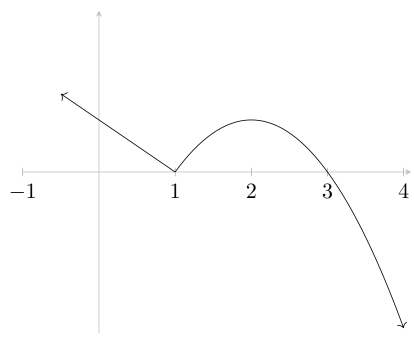
Name:

Quiz 3 Solutions

Instructions. The only tools you are permitted to use are pencils, pens, erasers, a handwritten sheet of notes, and your mind. No electronic devices! You have 1 hour. Good luck!

Problem 1 (True-false, 9 points)**.** You will get 1 point for a correct answer and 0 points for blank and incorrect answers.[1] No explanations are required.

(1) The function f whose graph is depicted below has exactly 1 critical point on $[0, 3]$. T **F**



(2) If f is a differentiable function, the absolute maximum of f on the closed interval $[1, 7]$ must occur at a critical point of f contained in $[1, 7]$. T **F**

(3) If f is a differentiable function and $f'(5) = 0$, then f must have either a local maximum or a local minimum at $x = 5$. T **F**

(4) If f is a differentiable function such that $f'(x) > 0$ for all x, then it must be the case that $\lim_{x\to\infty} f(x) = \infty$. T **F**

(5) There exists a continuous function defined on the closed interval $[-1, 1]$ that has an absolute minimum on the interval but does not have an absolute maximum. T **F**

(6) If f is an odd function, then $\int_{-2}^{2} f(x)\,dx = 0$. **T** F

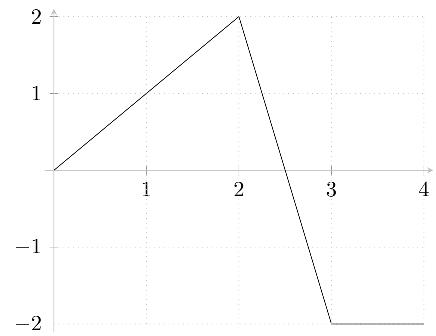
(7) $\sum_{k=0}^{2019}((k+1)^2 - k^2) = 2020^2$. **T** F

(8) $\lim_{x\to\infty} \frac{x^2+2}{e^x} = 0$. **T** F

(9) $\lim_{x\to 0} \frac{e^x}{e^x - 1} = 1$. T **F**

[1] During quiz revisions, you'll have the chance to meet with me one-on-one and convince me that you understand and can fully explain *up to 2 of the true-false questions that you left blank* to get full credit on those questions.

Problem 2 (3 points)**.** Let f be the function whose graph is depicted to the right.



(a) Calculate $\displaystyle\int_0^4 f(x)\,dx$.

Solution. The triangle from 0 to 2 has area $\frac{1}{2} \cdot (2)(2) = 2$. The two triangles from 2 to 3 cancel each other out. The rectange from 3 to 4 has area 2, which cancels the positive area of the triangle from 0 to 2. Thus the integral is 0.

(b) Calculate $\displaystyle\int_0^4 |f(x)|\,dx$.

Solution. The absolute value flips everything that's below the x-axis to above the x-axis. The two triangles from 2 to 3 together form a rectangle of area 1, so the total area is 5.

(c) Find a such that $0 \leq a \leq 4$ and $\displaystyle\int_0^a f(x)\,dx$ is as large as possible.

Solution. The maximum happens when $a = 2.5$. After that, the graph dips below the x-axis and the integral starts decreasing.

Problem 3 (3 points)**.** You want to enclose a rectangular garden of area 1000 m^2. The north and south walls of the garden will have brick walls costing \$50/m and the east and west walls will have metal fences costing \$20/m. Each wall should be at least 1 m in length. What should the length of the north wall be in order to minimize cost of the garden walls? What length should it be to maximize this cost?

Solution. Let x and y denote the lengths of the brick and metal sides, respectively. Then $xy = 1000$ is the constraint, and we are trying to optimize the cost

$$C = 100x + 40y = 100x + \frac{40\,000}{x}.$$

Note that $x \geq 1$, and the condition that $y \geq 1$ translates to $x \leq 1000$. So we want to find the absolute minimum and the absolute maximum of C for values of x in the closed interval $[1, 1000]$.

We calculate the critical points in $[1, 1000]$.

$$\begin{aligned} \frac{dC}{dx} = 100 - \frac{40\,000}{x^2} &= 0 \\ 100x^2 &= 40\,000 \\ x^2 &= 400 \\ x &= \pm 20 \end{aligned}$$

Thus there is only one critical point in $[1, 1000]$, which is $x = 20$.

We then test the critical point and the endpoints of the interval.

x	C
1 m	\$40 100
20 m	\$4 000
1000 m	\$100 040

Thus having the brick wall of length $x = 20$ m minimizes cost, and having a brick wall of length $x = 1000$ m maximizes cost.

There are a few other possibilities for deciding that $x = 20$ is a minimum. One possibility is to notice that $dC/dx < 0$ for $0 < x < 20$ and $dC/dx > 0$ for $x > 20$, so $x = 20$ is a global minimum of the function on $(0, \infty)$. In particular, it is a minimum on the interval $[1, 1000]$.

Another possibility is to compute the second derivative. We find that $d^2C/dx^2 = 80\,000/x^3$, which is positive for $x > 0$. In other words, C is always concave up, so again we find that $x = 20$ is a global minimum of the function on $(0, \infty)$. In particular, it must be the absolute minimum on $[1, 1000]$.

Problem 4 (3 points)**.** Sketch the graph of the function f defined by

$$f(x) = \frac{1}{x^2 - 1}.$$

Some things you might keep in mind as you make your sketch include: the domain of the function, horizontal and vertical asymptotes, critical points, intervals of increase/decrease, local minima/maxima, points of inflection, and concavity. In case you find it useful, here are the first and second derivatives of f.

$$f'(x) = \frac{-2x}{(x^2-1)^2} \qquad f''(x) = \frac{6x^2+2}{(x^2-1)^3}$$

Solution. The domain of the function is all real numbers except ± 1. We have the following limits.

$$\lim_{x \to \pm\infty} f(x) = 0$$
$$\lim_{x \to -1^-} f(x) = \infty \quad \text{and} \quad \lim_{x \to -1^+} f(x) = -\infty$$
$$\lim_{x \to 1^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \infty$$

Thus the x-axis is a horizontal asymptote, and the lines $x = \pm 1$ are vertical asymptototes.

Then note that $f'(x) = 0$ when $x = 0$, and $f'(x)$ is again undefined at $x = \pm 1$.

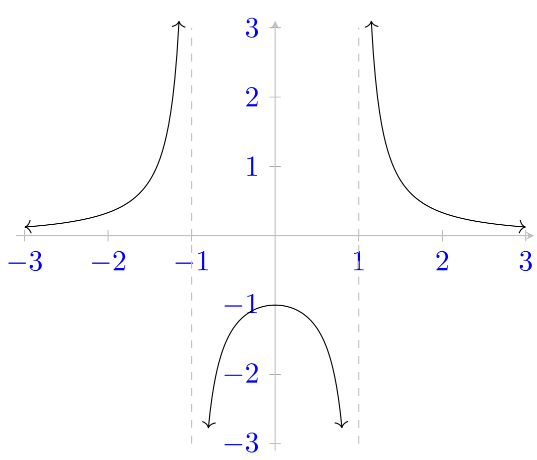
Interval	Sign of f'	Increasing/Decreasing
$(-\infty, -1)$	$+$	Increasing
$(-1, 0)$	$+$	Increasing
$(0, 1)$	$-$	Decreasing
$(1, \infty)$	$-$	Decreasing

At the critical point $x = 0$, we have $f(0) = -1$.

Then note that $f''(x)$ never equals zero, but it is undefined at ± 1.

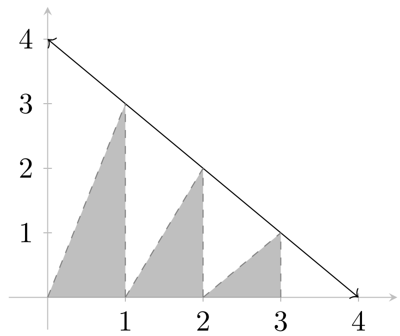
Interval	Sign of f''	Concavity of f
$(-\infty, -1)$	$+$	Up
$(-1, 1)$	$-$	Down
$(1, \infty)$	$+$	Up

Putting all of this together, we make a plot.



Not all of this was strictly necessary to end up with an accurate picture.

Problem 5 (Extra credit! 2 points, no partial credit)**.** You've decided to test out a new mathematical theory of integration. Consider the function $f(x) = 4 - x$ on a closed interval $[0, 4]$. You divide the interval $[0, 4]$ into n pieces. Over each of these n subintervals, you draw a right triangle whose height is the value of f at the right endpoint of the subinterval. You decide to call the area enclosed by all of these triangles the *right triangular sum* of f, and you denote it by T_n. The case $n = 4$ is depicted below.



What is the value of $\lim_{n\to\infty} T_n$?

Solution. Observe that T_n is always exactly half of R_n, where R_n is the right Riemann sum with n subintervals. Thus

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \frac{R_n}{2} = \frac{1}{2} \lim_{n\to\infty} R_n = \frac{1}{2} \int_0^4 f(x)\,dx.$$

The area represented by the integral $\int_0^4 f(x)\,dx$ is a triangle of height and width both equal to 4, so the value of the integral is 8. Thus

$$\lim_{n\to\infty} T_n = 4.$$

Honor code. If you have neither given nor received any unauthorized aid on this quiz, please write either "HCU" or "Honor Code Upheld" below, and sign your name next to it.

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