Intro to Probability

Computer Science Mentors 70

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Roast us here: `https://tinyurl.com/csm70-feedback20`

Introduction to Probability

1. Suppose two integers a and b are drawn uniformly (with repl.) from $[-n..n]$, that is $a, b \in \mathbb{Z}$ and $-n \le a, b \le n$.

   (a) Define a probability space for (a, b). Does each sample point occur with uniform probability?

   $\Omega = \{(i,j) : -n \le i \le n, -n \le j \le n\}$ $\quad Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{(2n+1)^2}$

   $\omega \in \Omega$ $\quad (3,5)$

   (b) Find the probability that $\underbrace{\max\{0, a\} = \min\{0, b\}}_{E}$.

   $\underbrace{a \le 0}$ $\quad \underbrace{b \ge 0}$

   $\#$ fav outcomes $= (n+1)(n+1)$

   $\#$ total outcomes $= |\Omega| = (2n+1)^2$

   $Pr[E] = \frac{|E|}{|\Omega|} = \frac{(n+1)^2}{(2n+1)^2}$

   (c) Find the probability that $\underbrace{|a - b| \le k}_{E}$. You may assume $k < \frac{n}{2}$.

   $Pr[E] = \frac{|E|}{|\Omega|} = \frac{\text{blah}}{(2n+1)^2}$

   $E \subseteq \Omega$

   $E = \{(a,b) : |a-b| \le k\}$

   $|E| = $ blah

2. Alex and Shruti are playing Yahtzee, a game involving rolling 5 dice.

   (a) First, define a probability space representing the possible outcome of Alex or Shruti's rolls of the 5 dice. Assume all dice are fair and labeled 1 through 6.

   Alex and Shruti each roll 1 die to see who goes first. The person with the higher roll goes first, and in case of a tie, they both roll their die again.

   (b) What's the chance Shruti rolls a higher number on the first roll?

   (c) What's the chance Shruti goes first?

   (d) They finally begin playing. Partway through the game, Alex is missing the "three of a kind" category while Shruti is missing the "four of a kind" category. What is the probability of rolling...

   1. exactly 3 of a kind?
   2. exactly 4 of a kind?
   3. Which one is more likely? 3 of a kind or 4 of a kind?

Inclusion-Exclusion Principle, Bayes' Theorem

3. Tri-State Area is experiencing bad weather because of Doctor Doofenshmirtz "Gloomy - inator". It is always at least rainy, cloudy or windy, but because the inator is random we don't exactly know what it would be like. It rains with 0.5 probability, gets windy

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$) = \frac{1}{6^5}$

(b) 1, 2, 3, 4, 5, 6
1, 2, 3, 4, 5, 6

$\frac{15}{36}$

Sagnik H

Aaron T

H — T

p = prob that Shruti rolls higher

prob that Alex rolls higher = p

$\therefore$ Prob of tie $= 1-2p = \frac{6}{36} = \frac{1}{6}$

$\Rightarrow p = \frac{5}{12}$

S A	A S
(5, 5)	(5, 5)
(4, 4)	(4, 4)
(6, 6)	(6, 6)
(1, 1)	(1, 1)
(3, 6)	(3, 6)

(c) $Pr[1 \text{ roll}] = \frac{5}{12}$

$Pr[2 \text{ rolls}] = \frac{1}{6} \cdot \frac{5}{12}$

$Pr[3 \text{ rolls}] = \frac{1}{6^2} \cdot \frac{5}{12}$

$\vdots$

$Pr[i \text{ rolls}] = \frac{1}{6^{i-1}} \cdot \frac{5}{12}$

$$Pr[\text{Shruti starts}] = \sum_{i=1}^{\infty} Pr[\text{Shruti starts in } i \text{ moves}]$$

$$= \sum_{i=1}^{\infty} \frac{1}{6^{i-1}} \cdot \frac{5}{12}$$

$$= \frac{5}{12} \cdot \frac{1}{1-1/6} = \frac{5}{12} \cdot \frac{6}{5} = \boxed{\frac{1}{2}}$$

with 0.65 probability and gets cloudy with 0.45 probability. We experience at least 2 of these together with probability 0.45. Help Agent P find the probability that all 3 of these happen together.

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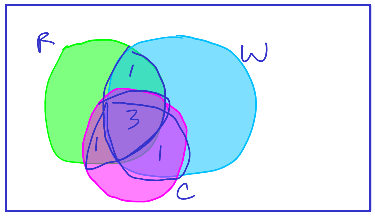
$Pr[R] = 0.5$

$Pr[W] = 0.65$

$Pr[C] = 0.45$



$Pr[R \cap W \cap C]$

$$Pr[R \cup W \cup C] = \underline{Pr[R] + Pr[W] + Pr[C]} - (Pr[R \cap W] + Pr[W \cap C] + Pr[C \cap R]) + Pr[W \cap C \cap R]$$

$$= 0.5 + 0.65 + 0.45 - 2\, Pr[W \cap C \cap R] - 0.45 + Pr[W \cap C \cap R] = 1$$

$$\Rightarrow Pr[W \cap C \cap R] = \boxed{0.15}$$

$$Pr[R \cap W] + Pr[W \cap C] + Pr[C \cap R] = 0.45 + 2\, Pr[W \cap C \cap R]$$

$$Pr[B|A] = \frac{Pr[A|B]\,Pr[B]}{Pr[A]} \Rightarrow Pr[B|A]\,Pr[A] \stackrel{?}{=} Pr[A|B]\,Pr[B] = Pr[A \cap B]$$

4. **Go Bears!** Oski the bear has lost his dog in either forest A (with a priori probability 0.4) or in forest B (with a priori probability 0.6).
   On any given day, if the dog is in A and Oski spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and Oski spends a day looking for it there, the conditional probability that he will find the dog that day is 0.15.
   The dog cannot go from one forest to the other. Oski can search only in the daytime, and he can travel from one forest to the other only at night.

(a) In which forest should Oski look to maximize the probability he finds his dog on the first day of the search?

$$Pr[\text{dog in } A \cap \text{Oski finds dog}] = Pr[\text{Oski finds} \mid \text{dog in } A]\, Pr[\text{dog in } A] = 0.1$$

$$Pr[\text{dog in } B \cap \text{Oski finds dog}] = 0.09$$

$$\underbrace{Pr[\text{Fair} \mid 8/10\,H]}_{\text{posterior}} = \frac{\overbrace{Pr[8/10\,H \mid \text{Fair}]}^{\text{likelihood}}}{Pr[8/10\,H]} \underbrace{Pr[\text{Fair}]}_{\text{prior}}$$

Search in forest A

(b) Given that Oski looked in A on the first day but didn't find his dog, what is the probability that the dog is in A?

$$Pr[\text{dog in } A \mid \text{didn't find in } A] = \frac{(1 - Pr[\text{find in } A \mid \text{dog in } A])}{Pr[\text{didn't find in } A]} Pr[\text{dog in } A] = \frac{0.75}{0.9} 0.4 = \boxed{\frac{1}{3}}$$

$$Pr[\text{didn't find in } A] = 1 - Pr[\text{find in } A] = 0.9 \qquad Pr[\text{find in } A] = Pr[\text{find in } A \mid \text{dog in } A]\,Pr[\text{dog in } A] = 0.1$$

(c) If Oski flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?

$$Pr[\text{looked in } A \mid \text{finds dog}] = \frac{Pr[\text{finds dog} \mid \text{looked in } A]}{Pr[\text{finds dog}]} Pr[\text{looked in } A] = \boxed{\frac{0.25 \times 0.5}{0.19}}$$

$$Pr[\text{finds dog}] = Pr[\text{dog in } A \cap \text{Oski finds dog}] + Pr[\text{dog in } B \cap \text{Oski finds dog}] = 0.19$$

(d) If the dog is alive and not found by the Nth day of the search, it will die that evening with probability $\frac{N}{N+2}$. Oski has decided to look in A for the first two days. What is the probability that he will find a live dog for the first time on the second day?

$$Pr[\text{no dog on day 1} \cap \text{live dog on day 2}] = Pr[\text{live dog on day 2} \mid \text{no dog on day 1}]\,Pr[\text{no dog on day 1}]$$
$$= 0.05 \times 0.9$$
$$= \boxed{0.045}$$

$$\frac{2}{2+2} = \frac{1}{2} \qquad Pr[\text{find on day 1 in } A \cap \text{dog in } A] = 0.25 \times 0.4 = 0.1$$

5. You own a pizzeria. You observe that one of your customers, Andy, buys a cheese pizza on Saturday with probability 0.3 and on Sunday with probability 0.6. Sa Su

   (a) If Andy's pizza purchasing habits on Sunday is independent from his pizza purchasing habits on Saturday, what is the probability that he buys pizza on a given weekend?

   $$Pr[Sa \cup Su] = Pr[Sa] + Pr[Su] - Pr[Sa \cap Su] = 0.72$$
   $$Pr[Su \mid Sa]\,Pr[Sa]$$

   Sa ✓ ✓ ✓
   Su ✓ ✓ ✓ ✓ ✓

   (b) If Andy buying a pizza on Saturday means that he will not buy pizza on Sunday, what is the probability that he buys pizza on a given weekend (i.e if he buys pizza on one day, he is guaranteed to not buy a pizza the next day)? Note that the probability that he buys a pizza on Sunday, 0.6, is *not* a conditional probability, i.e. it is not conditioned on whether he buys a pizza on Saturday. $Pr[Sa] = 0.3$ $Pr[Su \mid Sa] = 0$

   Sa ✓ ✓ ✓
   Su ✓ ✓ ✓ ✓ ✓ ✓

   $$Pr[Sa \cup Su] = 0.3 + 0.6 - 0 = 0.9$$

   (c) Suppose we don't know how Andy's pizza purchasing habits on Sundays depends on whether he bought a pizza on the preceding Saturday. Given that Andy buys pizza on a given weekend with probability 0.65, what is the probability that he buys pizza both days?

   $$Pr[Sa \cup Su] = Pr[Sa] + Pr[Su] - Pr[Sa \cap Su]$$
   $$0.65 = 0.3 + 0.6 - Pr[Sa \cap Su] \Rightarrow Pr[Sa \cap Su] = 0.25$$