete

$Pr[X=x]=0$ X is continuous

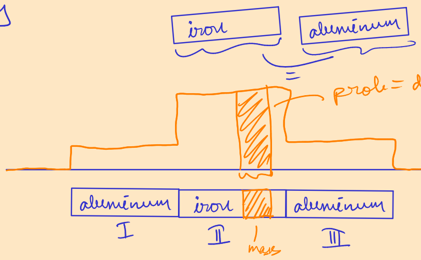
$f_X(x)$ — prob density

$\int_{-\infty}^{\infty} Pr[X=x]\,dx = \infty$



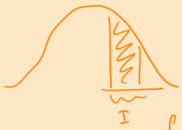
density ⟷ prob density

mass ⟷ prob

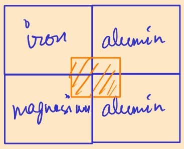


= prob = density × length of interval

= density of rod × volume of metal



$\int_I f(x)\,dx$



density × volume

$\iint$

$f_{XY}(x,y)$ = prob density at (x,y)

$Pr[x_1 \le X \le x_2, y_1 \le Y \le y_2]$

$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

NOT Prob $\iint$ to find prob

$f_{XY}(x,y) \cancel{\le} 1$

$\iint f_{XY}(x,y)\,dx\,dy = 1$

19300 kg m^{-3}

1 kg

$\frac{1}{19300}$ m^{-3}

$x=1,\ y=2\quad f(x,y)=2$

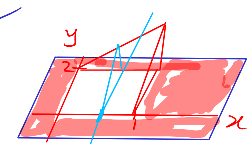
1 Continuous Joint Densities

$f(x,y)=\begin{cases} Cxy, & x\in[0,1]\wedge y\in[0,2] \\ 0, & o/w\end{cases}$

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

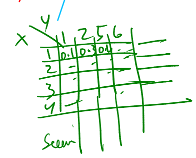
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^2 Cxy\,dy\,dx = \int_0^1 Cx\left[\frac{y^2}{2}\right]_0^2 dx = C\int_0^1 x\cdot 2\,dx = C\left[x^2\right]_0^1 = C = 1$



(b) Find $f_X(x)$, the marginal distribution of X.

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^2 xy\,dy = \begin{cases} 2x & \text{if } x\in[0,1] \\ 0 & o/w\end{cases}$

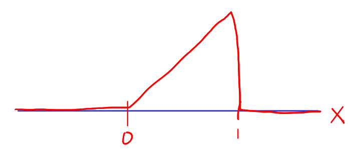


(c) Find the conditional distribution of Y given $X = x$.

$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{xy}{2x} = \begin{cases} \frac{y}{2} & x\in[0,1]\wedge y\in[0,2] \\ 0 & o/w\end{cases}$

$Pr[Y|X] = \frac{Pr[X,Y]}{Pr[X]}$

(d) Are X and Y independent?



Yes

$f_X(x) = 2x,\quad f_Y(y) = \int_0^1 xy\,dx = \frac{y}{2},\quad f_X(x)f_Y(y) = f_{XY}(x,y)$

→joint is prod of marginals

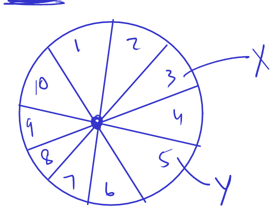
→$f_{Y|X}$ does not dep on X ⇒ $f_{X|Y}$ does not dep on Y

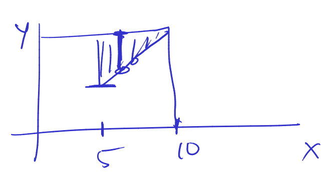
2 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \ge 5$, given that $Y \ge X$?

$Pr[X\ge 5 \mid Y\ge X]$

$Pr[X\ge 5 \cap Y\ge X]\quad \int_5^{10}\int_x^{10} \frac{1}{100}\,dy\,dx = \frac{1}{8}$





3 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a circle of radius r around the center. Alex's aim follows a uniform distribution over a circle of radius $2r$ around the center.

(a) Let the distance of Michelle's throw be denoted by the random variable X and let the distance of Alex's throw be denoted by the random variable Y.

- What's the cumulative distribution function of X?
- What's the cumulative distribution function of Y?
- What's the probability density function of X?
- What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \min\{X, Y\}$?

(d) What's the cumulative distribution function of $V = \max\{X, Y\}$?

(e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X - Y|]$? [*Hint*: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z] = \int_0^\infty P(Z \geq z)dz$.]