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7.19. For part (a), choose $\epsilon > 0$. Since $\lim a_n = 0$, there exists an N such that $|a_n| \leqslant \epsilon$ for all $n \geqslant N$. Thus

$$|f_n(z) - 0| = |f_n(z)| \leqslant a_n \leqslant |a_n| \leqslant \epsilon$$

for all $z \in G$ and $n \geqslant N$, so f_n converges uniformly to 0. For part (b), observe that $|f_n(z)| = |z^n| \leqslant 2^{-n}$ for all $z \in D[0, 1/2]$ and $\lim 2^{-n} = 0$, so part (a) implies that the functions f_n converge uniformly to 0.

7.28(a). Observe that

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{k=0}^{\infty}(1-z)^n = \sum_{k=0}^{\infty}(-1)^n(z-1)^n.$$

Note that $\lim \sqrt[n]{|(-1)^n|} = 1$, so by the root test, the radius of convergence of this power series is 1. In other words, we have $1/z = \sum(-1)^n(z-1)^n$ only when $|z-1| < 1$.

2.20. Let $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. If f is real-valued, then $v = 0$. The Cauchy-Riemann equations thus imply that $u_x = v_y = 0$ and $u_y = -v_x = 0$, so u must be constant. Thus $f = u + iv = u$ is also constant.

2.25(a). If $f = u + iv$ is holomorphic for $u = x^2 - y^2$, the Cauchy-Riemann equations imply that

$$v_y = u_x = 2x \implies v = \int 2x\, dy = 2xy + a(x)$$
$$v_x = -u_y = 2y \implies v = \int 2y\, dx = 2xy + b(y)$$

where $a(x)$ and $b(y)$ depend only on x and y, respectively. But then we must have $a(x) = b(y)$, and the only functions that are independent of both x and y are constant. In other words, we must have

$$v(x+iy) = 2xy + c$$

for some real number c. Then for $z = x + iy$, we have

$$f(z) = f(x+iy) = (x^2-y^2) + i(2xy+c) = (x^2 + 2ixy - y^2) + ic = (x+iy)^2 + ic = z^2 + ic.$$

3.45(c). Let $z = x + iy$. Then

$$i\pi = \exp(z) = e^x e^{iy} = e^x\cos(y) + ie^x\sin(y)$$

so, matching real and imaginary parts, we need to have $e^x\cos(y) = 0$ and $e^x\sin(y) = \pi$. Since e^x is always nonzero, the only way to have $e^x\cos(y) = 0$ is if $\cos(y) = 0$, which happens when $y = \pi/2 + k\pi$ for some integer k. Then $\sin(y) = \sin(\pi/2 + k\pi) = (-1)^k$. Since $e^x > 0$, in order to have $e^x\sin(y) = \pi$, we need for k to be even (ie, $k = 2n$ for some integer n), and we need $e^x = \pi$, ie, $x = \ln\pi$. Thus the solutions to the equation $\exp(z) = \pi i$ are

$$z = \ln\pi + i\left(\frac{\pi}{2} + 2\pi n\right)$$

for any integer n.

7.34(a). Observe that, if $w = z^2$, we have

$$\sum_{k\geqslant 0}\frac{z^{2k}}{k!} = \sum_{k\geqslant 0}\frac{w^k}{k!} = \exp(w) = \exp(z^2).$$

Thus the power series represents $\exp(z^2)$.

3.40. For part (a), we have $\operatorname{Log}(2i) = \ln 2 + i(\pi/2)$. For part (b), we have

$$(-1)^i = \exp(i\operatorname{Log}(-1)) = \exp(i\cdot i\pi) = \exp(-\pi).$$

Finally, for part (c), we have $\operatorname{Log}(-1+i) = \ln\sqrt{2} + i(3\pi/4)$.

3.44(c). Log, defined using the argument taking values in $[0, 2\pi)$, is holomorphic on $\mathbb{C}$ minus the non-negative real axis. Observe that $z - 2i + 1 = a$ is a non-negative real if and only if $z = (a-1) + 2i$, and the set of such z defines a horizontal ray with imaginary part 2 emanating to the right from $-1+2i$. Thus $\operatorname{Log}(z-2i+1)$ is holomorphic away from that ray.

4.4. We parametrize $C[w,r]$ as $\gamma(t) = w + e^{it}$ for $t \in [0, 2\pi]$. Then $\gamma'(t) = ie^{it}$, so

$$\int_{C[w,r]} \frac{dz}{z-w} = \int_0^{2\pi} \frac{ie^{it}\,dt}{e^{it}} = \int_0^{2\pi} i\,dt = 2\pi i.$$

4.16. Observe that $z\exp(z^2)$ is the derivative of $\exp(z^2)/2$ everywhere on $\mathbb{C}$. Since it has a global antiderivative, the integral over any closed path in $\mathbb{C}$ must be 0.

4.25. Let γ_1 and γ_2 be any two paths in $\mathbb{C}$. We may assume without loss of generality that both paths are parametrized on the unit interval $[0,1]$. Then define $h : [0,1] \times [0,1] \to \mathbb{C}$ by

$$h(t,s) = (1-s)\gamma_1(t) + s\gamma_2(t).$$

Observe that $h(t,0) = \gamma_1(t)$ and $h(t,1) = \gamma_2(t)$, and h is evidently continuous as it is built out of sums and products of continuous functions. In other words, h is a homotopy from γ_1 to γ_2.

Both parts follow from this. For part (a), we can take γ_1 to be a closed path which starts and ends at some point $a \in \mathbb{C}$, and we take γ_2 to be the constant path on a. For part (b), we take γ_1 and γ_2 to be any two closed paths.

4.27. Let $f(z) = 1/(z-a)$ and observe that f is holomorphic for all $z \neq a$.

$$I(r) = \int_{C[0,r]} \frac{dz}{z-a}.$$

If $r < |a|$, the path $C[0,r]$ over which we are integrating is null-homotopic in the domain of holomorphy of the integrand, so $I(r) = 0$. If $r > |a|$, then $C[0,r]$ is homotopic to $C[a,1]$, so

$$I(r) = \int_{C[a,1]} \frac{dz}{z-a} = 2\pi i$$

by exercise 4.4.

4.34. By Cauchy's integral formula, we have

$$f(w) = \int_\gamma \frac{f(z)\,dz}{z-w} = \int_\gamma \frac{g(z)\,dz}{z-w} = g(w).$$

5.14. Suppose f is entire and there exists $M > 0$ such that $|f(z)| \geqslant M$ for all $z \in \mathbb{C}$. In particular, $f(z) \neq 0$ for all $z \in \mathbb{C}$, so $1/f$ is also entire. Moreover, we have $|1/f(z)| \leqslant 1/M$ for all $z \in \mathbb{C}$, so by Liouville's theorem, there exists $c \in \mathbb{C}$ such that $1/f(z) = c$. But then $f(z) = 1/c$ for all $z \in \mathbb{C}$, proving that f is constant.

8.12. Suppose f is holomorphic and nonconstant in a region G and that $f(z) \neq 0$ for all $z \in G$. Suppose for a contradiction that there exists a weak relative minimum a of f in G. Then $1/f$ is also holomorphic and nonconstant, and a is a weak relative maximum of $1/f$ in G, but this contradicts the maximum modulus principle.

8.37. Observe that

$$\exp(z) = \exp(z+1-1) = \exp(z+1)e^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!e}(z+1)^n,$$

so that is the power series of exp centered at $z = -1$. Letting c_n be the nth term of this power series expansion, observe that theorem 8.8 tells us that

$$\int_{C[-2,2]} \frac{f(z)\,dz}{(z+1)^{34}} = 2\pi i c_{33} = \frac{2\pi i}{33!e}.$$

8.19. For all $0 < |z+1| < \infty$ (ie, all $z \neq -1$), observe that

$$\frac{z-2}{z+1} = \frac{z+1-3}{z+1} = 1 - \frac{3}{z+1}.$$

This is the two-term Laurent series we're after.

8.23. Observe that, if $|z-1| > 1$, then $|1/(z-1)| < 1$, so

$$\frac{z-1}{z-2} = \frac{z-1}{(z-1)-1} = \frac{1}{1-1/(z-1)} = \sum_{n=0}^{\infty} (z-1)^{-n},$$

which is what we wanted to show.

9.1. Suppose f has a zero of multiplicity m at a. Then there exists a function g, holomorphic and non-vanishing at a, such that $f(z) = (z-a)^m g(z)$ for all z near a. Then

$$\lim_{z\to a} \frac{(z-a)^{m+1}}{f(z)} = \lim_{z\to a} \frac{(z-a)^{m+1}}{(z-a)^m g(z)} = \lim_{z\to a} \frac{z-a}{g(z)} = 0,$$

using the limit law for division (since $g(a) \neq 0$). Applying proposition 9.5, we see that $1/f$ has a pole of order m at a.

9.3. Suppose f has an essential singularity at z_0. If $1/f$ has a pole at z_0, then $\lim_{z\to z_0} f(z) = \infty$, so $\lim_{z\to z_0} f(z) = 0$. But then

$$\lim_{z\to z_0} (z-z_0)f(z) = \left(\lim_{z\to z_0} (z-z_0)\right)\left(\lim_{z\to z_0} f(z)\right) = 0\cdot 0 = 0$$

using the limit law for multiplication, so proposition 9.5 implies that f has a removable singularity at 0. This is a contradiction. Next, suppose $1/f$ has a removable singularity at z_0. Let h be a holomorphic function extending $1/f$ across z_0. If $h(z_0) \neq 0$, then

$$0 \neq h(z_0) = \lim_{z\to z_0} h(z) = \lim_{z\to z_0} \frac{1}{f(z)}$$

implies that

$$\lim_{z\to z_0} f(z) = 1/h(z_0).$$

Then we apply proposition 9.5 again, as above, to see that f has a removable singularity at z_0, which is again a contradiction. The final case is when $h(z_0) = 0$. But then we have

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{1}{h(z)} = \infty$$

so f has a pole at z_0, which is yet again a contradiction. Having reached a contradiction in all cases, we conclude that $1/f$ must have an essential singularity at z_0.

9.5. For (c), observe that the function has simple poles at $z = -4, \pm i$. Only $\pm i$ are inside $C[0,3]$, so we only need to compute the residues at these two points. Using proposition 9.11, we have

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} \frac{z-i}{(z+4)(z^2+1)} = \lim_{z\to i} \frac{1}{(z+4)(z+i)} = \frac{1}{2i(4+i)}.$$

Similarly,

$$\operatorname*{Res}_{z=-i} f(z) = \lim_{z\to -i} \frac{z+i}{(z+4)(z^2+1)} = \lim_{z\to i} \frac{1}{(z+4)(z-i)} = \frac{-1}{2i(4-i)}.$$

Thus, the residue theorem implies that the integral is

$$\int_{C[0,3]} f(z)\,dz = 2\pi i \cdot \left(\frac{1}{2i(4+i)} + \frac{-1}{2i(4-i)}\right) = -\frac{2\pi i}{17}.$$

For (d), observe that

$$z^2 \exp(1/z) = z^2 \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n+2}}{n!}$$

so the residue is $1/3! = 1/6$. Thus the residue theorem implies that the integral is

$$\int_{C[0,3]} f(z)\,dz = \frac{2\pi i}{6} = \frac{\pi i}{3}.$$

9.7. For (c), observe that
$$\frac{z^2+4z+5}{z^2+z} = \frac{1}{z}\cdot\frac{z^2+4z+5}{z+1}$$
and the latter term is holomorphic at 0. Thus, applying exercise 9.6, we see that
$$\operatorname*{Res}_{z=0}\frac{1}{z}\cdot\frac{z^2+4z+5}{z+1} = \left(\operatorname*{Res}_{z=0}\frac{1}{z}\right)\cdot\frac{0^2+4\cdot 0+5}{0+1} = 1\cdot 5 = 5.$$
For (d), we compute the residue using the Laurent series. Observe that
$$\exp(1-1/z) = e\exp(-1/z) = \sum_{n=0}^{\infty}\frac{(-1)^n e}{n!}z^{-n}$$
so the residue is $-e$.