Every couple prefers to have a girl over a boy. There's a 50% chance that a child is a girl, 50% chance of being a boy. Children's genders are mutually indep.

Every couple will continue having children until they have one girl.
After a long time, what will be the proportion of girls to boys in this society?

Random Variable

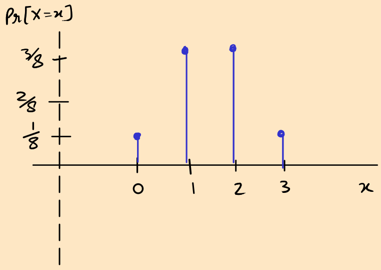
| | X = # of heads | Pr[X = x] |
|---|---|---|
| TTT | X = 0 | Pr[X = 0] = 1/8 |
| HTT THT TTH | X = 1 | Pr[X = 1] = 3/8 |
| HHT HTH THH | X = 2 | Pr[X = 2] = 3/8 |
| HHH | X = 3 | Pr[X = 3] = 1/8 |

distribution of X

$$Pr(X=x) = \binom{3}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-x} = \frac{1}{8}\binom{3}{x}$$

$x \in \{0, 1, 2, 3\}$

$\boxed{X \sim Bin(3, 1/2)}$ ✓




what is the distr of X?

1. What values can X take on?
2. For each of those vals, what's the Pr[X = val]?

# 1 Balls and Bins

Throw $n$ balls into $n$ labeled bins one at a time.

(a) What is the probability that the first bin is empty?

$Pr[\text{ball } i \text{ doesn't land in bin } 1] = \frac{n-1}{n}$

$Pr[\text{bin } 1 \text{ empty}] = \left(\frac{n-1}{n}\right)^n$

(b) What is the probability that the first $k$ bins are empty?

$\left(\frac{n-k}{n}\right)^n$

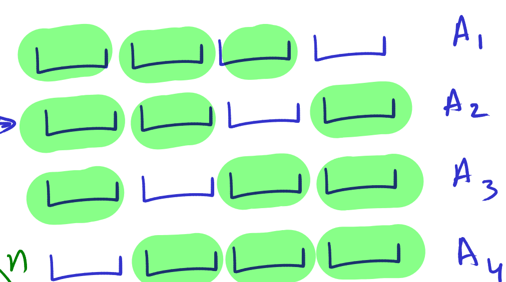
$n = 5$
$k = 3$

$Pr[A_i]$

(c) Let $A$ be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of $k$ bins out of the total $n$ bins. If we assume $A_i$ is the event that the $i^{\text{th}}$ set of $k$ bins is empty. Then we can write A as the union of $A_i$'s.

$n = 4$
$k = 3$

$$A = \bigcup_{i=1}^{m} A_i.$$




Write the union bound for the probability $A$.

$Pr[A \cup B] = Pr[A] + Pr[B] - \cancel{Pr[A \cap B]}$
$\leq Pr[A] + Pr[B]$

$$Pr\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} Pr[A_i] = m\left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n$$

$m = \binom{n}{k}$

(d) Use the union bound to give an upper bound on the probability $A$ from part (c).

$Pr[A_i] = \boxed{\left(\frac{n-k}{n}\right)^n}$, $1 \leq i \leq m$ by symm

(e) What is the probability that the second bin is empty given that the first one is empty?

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent?

## 2 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = \boxed{2/5}$. Suppose you flip the coin $\underline{20 \text{ times}}$, and define $X$ to be the number of heads.

$\underbrace{\text{Bernoulli}(p)}_{\text{1 coin toss}} \equiv \text{Bin}(1,p)$

(a) Name the distribution of $X$ and what its parameters are.

$Pr[H] = p$ flip $n$ times

$\underline{Pr[X=x]} = \binom{n}{x} p^x (1-p)^{n-x}$

$X$ is a $\underbrace{\text{Binomial}(n,p)}_{n \text{ coin tosses}}$

(b) What is $\mathbb{P}(X=7)$?

$$\binom{20}{7}\left(\frac{2}{5}\right)^7\left(1-\frac{2}{5}\right)^{20-7}$$

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

$$Pr[X \geq 1] = 1 - Pr[X=0]$$
$$= 1 - \binom{20}{0}\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^{20} = 1 - \left(\frac{3}{5}\right)^{20}$$

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

$$Pr[12 \leq X \leq 14] = \sum_{x=12}^{14} Pr[X=x] = Pr[X=12] + Pr[X=13] + Pr[X=14]$$
$$0 = \left( - Pr[X=12 \cap X=13] - Pr[X=13 \cap X=14] - Pr[X=12 \cap X=14] + Pr[X=12 \cap X=13 \cap X=14] \right.$$
$$= \text{Do it.}$$

## 3 Exploring the Geometric Distribution

Suppose $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ are independent. Find the distribution of $\min\{X,Y\}$ and justify your answer.