Problem Set 6

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1. (1 point) Let X and Y be metric spaces and let E be a dense subset of X. Show that if f and g are both continuous functions $X \to Y$ such that $f(x) = g(x)$ for all $x \in E$, then in fact $f(x) = g(x)$ for all $x \in X$.

Problem 2. (1 point) Let X, X', Y and Y' be metric spaces and suppose $f : X \to Y$ and $f' : X' \to Y'$ are continuous functions. Show that the function $f \times f' : X \times X' \to Y \times Y'$ defined by

$$(f \times f')(x, x') = (f(x), f'(x'))$$

is also continuous. (Regard $X \times X'$ and $Y \times Y'$ as metric spaces using the product metric defined in problem 10 of problem set 3.)

Problem 3. (1 point) Let X be a metric space. Show that the metric $d_X : X \times X \to \mathbb{R}$ is *uniformly* continuous (when $X \times X$ is given the product metric defined in problem 10 of problem set 2 and $\mathbb{R}$ is given the euclidean metric).

Problem 4. (1 point) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\limsup |a_n| = 0$. Let $X := [0, 1]$ and for each $n \in \mathbb{N}$ consider the function $f_n : X \to \mathbb{R}$ defined as follows.

$$f_n(x) = (x + a_n)^2$$

Does this sequence of functions converge uniformly?

Problem 5. (1 point) Let X be a metric space. Given a pair of points $x, y \in X$, a *path from x to y* is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Then X is *path-connected* if, for every pair of points $x, y \in X$, there exists a path from x to y.

Show that, if X is path-connected, then it is also connected.

Problem 6. (3 points) Let X be an open subset of $\mathbb{R}^2$. Show that X is connected if and only if it is path-connected. *Hint.* When X is nonempty, fix a point $a \in X$ and let U be the set of $x \in X$ such that there exists a path from a to x. Show that U is open and closed in X.

Remark. It is not true for general subsets of $\mathbb{R}^2$ that connectedness implies path-connectedness. See http://math.stanford.edu/~conrad/diffgeomPage/handouts/sinecurve.pdf for a description of a famous counterexample.

Problem 7. (3 points) Let X be a metric space and suppose $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence of uniformly continuous functions on X. Show that $f := \lim f_n$ is also uniformly continuous.

Problem 8. (3 points) Let X be a metric space and E a dense subset. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on X which converges uniformly on E. Then $(f_n)_{n\in\mathbb{N}}$ also converges uniformly on X. *Hint.* Show that $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on X.

Problem 9. Let

$$X := \mathbb{R} \setminus \left(\left\{-\frac{1}{n^2} : n = 1, 2, \dots\right\}\right)$$

and for each positive integer n, define the function $f_n : X \to \mathbb{R}$ as follows.

$$f_n(x) = \frac{1}{1 + n^2 x}$$

You should be able to verify that the series $\sum f_n(x)$ converges for all $x \in X \setminus \{0\}$.

(a) (3 points) Describe all subsets $S \subseteq X$ such that the series of functions $\sum f_n$ is uniformly convergent on S. *Hint.* Use problem 8 to rule out some possibilities.

(b) (1 point) Let $f : X \setminus \{0\} \to \mathbb{R}$ be the pointwise limit of the series of functions $\sum f_n$. Show that f is continuous.