Examples of Metric Spaces

Example 1. Let $X := \mathbb{R}$ and let $d(x, y) := |x - y|$. This is a metric. I will leave it to you to verify that it axioms (M1), (M2) and (M4). The fact that it satisfies the triangle inequality axiom (M3) is the usual triangle inequality for $\mathbb{R}$.

Example 2. Let $X := \mathbb{Q}$ and let $d(x, y) := |x - y|$ again. This is again a metric, for the same reasons as above. Thus $\mathbb{Q}$ is also a metric space.

Example 3 (Euclidean metric)**.** Generalizing the previous ex, for $X := \mathbb{R}^n$, we let

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are elements of $\mathbb{R}^n$. Again I leave it to you to verify (M1), (M2) and (M4), and you should be familiar with the triangle inequality (M3) from math 54 or some other equivalent linear algebra class. This is often called the *standard metric* or *euclidean metric* on $\mathbb{R}^n$.

Example 4 (Manhattan metric)**.** Let $X := \mathbb{R}^2$ and define

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. There is a nice accompanying picture here... I will leave it to you to check that (M1), (M2) and (M4) are satisfied for this. For (M3), notice that

$$d(\mathbf{x}, \mathbf{z}) = |x_1 - z_1| + |x_2 - z_2| \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$. Thus the triangle inequality is satisfied as well. This is called the *Manhattan metric* on $\mathbb{R}^2$. It admits a straightforward generalization to $\mathbb{R}^n$ as well.

Example 5 (Maximum metric)**.** Consider $X := \mathbb{R}^2$ and define

$$d(\mathbf{x}, \mathbf{y}) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Again, I leave it to you to verify that (M1), (M2) and (M4) are satisfied here. For the triangle inequality (M3), suppose we have $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$. Then

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) &= \max\{|x_1 - z_1|, |x_2 - z_2|\} \\ &\leq \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\} \\ &= |x_i - y_i| + |y_i - z_i| \end{aligned}$$

where i is whichever value (1 or 2) achieves the maximum. But notice that

$$|x_i - y_i| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} \text{ and } |y_i - z_i| \leq \max\{|y_1 - z_1|, |y_2 - z_2|\}$$

which means that

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) &\leq |x_i - y_i| + |y_i - z_i| \\ &\leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} \\ &= d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}). \end{aligned}$$

This completes the proof of the triangle inequality.

Example 6 (Supremum metric)**.** Let S be a set and let $\mathcal{B}(S)$ be the set of bounded functions $f : S \to \mathbb{R}$. Recall that $f : S \to \mathbb{R}$ is *bounded* if there exists some real number R such that $|f(s)| \leq R$ for all $s \in S$. Define

$$\|f\|_{\text{sup}} := \sup_{s \in S} f(s).$$

Notice that if f and g are both bounded functions, then

$$|f(s) - g(s)| \leq |f(s)| + |g(s)|$$

for all $s \in S$ by the triangle inequality for real numbers. In other words, if $f - g$ denotes the function $s \mapsto f(s) - g(s)$, then $f - g$ is also bounded. The *supremum metric* on $\mathcal{B}(S)$ is defined by

$$d(f, g) = \|f - g\|_{\text{sup}}.$$

I will leave it to you to prove that this is actually a metric.

Example 7 (Discrete metric)**.** Let X be any set at all and define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

This is a metric, called the *discrete metric*. I will leave it to you to verify (M1), (M2) and (M4). For (M3), suppose we have $x, y, z \in X$. We have a total of 5 cases. Here are the first 2, and the remaining 3 are left to you.

Case 1. Suppose $x = y = z$. Then $d(x, z) = 0 \leq 0 + 0 = d(x, y) + d(y, z)$.

Case 2. Suppose $x = y$ and z is distinct. Then $d(x, z) = 1 \leq 0 + 1 = d(x, y) + d(y, z)$.

Example 8 (Hamming distance)**.** Fix an integer n and consider the set $X := \{0, 1\}^n$ of binary strings of length n. Given two binary strings x and y, let $d(x, y)$ be the number of positions in which the strings x and y differ. For example, when $n = 5$, we have

$$d(10001, 10011) = 1 \text{ and } d(11001, 11110) = 3.$$

This is a metric, called the *Hamming distance*. The proof that this is a metric is an exercise.

Example 9 (L^2 pseudometric)**.** Let X be the set of integrable functions on $[0,1]$ and let

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, dx.$$

One can check that this verifies axioms (M1), (M2) and (M3). We will probably prove this later in the course. But it does not satisfy axiom (M4). To see this, consider the functions

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

and $g(x) = 0$ constantly. Then it is easy to see that $d(f,g) = 0$, even though $f \neq g$.

Example 10 (Metric on the leaves of a tree)**.** This one is hard to explain without pictures, so I'm omitting it from this PDF document. Come ask me about it in person if you need to know more about this example.

Example 11 (p-adic metric on $\mathbb{Q}$)**.** Let p be a prime number. For any nonzero integer a, we can write $a = p^n u$ for some nonnegative integer n and some integer u which is not divisible by p. Thus, for any nonzero rational number r, we can write $r = p^n u$ where n is some integer and u is some rational number whose numerator and denominator are both not divisible by p. We then define

$$|a/b|_p := p^{-r},$$

and we define $|0|_p := 0$. It is an exercise to prove that the function $d(r,s) = |r - s|_p$ is a metric on $\mathbb{Q}$, called the *p-adic metric*.