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So Much Counting

Computer Science Mentors 70

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Roast us here: *https://tinyurl.com/csm70-feedback20*

Prepared by: Aishani Sil, Austin Lei, Agnibho Roy, Debayan Bandyopadhyay, Abinav Routhu

Rules of Counting

First Rule of Counting:

$$A = \{a_1, a_2, \ldots, a_m\} \qquad B = \{b_1, b_2, \ldots, b_n\}$$

How many ways are there to choose (a, b) with $a \in A$ and $b \in B$?
There are $|A| \times |B| = mn$ ways to choose (a, b).

Corollary:

How many ways are there to choose c where $c \in A$ or $c \in B$?

$$C = A \bigcup B = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$$

There are $|C| = m + n$ ways to choose c.

Second Rule of Counting:

$$A = \{a_1, a_2, \ldots, a_m\} \qquad B = \{b_1, b_2, \ldots, b_n\}$$

How many ways are there to choose an unordered pair (a, b) with $a \in A$ and $b \in B$?
There are $\frac{|A|\times|B|}{2} = \frac{1}{2}mn$ ways to choose an unordered pair of elements, one from each set.

1 Intro to Counting

1. Say I have a standard 6-sided die. I generate a sequence of numbers by tossing this die 5 times.

   (a) How many distinct sequences of 5 numbers can I generate this way?

   (b) How many sequences have the form $(6, 6, 6, 6, 6)$? How many have the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$?

   (c) How many sequences contain at least one 3?

2. Austin is deciding on what courses to take for Spring 2021, and he must choose from 4 math courses, 3 CS courses, and 5 non-math/CS courses. Austin decides that he will take 2 math or CS courses, and one non-math/CS course. How many choices does Austin have?

3. How many of the first 1000 positive integers are neither perfect squares nor perfect cubes?

2 Applying Counting Techniques

1. Leanne has 9 songs she wants to sing at a concert: 6 old songs and 3 new songs. However, she does not want to sing 2 new songs back to back. If Leanne sings each song exactly once, how many possible orderings of the songs are possible?

2. How many length 7 bitstrings have more zeroes than ones?

3. How many length 8 bitstrings have more zeroes than ones?

4. How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

3 S U P E R M A N

1. How many ways are there to arrange the letters of the word "SUPERMAN"...
   (a) ...on a straight line?
   (b) ...on a straight line, such that "SUPER" occurs as a substring?
   (c) ...on a circle? Note: If we arrange elements on a circle, all permutations that are “shifts” are equivalent (i.e. SUPERMAN and UPERMANS).
   (d) ...on a circle, such that "SUPER" occurs as a substring? Reminder: SUPER can occur anywhere on the circle!

2. Now how many ways are there to arrange the letters of the word "SUPERMAN"...
   (a) ...on a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?
   (b) ...on a circle, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

4 Ready For A Challenge?

1. How many positive factors of 2020 are there? (Hint: Consider prime factorization.)

2. How many positive integers less than or equal to 105 are relatively prime to 105?

3. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order. (Source: 2020 AIME I)

5 Combinatorial Proofs

1. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

   Which of the following are valid ways of counting the number of squares in an $n \times n$ grid?

   (a) In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.

   (b) We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n - 1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n - 2$ squares on the hypotenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

   (c) Take the $(n-1) \times (n-1)$ subgrid that is the upper-left corner of this grid. This subgrid has $n - 1$ rows, each of which has $n - 1$ squares, so this part contributes $(n-1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n-1)^2 + 2n - 1$ squares in an $n \times n$ grid.

   (d) First, we peel off the leftmost column, and topmost row, removing exactly $2n - 1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n-1) - 1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n-1) + (2n-3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

2. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

3. Provide a combinatorial proof for the following:

$$\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$$

4. Prove $\binom{n}{a} a(n-a) = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.