Plancherel Formula

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0. Introduction

Let $G = G(F)$ be the reductive group as well as its F points, where F is a non-archimedian local field. Plancherel's formula describes the Schwartz functions on G in terms of the action of the tempered representation of G on it.

Let P be a parabolic subgroup of G and M its levi component. $\mathrm{Im}(X(M))$ denote the group of unramified, unitary characters on M. This group acts by torsion on the space of irreducible, square-integrable representations of M.
Let us fix an orbit $\mathcal{O}$ of this action. This is a compact real analytic variety. For $\omega \in \mathcal{O}$ we denote E_ω and resp. $E_\omega^\vee$ to be the spaces in which ω and its contragredient is realized. We have representations of G or $G \times G$ in different spaces, for example $\mathrm{Ind}_P^G(E_\omega)$, $\mathrm{Ind}_P^G(E_\omega^\vee)$, $L(\omega, P) = \mathrm{Ind}_P^G(E_\omega) \bigotimes_{\mathbb{C}} \mathrm{Ind}_P^G(E_\omega^\vee)$. When ω varies in $\mathcal{O}$, we can group these representations so that they form smooth fibers. Let f be the Schwartz-Harishchandra function on G. Let $f^\vee(g) = f(g^{-1})$. For $\omega \in \mathcal{O}$, $f^\vee$ naturally defines an endomorphism of $\mathrm{Ind}_P^G(E_\omega)$ (by scalar multiplication). Because f is sufficiently regular, this endomorphism belongs to the image of the natural injection:

$$L(\omega, P) \hookrightarrow \mathrm{End}(\mathrm{Ind}_P^G(E_\omega))$$

This defines an element of $L(\omega, P)$. We denote $\psi_f[\mathcal{O}, P]_\omega$ to be the product of this element by the formal degree $d(\omega)$. WE show that the function $\omega \mapsto \psi_f[\mathcal{O}, P]_\omega$ defined on $\mathcal{O}$ is a smooth section of the fiber $L(\cdot, P)$.

Conversely, let $\omega \mapsto \psi_\omega$ be a smooth section of this fiber. for $\omega \in \mathcal{O}$ and $g \in G$, we define $(E_P^G \psi_\omega)(g) \in \mathbb{C}$ in the following way...
Let π_ω be the induced representation of G in $\mathrm{Ind}_P^G(E_\omega^\vee)$, There is a natural pairing $\langle \cdot, \cdot \rangle$ between $\mathrm{Ind}_P^G(E_\omega)$ and $\mathrm{Ind}_P^G(E_\omega^\vee)$. Let's write

$$(E_P^G \psi_\omega)(g) = \sum_i \langle \pi_\omega(g) v_i, v_i^\vee \rangle$$

since the element g is fixed, the function $\omega \mapsto (E_P^G \psi_\omega)(g)$ is $\mathcal{C}^\infty$ on $\mathcal{O}$. On the other hand, we define the Harishchandra function μ on $\mathcal{O}$. For sufficiently regular $\omega \in \mathcal{O}$ we define the intertwining operator:

$$J_{\bar{P}|P}(\omega) : \mathrm{Ind}_P^G(E_\omega) \to \mathrm{Ind}_{\bar{P}}^G(E_\omega) \tag{0.1}$$

$$J_{P|\bar{P}}(\omega) : \mathrm{Ind}_{\bar{P}}^G(E_\omega) \to \mathrm{Ind}_P^G(E_\omega) \tag{0.2}$$

where $\bar{P}$ is the opposite parabolic of P.Their product is multiplication by a scalar $j(\omega)$. Then $\mu(\omega)$ is product of $j(\omega)^{-1}$ by an explicit constant. Th function μ this defined on an open subset of $\mathcal{O}$ and extends to a smooth function on $\mathcal{O}$.WE then define a function f_ψ on G by the following integral;

$$f_\psi(g) = \int_{\mathcal{O}} \mu(\omega)(E_P^G \psi_\omega)(g)\ d\omega$$

Plancherel formula then states that the above two operations are inverses of each other. More precisely, let f be a Schwartz-Harishchandra function on G. Then:

(1) The set of pairs $\mathcal{O}, P$ which satisfies the above conditions. and are such that $\psi_f[\mathcal{O}, P] \neq 0$ are finite up to conjugation.

(2) We have the equality $f = \sum_{\mathcal{O},P} c(P) f_{\psi_f[\mathcal{O},P]}$

To prove (1) Harishchandra demonstrates the following important result:

(3) if H is an open subgroup of G the set of classes of irreducible, square-integrable representations of G which have non-zero H invariant subgroup, is finite up to multiplications by element of $\mathrm{Im}(X(G))$.

1. Basic Definitions and Notations

I.1. Group structure. Throughout the notes, we will use the following notations:

- F = Non-Archimedean local field.
- G = Reductive group over F.
- A_0 = Maximal split torus.
- M_0 = Centralizer of A_0.
- If M is a levi subgroup, then A_M is the largest split torus in the *center* of M. We sometimes call it the split component of M.
- **Semi-Standard Parabolics:** A parabolic P is *semi-standard* if it contains A_0. In such cases, there is a unique Levi subgroup M of P containing A_0. This unique Levi is defined as *semi-standard Levi.*
- Given a semi-standard Levi M, we define $\mathcal{P}(M)$ to be all the parabolics containing M_0 as its Levi component.
- For any algebraic group H, we define the $\mathrm{Rat}(H)$ to be the group of characters on H rational over F.
- $a_0 := \left(\mathrm{Rat}(A_0) \bigotimes_{\mathbb{Z}} \mathbb{R}\right)^*$
- $a_M := \left(\mathrm{Rat}(A_M) \bigotimes_{\mathbb{Z}} \mathbb{R}\right)^*$
- $\Sigma(A_M)$ denotes the set of roots of A_M acting on the $\mathrm{Lie}(G) = \mathfrak{g}$. this is a subset of a_M^*.

- For $P \in \mathcal{P}(M)$, $\Sigma(P)$ denotes the set of positive roots relative to P.
  $\Sigma_{red}(P)$ denotes the subset of $\Sigma(P)$ consisting of reduced roots, i.e. roots α whose only multiple in the root system $\Sigma(P)$ are $\pm\alpha$.
  $\Delta(P)$ denotes the simple roots wrt. P.
- $\left({}^{+}a_P^G\right)^* := \{\chi \in a_M^* \mid \chi = \sum_{\alpha\in\Delta(P)} x_\alpha \alpha,\ x_\alpha > 0\}$
  $\overline{\left({}^{+}a_P^G\right)^*} := \{\chi \in a_M^* \mid \chi = \sum_{\alpha\in\Delta(P)} x_\alpha \alpha,\ x_\alpha \geq 0\}$
- Throughout we fix a minimal parabolic P_0. Any semi-standard parabolic containing P_0 will be defined as a *Standard Parabolic* subgroup. Then Δ_0^M is the set of simple roots of A_M acting on $\mathrm{Lie}(M)$, relative to P_0, and Δ_0 will define the simple roots corresponding to P_0.

Example 1.1. *Given a quadratic form $Q(x_1, ..., x_n) = x_1x_n + ... + x_q x_{n-q+1} + Q_0(x_{q+1,...,n-q})$, with Q_0 being a non-degenerate quadratic form Q_0 over F. Consider the group of matrices associated with Q, $\mathrm{SO}(Q)$. For example, let us choose $Q(x_i) = x_1x_6 + x_2x_5 + x_3^2 + x_4^2$. Then the corresponding special orthogonal group will be*

$$\mathrm{SO}_6(F) = \{g \in GL_n(F) : g^t J g = J\}$$

where $J = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & 1 & & & \\ & & & 1 & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$ *Then the maximal torus in this case would be*

$A_0 = \left\{\begin{pmatrix} t_1 & & & & & \\ & t_2 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & t_2^{-1} & \\ & & & & & t_1^{-1} \end{pmatrix}\right\}$. *Then centralizer of A_0 is* $M_0 = \left\{\begin{pmatrix} t_1 & & & & & \\ & t_2 & & & & \\ & & a & b & & \\ & & c & d & & \\ & & & & t_2^{-1} & \\ & & & & & t_1^{-1} \end{pmatrix}\right\}$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}_2(F)$. *Thus it is evident that $M_0 = A_0 \times \mathrm{SO}(Q_0) = A_0 \times \mathrm{SO}_{n-2q}(F)$. The standard parabolic subgroup would be* $\begin{pmatrix} B_2(F) & & * \\ & \mathrm{SO}_2(F) & \\ 0 & & B_2(F) \end{pmatrix} = M_0 \begin{pmatrix} N_2(F) & & * \\ & I_2 & \\ 0 & & N_2(F) \end{pmatrix}$. *Then $\Sigma(P_0) = \{e_i \pm e_j : 1 \leq i \leq j \leq 2\} \sqcup \{e_i : 1 \leq i \leq 2\}$, $\Sigma(P_0)_{red} = \{e_i \pm e_j : 1 \leq i < j \leq 2\} \sqcup \{e_i : 1 \leq i \leq 2\}$, where $e_i : A_0 \to \mathbb{G}_m$, sending $(t_1, t_2) \mapsto t_i$. The simple roots in the case would be*

$$\Delta_0 = \Delta(P_0) = \{e_1 - e_2, e_2\}$$

Here $a_0 := \left(\mathrm{Rat}(A_0) \otimes_{\mathbb{Z}} \mathbb{R}\right)^* = \left(\mathbb{Z}^2 \otimes \mathbb{R}\right)^* \cong \mathbb{Z}^2 \otimes \mathbb{R}$. $A_{M_0} = \left\{\begin{pmatrix} t & & & & & \\ & t & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & t^{-1} & \\ & & & & & t^{-1} \end{pmatrix}\right\}$

Then $a_{M_0} = \left(\mathrm{Rat}(A_{M_0}) \otimes_{\mathbb{Z}} \mathbb{R}\right)^* = (\mathbb{Z} \otimes \mathbb{R})^* \cong \mathbb{Z} \otimes \mathbb{R}$. *So clearly $a_{M_0} \subset a_0$. This* *is not difficult to see as $A_{M_0} \subset A_0$. As a quick remark, we can note that an element $(m,n)\otimes s \in \mathrm{Rat}(A_0)\otimes\mathbb{R}$ can be viewed as a group homomorphism from $A_0 \to \mathbb{C}$ defined as*

$$\begin{pmatrix} t_1 &&&&& \\ & t_2 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& t_2^{-1} & \\ &&&&& t_1^{-1} \end{pmatrix} \mapsto |t_1|^{ms}|t_2|^{ns}.$$

$\mathfrak{g}$ = {*skew-symmetric matrices*}, $\mathrm{Lie}(A_M) = \left\{ \begin{pmatrix} D_2(F) && \\ & \mathfrak{so}_2(F) & \\ && D_2(F) \end{pmatrix} \right\}$ $\Sigma(A_{M_0}) =$

$\Delta_0^M =$

More Notations:

- $\mathrm{Hom}(G,\mathbb{C}^\times)$ is the set of continuous homomorphism from G to $\mathbb{C}^\times$. If $\chi \in \mathrm{Rat}(G)$, then $|\chi| : g \mapsto \big|\chi(g)\big|_F$ is an element in $\mathrm{Hom}(G,,\mathbb{C}^\times)$.
- $G^1 := \bigcap_{\chi\in\mathrm{Rat}(G)} \ker(|\chi|_F)$
- $X(G) = \mathrm{Hom}(G/G^1, \mathbb{C}^\times)$.

Example 1.2. *If $G = GL_n$, then every element of $\mathrm{Rat}(G) \cong \mathbb{Z}$ is of the form $g \mapsto |\det g|^n$. Thus $G^1 = \{g \in GL_n : |\det g| = 1\}$.*

- There is a surjection

$$\begin{aligned} (a_G^*)_{\mathbb{C}} := (a_G^* \otimes_{\mathbb{R}} \mathbb{C}) &\to X(G) \\ \chi \otimes s &\mapsto |\chi|^s \end{aligned}$$

The kernel of this surjective map is of the form $\frac{2\pi i}{\log q}R$, where R is a lattice in $\mathrm{Rat}(G)\otimes\mathbb{Q} \subset a_G^*$, which means we get a complex variety structure on $X(G) \cong \mathbb{C}^d$, where $d = \dim_{\mathbb{R}} a_G$. Note that here $a_G^* = \mathrm{Rat}(G)\otimes\mathbb{R} \cong \mathrm{Rat}(A_G)\otimes\mathbb{R}$, where A_G is the split component of G, i.e. largest split torus inside the center of G.

Example 1.3. *$G = GL_n$, then $a_G^* = \mathbb{Z}\otimes\mathbb{R}$. Note that we have an isomorphism $\mathrm{Rat}(A_G)\otimes\mathbb{R} \to \mathrm{Rat}(G)\otimes\mathbb{R}$, where we send the character*

$$\begin{aligned} \chi : \mathbb{G}_m = A_G(F) &\to F^\times \\ x &\mapsto x^m \end{aligned}$$

to

$$\begin{aligned} \chi : GL_n(F) = G(F) &\to F^\times \\ g &\mapsto (\det g)^{\frac{m}{n}} \end{aligned}$$

such that when we consider the diagonal embedding of $\mathbb{G}_m$ into GL_n, the characters agree. However, we note that under the projection map

$$\begin{aligned} a_G^* &\to X(G) \\ \chi\otimes s &\mapsto (g \mapsto |\chi(g)|^s) \end{aligned}$$

Thus the kernel of this map will be $\{\chi \otimes s : \left|\chi(g)\right|^s = 1, \forall g \in G/G^1\}$. *Now assume that the character* χ *is of the above form, i.e. it sends* $g \mapsto \det(g)^{\frac{m}{n}}$. *Thus if* $\chi \otimes s$ *is inside the kernel we must have that*

$$\begin{aligned} &\left|\chi(g)\right|^s = 1, \forall g \in GL_n(F)/G^1 \\ \implies &e^{-\mathrm{ord}(\det g) \ln q \frac{m}{n} s} = 1 \end{aligned}$$

So the kernel is $\{m \otimes \frac{2\pi i}{\ln q} l \mid \ m, l \in \mathbb{Z}\} = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$ *Thus* $X(G) = \mathbb{C}/Z \cong \mathbb{C}^*$ *(because, we can consider the exponential map* $e^{2\pi\circ}\mathbb{C} \to \mathbb{C}^*$, *the kernel of which is* $\mathbb{Z}$*).*

- For every $\chi \in X(G)$ let $\lambda \in (a_G^*)_{\mathbb{C}}$ be a pre-image of χ. Then $\mathrm{Re}(\lambda) \in a_G^*$ is independent of λ and denote it by $\mathrm{Re}(\chi)$.
- If $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ then $|\chi|_F \in X(G)$. Similarly, if $\chi \in \mathrm{Hom}(A_G, \mathbb{C}^\times)$, then $|\chi|$ uniquely extends to $X(G)$, we will denote $\mathrm{Re}(\chi) :=$ Real part of extension of $|\chi|$.

$$\mathrm{Im}(X(G)) = \{\chi \in X(G) : \mathrm{Re}\chi = 0\} = \mathrm{Hom}(G/G^1, S^1)$$

- Define the Harishchandra map

$$\begin{aligned} &H_G : G \to a_G \\ &|\chi|(g) = q^{-\langle H_G(g), \chi\rangle}, \ \forall \chi \in \mathrm{Rat}(G) \end{aligned}$$

  For example in the case of GL_n, $H_G(g) = \mathrm{ord}(\det g) = -\log_q(|\det g|) \in a_G$.
- R is the kernel of the aforementioned $(a_G)^*_{\mathbb{C}} \to X(G)$ which is also the annihilator of $H_G(G)$.
- $\bar{M_0}^+ = H_{M_0}^{-1}(\bar{a_0^+})$. Also, $\bar{A_0}^+ = \bar{M_0}^+ \cap A_0$

Example 1.4. *Let's consider* $G = GL_n$*. then for each* $\chi = \det(g)^{\frac{m}{n}} \in \mathrm{Rat}(G) = \mathbb{Z}$, *we have* $|\chi|(g) = \left|\det(g)\right|^{\frac{m}{n}} = q^{-\mathrm{ord}(\det g)\frac{m}{n}}$

Then H_G *is defined as follows;*

$$\begin{aligned} H_G : GL_n &\to a_G = (\mathrm{Rat}(G) \otimes \mathbb{R})^* \\ g &\mapsto \mathrm{ord}(\det g) \end{aligned}$$

When we consider the standard Levi $M_0 = \mathrm{diag}(a_1, ..., a_n)$, *then* $a_{M_0}^* = \bigoplus_{i \leq n} \mathbb{Z}$, *where each rational character* $\chi = \chi_{(m_1,...,m_n)} \in \mathbb{Z}^n$ *sends* $\mathrm{diag}(a_i) \mapsto \prod a_i^{m_i}$. *Then* $|\chi|(\mathrm{diag}(a_i)) = \prod |a_i|^{m_i}$. *Then clearly*

$$\begin{aligned} H_{M_0} : M_0 &\to a_{M_0} \\ \mathrm{diag}(a_i) &\mapsto (\mathrm{ord}(a_i))_{i=1}^n \in \mathbb{Z}^n \end{aligned}$$

Recall that $\overline{a_0^+}$ *are the elements of* a_0 *with positive coefficient wrt. dual elements of the simple roots. Then clearly* $\bar{M_0^+}$ *is consisted of* $\{\mathrm{diag}(a_1, ..., a_n) : |a_1| \leq |a_2| \leq ... \leq |a_n|\}$.

Measures. Fix a maximal open compact subgroup K of G, which we assume to be a stabilizer of a special point of the apartment associated with A_0. For example: for $G = \mathrm{GL}_n$ we have $K = \mathrm{GL}_n(\mathcal{O})$ and $G = \mathrm{SO}_n$ we have $K = \mathrm{SO}_n(\mathcal{O})$. For any closed subgroup H of G, we choose a haar measure of H such that $\mathrm{meas}(H \cap K) = 1$.

If $P = MU$ is standard we have $G = PK$. For every $g \in G$ we choose $u_P(g) \in U, m_P(g) \in M, k_P(g) \in K$ such that $g = u_P(g)m_P(g)k_P(g) \in UMK$. For standard choice P_0, we replace each $\circ_P = \circ_0$. Then under the above decomposition, we have for any smooth f the following integral decomposition

(1) $\int_G f(g)\ dg = \int_{UMK} f(umk)\delta_P^{-1}(m)\ dk\ dm\ du$

(2) $\int_G f(g)\ dg = \gamma^{-1}(P)\int_{UM\bar{U}} f(um\bar{u})\delta_P^{-1}(m) d\bar{u}\ dm\ du$ where $\gamma(P) = \int_{\bar{U}} \delta_P(m_P(\bar{u}))\ d\bar{u}$, and clearly this depends on M, and we may define it also by the symbol $\gamma(G \mid M)$

Definition 1.5. *Fro $\alpha \in \Sigma_{red}(P)$, let A_α be the identity component of the kernel of $A_\alpha \xrightarrow{\alpha} \mathbb{G}_m$. Let M_α be the centralizer of A_α. Also*

$$c(G \mid M) := \gamma(G \mid M)^{-1} \prod_{\alpha\in\Sigma_{red}(P)} \gamma(M_\alpha \mid M)$$

For example if we consider $G = \mathrm{GL}-n$ and $P = MU$ standard parabolic with partition $n = \sum_{i=1}^r n_i$ Let $H = I_n + \varpi M_n(\mathcal{O}_F)$. We have the Iwahori decomposition

$$H = (H \cap \bar{U})(H \cap M)(H \cap U)$$

A group admitting such a decomposition is defined as "Groups with good position". However, let $f = 1_H$. then as we have seen before

$$\begin{aligned}\mathrm{meas}(H) &= \int_G 1_H\ dg \\ &= \gamma(P)^{-1}\mathrm{meas}(U \cap H)\mathrm{meas}(M \cap H)\mathrm{meas}(\bar{U} \cap H)\end{aligned}$$

Now $\mathrm{meas}(\bar{U} \cap H) = \mathrm{meas}(U \cap H) = q^{-R}$ where R =number of positive roots of $G-$ number of negative roots of $G = \sum_{1\le i<j\le r} n_i n_j$. Now we have an exact sequence

$$1 \longrightarrow H \longrightarrow K \longrightarrow \mathrm{Gl}_n(\mathbb{F}_q)$$

Since $\mathrm{meas}(K) = 1$, we have $\mathrm{meas}(H) = \frac{1}{\#\,\mathrm{GL}_n(\mathbb{F}_q)}$, $\mathrm{meas}(H \cap M) = \frac{1}{\prod_{i=1}^r \#\,\mathrm{GL}_{n_i}(\mathbb{F}_q)}$, and finally $\gamma(G \mid M) = q^{-2\sum n_i n_j} \frac{\#\,\mathrm{GL}_n(\mathbb{F}_q)}{\prod_{i=1}^r \#\,\mathrm{GL}_{n_i}(\mathbb{F}_q)}$

- **Cartan Decomposition:** We have

$$G = \bigsqcup_{m\in \bar{M}_0^+/M_0^1} KmK$$

Proposition 1.6. *There exists $C_1, C_2 > 0$ such that $\forall m \in \bar{M}_0^+$ we have*

$$C_1\delta_0(m)^{-1} \le \mathrm{meas}(KmK) \le C_2\delta_0(m)^{-1}$$

Here $\delta_{P_0} = \delta_0$.

Proof. ∎

Fix an algebraic embedding $\tau : G \to \mathrm{GL}_n(F)$. Assume $\tau(K) \subset \mathrm{GL}_n(\mathcal{O}_F)$. for every $g \in G$ we write $\tau(g) = (a_{ij})_{1\le i,j\le n}$ and $\tau(g)^{-1} = (b_{ij})$. Define the sup norm on G, as in

$$\|g\| = \sup\{|a_i j|_F, |b_{ij}|_F\}$$

one can verify that

$$\|g\| \ge 1,\ \|g_1 g_2\| \le \|g_1\|\|g_2\|$$

for all $g_i \in G$.and $\|k_1 g k_2\| = \|g\|$.

Let $\sigma(g) = \log\|g\|$.
Equip a_0 with Euclidean norm which is invariant under the action of W^G (the Weyl group). There exists constants $C_1, C_2 > 0$ such that for all $m \in M_0$ we have

$$C_1(1 + |H_0(M)|) \leq 1 + \sigma(m) \leq C_2(1 + |H_0(m)|)$$

Example 1.7. *One can check that for $G = GL_n$ the constants $C_1 = C_2 = 1$ suffices the purpose.*

I.2. Analysis on $\mathcal{C}^\infty(A_M)$. Let M be a semi-standard Levi subgroup. ρ be the representation of A_m on $\mathcal{C}^\infty(A_m)$ by right translation. Let V be a finite dimensional subspace of $\mathcal{C}^\infty(A_m)$, stable by the action of ρ. Then there exists a finite subset $\mathcal{X} \subset \mathrm{Hom}(A_M, \mathbb{C}^*)$ and $d \in \mathbb{N}$ such that for any $f \in V$ and any $\chi \in \mathcal{X}$ there exists a polynomial $P_{\chi,f}$ on a_M with complex coefficient and degree $\leq d$ such that for all $a \in A_M$ we have

$$f(a) = \sum_{\chi \in \mathcal{X}} \chi(a) P_{\chi,f}(H_M(a))$$

Suppose $\mathcal{X}$ is minimal, i.e. for all $\chi \in \mathcal{X}$ there exists $f \in V$ such that $P_{\chi,f} \neq 0$. then for all $\chi \in \mathcal{X}$ the functions

$$a \mapsto \chi(a), \quad \text{and} \quad a \mapsto \chi(a) P_{\chi,f}(H_M(a))$$

belong to V.
Reason: On $\mathbb{Z}$ if V is a finite-dimensional representation of $\mathbb{Z} \in \mathcal{C}^\infty(\mathbb{Z})$ and assume the map $n \mapsto n^d$ to be an element of V, then the map $n \mapsto n^d - (n-1)^d$ is also in V. ?????????????

Proposition 1.8. *TFAE:*

(1) *There exists $n \in \mathbb{N}$ and, for any $f \in V$ there exists $C > 0$ such that for any $a \in A_M$ we have the inequality*

$$|f(a)| \leq C(1 + \sigma(a))^n$$

(2) *For all $\chi \in \mathcal{X}$,* $\mathrm{Re}(\chi) = 0$.

Let $\mathcal{Y} \in \mathrm{Hom}(A_M, \mathbb{C}^*)$ be a finite subset and D be an integer ≥ 1. Suppose that for all $a \in A_M$ the operator,

$$\prod_{x \in \mathcal{Y}} (\rho(a) - \chi(a))^D$$

vanishes on V. then due to minimality condition on $\mathcal{X}$ we have $\mathcal{X} \subset \mathcal{Y}$, and we can choose $d \leq D - 1$.

For any $f \in \mathcal{C}^\infty(A_M)$, let V_f be the subspace of $\mathcal{C}^{\infty(A_M)}$ spanned by $\rho(a)f$ for all $a \in A_M$. Then Condition 1 $\Leftrightarrow$ Condition 2 $\Leftrightarrow$ $\exists n, C > 0$ such that for all $a \in A_M$

$$|f(a)| \leq C(1 + \sigma(a))^n$$

I.3. Representation Theory. Let (π, V) be an admissible representation of G. Let $\chi \in \mathrm{Hom}(A_G, \mathbb{C}^*)$. Let

$$V_\chi := \{v \in V : \exists d \in \mathbb{N}, \text{ s.t. } \forall a \in A_G, (\pi(a) - \chi(a))^d v = 0\}$$

The exponent of π is a character χ such that $V_\chi \neq 0$. Then

$$V = \bigoplus_{\chi \in \mathrm{Exp}(\pi)} V_\chi$$

WHY?

Definition 1.9. *Let $P = MU$ be a semi-standard parabolic subgroup. V_P is the Jacquet Module of V relative to P defined as $V_P := V/\mathrm{Span}\{v - \pi(u)v : u \in U\}$ and $j_P : V \to V_P$, and $j_P : V \to V_P$ be the natural projection. Then M acts on V_P by the following action*

$$\pi_P(m) j_P(v) := \delta_P^{-\frac{1}{2}}(m) j_P(\pi(m)v)$$

The representation (π_P, V_P) is also admissible.

Definition 1.10. *For any admissible representation (π, V) of M, we define a representation of GL_n, $(I_P^G \pi, I_P^G V)$, where $I_P^G V := \{f : G \to V : f(mug) = \delta_P^{\frac{1}{2}}(m)\pi(m)f(g)\}$ and the action is by right translation.*

Proposition 1.11. *We have Frobenius reciprocity for any admissible representations (π, V) of M and (π', V') of G, given by*

$$\mathrm{Hom}_G(V', I_P^G V) = \mathrm{Hom}_M(j_P V', V)$$

Proposition 1.12. *We have*

$$\mathrm{I}_P^G(\check{V}) = \widetilde{\mathrm{I}_P^G V}$$

Proof. Consider the pairing

$$I_P^G V \times I_P^G \check{V} \to \mathbb{C}$$
$$(f, \check{f}) \mapsto \int_{G/P} \langle f(g), \check{f}(g) \rangle \, dg$$

$\square$

Proposition 1.13. *Let $P' = M'N' \supset P$ be another semi-standard parabolic, then*

$$I_P^G V = \mathrm{I}_{P'}^G \left(I_{P \cap M'}^{M'} \left(V \Big|_{P \cap M'} \right) \right)$$

Geometric Theory: let $P = MU$ and $P' = M'U'$ be two arbitrary semi standard parabolics. Define

$${}^{P'}W^P := \left\{ w \in W^G \mid w^{-1}(M' \cap P_0)w \subset P_0 \text{ and } w(M \cap P_0)w^{-1} \subset P_0 \right\}$$

Lemma 1.14. *${}^{P'}W^P$ is a set of representative of $W^{M'} \backslash W^G / W^M$.*

Then according to the Bruhat decomposition, we have an isomorphism

$$W^{M'} \backslash W^G / W^M \longrightarrow P \backslash G / P'$$
$$w \mapsto P w^{-1} P'$$

Example 1.15. *Consider $G = GL_3$. Consider $P' = P_{2,1}$ and $P = P_{1,1,1} = P_0$. then $M' = GL_2 \times GL_1$ and $M = GL_1 \times GL_1 \times GL_1$.*
Then $M' \cap P_0 = B_2 \times GL_1$. $W^{M'}$ =subgroup of S_n generated by the simple reflection $s_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $W^M = W^G$. Thus $W^{M'} \backslash W^G / W^M = \{W^{M'} e W^M\}$ which will correspond to PeP'

Contragredient of Jacquet Functor: Let (π, V) be an admissible representation of G. Let $(\check{\pi}, \check{V})$ be the contragredient. P be a standard parabolic and (π_P, V_P) be the Jacquet module. We also have $(\check{\pi}_P, \check{V}_P)$

Theorem 1.16. $\exists$ *non degenerate M-invariant bilinear form*

$$\langle , \rangle_P : V_P \times \check{V}_P \to \mathbb{C}$$

such that for any $v \in V$ and $\check{v} \in \check{V}$, $\exists \epsilon > 0$ such that for any $a \in A_M$ with $|\alpha|_F$ small enough for all $\alpha \in \Delta(P)$, we have the following identity

$$\langle \pi_P(a) j_P(v), \check{j}_{\bar{P}}(\check{v}) \rangle = \delta_P^{-\frac{1}{2}}(a) \langle \pi(a) v, \check{v} \rangle$$

Proof. We will prove a more general version of this in the next section.

□

Corollary 1.17. $(\check{\pi}_{\bar{P}}, \check{V}_{\bar{P}})$ *is the contragredient of* (π_P, V_P).

Definition 1.18. *Let $m \in \bar{M}_0^+$, $t \in \mathbb{R}$. Define a standard parabolic $P_{m,t} = M_{m,t} U_{m,t}$ by requiring*

$$\Delta_0^{M_{m,t}} := \{\alpha \in \Delta_0, \langle \alpha, H_0(m) \rangle \le t\}$$

Example 1.19. *In the case $G = GL_n$ and $M_0 = \{\mathrm{diag}(a_1, ..., a_n)\}$. If $m = \mathrm{diag}(a, ..., a)$ $\Delta_0^{M_{m,t}} = \Delta_0$. If $m = \mathrm{diag}(a_1, ..., a_n)$ such that $a_1 - a_2 \le 1$ and $a_i - a_{i+1} > 1$, then $\Delta_0^{M_{m,1}} = \{e_1 - e_2\}$. Then $P_{m,1}$ is the parabolic of type $\{2, 1, ..., 1\}$.*

Then, we have the following proposition

Proposition 1.20. $\forall v \in V, \check{v} \in \check{V}$, $\exists t > 0$ *such that $m \in \bar{M}_0^+$ we have the equality*

$$\delta_P^{-\frac{1}{2}} \langle \pi(m) v, \check{v} \rangle = \langle \pi_P(m) j_P(v), \check{j}_{\bar{P}}(\check{v}) \rangle$$

where $P = P_{m,t}$.

Corollary 1.21. $\forall v \in V, \check{v} \in \check{V}$, $\exists d \in \mathbb{N}, C > 0$ *such that $\forall g \in G$ we have*

$$\left| \langle \pi(g) v, \check{v} \rangle \right| \le C |g|^d$$

I.5. Let B be a fin. generated commutative algebra over $\mathbb{C}$.

Definition 1.22. *An algebraic B-family of admissible representation of G is a pair (π, V) where, V is a B-module and $\pi : G \to \mathrm{Aut}_B(V)$ such that*

1. *$\forall v \in V$, $G_v \subset G$ is open. Here obviously G_v denotes the stabilizer of v under G action.*
2. *$\forall$ open compact $H \subset G$ V^H is a finitely generated B-module.*

For such a B family we can also define its $B-$contragredient: $(\check{\pi}, \check{V})$ where $\check{V}^B$ is the smooth part of V under the B action.

If $P = MU$ is again a semi-standard parabolic then we can define the notion of Jacquet modules, as well as Parabolic induction analogously as in the previous cases.
For all $\epsilon > 0$, let $A_P(\epsilon) := \{a \in A_M \mid \left|\alpha(a)\right|_F < \epsilon, \forall \alpha \in \Delta(P)\}$. Let H be an open compact subgroup at a good position, i.e. $H = (H \cap U)(H \cap M)(H \cap \bar{U})$. If ϵ is small enough, we have the following inclusions

$$a(H \cap U)a^{-1} \subset H \cap U \tag{1.1}$$

$$a^{-1}(H \cap \bar{U})a \subset H \cap \bar{U} \tag{1.2}$$

Now for all $a \in A_M$ define

$$\varphi_a^H(g) := \begin{cases} 0, & \text{if, } g \notin HaH \\ \delta_P^{\frac{1}{2}}(a)\text{meas}(H)^{-1} & \text{otherwise} \end{cases}$$

Here are some basic properties of φ_a^H:

Lemma 1.23. $\varphi_a^H * \varphi_{a'}^H = \varphi_{aa'}^H$

Lemma 1.24. $\forall v \in V^H$ $a \in A_P(\epsilon)$ *we have*

$$j_P(\pi(\varphi_a^H)v) = \pi_P(a)j_P(v)$$

Proof. Since $v \in V^H$ $HaH = (H \cap U)aH$. Then

$$\pi(\varphi_a^H)v = \delta_P^{\frac{1}{2}}(a)\text{meas}(H)^{-1} \int_{H \cap U} \pi(ua)v \; du \frac{\text{meas}(HaH)}{\text{meas}(H \cap U)}$$

$$\begin{aligned} \implies j_P(\pi(\varphi_a^H)v) &= \delta_P^{\frac{1}{2}}(a)\delta_P^{\frac{1}{2}}(a)\pi_P(a)j_P(v)\text{meas}(HaU) \\ &= \pi_P(a)j_P(v) \end{aligned}$$

□

2. Bounds:

II.1: Spherical Integrals and its Bound: Let $(1, \mathbb{C})$ denote the trivial representation of M_0. Let $(\pi, V) = (I_{P_0}^G 1, I_{P_0}^G \mathbb{C})$ and e is the unique K invariant element of V, such that $e(1) = 1$. for $g \in G$, we define

$$\Xi(g) = \langle \pi(g)e, e \rangle$$

Note that $(\check{\pi}, \check{V}) \cong (\pi, V)$. This is because $\mathbb{C} \cong \check{\mathbb{C}}$ where we send $c \in \mathbb{C}$ to the dual element $c \mapsto cx$ inside $\check{\mathbb{C}}$. Then $I_{P_0}^G \mathbb{C} \cong I_{P_0}^G \check{\mathbb{C}} \cong \widetilde{I_{P_0}^G \mathbb{C}}$ We have the following equality,

$$\Xi(g) = \int_K \delta_0(m_0(kg))^{\frac{1}{2}} \; dk$$

It is not difficult to see the above equality. In fact, note that $G = P_0 K$, and recall that u_0, m_0, k_0 denotes the component one gets after performing the Iwasawa decomposition of

$g = u_0(g)m_0(g)k_0(g)$. Then

$$\begin{aligned}\Xi(g) &= \langle \pi(g)e, e\rangle \\ &= \int_K e(k)e(kg)\ dk \\ &= \int_K e\left(u_0(kg)m_0(kg)k_0(kg)\right)\ dk; \text{ (note that } e \text{ is right } K \text{ invariant and } e(1)=1) \\ &= \int_K \delta_0(m_0(kg))^{\frac{1}{2}}\end{aligned}$$

Before beginning with the next lemma we remind you of the definition of $\bar{M}_0^+$. In fact, we would like to refer the reader to *I.1*, Example 1.4 and 1.3.

Lemma 2.1. *The function Ξ is K bi-invariant. there exists $C_1, C_2 > 0$ and $d \in \mathbb{N}$, such that for all $m \in \bar{M}_0^+$, we have*

$$C_1\delta_0(m)^{\frac{1}{2}} \leq \Xi(m) \leq C_2\delta_0(m)^{\frac{1}{2}}(1+\sigma(m))^d$$

Proof. The first assertion is clear. Let $t > 0$ such that the Proposition 1.20 (Prop I.4.3) holds for the pair (e, e). Fix this t. For any standard parabolic $P = MU$, let $\bar{M}_0^+(P) = \{m \in \bar{M}_0^+; P_{m,t} = P\}$. To demonstrate the second inequality, we can fix P and suppose $m \in \bar{M}_0^+(P)$. There exists a compact subset C of M_0 such that $\bar{M}_0^+(P) \subset A_M C$. To see this, we provide an example. Assume M_0 to be the maximal torus of GL_4 and P is the parabolic subgroup of type $2, 2$. Then, one can note that, $\bar{M_0}^+(P) = \{m = \mathrm{diag}(a_1, ..., a_4) \in \bar{M_0}^+ : \mathrm{ord}(a_1) - \mathrm{ord}(a_2) \leq t, \mathrm{ord}(a_3) - \mathrm{ord}(a_4) \leq t\}$. Then clearly $\bar{M_0}^+(P)$ consists of elements of the form $\begin{pmatrix} \alpha\varpi^d\mathcal{O}_F^\times & & & \\ & \alpha & & \\ & & \beta\varpi^{d'}\mathcal{O}_F^\times & \\ & & & \beta \end{pmatrix}$ where $d, d' \leq t-1$.

We can therefore fix $h \in C$, and assume that $m \in \bar{M}_0^+(P) \cap A_M h$, and hence $m = ah$. We know how to calculate π_P for example, see I.3. Due to the Geometric Lemma we see that there exists an integer $d \in \mathbb{N}$ such that for all $a \in A_M$, $(\pi_P(a) - 1)^d$ annihilates V_P. This follows from the decomposition we encountered at the beginning of section I.3 of the paper. Let's explain this. First of all note that for each $v \in V_P \cong \check{V}_P$, we can consider the smooth function on A_M as follows:

$$\begin{aligned}F_v : A_M &\to \mathbb{C} \\ a &\mapsto \langle \pi_P(ah) j_P(v), \check{j}_{\bar{P}}(v)\rangle_P\end{aligned}$$

So we get a map from $V_P \to \mathbb{C}^\infty(A_M)$. Also note that for any such F_v in the image, $\rho(b)F_v = F_v$ for all $b \in A_M$. In particular, we can consider the function F_e spanned by $\rho(a)F_e = F_e$ for all a. This is a finite-dimensional subspace of $\mathbb{C}^\infty(A_M)$. Therefore there exists a polynomial Q on a_M of degree $\leq d$ such that

$$\langle \pi_P(ah) j_P(e), \check{j}_{\bar{P}}(e)\rangle_P = Q(H_M(a))$$

for all $a \in A_M$. From this equality and Proposition I.4.3. the upper bound on $m \in \bar{M}_0^+(P) \cap A_M h$ follows.

Let H be a compact open subgroup of K such that $H = (H \cap U_0)(H \cap M_0)(H \cap \bar{U}_0)$. For $m \in \bar{M}_0^+$ we have $m^{-1}(H \cap \bar{U}_0)m \subset K$, and hence $Hm \subset U_0 m K$ and $\delta_0(m_0(hm)) = \delta_0(m)$ for all $h \in H$. Then

$$\Xi(m) \geq \int_H \delta_0(m_0(hm))^{\frac{1}{2}}\ dh = \text{meas}(H)\delta_0(m)^{\frac{1}{2}}$$

□

Lemma 2.2. *There exists $d \in \mathbb{N}$ and for all $g_1, g_2 \in G$ there exists $C > 0$ such that for all $g \in G$ we have*

$$\Xi(g_1 g g_2) \leq C\Xi(g)(1 + \sigma(g))^d$$

Proof. Let $k_1, k_2 \in K$. for $g \in G$ we have

$$\Xi(g_1 k_1 g k_2 g_2) = \langle \pi(g) v_2, v_1 \rangle,$$

where $v_2 = \pi(k_2 g_2)e,\ v_1 = \check{\pi}(k_1^{-1} g_1^{-1})e$. The same reasoning as in the previous proof proves the existence of $d \in \mathbb{N}$, and $C > 0$ such that

$$\Xi(g_1 k_1 m k_2 g_2) \leq C\delta_0(m)^{\frac{1}{2}}(1 + \sigma(m))^d$$

for all $m \in \bar{M}_0^+$. We can choose d independently of g_1, g_2, k_1, k_2 and we can choose C independent of k_1, k_2. Replacing $k_1 m k_2$ by g in the above inequality, we obtain

$$\Xi(g_1 g g_2) \leq CC_1^{-1}\Xi(g)(1 + \sigma(g))^d$$

where C_1, is the constant from the preceding Lemma. It remains to apply I.1(4).

□

Lemma 2.3. *For all $g_1, g_2 \in G$ we have*

$$\int_K \Xi(g_1 k g_2)\ dk = \Xi(g_1)\Xi(g_2)$$

Proof. Let $v = \int_K \pi(kg_2)e\ dk$. Since v is invariant by K, it is proportional to e. We calculate the proportionality factor by computing $\langle v, e \rangle$.

We have $\langle v, e \rangle = \int_K \langle \pi(kg_2)e, e \rangle\ dk = \int_K \langle \pi(g_2)e, e \rangle\ dk = \langle \pi(g_2)e, e \rangle = \Xi(g_2)$. Thus, we obtain $v = \Xi(g_2)e$. The LHS of the equation is equal to the $\langle \pi(g_1)v, e \rangle$, i.e. $\Xi(g_2)\langle \pi(g_1)e, e \rangle$, i.e. $\Xi(g_1)\Xi(g_2)$.

□

Lemma 2.4. *For any $g \in G$, $\Xi(g) = \Xi(g^{-1})$.*

Proof. The above lemma $\pi = \check{\pi}$. □

Lemma 2.5. *There exists $d \in \mathbb{N}$ such that the integral*

$$\int_G \Xi(g)^2 (1 + \sigma(g))^{-d}\ dg$$

is convergent.

Proof. From I.1.(4) it suffices to prove that there exists $d \in \mathbb{N}$ such that the series

$$\sum_{m \in \bar{M}_0^+/M_0^1} \text{meas}(KmK)\Xi(m)^2(1 + \sigma(m))^{-d}$$

is convergent. Due to I.1.(5) and Lemma II.1.1, we can prove that the series

$$\sum_{m \in \bar{M}_0^+/M_0^1} (1+\sigma(m))^{-d}$$

is convergent for d large enough. Or according to I.1.(6) the series

$$\sum_{H \in H_0(\bar{M}_0^+)} (1+|H|)^{-d}$$

This is clear since $H_0(M_0)$ is a lattice of a_0.

□

Note that Ξ is independent of the choice of the minimal parabolic P_0 (they are all conjugate under K). Let $P = MU$ be a semi-standard parabolic subgroup of G. We define a function Ξ^M on M we have just defined Ξ on G.

More precisely, we have that $V^M = \mathrm{Ind}_{P_0 \cap M}^M \mathbb{C}$. Recall that $V = I_{P_0}^G(\mathbb{C}) \cong I_P^G\left(I_{P_0\cap M}^M(\mathbb{C})\right) = I_P^G(V^M)$.

Lemma 2.6. *For all $g \in G$ we have the equality*

$$\Xi(g) = \int_K \delta_P(m_P(kg))^{\frac{1}{2}} \Xi^M(m_P(kg))\ dk$$

Proof. Even if we change P_0 we can assume $P \supset P_0$. Let us denote (π^M, V^M), e^M, the analogues of $(\pi, V), e$ for M. Let us identify V with $I_P^G V^M$. Then e is identified with the element of $I_P^G V^M$ such that $e(k) = e^M$, for all $k \in K$. We have

$$\Xi(g) = \langle \pi(g)e, e\rangle = \int_{P\backslash G} \langle e(hg), e(h)\rangle\ dh = \int_K \langle e(kg), e(k)\rangle\ dk$$

But, for $k \in K$,

$$\begin{aligned}\langle e(kg), e(k)\rangle &= \delta_P(m_P(kg))^{\frac{1}{2}} \langle \pi^M(m_P(kg))e^M, e^M\rangle \\ &= \delta_P(m_P(kg))^{\frac{1}{2}} \Xi^M(m_P(kg))\end{aligned}$$

□

II.2: A Special Finite Dimensional Representation: Let $P = MU$ be a standard parabolic subgroup of G. There exists a finite-dimensional space E_P over F., a basis $(e_i)_{i=1,\dots,n}$ of E_P and an algebraic representation τ_P of G in E_P satisfying the following properties:

(1) For all $m \in M, u \in U$, $\tau_P(mu)e_1 = \delta_P^{alg}(m)e_1$, where $\delta_P^{alg}(m)$ is the determinant of the adjoint action $\mathrm{Ad}(m)$ in the Lie algebra of U (we have $\delta_P(m) = \left|\delta_P^{alg}(m)\right|_F$);

(2) for all $i = 1, ..., n$ there exists $\chi_i \in \mathrm{Rat}(A_0)$ such that

$$\tau_P(a)e_i = \delta_P^{alg}(a)\chi_i(a)e_i$$

for all $a \in A_0$.

(3) if $i \geq 2$, there exists $\alpha_i \in \Delta_0 - \Delta_0^M$ such that $-\mathrm{Re}(\chi) \in C_i\alpha_i + {}^+\bar{a}_0{}^{G*}$

(4) The maps

$$\bar{U} \to E_P, \quad \bar{u} \mapsto \tau_P(\bar{u})e_i$$

is injective, with values in $e_1 + E'_p$ where E'_P is generated by $e_2, ..., e_n$

Indeed, let's denote $F[G]$ denote the space of polynomials on G defined on F and E_P is the subspace consisting of $Q \in F[G]$ such that

$$Q(m\bar{u}g) = \delta_P^{alg}(m)Q(g)$$

for all $m \in M, \bar{u} \in \bar{U}, g \in G$. Let τ_P be the representations of G on E_P by translation on the right. Then the pair (τ_P, E_P) satisfies the above conditions.

With these data fixed, we define a height on E_P by

$$\|e\| = \sup\left\{|x_i|_F : i = 1, ..., n\right\}$$

for all $e = \sum_{i=1}^n x_i e_i \in E$. Thus we have the following:

(5) There exists $C_1, C_2 > 0$ such that for all $k \in K$, $e \in E$,

$$C_1\|e\| \le \left\|\tau_P(k)e\right\| \le C_2\|e\|$$

We also have

(6) There exists $C_1, C_2 > 0$, such that for all $\bar{u} \in \bar{U}$,

$$C_1(1 + \log\left\|\tau_P(\bar{u})e_1\right\|) \le 1 + \sigma(\bar{u}) \le C_2(1 + \log\left\|\tau_P(\bar{u})e_1\right\|)$$

Proof. Since the coefficients of $\tau_P(\bar{u})e$ are polynomials on $\bar{U}$, there exists $C > 0$ and an integer $N \ge 0$, such that

$$\left\|\tau_P(\bar{u})e_1\right\| \le C\|\bar{u}\|^N$$

for all $\bar{u} \in \bar{U}$.
Let θ be the function defined in (4). Its image is an orbit of an unipotent group acting on an affine variety. It is therefore an algebraic subset. On the other hand, θ is injective. If the characteristic of F is 0, then θ is an isomorphism of $\bar{U}$ onto its image. For any polynomial Q on $\bar{U}$, there exists a polynomial Q' on E such that $Q = Q' \circ \theta$. If the characteristic of F is $p > 0$ the set of rational functions on $\bar{U}$ is a purely inseparable extension of the set of rational functions on the image of θ. there exists a polynomial Q' on E such that $Q^{p^d} = Q' \circ \theta$. In both cases, we can deduce that for any polynomial Q on U there exists $C > 0$ and a natural number $N \ge 0$ such that $\left|Q(\bar{u})\right|_F \le C\left\|\tau_P(\bar{u})e_1\right\|^N$ for all $\bar{u} \in \bar{U}$. There are C and N such that

$$\|\bar{u}\| \le C\left\|\tau_P(\bar{u})e_1\right\|^N$$

for all $\bar{u} \in \bar{U}$. Note that we always have $\sigma(\bar{u}) \ge 0$ and $\log\left\|\tau_P(\bar{u})e_1\right\| \ge 0$ from (4). Inequalities (7), and (8) then imply the assertion.

□

II.3: Application of the Representation Datum:

Lemma 2.7. *For a semi-standard parabolic subgroup $P = MU$, there exists $C_1, C_2 > 0$ such that for all $m \in M, u \in U$ we have*

$$C_1(1 + \sigma(mu)) \le 1 + \sup(\sigma(m), \sigma(u)) \le C_2(1 + \sigma(mu))$$

Proof. The first inequality follows form $\sigma(mu) \leq \sigma(m) + \sigma(u)$. Fix $a \in A_M$ such that $\left|\alpha(a)\right| < 1$, for all $\alpha \in \Delta(P)$. Since A_M is a torus, $\tau(a)$ is diagonalizable for any $a \in A_M$. Recall that τ is an embedding of G into GL_n. WLOG we can assume that $\tau(a)$ is diagonal and let $(a_i)_{i=1}^n$ denote the matrix $\tau(a)$, and we have $|a_1| \leq ... \leq |a_n|$. Define a decomposition of n such that

$$\begin{aligned} |a_1| = ... = |a_{n_1}| < |a_{n_1+1}| = ... \\ = |a_{n_1+n_2}| < ... < \left|a_{n_1+...+n_{t-1}+1}\right| = ... = |a_n| \end{aligned}$$

Let $P' = M'U'$ be the subgroup of parabolic of type $n_1, ..., n_t$. Since M commutes with $a, \tau(M) \subset M'$. Since $\mathrm{Ad}(a)$ restricts to U, we have $\tau(U) \subset U'$. But then for all $m \in M, u \in U$ the non-zero coefficient of $\tau(m)$ are coefficients of $\tau(mu)$. this leads to the inequality

$$\sigma(m) \leq \sigma(mu)$$

If $\sigma(m) \geq \frac{\sigma(u)}{2}$, we obtain

$$\sup(\sigma(m), \sigma(u)) \leq 2\sigma(mu)$$

If $\sigma(m) \leq \frac{\sigma(u)}{2}$, we use the relations $\sigma(u) \leq \sigma(m^{-1}) + \sigma(mu)$, and $\sigma(m) = \sigma(m^{-1})$. This leads to the inequality.

□

Lemma 2.8. *There exists $C > 0$, such that for all $m \in \bar{M_0}^+$, $g \in KmK$ such that we have the inequality*

$$\delta_0(m) \leq C\delta_0(m_0(g))$$

Proof. Let Γ be a compact subset of M_0 such that $\bar{M_0}^+ \subset \Gamma\bar{A_0}^+$. Let's introduce the representation $\tau_0 = \tau_{P_0}$ from II.2. Chose $C_1 > 0$ such that for all $e \in E_{P_0}, h \in K \cup K\Sigma$ such that $\left\|\tau_0(h^{-1})e\right\| \leq C_1\|e\|$. Let $m \in \bar{M_0}^+$ and $g \in KmK$. Introduce $a \in \bar{A_0}^+, h \in K\Gamma$ and $k \in K$ such that $ma^{-1} \in \Gamma$ and $u_0(g)m_0(g) = hak$. We have

$$\begin{aligned} \delta_0(m_0(g))^{-1} &= \left\|\tau_0(m_0(g)^{-1}u_0(g)^{-1}e_1)\right\| \\ &= \left\|\tau_0(k^{-1}a^{-1}h^{-1}e_1)\right\| \\ &\leq C_1\left\|\tau_0(a^{-1}h^{-1})e_1\right\| \end{aligned}$$

Let's define

$$\tau_0(h^{-1})e_1 = \sum_{i=1}^n x_i e_i$$

So,

$$\tau_0(a^{-1}h^{-1})e_1 = \delta_0^{alg}(a)^{-1}\sum_{i=1}^n x_i\chi_i(a)^{-1}e_i$$

Since, $a \in \bar{A_0}^+$, we have $\left|\chi_i(a)\right|^{-1} \leq 1$ for all i. Therefore,

$$\begin{aligned} \left\|\tau_0(a^{-1}h^{-1})e_1\right\| &= \delta_0(a)^{-1}\sup\left\{\left|x_i\chi_i(a)^{-1}\right| :\ i = 1, ..., n\right\} && (2.1) \\ &\leq \delta_0(a)^{-1}\sup\left\{|x_i| :\ i = 1, ..., n\right\} && (2.2) \\ &\leq \delta_0(a)^{-1}\left\|\tau_0(h)^{-1}e_1\right\| && (2.3) \\ &\leq C_1\delta_0(a)^{-1} && (2.4) \end{aligned}$$

Hence we have inequality

$$\delta_0(m_0(g))^{-1} \leq C_1^2 \delta_0(a)^{-1}$$

Since $a^{-1} \in m^{-1}\Gamma$, we have $\delta_0(a)^{-1} \leq \delta_0(m)^{-1} \sup \{\delta_0(h) : h \in \Gamma\}$.

□

Lemma 2.9. *There exists $C > 0$ such that for all $a \in \bar{A_0}^+, p \in P_0$*

$$1 + \sigma(apa^{-1}) \leq C(1 + \sigma(p))$$

Proof. By inverting the roles of P_0 and $\bar{P}_0$, it suffices to prove that for all $ain\bar{A_0}^+$ ad $\bar{p} \in \bar{P}_0$,

$$1 + \sigma(a^{-1}\bar{p}a) \leq C(1 + \sigma(\bar{p}))$$

Let $m \in M_0, u \in \bar{U}_0, a \in \bar{A_0}^+, \bar{p} = m\bar{u}$. We have $a^{-1}\bar{p}a = ma^{-1}\bar{u}a$, and hence

$$\sigma(a^{-1}\bar{p}a) \leq \sigma(m) + \sigma(a^{-1}\bar{u}a)$$

Let's introduce the representations $\tau_0 = \tau_{P_0}$ as in II.2. From II.2.(6) we have

$$1 + \sigma(a^{-1}\bar{u}a) \leq C_1(1 + \log\|\tau_0(a^{-1}\bar{u}a)e_1\|)$$

where C_1 like the other constants as in II.2 is independent of $m, \bar{u}, a$. Write,

$$\tau_0(\bar{u})e_1 = \sum_{i=1}^{n} x_i e_i$$

So,

$$\tau_0(a^{-1}\bar{u}a)e_1 = \sum_{i=1}^{n} \chi_i(a)^{-1} x_i e_i$$

For all $i \leq n$, we have

$$|\chi^{-1}(a)| = q^{\langle \mathrm{Re}\chi, H_0(a) \rangle}$$

Since $H_0(a) \in \bar{a_0}^+$, and $\mathrm{Re}\chi_i \in -{}^+\bar{a_0}^{G*}$ (II.2 (1) and (3)), we have $|\chi_i^{-1}(a)| \leq 1$. we have the inequality

$$\|\tau_0(a^{-1}\bar{u}a)e_1\| \leq \|\tau_0(\bar{u})e_1\|$$

According to II.2(6), $1 + \log\|\tau_0(\bar{u})e_1\| \leq C_2(1 + \sigma(\bar{u}))$. Then

$$1 + \sigma(a^{-1}\bar{u}a) \leq C_1 C_2(1 + \sigma(\bar{u})),$$

and hence,

$$1 + \sigma(a^{-1}\bar{p}a) \leq C_3(1 + \sigma(m) + \sigma(\bar{u}))$$

Now use Lemma II.3.1

□

Lemma 2.10. *Let $P = MU$ be a semi-standard parabolic subgroup of G. Then there exists constants $C_0, ..., C_3$ such that for all $\bar{u} \in \bar{U}$, we have*

$$C_1(1 + \sigma(\bar{u})) \leq C_0 - \log \delta_P(m_P(\bar{u})) \leq C_2(1 + \sigma(m_P(\bar{u}))) \leq C_2(1 + \sigma(\bar{u}))$$

Proof. Since δ_P^{alg} see II.2.(1) is a polynomial on M, there exists $C_4 > 0$ such that

$$-\log \delta_P(m) \leq C_4(1 + \sigma(m))$$

for all $m \in M$. We deduce the central inequality of the statement. For all $\bar{u} \in \bar{U}$, we have

$$1 + \sigma(m_P(\bar{u})) \leq C_5(1 + \sigma(u_P(\bar{u})m_P(\bar{u})))$$

from Lemma II.3.1, and

$$\sigma(u_P(\bar{u})m_P(\bar{u})) = \sigma(u_P(\bar{u}))m_P(\bar{u})k_P(\bar{u}) = \sigma(\bar{u})$$

This leads to the right hand side of the inequality. If we want to conjugate P, we can assume that P is standard. We define the representation τ_P of II.2. Then, due to II.2.(6)

$$1 + \sigma(\bar{u}) = 1 + \sigma(\bar{u}^{-1}) \leq C_6(1 + \log\|\tau_P(\bar{u}^{-1})e_1\|)$$

But,

$$\tau_P(\bar{u}^{-1})e_1 = \tau_P(k_P(\bar{u})^{-1}m_P(\bar{u})^{-1}u_P(\bar{u})^{-1}) = \delta_P^{alg}(m_P(\bar{u})^{-1})\tau_P(k_P(\bar{u})^{-1})e_1$$

Hence from ii.2.(5) we have

$$\|\tau_P(\bar{u}^{-1})e_1\| \leq C_7\delta_P(m_P(\bar{u}))^{-1}$$

and

$$1 + \log\|\tau_P(\bar{u}^{-1})e_1\| \leq C_8 - \log\delta_P(m_P(\bar{u}))$$

This proves the first inequality.

□

Lemma 2.11. *Let $P = MU$ be a maximal standard parabolic subgroup of G, $\alpha \in \Delta_0$ such that $\Delta_0^M = \Delta_0 - \{\alpha\}$. $\Gamma, \bar{\Gamma}$ resp. be open compact subgroup of U and $\bar{U}$. Then there exists $C > 0$ such that for all $a \in \bar{A_0}^+, g \in G$ if*

$$1 + \sigma(g) + \sigma(aga^{-1}) \leq C\langle\alpha, H_0(\alpha)\rangle$$

then $g \in a^{-1}\Gamma a M\bar{\Gamma}$.

Proof. Let us show that

(2) there exists $C_1 > 0$ such that for all $a \in \bar{A_0}^+, g \in K \cap \bar{U}P$, if

$$1 + \sigma(aga^{-1}) \leq C_1\langle\alpha, H_0(\alpha)\rangle$$

then $g \in \bar{\Gamma}P$.

We introduce the representation τ_P from II.2, and constants $C - 2, ..., C_5 > 0$ such that

1. for all $g \in G, \log\|\tau_P(g)e_1\| \leq C_2(1 + \sigma(g))$;
2. for all $g \in K$, $C_3 \leq \log\|\tau_P(g)e_1\|$;
3. for all $i \geq 2, a \in \bar{A_0}^+, C_4\langle\alpha, H_0(\alpha)\rangle \leq -\langle\mathrm{Re}(\chi_i), H_0(a)\rangle$.
4. for all $\bar{u} \in \bar{U}$ if we write $\tau_P(\bar{u})e_1 = \sum_{i=1}^n y_ie_i$. then

$$\sup\left\{\log|y_i| : 2 \leq i \leq n\right\} \leq -C_5 \implies \bar{u} \in \bar{\Gamma}$$

The existence of C_5 is because of the fact that the assignment θ in the proof of II.2.(6) is a homomorphism of $\bar{U}$ onto its image. Let's Define

$$C_1 = C_4(C_2 + C_3 + C_5)^{-1}\log q$$

Let a and g satisfy the assumptions of (2). Let's write $\tau_P(g)e_1 = \sum_{i=1}^n x_ie_i$. Then

$$\tau_P(aga^{-1})e_1 = x_1e_1\sum_{i=2}^n x_i\chi_i(a)e_i$$

For $2 \leq i \leq n$ we have

$$\begin{aligned}\log|x_i| + C_4 \log q \langle \alpha, H_0(\alpha)\rangle &\leq \log|x_i| - \langle \mathrm{Re}(\chi_i), H_0(a)\rangle \log q \\ &= \log\big|x_i \chi_i(a)\big| \\ &\leq \log\big\|\tau_P(aga^{-1})e_1\big\| \\ &\leq C_2(1+\sigma(aga^{-1})) \\ &\leq C_1 C_2 \langle \alpha, H_0(a)\rangle\end{aligned}$$

Hence,

$$\log|x_i| \leq (C_1C_2 - C_4 \log q)\langle \alpha, H_0(a)\rangle$$

Note that $C_1C_2 - C_4 \log q < 0$ and $\langle \alpha, H_0(\alpha)\rangle \geq C_1^{-1}$ according to the hypothesis. Hence,

$$\log|x_i| \leq C_2 - C_4 C_!^{-1} \log q = -C_3 - C_5$$

for all $2 \leq i \leq n$. On the other hand,

$$C_3 \leq \sup\big\{\log|x_i| : 1 \leq i \leq n\big\}$$

So,

$$C_3 \leq \log|x_1|$$

Let's write $g = \bar{u}mu$, with $\bar{u} \in \bar{U}, m \in M$ and $\tau_P(\bar{u})e_1 = \sum_{i=1}^n y_i e_i$. We have $x_i = y_i \delta_P^{alg}(m)$, for all i. For $2 \leq i \leq n$, we hence have

$$\log|y_i| = \log|y_i| - \log|y_1| = \log|x_i| - \log|x_1| \leq -C_5.$$

So $\bar{u} \in \bar{\Gamma}$, this proves (2).

Let's now prove the following:

(3)There exists $C_6 > 0$ such that for all $a \in \bar{A_0}^+, g \in G$, if

$$1 + \sigma(g)\sigma(aga^{-1}) \leq C_6 \langle \alpha, H_0(a)\rangle,$$

then $g \in \bar{\Gamma}P$.

Let's denote the constant C_7 in Lemma II.3.3 (we have $C_7 \geq 1$), let $C_6 = C_1 C_7^{-1}$. We write $g = kp$ with $k \in K, p \in P$. We have $aka^{-1} = aga^{-1}apa^{-1}$, hence

$$\begin{aligned}1 + \sigma(aka^{-1}) &\leq 1 + \sigma(aga^{-1}) + \sigma(apa^{-1}) \\ &\leq C_7(1 + \sigma(aga^{-1}) + \sigma(p)) \\ &= C_7(1 + \sigma(aga^{-1}) + \sigma(g)) \\ &\leq C_1\langle \alpha, H_0(a)\rangle\end{aligned}$$

Assume first that $g \in \bar{U}P$. Then $k \in \bar{U}P$, and so, according to (2), $k \in \bar{\Gamma}P$. In the general case, since $\bar{U}P$ is dense in G, we can choose $g' \in \bar{U}P$ as close as we like to g, and we shall prove the inequality for g'. Then $g' \in \bar{\Gamma}P$. So, g belongs to the closure of $\bar{\Gamma}P$. The latter being closed, we obtain $g \in \bar{\Gamma}P$.

By replacing P_0, with P and $\bar{P}_0$ with $\bar{P}$, we also see that there exists $C_8 > 0$ such that, for all $a \in \bar{A_0}^+, g \in G$, if

$$1 + \sigma(g) + \sigma(a^{-1}ga) \leq C_8\langle \alpha, H_0(a)\rangle,$$

then $g \in \Gamma\bar{P}$. Now, let $C = \inf(C_6, C_8)$. Let g and a satisfy the conditions of the statement. Let's apply the above relation and the resp. the relation (3) on aga^{-1} and g^{-1}. We obtain $aga^{-1} \in \Gamma\bar{P}, g^{-1} \in \bar{\Gamma}P$, i.e. $g \in a^{-1}\Gamma a\bar{P} \cap P\bar{\Gamma}$. But, $a^{-1}\Gamma a\bar{P} \cap P\bar{\Gamma} = a^{-1}\Gamma aM\bar{\Gamma}$. This completes the proof.

□

II.4:

Let $P = MU$ be a standard parabolic subgroup of G. For any $n \in \mathbb{N}$, define

$$\bar{U}(n) = \{\bar{u} \in \bar{U} : \delta_0(m_0(\bar{u})) = q^{-n}\}$$

Lemma 2.12. *(II.4.1) There exits $C > 0$ and $d \in \mathbb{N}$, such that for all $n \geq 1$*

$$\text{meas}(U(\bar{n})) \leq cq^{\frac{n}{2}}n^d$$

Proof. For all $a \in A_M$, let $\gamma(a) := \sup\Big\{\big|\alpha(a)\big| : \alpha \in \Delta(P)\Big\}$. Let's show that

(1) there exists $C_1 > 0$ and $r > 0$ such that for all $\bar{u} \in \bar{U}, a \in A_M \cap \bar{A_0}^+$,

$$\delta_0(m_0(a^{-1}\bar{u}a))^{-\frac{1}{2}} \leq C_1[1 + \gamma(a)^r\delta_0(m_0(\bar{u}))^{-\frac{1}{2}}]$$

Let τ_0 be the representation from II.2. Let $\bar{u} \in \bar{U}, a \in A_M$. Let

$$\tau_0(\bar{u}^{-1})e_1 = e_1 + \sum_{i=2}^{n} x_i e_i$$

So,

$$\tau_0(a^{-1}\bar{u}^{-1}a)e_1 = e_1 + \sum_{i=2}^{n} x_i\chi_i(a)^{-1}e_i$$

For $i \in \{2, ..., n\}$, we denote λ_i to be the projection of $-\text{Re}(\chi_i)$ on a_M^*. Then $\big|\chi_i(a)^{-1}\big| = q^{-\langle\lambda_i, H_M(a)\rangle}$. According to II.2.(3) λ_i is a linear combination of elements of $\Delta(P)$ with coefficients ≥ 0. We say that if $x_i \neq 0$, we have $\lambda_i \neq 0$.
Indeed, if we choose a such that $\gamma(a) < 1$, the conjugation by a^{-1}, contracts $\bar{U}$, and hence $\lim_{m\to\infty} a^{-m}\bar{u}^{-1}a^m = 1$. Then $\lim_{m\to\infty}\big|x_i\chi_i(a)^{-m}\big| = 0$ for all $i \geq 2$, and this implies the assertion.

Then there exists $r > 0$, independent of $\bar{u}, a$ such that $\big|\chi_i(a)^{-1}\big| \leq \gamma(a)^{2r}$. We deduce that

$$\begin{aligned}\big\|\tau_0(a^{-1}\bar{u}a)e_1\big\| &\leq \sup\big(1, \gamma(a)^{2r}\{\,|x_i| : 2 \leq i \leq n\}\big)\\ &\leq \sup\Big(1, \gamma(a)^{2r}\big\|\tau_0(\bar{u}^{-1})e_1\big\|\Big)\end{aligned}$$

Since,

$$\begin{aligned}\tau_0(\bar{u}^{-1})e_1 &= \tau_0(k_0(\bar{u}^{-1}))\tau_0(m_0(\bar{u}))^{-1}\tau_0(u_0(\bar{u}))^{-1}e_1\\ &= \delta_0^{alg}(m_0(u))^{-1}\tau_0(k_0(u)^{-1})e_1\end{aligned}$$

we have,

$$\big\|\tau_0(\bar{u}^{-1})e_1\big\| \leq C_2\delta_0(m_0(\bar{u}))^{-1}$$

where C_2 is independent of $\bar{u}$. Also,

$$\delta_0(m_0(a^{-1}\bar{u}a))^{-1} \leq \left\|\tau_0(a^{-1}\bar{u}^{-1}a)e_1\right\|$$

Thus, we deduce that there exists $C_1 > 0$ such that

$$\delta_0(m_0(a^{-1}\bar{u}a))^{-1} \leq C_1^2 \sup(1, \gamma(a)^{2r}\delta_0(m_0(\bar{u}))^{-1})$$

Thus we have the required inequality as in (1).

For all $a \in A_M$, we have the equality;

$$\Xi(a) = \gamma(G \mid M)^{-1}\delta_0(a)^{\frac{1}{2}} \int_{\bar{U}} \delta_0(m_0(a^{-1}\bar{u}a)m_0(\bar{u}))^{\frac{1}{2}}\ d\bar{u} \tag{2.5}$$

Using the notations of II.1, we note that

$$\Xi(a) = \int_{P_0\backslash G} e(ga)e(g)\ dg$$

From the integration formula in I.1, we have the equality

$$\begin{aligned}\Xi(a) &= \gamma(G \mid M)^{-1} \int_{\bar{U}} \int_{P_0\cap M\backslash M} e(m\bar{u}a)e(m\bar{u})\delta_P(m)^{-1}\ dmd\bar{u} \\ &= \gamma(G \mid M)^{-1} \int_{\bar{U}} \int_{K\cap M} e(k\bar{u}a)e(k\bar{u})\ dkd\bar{u}\end{aligned}$$

By the change of variable $\bar{u} \mapsto k^{-1}\bar{u}k$, and using the fact that e is right-invariant under K and commutes with M, we obtain

$$\Xi(a) = \gamma(G \mid M)^{-1} \int_{\bar{U}} e(\bar{u}a)e(\bar{u})\ d\bar{u}$$

Due to the definition $e(\bar{u}) = \delta_0(m_0(\bar{u}))^{\frac{1}{2}}, e(\bar{u}a) = \delta_0(m_0(\bar{u}a))^{\frac{1}{2}} = \delta_0(a)^{\frac{1}{2}}\delta_0(m_0(a^{-1}\bar{u}a))^{\frac{1}{2}}$, and we have the equality (2).

Using (1), (2) which is Equation 2.5, Lemma II.1.1, we see that there exists $C_3 > 0, r > 0$ and $d \in \mathbb{N}$ such that for all $a \in A_M \cap \bar{A_0}^+$,

$$\int_{\bar{U}} [1 + \gamma(a)^r\delta_0(m_0(\bar{u}))^{-\frac{1}{2}}]\delta_0(m_0(\bar{u}))^{\frac{1}{2}}\ d\bar{u} \leq C_3(1 + \sigma(a))^d \tag{2.6}$$

Fix $a \in A_M \cap \bar{A_0}^+$, such that $\gamma(a) < 1$. for all integer $n \geq 1$, let m be the smallest integer such that $\gamma(a)^{rm} \leq q^{-\frac{n}{2}}$. Replace a with a^m in the inequality Equation 2.6. Note that, since the function, we are integrating on the left-hand side is positive, the integration on $\bar{U}(n)$ is also less than the right hand side,

$$\int_{\bar{U}} [1 + \gamma(a)^{rm}\delta_0(m_0(\bar{u}))^{-\frac{1}{2}}]\delta_0(m_0(\bar{u}))^{\frac{1}{2}}\ d\bar{u} \leq C_3(1 + \sigma(a^m))^d \tag{2.7}$$

From the definition of m and $\bar{U}(n)$, the left-hand side is $\geq \frac{1}{2}q^{-\frac{n}{2}}\mathrm{meas}(\bar{U}(n))$, while there exists $C_4 > 0$ such that the right hand side of this inequality is $\leq C_4 n^d$. Hence we have the inequality of the statement.

$\square$

Lemma 2.13. *(Lemma II.4.2) There exists an integer $d \in \mathbb{N}$ such that the integral*

$$\int_{\bar{U}} \delta_0(m_0(\bar{u}))(1+\sigma(\bar{u}))^{-d}\ d\bar{u}$$

is convergent.

Proof. According to Lemma II.3.4., $\bar{U}$ is the union of the compact subsets $\bar{U}(n)$, for all $n \geq 1$. We can replace LHS with the sum of integrals on $\bar{U}(n)$. Now, if we note here that $d' \in \mathbb{N}$ from Lemma 2.12, Lemma II.3.3 and this lemma implies the existence of $C > 0$ such that for all $u \geq 1$,

$$\int_{\bar{U}(n)} \delta_0(m_0(\bar{u}))(1+\sigma(\bar{u}))^{-d}\ d\bar{u} \leq cn^{d'-d}$$

It suffices that $d \geq d' + 2$ to ensure the convergence of this sum over n

□

Lemma 2.14. *(Lemma II.4.3) There exists an integer $d \in \mathbb{N}$ such that the integral*

$$\int_{\bar{U}} \delta_P(m_P(\bar{u}))^{\frac{1}{2}} \Xi^M(m_P(\bar{u}))(1+\sigma(\bar{u}))^{-d}\ d\bar{u}$$

Proof. For $k \in K \cap M$, let's do the change of variable $\bar{u} \mapsto k\bar{u}k^{-1}$ in the integral of the previous lemma and then integrate in $k \in K \cap M$. We obtain the convergence of the following integral

$$\int_{\bar{U}} \int_{K\cap M} \delta_0(m_0(k\bar{u}k^{-1}))^{\frac{1}{2}}(1+\sigma(\bar{u}))^{-d}\ d\bar{u}\ dk$$

But $m_0(k\bar{u}k^{-1}) = m_0(km_P(\bar{u}))$, hence

$$\begin{aligned}\delta_0(m_0(k\bar{u}k^{-1})) &= \delta_0^M[m_0 k m_P(\bar{u})]\delta_P[m_0(km_P(\bar{u}))] \\ &= \delta_0^M[m_0 k m_P(\bar{u})]\delta_P[m_P(\bar{u})]\end{aligned}$$

Hence the following integral is convergent

$$\int_{\bar{U}} \delta_P(m_P(\bar{u}))^{\frac{1}{2}}(1+\sigma(\bar{u}))^{-d} \int_{K\cap M} \delta_0^M[m_0(km_P(\bar{u}))]^{\frac{1}{2}}\ dk\ d\bar{u}$$

The inner integral is $\Xi^M(m_p(\bar{u}))$.

□

Lemma 2.15. *(Lemma II.4.4) There exists $d \in \mathbb{N}$ and $c > 0$ such that for any $m \in M_0$ and any $u \in \bar{U}$, we have the inequality;*

$$\Xi(m\bar{u}) \leq c\delta_0(m)^{\frac{1}{2}}\delta_0(m_0(\bar{u}))^{\frac{1}{2}}(1+\sigma(m\bar{u}))^d$$

Proof. For $m \in M_0$ and $u \in \bar{U}$, let $h \in \bar{M_0}^+$ be such that $mu \in KhK$ (see I.1(4)). From Lemma II.1.1 we have,

$$\Xi(m\bar{u}) = \Xi(h) \leq C_1\delta_0(h)^{\frac{1}{2}}(1+\sigma(h))^d$$

The integer d, the constant C_1, and the constant C_2 in the equation below are independent of m and $\bar{u}$. We have $\sigma(h) = \sigma(m\bar{u})$. From Lemma II.3.2, we have

$$\delta_0(h) \leq C_2\delta_0(m_0(m\bar{u})) = C_2\delta_0(m)\delta_0(m_0(\bar{u}))$$

□

Proposition 2.16. *(Proposition II.4.5) Let $d \in \mathbb{N}$. Then there exists $d' \in \mathbb{N}$, and $c > 0$ such that for all $m \in M$ we have the inequality*

$$\delta_P(m)^{\frac{1}{2}} \int_{\bar{U}} \Xi(m\bar{u})(1+\sigma(m\bar{u}))^{-d'}\ du \le c\Xi^M(m)(1+\sigma(m))^{-d}$$

Proof. Define

$$\bar{M_0}^{M+} = \{m \in M_0 : \forall \alpha \in \Delta_0^M,\ \langle \alpha, H_0(m)\rangle \ge 0\}$$

We have the equality $M = (K \cap M)\bar{M_0}^{M+}(K \cap M)$. We can simply prove the inequality for $m \in \bar{M_0}^{M+}$. Let $m \in \bar{M_0}^{M+}$. We denote $X(m)$ to be the LHS of this statement. Due to Lemma 2.15, there exists $C_1 > 0$ and $D \in \mathbb{N}$ such that

$$X(M) \le C_1 \delta_0(m)^{\frac{1}{2}} \delta_P(m)^{-\frac{1}{2}} \int_{\bar{U}} \delta_0(m_0(\bar{u}))^{\frac{1}{2}} (1+\sigma(m\bar{u}))^{D-d'}\ d\bar{u}$$

(the constants are independent of m). From Lemma II.3.1, we have the inequality

$$1+\sigma(m\bar{u}) \ge C_2(1+\sup(\sigma(m),\sigma(\bar{u})))$$

We also have

$$1+\sup(\sigma(m),\sigma(\bar{u})) \ge (1+\sigma(m))^{\frac{1}{2}}(1+\sigma(\bar{u}))^{\frac{1}{2}}$$

Suppose $d' \ge D$. From the above relation we deduce that

$$(1+\sigma(m\bar{u}))^{D-d'} \le C_2^{D-d'}(1+\sigma(m))^{\frac{D-d'}{2}}(1+\sigma(\bar{u}))^{\frac{D-d'}{2}}$$

On the other hand, according to the Lemma II.1.1

$$\delta_0(m)^{\frac{1}{2}}\delta_P(m)^{-\frac{1}{2}} = \delta_0^M(m)^{\frac{1}{2}} \le C_3\Xi^M(m)$$

We obtain,

$$X(m) \le C_4 \Xi^M(m)(1+\sigma(m))^{\frac{D-d'}{2}} \int_{\bar{U}} \delta_0(m_0(\bar{u}))^{\frac{1}{2}}(1+\sigma(\bar{u}))^{\frac{D-d'}{2}}\ d\bar{u}$$

By choosing d' large enough we can make the above integral convergent (due to Lemma 2.13) and $\frac{1}{2}(D-d') \le -d$. Done!

□

3. Tempered Representation

III.1: Let (π, V) be an admissible representation of G admitting a *unitary* central character. We say that π is square integrable if the matrix coefficients of π are square-integrable on $A_G \backslash G$

Proposition 3.1 (Prop III.1.1)**.** *TFAE*

(i) *π is square-integrable;*

(ii) *For any semi-standard parabolic subgroup $P = MU$ of G and for any $\chi \in \mathcal{E}xp(\pi_P)$, $\mathrm{Re}(\chi) \in ({}^+a_P^G)^*$*

(iii) *For any standard parabolic subgroup $P = MU$ of G, proper and maximal, and $\chi \in \mathcal{E}xp(\pi_P)$, $\mathrm{Re}(\chi) \in ({}^+a_P^G)^*$*

Proof. See □

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