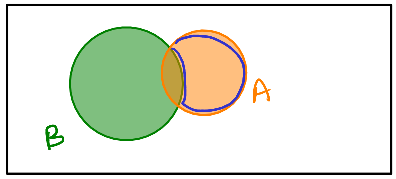
CS 70 Discrete Mathematics and Probability Theory
Fall 2020 DIS 10A

1 Probability Potpourri



$P(A) = P(A \setminus B) + \cancel{P(A \cap B)}$
$- P(B) = P(B \setminus A) + \cancel{P(A \cap B)}$
$P(A) - P(B) = P(A \setminus B) - P(B \setminus A) \leq P(A \setminus B)$

Prove a brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) If $|\Omega| = n$, how many distinct events does the probability space have? 2^n : # of subsets ∵ each outcome either in or not in E

(c) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

$P(D|C) = P(D|\overline{C}) \Rightarrow D \perp\!\!\!\perp C$

$$P(D) = P(C)P(D|C) + P(\overline{C})P(D|\overline{C}) = P(D|C)\underbrace{[P(C) + P(\overline{C})]}_{1} = P(D|C) \Rightarrow D \perp\!\!\!\perp C.$$

2 Aces

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

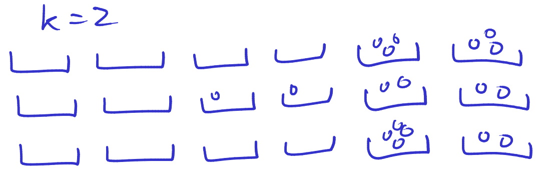
(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

3 Balls and Bins

Throw n balls into n labeled bins one at a time.

$\frac{(n-k)^k}{n^k}$ $\qquad n-(n-k)=k$

$k=2$



(a) What is the probability that the first bin is empty?

$\frac{(n-1)^n}{n^n} \qquad \left(\frac{n-1}{n}\right)^n$

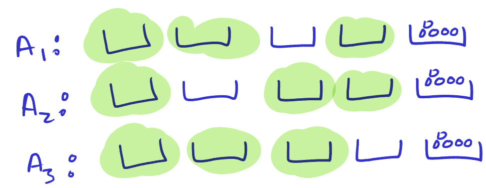
(b) What is the probability that the first k bins are empty?

$\frac{\#\text{ fav outcomes}}{\#\text{ of total outcomes}} \qquad \frac{(n-k)^n}{n^n} \qquad \left(\frac{n-k}{n}\right)^n$

(c) Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. If we assune A_i is the event that the i^{th} set of k bins is empty. Then we can write A as the union of A_i's.

$k=3$

$$A = \bigcup_{i=1}^{m} A_i.$$



Write the union bound for the probability A.

$$P(A) = P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$$

(d) Use the union bound to give an upper bound on the probability A from part (c).

$$\sum_{i=1}^{m} P(A_i) = \sum_{i=1}^{m}\left(\frac{n-k}{n}\right)^n = m\left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n \qquad P(A_i) = \left(\frac{n-k}{n}\right)^n \ \forall i$$

(e) What is the probability that the second bin is empty given that the first one is empty?

$$P(2^{nd}\text{ empty} \mid 1^{st}\text{ empty}) = \frac{P(1^{st}\text{ empty} \cap 2^{nd}\text{ empty})}{P(1^{st}\text{ empty})} = \left(\frac{n-2}{n}\right)^n \Big/ \left(\frac{n-1}{n}\right)^n = \left(\frac{n-2}{n-1}\right)^n.$$

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

$P(1^{st}\text{ empty} \mid 1^{st}\ \&\ 2^{nd}\text{ empty}) = 1$

$P(1^{st}\text{ empty}) = \left(\frac{n-1}{n}\right)^n$

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent?

NOT indep. $\quad P(2^{nd}\text{ empty}) = \left(\frac{n-1}{n}\right)^n \neq \left(\frac{n-2}{n-1}\right)^n = P(2^{nd}\text{ empty} \mid 1^{st}\text{ empty})$