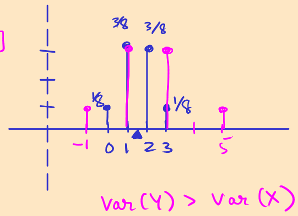
X = # of Heads Y = amt of money won

$X \sim Bin(3, 1/2)$ $Y = 2X - 1$

Y	X	$Pr[X=x] = Pr[Y=y]$
0	0	1/8
2	1	3/8
4	2	3/8
6	3	1/8



$\mathbb{E}[X] = 1.5$ $Var(Y) > Var(X)$

$\mathbb{E}[5] = 5$

$\mathbb{E}[\mu] = \mu$

$$Var(X) = \mathbb{E}\left[(X - \overset{\mu}{\mathbb{E}[X]})^2\right] \qquad W = (X - \mathbb{E}[X])^2$$
$$= \mathbb{E}[X^2 + \mu^2 - 2\mu X]$$
$$= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2$$
$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

5	$Pr[5=5]$
5	1

$\mathbb{E}[5] = 5 \cdot Pr[5=5] = 5$

X	X^2	Pr
0	0	1/8
1	1	3/8
2	4	3/8
3	9	1/8

$\mathbb{E}[X^2] = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{1}{8}$

$Var(\alpha X)$ α - any const

$= \alpha^2 Var(X)$

$$Var(\alpha X) = \mathbb{E}\left[(\alpha X - \mathbb{E}[\alpha X])^2\right]$$
$$= \mathbb{E}\left[(\alpha X - \alpha\mathbb{E}[X])^2\right]$$
$$= \mathbb{E}\left[\alpha^2 (X - \mathbb{E}[X])^2\right]$$
$$= \alpha^2 \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$
$$= \alpha^2 Var(X)$$

$$(\alpha a + \alpha b)^2$$
$$= \alpha^2 a^2 + 2\alpha^2 ab + \alpha^2 b^2$$
$$= \alpha^2 (a^2 + 2ab + b^2)$$
$$= \alpha^2 (a+b)^2$$

CS 70 Discrete Mathematics and Probability Theory
Fall 2021 DIS 11A

1 Variance

(a) Let X be a random variable representing the outcome of the roll of one fair 6-sided die. What is $\mathrm{Var}(X)$?

X	X^2	
1	1	1/6
2	4	1/6
3	9	
4	16	
5	25	
6	36	

$\mathbb{E}[X] = \frac{7}{2}$ $\mathbb{E}[X^2] = \frac{91}{6}$

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
$= \frac{35}{12}$

aside

Y	Y^2	P_r
−1	1	1/3
0	0	1/3
1	1	1/3

$\mathbb{E}[Y^2] = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}$
$= 1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3}$

(b) Let Z be a random variable representing the average of n rolls of a fair die 6-sided die. What is $\mathrm{Var}(Z)$?

Each roll is indep of all others.

$Y_1 \perp\!\!\!\perp Y_2$ $\mathrm{Var}(Y_1 + Y_2) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)$

not here $Y_1 \not\perp\!\!\!\perp Y_2$ $\mathrm{Var}(Y_1 + Y_2) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \mathrm{Cov}(Y_1, Y_2)$

$1 \le i \le n$
X_i = outcome of the i^{th} die roll

$Z = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{35}{12} = \frac{n}{n^2}\cdot\frac{35}{12}$

all X_i's are indep

$= \boxed{\frac{35}{12n}}$

2 Coupon Collector Variance

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\mathrm{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Try to break this problem down using indicators as with the coupon collector's problem. Are the indicators independent?]*

$X \sim Pois(\lambda)$ X = # of arrivals

arrival rate

3 Shuttles and Taxis at Airport

$\lambda \to \frac{\text{arrivals}}{\text{time}}$ $E[X] = \lambda \cdot 1\,\text{time}$

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson process. The shuttles arrive at a rate $\lambda_1 = 1/20$ (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate $\lambda_2 = 1/10$ (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

$\lambda_2 \Rightarrow \frac{\text{arr}}{10\text{ mins}}$ $\lambda_1 \Rightarrow \frac{\text{arr}}{20\text{ mins}}$ $1/20$

(a) What is the distribution of the following:

(i) The number of taxis that arrive between times 00:00 and 00:20? $Pois(20 \times \frac{1}{10}) = Pois(2) \sim T$

(ii) The number of shuttles that arrive between times 00:00 and 00:20? $Pois(20 \times \lambda_1) = Pois(1) \sim S$

(iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20? $N = T + S$, $N \sim Pois(1+2)$ $Pois(3)$

(b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?

(c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?

(d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

$X \sim Pois(\lambda)$

$Pr[X = x] = e^{-\lambda} \frac{\lambda^x}{x!}$

(b) $Pr[S = 1 \cap T = 3] = Pr[S=1]\,Pr[T=3 \mid S=1]$
$= Pr[S=1]\,Pr[T=3]$
$= \frac{1^1 e^{-1}}{1!} \cdot \frac{2^3 e^{-2}}{3!}$