Countability

$\{1, 2, 3\}$

$\{0, 1, 2, 3, \ldots\}$ $\quad \mathbb{N}$

S — finite? —

- Yes $\rightarrow$ Countable
- No $\rightarrow$ ~~bijection~~ injection from $S \to \mathbb{N}$?
  - Yes $\rightarrow$ Countable
  - No $\rightarrow$ Uncountable

injection from $S \to \mathbb{N}$

f is a **Bijection** between S, T

① f is an Injection $S \to T$
(one-to-one)

$f(s_1) = f(s_2) \Rightarrow s_1 = s_2$

$s_1 \neq s_2 \Rightarrow f(s_1) \neq f(s_2)$

② f is a Surjection $S \to T$
(onto)

$\forall t \in T, \exists s \in S : f(s) = t$

$S = \{1, 2\}$

$T = \{3, 4\}$

$f = \{(1,3), (2,4)\}$

$f = \{(1,3), (2,3)\}$
not a surjection

$f = \{(1,4)\}$
neither surj, nor inj

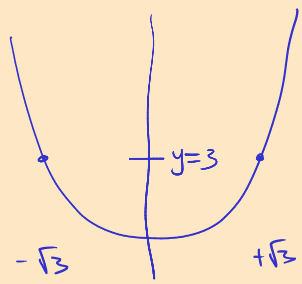
$S = \{1, 2\}$

$T = \{3, 4, 5\}$

inj $= \{(1,3), (2,4)\}$
$\{(1,4), (2,5)\}$

surj = No surj $S \to T$

$\{(1,3), (2,4), (2,5)\}$
Not a func!



https://tinyurl.com/sagnick-8a

CS 70 Fall 2020 — Discrete Mathematics and Probability Theory — DIS 8A

1 Graph Isomorphic

In graph theory, an isomorphism of graphs G and H is a bijection between the vertex sets of G and H

$$f : V(G) \to V(H)$$

such that any two vertices u, v of G are adjacent in G if and only if $f(u), f(v)$ are adjacent in H. Prove the following:

1. The degrees of corresponding nodes $u, f(u)$ are the same.
2. There is a bijection between edges.
3. If G is connected, then H is also connected.

2 Countability Practice

(a) Do $(0,1)$ and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$ (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.

(b) Is the set of strings over the English alphabet countable? (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.) If so, then provide a method for enumerating the strings. If not, then use a diagonalization argument to show that the set is uncountable.

(c) Consider the previous part, except now the strings are drawn from a countably infinite alphabet $\mathscr{A}$. Does your answer from before change? Make sure to justify your answer.

3 Python Functions

Prove that the set of all functions from N to {0,1} is uncountable.

(a) The set $\underbrace{F}_{\text{set of funcs}} = \{\underbrace{f}_{\text{func}} : \mathbb{N} \to \{0,1\}\}$ is not countable. $f(x) = \lfloor x \rfloor \bmod 2$ 0101010101- - - $f(x)=0$ 0000··· $f(x)=1$ 1111111 ···

(b) Prove that the set of all python functions that output $\{0,1\}$ is countable. (Python functions have the same power as Turing machines, but people are more familiar with python.)

$f(0)$

0101101 0 0101110100001 - - -

$f(5)$

$f(x) =$ if x==0: ___
elif x==1: ___
else: ___

0100000 0 0
1000000 0 0
11 00000 0 0
00 00000 0 0

3a

The set of all ∞-length bit strings is uncountable!

0	01011010 0101110100000 1 ...
1	00100111 0101110 00000 1 ...
2	01011010 01011111100000 1 ...
3	01011010 0101110100000 1 ...
4	010110110 0101110100000 1 ...
5	0101101011101111111110 0 1 ...
6	0101101010101011101000001 ...
⋮	

$\longrightarrow$ 1110010 - - - $= s'$

diff from every string in this enumeration

s' is diff from the n^{th} string

$s'[n] \neq n^{th}$ string $[n]$

WTS S is count.

find inj $g: F \to \mathbb{N}$

bij $g: \mathbb{N} \to F$

3b

Must write down a Python func.

Every Python func is a finite string.

```
def f(n):
  if n==1 or n== 4 or n== 6:
      return 1
  else:
    return 0
```

0100101 0000- - -

def\s f(n):\n if\s n== 1\s . . .

3c

Python funcs are finite-length strings.

(c) The set of Python functions that take in input x and output either 0 or 1 appears to be the same as F in (a), but the set of Python function is countable. Why?