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Conditional Expectation, Markov Chains, Continuous Random Variables and Distribution

Computer Science Mentors 70

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1 Markov Chains

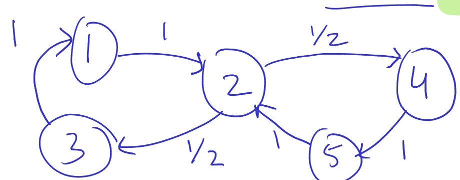
$X_1 = 1$ $X_2 = 3$ $X_{100} = 2$ 1 2 3

$\Pr[X_2 = 2 \mid X_1 = 1]$ $\Pr[X_{10} = 2 \mid X_9 = 1]$

1. **One State at a Time**

   (a) A sequence of random variables $X_0, X_1, X_2, X_3, \ldots$ is a Markov chain if $\Pr[X_{t+1} \mid X_t, X_{t-1}, \cdots, X_0] = \Pr[X_{t+1} \mid X_t]$ and $\Pr[X_{t+1} = x \mid X_t = y] = \Pr[X_{t'+1} = x \mid X_{t'} = y]$. (Fill in the blanks with equations) Markov prop.

   (b) Any irreducible Markov chain where one state has a self-loop is aperiodic. (True/False)

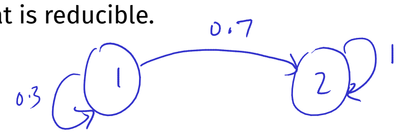


   (c) Given a Markov Chain, let the random variables $X_1, X_2, X_3, \ldots$, where X_t is the state visited at time t in the Markov Chain. Then $E[X_t|X_{t1} = x] = E[X_t|X_{t-1} = x,_{t-2} = x']$.(True/False)

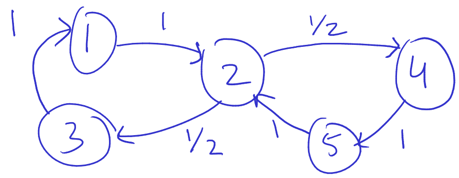
   $\mathbb{E}[X_t \mid X_{t-1} = x, X_{t-2} = x']$ Markov Prop

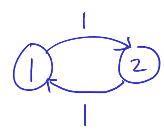
2. **Types of Markov Chains**

   (a) Draw a Markov Chain that is reducible.



   (b) From here on, we will exclusively discuss (and assume) irreducible Markov chains. Draw a periodic Markov Chain. Does it have an invariant distribution? Does it always converge to this invariant distribution? What does the invariant distribution represent?





   periodic $[0.5 \quad 0.5]$

   $\Pr[X_{t+1} = 1 \mid X_t = 2] = 1 \neq 0.5 \quad \forall t$

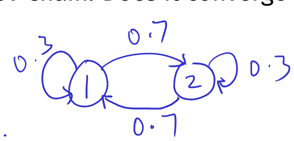
   $\pi_0 = [0.3 \quad 0.7]$

   $\pi_1 = [0.7 \quad 0.3]$

   $\pi_2 = [0.3 \quad 0.7]$

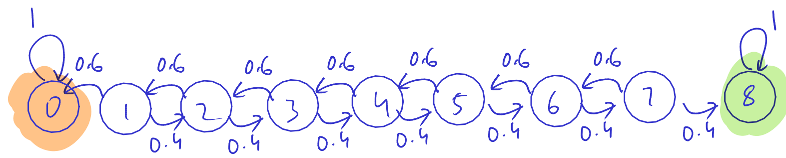
   (c) Draw an aperiodic Markov Chain. Does it converge to an invariant distribution?

   ALWAYS converges to inv dist.



3. Sylvia is stuck in jail with her life savings of \$3 and needs \$8 for bail to get out so she can teach her students again. Professor Rao gives her a chance to redeem herself through a series of bets – if Sylvia bets \$$A$, she wins \$$A$ with probability 0.4 and loses \$$A$ with probability 0.6. If she would like to get to \$8 without losing all her money, draw a Markov chain representing the following situations:

(a) Choosing to play it safe, Sylvia bets \$1 at a time. (Hint: the states are the possible dollar amounts Sylvia has at any point)



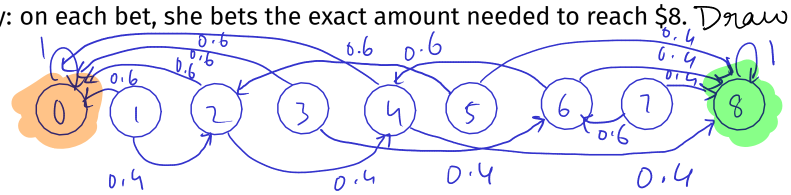
(b) Set up, but DO NOT SOLVE, the equations to determine the probability Sylvia reaches \$8 before she runs out of money.

$\alpha(i) = \Pr[\text{get to 8 before 0} \mid \text{currently at } i]$

$\alpha(0) = 0$ $\quad\quad \alpha(i) = 0.4\alpha(i+1) + 0.6\alpha(i-1) \quad\quad i \in [1,7]$

$\alpha(8) = 1$

(c) She realizes that with the strategy in the previous parts, she has a $< 10\%$ chance of getting out! She decides on a more risky strategy: on each bet, she bets the exact amount needed to reach \$8. Draw the Markov Chain.



(d) Given that the first strategy gives a $\approx 9.64\%$ chance of escaping, which strategy gives her a better chance of getting out?

$\alpha(0) = 0$

$\alpha(8) = 1$

$\alpha(3) = 0.4\alpha(6) + \cancel{0.6\alpha(0)}^{\,0} = \boxed{0.256}$

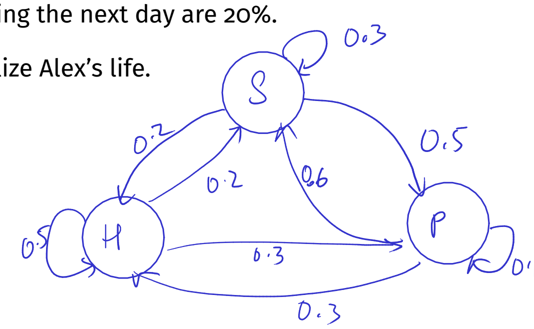
$\alpha(6) = 0.4\cancel{\alpha(8)}^{\,1} + 0.6\alpha(4) = 0.4 + 0.6\alpha(4) = 0.64$

$\alpha(4) = 0.4\alpha(8) + 0.6\alpha(0) = 0.4$

4. **Life of Alex**

Alex is enjoying college life. She spends a day either studying, partying, or looking for housing for the next year. If she is studying, the chances of her studying the next day are 30%, the chances of her partying the next day are 50%, and the chances of her looking for housing the next day are 20%. If she is partying, the chances of her partying the next day are 10%, the chances of her studying the next day are 60%, and the chances of her looking for housing the next day are 30%. If she is looking for housing, the chances of her looking for housing the next day are 50%, the chances of her partying the next day are 30% and the chances of her studying the next day are 20%.

(a) Draw a Markov chain to visualize Alex's life.



(b) Write out a matrix to represent this Markov chain (In this solution (and in CS 70), the rows represent the source and the columns represent the destination).

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} S & P & H \end{array} \\ \text{From} \begin{array}{c} S \\ P \\ H \end{array} & \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} = P \end{array}$$

(c) If Alex studies on Monday, what is the chance that she is partying on Friday? (Don't do the math, just write out the expression that you would use to find it.)

$\pi_0 = [0 \ 1 \ 0]$ $\quad \pi_4 = \pi_0 P^4$ $\quad \pi_{t+n} = \pi_t P^n$ $\quad \pi_{4,2}$

(d) What percentage of her time should Alex expect to spend looking for housing?

Prob of H in the inv dist. $\quad \pi = \pi P$

$$[\pi_S \ \pi_P \ \pi_H] = [\pi_S \ \pi_P \ \pi_H] \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \iff \begin{array}{l} \pi_S = 0.3\pi_S + 0.6\pi_P + 0.2\pi_H \\ \pi_P = 0.5\pi_S + 0.1\pi_P + 0.3\pi_H \\ \pi_H = 0.2\pi_S + 0.3\pi_P + 0.5\pi_H \\ \pi_S + \pi_P + \pi_H = 1 \end{array}$$

(e) If Alex parties on Monday, what is the chance of Alex partying again before studying?

$\alpha(\ ) = $ ____

$\alpha(\ ) = $ ____

5. **Riemann's Pontiac**

Your friend Riemann drives a 1994 Pontiac Firebird, an old car and that he hasn't taken care of it all too well. At every second in $\{t_1, \ldots, t_n\}$ with probability p he accelerates at 1 m/s^2 and with probability $1 - p$ his engine sputters and he stops immediately. Find the expected time it takes him to reach a velocity of n m/s.

6. **Tired of Balls and Urns Yet?**

An urn contains six balls, of which three are red and three are green. In each step, two balls are selected at random. If one of them is red, and the other is green, then we discard them and replace them by two blue balls, and if both of the balls are blue, then we replace those blue balls with an equal amount of red and green balls. Otherwise, we do not do anything. Find the probability that if we start with an equal number of balls of every color, what is the probability that we reach 6 blue balls before 0 blue balls in the bag?

2 Continuous RV and Distribution

Discrete vs Continuous Probability

Here is a table illustrating the parallels between discrete and continuous probability.

Discrete	Continuous
$P[X = k] = \sum_{\omega \in \Omega : X(\omega) = k} P(\omega)$	$P[k < X \leq k + \mathrm{d}x] = f_X(k)\mathrm{d}x$ (*)
$P[X \leq k] = \sum_{\omega \in \Omega : X(\omega) \leq k} P(\omega)$	$P[X \leq k] = F_X(k)$
$E[X] = \sum_{a \in A} a \cdot P[X = a]$	$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$
$E[\phi(X)] = \sum_{a \in A} \phi(a) \cdot P[X = a]$	$E[\phi(X)] = \int_{-\infty}^{\infty} \phi(x) \cdot f_X(x) dx$
$\sum_{\omega \in \Omega} P[\omega] = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$

(*) When solving problems with continuous distributions, you can think of $f_X(k)$ as being analogous to $P[X = k]$ in discrete distributions, but they are not equal.

1. **PDFs**

   Consider the following functions and determine whether or not they are valid probability density functions.

   (a) $f(x) = \sin(x)$

   (b) $f(x) = x$ for $0 \leq x \leq 1$, and $f(x) = 0$ everywhere else.

   (c) $f(x) = 1$ for $0 \leq x \leq 1$, and $f(x) = 0$ everywhere else.

(d) $f(x) = e^{-x}$ for $x \geq 0$, and $f(x) = 0$ everywhere else.

2. **Disk**

   Define a continuous random variable R as follows: we pick a point uniformly at random on a disk of radius 1; the value of R is distance of this point from the center of the disk. We will find the probability density function of this random variable.

   (a) Why is R not $U(0,1)$?

   (b) What is the probability that R is less than r, for any $0 \leq r \leq 1$? What is the CDF $F_R(r)$ of the random variable R?

   (c) What is the PDF $f_R(r)$ of the random variable R?

   (d) Now say that $R \sim U(0,1)$. Are you more or less likely to hit closer to the center than before?

3. **Joint Density**

   The joint density for the random variables X and Y is defined by $f(x,y) = \frac{2}{3}x + \frac{4}{3}y$ for $0 \leq x, y \leq c$ for some positive real number c, and $f(x,y) = 0$ for all other (x,y).

   (a) For what value of c is this a valid joint density?

   (b) Compute $P(X < Y)$.

(c) Compute $E[X|Y = y]$ for $0 \le y \le c$.

(d) Compute $E[XY]$.