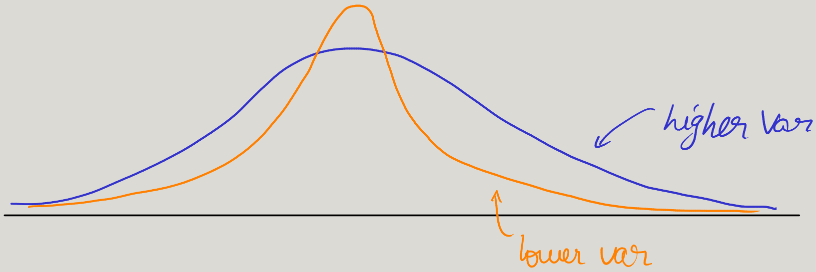


$\mathbb{E}[\text{const}] = \text{const}$

$$\begin{aligned}
Var(X) &= \mathbb{E}[(X-\mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]\,\mathbb{E}[X] + (\mathbb{E}[X])^2 \\
&= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}$$

Very useful indicators

$X = \sum_{i=1}^{n} X_i$, $X_i \sim Bern(p)$

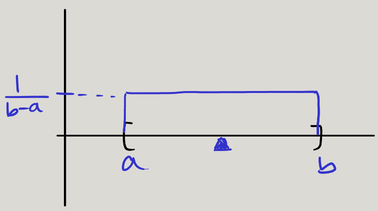
$X_i^2 \sim Bern(p)$

X_i	Pr	X_i^2	$Pr[X_i^2 = k]$
0	$1-p$	0	$1-p$
1	p	1	p

$X_iX_j \sim Bern(\tilde{p})$

$\mathbb{E}[X_iX_j] = Pr[X_iX_j = 1]\cdot 1 + \cancel{Pr[X_iX_j = 0]\cdot 0}$

$X \sim$ Dist	$\mathbb{E}[X]$	$Var(X)$
$Bern(p)$	p	$p(1-p)$
$Binom(n,p)$	np	$np(1-p)$
$Geom(p)$	$1/p$	$\frac{1-p}{p^2}$
$Poisson(\lambda)$	λ	λ
$Expo(\lambda)$	$1/\lambda$	$1/\lambda^2$
$\mathcal{N}(\mu,\sigma^2)$	μ	σ^2
$Unif[a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
discrete $Unif[a,b]$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$



$$\mathbb{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{3(b-a)}(b^3 - a^3)$$

$$Var(X) = \frac{1}{3(b-a)}(b^3-a^3) - \left(\frac{a+b}{2}\right)^2$$

$U \sim Unif[a,b]$

$$f_U(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & o/w \end{cases}$$

$U \sim$ disc. $Unif[a,b]$

$$Pr[U = x] = \begin{cases} \frac{1}{b-a+1}, & x \in \mathbb{Z}, x \in [a,b] \\ 0 & , o/w \end{cases}$$

Final Sp15 Q2

(d) X and Y are independent random variables modulo n. You don't know the distribution of X, but you know that Y is uniformly distributed. What can you say about the distribution of $Z = (X+Y) \bmod n$? Justify your answer. $[0, n-1]$ $X \in \{0, 1, \dots, n-1\}$

What vals can Z take on? Probs?

$z \in \{0, 1, \dots, n-1\}$

$$\Pr[Z = i] = \sum_{j=0}^{n-1} \Pr[X=j, Y=i-j]$$

$X \perp\!\!\!\perp Y$

$$= \sum_{j=0}^{n-1} \Pr[X=j] \Pr[Y = i-j \bmod n]$$

n cases

X	Y
0	i
1	$i-1$
2	$i-2$
$\vdots$	
$n-1$	$i-n+1$
n	$i-n$

$n = 6$

$$= \frac{1}{n} \sum_{j=0}^{n-1} \Pr[X=j]$$

$$= \frac{1}{n} \cdot 1 = \boxed{\frac{1}{n}}$$

Z is unif.

(e) X and Y are independent random variables with normal distribution with mean m_1 and m_2 respectively, and variance σ_1^2 and σ_2^2 respectively. Describe the distribution of $Z = X + Y$ (including mean and variance).

What if we are not told that X and Y are independent?

(1) $X \sim \mathcal{N}(m_1, \sigma_1^2)$ $X \perp\!\!\!\perp Y$

$Y \sim \mathcal{N}(m_2, \sigma_2^2)$

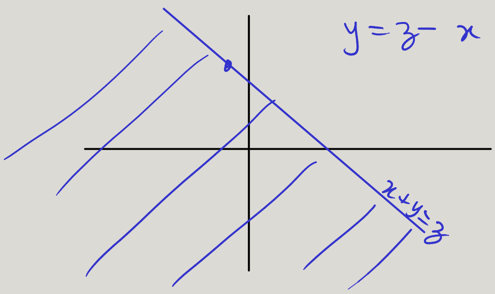
$Z \sim \mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$

$$\frac{Z - (m_1+m_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} \overset{\text{show}}{\sim} \mathcal{N}(0,1)$$

(2) $X \sim \mathcal{N}(m_1, \sigma_1^2)$ ~~$X \perp\!\!\!\perp Y$~~

$Y \sim \mathcal{N}(m_2, \sigma_2^2)$

$Z \sim$ NA



$Z = X + Y$

$$\Pr[Z \le z] = \Pr[X + Y \le z] = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y) \, dy \, dx$$

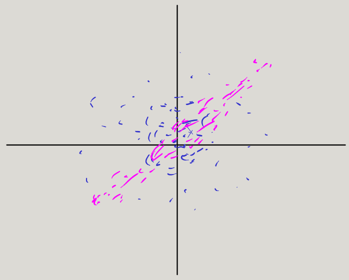
don't know

if $X \perp\!\!\!\perp Y$, $\int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y) \, dy \, dx$ know

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{z-x} f_Y(y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) F_Y(z - x) \, dx$$



Final Fa18 Q1h

True	False	
◯	●	For dependent random variables X, Y and constants a, b, it is possible that $\mathbb{E}[aX + bY] \neq a\mathbb{E}[X] + b\mathbb{E}[Y]$.
◯	●	$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ if and ✓ ~~only~~ if X and Y are independent.
◯	●	Consider two random variables X and Y with ranges $\mathcal{A}_X$ and $\mathcal{A}_Y$, respectively. If forall ~~there exist~~ $a \in \mathcal{A}_X$ and $b \in \mathcal{A}_Y$ such that $\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a]\mathbb{P}[Y = b]$, then X and Y are independent.

$$\overset{\overset{0}{\|}}{\mathrm{Cov}(X,Y)} = E[XY] - E[X]E[Y] = 0$$

$$\Rightarrow E[XY] = E[X]E[Y]$$

$$\mathrm{Cov}(X,Y) = 0 \not\Rightarrow X \perp\!\!\!\perp Y$$

$$X \perp\!\!\!\perp Y \Rightarrow \mathrm{Cov}(X,Y) = 0$$

Q2

(m) Let A and B denote two events such that $A \subset B$. Suppose $\mathbb{P}[A] = a$ and $\mathbb{P}[B] = b$, and let I_A and I_B denote the indicator random variables for A and B, respectively. Find $\mathrm{Cov}(I_A, I_B)$.

https://en.wikipedia.org/wiki/Normally_distributed_and_uncorrelated_does_not_imply_independent