You 100 fair coins

Friend 101 fair coins

$Pr[\#H_{You} < \#H_{Friend}]$

1 Aces

Consider a standard 52-card deck of cards: ♡ ♢ ♣ ♠ A 2 3 -- 10 K Q J

(a) Find the probability of getting an ace or a red card, when drawing a single card.

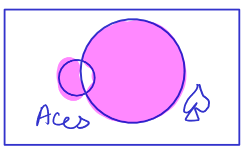
$P[\text{red or ace}] = \Pr[\text{red}] + \Pr[\text{ace}] - \Pr[\text{red and ace}]$

$\frac{26+4-2}{52}$

$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

$\Pr[\text{ace XOR spade}] = \underbrace{\frac{13}{52}}_{\text{spade}} + \underbrace{\frac{4}{52}}_{\text{ace}} - \underbrace{\frac{1}{52}}_{\text{ace of spades double-count}} - \underbrace{\frac{1}{52}}_{\text{ace of spades unwanted}} = \frac{15}{52}$ ✓



(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

$\frac{\cancel{51!}}{1\,\cancel{4!}\,\cancel{47!}} \cdot \frac{\cancel{47!}\,\cancel{5!}}{\cancel{52!}} = \frac{5}{52}$

$\frac{1 \cdot \binom{51}{4}}{\binom{52}{5}}$ ← choose 4 cards; $A^{\diamond}$; $\binom{52}{5}$ $|\Omega|$

$A^{\diamond}$ _ _ _ _ _

$\frac{5}{52}$

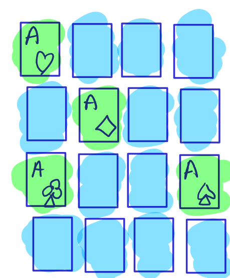
(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

$\frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}}$ $\quad \frac{\cancel{\binom{52}{4}}\binom{4}{2}}{\binom{52}{5}}$

$\underbrace{A^{\heartsuit} \quad 3^{\diamond} \quad 3^{\spadesuit} \quad 4^{\diamond}}$

$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{4!}{\cancel{3!}\,\cancel{2!}} \cdot \frac{\cancel{48!}}{\cancel{3!}\,\cancel{45!}} \cdot \frac{\cancel{5!}\,\cancel{47!}}{\cancel{52!}}$ $\quad 5 \cdot \cancel{4} \cdot 47 \cdot 46$ $\quad 52 \cdot 51 \cdot 50 \cdot 49$

$= \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 48 \cdot 47 \cdot 46}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$



(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

$1 - \Pr[\text{no aces in 5 cards}]$

$= 1 - \frac{\binom{48}{5}}{\binom{52}{5}}$ ✓

$\frac{\binom{4}{1} \cdot \binom{51}{4}}{\binom{52}{5}}$ ✓ 0, 1, 2, 3 aces

$\binom{52}{5} - \binom{48}{5}$
$\shortparallel$
$\binom{4}{1} \cdot \binom{51}{4}$

Comb proof

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

$\Pr[\geq 1 \text{ ace OR } \geq 1 \text{ heart}]$

$= 1 - \Pr[0 \text{ aces AND } 0 \text{ hearts}]$

$= 1 - \frac{|E|}{|\Omega|} = 1 - \frac{\binom{36}{5}}{\binom{52}{5}}$

$(A \cup B)^c$

$= A^c \cap B^c$

4 aces

13 hearts

1 ace of hearts

$13+4-1$

$=16$

aces or

hearts

$52-16=36$

2 Box of Marbles

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly, and pick a marble what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

3 Mario's Coins

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

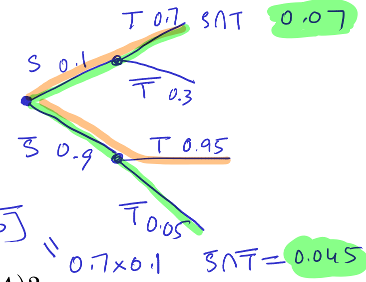
(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

4 Duelling Meteorologists

$Pr[T|S]$

$Pr[\text{Tom pred snow}]$
$= Pr[S \cap T] + Pr[\bar{S} \cap T]$
$= 0.115$

Tom is a meteorologist in New York. On days when it snows, Tom correctly predicts the snow 70% of the time. When it doesn't snow, he correctly predicts no snow 95% of the time. In New York, it snows on 10% of all days.



(a) If Tom says that it is going to snow, what is the probability it will actually snow?

$Pr[S] = 0.1$

$Pr[T|S] = 0.7 \qquad Pr[\bar{T}|S] = 0.3$

$Pr[T|\bar{S}] = 0.95 \qquad Pr[\bar{T}|\bar{S}] = 0.05$

$Pr[S \mid \text{Tom pred snow}] = \dfrac{Pr[T|S]\,Pr[S]}{Pr[\text{Tom pred snow}]} = \dfrac{0.7 \times 0.1}{0.115} \approx$

(b) Let A be the event that, on a given day, Tom predicts the weather correctly. What is $\mathbb{P}(A)$?

$Pr[T] = Pr[T|S]\,Pr[S] + Pr[T|\bar{S}]\,Pr[\bar{S}]$

$Pr[A] = Pr[T \cap S] + Pr[T \cap \bar{S}]$

$Pr[\bar{S} \cap \bar{T}] = Pr[\bar{S}]\,Pr[\bar{T}|\bar{S}]$
$= 0.9 \times 0.05 = 0.045$

(c) Tom's friend Jerry is a meteorologist in Alaska. Jerry claims that she is a better meteorologist than Tom even though her overall accuracy is lower. After looking at their records, you determine that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny day. Give an instance of the situation described above. *Hint: what is the weather like in Alaska?*