CS 70 Discrete Mathematics and Probability Theory
Fall 2020 DIS 6B

1 Count it

1 0 1 0 1 1 0 . . . $\underbrace{\qquad}_{15}$

2^{11} ≥ 4 heads

Let's get some practice with counting!

15 spots

(a) How many sequences of 15 coin-flips are there that contain exactly 4 heads? $\binom{15}{4} = \binom{15}{11} = \frac{15!}{4!\,11!}$

(b) An anagram of HALLOWEEN is any re-ordering of the letters of HALLOWEEN, i.e., any string made up of the letters H, A, L, L, O, W, E, E, N in any order. The anagram does not have to be an English word. HALOWEN
How many different anagrams of HALLOWEEN are there? $\frac{9!}{1!1!2!1!1!1!2!1!1!} = \frac{9!}{2!2!}$

$n+k$

person — fixed — $

(c) How many solutions does $\underbrace{\boxed{y_0} + y_1 + \cdots + \boxed{y_k}}_{\text{var}} = n$ have, if each y must be a non-negative integer? $\binom{n+k}{k}$

(d) How many solutions does $y_0 + y_1 = n$ have, if each y must be a positive integer? $\binom{n-1}{1}$

(e) How many solutions does $\underbrace{y_0 + y_1 + \cdots + y_k}_{k+1} = n$ have, if each y must be a positive integer? $\binom{n-(k+1)+k}{k}$

Give everyone a ☆

Now, we first put 1 item in each of k+1 boxes.

☆ ☆ ☆ ☆☆☆

2 Inclusion and exclusion

What is total number of positive numbers that smaller than 100 and coprime to 100?

3 Identities

(a) $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$

combinatorial proof

(b) $\sum_{i=0}^{n} \binom{r+i}{i} = \binom{r+n+1}{n}$ $\frac{(r+n+1)!}{n!\,(r+1)!} = \sum_{i=0}^{n} \frac{(r+i)!}{r!\,i!}$

(c) $\sum_{i=0}^{n} \binom{r}{i}\binom{s}{n-i} = \binom{r+s}{n}$ (Note: Assuming $r > n, s > n$)

4 Largest binom

For which value(s) of k is $\binom{n}{k}$ maximum? Prove your answer.

3.(b) ① Identify what we're counting / come up with a story.

② Prove one side (LHS/RHS). ~ easy

③ Prove the other side.

$$\sum_{i=0}^{n} \binom{r+i}{i} = \binom{n+r+1}{n}$$

RHS: $\{1, 2, \dots, r, r+1, \dots r+n+1\}$, choose n elements.

LHS:

$$\binom{r+i}{i} \qquad \sum_{j=0}^{n} \binom{r+n+1-j}{n-j+1}$$

$$\binom{r+n+1-1}{n} \qquad \binom{r+i}{i}$$

i is the smallest elt NOT in my subset

$\Updownarrow$

$1, 2, 3, \dots, i-1$ in my subset

AND

i NOT in my subset

$$\binom{r+0}{0} + \binom{r+1}{1} + \dots \binom{r+n}{n}$$

$$\binom{r+n+1-2}{n-1}$$

$$\binom{r+n+1-i}{n-i+1} \qquad \binom{r+n+1-i}{n-i+1} = \binom{r+1+(n-i)}{r}$$

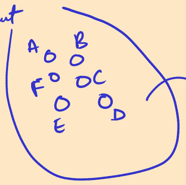
$n+1$ is the smallest elt NOT in my subset

$\Updownarrow$

$1, \dots, n$ ✓ AND n is not

$$\binom{r+n+1-n-1}{0} = \binom{r}{0}$$

Sampling w/ replacement

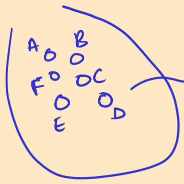


n balls
m samples of size 1

$\underline{n^m}$

$\underbrace{n \cdot n \cdot n \cdots}_{m \text{ times}}$

$\underset{1}{\sqcup}\ \underset{2}{\sqcup}\ \underset{3}{\sqcup}\ \underset{4}{\sqcup}$



n balls
$\underline{\underline{1}}$ sample of size m

(or m samples of size 1 w/o repl)

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

\# of perms of accept

\# perms of rej

$$\overbrace{\underbrace{o\,o\,o\,o}_{m}\,|\,o\,o}^{n}$$

"chosen" "rejected"

How many poker hands?

Out of 52 cards, how many subsets of 5?

$\binom{52}{5}$

n − b

o o o o

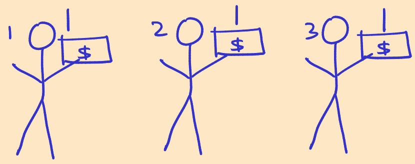
m 1's
k 0's

0 1 0

$

$\underset{1}{\sqcup^{0}}\ \underset{2}{\sqcup^{0}}\ \underset{3}{\sqcup^{0}}$

p



Divide m \$ among p ppl:

$k = p - 1$

$$\binom{m+k}{k} = \binom{m+k}{m} = \frac{(m+k)!}{k!\; m!}$$

$$\binom{m+p-1}{p-1} = \binom{m+p-1}{m}$$

"Stars & bars"

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| o o o | o