Every couple prefers to have a girl over a boy. There's a 50% chance that a child is a girl, 50% chance of being a boy. Children's genders are mutually indep.

Every couple will continue having children until they have one girl.
After a long time, what will be the proportion of girls to boys in this society?

1: B B G

2: G

3: B G

X_B = # of boys for a couple

X_G = " " girls " " "

$\mathbb{E}[X_G] = 1 \cdot 1 = 1$

$\mathbb{E}[X_B] = \sum_{i=0}^{\infty} \left[\left(\frac{1}{2}\right)^{i+1} \cdot i \right] = 1$

X_G	$\Pr[X_G = 1]$
1	1

X_B	$\Pr[X_B = x]$
0	$1/2$
1	$1/2 \cdot 1/2 = \frac{1}{4}$
2	$(1/2)^2 \cdot 1/2 = \frac{1}{8}$

1 How Many Marbles?

Leanne has 6 marbles, 2 red, 2 blue, and 2 green. She picks three marbles uniformly at random without replacement. Let X denote the number of blue marbles she draws.

(a) What is $\mathbb{P}[X = 0]$, $\mathbb{P}[X = 1]$, and $\mathbb{P}[X = 2]$?

3 red, 2 bl, 2 gr

$$\frac{\binom{5}{3}}{\binom{7}{3}} = \frac{10}{35} = \frac{2}{7}$$

$$\frac{2 \cdot 10}{35} = \frac{4}{7}$$

$$\frac{1 \cdot 7}{35} = \frac{1}{7}$$

$$\Pr[X=0] = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{4}{20} = \frac{1}{5}$$

$$\Pr[X=1] = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{2 \cdot 6}{20} = \frac{3}{5}$$

$$\Pr[X=2] = \frac{\binom{2}{2}\binom{4}{1}}{|\Omega| = \binom{6}{3}} = \frac{1 \cdot 4}{20} = \frac{1}{5}$$

(b) What do your answers you computed in part (a) add up to?

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(c) Compute $\mathbb{E}[X]$ from the definition of expectation.

$$\mathbb{E}[X] = \sum_{\substack{\text{possible} \\ x}} x \Pr[X = x] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{3}{5} + 2 \cdot \frac{1}{5} = \boxed{1}$$

(d) Suppose we define indicators X_i, $1 \le i \le 3$, where X_i is the indicator variable that equals 1 if the ith marble is a blue marble and 0 otherwise. Compute $\mathbb{E}[X]$ using linearity of expectation.

(e) Are the X_i indicators independent? Does this affect your solution to part (d)?

$X_1=0 \quad X_2=0 \quad X_3=1$

O O O

O O O

$X_1=1$
$X_2=0$
$X_3=1$

(d) $X = \sum_{i=1}^{3} X_i$ $\qquad X_i = \mathbb{1}\{i^{th} \text{ marble is blue}\}$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{3} X_i\right] = \sum_{i=1}^{3} \mathbb{E}[X_i] = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = 1$$

$$\mathbb{E}[\mathbb{1}\{E\}] = 1 \cdot \Pr[E] + 0 \cdot \Pr[\bar{E}] = \Pr[E]$$

$$\mathbb{1}\{E\} = \begin{cases} 1 & \text{if } E \text{ happens} \\ 0 & \text{o/w} \end{cases}$$

$\mathbb{E}[X_2 \mid X_1 = 1] = \frac{1}{5}$ $\qquad \mathbb{E}[X_2] = \frac{2}{6}$

$\mathbb{E}[X_2 \mid X_1 = 0] = \frac{2}{5}$

2 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

$1{,}000{,}000 - 3 = 999\,997$ $\qquad$ 7 letters

$$\mathbb{E}[\#\text{ of "book"}] = \mathbb{E}\left[\sum_{i=1}^{999997} A_i\right] = \frac{999997}{26^4} \approx \boxed{2.17}$$

$\underbrace{\_\ \_\ \_}_{7-3}\ \underline{b}\ \underline{o}\ \underline{o}\ \underline{k}$

$$A_i = \begin{cases} 1 & \text{if "book" starts at } i \\ 0 & \text{o/w} \end{cases}$$

3 Ball in Bins

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

(a) What is $\mathbb{E}[X_i]$?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when a ball lands in a nonempty bin (if there are n balls in a bin, count that as $n-1$ collisions). What is the expected number of collisions?