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# Countability

Computer Science Mentors 70

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1. Show that for any positive integer $n$, an injective (one-to-one) function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ must be a bijection.

> **Solution:** Since $f$ is an injection, every element must send to a distinct element. Thus, the cardinality of the image of the function $f$ must also be $n$, which is equal to the cardinality of the range. Thus, the range and the image must be the same set, so the function is a surjection. Thus, $f$ is a bijection.

2. Find a bijection between N and the set of all integers congruent to $1 \bmod n$, for a fixed $n$.

> **Solution:** The set of integers congruent to 1 mod $n$ is $A = \{1 + kn \mid k \in \mathbb{Z}\}$. Define $g : \mathbb{Z} \to A$ by $g(x) = 1 + x \cdot n$; this is a bijection because it is clearly one-to-one, and is onto by the definition of $A$. We can combine this with the bijective mapping $f : \mathbb{N} \to \mathbb{Z}$ from the notes, defined by $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{-(x+1)}{2} & \text{if } x \text{ is odd} \end{cases}$. Then $f \circ g$ is a function from $\mathbb{N}$ to $A$, which is a bijection.

3. Are these sets countably infinite/uncountably infinite/finite? If finite, what is the order of the set? Reminder: A bit string is a sequence of digits where each digit corresponds to either a 1 (on) or a 0 (off).

   (a) Finite bit strings of length n.

   > **Solution:** Finite. There are 2 choices (0 or 1) for each bit, and n bits, so there are $2 \times 2 \times \ldots \times 2 = 2^n$ such bit strings.

   (b) All finite bit strings of length 1 to n.

   > **Solution:** Finite. By part (a), there are $2^1$ bit strings of length 1, $2^2$ of length 2, etc. Thus, there are $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$.

   (c) All finite bit strings

   > **Solution:** Countably infinite. We can list these strings as follows:
   > $\{0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots\}$. This gives us a bijection with the (countable) natural numbers, so these are countably infinite.

   (d) All infinite bit strings

   > **Solution:** Uncountably infinite. We can construct a bijection between this set and the set of real numbers between 0 and 1. We can represent these real numbers using binary— e.g. they are of the form $0.0110001010110\ldots$. By diagonalization, the set of real numbers between 0 and 1 is uncountably infinite; therefore, so is this set.

   (e) All finite or infinite bit strings.

**Solution:** Uncountably infinite. This is the union of a countably infinite set (part c) and an uncountably infinite set (part d), so it is uncountably infinite.

(f) Suppose that we try to show that infinite bit strings are countable by induction. We show that for any positive integer $n$, a bit string of length $n$ is countable. Why does this not work for infinite strings?

**Solution:** Induction applies only to finite values.

4. Is the power set $\mathcal{P}(S)$, where $S$ is countably infinite, finite, countably infinite, or uncountably infinite? Provide a proof for your answer. Reminder: the power set of a set is the set of all possible subsets of that set. Ex: $S = \{A, B\}$, $\mathcal{P}(S) = \{\{\}, \{A\}, \{B\}, \{A, B\}\}$

**Solution:** The power sets of a countably infinite set are uncountably infinite. There is a bijection between the set $2^{\mathbb{N}}$ and $2^S$, as $S$ and $\mathbb{N}$ have the same cardinality. The set $2^{\mathbb{N}}$ is uncountable. We prove this through contradiction. We assume the set $2^{\mathbb{N}}$ is countably infinite. This means we can list the subsets of $\mathbb{N}$ such that every subset is $N_i$ for some $i$. We define another set $A = \{i | i \geq 0 \text{ and } i \notin N_i\}$ which contains integers $i$ not part of $N_i$. But, $N$ is a subset of $\mathbb{N}$ so we must have $N = N_j$ for some $j$. This means that if $j \in N$, then $j \notin N$, and if $j \notin N$, then $j \in N$. This is a contradiction since $j$ is either in $N$ or not, so the set is not countably infinite.