GRAPHS

Computer Science Mentors 70

Sept 14 - Sept 18
Fall 2020

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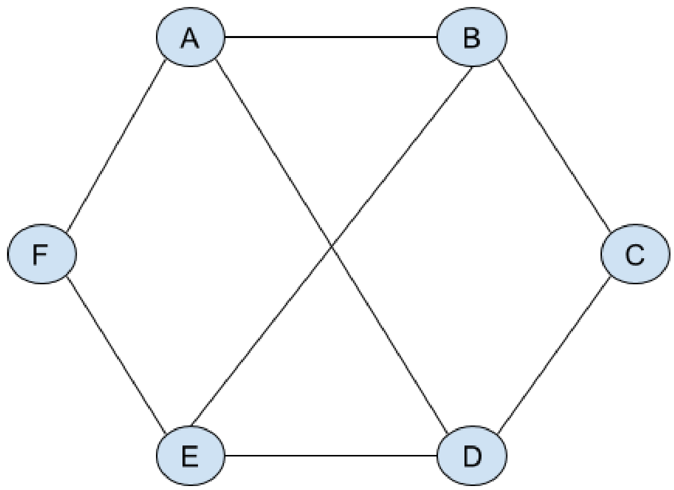
1 Graph 101



1. Take a look at the following undirected graph.

   (a) How many vertices are in this graph? 6

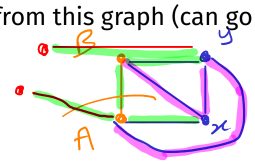
   (b) What is the degree of vertex B? 3

   (c) What is the total degree of this graph? 16

   (d) Consider the traversal $A \to B \to C \to D \to A$. How would you categorize it (walk / cycle / path / tour)?

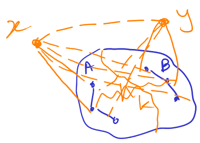
   (e) Give an example of a simple path of length 4.

   (f) Is it possible to construct a traversal that is a tour but not a cycle from this graph (can go through vertices twice, but not edges)? Why or why not? Need a vert of deg ≥ 4.



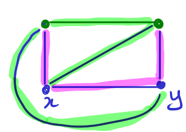
2. Consider all complete undirected graphs on an even number of vertices. Prove that such graphs can be partitioned into $\frac{n}{2}$ spanning trees that share no edge with another spanning tree.

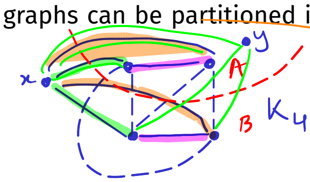
IS: We have K_{k+2}. Remove 2 verts, get K_k. Apply IH on K_k.



$\frac{k+2}{2} = \frac{k}{2} + 1$

$\frac{k}{2} + 1 = \frac{k+2}{2}$





$\frac{k(k+1)}{2}$ K_{k+2}

$k+1$ new $|V| = k+2$ $(k+2)(k+1)/2$

Base case: K_2 K_n

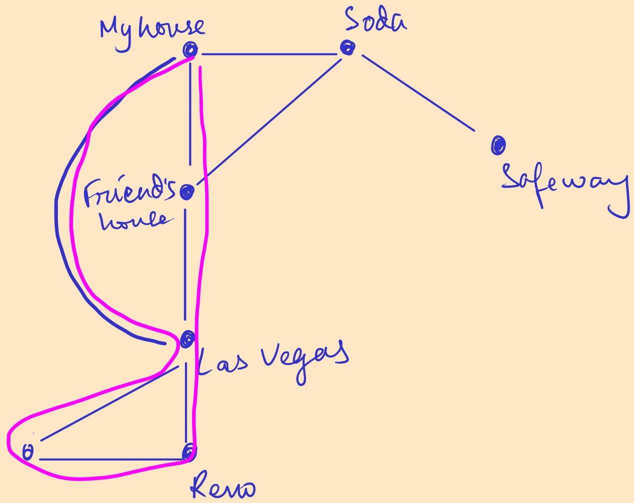
IH: Even k, K_k can be partitioned into $k/2$ STs.

3. Draw a simple bipartite graph with 6 vertices and 8 edges. What is the most edges you could have in a bipartite graph with 6 vertices? With $2n$?

Walk Tour

Cycle Path (simple)

walk but don't rep edges



4. Which of these graphs have Eulerian tours?

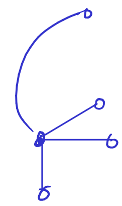
   (a) The complete graph on 5 vertices (K_5). NO

   (b) The complete graph on 6 vertices (K_6). Yes

   (c) The complete graph on 7 vertices (K_7). NO

   (d) The 3-dimensional hypercube. NO

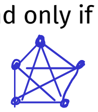
   (e) The 4-dimensional hypercube. Yes

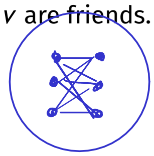


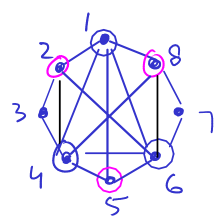
2 Planarity & Coloring

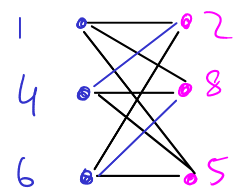
1. Consider a group of 8 friends sitting at a round table. Any one person is friends with the two individuals next to them and the person sitting directly accross from them. Consider the graph where each individual is a vertex and an edge exists between person u and v if and only if u and v ~~are friends~~. Prove that this graph is non-planar.

K_5 or $K_{3,3}$

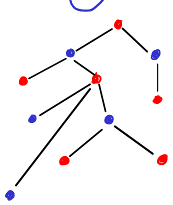


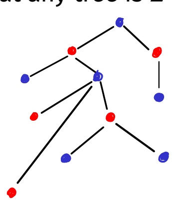


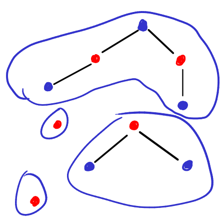




2. Show that any tree is 2-colorable. Color verts







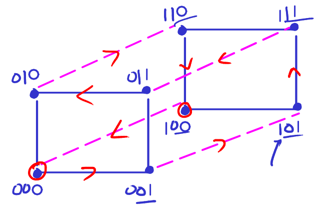
Base: 1 vert → 1 col
IH: ≤ n verts → n cols
IS:

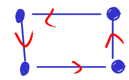
3. You are hosting a very exclusive party such that a guest is only allowed to come in if they are friends with you or someone else already at the party. After everyone has showed up, you notice that there are n people (including yourself); each person has at least one friend (of course), but no one is friends with everyone else. It is still quite a sad party, because among all the possible pairs of people, there are only a total of $n-1$ friendships. You want to play a game with two teams, and in order to kindle new friendships, you want to group the people (including yourself) such that within each team, no one is friends with each other. Is this possible? *(Hint: How might the previous question be useful?)*

4. Two knights are placed on diagonally opposite corners of a chessboard, one white and the other black. The knights take turn moving as in standard chess, with white moving first.

   (a) Let every square on a chessboard represent a vertex of a graph, with edges between squares that are a knight's move away. Describe a 2-coloring of this graph.

   (b) Show that the black knight can never be captured, even if it cooperates with the white knight.





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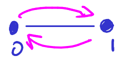
3 Hypercubes

1. Austin has n lightbulbs in a row, all turned off. Every second Austin performs a **move**, where Austin either turns a lightbulb on or turns a lightbulb off. Show that for any n, there is a a sequence of moves that Austin can make such that each possible configuration of lightbulbs being on or off has occurred exactly once, and such that the last move causes all the lightbulbs to be off.

Hamiltonian Cycle

Base:  $n=1$ ← dim of hypercube

IH: $n=k$, Ham Cycle exists.

IS: Split H_{k+1} into two copies of H_k, apply IH.

H_k^L, H_k^R

000, 001, ... 100, 101

000 → 001 → 101 → 111 → 011

2. You wish to color the *edges* of a n-dimension hypercube, such that edges that share a vertex are different colors. (Note: This differs from the normal definition of graph coloring, where you color vertices so that vertices that share an edge are different colors). Prove that n colors are both sufficient and necessary; that is, you can do this with n colors, but not with $n-1$ colors.