# Problem Set 2

**Problem 1.** (1 point) Let $X$ be a set and let $d_1$ and $d_2$ be two metrics on $X$. Then $d_1$ and $d_2$ are *equivalent* if a subset $E$ of $X$ is open with respect to $d_1$ if and only if it is open with respect to $d_2$.

Show that on $X := \mathbb{R}^2$, the euclidean metric, the Manhattan metric, and the maximum metric are all equivalent to each other.

**Problem 2.** (1 point) Determine the closures of the following subsets of $\mathbb{R}$ with the euclidean metric. You must prove your assertions.

(a) $E_1 = \mathbb{Z}$.

(b) $E_2 = \{1/n : n = 1, 2, 3, \ldots\}$.

(c) $E_3 = \{r \in \mathbb{Q} : r^2 \leq 2\}$.

**Problem 3.** (1 point) Let $E$ be a subset of a metric space $X$.

(a) Must $E$ and its closure $\bar{E}$ have the same interiors? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

(b) Must $E$ and its interior $E^\circ$ have the same closures? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

**Problem 4.** (1 point) Let $E$ be a connected subset of a metric space.

(a) Must its closure $\bar{E}$ be connected? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

(b) Must its interior $E^\circ$ be connected? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

**Problem 5.** (1 point) Does there exist a compact subset of $\mathbb{R}$ whose set of limit points is countably infinite? If so, provide an example and prove that it is an example. If not, prove that no such set exists.

**Problem 6.** (3 points)

(a) A metric space $X$ is *separable* if it contains a countable dense subset. Give an example of a metric space which is *not* separable.

(b) A *base* for a metric space $X$ is a collection $\mathcal{U}$ of open subsets of $X$ such that, for every open set $G \subseteq X$, there exists a collection of open sets $U_\alpha \in \mathcal{U}$ such that

$$G = \bigcup_\alpha U_\alpha.$$

Give an example for a base for the metric space $\mathbb{R}^2$.

(c) Show that a metric space $X$ is separable if and only if it has a countable base.

**Problem 7.** (5 points) Let $X$ be a metric space in which every infinite subset has a limit point.

(a) For any $\varepsilon \gneq 0$, show that $X$ can be covered by finitely many open balls of radius $\varepsilon$.

(b) Show that $X$ is separable.

(c) Show that $X$ is compact.

**Problem 8.** (5 points) Let $(X, d)$ be a metric space and let $\mathcal{C}$ be the set of all nonempty, bounded and closed subsets of $X$. Define $f : X \times \mathcal{C} \to \mathbb{R}$ by

$$f(x, B) := \inf_{b \in B} d(x, b)$$

and define $g : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ by

$$g(A, B) := \sup_{a \in A} f(a, B).$$

(a) Show that $g$ need not be a metric on $\mathcal{C}$.

(b) Let $h : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be defined by

$$h(A, B) := \max\{g(A, B), g(B, A)\}.$$

Show that $h$ is a metric on $\mathcal{C}$.