1 Counting Strings

(a) How many bit strings of length 10 contain at least five consecutive 0's?

(b) How many different ways are there to rearrange the letters of DIAGONALIZATION (15 letters with 3 A's, 3 I's, 2 N's, and 2 O's) without the two N's being adjacent?

2 Teams and Leaders

Prove the following identities using a combinatorial proof.

1. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

2. $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

3 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

Story: Choose 2 directors out of $2n$.

LHS: By def.

RHS:

OOO Exp n

OOO New n

$2n+1$

$n(n+1)$

$+\binom{n}{2} + \binom{n+1}{2}$

$\binom{n}{2} + \binom{n}{2} + n^2$

(b) Edward would now like to select a crew k out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called Pascal's Identity)

X

O O O O O O O O O O

$n-1$

k

friend

chosen: $\binom{n-1}{k-1}$

$+$

not chosen: $\binom{n-1}{k}$ $= \binom{n}{k}$

$\binom{n}{k} = \binom{n}{n-k}$

$$\frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,(k-n+n)!}$$

$$= \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \binom{n}{n-k}$$

Story: choose k out of n

LHS By def$^{\underline{n}}$

RHS

$k = 4$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ (1) Choose k

(1) cast

(2) lead

LHS: $\binom{n}{k}$

k=4

RHS: lead: n

others: 2^{n-1}

$n2^{n-1}$ ✓

(1) lead

(2) others

n−1

$\underbrace{2\times 2\times 2\times 2\times 2 \cdots}_{n-1}$

2^{n-1}

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

j lead actors

others

4 Countability: True or False

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ $\mathbb{R}$

(a) The set of all irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) f and g are injective (one-to-one) $\implies$ $f \circ g$ is injective (one-to-one).

(e) f is surjective (onto) $\implies$ $f \circ g$ is surjective (onto).

$\{1, 2, 3\} \setminus \{1, 3\} = \{2\}$

(a) uncountable

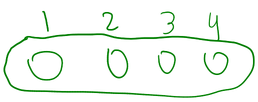
$(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$

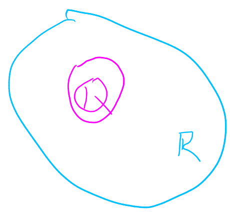
must be unc

count

unc

$X \hookrightarrow \mathbb{N}$





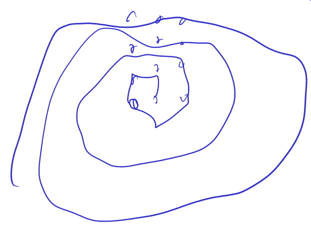
$\mathbb{R} \setminus \mathbb{Q}$ $\cup\ \mathbb{Q}$

$\mathbb{N} \times \mathbb{N}$ count

$\mathbb{Q}$

$\uparrow\uparrow$

$\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{Q}$



$\frac{3}{5}$ $\frac{-10}{7}$

$\frac{1}{2}$ $\frac{2}{4}$ $\frac{3}{6}$

$X \hookrightarrow Y \implies |X| \leq |Y|$

$X \twoheadrightarrow Y \implies |X| \geq |Y|$