1 First Exponential to Die

$X \sim \text{Expo}(\lambda)$ $\quad f_X(x) = \lambda \exp(-\lambda x) = \lambda e^{-\lambda x}$ $\quad F_X(x) = 1 - \exp(-\lambda x) = \Pr[X \le x]$

$X \ge 0$

Let X and Y be Exponential(λ_1) and Exponential(λ_2) respectively, independent. What is

$\lambda = \frac{\text{arrivals}}{\text{time}}$

$$\mathbb{P}(\min(X,Y) = X),$$

$\min(X,Y) = X$

$\min(5,3) = 3$

3 vs 5 ?

the probability that the first of the two to die is X?

$$\Pr[\min(X,Y) = X] = \Pr[X < Y]$$
$$= \int_0^\infty \Pr[Y > X \mid X = x]\, f_X(x)\, dx$$
$$= \int_0^\infty (1 - F_Y(x))\, f_X(x)\, dx$$
$$= \int_0^\infty \exp(-\lambda_2 x)\, \lambda_1 \exp(-\lambda_1 x)\, dx$$
$$= \int_0^\infty \lambda_1 \exp(-(\lambda_1 + \lambda_2)x)\, dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

$\Pr[Y > X]$

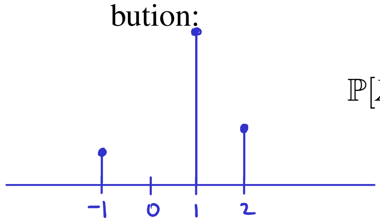
$= \sum_{x \in X} \Pr[Y > x \cap X = x]$

$= \sum_{x \in X} \Pr[Y > X \mid X = x] \Pr(X = x)$

$\int_{x=0}^{x=\infty} \Pr[Y > x]\, f_X(x)\, dx$

2 Chebyshev's Inequality vs. Central Limit Theorem

Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be $\boxed{\text{i.i.d.}}$ random variables with the following distribution:



$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \qquad \mathbb{P}[X_i = 1] = \frac{9}{12}; \qquad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1, $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

$\mathbb{E}[X_1] = -1 \cdot \frac{1}{12} + 1 \cdot \frac{9}{12} + 2 \cdot \frac{2}{12} = 1$

$\text{Var}(X_1) = \mathbb{E}[(X - \mathbb{E}[X])^2] = (-2)^2 \cdot \frac{1}{12} + 0^2 \cdot \frac{9}{12} + 1^2 \cdot \frac{2}{12} = \frac{1}{2}$

$\mathbb{E}[Z_n] = 0$

$\text{Var}(Z_n) = \text{Var}\left(\frac{1}{\sqrt{n/2}} \sum (X_i - \mathbb{E}[X_i])\right) = \frac{2}{n} \cdot \frac{n}{2} = 1$

$\mathbb{E}\left[\sum_{i=1}^n X_i\right] = n$

$\mathbb{E}\left[\sum_{i=1}^n (X_i - \mathbb{E}[X_i])\right] = \mathbb{E}\left[\sum_{i=1}^n (X_i - 1)\right] = \left(\sum \mathbb{E}[X_i]\right) - n = 0$

$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum \text{Var}(X_i) = \frac{n}{2}$

$\text{Var}(-n + \sum X_i)$

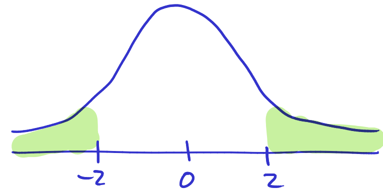
$\text{Var}\left(\sum_{i=1}^n (X_i - \cancel{\mathbb{E}[X_i]}^{1})\right) = \sum_{i=1}^n \text{Var}(X_i - \cancel{\mathbb{E}[X_i]}^{1}) = \sum_{i=1}^n \text{Var}(X_i) = \frac{n}{2}$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \ge 2]$.

$\Pr[|Z_n| \ge 2] = \Pr[|Z_n - 0| \ge 2] \le \frac{\text{Var}(Z_n)}{2^2} = \frac{1}{4}$

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum X_i - \mathbb{E}[X_1]\right)}{\sqrt{1/2}}$$

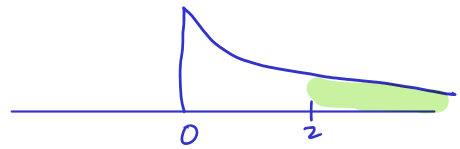
$$Z_n = \frac{\sum (X_i - \mathbb{E}[X_1])}{\sqrt{n/2}} = \frac{\frac{1}{\sqrt{n}}\sum X_i - \mathbb{E}[X_1]}{\sqrt{1/2}}$$



(c) Can you use b to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

$\Pr[Z_n \geq 2] \leq \Pr[|Z_n| \geq 2] \leq \frac{1}{4} = 0.25$

$\Pr[Z_n \leq -2] \leq \Pr[|Z_n| \geq 2] \leq \frac{1}{4}$



$\frac{\bar{X} - \mathbb{E}[X_1]}{\sqrt{Var(X_1)}} \to \mathcal{N}(0,1)$ n iid $X_1, \ldots, X_n$

(d) As $n \to \infty$, what is the distribution of Z_n? CLT

$\frac{(\bar{X} - E[X_i])}{\sqrt{Var(X_i)}} \to \mathcal{N}(0,1)$

(e) We know that if $Z \sim \mathcal{N}(0,1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \to \infty$, can you provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

$\frac{\frac{1}{n}(\sum X_i - \mathbb{E}[X_i])}{\sqrt{Var(X_i)}}$

$\bar{X} = \frac{1}{n} \sum X_i$

Can use symmetry!

$\Pr[|Z_n| \geq 2] = 1 - 0.9545 = 0.0455$

$\Pr[Z_n \geq 2] = \Pr[Z_n \leq -2] = \frac{0.0455}{2} = 0.02275$

3 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. One approach is to start with the cumulative distribution function of Y and use it to derive the probability density function of Y.

[1.You can use without proof that the pdf for any gaussian with mean and sd is given by the formula $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean value for X and σ^2 is the variance. 2. The drivative of CDF gives PDF.]

$\mathbb{E}[Y] = a\mu + b$ $\quad Var(Y) = a^2\sigma^2$

$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$