Seeing Calculus Through Algebra!

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December 3, 2024

1 Differentiation and Integration are Inverse Operations!

We will show how differentiation and integration are linear maps and then we'll show how they are inverses of each other.

$\boxed{\textbf{Definition:}}$ A **linear map** from V to W is a function from $T : V \to W$ that satisfy the following properties:

- $T(u + v) = Tu + Tv \ \forall u, v \in V$
- $\mathrm{T}(\alpha v) = \alpha Tv \ \forall \alpha \in F, \forall v \in V$

Let's see how differentiation is a linear map.

Take V to be $\mathbb{P}_4$ and W to be $\mathbb{P}_3$ where $\mathbb{P}_n$ denotes the set of polynomials with degree at most n.
$F = \mathbb{R}$.

It's easy to see that
$\mathbb{P}_4 = (a_0 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)$
$\mathbb{P}_3 = (b_0 1 + b_1 x + b_2 x^2 + b_3 x^3)$ where $b_i, a_i \in \mathbb{R}$, $i \in \mathbb{N}$.

$$\frac{d(f(x)+g(x))}{dx} = \frac{d(f(x))}{dx} + \frac{d(g(x))}{dx}$$
$$\frac{d(\alpha f(x))}{dx} = \alpha \frac{d(f(x))}{dx}$$

$\therefore$ Differentiation is a linear map from $\mathbb{P}_4$ to $\mathbb{P}_3$.
Denote it by $\mathcal{D}$

$\mathcal{D}1 = 0.1 + 0.x + 0.x^2 + 0.x^3$
$\mathcal{D}x = 1 + 0.x + 0.x^2 + 0.x^3$
$\mathcal{D}x^2 = 0.1 + 2.x + 0.x^2 + 0.x^3$
$\mathcal{D}x^3 = 0.1 + 0.x + 3.x^2 + 0.x^3$
$\mathcal{D}x^4 = 0.1 + 0.x + 0.x^2 + 4x^3$

$$\therefore \mathcal{D}=\begin{bmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\end{bmatrix}$$

Now let's see whether this matrix really gives the differentiated function

Let $p = a_0 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

Thus the coefficient matrix of p can be written as a column matrix as:

$$\begin{bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\end{bmatrix}$$

$$\therefore \mathcal{D}p = \begin{bmatrix}0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\end{bmatrix} \cdot \begin{bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\end{bmatrix} = \begin{bmatrix}a_1\\2a_2\\3a_3\\4a_4\end{bmatrix}$$

$$\text{p'}=\begin{bmatrix}a_1 & 2a_2 & 3a_3 & 4a_4\end{bmatrix} \cdot \begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

Indeed $\mathcal{D}$ is the differentiation operator!

Coming to Integration now....

Indefinite integration cannot be treated as a linear map, so we have to define integration as follows:

$$\mathcal{I} = \int_0^x f(x)\,dx$$

It is easy to see $\mathcal{I}$ is a linear map from $\mathbb{P}_3$ to $\mathbb{P}_4$.
Working as before,
$\mathcal{I}1 = 0.1 + 1.x + 0.x^2 + 0.x^3 + 0.x^4$
$\mathcal{I}x = 0.1 + 0.x + \frac{1}{2}.x^2 + 0.x^3 + 0.x^4$
$\mathcal{I}x^2 = 0.1 + 0.x + 0.x^2 + \frac{1}{3}.x^3 + 0.x^4$
$\mathcal{I}x^3 = 0.1 + 0.x + 0.x^2 + 0.x^3 + \frac{1}{4}.x^4$

$$\therefore \mathcal{I} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\therefore \mathcal{D}\mathcal{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$$\mathcal{D}\mathcal{I}p = Ip = p$$

Therefore, a polynomial in $\mathbb{P}_3$ when integrated and differentiated gives itself which is true!

Similarly,

$$\therefore \mathcal{I}\mathcal{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5$$

$$\mathcal{I}\mathcal{D}p = Ip = p$$

Therfore, a polynomial in $\mathbb{P}_4$ when differentiated and integrated again gives itself which can be verified easily by manual calculations.

Thus, we have been able to show that $\mathcal{D}$ and $\mathcal{I}$ are respectively the right and left inverses of $\mathcal{I}$ and $\mathcal{D}$. Therefore, the operations- differentiaion and intgeration are really inverse functions of each other subject to some constraints.

Caution:
Here I have used $\mathbb{P}_3$ and $\mathbb{P}_4$ but in general you can use $\mathbb{P}_n$ and $\mathbb{P}_m$ as long as n, m are countable. If they become uncountably infinite, the matrices $\mathcal{D}$ and $\mathcal{I}$ will have infinite rows and columns and well... you really can't multiply matrices with infinite elements.

[0]Credits to Gilbert Strang and his book "Linear Algebra and its Applications" for this idea.