55 coins

550 g if all correct

$550 \pm i$

10g	10g	10g	10g	11g
1	2	3	4	5

$10 + 20 + 30 + 40 + 55g$

$150 + \underline{5}\ g$

1000 coins

$x = 405 \quad \times \quad y = 595$

$x_1 \quad x_2 \quad y_1 \quad y_2$

1 1

$x \cdot y$
$+ x_1 x_2$
$+ y_1 y_2$

What's the final sum?

$\boxed{\text{Hint:}}$ Start with small cases.

split_sum(5)

$= (4 \times 1) + ss(4) + ss(1)$

$= (2 \times 3) + ss(2) + ss(3)$

2 → 1, 1 $\quad 1 \times 1 = 1$

3 → 2, 1; 2 → 1, 1 $\quad (2 \times 1) + (1 \times 1) = 3$

4 → 2, 2; 2 → 1, 1; 2 → 1, 1
4 → 3, 1; 3 → 2, 1; 2 → 1, 1

$(2 \times 2) + (1 \times 1) + (1 \times 1) = 6$

$(3 \times 1) + (2 \times 1) + (1 \times 1) = 6$

5 → 4, 1 $\quad$ 5 → 2, 3

$(2 \times 3) + \underline{1} + \underline{3} = ?\ 10$

$(4 \times 1) + \underline{\quad} = ?$

Notice a pattern? Can you generalize this?

1 3 6 10 $\dfrac{n(n-1)}{2}$

n+1 → $x \times n+1-x$

Base: 2: $\dfrac{2(1)}{2} = 1$

IH: $ss(k) = \dfrac{k(k-1)}{2}$ for all $n \geq k \geq 2$.

IS: $ss(n+1) = \dfrac{(n+1)n}{2}$

Split into x, $n+1-x$.

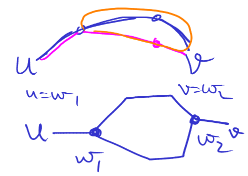
$$ss(n+1) = \underbrace{(x)(n+1-x)} + \frac{x(x-1)}{2} + \frac{(n+1-x)(n-x)}{2}$$

$$= \frac{(n+1)n}{2}.$$

$$ss(1000) = \frac{1000 \times 999}{2}$$

1 True or False

≥ 1 ≤ 1
connected $\wedge$ no cycles



(a) Any pair of vertices in a tree are connected by exactly one path.

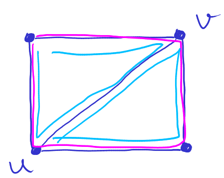
True: Connected $\Rightarrow$ ≥ 1 path.
Assume there are 2 paths between u and v.
Let w_1 be the branching pt. and w_2 be the merging pt.
Cycle involving w_1 & w_2. Contradiction.

(b) A simple graph obtained by adding an edge between two vertices of a tree creates a cycle.

True: Let G be a tree. Pick any two vertices u and v. By part (a) we know there is 1 path between u and v, $\{(u,v_1),(v_1,v_2),\ldots(v_k,v)\}$. Now we add an edge (u,v).
Cycle: $\{(v,v_k),\ldots,(v_2,v_1),(v_1,u),(u,v)\}$.

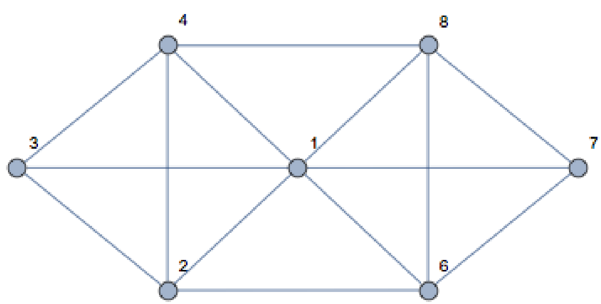
(c) Adding an edge in a connected graph creates exactly one new cycle.

False:



2 new cycles

2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$. Hint: Sum of an even # of odds is even.

(ii) Induction on $m = |E|$ (number of edges)

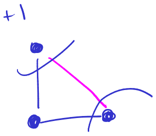
(iii) Induction on $n = |V|$ (number of vertices)

(i) $\sum_{v \in V} \deg(v) = \underbrace{2m} + \sum_{v \in V_{odd}(G)} \deg(v)$ $\quad V_{odd}(G) = \{\text{set of odd-deg verts in } G\}$

(ii) Base: $|E| = 0$. All verts have deg 0. 0 is even, 0 verts w/ odd deg.

IH: $|E| = k$, even # of verts w/ odd deg.

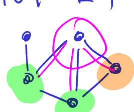
IS: $k+1$ $\quad +2, 0, -2$



(iii) Base: $|V| = 1$. 0 edges. 0 verts w/ odd deg. 0 even ✓

IH: $|V| = k$, holds.

IS:



# of verts w/ odd deg

# of verts w/ even

4 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]