6 So Much Counting

Computer Science Mentors 70

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1 Intro to Counting

1. Say I have a standard 6-sided die. I generate a sequence of numbers by tossing this die 5 times.

   (a) How many distinct sequences of 5 numbers can I generate this way?

   Solution: $6^5 = 7776$

   (b) How many sequences have the form $(6, 6, 6, 6, 6)$? How many have the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$?

   Solution: 1 and 32. For the first one, there is one choice for each of the digits five times, $1^5 = 1$. In the second one, there are two choices for each of the digits, $2^5 = 32$.

   (c) How many sequences contain at least one 3?

   Solution: $6^5 - 5^5 = 4651$. It is easier to calculate the number of sequences that don't have any 3s and then subtract from the total number of possible sequences. If a sequence does not have any 3s, all of its digits must be in $\{1, 2, 4, 5, 6\}$; following the first rule of counting, we get 5^5.

2. Austin is deciding on what courses to take for Spring 2021, and he must choose from 4 math courses, 3 CS courses, and 5 non-math/CS courses. Austin decides that he will take 2 math or CS courses, and one non-math/CS course. How many choices does Austin have?

   Solution: There are $\binom{4+3}{2} = 21$ choices for the math/CS courses, and $\binom{5}{2} = 10$ choices for the non-math/CS courses, so there are $21 \cdot 10 = 210$ total choices.

3. How many of the first 1000 positive integers are neither perfect squares nor perfect cubes?

   Solution: It would be easier to first count the number of perfect squares and perfect cubes. We know that $31^2 = 961$ and $32^2 = 1024$. Thus, there are 31 perfect squares. We know that $10^3 = 1000$ so there are 10 perfect cubes. This means that the answer is 1000 - 31 - 10 = 959. However, we have over counted and subtracted the cases where numbers are are both perfect squares and perfect cubes (perfect sixths) twice, which would be $2^6 = 64$ and $3^6 = 729$. Thus, we add two back to our solution to get 961.

2 Applying Counting Techniques

1. Leanne has 9 songs she wants to sing at a concert: 6 old songs and 3 new songs. However, she does not want to sing 2 new songs back to back. If Leanne sings each song exactly once, how many possible orderings of the songs are possible?

   Solution:

   Solution 1: Let the three new songs be the dividers, creating four bins, one for each spot before, between, or after the new songs. There are 3! ways to order the new songs.

We now determine the position of the old songs relative to the new songs. We use balls and bins, where the old songs are the balls and the three new songs are the dividers. First, since, the second and third bins need to have at least one ball, we give the two bins one ball. Thus, we need to allot four balls among four bins. By balls and bins, there are $\binom{4+4-1}{4} = \binom{7}{4}$ ways to do so.
Finally, we need to decide on the order for the old songs, and there are 6! ways to do so. Thus, there are $6! \cdot \binom{7}{4} \cdot 3!$ different orderings.
Solution 2: Let the six old songs be the dividers, creating seven bins, one for each spot before, between, or after the old songs. There are 6! ways to order the old songs. Then, we must choose a different bin for each of the new songs (since two new songs can't be back to back), giving $\binom{7}{3}$ choices for where the old songs can go. Finally, there are 3! ways to choose the order of the new songs, so combining there are $6! \cdot \binom{7}{3} \cdot 3!$ different orderings.

2. How many length 7 bitstrings have more zeroes than ones?

Solution: There are a total of 2^7 bitstrings, since each bit can be either 0 or 1. Since the bitstring is of odd length, it needs to either have more zeroes than ones or vice versa. One can take a bitstring that has more zeroes than ones, and flip all of the bits so that there are now more ones than zeroes, so there are an equal amount of either type. $2^7/2 = 64$.

3. How many length 8 bitstrings have more zeroes than ones?

Solution: Similar approach as above, but we must also consider the bitstrings that have an equal number of 0s and 1s. There are $\binom{8}{4}$ of these as we pick 4 positions to place the 1s in, and the rest of the bitstrings either have more zeroes than ones or vice versa. $(2^8 - \binom{8}{4})/2 = 93$.

4. How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

Solution: We can think of this in terms of stars and bars. We have two bars between the variables x, y, and z, and our stars are the 10 1s we have to distribute among them. Since all variables must be positive integers, x, y, and z will each be at least 1. So, we have 7 1s left to distribute. So we have 7 stars, 2 bars. Answer = $\binom{9}{2} = 36$.

3 SUPERMAN

1. How many ways are there to arrange the letters of the word "SUPERMAN"...

   (a) ...on a straight line?

   Solution: 8!

   (b) ...on a straight line, such that "SUPER" occurs as a substring?

   Solution: 4! Treat "SUPER" as one character.

   (c) ...on a circle? Note: If we arrange elements on a circle, all permutations that are "shifts" are equivalent (i.e. SUPERMAN and UPERMANS).

   Solution: 7! Anchor one element, arrange the other 7 around in a line.

   (d) ...on a circle, such that "SUPER" occurs as a substring? Reminder: SUPER can occur anywhere on the circle!

Solution: 3! Treat "SUPER" as a single character, anchor one element, and arrange the other 3 around in a line.

2. Now how many ways are there to arrange the letters of the word "SUPERMAN"...

   (a) ...on a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   Solution: 3! * $\binom{8}{3}$ There are two ways to think about the problem.

   We can arrange the letters of SUPERMAN 8! ways, but divide by 5! because we have arranged SUPER in any of 5! ways, when we only want one way. This gives us 8! / 5!.

   Alternatively, we can think about picking three slots for "MAN" and then permute them. Then "SUPER" should fill in the other 5 slots with S going first and then U, P, E, R as they have to appear as a subsequence. This gives us 3! * $\binom{8}{3}$.

   You're encouraged to check that the above two methods give the same answer.

   (b) ...on a circle, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   Solution: $\binom{7}{3} * 3!$ = $\binom{7}{2} * 2! * 5 = 210$.

   There are two methods. Method 1: anchor one of S, U, P, E, R. Choose which 3 places to put the M, A, and N (7 choose 3) and allow them to be shuffled (3!). Then the U, P, E, R must fill in the remaining slots in order. We get $\binom{7}{3} * 3!$.

   Method 2: anchor one of M, A, N. Choose which of the 2 remaining 8 spots to place the A and N, allowing shuffles (7 choose 2 * 2!). Then, choose which of the 5 remaining spots to place the S (the other letters must follow in order after the S). We get $\binom{7}{2} * 2! * 5$.

   You are encouraged to check the two answers above are equivalent.

4 Ready For A Challenge?

1. How many positive factors of 2020 are there? (Hint: Consider prime factorization.)

   Solution: The prime factorization of 2020 is $2^2 \cdot 5 \cdot 101$. A positive factor of 2020 must be of the form $2^a \cdot 5^b \cdot 101^c$, where $0 \leq a \leq 2, 0 \leq b \leq 1$, and $0 \leq c \leq 1$. Thus, there are 3 choices for a, 2 choices for b, and 2 choices for c, giving $3 \cdot 2 \cdot 2 = 12$ positive factors.

2. How many positive integers less than or equal to 105 are relatively prime to 105?

   Solution: We first count the complement: the positive integers less than or equal to 105 that share a common factor with 105. Since $105 = 3 \cdot 5 \cdot 7$, a number that shares a common factor with 105 is divisible by 3, 5, or 7.
   Let A, B, and C be the sets of positive integers less than or equal to 105 that are divisible by 3, 5, and 7, respectively. We wish to compute $|A \cup B \cup C|$, which by the Principle of Inclusion Exclusion is

   $$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

   There are $\frac{105}{3} = 35$ elements in A, $\frac{105}{5} = 21$ elements in B, and $\frac{105}{7} = 15$ elements in C. Moreover, $|A \cap B|$ is the number of positive integers less than or equal to 105 divisible by $3 \cdot 5 = 15$, so there are $\frac{105}{15} = 7$ such numbers. Similarly, $|B \cap C| = \frac{105}{35} = 3$, $|C \cap A| = \frac{105}{21} = 5$, and $|A \cap B \cap C| = \frac{105}{105} = 1$. Thus, the number of positive integers less than or equal to 105 not relatively prime to 105 is

$$35 + 21 + 15 - 7 - 3 - 5 + 1 = 57,$$

so the number of positive integers relatively prime to 105 is

$$105 - 57 = 48.$$

3. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order. (Source: 2020 AIME I)

Solution: We can deal with one of the cases first, say the ascending case, and multiply by 2 since we know that for each ascending case there is a unique corresponding decreasing case. All valid sequences can be constructed by taking all numbers but one, order them in an increasing sequence, and then insert the final number, say some number k, at any location. This is valid because k can be deleted and the sequence is still ascending. Since there are 6 possible values of k, and 6 locations that they can be inserted into, we have a total of 6×6 choices.
However, we have overcounted here. Let us visualize by expanding on the cases of inserting 1, 2, and 3:

inserting 1	inserting 2	inserting 3
123456	213456	312456
213456	123456	132456
231456	132456	123456
234156	134256	124356
234516	134526	124536
234561	134562	124563

Each time, there are two sequences that are shared with each adjacent list (and one of these sequences is always 123456). This is because, for each pair $\{k, k+1\}$, we want to include the sequences $\ldots k, k+1, \ldots$ and $\ldots k+1, k, \ldots$. These sequences will both appear twice, once when k and $k + 1$ each are being inserted, but we want to only keep one of each. Thus, we subtract 2 for every adjacent pair of numbers, of which there are 5. We can also think about it as taking out the extra five 123456's that we end up with, and also taking out the repeat instance of having consecutive digits next to each other.
After accounting for overcounting and doubling the count since we can have decreasing sequences as well, the answer should be $(6 \times 6 - 10) \times 2 = 52$.

5 Combinatorial Proofs

1. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

Which of the following are valid ways of counting the number of squares in an $n \times n$ grid?

(a) In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.

(b) We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n - 1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n - 2$ squares on the hypothenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

(c) Take the $(n - 1) \times (n - 1)$ subgrid that is the upper lefthand corner of this grid. This subgrid has $n - 1$ rows, each of which has $n - 1$ squares, so this part contributes $(n - 1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n-1)^2 + 2n - 1$ squares in an $n \times n$ grid.

(d) First, we peel off the leftmost column, and topmost row, removing exactly $2n - 1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n-1) - 1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n-1) + (2n-3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

> **Solution:** All of the above methods are valid ways of counting!

2. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

> **Solution:** Choose a team of k players where one of the players is the captain.
> LHS: Pick a team with k players. This is $\binom{n}{k}$. Then make one of the players the captain. There are k options for the captain so we get $k \times \binom{n}{k}$.
> RHS: Pick the captain. There are n choices for the captain. Now pick the last $k-1$ players on the team. There are now $n-1$ people to choose from. So we get $n \times \binom{n-1}{k-1}$.

3. Provide a combinatorial proof for the following:

$$\sum_{k=1}^{n-1} 2^k = 2^n - 1$$

> **Solution:**
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> **Right Side**: This is the number of ways to pick a non-empty team from a group of n people. 2 choices for each individual: either choose them or not. This can also be thought of as the number of nonempty subsets of a set.
>
> **Left Side**: Number all individuals from 1 to n. Let k be some integer and let our new problem be the number of teams that can be made, with the restriction that k is the highest numbered individual that can be selected in our team. Then, we have 2^{k-1} total choices for a team, as we have 2 choices for each individual 1 to $k-1$: either to be in the team or not. Summing over all possible k, we get the left side summation.

4. Prove $\binom{n}{a}a(n-a) = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

> **Solution:** Suppose that you have a group of n players and want to pick a team of a with a captain, as well as a reserve player from the remaining $n-a$ players.
> LHS: Number of ways to pick a team of a of these players ($\binom{n}{a}$ ways), designate one member of the team as captain (a ways), and then pick one reserve player from the remaining $n-a$ people ($n-a$ ways).
> RHS: The right-hand side is the number of ways to pick the captain (n ways), then the reserve player ($n-1$ ways), and then the other $a-1$ members of the team ($\binom{n-2}{a-1}$ ways).