2 ropes. 1 hr to burn. Measure 45 mins?

CS 70 Fall 2021 — Discrete Mathematics and Probability Theory — DIS 1B

1 Prove or Disprove

odd + odd = even

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

(a) For all natural numbers n, if n is odd then $n^2 + 3n$ is even.

(b) For all real numbers a, b, if $a + b \geq 20$ then $a \geq 17$ or $b \geq 3$.

(c) For all real numbers r, if r is irrational then $r + 1$ is irrational.

(d) For all natural numbers n, $10n^3 > n!$.

(e) For all natural numbers a where a^5 is odd, then a is odd.

(a) $\underbrace{n \cdot n}_{\text{odd}} + \underbrace{3n}_{\text{odd}}$

$(2k+1)+(2l+1)$

$= 2\underbrace{(k+l+1)}_{\in \mathbb{Z}}$

(b) contrapos: $a<17 \wedge b<3 \Rightarrow a+b<20$

$a+b < 17+3 = 20$

(c) contradict: Assume $r+1$ rational: $r+1 = \frac{a}{b}$, $a, b \in \mathbb{Z}$

$\Rightarrow r = \frac{a-b}{b}$, $a-b, b \in \mathbb{Z} \Rightarrow r \in \mathbb{Q}$ contradict!

$\neg P \Rightarrow \neg Q$

$P \Rightarrow Q$

$P \Rightarrow \neg Q \Rightarrow$ (Absurd) $\neg P$

(d) $n = 10$: $10n^3 = 10000 < 3628800$, $n = 0$ ✓

(e) contrapos: a even $\Rightarrow a^5$ even

$a = (2k)$

$a = (2k)^5 = 32k^5 = 2(16k^5)$

2 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

r irrational $\Rightarrow$ $r+1$ irrational

"not rational"

$P \Rightarrow Q$

$\neg Q \Rightarrow \neg P$

Contraposition

$r+1$ rational $\Rightarrow$ ~~r rational~~

Assume $r+1$ rational

$\exists\, a, b \in \mathbb{Z} : r+1 = \frac{a}{b} \Rightarrow r = \frac{a}{b} - 1 = \frac{a-b}{b}$

$a - b \in \mathbb{Z},\ b \in \mathbb{Z}$

$\therefore$ r rational. (But this contr. r is irr. $\therefore$ $r+1$ must be irr.

$\therefore$ The contrapos is also true.

3 Induction

Base case : $P(0)$

$P(k) = 2^k > 2k+1$ Ind. hypothesis : $P(k)$ is true.

Prove the following using induction:

Ind. Step : $P(k) \Rightarrow P(k+1)$

$2k > 1$

(a) For all natural numbers $\boxed{n > 2}$, $2^n > 2n+1$.

Base : When $n=3$, $2^n = 8 > 6+1 = 7$.

IH : For $n = k > 2$, $2^k > 2k+1$.

IS : $2^{k+1} = 2 \cdot 2^k > 2 \cdot (2k+1) = 4k+2 = 2k+2k+2 > 2k+1+2 = (2k+2)+1 = 2(k+1)+1$.

$n \geq 1$

(b) For all positive integers n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Base : $n=1$: $1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$.

IH : $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

IS : $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \{\text{algebra}\} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$.

(c) For all positive natural numbers n, $\frac{5}{4} \cdot 8^n + 3^{3n-1}$ is divisible by 19.

4 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{(2^n - 1)}$ using induction.

(c) Why does the hypothesis in part (b) imply the conclusion from part (a)?