Finite $\Rightarrow$ Countable

S   Infinite $\rightarrow$ Countable
             $\rightarrow$ Not Countable $\rightarrow$ $\mathbb{R}$

Assume $\mathbb{R}$ is countable.

$\therefore \exists f: \mathbb{N} \to \mathbb{R}$ which is surjective.

f is called an "enumeration".

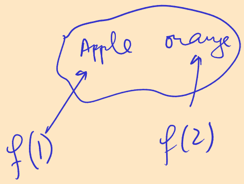
$f(1) = 0.3775100\,39\ldots$

$f(2) = 0.999888\,8\,79\ldots$

$f(3) = 7.853210\,12\ldots$

bijection betn S & $\mathbb{N}$.

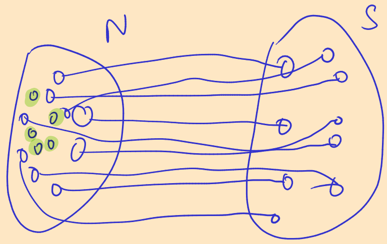
$f(i) =$



surjection from $\mathbb{N} \to S$

$\Downarrow$

$|S| \leq |\mathbb{N}|$



1 | $0.3775100\,39\ldots$
2 | $0.999888\,8\,79\ldots$
3 | $0.853210\,12\ldots$
4 | $0.0000000\,11\ldots$
5 | $0.7777778\,55\ldots$
6 | $0.4041873\,3\ldots$

$r^* = 0.404188\ldots$

r^* is not in the enumeration.

$\Downarrow$

$\nexists\, i \in \mathbb{N} : f(i) = r^*$. Contra.

A, B, $\exists$ surj $A \to B$

$\Downarrow$

$|A| \geq |B|$

$\exists$ inj $A \to B \Rightarrow |A| \leq |B|$

$\underbrace{\exists \text{ inj \& surj}}_{\text{bij}}$ A & B

$|A| \leq |B| \wedge |A| \geq |B|$

$\Downarrow$

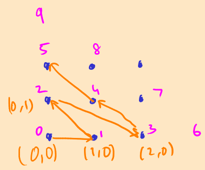
$|A| = |B|$

$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

$\uparrow$

bijection

(x, y)



$f(x, y) =$ ________

(x, y)

$$\frac{(x+y+1)(x+y)}{2}$$

6 Counting, Combinatorics, Countability

Computer Science Mentors 70

Mar 08 - Mar 12
Spring 2021

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Roast us here: `https://tinyurl.com/csm70-feedback20`

1 Counting

1. How many length 7 bitstrings have more zeroes than ones?

2. How many length 8 bitstrings have more zeroes than ones?

3. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order. (Source: 2020 AIME I)

4. How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

5. How many solutions does $x_1 + x_2 + x_3 + x_4 = 20$ have, if all variables must be non-negative integers, $x_1 < 8$, and $x_2 < 8$?

6. Consider a little game. Alice and Bob arrange 8 coins in a circle with alternating faces up (i.e. Heads, Tails, Heads, Tails, ...), and they each take turns flipping coins according to the following rules:

   Alice chooses any coin with heads facing up and flips it. Then, she sets the coin adjacent to it in the clockwise direction to heads.
   Bob chooses any coin with tails facing up and flips it. Then, he sets the coin two spots away (index $i + 2$) in the clockwise direction to tails.

   Assume Alice always plays the first move.

   (a) How many distinct ways are there for Alice and Bob to play the first two moves (hint: how many unique tuples (i, j) exist where i is the index of the first coin being flipped and j is the index of the second).

   (b) Suppose the game ends when there are no legal moves for the next player. Prove that no matter what Alice and Bob play, the game will never end.

   (c) Use parts (a) and (b) to provide an upper bound to the number of distinct ways for Alice and Bob to play the first k moves. Assume k is even.
   Hint: to find an upper bound, think about the maximum number of choices Alice/Bob can make at each step.

(d) How many distinct arrangements are there for the coins after 2 moves have been played? Assume that any arrangement of coins that can be reached by rotating all the coins by $1 \leq i \leq 7$ indices is equivalent.

2 Combinatorial Proofs

1. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

   Which of the following are valid ways of counting the number of squares in an $n \times n$ grid?

   (a) In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.

   (b) We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n-1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n-2$ squares on the hypotenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

   (c) Take the $(n-1) \times (n-1)$ subgrid that is the upper-left corner of this grid. This subgrid has $n-1$ rows, each of which has $n-1$ squares, so this part contributes $(n-1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n-1)^2 + 2n - 1$ squares in an $n \times n$ grid.

   (d) First, we peel off the leftmost column, and topmost row, removing exactly $2n-1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n-1)-1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n-1) + (2n-3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

2. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

3. Prove $\binom{n}{a}a(n-a) = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

3 Countability

1. Show that for any positive integer n, an injective (one-to-one) function $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ must be a bijection.

$\forall a,b\ ,\ f(a) = f(b) \Rightarrow a = b$ $\qquad \forall c \in C\ \exists d \in D: f(d) = c$ $\qquad \{1, 2, \ldots, n-1\}$

$\forall a,b\ ,\ a \neq b \Rightarrow f(a) \neq f(b)$ $\qquad \exists c \in C\ \forall d \in D: f(d) \neq c$

2. Find a bijection between $\mathbb{N}$ and the set of all integers congruent to 1 mod n, for a fixed n. $A = \{1 + kn : k \in \mathbb{Z}\}$

$g(n) = 1 + kn,\ n \in \mathbb{Z}$ $\qquad g : \mathbb{Z} \to A$

$a \in A$ $\quad g^{-1}(a) = \frac{a-1}{k}$ $\qquad (g \circ f) : \mathbb{N} \to A$

$f(i) = \begin{cases} i/2 & \text{if } i \text{ even} \\ -\frac{(i+1)}{2} & \text{if } i \text{ odd} \end{cases}$ $\qquad f : \mathbb{N} \to \mathbb{Z}$ $\qquad g(f(i))$

3. Are these sets countably infinite/uncountably infinite/finite? If finite, what is the order of the set? Reminder: A bit string is a sequence of digits where each digit corresponds to either a 1 (on) or a 0 (off).

   (a) Finite bit strings of length n.

   2^n

0	0	1	–	–	—
0	1	0	–	–	—
1	1	1	–	.	–

   (b) All finite bit strings of length 1 to n.

   $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$

   (c) All finite bit strings

   $\{0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$ countably ∞

   (d) All infinite bit strings (A)

   diagonalize / bij with $\mathbb{R}$ (show inj $A \to \mathbb{R}$, inj $\mathbb{R} \to A$) uncountably ∞

   (e) All finite or infinite bit strings.

   (c) $\cup$ (d)

   (f) Suppose that we try to show that infinite bit strings are countable by induction. We show that for any positive integer n, a bit string of length $\boxed{n}$ is countable. Why does this not work for infinite strings?

   $n = \infty$

   $n \to \infty$

   $\cancel{n = \infty}$

4. If S is countably infinite, is the power set $\mathcal{P}(S)$ finite, countably infinite, or uncountably infinite? Provide a proof for your answer. Reminder: the power set of a set is the set of all possible subsets of that set. Ex: $S = \{A, B\}, \mathcal{P}(S) = \{\{\}, \{A\}, \{B\}, \{A, B\}\}$