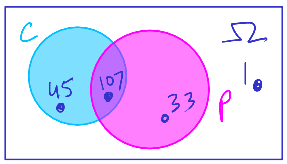
1 Venn Diagram

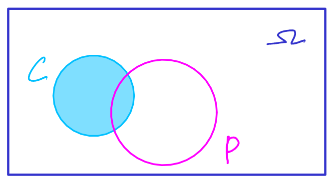
Out of 1,000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.



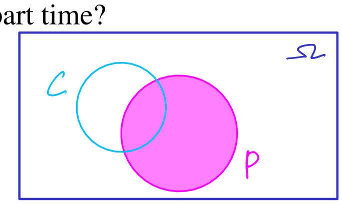
(b) What is the probability that the student belongs to a club?

$$\Pr[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000}$$



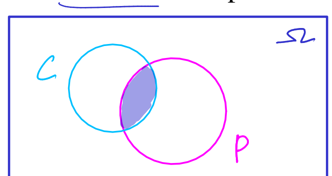
(c) What is the probability that the student works part time?

$$\Pr[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000}$$



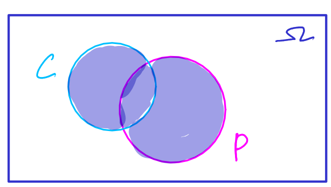
(d) What is the probability that the student belongs to a club AND works part time?

$$\Pr[C \cap P] = \frac{|C \cap P|}{|\Omega|} = \frac{50}{1000}$$



(e) What is the probability that the student belongs to a club OR works part time?

$$\Pr[C \cup P] = \frac{|C \cup P|}{|\Omega|}$$



$$= \frac{|C| + |P| - |C \cap P|}{|\Omega|} = \frac{850}{1000}$$

$$= \frac{|C|}{|\Omega|} + \frac{|P|}{|\Omega|} - \frac{|C \cap P|}{|\Omega|}$$

$$= \Pr[C] + \Pr[P] - \Pr[C \cap P]$$

2 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(HHT) (THT) (HHH)
(HTT)
~~(HH)~~

(a) What is the *sample space* for our experiment?

$\Omega = \{ (HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT) \}$ — each $1/8$

set of all possible outcomes

(b) Which of the following are examples of *events*? Select all that apply.

$E \subseteq \Omega$

- ✗ $\{(H,H,T), (H,H), (T)\}$
- E_1 ✓ $\{(T,H,H), (H,T,H), (H,H,T), (H,H,H)\}$
- E_2 ✓ $\{(T,T,T)\}$
- E_3 ✓ $\{(T,T,T), (H,H,H)\}$
- E_4 ✓ $\{(T,H,T), (H,H,T)\}$

✓ $\{\}$

$(HHH) \in E_1$, $(HHH) \in E_3$

(c) What is the complement of the event $E = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,T,T)\}$?

$A \subseteq \mathcal{U}$

$A^c = \mathcal{U} \setminus A$

$E^c = \Omega \setminus E = \{(THT), (TTH), (THH)\}$

$|E^c| = |\Omega \setminus E| = |\Omega| - |E| = 8 - 5 = 3$ ✓

$E \cap E^c = \emptyset$

$Pr[E] = 1 - Pr[E^c]$

$Pr[\Omega] = 1$

$Pr[E \cup E^c] = 1$

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

$A = \{(TTT)\}$

$B = \{(HHT), (HTH), (THH)\}$

(e) What is the probability of the outcome (H,H,T)?

$Pr[\{(H,H,T)\}] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$\shortparallel$

$Pr[H \text{ AND } H \text{ AND } T]$ independent events

$Pr[E_1 \text{ AND } E_2] = Pr[E_1] Pr[E_2 | E_1]$

$Pr[E_1 \text{ OR } E_2] = Pr[E_1] + Pr[E_2] - Pr[E_1 \cap E_2]$

(f) What is the probability of the event that our outcome has exactly two heads?

$Pr[B] = \frac{|B|}{|\Omega|} = \boxed{\frac{\# \text{success}}{\# \text{possibilities}}} = \frac{3}{8}$

$\shortparallel$

$Pr[HHT \text{ OR } HTH \text{ OR } THH] = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ mutually exclusive events

(g) What is the probability of the event that our outcome has at least one head? E

$Pr[E] = \frac{|E|}{|\Omega|} = \frac{7}{8}$ ✓

$\shortparallel$

$1 - Pr[E^c] = 1 - \frac{1}{8} = \frac{7}{8}$

$\frac{7}{8}$

HTT THT ...

$E^c = \{(TTT)\}$

$|E| = |\Omega| - |E^c| = 8 - 1 = 7$

3 Sampling

Suppose you have balls numbered $1, \ldots, n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

$\Pr[(1,2)]$

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

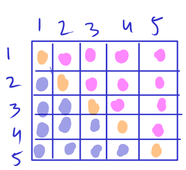
(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

	Replace	No Replacement
Ω	$\{(1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\}$	$\{(1,2)\ (1,3)\ (2,1)\ (2,3)\}$
$\lvert\Omega\rvert$	$n \cdot n = n^2$	$n \cdot (n-1)$

(a) $\Pr[X_1 = 1, X_2 = 2]$ $\quad |E| / |\Omega| = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n}$ $\quad \frac{1}{n(n-1)} = \Pr[1 \text{ AND } 2] = \Pr[1] \Pr[2 \mid 1] = \frac{1}{n} \cdot \frac{1}{n-1}$

$E = \{(1,2)\}$

(b) 

$$\Pr[\bullet] = \frac{\left(\frac{n^2-n}{2}\right)}{n^2}$$

$E = \{(i,j) : i > j\}$

$$|E| = \frac{n^2 - n}{2} = \binom{n}{2}$$

(c) $(1,1), (2,2), \ldots, (n,n)$

$i > j$

$i = j$

$i < j$