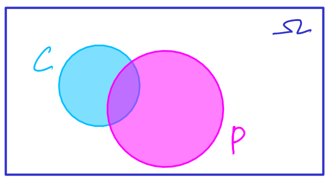
CS 70 Discrete Mathematics and Probability Theory
Fall 2021 **DIS 8A**

1 Venn Diagram

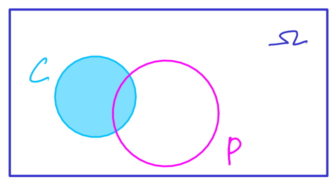
Out of 1,000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.



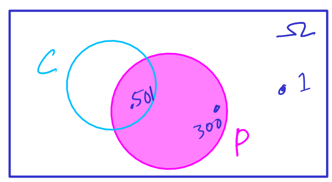
(b) What is the probability that the student belongs to a club?

$$\Pr[C] = \frac{\#(C)}{\#(\Omega)} = \frac{400}{1000}$$



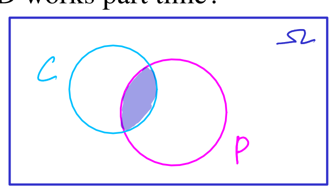
(c) What is the probability that the student works part time?

$$\Pr[P] = \frac{\#(P)}{\#(\Omega)} = \frac{500}{1000}$$



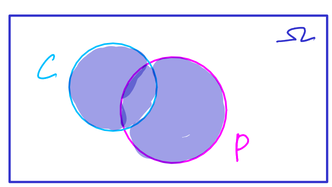
(d) What is the probability that the student belongs to a club AND works part time?

$$\Pr[C \cap P] = \frac{\#(C \cap P)}{\#(\Omega)} = \frac{50}{1000}$$



(e) What is the probability that the student belongs to a club OR works part time?

$$\Pr[C \cup P] = \frac{\#(C \cup P)}{\#(\Omega)}$$



$$= \frac{\#(C) + \#(P) - \#(C \cap P)}{\#(\Omega)}$$

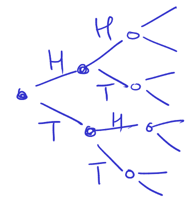
$$= \frac{\#(C)}{\#(\Omega)} + \frac{\#(P)}{\#(\Omega)} - \frac{\#(C \cap P)}{\#(\Omega)}$$

$$= \Pr[C] + \Pr[P] - \Pr[C \cap P]$$

2 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?



$$\Omega = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$$

Set of outcomes

(b) Which of the following are examples of *events*? Select all that apply.

$E \subseteq \Omega$

$E_4 \cap E_2 = \{(HHH)\}$

- $\{(H,H,T), (H,H), (T)\}$
- $\{(T,H,H), (H,T,H), (H,H,T), (H,H,H)\}$ — E_2
- $\{(T,T,T)\}$
- $\{\cancel{(T,T,T)}, (H,H,H)\}$ E_4
- $\{(T,H,T), (H,H,T)\}$

$E_3 = \{(HHT), (HHH)\}$ $\quad (TTH) \notin E_2$

$E_3 \subseteq E_2$ $\quad (HHH) \in E_2$

(c) What is the complement of the event $\{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,T,T)\}$? E

$A \subseteq U$ $\quad A^c \stackrel{def}{=} U \setminus A$

$E^c = \Omega \setminus E$
$= \{(THH), (THT), (TTH)\}$

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

$A = \{(TTT)\}$

$B = \{(HHT), (HTH), (THH)\}$

$1 - Pr[\{(HHT)\}^c]$

(e) What is the probability of the outcome (H,H,T)?

$$Pr[\{(HHT)\}] = \frac{\#(\{(HHT)\})}{\#(\Omega)} = \frac{1}{8}$$

H H T
$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
$= \frac{1}{8}$

(f) What is the probability of the event that our outcome has exactly two heads?

$$Pr[B] = \frac{\#(B)}{\#(\Omega)} = \frac{3}{8}$$

success

total # of outcomes

HHT HTH THH
$\frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

(g) What is the probability of the event that our outcome has at least one head? E

$E^c = \{(TTT)\}$

$Pr[E] = 1 - Pr[E^c]$

$= 1 - \frac{1}{8} = \frac{7}{8}$

3 Sampling

Suppose you have balls numbered $1, \ldots, n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

$\{(i,j) : i \in [1,n], j \in [1,n]\}$

$(1,1), (1,2), \ldots$
$(2,1), \ldots$

(a) What is the probability that the first ball is 1 and the second ball is 2?

$Pr[X_1=1, X_2=2] = 1/n^2$

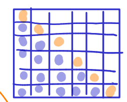
$Pr[X_1=1] Pr[X_2=2] = \frac{1}{n} \cdot \frac{1}{n} = \boxed{\frac{1}{n^2}}$

(d) $= Pr[X_1=1] Pr[X_2=2 \mid X_1=1] = \frac{1}{n} \cdot \frac{1}{n-1} = \boxed{\frac{1}{n(n-1)}}$

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

$\cancel{\frac{1}{2n^2}}$ $\cancel{\frac{n}{n^2}}$

X_1



X_2

$\left(\frac{n^2-n}{2}\right) / n^2 = \frac{|E|}{|\Omega|}$

$\boxed{\frac{n-1}{2n}}$

$(5,3)$
$(5,4)$

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

$(4,5)$

$\cancel{(3,5)}$

$(1,1)$ $(2,2)$, $(3,3)$, $\ldots$, (n,n)

$E = \{(1,2), (2,3), (3,4), \ldots, (n-1,n)\}$

$Pr[E] = \frac{|E|}{|\Omega|} = \frac{n-1}{n^2}$

2

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

	Replace	Don't
$Pr[X_1=1, X_2=1]$	$\frac{1}{n^2}$	0
$Pr[X_2=1 \mid X_1=1]$	$\frac{1}{n}$	$\frac{0}{n}=0$