What's a conf interval?

$$p = \frac{300}{1000}$$

31 100

What's p?

How sure are you abt that? $\hat{p} = 0.32$ $\frac{32}{100} = \frac{320}{1000}$

$p \notin [0.30, 0.32]$

80%

$\mathbb{E}[X]$ $\mathrm{Var}(X)$ X, Y

$$Z = X|Y$$

$Y \backslash X$	0	1	2	3
0	0	0	1/3	0
1	0	1/3	1/6	0
2	1/12	0	0	1/12

$Z_0 = X|(Y=0)$

$Z_1 = X|(Y=1)$

Z_2

0	1	2	3
0	0	1	0
0	2/3	1/3	0
1/2	0	0	1/2

$$\mathbb{E}[X|Y=y] = \mathbb{E}[Z_y]$$

$$\mathbb{E}[X|Y=0] = \mathbb{E}[Z_0] = 2$$

$$\mathbb{E}[X|Y=1] = \mathbb{E}[Z_1] = 4/3$$

Prob 140 Text

11 Confidence Intervals, Conditional Expectation, LLSE

Computer Science Mentors 70

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Roast us here: `https://tinyurl.com/csm70-feedback20`

1 Confidence Intervals

1. Define i.i.d. variables $A_k \sim \text{Bern}(p)$ where $k \in \{1, \cdots, n\}$. Assume we can declare that $\Pr\left[\left|\frac{1}{n}\sum_{k=1}^{n} A_k - p\right| \geq 0.25\right] = 0.01$.

(a) Please give a 99% confidence interval for p given A_k.

$\underbrace{\text{point estimate}}_{\text{mean}} \pm \underbrace{\text{margin of error}}_{\text{standard dev} = \sqrt{\text{Var}}}$

(b) We know that the variables $X_1, \ldots, X_n$, are i.i.d. random variables and have variance σ^2. We also have the observation that $A_n = \frac{X_1 + \ldots + X_n}{n}$. We want to estimate the mean, μ, of each X_i.

Prove that we have 95% confidence that μ lies in the interval $\left[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}\right]$

That is, $\Pr\left[A_n - 4.5\frac{\sigma}{\sqrt{n}} \leq \mu \leq A_n + 4.5\frac{\sigma}{\sqrt{n}}\right] \geq 95\%$

(c) Give the 99% confidence interval for μ.

2. We have a die with 6 faces labeled $1, 2, 3, 4, 5, 6$.

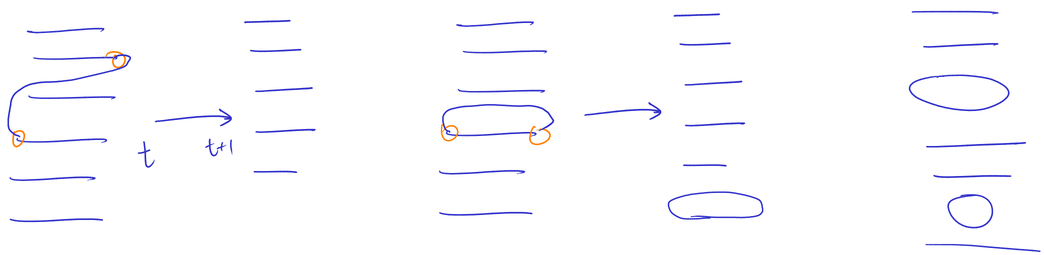
(a) Develop a 99% confidence interval for the sum of n samples.

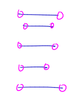
(b) Now, suppose the die's face values are just 6 consecutive integers $k+1, k+2, \ldots, k+6$, but we do not know k. For example, if $k = 6$, the die faces would take on the values $7, 8, 9, 10, 11, 12$. If we observe that the average of the n samples is 15.5, develop a 99% confidence interval for the value of k.

2 Conditional Expectation

1. **Looping Ropes**

   Frobenius has n ropes in his backyard, which he likes a lot. But when he goes to work everyday, they grow a mind of their own and begin behaving weirdly. At every timestep t, two ends of a rope(s) are uniformly chosen at random and knotted together. If the two ends are from the same rope, they form a loop. If the two ends are from different ropes, they join together to form a new rope. By the time Frobenius comes home, this process has completed (meaning no more loose ends are left). How many loops can Frobenius expect to see? **Bonus:** Does this converge as $n \to \infty$?





X_n is # of loops when n ropes are left.

I is indicator of loop formation.

① $E[X_n] = E[X_n \mid I=1] Pr[I=1] + E[X_n \mid I=0] Pr[I=0]$

$= (E[X_{n-1}] + 1) Pr[I=1] + (E[X_{n-1}]) Pr[I=0]$

$= E[X_{n-1}] + Pr[I=1]$

$= E[X_{n-1}] + \frac{1}{2n-1}$

② $E[X_1] = 1$

③ $E[X_3] = E[X_2] + \frac{1}{5}$

$= E(X_1) + \frac{1}{3} + \frac{1}{5}$

$= 1 + \frac{1}{3} + \frac{1}{5}$

$$\boxed{E[X_n] = \sum_{i=1}^{n} \frac{1}{2i-1}}$$

2. If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a cat, I stop and pet the cat and chat to the owner. The number Y of cats I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the cat and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x} \, \mathbb{1}\{x \geq 0\}$$

   (a) If I see a single cat, what is the expectation and variance of the time spent petting the cat and chatting to its owner?

   $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_0^{\infty} x \cdot (0.5 e^{-0.5x})\, dx = \boxed{2}$

   $E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_0^{\infty} x^2 \cdot (0.5 e^{-0.5x})\, dx = 8$

   $Var(X) = E[X^2] - E^2[X] = 8 - 2^2 = \boxed{4}$

   (b) What is the conditional expectation $E[X \mid Y]$ of the total time spent petting cats and chatting to their owners, as a function of Y?

   (c) Using the smoothing law (or law of iterated expectation, or law of total expectation), calculate $E[X]$.

3. **Rolling Chopsticks**

The content mentors were trying to eat noodles in a new way. Rather than eating noodles by chopsticks directly, they tried eating noodles by rolling one noodle on the chopstick and eat it. This is seemingly a hard way to eat noodles so the probability they successfully eat a noodle on each attempt is p.

(a) Suppose they start attempting to eat a noodle, and eat the noodle on the attempt X. What is the distribution of X? What is the distribution of unsuccessful attempts to eat that noodle, X' in terms of X?

$X \sim \text{Geom}(p)$

$X' = X - 1$

$\Pr[X' = k] = \Pr[X = k+1]$

① Values ② Prob

(b) Let Y be the unsuccessful attempts that they will make trying to eat 2 noodles. What is the distribution of Y ?

X_1', X_2', where $(X_1'+1) \sim \text{Geom}(p)$ $X_1' \perp\!\!\!\perp X_2'$

$(X_2'+1) \sim$

$Y \geq 0$

$$\Pr[Y = k] = \Pr[X_1' + X_2' = k]$$

$$= \sum_{w=0}^{k} \Pr[X_1' = w \cap X_2' = k-w]$$

$(k-1)(1-p)^{k-2}p^2$

$\cancel{(1-p)^w (1-p)^{-w}}$

$$= \sum_{w=0}^{k} \Pr[X_1 = w+1 \cap X_2 = k-w+1]$$

$$= \sum_{w=0}^{k} (1-p)^w p (1-p)^{k-w} p = (1-p)^k p^2 (k+1)$$

(c) Content with their distribution Y and eating 2 noodles, the content mentors want to find the distribution Z for the total unsuccessful attempts of eating the whole bowl of R noodles. They were planning to proceed as part b) but then Aekus, a random variable distribution enthusiast, suggested to use $P(Z = k) = \binom{r+k-1}{k}(1-p)^k p^r$ where $r = R$.

The distribution Z is defined by 2 parameters: 1) the number of successful attempts r and 2) the probability of a successful attempt$a$ so we will write Z as $Z(r, p)$.

Show that Z is the sum of independent X' random variables by using induction on r where $r = 1$ is the base case, and the content mentors can use Z as their distribution.
(*Hint*: Remember the "Hockey stick" identity $\sum_{i=0}^{k-1} \binom{n+i}{i} = \binom{n+k}{k-1}$)

(d) What is the expected value of total unsuccessful attempts of eating the whole bowl of R noodles, the random variable Z?.

3 Linear Least-Squares Estimation

1. **Linear Least Squares Estimate**

   height weight

   Linear Least Squares Estimate (LLSE) The LLSE of Y given X, denoted by $L[Y|X]$, is the linear function $a + bX$ that minimizes

   $Y = a + bX$

   $$C(g) = E(|Y - a - bX|^2).$$

   $X_1 |(a+bx_1) - Y_1|^2 +$
   $X_2 |(a+bx_2) - Y_2|^2$
   $\vdots$

   Let's try to derive a formula for $L[Y|X]$ in the form of properties of distribution of X and Y

   (a) Write $C(g)$ as linear fuunction of $E(Y^2)$, $E(X^2)$, $E(Y)$, $E(X)$ and $E(XY)$

   (b) Find the values of a and b that minimize the expression in part a. To simplify the calculation use

   $$Cov(X,Y) = E(YX) - E(Y)E(X) \text{ and } Var(X) = E(X^2) - E(X)^2.$$

   (c) Let's put everything together and find the formula for $L[Y|X]$

(a) $E[|Y - a - bX|^2] = E(Y^2) + a^2 + b^2 E[X^2] - 2aE[Y] - 2bE[YX] + 2abE[X]$

(b) $\frac{\partial}{\partial a} E[|Y - a - bX|^2] = -2E[Y - a - bX] = 0 \Rightarrow \boxed{a = E[Y] - bE[X]}$

$$\begin{aligned}
\frac{\partial}{\partial b} C(g) &= 2bE[X^2] - 2E[XY] + 2aE[X] \\
&= 2bE[X^2] - 2E[XY] + 2(E[Y] - bE[X])E[X] \\
&= 2b(E[X^2] - E^2[X]) - 2(E[XY] - E[Y]E[X]) \\
&= 2bVar(X) - 2Cov(X,Y) = 0
\end{aligned}$$

$$\Rightarrow \boxed{b = \frac{Cov(X,Y)}{Var(X)}}$$

(c) $L(Y|X) = a + bX = \left(E[Y] - \left(\frac{Cov(X,Y)}{Var(X)}\right)E[X]\right) + \left(\frac{Cov(X,Y)}{Var(X)}\right)X$

$$= E[Y] + \left(\frac{Cov(X,Y)}{Var(X)}\right)(X - E[X])$$

unconditional mean

correction term

MMSE

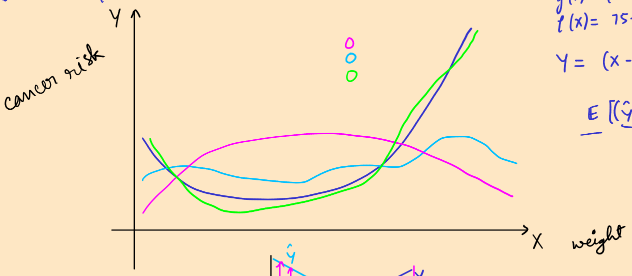
Min Mean Squared Error estimator. $\hat{Y} = Y | X = x$

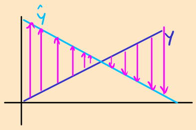
$g(x) = (x-150)^2$

$f(x) = 75 + x$

$Y = (x-150)^2$

$E[(\hat{Y} - Y)^2]$





Lemma 20.1

(a) $\forall \phi$ $\quad \mathbb{E}[(Y - E[Y|X])\phi(X)] = 0$

(b) If we have $g(X)$:

$$\mathbb{E}[(Y - g(X))\phi(X)] = 0 \quad \forall \phi,$$

then $g(X) = \mathbb{E}[X|Y]$