Week 6: Counting

1 Intro

1. I have a 6-sided die. I roll it 5 times. ABCDE 5!

(a) How many sequences of 5 numbers?

6^5 _ _ _ _ _ 6 choices per spot 12113 51543 11213, 55143, 65423, ...

(b) How many sequences like 66666? 1

like $x_1 x_2 x_3 x_4 x_5$, $x_i \in \{5,6\}$? 2^5

_ _ _ _ _ 55565 ✓ 66656 ✓ 46654 X

(c) How many contain at least one 3?

$6^5 - 5^5 = 4651$ 31333 ✓ 66653 ✓ 12456 X

$5 \cdot 6^4 = 5 \cdot 1296 = 6480$

_ _ _ _ 3

33333

3. How many numbers from $1, \ldots, 1000$ are neither perfect squares nor perfect cubes?

2 Applying Counting Techniques

1. Lea[redacted] has 9 songs she wants to sing at a concert: ~~6~~ 4 old songs old songs and 3 new songs. However, she does not want to sing 2 new songs back to back. If Leanne sings each song exactly once, how many possible orderings of the songs are possible?

_ _ N O N O _ N _ → (3 div) 4 bins

balls (old songs) $\binom{4+3}{3} \cdot 3! \cdot 6!$ div $= \binom{7}{3} \cdot 3! \, 6!$

2. How many length 7 bitstrings have more zeroes than ones?

1110000 ✓
1011100 X

2^6

1101000
0010111

Bijection between S, T
flip bits

more 0's than 1's, more 1's than 0's $\Rightarrow |S| = |T|$. $|S| + |T| = 2^7$, $\therefore |S| = \frac{2^7}{2} = 2^6$

$\Rightarrow 2|S| = 2^7 \Rightarrow$

3. How many length 8 bitstrings have more zeroes than ones?

R same # of 1's & 0's

$|S| = |T|$ $|S| + |T| = 2^8 - \binom{8}{4}$

$\Rightarrow 2|S| = 2^8 - \binom{8}{4}$ $\Rightarrow |S| = \frac{1}{2}\left(2^8 - \binom{8}{4}\right)$

1 0 1 0 1 0 0 1 11110110
00001001

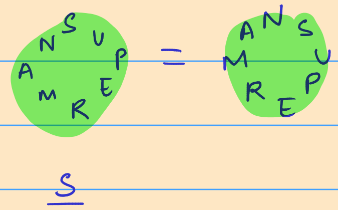
3 S U P E R M A N

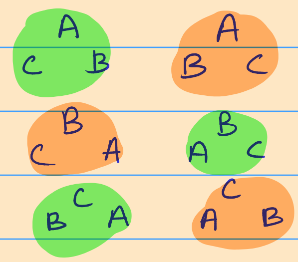
1.(a) arrange on a straight line : $8!$

1. (b) arrange on a straight line, s.t. "SUPER" is a substring : $4 \cdot 3! = 4!$

— — — — MSUPERAN ✓ AMSUPERN ✓ ASMUPENR ✗

1.(c) arrange on a circle : $\frac{8!}{8}$ $(=7!)$





$\frac{3!}{3} = 2!$

1.(d) arrange on a circle, s.t. "SUPER" is a substring : $\frac{4!}{4}$ $(=3!)$

2.(a) arrange on a straight line, s.t. "SUPER" is a subsequence :

MSUPERAN ✓ AMSUPERN ✓ ASMUPENR ✓ ASMPUENR ✗

$\binom{8}{5} \cdot 3!$

perms of MAN

— — — — — — — —

S U P E R

A M N

$\binom{8}{5}$

2.(b) arrange on a circle, s.t. "SUPER" is a subsequence :

S

S

$\binom{3+4}{3} \cdot 3!$

R N A E P U M

$\binom{7}{3} = \binom{7}{4}$

5 Combinatorial Proofs

2. Show $k\binom{n}{k} = n\binom{n-1}{k-1}$

$\binom{n}{k}$

$= \frac{n!}{k!(n-k)!}$

$= \frac{n!}{(n-k)!(n-(n-k))!}$

$k = n-(n-k)$

$= \binom{n}{n-k}$

to take

Choosing k items to take out of n is the same as choosing $n-k$ items not to take out of n.

1. Story: The thing you're counting.
2. LHS counts story
3. RHS counts story

1. Story: Choose a team of k players, one player capt.
2. LHS counts story: choose k players out of n $\binom{n}{k}$
   Choose 1 capt. out of k $\binom{k}{1} = k$
3. RHS counts story: Choose 1 capt. out of n $\binom{n}{1} = n$
   choose remaining $k-1$ players out of $n-1$ $\binom{n-1}{k-1}$

☆ $$\binom{2n}{n} = \sum_{i=0}^{n}\left[\binom{n}{i}\binom{n}{n-i}\right]$$

http://discrete.openmathbooks.org/dmoi2/sec_comb-proofs.html

1. Story: $2n$ toppings available, can choose only n.
2. LHS trivial.
3. $\underbrace{\circ\ \circ\ \circ\ \circ\ \circ\ \circ}_{n \text{ veg}}\ \underbrace{\circ\ \circ\ \circ\ \circ\ \circ\ \circ}_{n \text{ non-veg}}$

$\binom{n}{i}\binom{n}{n-i}$