Integration

1 Partitions

Let $a \leq b$ be real numbers. A *partition* of the interval $[a, b]$ is a finite subset P of $[a, b]$ containing both a and b. We'll write P_0 for the smallest element of P, P_1 for the second smallest, and so forth. In other words, $P = \{P_0, P_1, \ldots, P_n\}$ and

$$a = P_0 \lneq P_1 \lneq \cdots \lneq P_n = b.$$

We define the *mesh* of P by

$$\operatorname{mesh}(P) := \max\{P_k - P_{k-1} : k = 1, \ldots, n\}.$$

If P and Q are both partitions, we say that Q is a *refinement* of P if $Q \supseteq P$. Clearly we then have $\operatorname{mesh}(Q) \leq \operatorname{mesh}(P)$.

Example 1.1. The following are all partitions of the unit interval $[0, 1]$.

$$\begin{aligned} P &:= \{0, 1\} \\ Q &:= \left\{0, \frac{1}{2}, 1\right\} \\ R &:= \left\{0, \frac{1}{2}, \frac{2}{3}, 1\right\} \\ S &:= \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{5}{6}, 1\right\} \end{aligned}$$

Notice that Q, R and S are all refinements of P, R is also a refinement of Q, S is not a refinement of either Q or R. We have $\operatorname{mesh}(P) = 1$, and the mesh of all of remaining partitions is $1/2$.

If P and Q are both partitions of $[a, b]$, a *common refinement* of P and Q is a partition which is a refinement of both P and Q simultaneously. Notice that the union $P \cup Q$ is always a common refinement of P and Q. In other words, common refinements always exist.

Remark 1.2. Observe the following properties of the relation "is a refinement of" on the set of all partitions of $[a, b]$.

(1) Any partition P is a refinement of itself.

(2) If P is a refinement of a Q, and Q is also a refinement of P, then $P = Q$.

(3) If P is a refinement of Q, and Q is a refinement of R, then P is a refinement of R.

(4) If P and Q are partitions, there exists a partition R which is a refinement of both P and Q.

The first three properties say exactly that the relation "is a refinement of" is reflexive, antisymmetric, and transitive. In other words, the relation "is a refinement of" is an example of a *partial order*, and the set of all partitions, equipped with this relation, is a *partially ordered set*. Notice that this partial order isn't totally ordered, since there are partitions neither of which is a refinement of the other. But, even though its not a totally ordered set, it is a *directed set* because it also satisfies property 4 listed above.

Anyway, this is all just terminology and we won't use this terminology again. We will, however, see some of the above properties recur as we start proving things about integrals.

2 Upper and Lower Darboux Sums

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let P be a partition of $[a, b]$. We define the *upper Darboux sum* of f with respect to P to be the quantity

$$U(f, P) := \sum_{k=1}^{n} (P_k - P_{k-1}) \sup f([P_{k-1}, P_k])$$

and analogously we define the *lower Darboux sum* of f with respect to P to be the quantity

$$L(f, P) := \sum_{k=1}^{n} (P_k - P_{k-1}) \inf f([P_{k-1}, P_k]).$$

These quantities have a very natural interpretation in terms of areas of rectangles. I'll draw a picture in class.

The following lemma says that taking refinements "squeezes" the Darboux sums closer together.

Lemma 2.1. *Suppose P is a partition of $[a, b]$ and Q is a refinement of P. Then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. The middle inequality is obvious. We'll prove the inequality $L(f, Q) \leq L(f, P)$, and the proof that $U(f, P) \leq U(f, Q)$ is analogous. First consider the case when Q has exactly one more point than P. In other words, the elements of Q are

$$P_0 \lneq P_1 \lneq P_{k-1} \lneq Q_k \lneq P_k \lneq \cdots \lneq P_n.$$

Then the fact that $[P_{k-1}, P_k]$ contains both $[P_{k-1}, Q_k]$ and $[Q_k, P_k]$ means that

$$\inf f([P_{k-1}, Q_k]), \inf f([Q_k, P_k]) \geq \inf f([P_{k-1}, P_k]).$$

We want to show that $L(f, Q) - L(f, P) \geq 0$. Notice that the only terms that don't cancel out in this difference are the following.

$$\inf f([P_{k-1}, Q_k])(Q_k - P_{k-1}) + \inf f([Q_k, P_k])(P_k - Q_k) - \inf f([P_{k-1}, P_k])(P_k - P_{k-1})$$

To see that this is nonnegative, observe the following.

$$\begin{aligned}\inf f([P_{k-1}, P_k])(P_k - P_{k-1}) &= \inf f([P_{k-1}, P_k])(P_k - Q_k + Q_k - P_{k-1}) \\ &= \inf f([P_{k-1}, P_k])(Q_k - P_{k-1}) + \inf f([P_{k-1}, P_k])(P_k - Q_k) \\ &\leq \inf f[P_{k-1}, Q_k](Q_k - P_{k-1}) + \inf f([Q_k, P_k])(P_k - Q_k)\end{aligned}$$

This completes the proof. In general, if Q has m more elements than P, then we just apply the above argument m times, adding one extra element of Q each time. □

Corollary 2.2. *If P and Q are any partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.*

Proof. Let R be a common refinement of P and Q. Then lemma 2.1 tells us that

$$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q).$$ □

3 Integrability

For a bounded function $f : [a, b] \to \mathbb{R}$, we define the following.

$$\begin{aligned}U(f) &:= \inf\{U(f, P) : P \text{ is a partition of } [a, b]\} \\ L(f) &:= \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}\end{aligned}$$

Lemma 3.1. *For any partition P of $[a, b]$, we have*

$$L(f, P) \leq L(f) \leq U(f) \leq U(f, P).$$

In particular, $L(f)$ and $U(f)$ are real numbers.

Proof. Both the first and third inequalities are trivial. We only need to show that $L(f) \leq U(f)$. Fix a partition Q. Then corollary 2.2 guarantees that $L(f, Q) \leq U(f, R)$ for all partitions R, which means that $L(f, Q)$ is a lower bound for the same set for which $U(f)$ is an infimum. Thus $L(f, Q) \leq U(f)$. But now, since Q is an arbitrary partition, we see that $U(f)$ is an upper bound for the same set for which $L(f)$ is a supremum, so $L(f) \leq U(f)$. □

If $L(f) = U(f)$, then the function f is said to be *integrable* and the common value $L(f) = U(f)$ is called the *integral* of f on $[a, b]$ and is denoted

$$\int_a^b f \text{ or } \int_a^b f(x)\, dx.$$

Proposition 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if, for each $\varepsilon \gneq 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) \leq \varepsilon$.*

Proof. Suppose first that f is integrable and fix $\varepsilon \gneq 0$. Then, by definition of supremums and infimums, there exist partitions P and Q such that

$$L(f, P) \geq L(f) + \frac{\varepsilon}{2} \text{ and } U(f, Q) \leq U(f) + \frac{\varepsilon}{2}.$$

Let R be a common refinement of P and Q. Then, using lemma 2.1, we see that

$$U(f, R) - L(f, R) \leq U(f, Q) - L(f, P) \leq U(f) - L(f) + \varepsilon = \varepsilon.$$

Conversely, fix some $\varepsilon \gneq 0$ and let P be a partition such that $U(f, P) - L(f, P) \leq \varepsilon$. Observe that

$$U(f) \leq U(f, P) = U(f, P) - L(f, P) + L(f, P) \leq \varepsilon + L(f, P) \leq \varepsilon + L(f).$$

Since $\varepsilon \gneq 0$ is arbitrary, we conclude that $U(f) \leq L(f)$, which proves that $U(f) = L(f)$ using the middle inequality of lemma 3.1. □

Our next task is to prove that a large class of functions are automatically integrable.

Lemma 3.3. *Let $f : [a, b] \to \mathbb{R}$ be a function.*

(a) If f is monotonic, then it is integrable.

(b) If f is continuous, then it is integrable.

Proof. We'll prove that f is integrable when it is monotonically increasing, and the monotonically decreasing case is analogous. If $f(a) = f(b)$, then f is a constant function and we are done immediately, so suppose that $f(a) \lneq f(b)$. Notice that

$$f(a) \leq f(x) \leq f(b)$$

for all $x \in [a, b]$, so f is bounded. We'll show that f is integrable using proposition 3.2. Fix $\varepsilon \gneq 0$ and let P be any partition such that

$$\operatorname{mesh}(P) \leq \frac{\varepsilon}{f(b) - f(a)}.$$

Then

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^{n} (\sup f([P_{k-1}, P_k]) - \inf f([P_{k-1}, P_k]))(P_k - P_{k-1}) \\
&= \sum_{k=1}^{n} (f(P_k) - f(P_{k-1}))(P_k - P_{k-1}) \\
&\leq \sum_{k=1}^{n} (f(P_k) - f(P_{k-1})) \cdot \frac{\varepsilon}{f(b) - f(a)} \\
&= \frac{\varepsilon}{f(b) - f(a)} \sum_{k=1}^{n} (f(P_k) - f(P_{k-1})) \\
&= \varepsilon.
\end{aligned}$$

This completes the proof of (a). For (b), if f is continuous, the fact that $[a, b]$ is compact means that f is automatically bounded and uniformly continuous. We'll again use proposition 3.2 to prove integrability. Fix $\varepsilon \gneq 0$ and let $\delta \gneq 0$ be such that $|x - y| \leq \delta$ implies

$$|f(x) - f(y)| \leq \frac{\varepsilon}{b-a}.$$

Fix any partition P such that $\operatorname{mesh}(P) \leq \delta$. Since f assumes its maximum and minimum on $[P_{k-1}, P_k]$ for all k, we see that

$$\sup f([P_{k-1}, P_k]) - \inf f([P_{k-1}, P_k]) \leq \frac{\varepsilon}{b-a}.$$

This means that

$$U(f, P) - L(f, P) \leq \sum_{k=1}^{n} \frac{\varepsilon}{b-a} \cdot (P_k - P_{k-1}) = \varepsilon. \qquad \square$$

Lemma 3.4. *Let f and g be integrable functions $[a, b] \to \mathbb{R}$ and let c be a real number.*

(a) cf is integrable and

$$\int_a^b cf = c \int_a^b f.$$

(b) $f + g$ is integrable and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(c) $|f|$ is integrable, where $|f|$ is given by $x \mapsto |f(x)|$.

Proof. For (a), there is nothing to show when $c = 0$. First consider the case when $c \gneq 0$. Let P be a partition of $[a, b]$. Clearly

$$\sup cf([P_{k-1}, P_k]) = c \cdot \sup f([P_{k-1}, P_k])$$

which means that $U(cf, P) = cU(f, P)$. Thus it follows that $U(cf) = cU(f)$. Similarly, we also have $L(cf) = cL(f)$. Then

$$U(cf) = cU(f) = cL(f) = L(cf)$$

so cf is integrable and

$$\int_a^b cf = c \int_a^b f.$$

Now to prove the case when $c \lneq 0$, it suffices to consider the case when $c = -1$, since then for general $c \lneq 0$, we have

$$\int_a^b cf = \int_a^b -(-c)f = -\int_a^b (-c)f = -(-c)\int_a^b f = c\int_a^b f$$

using the case $c = -1$ for the second equality and the case $c \gneq 0$ for the third. Thus we assume that $c = -1$. Then its easy to see that $U(-f, P) = -L(f, P)$ for all partitions P. Thus

$$U(-f) = \inf\{U(-f, P) : P \text{ is a partition}\} = -\sup\{L(f, P) : P \text{ is a partition}\} = -L(f).$$

Similarly, we also get $L(-f) = -U(f)$. Thus, since f is integrable, we have

$$U(-f) = -L(f) = -U(f) = L(-f)$$

so $-f$ is integrable and

$$\int_a^b (-f) = -\int_a^b f.$$

This completes the proof of (a).

For (b), we'll use proposition 3.2 again. Fix $\varepsilon \gneq 0$. Since f is sintegrable, there exists a partition with respect to which the upper Darboux sum and lower Darboux sum of f differ by at most $\varepsilon/2$. There exists another partition which does the same thing for g, and then by taking a common refinement of these two partitions, we find a partition P such that

$$U(f,P) - L(f,P) \leq \frac{\varepsilon}{2} \text{ and } U(g,P) - L(g,P) \leq \frac{\varepsilon}{2}.$$

Observe that for any $S \subseteq [a,b]$, we have

$$\inf(f+g)(S) \geq \inf f(S) + \inf g(S)$$

which implies that

$$L(f+g,P) \geq L(f,P) + L(g,P)$$

and similarly we have

$$U(f+g,P) \leq U(f,P) + U(g,P).$$

Thus

$$U(f+g,P) - L(f+g,P) \leq (U(f,P) + U(g,P)) - (L(f,P) + L(g,P)) \leq \varepsilon.$$

Thus $f+g$ is integrable by proposition 3.2. Moreover, observe that

$$\int_a^b (f+g) \leq U(f+g,P) \leq U(f,P) + U(g,P) \leq L(f,P) + L(g,P) + \varepsilon \leq \int_a^b f + \int_a^b g + \varepsilon$$

and similarly

$$\int_a^b (f+g) \geq L(f+g,P) \geq L(f,P) + L(g,P) \geq U(f,P) + U(g,P) - \varepsilon = \int_a^b f + \int_a^b g - \varepsilon.$$

Combining these inequalities, we find that

$$\int_a^b f + \int_a^b g - \varepsilon \leq \int_a^b (f+g) \leq \int_a^b f + \int_a^b g + \varepsilon$$

so, since ε is arbitrary, we are done.

For (c), first observe that

$$\sup |f|(S) - \inf |f|(S) \leq \sup f(S) - \inf f(S)$$

for any subset $S \subseteq [a, b]$. Indeed, observe that for any $x, y \in S$, we have

$$|f(x)| - |f(y)| \leq |f(x) - f(y)| \leq |\sup f(S) - \inf f(S)| = \sup f(S) - \inf f(S).$$

Now taking the supremum over all $x \in S$ and then the infimum over all $y \in S$ to complete the proof. Now notice that this inequality tells us that

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$$

for any partition P. Thus, for any $\varepsilon \gneq 0$, there exists a partition P such that $U(f,P) - L(f,P) \leq \varepsilon$ since f is integrable, and then we have

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) \leq \varepsilon$$

so proposition 3.2 guarantees integrability. $\square$

Lemma 3.5. *If $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence of integrable functions $[a, b] \to \mathbb{R}$ and $f = \lim f_n$, then f is integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Let $\varepsilon_n := \|f_n - f\|_{\sup}$ so that

$$f_n(x) - \varepsilon_n \leq f(x) \leq f_n(x) + \varepsilon_n$$

for all $n \in \mathbb{N}$ and $x \in [a, b]$. This means that, for any partition P, we have

$$L(f_n - \varepsilon_n, P) \leq L(f, P) \text{ and } U(f, P) \leq U(f_n + \varepsilon_n, P)$$

so taking the supremum and the infimum over all P, we find that

$$\int_a^b (f_n - \varepsilon_n) = L(f_n - \varepsilon_n) \leq L(f) \leq U(f) \leq U(f_n + \varepsilon_n) = \int_a^b (f_n + \varepsilon_n). \tag{1}$$

This means that

$$0 \leq U(f) - L(f) \leq \int_a^b (f_n + \varepsilon_n) - \int_a^b (f_n - \varepsilon_n) = \int_a^b (2\varepsilon_n) = 2\varepsilon_n(b - a).$$

Now observe that this is true for all n and that $\lim \varepsilon_n = 0$, so we conclude that $U(f) = L(f)$. Thus f is integrable. Now observe that (1) tells us that

$$\left(\int_a^b f_n\right) - \varepsilon_n(b - a) \leq \int_a^b f \leq \left(\int_a^b f_n\right) + \varepsilon_n(b - a).$$

In other words,

$$\left|\int_a^b f - \int_a^b f_n\right| \leq \varepsilon_n(b - a)$$

so letting n tend to infinity completes the proof. $\square$

4 Properties of Integrals

Lemma 4.1. *Let f and g be integrable functions $[a,b] \to \mathbb{R}$. If $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. Observe that $h = g - f$ is integrable due to lemma 3.4 and $h(x) \geq 0$ for all $x \in X$. This clearly means that $L(h,P), U(h,P) \geq 0$ for all partitions P, so

$$\int_a^b g - \int_a^b f = \int_a^b h \geq 0. \qquad \square$$

Corollary 4.2. *If $f : [a,b] \to \mathbb{R}$ is integrable, then*

$$\left| \int_a^b f \right| \leq \int_a^b |f|,$$

where $|f|$ denotes the function $x \mapsto |f(x)|$.

Proof. We know from lemma 3.4 that $|f|$ is integrable. Observe that $-|f|(x) \leq f(x) \leq |f|(x)$ for all x. This means that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

using lemma 4.1, and this proves the result. $\square$

Lemma 4.3. *If $f : [a,b] \to \mathbb{R}$ is continuous and nonnegative and*

$$\int_a^b f = 0,$$

then $f(x) = 0$ for all $x \in [a,b]$.

Proof. For any partition P, the fact that $f(x) \geq 0$ means that $L(f,P) \geq 0$. This means that

$$0 \leq L(f,P) \leq L(f) = \int_a^b f = 0$$

so actually $L(f,P) = 0$. But

$$L(f,P) = \sum_{k=1}^{n} \inf f([P_{k-1}, P_k])(P_k - P_{k-1})$$

and $P_k - P_{k-1} \gneq 0$, so actually we must have

$$\inf f([P_{k-1}, P_k]) = 0.$$

This is true for any partition P. Then for any compact interval $[s,t]$ contained in $[a,b]$, we can consider the partition $P = \{a,s,t,b\}$ and the above shows that $\inf f([s,t]) = 0$.

Now suppose for a contradiction that $f(x) \neq 0$ for some x. Since f is nonnegative, we have $f(x) \gneq 0$. Since f is continuous, there exists some $\delta \gneq 0$ such that $f(y) \gneq 0$ whenever $|x-y| \leq \delta$. But then we clearly cannot have $\inf f([x-\delta, x+\delta]) = 0$, since f must achieve its minimum on the compact set $[x-\delta, x+\delta]$ but its also strictly positive on this entire compact set. This is a contradiction. $\square$

5 Sample Problems

Problem 1. Show that the following function $f : [0,1] \to \mathbb{R}$ is not integrable.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Problem 2. Give an example of a sequence of integrable functions $(f_n)_{n\in\mathbb{N}}$ which converges pointwise to a non-integrable function.

Hint. Find a sequence of integrable functions converging to the function in problem 1.

Problem 3. Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in [a,b]$ some value such that f is integrable on $[a,c]$ and $[c,b]$. Then f is integrable on $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Fix $\varepsilon \gneq 0$. Then there exists a partition P of $[a,c]$ such that $U(f,P) - L(f,P) \leq \varepsilon/2$. Similarly there exists a partition Q of $[c,b]$ such that $U(f,Q) - L(f,Q) \leq \varepsilon/2$. But then observe that $R = P \cup Q$ is a partition of $[a,b]$, and clearly

$$U(f,R) = U(f,P) + U(f,Q) \text{ and } L(f,R) = L(f,P) + L(f,Q)$$

so then

$$U(f,R) - L(f,R) = U(f,P) - L(f,P) + U(f,Q) - L(f,Q) \leq \varepsilon.$$

Thus f is integrable. Moreover, observe that

$$\int_a^b f = U(f) \leq U(f,R) = U(f,P) + U(f,Q) \leq L(f,P) + L(f,Q) + \varepsilon \leq \int_a^c f + \int_c^b f + \varepsilon$$

and analogously we also have

$$\int_a^c f + \int_c^b f - \varepsilon \leq \int_a^b f$$

so, since $\varepsilon \gneq 0$ is arbitrary, we are done. $\square$

Problem 4. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that there exists some $c \in [a,b]$

such that

$$f(c) = \frac{1}{b-a} \int_a^b f.$$

Hint. Use the intermediate value theorem.