Math 103A — Modern Algebra I

Instructor: Sunny (Shishir Agrawal)

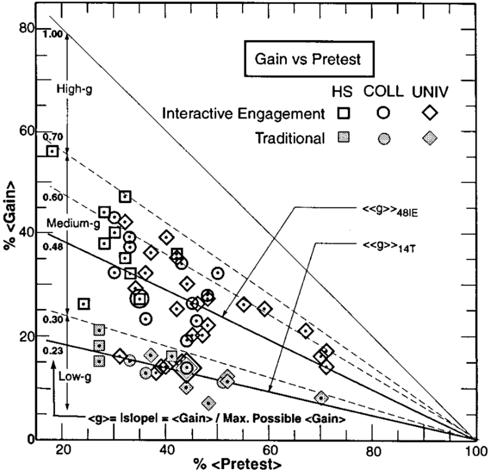
Common Ground

Turn to someone sitting near you who you don't already know. Take about 5 minutes to find at least *two* things that you have in common with your partner.

(Try to go beyond "We're both taking Math 103A this quarter," but it doesn't have to anything deeply personal.)

About Me

Pedagogy Data



Hake, doi:10.1119/1.18809

Class Structure

https://sagrawalx.github.io/teaching/fa22_math103a/

Group Theory

A "group" is a gadget that helps us understand symmetries.

History

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Here’s a story about how the concept started to solidify.

Quadratic Formula

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- ...

Cubic Formula

The solutions to

$ax^3 + bx^2 + cx + d = 0$ are:

$$x_k = \frac{-1}{3a}\left(b + \xi^k C + \frac{\Delta_0}{\xi^k C}\right)$$

for $k = 0, 1, 2$, where:

$$\xi = (-1 + \sqrt{-3})/2$$

$$\Delta_0 = b^2 - 3ac$$

$$\Delta_1 = 2b^3 - 9abc + 27a^2 d$$

$$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

Gerolamo Cardano (Italy, 1501–1576)

Quartic Formula

Lodovico Ferrari (Italy, 1522–1565) and Gerolamo Cardano (Italy, 1501–1576)

$$r_1 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}(b^2-3ac+12d)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}(b^2-3ac+12d)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}}{54}\right)^{\frac{1}{3}} - \frac{-a^3+4ab-8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}(b^2-3ac+12d)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}}{54}\right)^{\frac{1}{3}}}}}$$

$$r_2 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}(b^2-3ac+12d)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}(b^2-3ac+12d)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}}{54}\right)^{\frac{1}{3}} - \frac{-a^3+4ab-8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}(b^2-3ac+12d)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4(b^2-3ac+12d)^3+(2b^3-9abc+27c^2+27a^2d-72bd)^2}}{54}\right)^{\frac{1}{3}}}}}$$

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What about the quintic?

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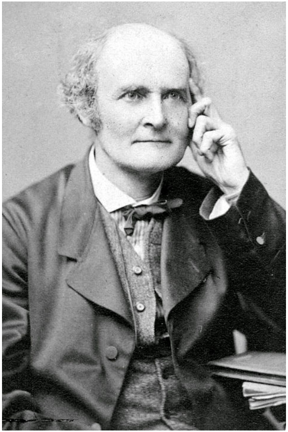
300 years go by...

Insolvability of the Quintic

Niels Henrik Abel (Norway, 1802–1829)

Évariste Galois (France, 1811–1832)

Modern Definition of Groups



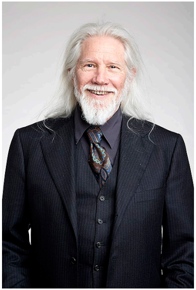
Arthur Cayley (UK, 1821–1895)

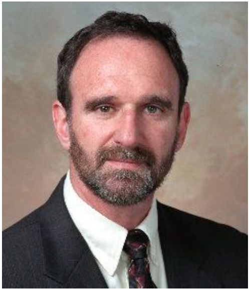
First Isomorphism Theorem, Applications in Physics

Amalie Emmy Noether (Germany, 1882–1935)

Applications in Cryptography

Malcolm Williamson (UK and USA, 1950–2015)





Whitfield Diffie (USA, 1944–) and Martin Hellman (USA, 1945–)

Let's do some math!

1. How many positive integers less than 12 are relatively prime to 12?

(A) 1

(B) 4

(C) 5

(D) None of the above

2. What is the greatest common divisor (gcd) of 12 and 30?

(A) 2

(B) 3

(C) 4

(D) None of the above