B You toss ~~100~~ n coins

A Friend tosses ~~101~~ $n+1$ coins

$Pr[\#H_A > \#H_B]$ ⊛

In the first 100 coins

E_1: $\#H_A > \#H_B$ in first 100 tosses

E_2: $\#H_A = \#H_B$ in first 100 tosses

E_3: $\#H_A < \#H_B$ in first 100 tosses

$E_1 \quad E_3$

bij: flip all coins

$Pr[E_1] = Pr[E_3] = p$

$Pr[E_2] = q$

$p + q + p = q + 2p = 1 \Rightarrow q = 1 - 2p$

Last coin

$E_1 \xrightarrow{1}$ win for A

$E_2 \xrightarrow{1/2}$ win for A

$E_2 \xrightarrow{1/2}$ win for B

$E_3 \xrightarrow{1}$ win for B

$$Pr[win] = Pr[E_1] \cdot 1 + Pr[E_2]\frac{1}{2}$$

$$= p + \frac{1}{2}q$$

$$= p + \frac{1}{2} - p = \frac{1}{2}$$

⊛

$$Pr[win] = Pr[win \cap E_1] + Pr[win \cap E_2] + Pr[win \cap E_3]$$

$$= Pr[win | E_1] Pr[E_1] + Pr[win | E_2] Pr[E_2] + Pr[win | E_3] Pr[E_3]$$

$$= 1 \cdot Pr[E_1] + \frac{1}{2} \cdot Pr[E_2] + 0 \cdot Pr[E_3]$$

CS 70 Discrete Mathematics and Probability Theory
Fall 2021 **DIS 9A**

1 Probability Potpourri

$A \perp\!\!\!\perp B$
$\uparrow$
indep

$\Pr[A|B] = \Pr[A]$
$\Pr[B|A] = \Pr[B]$
$\Pr[A \cap B] = \Pr[A]\Pr[B]$

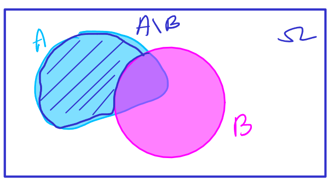
Provide brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

$\{1,2,3\} \setminus \{1,3\} = \{2\}$
4

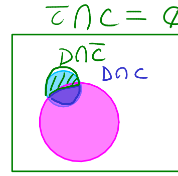
$\Pr[A \setminus B] = \Pr[A] - \Pr[A \cap B] \geq \Pr[A] - \Pr[B]$

$\Pr[B] \geq \Pr[A \cap B] \rightarrow \Pr[A \cap B] = \Pr[B]\underbrace{\Pr[A|B]}_{\leq 1}$



(b) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

$\overline{C} \cup C = \Omega$ ✓
$\overline{C} \cap C = \emptyset$ ✓



$\Pr[A \cap B] = \Pr[A] P[B|A]$
$\Pr[A]\Pr[B]$

$$\Pr[D] = \Pr[D \cap C] + \Pr[D \cap \overline{C}]$$

law of total prob

$$= \Pr[C]\Pr[D|C] + \Pr[\overline{C}]\Pr[D|\overline{C}]$$
$$= \Pr[C]\Pr[D|C] + \Pr[\overline{C}]\Pr[D|C]$$
$$= \Pr[D|C]\left(\Pr[C] + \Pr[\overline{C}]\right)$$

$= \Pr[C] + \Pr[\overline{C}] - \Pr[C \cap \overline{C}]$
$= \Pr[C \cup \overline{C}]$
$= \Pr[\Omega] = 1$

$\Pr[C]$
$= 1 - \Pr[\overline{C}]$

1 0 0 0 0 0

0

$\Pr[D] = \Pr[D|C]$

$$= \Pr[D|C]$$

(c) If A and B are disjoint, does that imply they're independent? no, implies dependent almost always (except if $\Pr[B] = \Pr[A] = 0$)

$A \cap B = \emptyset$
$B \subseteq \overline{A}$

NO

$\Pr[A] = 0.99$
$\Pr[A|B] = 0$

$A \cap B = \emptyset$
H T

2 Symmetric Marbles

A bag contains 4 red marbles and 4 blue marbles. Leanne and Sylvia play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Leanne wins if there are more red than blue marbles, and Sylvia wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent? No

$$\Pr[A_2] = 1/2$$

$$\Pr[A_2 | A_1] = 3/7 < 4/8$$

(b) What is the probability that Leanne wins the game?

bij: blue → red, red → blue

symm

$$\Pr[\text{Leanne W}] = p = \boxed{\frac{17}{70}}$$

$$\Pr[\text{Leanne L}]$$

$$\Pr[\text{Tie}] = 1 - 2p = \frac{\overbrace{\binom{4}{2}\binom{4}{2}}^{|\text{Tie}|}}{\underbrace{\binom{8}{4}}_{|\Omega|}} = \frac{36}{70}$$

(c) Given that Leanne wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the six drawn, Leanne wins if the third marble is red, and Sylvia wins if the third marble is blue.

(d) What is the probability that the third marble is red?

(e) Given that there are k red marbles among the four drawn, where $0 \leq k \leq 4$, what is the probability that the third marble is red? Answer in terms of k.

(f) Given that Leanne wins the game, what is the probability that the third marble is red?

(g) Given that the third marble is red, what is the probability that Leanne wins the game?

3 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?