1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

P	Q	$P\wedge(Q\vee P)$	$P\wedge Q$
F	F	F	F
F	T	'	'
T	F	'	'
T	T	'	'

(a) $P\wedge(Q\vee P) \equiv P\wedge Q$

(b) $(P\vee Q)\wedge R \equiv (P\wedge R)\vee(Q\wedge R)$

(c) $(P\wedge Q)\vee R \equiv (P\vee R)\wedge(Q\vee R)$

2 Converse and Contrapositive

$(\forall n \in \mathbb{N})$

$\cancel{(P \Rightarrow Q)}$

if then

$\Rightarrow$

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

$\neg P \Rightarrow \neg Q$ $\quad (\forall n \in \mathbb{N})\ ((4 \mid n) \Rightarrow (2 \mid n))$ $\quad x = 4k,\ k \in \mathbb{N} \Rightarrow x = 2(2k),\ 2k \in \mathbb{N}.$

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

$(\forall n \in \mathbb{N})\ (4 \nmid n \Rightarrow 2 \nmid n)$ $\quad 4 \nmid 2$ but $2 \mid 2$

$\exists m \in \mathbb{N}: m > n$

$\forall m \in \mathbb{N}: m \leq n$

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

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$P \Rightarrow Q$
$\equiv \neg P \vee Q$

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. $(\forall n \in \mathbb{N})\ (2 \mid n \Rightarrow 4 \mid n)$ $\quad P \Rightarrow Q \rightsquigarrow Q \Rightarrow P$

$\neg Q \Rightarrow \neg P$
$\equiv Q \vee \neg P$
$\equiv \neg P \vee Q$

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

$P \Rightarrow Q \rightsquigarrow \neg Q \Rightarrow \neg P$

$(\forall n \in \mathbb{N})\ (2 \nmid n \Rightarrow 4 \nmid n)$

$2k + 1 = 4r$

$\Rightarrow 1 = 4r - 2k \Rightarrow \underbrace{0.5}_{\notin \mathbb{Z}} = \underbrace{2r - k}_{\in \mathbb{Z}}$

3 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

$f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

$f(A) = \{f(a) : a \in A\}$
$f(x) = 3x$
$f(\{1,2,3\}) = \{3,6,9\}$
$f(x) = x^2$
$f^{-1}(\{2,3,4\}) = \{\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, 2, -2\}$

$y \in f^{-1}(A \cup B) \Rightarrow (\exists x \in A \cup B)(f(x) = y)$
$\Rightarrow (\exists x \in A)(f(x) = y) \vee (\exists x \in B)(f(x) = y)$
$\Rightarrow (x \in f^{-1}(A)) \vee (x \in f^{-1}(B))$
$\Rightarrow y \in f^{-1}(A) \cup f^{-1}(B)$

(a) $f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$.

(b) $f(A\cup B) = f(A)\cup f(B)$.

$y \in f^{-1}(A) \cup f^{-1}(B) \Rightarrow y \in f^{-1}(A \cup B)$

$f^{-1}(4) = 2? -2?$

$X = Y$
$\{ X \subseteq Y$
$\{ Y \subseteq X$

sets-are-cool

$X \subseteq Y$ $\quad Y \subseteq X$

$x \in X \Rightarrow x \in Y$ $\quad y \in Y \Rightarrow y \in X$

4 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)