# Series

## 1 Series

Given a sequence $(a_n)_{n\in\mathbb{R}}$ of real numbers, the symbol

$$\sum_{n=0}^{\infty} a_n,$$

or, more succinctly, $\sum a_n$, is called a *series* and is shorthand for the sequence $(s_n)_{n\in\mathbb{N}}$, where

$$s_n := a_0 + \cdots + a_n$$

is the *nth partial sum* of the series for all $n \in \mathbb{N}$. For example, saying that the series $\sum a_n$ *converges* means that the sequence $(s_n)_{n\in\mathbb{N}}$ converges. If $\sum a_n$ converges or if $\lim s_n = \pm\infty$, then the same symbol $\sum a_n$ is also used as shorthand to denote the value $\lim s_n$.

**Example 1.1** (Geometric series)**.** Let $r$ be a real number and let $a_n := r^n$ for all $n \in \mathbb{N}$. Then the series

$$\sum_{n=0}^{\infty} r^n$$

is called a *geometric series*. The $n$th partial sum of this series is

$$s_n := 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

since we can verify directly by multiplication that

$$(1 - r)(1 + r + \cdots + r^n) = 1 - r^{n+1}.$$

If $|r| \lneq 1$, then we know that $\lim r^{n+1} = 0$, so

$$\sum_{n=0}^{\infty} = \lim_{n\to\infty} \left(\frac{1 - r^{n+1}}{1 - r}\right) = \frac{1}{1 - r}.$$

We will show a bit later that the geometric series $\sum r^n$ does not converge when $|r| \geq 1$.

**Example 1.2.** Recall that, when we were discussing ternary expansions for the Cantor set, we stated that $0.9999\cdots$ is a decimal expansion for the number 1. We can now make sense of this.

Notice that

$$0.9999\cdots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \sum_{n=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^n.$$

Notice that the $n$th partial sum $s_n$ for the series $\sum(9/10)(1/10)^n$ is

$$s_n = \frac{9}{10}\left(\frac{1}{10}\right)^0 + \frac{9}{10}\left(\frac{1}{10}\right)^1 + \cdots + \frac{9}{10}\left(\frac{1}{10}\right)^n = \frac{9}{10}\left(1 + \frac{1}{10} + \cdots + \left(\frac{1}{10}\right)^n\right) = \frac{9}{10} \cdot t_n$$

where $t_n$ is the $n$th partial sum for the geometric series $\sum(1/10)^n$. Since $|1/10| \lneq 1$, by our calculation in example 1.1 we see that

$$\lim_{n\to\infty} t_n = \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{1-(1/10)} = \frac{10}{9},$$

which means that

$$0.9999\cdots = \sum_{n=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{9}{10}\cdot t_n\right) = \frac{9}{10}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = 1.$$

**Lemma 1.3** (Cauchy criterion)**.** *The series $\sum a_n$ converges if and only if, for every $\varepsilon \gneq 0$, there exists $N \in \mathbb{N}$ such that*

$$|a_m + a_{m+1} + \cdots + a_{n-1} + a_n| \lneq \varepsilon$$

*whenever $n \geq m \geq N$.*

*Proof.* Let $s_n := a_0 + \cdots + a_n$ be the $n$th partial sum of the series. Then $\sum a_n$ converges if and only if the sequence $(s_n)_{n\in\mathbb{N}}$ converges, if and only if the sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy (since $\mathbb{R}$ is a complete metric space). Now the sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy if and only if, for all $\varepsilon \gneq 0$, there exists $N \in \mathbb{N}$ such that

$$d(s_n, s_{m-1}) = |s_n - s_{m-1}| \lneq \varepsilon$$

for all $n \geq m-1 \geq N$. But notice that

$$s_n - s_{m-1} = (a_0 + \cdots + a_n) - (a_0 + \cdots + a_{m-1}) = a_m + \cdots + a_n$$

so we are done. $\square$

**Corollary 1.4.** *If a series $\sum a_n$ converges, then $\lim a_n = 0$.*

*Proof.* Fix an $\varepsilon \gneq 0$. The Cauchy criterion 1.3 says there exists some $N$ such that $|a_m + \cdots + a_n| \lneq \varepsilon$ for all $n \geq m \geq N$. In particular, taking $m = n$, we see that we have $|a_n| \lneq \varepsilon$ for all $n \geq N$. $\square$

**Example 1.5.** The geometric series $\sum r^n$ does not converge when $|r| \geq 1$. Indeed, notice that when $|r| \geq 1$, clearly the sequence $(r^n)_{n\in\mathbb{N}}$ does not converge to 0 (if $r = 1$, then it converges to 1, and if $r \neq 1$, then it does not converge at all), so the contrapositive of corollary 1.4 proves that $\sum r^n$ does not converge.

**Example 1.6.** The converse of corollary 1.4 is not true: for example, we have $\sum 1/n = \infty$ even though $\lim 1/n = 0$. We will prove that $\lim 1/n = \infty$ a bit later.

# 2 Special Types of Series

## 2.1 Series of Nonnegative Terms

**Lemma 2.1.** *Suppose $\sum a_n$ is a series where $a_n \geq 0$ for all $n$. Then $\sum a_n$ equals the supremum of its set of partial sums. In particular, $\sum a_n$ converges if and only if its set of partial sums is bounded.*

*Proof.* If $s_n$ is the $n$th partial sum of the series, notice that

$$s_{n+1} = a_0 + \cdots + a_n + a_{n+1} = s_n + a_{n+1} \geq s_n$$

for all $n \in \mathbb{N}$, so the sequence $(s_n)_{n\in\mathbb{N}}$ is monotonically increasing. Thus

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = \sup\{s_n : n \in \mathbb{N}\}. \qquad \square$$

## 2.2 Alternating Series

An *alternating series* is a series of the form $\sum(-1)^n a_n$ where $a_n \geq 0$ for all $n \in \mathbb{N}$. In other words, the terms of an alternating series "alternate" between being nonnegative and nonpositive.

**Lemma 2.2** (Alternating series test)**.** *If $(a_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence with $\lim a_n = 0$, then the alternating series $\sum(-1)^n a_n$ converges.*

*Proof.* Fix $n \geq m$ and let

$$\alpha := a_m - a_{m+1} + \cdots + (-1)^{n-m} a_n. \tag{1}$$

We claim that $0 \leq \alpha \leq a_m$. To prove this, we consider two cases.

*Case 1.* First, suppose $n - m$ is odd. Then

$$\alpha = a_m - a_{m+1} + \cdots + a_{n-1} - a_n = (a_m - a_{m+1}) + (a_{m+2} - a_{m+3}) + \cdots + (a_{n-1} - a_n) \geq 0,$$

since the fact that $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing means that each parenthetical term is nonnegative, so $\alpha$ is obtained by adding together a whole bunch of positive numbers. On the other hand, notice also that

$$\alpha = a_m - (a_{m+1} - a_{m+2}) - (a_{m+3} - a_{m+4}) - \cdots - (a_{n-2} - a_{n-1}) - a_n \leq a_m$$

since we are subtracting a whole bunch of positive numbers from $a_m$.

*Case 2.* Next, suppose $n - m$ is even. Then

$$\alpha = (a_m - a_{m+1}) + (a_{m+2} - a_{m+3}) + \cdots + (a_{n-2} - a_{n-1}) + a_n \geq 0$$

since we are adding together a bunch of nonnegative reals. Also, we have

$$\alpha = a_m - (a_{m+1} - a_{m+2}) - \cdots - (a_{n-1} - a_n) \leq a_m$$

since we are subtracting a whole bunch of positive numbers from $a_m$ again.

This proves our claim that $0 \leq \alpha \leq a_m$. Now, since $\lim a_n = 0$, for any $\varepsilon \gneq 0$ there exists some $N \in \mathbb{N}$ such that $|a_m| \lneq \varepsilon$ for all $m \geq N$. Then for all $n \geq m \geq N$, if we let $\alpha$ be defined as in equation (1), we have

$$\left|(-1)^m a_m + (-1)^{m+1} a_{m+1} + \cdots + (-1)^n a_n\right| = |(-1)^m \alpha| = \alpha \leq a_m = |a_m| \lneq \varepsilon.$$

Thus the Cauchy criterion 1.3 guarantees convergence of $\sum(-1)^n a_n$. □

**Example 2.3.** The alternating series test 2.2 immediately implies that $\sum(-1)^n/n$ converges.

# 3 Absolute Convergence

A series $\sum a_n$ *converges absolutely* if the series $\sum |a_n|$ converges. (Note that $\sum |a_n|$ is a series of nonnegative terms, so its limit is always defined: if it converges then it is a real number, and otherwise it is $\infty$.)

**Lemma 3.1.** *If a series $\sum a_n$ converges absolutely, it it converges.*

*Proof.* Fix $\varepsilon \gneq 0$. Since $\sum a_n$ converges absolutely, the series $\sum |a_n|$ converges, so the Cauchy criterion 1.3 implies there exists some $N \in \mathbb{N}$ such that

$$||a_m| + \cdots + |a_n|| \lneq \varepsilon.$$

By repeatedly applying the triangle inequality, we see that

$$|a_m + \cdots + a_n| \leq |a_m| + |a_{m+1} + \cdots + a_n| \leq \cdots \leq |a_m| + \cdots + |a_n| = ||a_m| + \cdots + |a_n|| \lneq \varepsilon$$

so the Cauchy criterion 1.3 guarantees that $\sum a_n$ converges as well. □

**Example 3.2.** Recall that a geometric series $\sum r^n$ converges if and only if $|r| \lneq 1$. But if $|r| \lneq 1$, then the series $\sum |r^n| = \sum |r|^n$ also converges since $||r|| = |r| \lneq 1$. Thus, if the geometric series $\sum r^n$ converges at all, it converges absolutely.

**Example 3.3.** As we noted in example 2.3, the alternating series test implies that $\sum(-1)^n/n$ converges. But we will prove below that $\sum |(-1)^n/n| = \sum 1/n$ does not converge, so $\sum(-1)^n/n$ is an example of a series which converges but not absolutely.

One important reason to care about absolute convergence is that absolutely convergent series are invariant under rearrangement. More precisely, let $\sum a_n$ be a series. Then a *rearrangement* of this series is a series of the form $\sum a_{\sigma(n)}$ where $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection.

**Example 3.4.** Consider the following series.

$$\underbrace{1-1}_{\text{2 terms}}+\underbrace{\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}}_{\text{4 terms}}+\underbrace{\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}}_{\text{8 terms}}+\underbrace{\frac{1}{8}-\frac{1}{8}+\cdots-\frac{1}{8}}_{\text{16 terms}}+\cdots$$

Clearly this sequence does not converge absolutely. On the other hand, it is easy to see from the alternating series test 2.2 that this converges, but the alternating series test does not calculate this limit for us. To calculate the limit, observe that the sequence of partial sums for this series is

$$\left(1,0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{4},0,\frac{1}{4},0,\frac{1}{4},0,\frac{1}{4},0,\frac{1}{8},0,\dots\right)$$

and now you should be able to prove that this converges to 0. On the other hand, we can rearrange this series as follows.

$$1+\frac{1}{2}+\frac{1}{2}-1+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}+\frac{1}{8}+\frac{1}{8}-\frac{1}{4}+\cdots$$

The sequence of partial sums of this rearrangement is

$$\left(1,\frac{3}{2},2,1,\frac{5}{4},\frac{3}{2},1,\frac{5}{4},\frac{3}{2},1,\frac{9}{8},\frac{5}{4},1,\dots\right)$$

and you should be able to prove that this converges to 1.

**Lemma 3.5.** *Let $\sum a_n$ be an absolutely convergent series. Then for any bijection $\sigma:\mathbb{N}\to\mathbb{N}$, the rearrangement $\sum a_{\sigma(n)}$ also converges absolutely and*

$$\sum_{n=0}^{\infty} a_n=\sum_{n=0}^{\infty} a_{\sigma(n)}.$$

*Proof.* Let $s_n$ and $t_n$ be the $n$th partial sum of $\sum a_n$ and $\sum a_{\sigma(n)}$, respectively. Fix $\varepsilon \gneq 0$. Then there exists an integer $N$ such that

$$|a_m|+\cdots+|a_n| \lneq \varepsilon \tag{2}$$

whenever $n\geq m\geq N$. Now let

$$N' := \max \sigma^{-1}(\{0,\dots,N\}),$$

and observe that for any $k=0,\dots N$ we have $\sigma^{-1}(k)\leq N'$, which means that $k=\sigma(k')$ for some $k'\leq N$. In other words, $\{0,\dots,N\}\subseteq\{\sigma(0),\dots,\sigma(N')\}$. Now for any $n\geq N$, notice that all of the terms $a_0,\dots,a_N$ cancel out in the difference

$$s_n-t_n=(a_0+\cdots+a_n)-(a_{\sigma(0)}+\cdots+a_{\sigma(n)}).$$

Some other terms might cancel out as well, but what remains behind is the sum of a bunch of terms of the form $\pm a_k$ for distinct $k\gneq N$. Then the triangle inequality says that $|s_n-t_n|$ is less than the sum of the absolute values of a bunch of terms of the form $|a_k|$ where $k \gneq N$, and then equation (2) shows that this sum must be less than $\varepsilon$. In other words, for all $n \geq N'$, we have $|s_n - t_n| \lneq \varepsilon$, which means that $\lim |s_n - t_n| = 0$. You should now be able to prove why this implies that $(t_n)_{n\in\mathbb{N}}$ converges to the same limit as $(s_n)_{n\in\mathbb{N}}$. $\square$

**Remark 3.6.** In fact, the situation regarding rearrangements for non-absolutely convergent series is basically as bad as can be. The "Riemann rearrangement theorem" says that if $\sum a_n$ is a series which converges, but not absolutely, then for any $\alpha, \beta$ such that

$$-\infty \leq \alpha \leq \beta \leq \infty,$$

there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that, if $t_n$ is the $n$th partial sum of the rearrangement $\sum a_{\sigma(n)}$ for all $n \in \mathbb{N}$, then

$$\liminf_{n\to\infty} t_n = \alpha \text{ and } \limsup_{n\to\infty} t_n = \beta.$$

In particular, by taking $\alpha = \beta$, we see that there exist rearrangements which converge to any possible value in the extended real number system! The proof of this theorem is conceptually not very difficult (and I will draw a picture in class), but the notation is difficult to keep track of. See theorem 3.54 in Rudin for a formal proof.

# 4 Tests for Absolute Convergence

## 4.1 Comparison Test

**Lemma 4.1** (Comparison test)**.** *Let $\sum a_n$ be a series with $a_n \geq 0$ for all $n \in \mathbb{N}$.*

*(a) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all $n \in \mathbb{N}$, then $\sum b_n$ converges absolutely.*

*(b) If $\sum a_n = \infty$ and $b_n \geq a_n$ for all $n \in \mathbb{N}$, then $\sum b_n = \infty$.*

*Proof.* Fix $\varepsilon \gneq 0$. By the Cauchy criterion 1.3, the convergence of $\sum a_n$ implies that there exists an $N \in \mathbb{N}$ such that

$$|a_m + \cdots + a_n| \lneq \varepsilon$$

for all $n \geq m \geq N$. Then

$$||b_m| + \cdots + |b_n|| = |b_m| + \cdots + |b_n| \leq a_m + \cdots + a_n = |a_m + \cdots + a_n| \lneq \varepsilon.$$

Thus applying the Cauchy criterion 1.3 again shows that $\sum |b_n|$ converges, proving part (a). For part (b), notice that $b_n \geq a_n \geq 0$ and

$$b_0 + \cdots + b_n \geq a_0 + \cdots + a_n.$$

Since $\sum a_n = \infty$, its set of partial sums is unbounded, so $\sum b_n$ is a series of non-negative terms with unbounded partial sums, so $\sum b_n = \infty$ also. $\square$

**Example 4.2** (Harmonic series)**.** Let us prove that the *harmonic series* $\sum(1/n)$ diverges to $\infty$. To do this, let us define the following.

$$\begin{aligned}
b_1 &= 1 \\
b_2 &= \frac{1}{2} \\
b_3 &= \frac{1}{3} + \frac{1}{4} \\
b_4 &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
b_5 &= \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{15} + \frac{1}{16} \\
&\vdots \\
b_n &= \frac{1}{2^{n-2}+1} + \cdots + \frac{1}{2^{n-1}} \\
&\vdots
\end{aligned}$$

Notice that then the sequence of partial sums of $\sum b_n$ is a *subsequence* of the sequence of partial sums of $\sum(1/n)$. But the sequence of partial sums of $\sum(1/n)$ is monotonically increasing, so if one of its subsequence diverges to $\infty$, then it must itself diverge to $\infty$. In other words, it suffices to show that $\sum b_n = \infty$. To do this, observe the following.

$$\begin{aligned}
b_3 &= \frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\
b_4 &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \\
&\vdots \\
b_n &= \frac{1}{2^{n-2}+1} + \cdots + \frac{1}{2^{n-1}} \geq \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^{n-1}} = \frac{2^{n-2}}{2^{n-1}} = \frac{1}{2}.
\end{aligned}$$

In other words, if we let $a_n := 1/2$ for all $n \in \mathbb{N}$, we see that $b_n \geq a_n$ for all $n \in \mathbb{N}$. Thus, by the comparison test 4.1, it suffices to show that $\sum a_n = \infty$. This is easy to check directly from the definition, since for any real number $R$, if $N$ is any positive integer such that $N \gneq 2R - 1$, then

$$a_0 + \cdots + a_n \geq a_0 + \cdots + a_N = \frac{N+1}{2} \gneq R.$$

This completes the proof that $\sum(1/n) = \infty$. In particular, $\sum(1/n)$ does *not* converge, even though $\lim(1/n) = 0$. Thus the harmonic series shows that the converse to corollary 1.4 is not true: even if the terms of a series get arbitrarily small, the series need not converge.

One can generalize this procedure slightly and prove the following proposition 4.3. You can find this generalization carried out in the proofs of theorems 3.27 and 3.28 in Rudin (note that theorem 3.27 of Rudin is sometimes also called the "Cauchy condensation test").

**Proposition 4.3.** *Let $p$ be a real number. Then the series $\sum(1/n^p)$ converges absolutely if $p \gneq 1$*

*and diverges to* $\infty$ *if* $p \leq 1$.

**Remark 4.4.** You can also prove proposition 4.3 using the "integral test" like in theorem 15.1 in Ross, but we haven't defined integrals yet so this feels like cheating to me.

**Example 4.5.** Consider the series $\sum 1/(n^2+1)$. Then $\sum 1/n^2$ converges by proposition 4.3 and $1/(n^2+1) \leq 1/n^2$ for all $n \in \mathbb{N}$, so the comparison test 4.1(a) shows that $\sum 1/(n^2+1)$ converges.

## 4.2 Root Test

**Lemma 4.6** (Root test)**.** *Let* $\sum a_n$ *be a series and let* $\alpha := \limsup |a_n|^{1/n}$.

*(a) If* $\alpha \lneq 1$*, then the series* $\sum a_n$ *converges absolutely.*

*(b) If* $\alpha \gneq 1$*, then the series* $\sum a_n$ *does not converge.*

*Proof.* Suppose $\alpha \lneq 1$, and let $r$ be such that $\alpha \lneq r \lneq 1$. Then there exists an $N$ such that $|a_n|^{1/n} \lneq r$ for all $n \geq N$. In other words, we have $|a_n| \lneq r^n$, and the geometric series $\sum r^n$ converges, so the comparison test 4.1(a) guarantees that $\sum a_n$ converges absolutely.

On the other hand, if $\alpha \gneq 1$, then $|a_n|^{1/n} \gneq 1$ for infinitely many $n$, which means that $|a_n| \gneq 1$ for infinitely many $n$. Thus we cannot have $\lim a_n = 0$, so corollary 1.4 proves that $\sum a_n$ cannot converge. $\square$

**Example 4.7.** If it turns out that $\alpha := \limsup |a_n|^{1/n} = 1$, then the root test tells us nothing about the convergence of the series $\sum a_n$. For example, notice that if $a_n = 1/n$, then, since $\lim n^{1/n} = 1$, we have $\alpha = \limsup |1/n|^{1/n} = \lim |1/n|^{1/n} = 1$, and we know from example 4.2 that $\sum 1/n$ diverges to $\infty$. On the other hand, notice that if $a_n = 1/n^2$, then $\lim (n^2)^{1/n} = 1^2 = 1$, so again we have $\alpha = \limsup |1/n^2|^{1/n} = \lim |1/n^2|^{1/n} = 1$, even though we know from proposition 4.3 that $\sum 1/n^2$ converges absolutely.

## 4.3 Ratio Test

**Lemma 4.8** (Ratio test)**.** *Let* $\sum a_n$ *be a series with* $a_n \neq 0$ *for all* $n \in \mathbb{N}$*. Define*

$$\alpha := \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \quad \textit{and } \beta := \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

*(a) If* $\beta \lneq 1$*, then the series* $\sum a_n$ *converges absolutely.*

*(b) If* $\alpha \gneq 1$*, then the series* $\sum a_n$ *does not converge.*

*Proof.* For (a), choose some constant $r$ such that $\beta \lneq r \lneq 1$. Then there exists $N$ such that $|a_{n+1}/a_n| \lneq \beta$ for all $n \geq N$. Then

$$\begin{aligned}
|a_{N+1}| &\leq r\,|a_N| \\
|a_{N+2}| &\leq r\,|a_{N+1}| \leq r^2\,|a_N| \\
&\vdots \\
|a_{N+k}| &\leq r^k\,|a_N|
\end{aligned}$$

But $r \lneq 1$, so we know that the geometric series $\sum r^k$ converges. Then $\sum |a_N| r^k$ also converges, which means that the series $\sum a_{N+k}$ also converges absolutely by the comparison test. Now clearly

$$\sum_{n=0}^{\infty} a_n = a_0 + \cdots + a_{N-1} + \sum_{k=0}^{\infty} a_{N+k},$$

so we have proved (a). For (b), notice that $\alpha \gneq 1$ means that there exists an $N$ such that $|a_{n+1}| \geq |a_n|$ for all $n \geq N$. This means that it is impossible to have $\lim a_n = 0$, so corollary 1.4 again implies that $\sum a_n$ cannot converge. □

**Example 4.9.** Keeping notation as in the statement of the ratio test 4.8, we claim that the test is inconclusive if $\alpha \leq 1 \leq \beta$. To see this, notice that for both of the series $\sum 1/n$ and $\sum 1/n^2$, we have $\alpha = \beta = 1$, but the former diverges to $\infty$ while the former converges.

**Remark 4.10.** An interesting and sometimes useful fact is that, for any sequence of positive real numbers $(a_n)_{n\in\mathbb{N}}$, we have

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

This means that, if the ratio test guarantees convergence, then the root test also guarantees convergence. In other words, as a test for convergence, the root test is *stronger* than the ratio test. That said, notice that another consequence of this chain of inequalities is that, if $\lim(a_{n+1}/a_n)$ exists, so that the limit inferior and limit superior on both sides of the above inequalities are equal, so actually $\lim a_n^{1/n}$ must exist also (here we are using a problem on problem set 4). This means that, even if we're having trouble calculating $\limsup a_n^{1/n}$, if we find that $\lim(a_{n+1}/a_n)$ exists, then its value must equal $\limsup a_n^{1/n}$. You can find a proof of this chain of inequalities in Rudin, theorem 3.37.

# 5 Power Series

Given a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers and a variable $x$, a *power series* is an expression of the form

$$\sum_{n=0}^{\infty} a_n x^n.$$

We want to treat such an expression as a function, where we plug in various real numbers $x$ and calculate the value of the resulting series to determine the output of the function. Of course, there's no reason that we have to be able to do this for all $x$.

**Lemma 5.1.** *For a power series $\sum a_n x^n$, let*

$$\alpha := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

*and let $R := \alpha^{-1}$. If $\alpha = 0$, then we define $R := \infty$, and if $\alpha = \infty$, we define $R := 0$.*

*(a) For all $|x| \lneq R$, the series $\sum a_n x^n$ converges absolutely.*

*(b) For all $|x| \gneq R$, the series $\sum a_n x^n$ diverges.*

*The number $R$ is called the* radius of convergence *of the power series $\sum a_n x^n$.*

*Proof.* This is an easy consequence of the root test. Let

$$\alpha_x := \limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \limsup_{n\to\infty} |x| \sqrt[n]{|a_n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x|\,\alpha.$$

The third equality is justified by problem 7. We now consider three cases.

*Case 1.* Suppose $0 \lneq R \lneq \infty$. Then $\alpha_x = |x|/R$. If $|x| \lneq R$, then $\alpha_x \lneq 1$ so the series converges absolutely by the root test. Similarly, if $|x| \gneq R$ then $\alpha_x \gneq 1$ so the series diverges by the root test.

*Case 2.* Suppose $R = 0$. Then $\alpha = \infty$ so, for all $x \neq 0$ we have $\alpha_x = \infty$, so the series diverges by the root test. For $x = 0$, however, clearly the series converges.

*Case 3.* Suppose $R = \infty$. Then $\alpha = 0$, so $\alpha_x = 0 \lneq 1$ for all $x$, so the series converges for all $x$ by the root test. □

Thus, if $R \gneq 0$ is the radius of convergence of the power series $\sum a_n x^n$, then this power series gives rise to a well-defined function $B(0, R) \to \mathbb{R}$.

# 6 Sample Problems

**Problem 1.** Determine whether or not the following series converge.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$
$$\sum_{n=0}^{\infty} (-1)^n$$
$$\sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

*Hint for (c).* This is a "telescoping sum." Note that the $n$th partial sum is

$$\left(\sqrt{1} - \sqrt{0}\right) + \left(\sqrt{2} - \sqrt{1}\right) + \cdots + \left(\sqrt{n+1} - \sqrt{n}\right) = \sqrt{n+1}.$$

**Problem 2.** Prove that $0.2222\cdots$ is a ternary expansion for 1.

**Problem 3.** Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}.$$

(a) Show that this series converges to a positive number.

(b) Find a rearrangement of this series which converges to a negative number.

*Hint.* For (a), notice that we can group terms of the series as follows.

$$\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots$$

For (b), consider the following rearrangement.

$$\left(\frac{1}{2}-\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{7}-\frac{1}{9}\right)+\left(\frac{1}{6}-\frac{1}{11}-\frac{1}{13}\right)+\cdots$$

**Problem 4.** Consider the series $\sum a_n$ where $a_n = n3^{-n}$.

(a) Use the ratio test to prove that $\sum a_n$ converges absolutely.

(b) Use the root test to prove that $\sum a_n$ converges absolutely.

*Reference.* Ross, section 14, example 6.

**Problem 5.** Consider the series $\sum a_n$ where $a_n = 2^{(-1)^n - n}$ for all $n \in \mathbb{N}$.

(a) Show that $\liminf |a_{n+1}/a_n| = 1/8$.

(b) Show that $\limsup |a_{n+1}/a_n| = 2$.

(c) Show that $\limsup |a_n|^{1/n} = 1/2$.

What does the root test say about convergence of the series $\sum a_n$? What does the ratio test say?

**Problem 6.** Determine whether or not each of the following series converges.

$$\sum_{n=0}^{\infty}\left(\sqrt[n]{n}-1\right)^n$$
$$\sum_{n=0}^{\infty}\frac{n}{n^2+3}$$
$$\sum_{n=0}^{\infty}\left(\frac{2}{(-1)^n-3}\right)^n$$
$$\sum_{n=0}^{\infty}\left(\frac{\sqrt{n+1}-\sqrt{n}}{n}\right)$$

*Hints and references.* If you're stuck on the second and third one, see Ross, section 15, examples 4 and 7. For the first one, use the root test. Hint for the the fourth one: find a way to compare the series to the series $\sum n^{-3/2}$.

**Problem 7.** If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ and $c$ is some nonnegative real constant, show that

$$\limsup_{n\to\infty} cx_n = c\cdot\limsup_{n\to\infty} x_n.$$

*Hint.* If $c = 0$, we are done immediately, so consider $c \gneq 0$. For each side, choose a subsequence whose limit equals the corresponding limit superior, and then use properties of limits...

**Problem 8.** What is the radius of convergence of each of the following power series?

$$\sum_{n=0}^{\infty} x^n$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n}$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n^2}$$
$$\sum_{n=0}^{\infty} n!x^n$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sum_{n=0}^{\infty} \frac{x^{3n}}{2^n}$$

Also, for each of the above in which the radius of convergence $R$ satisfies $0 \lneq R \lneq \infty$, try to determine what happens when $|x| = R$.

*Hint.* For some of these, it may be easier to calculate $\alpha := \limsup |a_n|^{1/n}$ by instead showing that $\lim |a_{n+1}/a_n|$ exists and then using remark 4.10 to conclude that that $\alpha$ must equal this limit. If you're stuck, check out examples 1 through 6 in section 23 of Ross.