1. True or False?

There exists a non-constant smooth function $f : \mathbb{R} \to \mathbb{R}$ whose derivative f' has support $[-1, 1]$.

2. True or False?

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0, $f(0) = 0$, and $f'(0) = 1$. Then $|f(x) - x| = o(|x|)$ as $x \to 0$.

3. True or False?

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and let

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : y = f(x), x \in \mathbb{R}\}.$$

Then Γ is a submanifold of $\mathbb{R}^2$ and $(\mathbb{R}^2, h)$ is a chart that is adapted to Γ, where h is given by

$$h(x, y) = (f(x) - y, x).$$

4. True or False?

Suppose $\ell, \ell' : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear maps. Then

$$\|\ell \circ \ell'\| = \|\ell\| \cdot \|\ell'\|.$$

5. True or False?

Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^k for some $k \geq 2$,

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$$

and $f^{(k)}(0) > 0$. Then 0 is a local minimum of f.

6. Let C be the infinitely tall cylinder of radius r centered around the z-axis in $\mathbb{R}^3$. In other words, C is a tube of radius r around the entire z-axis.

Prove that C is a submanifold of $\mathbb{R}^3$.

7. True or False?

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and

$$\lim_{x \to 0} f'(x)$$

exists. Then f' is continuous at 0.

8. True or False?

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and bijective, then f is étale.

9. Suppose $f : (0, \infty) \to \mathbb{R}$ is a differentiable function and the derivative f' is bounded (ie, there exists an $R > 0$ such that $|f'(x)| \leq R$ for all x). Show that

$$\lim_{n \to \infty} f(1/n)$$

exists.

Hint. Show that the sequence is Cauchy.

10. Let T be the torus inside $\mathbb{R}^3$ centered around the z-axis, where the distance from the origin to the center of the tube is R, and the radius of the tube is r, and $0 < r < R$. In other words, T is the tube of radius r that goes around the circle of radius of R centered at the origin that lies on the xy-plane.

Find a smooth submersive function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^3$, such that

$$T = \{(x, y, z) \in U : f(x, y, z) = 0\}.$$

Use the regular level set theorem to conclude that T is a submanifold of $\mathbb{R}^3$.