Problem Set A – Partial Solutions

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Problem 1. Let T be the set of equilateral triangles with integer side lengths. Let T_1 be the subset of T consisting of triangles with even side lengths, and let T_2 be the subset of T consisting of triangles with even perimeter. Is it true that $T_1 = T_2$? If so, prove it. If not, give an explicit example of a triangle that is in one of the sets but not the other.

Solution. First we show that $T_1 \subseteq T_2$. Suppose $t \in T_1$. This means that t is an equilateral triangle whose side length, a, is even. Then the perimeter of t is $3a$, which must also be even since a is even. Thus $t \in T_2$.

Next, we show that $T_2 \subseteq T_1$. Suppose $t \in T_2$. This means that t is an equilateral triangle with integer side lengths whose perimeter is even. Let a be the side length of t. Then the perimeter is $3a$. In order for $3a$ to be even, it must be that a is even (if a were odd, then $3a$ would be a product of two odd numbers which would also be odd). Thus $t \in T_1$.

Since $T_1 \subseteq T_2$ and $T_2 \subseteq T_1$, we conclude that $T_1 = T_2$.

Problem 4. Suppose that the postal office only issued 3-cent stamps and 7-cent stamps. Give an *explicit* description the set of all postage amounts that can be created using stamps of these two types. Prove that your description is correct.

Solution. I claim that the set of possible postage amounts is 3, 6, 7, 9, 10, and then all integers $n \geq 12$. To prove this, I first check the first several possible amounts by hand.

amount	combination
1	impossible
2	impossible
3	$1 \cdot 3 + 0 \cdot 7$
4	impossible
5	impossible
6	$2 \cdot 3 + 0 \cdot 7$
7	$0 \cdot 3 + 1 \cdot 7$
8	impossible
9	$3 \cdot 3 + 0 \cdot 7$
10	$1 \cdot 3 + 1 \cdot 7$
11	impossible
12	$4 \cdot 3 + 0 \cdot 7$
13	$2 \cdot 3 + 1 \cdot 7$
14	$0 \cdot 3 + 2 \cdot 7$

To prove that every amount $n \geq 12$ is possible, I will use strong induction (ie, the "second principle of induction"). The base cases for this induction are $n = 12, 13, 14$, which we've shown are possible to make in the table above. For the inductive step, we assume that it is possible to make all amounts $12, 13, \ldots, k-1, k$ for some integer $k \geq 14$. We want to use this to show that we can also make $k+1$. Since we can make $k-2$, we can also make $(k-2)+3$, just by adding one 3-cent stamp to the combination that we used to make $k-2$. But $(k-2)+3 = k+1$, so it is possible to make $k+1$. This completes the induction.

Here's an alternative strategy that avoids the use of induction. We start off by calculating the same table that we calculated above, again going up to 14. Then we argue as follows: Suppose $n \geq 14$. By the division algorithm, there are three possible cases when n is divded by 3 — n can be of the form $3q$, or $3q+1$, or $3q+2$. We consider each of these cases in turn.

- Case 1. If $n = 3q$, then n can be made using q 3-cent stamps.
- Case 2. Suppose $n = 3q + 1$. Notice that

  $$n = 3q + 1 = 3(q-2) + 7.$$

  Moreover, since $n \geq 14$, we have $3q = n - 1 \geq 13$ which means that $q \geq 4$, which means that $q - 2$ is a non-negative integer. Thus the equation above shows that we can form a postage amount of n using $q - 2$ 3-cent stamps and one 7-cent stamp.
- Case 3. Suppose $n = 3q + 2$. Notice that

  $$n = 3q + 2 = 3(q-4) + 2 \cdot 7.$$

  Since $n \geq 14$, we have $3q = n - 2 \geq 12$ which means that $q \geq 4$, which means that $q - 4$ is a non-negative integer. Thus the above equation shows that we can form a postage amount of n using $q - 4$ 3-cent stamps and two 7-cent stamps.

Problem 5. Prove that, for any integer $n \geq 1$, the binomial coefficient $\binom{2n}{n}$ is even.

Solution. There are a variety of approaches here. Here's one that just uses the definition of binomial coefficients. Notice that

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{2n \cdot (2n-1)!}{n \cdot (n-1)!n!} = 2 \cdot \frac{(2n-1)!}{(n-1)!n!} = 2 \cdot \binom{2n-1}{n}.$$

Since the binomial coefficient $\binom{2n-1}{n}$ is an integer, we see that $\binom{2n}{n}$ is even.