1. Meanwhile, at the Unfair Coin Factory...

You are given a bucket that contains 1000 coins. 999 of these are fair coins, but one of them is a trick coin that always comes up heads. You select one coin from this bucket at random. Let T be the event that you select the trick coin. This means that $P(T) = 0.001$.

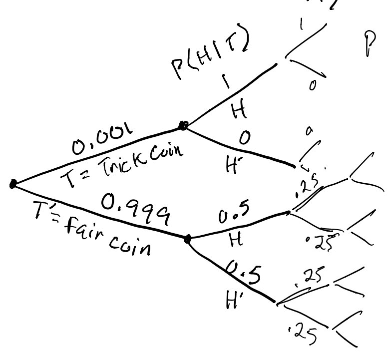
a. Suppose you flip the coin k times. Let H_k be the event that the coin comes up heads all k times. If you see this occur, what is the conditional probability that you have the trick coin? In other words, what is $P(T|H_k)$. (3 points)

b. How many heads in a row would you need to observe in order for the conditional probability that you have the trick coin to be higher than 99.9%? (3 points)

1a) $P(T) = 0.001 \qquad P(T') = 0.999$

H T

$$P(T|H_K) = \frac{P(T \cap H_K)}{P(H_K)}$$



T = Trick Coin $\qquad$ T' = Fair Coin

H = Heads $\qquad$ H' = Tails

$$P(A \cap B) = P(A) \cdot P(B)$$

$P(T) = 0.001 \qquad P(H_k|T) = 1 \qquad P(H_K'|T) = 0$

$P(T') = 0.999 \qquad P(H_k|T') = 0.5^k \qquad P(H_K'|T') = 0.5^k$

$$P(B/A) = \frac{P(A|B)\,P(B)}{P(A)}$$

$$P(T|H_K) = \frac{P(H_K|T)\,P(B)}{P(H_K)}$$

$$P(H_K|T) = \frac{1}{1000} \qquad P(T) = \frac{1}{1000} \qquad P(T') = \frac{999}{1000}$$

$P(T|H_K) =$

$P(T|H_K) =$ $\qquad P(H_K|T') = \left(\frac{1}{2}\right)^k$

Total Probability

$$P(B) = P(B|A_1)P(A_1) + \dots + P(B|A_K)P(A_K) = \sum_{i=1}^{K} P(B|A_i)P(A_i)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}$$

$$P(T|H_K) = \frac{P(H_K|T)\,P(T)}{P(H_K|T)P(T) + P(H_K|T')P(T')} = \frac{1 \cdot \frac{1}{1000}}{1 \cdot \frac{1}{1000} + \left(\frac{1}{2}\right)^K \frac{999}{1000}}$$

$$= \frac{\frac{1}{1000}}{\frac{1}{1000} + \left(\frac{1}{2}\right)^K \frac{999}{1000}} = \frac{1}{1000\left(\frac{1}{1000} + \left(\frac{1}{2}\right)^K \frac{999}{1000}\right)} = \frac{1}{1 + 999\left(\frac{1}{2}\right)^K} \cdot \frac{2^K}{2^K}$$

$$\boxed{P(T|H_k) = \frac{2^k}{2^k + 999}}$$

1b) $\frac{2^k}{2^k+999} > 0.999 \rightarrow 2^k > 0.999(2^k + 999) \rightarrow 1000 \cdot 2^k > 999(2^k+999)$

$\rightarrow 1000 \cdot 2^k > 999 \cdot 2^k + 998001 \rightarrow 1000 \cdot 2^k - 999 \cdot 2^k > 998001$

$\rightarrow 2^k > 998001 \rightarrow k \ln(2) \geq \ln(998001) \rightarrow k > \frac{\ln(998001)}{\ln(2)}$

$k > 19.93$

$\boxed{\text{20 heads in a row}}$

2. Broken Rulers

You have a ruler of length 4 and you choose a place to break it using a uniform probability distribution. Let random variable X represent the length of the left piece of the ruler. X is distributed uniformly in $[0, 4]$. You take the left piece of the ruler and once again choose a place to break it using a uniform probability distribution. Let random variable Y be the length of the left piece from the second break.

a. Draw a picture of the region in the X-Y plane for which the joint density of X and Y is non-zero. (3 points)

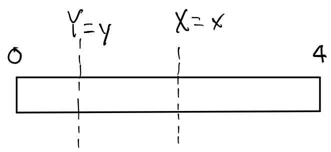
b. Find the conditional expectation of Y given X, $E(Y|X)$. (3 points)

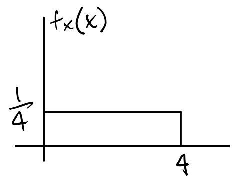
c. Find the unconditional expectation of Y. (3 points)

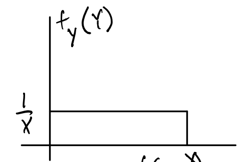
d.* Give a complete expression for the conditional distribution of X, conditional on Y. (3 points)

e.* Compute $cov(X, Y)$. (3 points)

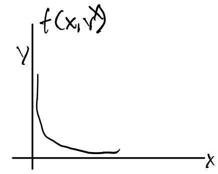
2a)







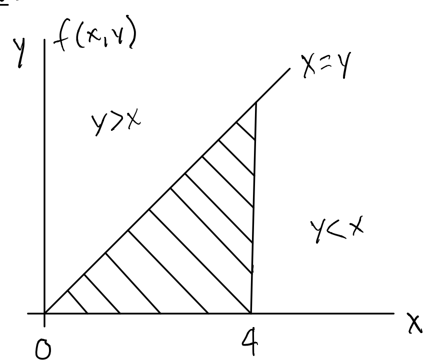
$$f_X(x) = \begin{cases} \frac{1}{4} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases} \qquad f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$



$$f(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \frac{1}{x} \cdot \frac{1}{4} = \begin{cases} \frac{1}{4x} & 0 \le y < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \frac{4-0}{2} = 2 \qquad E(Y|X) = \frac{X-0}{2} = \frac{X}{2}$$ normal distribution

<u>Solution:</u>



2b) $$E(Y|X) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_0^x y \cdot \frac{1}{x}\, dy = \frac{y^2}{2x}\Big|_0^x = \frac{x^2}{2x} = \frac{x}{2}$$

$$\boxed{E(Y|X) = \frac{X}{2}}$$

2c) $E(Y) = E[E(Y|X)] = E(\frac{X}{2}) = \frac{1}{2}E(X) = \frac{1}{2} \cdot 2 = 1$

OR

$E(Y) = \int_{-\infty}^{\infty} h(y) f_Y(y)\,dy \qquad E(X) = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$

$E(Y) = E(\frac{X}{2}) \rightarrow E(\frac{X}{2}) = \int_0^4 \frac{x}{2} \cdot \frac{1}{4}\,dx = \int_0^4 \frac{x}{8}\,dx = \frac{x^2}{16}\Big|_0^4 = \frac{16}{16} - 0 = 1$

$\boxed{E(Y) = 1}$

2d) $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

$\int \frac{1}{x}dx = \ln(x)$

$f(x,y) = \frac{1}{4x} \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^4 \frac{1}{4x}\,dx = \frac{1}{4}\int_y^4 \frac{1}{x}\,dx = \frac{1}{4}\ln(|x|)\Big|_y^4$

$= \frac{\ln(4)}{4} - \frac{\ln(y)}{4} = \frac{\ln(4) - \ln(y)}{4} = \frac{\ln(4/y)}{4} = f(y)$

$f_{X|Y}(x|y) = \frac{\frac{1}{4x}}{\frac{\ln(4) - \ln(y)}{4}}$

$\log_c(a) - \log_c(b) = \log_c(\frac{a}{b})$

can I use ln?

$\frac{\frac{1}{4x}}{\frac{\ln(4/y)}{4}} = \frac{1}{4x} \cdot \frac{4}{\ln(4/y)} = \frac{1}{x\ln(4/y)}$

$$\boxed{f_{X|Y}(x|y) = \begin{cases} \frac{1}{x\ln(4/y)} & y < x < 4 \\ 0 & \text{otherwise} \end{cases}}$$

2e) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

<u>Solve for $E(XY)$</u>

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_{0}^{4}\int_{y}^{4} xy\,\frac{1}{4x}\,dx\,dy$$

OR

$$E(XY) = \int_{0}^{4}\int_{0}^{x} xy\,\frac{1}{4x}\,dy\,dx = \int_{0}^{4}\int_{0}^{x} \frac{1}{4}\,y\,dy\,dx = \int_{0}^{4} \frac{y^2}{8}\Big|_{0}^{x}\,dx$$

$$= \int_{0}^{4} \frac{x^2}{8}\,dx = \frac{x^3}{24}\Big|_{0}^{4} = \frac{64}{24} = \frac{8}{3}$$

OR

$$E(XY) = E(E(XY|X)) = E(X\,E(Y|X)) = E(X\cdot\tfrac{X}{2}) = E(\tfrac{X^2}{2})$$

$$E[h(x)] = \int_{-\infty}^{\infty} \frac{x^2}{2}\, f_X(x)\,dx = \int_{0}^{4} \frac{x^2}{2}\,\frac{1}{4}\,dx = \int_{0}^{4} \frac{x^2}{8}\,dx = \frac{x^3}{24}\Big|_{0}^{4} = \frac{64}{24} = \frac{8}{3}$$

<u>Solve for $\text{cov}(X,Y)$</u>

$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = \frac{64}{24}$ $\quad E(X) = 2$ $\quad E(Y) = 1$

$$\text{cov}(X,Y) = \frac{64}{24} - (2)(1) = 2\frac{16}{24} - 2 = \frac{16}{24} = \frac{2}{3}$$

$$\boxed{\text{cov}(X,Y) = \frac{2}{3}}$$

$P(W_{i+1} \mid W_i) = 0.2$ for all i $\qquad$ W_1 patient is well

L is the random variable representing the number of checkups the patient is well

Patients stay in current state or get sicker

$P(W_{i+1} \mid I_i) = P(W_{i+1} \mid D) = P(I_{i+1} \mid D_i) = 0 \qquad P(D_{i+1} \mid D_i) = 1$

$P(W_1) = 100\% \qquad P(W_2 \mid W_1) = 0.2 = P(W_{i+1})\ P(I_2 \mid W_1) = \dfrac{P(W_1 \mid I_2)\ P(I_2)}{P(W_1)}$

$P(I_2 \mid W_1) = P(I_2)$ Independent

3a) $pmf = p(L)$ has $p_I = 0.8$

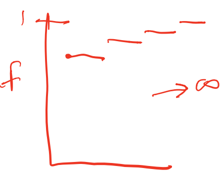
$P_I = 1 - P(W)$

$p(L)$ number of check ups until patient becomes ill

Solution

$$p(L) = \begin{cases} 0.8(.2)^{L-1} & L = 1, 2, 3 \ldots \infty \\ 0 & \text{otherwise} \end{cases}$$

cdf



$.8 + .16 + .032$

3b) $p_I = 0.8$

$$E(L) = \sum_D x\, p(x) = \sum_{l=1}^{\infty} l\, p(1-p)^{l-1} = \sum_{l=1}^{\infty} 0.8\, l\, (1-0.8)^{l-1} = 0.8 \sum_{l=1}^{\infty} l (.2)^{l-1}$$

$$\sum_{l=1}^{\infty} l \cdot z^{l-1} = \frac{1}{(1-z)^2}$$

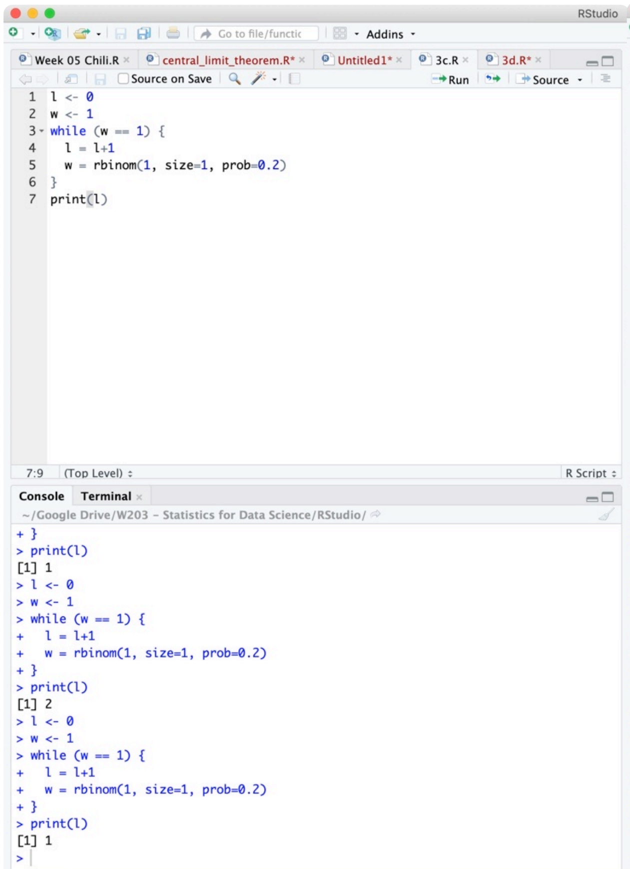
$$\frac{0.8}{(1-.2)^2} = \frac{0.8}{0.8^2} = \frac{1}{0.8} = 1.25$$

$$\frac{1}{0.8} \rightarrow \frac{1}{P_I}$$

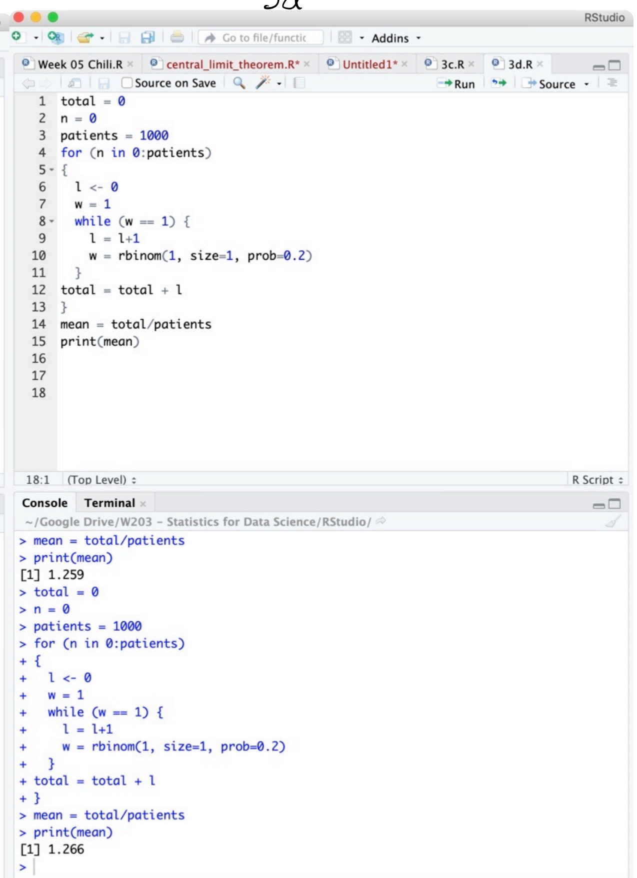
	1	$.8 \cdot .2^1$	$.8 \cdot .2^2$	$.8 \cdot .2^3$	$.8 \cdot .2^n$	$= 1.25$
$L =$	1,	2	3	4	$\ldots\ n$	

$\boxed{E(L) = 1.25}$ checkups

3c



3d



3e) $p = 0.5$

$$E(L) = \sum_{D} x\, p(x) = \sum_{l=1}^{\infty} l\, p(1-p)^{l-1} = \sum_{l=2}^{\infty} 0.5\, l\,(1-0.5)^{l-1} = 0.5 \sum_{l=2}^{\infty} l\,(.5)^{l-1}$$

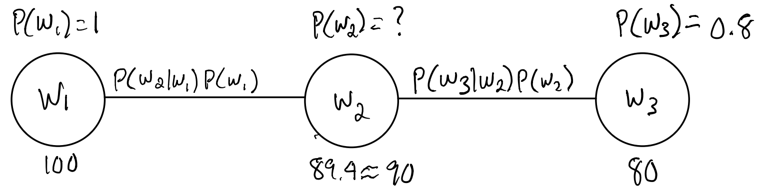
$$\sum_{l=1}^{\infty} l \cdot z^{l-1} = \frac{1}{(1-z)^2} \rightarrow \frac{.5}{(1-.5)^2} = \frac{0.5}{0.5^2} = \frac{1}{0.5} = \underline{\underline{2}}$$

$$\frac{1}{0.5} \downarrow \frac{1}{p}$$

$2 - 1.25$

$\boxed{\text{Walker adds .75 checkups}}$

3f)



$$P(w_3|w_2) = \frac{P(w_2|w_3)\,P(w_3)}{P(w_2)}$$

$$P(w_2|w_1) = \frac{P(w_1|w_2)\,P(w_2)}{P(w_1)} \rightarrow P(w_2|w_1) = \frac{P(w_2)}{P(w_1)}$$

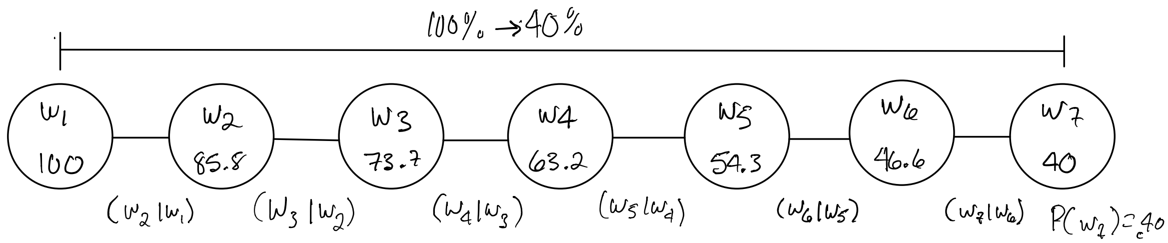
$$P(w_2) = P(w_2|w_1)\,P(w_1)$$

$$P(w_3|w_2) = \frac{P(w_2|w_3)\;P(w_3)}{P(w_2|w_1)\,P(w_1)} = P(w_3|w_2)P(w_2|w_1)\,P(w_1) = \frac{(1)(0.8)}{P(w_2|w_1)}$$

$$P(w_3|w_2)\cdot P(w_2|w_1) = 0.8 \rightarrow P(w_{i+1}|w_i)^2 = 0.8$$

$$\boxed{P(w_{i+1}|w_i) = \sqrt{0.8} \approx 0.894}$$

3g)



$$P(w_7|w_6)P(w_6) = w_7 - 0.40$$
$$P(w_7|w_6)\,P(w_6|w_5)P(w_5) = 0.40$$
$$P(w_7|w_6)P(w_6|w_5)P(w_5|w_4)P(w_4) = 0.40$$
$$\vdots$$

$$P(w_7|w_6)\,P(w_6|w_5)\,P(w_5|w_4)\,P(w_4|w_3)\,P(w_3|w_2)\,P(w_2|w_1)\,P(w_1) = 0.40$$

$$P(w_{i+1}|w_i)^6\cdot P(w_1) = 0.40 \rightarrow P(w_{i+1}|w_i)^6\cdot 1 = 0.40$$

$$\boxed{P(w_{i+1}|w_i) = (0.40)^{1/6} = 0.858}$$

$$b(x;n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0,1,2\ldots n \\ 0 & \text{o.w.} \end{cases}$$

$$W_1,\ W_2,\ I_3 = (.2)^2(1-.2)^1 = .032$$

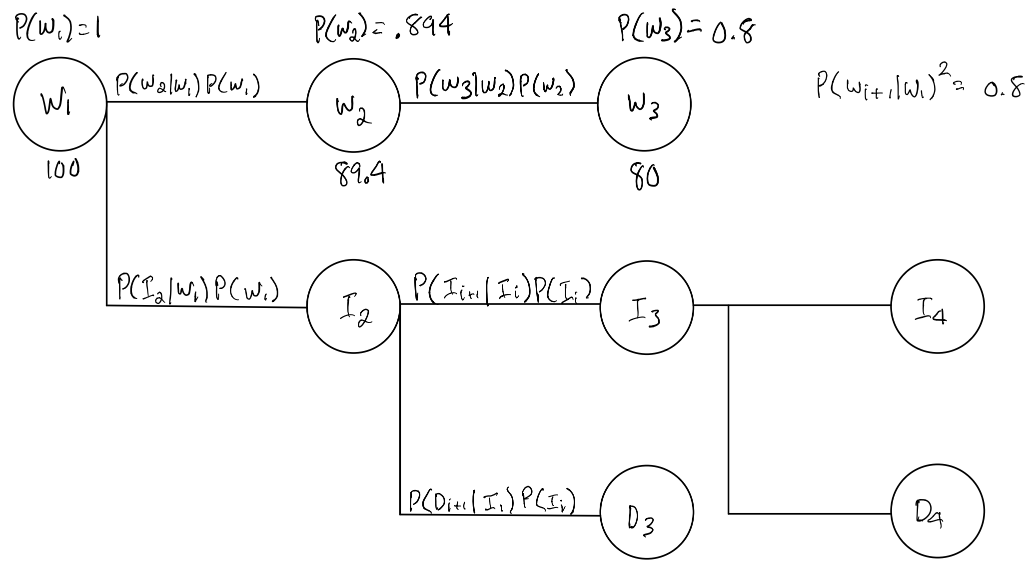
3h

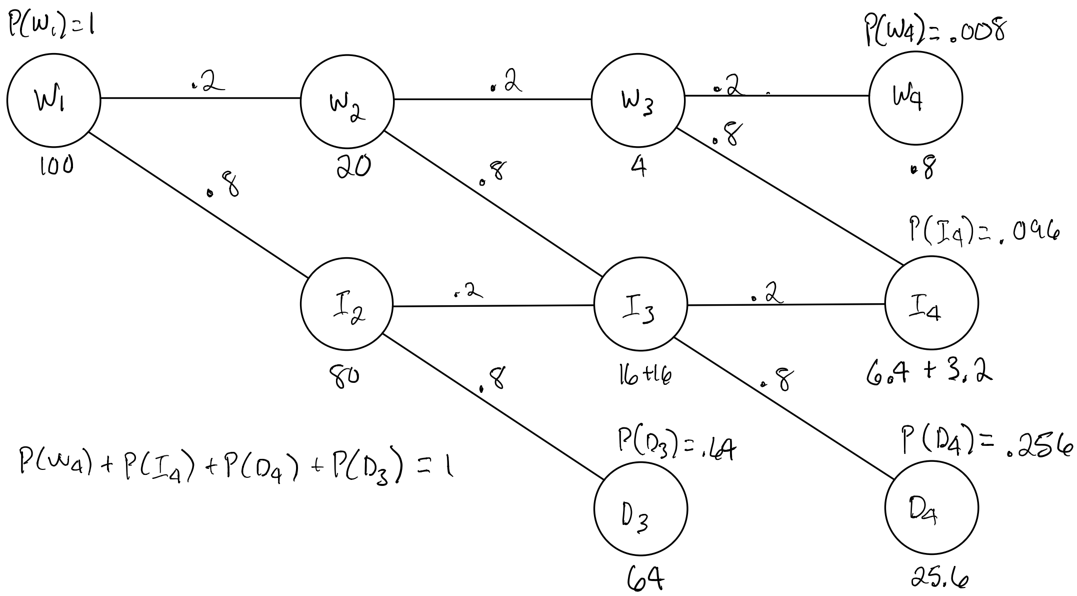
The transition probability is used to calculate the probability for any checkup after the first year. Therefore, it can be used to determine the probability of any subsequent checkup from checkup 1. In example 3f the probability of being well at checkup 3 is 80%. In the example 3g the probability of being well at checkup 3 is 73.7%.

Although this information is not specified in the problem statement, it can be derived from the information given. In problem 3g we are told that there are 100 patients. At checkup 7, 40 patients are well. In order to calculate the transition probabilities, we use $P(W_{i+1} \mid W_i)^6 = 0.40$. Solving for the transition probability, we get 0.858. This can be used to calculate any checkup greater than checkup 1. For example, checkup 3 probability is $(0.858)^2 = 73.7\%$.

Despite an average of 10 patients becoming ill per year, this is not actually what happened. In example 3f $W_1 = 100$, $W_2=89$, and $W_3=80$. In example 3g $W_1=100$, $W_2=86$, $W_3=74$, $W_4=63$, $W_5=54$, $W_6=47$, and $W_7=40$. Writing this out, it is clear that although the average ill patients per year are 10, the actual amount from year to year is different because the transition probabilities are different. The statement in 3h leads the reader to believe the transition probability should be the same. When in reality, they are not. Another observation is that as the years go on, fewer patients become ill year to year because fewer patients are well.

3i)





$P(W_4) + P(I_4) + P(D_4) + P(D_3) = 1$

64

$E(A) = E(W + I) = E(W) + E(I)$

Solve for $E(W)$ how long untill ill

$E(W) = \frac{1}{P_I} = \frac{1}{.8} = 1.25$

Solve for $E(I)$ how long until dead

$E(I) = \frac{1}{P_D} = \frac{1}{.8} = 1.25$

$E(A) = 1.25 + 1.25$

$\boxed{E(A) = 2.5}$ checkups

4a) $E(X) = E(U_1 + U_2) = E(U_1) + E(U_2)$

Solve for $E(U_1)$

$E(U_1) = \sum x\, p(x) = (0)(.25) + (1)(.75)$

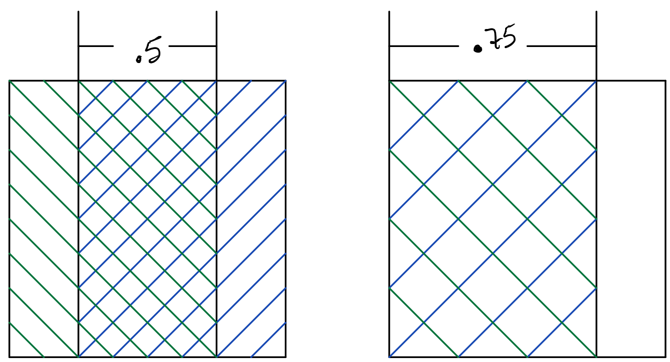
$E(U_1) = 0.75$

$E(x) = 0.75 + 0.75 \rightarrow \boxed{E(x) = 1.5}$

Solve for $E(U_2)$

$E(U_2) = (0)(.25) + (1)(.75)$

$E(U_2) = 0.75$



$P(U_1 \cup U_2) = P(U_1) + P(U_2) - P(U_1 \cap U_2)$

$P(U_1) + P(U_2) - 1 \leq P(U_1 \cap U_2) \leq \min[P(U_1), P(U_2)]$

$0.5 \leq P(U_1 \cap U_2) \leq 0.75$

$P(U_1 | U_2)_{max} = \dfrac{P(U_1 \cap U_2)}{P(U_1)} = \dfrac{0.75}{0.75} = 1$ $P(U_1 | U_2)_{min} = \dfrac{.5}{.75} = \dfrac{2}{3}$

$0.75 \leq P(U_1 \cup U_2) \leq 1$

$Y_1 = Y_2$ $Y_1 = 1 - Y_2$

4b+c)

Bernoulli rv

$$p(x;\alpha) = \begin{cases} 1-\alpha & \text{if } x=0 \\ \alpha & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

$$p(C_1) = \begin{cases} P(U_1') = 0.25,\ 0 \\ P(U_1) = 0.75,\ 1 \\ 0 \quad \text{otherwise} \end{cases} \qquad p(C_2) = \begin{cases} P(U_2') = 0.25,\ 0 \\ p(U_2) = 0.75,\ 1 \\ 0 \quad \text{otherwise} \end{cases}$$

$$var(X) = var(C_1 + C_2) = var(C_1) + var(C_2) + 2cov(C_1, C_2)$$

$$var(X) = E(C_1^2) - [E(C_1)]^2 + E(C_1^2) - [E(C_2)]^2 + 2[E(C_1C_2) - E(C_1)E(C_2)]$$

Solve for $E(C_1^2) = E(C_2^2)$

$$E[h(x)] = \sum_D h(x) \cdot p(x) = (0)^2(0.25) + (1)^2(0.75) = 0.75 = 3/4$$

$h(x) = C^2$

Solve for $[E(C_1)]^2 = [E(C_2)]^2$

$$[E(C_1)]^2 = (3/4)^2 = 9/16$$

Solve for $E(C_1C_2)$

$$E(C_1C_2) = \sum_{C_1}\sum_{C_2} C_1C_2\ p(C_1, C_2) = (0)(0)P(U_1' \cap U_2') + (0)(1)P(U_1 \cap U_2') + (1)(0)P(U_1' \cap U_2) + (1)(1)P(U_1 \cap U_2)$$

$$E(C_1C_2) = P(U_1 \cap U_2)$$

$P(U_1 \cap U_2)_{max} = 0.75$ $\quad$ $P(U_1 \cap U_2)_{min} = 0.5$

$E(C_1C_2)_{max} = 3/4$ $\quad$ $E(C_1C_2)_{min} = 1/2$

$$var(X)_{max} = \frac{3}{4} - \frac{9}{16} + \frac{3}{4} - \frac{9}{16} + 2\left(\frac{3}{4} - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

$$var(X)_{min} = \frac{3}{8} + 2\left(\frac{1}{2} - \frac{9}{16}\right) = \frac{1}{4}$$

$$\boxed{\text{4b } var(X)_{max} = 3/4 \qquad \text{4c } var(X)_{min} = 1/4}$$

4d) $var(X) = var(C_1 + C_2 + C_3 + C_4)$

$= var(C_1) + var(C_2) + var(C_3) + var(C_4) + 2cov(C_1, C_2) + 2cov(C_1, C_3) + 2cov(C_1, C_4) \ldots$

$\ldots + 2cov(C_2, C_3) + 2cov(C_2, C_4) + 2cov(C_3, C_4)$

$var(X)_{max} = 4\,var(C_i) + 12\,cov(C_1, C_1) = 4(3/16) + 12(3/16)$

$\boxed{var(X)_{max} = 3}$

$var(X)_{min} = 4(3/16) + 12(-1/16)$

$\boxed{var(x)_{min} = 0}$

4a practice

$X=0 \quad = P(U_1' \cap U_2') = 0.0625$

$X=1 \quad = P(U_1 \cap U_2') + P(U_1' \cap U_2) = .1875 + .1875 = .375$

$X=2 \quad = P(U_1 \cap U_2) = .5625$

$E(X) = 0 \cdot 0.0625 + 1 \cdot .375 + (2)(.5625) = 1.5$

4b practice

$P(C_1=1) = P(U_1) = 3/4$ | $P(C_2=1) = P(U_2) = 3/4$

$P(C_1=0) = P(U_1') = 1/4$ | $P(C_2=0) = P(U_2) = 1/4$

$\text{var}(X) = \text{Var}(C_1 + C_2) = \text{Var}(C_1) + \text{var}(C_2) + 2\,\text{cov}(C_1, C_2)$

$\text{var}(C_1) = p(1-p) = .75(1-.75) = 3/16$

$\text{var}(C_2) = .75(1-.75) = 3/16$

$\text{cov}(C_1, C_2) = E(C_1C_2) - E(C_1)E(C_2)$

$$\text{corr}(C_1, C_2) = \frac{\text{cov}(C_1, C_2)}{\sqrt{\text{var}(C_1)\,\text{var}(C_2)}}$$

$$P_{C_1}(C_1) = \begin{cases} .25 & c=0 \\ .75 & c=1 \end{cases}$$

$$P_{C_2}(C_2) = \begin{cases} .25 & c=0 \\ .75 & c=1 \end{cases}$$

$$= \begin{cases} \frac{1}{16} & C_1=0, C_2=0 \\ \frac{3}{16} & C_1=1, C_2=0 \\ \frac{3}{16} & C_1=0, C_2=1 \\ \frac{9}{16} & C_1=1, C_2=1 \end{cases}$$

$P_{C_1,C_2}(C_1,C_2)$

marginals. The correlation between two Bern(p) random variables belongs to the interval [$\rho_{\min}$,1]. Maximum correlation in case of equal marginals, always equals to 1 and the minimum correlation $\rho_{\min}$ can be calculated using Fréchet-Hoeffding bounds (Fréchet 1951; Hoeffding 1940)

$$\rho_{\min} = \begin{cases} -(1-p)/p, & \text{for } p \geq 1/2 \\ -p/(1-p), & \text{for } p \leq 1/2. \end{cases}$$

It is clear now that only for $p=1/2$, $\rho_{\min}=-1$ and possible correlations equal to the entire interval [−1,1], while for any other value of p it is a strict subinterval of [−1,1]. For example, for $p=3/4$, $-1/3 \leq \rho \leq 1$.

$\text{corr}(C_1, C_2)_{max} = 1$

If $\text{corr}(C_1, C_2) = 1$

$$1 = \frac{\text{cov}(C_1, C_2)}{\sqrt{\frac{3}{16} \cdot \frac{3}{16}}} = \frac{\text{cov}(C_1, C_2)}{3/16} = \frac{16}{3}\text{cov}(C_1, C_2)$$

$\frac{16}{3}\text{cov}(C_1, C_2) = 1 \rightarrow \text{cov}(C_1, C_2) = 3/16$

$\text{Var}(X)_{max} = 3/16 + 3/16 + 2(3/16) = 3/4$

$\text{var}(X)_{max} = 3/4$

4c practice

$$corr(c_1,c_2)_{min} = \frac{-(1-\rho)}{\rho} = \frac{-(1-.75)}{.75} = -\frac{1}{3}$$

If $corr(C_1,C_2) = -1/3$

$\underline{16}\, cov(C_1,C_2) = -1/3 \rightarrow cov(C_1,C_2) = \frac{-1}{16}$

$$var(x)_{min} = \frac{3}{16} + \frac{3}{16} + 2\left(\frac{-1}{16}\right) = \frac{1}{4}$$

$$var(x)_{min} = \frac{1}{4}$$