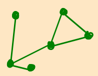
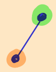
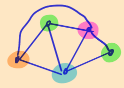
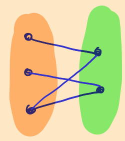
4-color Thm: Any planar graph can be colored with ≤ 4 colors.

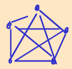




Bi partite
2 part

K ← complete



partite sets



Assume H_n has $n2^{n-1}$.

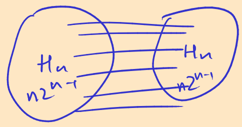
H_{n+1}

H_1

$2n2^{n-1} = n2^n$

$+ \; 2^n \quad = (n+1)2^n$ ✓

H_{n+1}

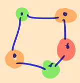


Extra Problem:

planar => 5-colorable

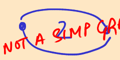
≤ 5 cols needed

≤ 4 cols needed



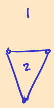
>4 cols needed => $\neg$ planar

$e \leq 3v - 6$



NOT A SIMP GRAPH

$v - e + f = 2$



$|E| = e$

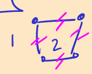
Every face must touch ≥ 3 edges

Face-edge ~~adjacency~~ touching

$2|E|$

$4+4=8$

$4 \times 2 = 8$



FET = 2e

FET $\geq 3f$

$v - e + f = 2$

$2e \geq 3f \Rightarrow f \leq \frac{2}{3}e$

$v - e + \frac{2}{3}e \geq 2$

$\Rightarrow e \leq 3v - 6$

1 Short Answers

$v - e + f = 2$ $v + 5 = e$ $\Rightarrow v = e - 5$ f ?

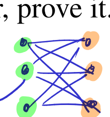
(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have? 7

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

$\cancel{e} - 5 - \cancel{e} + f = 2 \Rightarrow f = 7$

2 Always, Sometimes, or Never

In each part below, you are given some information about the so-called original graph, *OG*. Using only the information in the current part, say whether *OG* will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.



(a) *OG* can be vertex-colored with 4 colors. Planar ✓ Nonplanar ✓

(b) *OG* requires 7 colors to be vertex-colored. >4 colors needed ⇒ nonplanar ✓

(c) $e \leq 3v - 6$, where e is the number of edges of *OG* and v is the number of vertices of *OG*. ✓planar; planar ✓ nonplanar ✓ $K_{3,3}$

(d) *OG* is connected, and each vertex in *OG* has degree at most 2.

(e) Each vertex in *OG* has degree at most 2.

3 Trees and Components

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph? Please provide an explanation.

(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph? Please provide an explanation.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.