1 Beast Arcade

One day you find yourself inside the Mr. Beast Arcade, which is full of games that pay YOU to play them!

(a) In the first game, Chandler hands you a crisp \$20 bill. Then, he flips a coin that shows heads with probability p until a heads comes up. You receive an additional dollar for each flip. How much money will you get in expectation?

$20+\frac{1}{p}$ $X = $ # of flips before a H

~~$X \sim \text{Expo}(p)$~~ continuous

$$\mathbb{E}[\#\text{ of dollars}] = 20 + \mathbb{E}[X] = 20 + \frac{1}{p}$$

$X \sim \text{Geom}(p)$

(b) In the next game, Karl rolls a fair 6-sided die. He then calculates 2^x, where x is the result of that die and hands you that much money. What is the expected amount of money you'll receive?

$X = $ outcome of fair die roll

$Y = 2^X = g(X)$, $g(x) = 2^x$

$\mathbb{E}[X] = 3.5$

$\mathbb{E}[2^X] = 2^{\mathbb{E}[X]} = 2^{3.5} \approx 11$

$$\mathbb{E}[Y] = \frac{1}{6}(2+4+8+16+32+64) = 21$$

$$\mathbb{E}[g(X)] = \frac{1}{6}(g(1)+g(2)+g(3)+g(4)+g(5)+g(6))$$

X	Y	$Pr[Y=y]$
1	2^1	1/6
2	2^2	1/6
3	2^3	1/6
4	2^4	1/6
5	2^5	1/6
6	2^6	1/6

(c) For the last game, Jimmy makes your friend flip a fair coin 10,000 times in a row, keeping track of the number of heads that show up. He then hands you a briefcase filled with \$1,000 and says he will also pay you \$5 for each head that comes up. Let X be a random variable representing the number of heads your friend flips. Use it to come up with an expression for Y, a random variable representing the total amount of money you'll receive.

$Y = 1000 + 5X$

$\mathbb{E}[Z] = np$, $Z \sim \text{Bin}(n,p)$

$$\mathbb{E}[Y] = \mathbb{E}[1000+5X] = 1000 + 5\,\mathbb{E}[X] = 1000 + 5 \cdot 5000 = 26000$$

$X \sim \text{Bin}(10000, \tfrac{1}{2})$

(d) What is $E[Y]$? What about $P[Y = 26,000]$?

$$Pr[Y=26000] = Pr[1000+5X = 26000] = Pr[X = 5000] = \binom{10000}{5000}\underbrace{\left(\tfrac{1}{2}\right)^{5000}}_{H}\underbrace{\left(1-\tfrac{1}{2}\right)^{5000}}_{T} = \binom{10000}{5000}\left(\tfrac{1}{2}\right)^{10000}$$

2 Student Life

In an attempt to avoid having to do laundry often, Marcus comes up with a system. Every night, he designates one of his shirts as his dirtiest shirt. In the morning, he randomly picks one of his shirts to wear. If he picked the dirtiest one, he puts it in a dirty pile at the end of the day (a shirt in the dirty pile is not used again until it is cleaned). When Marcus puts his last shirt into the dirty pile, he finally does his laundry, and again designates one of his shirts as his dirtiest shirt (laundry isn't perfect) before going to bed. This process then repeats.

(a) If Marcus has n shirts, what is the expected number of days that transpire between laundry events? Your answer should be a function of n involving no summations.

(b) Say he gets even lazier, and instead of organizing his shirts in his dresser every night, he throws his shirts randomly onto one of n different locations in his room (one shirt per location), designates one of his shirts as his dirtiest shirt, and one location as the dirtiest location. In the morning, if he happens to pick the dirtiest shirt, *and* the dirtiest shirt was in the dirtiest location, then he puts the shirt into the dirty pile at the end of the day and does not throw any future shirts into that location and also does not consider it as a candidate for future dirtiest locations (it is too dirty). What is the expected number of days that transpire between laundry events now? Again, your answer should be a function of n involving no summations.

3 Alternating Technicians

A faulty machine is repeatedly run and on each run, the machine fails with probability p independent of the number of runs. Let the random variable X denote the number of runs until the first failure. Now, two technicians are hired to check on the machine every run. They decide to take turns checking on the machine every run. What is the probability that the first technician is the first one to find the machine broken? (Your answer should a be closed-form expression.)

cannot be an ∞-sum

$\cancel{\sum_{i=1}^{\infty}}$

Tech 1 obs failure $= (X=1) \cup (X=3) \cup (X=5) \cup \cdots$

$X \sim \text{Geom}(p)$

Tech 1: 1 3 5 7 ...

Tech 2: 2 4 6 8 ...

$$\begin{aligned}
&\Pr[\text{Tech 1 obs failure}] \\
&= \Pr[X=1] + \Pr[X=3] + \Pr[X=5] + \cdots \\
&= \sum_{k=0}^{\infty} \Pr[X = \underbrace{2k+1}] \\
&= \sum_{k=0}^{\infty} p(1-p)^{2k+1-1} \\
&= p\sum_{k=0}^{\infty} (1-2p+p^2)^k \\
&= \frac{p}{1-1+2p-p^2} = \boxed{\frac{1}{2-p}}
\end{aligned}$$

$\Pr[X=x] = p(1-p)^{x-1}$, $X \sim \text{Geom}(p)$

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

A gambler who starts with i dollars.
Win \$1 with probability p, $0 < p < 1$.
Lose \$1 with probability $1-p$.

Target of N dollars. If they reach \$N, then stop gambling.
Else if the gambler loses all their money, then stop gambling.
What is the probability of ending this game with \$N?