X = body fat %age

Y = prob of heart attack

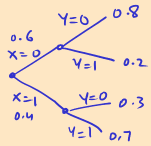
0.05	15–20%
0.06	20–25%
0.08	25–30%

$E[Y|X=23\%]$ 23%

$E[Y|X]$

$$Y = \begin{cases} 0 & \text{no rain} \\ 1 & \text{rain} \end{cases}$$

$$X = \begin{cases} 0 & \text{no clouds} \\ 1 & \text{clouds} \end{cases}$$



Y	
0	0.6
1	0.4

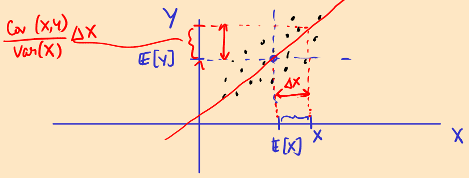
$Y \backslash X$	0	1
0	0.48	0.12
1	0.12	0.28

0.3
0.7

$$E[(Y-\hat{Y})^2] = E[(Y-E[Y])^2] = Var(Y) \quad \text{IF } \hat{Y} = E[Y]$$

Note 20

$$E[Y|X=x] = E[Y] + \underbrace{\frac{Cov(X,Y)}{Var(X)}(X-E[X])} = L(Y|X)$$



$Var(X) = Cov(X,X)$

$E[(X-E[X])^2]$

$= E[(X-E[X])(X-E[X])]$

$= Cov(X,X)$

$Var(\alpha X) = Cov(\alpha X, \alpha X)$

Cov is bilinear

$\alpha, \beta \in \mathbb{R}$

1. $Cov(\alpha X, \beta Y)$
   $= \alpha Cov(X, \beta Y)$
   $= \beta Cov(\alpha X, Y)$
   $= \alpha\beta Cov(X,Y)$

2. $Cov(X_1+X_2, Y)$
   $= Cov(X_1, Y) + Cov(X_2, Y)$

$Cov(X_1+X_2, Y_1+Y_2)$
$= Cov(X_1, Y_1)$
$+ Cov(X_1, Y_2)$
$+ Cov(X_2, Y_1)$
$+ Cov(X_2, Y_2)$

$Cov(X,Y)$
$= Cov(Y,X)$

ALWAYS TRUE

$Var(X_1+X_2)$

$= Cov(X_1+X_2, X_1+X_2)$

$= Cov(X_1, X_1)$
$+ Cov(X_2, X_2)$
$+ Cov(X_2, X_1)$
$+ Cov(X_1, X_2)$

$= Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$

If $Cov(X_1, X_2) = 0$

Then $Var(X_1+X_2) = Var(X_1) + Var(X_2)$

1.(c) $Cov(X, Y)$

$= Cov(X_1+X_2+X_3, X_4+X_5+X_6)$

$= 9 \cdot Cov(X_1, X_4)$ bilinearity of Cov; $X_1, X_2, \dots, X_6$ have same dist

$= 9 \cdot \frac{1}{144}$

$= \frac{9}{144}$

1.(d) $L(Y|X) = \frac{7}{4} + \underbrace{\frac{9/144}{111/144}\left(X - \frac{7}{4}\right)}$

out in

function of x.

input x

output : predicted value of Y.

CS 70 Discrete Mathematics and Probability Theory
Fall 2021 **DIS 14B**

1 LLSE

We have two bags of balls. The fractions of red balls and blue balls in bag A are $2/3$ and $1/3$ respectively. The fractions of red balls and blue balls in bag B are $1/2$ and $1/2$ respectively. Someone gives you one of the bags (unmarked) uniformly at random. You then draw 6 balls from that same bag with replacement. Let X_i be the indicator random variable that ball i is red. Now, let us define $X = \sum_{1 \le i \le 3} X_i$ and $Y = \sum_{4 \le i \le 6} X_i$.

A B

$\underbrace{X_1 \quad X_2 \quad X_3}_{X = \#\text{red}}$ $\underbrace{X_4 \quad X_5 \quad X_6}_{Y = \#\text{red}}$

(a) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(b) Compute $\text{Var}(X)$.

(c) Compute $\text{cov}(X,Y)$. (*Hint*: Recall that covariance is bilinear.)

(d) Compute $L(Y \mid X)$, the best linear estimator of Y given X. (*Hint*: Recall that

$$L(Y \mid X) = \mathbb{E}[Y] + \frac{\text{cov}(X,Y)}{\text{Var}(X)}(X - \mathbb{E}[X]).$$

)

X_1 has same dist as $X_2, X_3, \dots, X_6$

(a) $\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + X_3] = 3\,\mathbb{E}[X_1] = 3\Pr[X_1 = 1] = 3 \cdot \left(\frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{2}\right) = \frac{7}{4}$

$\mathbb{E}[Y]$ //

$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

$X_2 \backslash X_1$	0	1
0	0	0
1	0	1

(b) $\text{Var}(X) = \text{Cov}(X, X)$

$= \text{Cov}(X_1 + X_2 + X_3, X_1 + X_2 + X_3)$

$= \sum_{1 \le i,j \le 3} \text{Cov}(X_i, X_j)$ bilinearity of Cov

$= \text{Cov}(X_1, X_1) + \text{Cov}(X_2, X_2) + \text{Cov}(X_3, X_3)$
$+ \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3) + \text{Cov}(X_2, X_1)$
$+ \text{Cov}(X_2, X_3) + \text{Cov}(X_3, X_1) + \text{Cov}(X_3, X_2)$

$= 3\text{Var}(X_1) + 6\,\text{Cov}(X_2, X_1)$

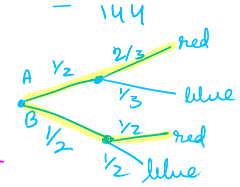
$= 3 \cdot \frac{35}{144} + 6 \cdot \frac{1}{144}$

$= \frac{111}{144}$

$\text{Cov}(X_1, X_2)$
$= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$
$= \frac{50}{144} - \frac{49}{144} = \frac{1}{144}$

$\text{Var}(X_1)$
$= \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2$
$= \mathbb{E}[X_1] - (\mathbb{E}[X_1])^2$
$= \frac{7}{12} - \frac{49}{144}$
$= 35/144$

$\mathbb{E}[X_1 X_2] = \Pr[X_1 X_2 = 1]$
$= \Pr[X_1 = 1]\Pr[X_2 = 1 \mid X_1 = 1]$
$= \frac{50}{144}$



X_i^2	X_i	Pr
0	0	$1-p$
1	1	p

X_i is indicator
X_i^2 has same dist as X_i

2 Balls in Bins Estimation

We throw $n > 0$ balls into $m \geq 2$ bins. Let X and Y represent the number of balls that land in bin 1 and 2 respectively.

(a) Calculate $\mathbb{E}[Y \mid X]$. [*Hint*: Your intuition may be more useful than formal calculations.]

(b) What is $L[Y \mid X]$ (where $L[Y \mid X]$ is the best linear estimator of Y given X)? [*Hint*: Your justification should be no more than two or three sentences, no calculations necessary! Think carefully about the meaning of the conditional expectation.]

(c) Unfortunately, your friend is not convinced by your answer to the previous part. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Compute $\text{Var}(X)$.

(e) Compute $\text{cov}(X,Y)$.

(f) Compute $L[Y \mid X]$ using the formula. Ensure that your answer is the same as your answer to part (b).

3 Continuous LLSE

Suppose that X and Y are uniformly distributed on the shaded region in the figure below.

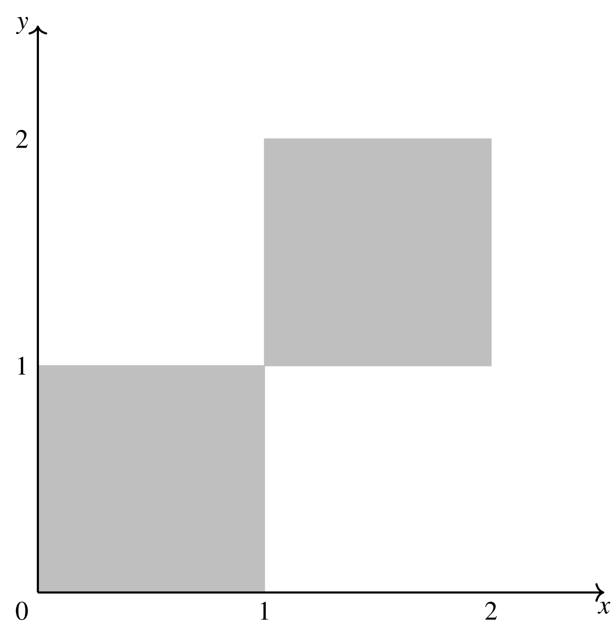


Figure 1: The joint density of (X,Y) is uniform over the shaded region.

That is, X and Y have the joint distribution:

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & 0 \le x \le 1, 0 \le y \le 1 \\ 1/2, & 1 \le x \le 2, 1 \le y \le 2 \end{cases}$$

(a) Do you expect X and Y to be positively correlated, negatively correlated, or neither?

(b) Compute the marginal distribution of X.

(c) Compute $L[Y \mid X]$, the best linear estimator of Y given X.

(d) What is $\mathbb{E}[Y \mid X]$?