$\{(T), (HT), (HH)\}$
$x=0 \quad x=1 \quad x=2$

Toss a coin: → If H, then toss another, then stop.
→ If T, then stop

X	$Pr[X=x]$
0	1/2
1	1/4
2	1/4

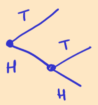
$\mathbb{E}[\underbrace{\text{total \# of Heads}}_{X}]$

$Y = \begin{cases} 0 & \text{if } 1^{st}\ T \\ 1 & \text{if } 1^{st}\ H \end{cases}$

$Y=0 \quad Y=1$

mut excl
coll exhaustive

$$\boxed{Pr[X=x] = Pr[X=x \cap Y=0] + Pr[X=x \cap Y=1] = Pr[X=x \mid Y=0]\,Pr[Y=0] + Pr[X=x \mid Y=1]\,Pr[Y=1]}$$



$Pr[X=0 \mid Y=0] = 1$

$Pr[Y=0] = \frac{1}{2}$

$Pr[X=0 \mid Y=1] = 0$

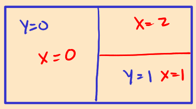
$\mathbb{E}[X]$

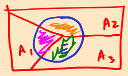
$= \sum_{x=0}^{2} x\, Pr[X=x]$ def^n of $\mathbb{E}$

$= \sum_{x=0}^{2} x \left(Pr[X=x \mid Y=0]\,Pr[Y=0] + Pr[X=x \mid Y=1]\,Pr[Y=1] \right)$ law of tot prob

$= Pr[Y=0] \sum_{x=0}^{2} x\, Pr[X=x \mid Y=0] + Pr[Y=1] \sum_{x=0}^{2} x\, Pr[X=x \mid Y=1]$ alg

$= Pr[Y=0]\,\mathbb{E}[X \mid Y=0] + Pr[Y=1]\,\mathbb{E}[X \mid Y=1]$ def^n of $\mathbb{E}$





$Pr[E] = Pr[E \cap A_1] + Pr[E \cap A_2] + Pr[E \cap A_3]$

$(A_1 + A_2) + (B_1 + B_2)$

$(A_1 + B_1) + (A_2 + B_2)$

CS 70 Fall 2021 Discrete Mathematics and Probability Theory DIS 12A

1 Number Game

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0,100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let S be Sinho's number and V be Vrettos' number.

$S=0 \quad S=1 \quad S=2 \quad S=3 \quad \vdots \quad S=100$

$\frac{1}{101} \times \frac{100(100+1)}{2} = 50$

(a) What is $\mathbb{E}[S]$?

$S = 0 \; 1 \; 2 \; 3 \cdots 100$

$U = \text{Unif}[a,b]$

$E[U] = \frac{a+b}{2}$

$S = 35$

$35 \; 36 \; 37 \cdots 100$

$E[S] \quad \frac{0+100}{2} = \boxed{50}$

"average value"

(b) What is $\mathbb{E}[V|S=s]$, where s is any constant such that $0 \le s \le 100$?

$V|(S=s)$ $\quad$ $E[W]$

$\sim \text{Unif}[s,100]$

$E[V|(S=s)] = \boxed{\frac{s+100}{2}} = \sum_{v=s}^{100} v \Pr[V=v|S=s]$ $\quad$ $\frac{1}{101-s}$

event

takes on vals

(c) What is $\mathbb{E}[V]$?

$E[V] = \Pr[S=0]E[V|S=0] + \Pr[S=1]E[V|S=1] + \cdots + \Pr[S=100]E[V|S=100]$

$E[V] = \sum_{s=0}^{100} \Pr[S=s]\, \boxed{E[V|S=s]}$

??

$= \sum_{s=0}^{100} \frac{1}{101} \cdot \boxed{\frac{s+100}{2}}$

$= \frac{1}{202}\left[\left(\sum_{s=0}^{100} s\right) + \left(\sum_{s=0}^{100} 100\right)\right] = \frac{1}{202}\left(\frac{100(100+1)}{2} + 101 \cdot 100\right)$

$= \boxed{75}$

Toss 1 coin.

$X = \#$ of heads

$Y = \#$ of tails

$X \sim \text{Bern}(1/2)$

$Y \sim \text{Bern}(1/2)$

$\Pr[X=1] = 1/2$

$\Pr[Y=1] = 1/2$

$\Pr[X=1 \cap Y=1] = 0$

$\Rightarrow$ dependent

2 Joint Distributions

(a) Give an example of discrete random variables X and Y with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of X and Y.

$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \text{Cov}(X,Y)$

Y \ X	0	1
0	0	1/2
1	1/2	0

$\mathbb{E}[XY] = 0 \neq \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

X	Y
0	1
1	0

(b) Give an example of discrete random variables X and Y that (i) are *not independent* and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of X and Y.

3 Inequality Practice

(a) X is a random variable such that $X \geq -5$ and $\mathbb{E}[X] = -3$. Find an upper bound for the probability of X being greater than or equal to -1.

(b) Y is a random variable such that $Y \leq 10$ and $\mathbb{E}[Y] = 1$. Find an upper bound for the probability of Y being less than or equal to -1.

(c) You roll a die 100 times. Let Z be the sum of the numbers that appear on the die throughout the 100 rolls. Compute $\text{Var}(Z)$. Then use Chebyshev's inequality to bound the probability of the sum Z being greater than 400 or less than 300.