Name:

Quiz 2 Solutions

Instructions. The only tools you are permitted to use are pencils, pens, erasers, a sheet of handwritten notes, and your mind. No electronic devices! You have 1 hour. Good luck! ☺

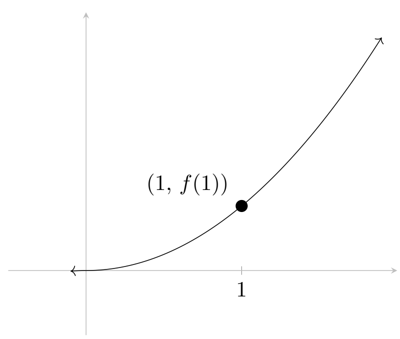
Problem 1 (True-false, 8 points)**.** You will get 1 point for a correct answer and 0 points for blank and incorrect answers.[1] No explanations are required.

(1) If $f(x) = |x^2 - 1|$, then there are exactly two values of x such that $f'(x)$ does not exist. **T** F

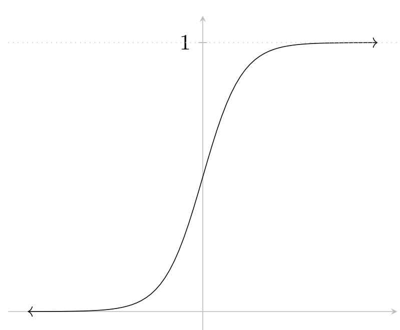
(2) The function $f(x) = \sqrt[3]{x}$ is differentiable at $x = 0$. T **F**

(3) If f and g are differentiable functions such that $f'(0) = 7$ and $g'(0) = 2$, then it must be the case that $(fg)'(0) = 14$. T **F**

(4) For the function f whose graph is depicted below, $f'(1)$ is greater than the slope of the secant line passing through $(1, f(1))$ and $(1+h, f(1+h))$ for all $h > 0$. T **F**



(5) For the function f whose graph is depicted below, we have $\lim_{x\to\infty} f'(x) = 1$. T **F**



(6) If $f(x) = \sin(x)$, then $f^{(2019)}(x) = \sin(x)$. T **F**

(7) If f is differentiable and odd, then f' is even. **T** F

(8) If $f(x) = x^{\sin(x)}$, then $f'(\pi/2) = 1$. **T** F

[1] During quiz revisions, you'll have the chance to meet with me one-on-one and convince me that you understand and can fully explain *up to 2 of the true-false questions that you left blank* to get full credit on those questions.

Problem 2 (3 points)**.** For each of the following functions f, calculate $f'(0)$. Show your work.

(a) $f(x) = x\cos(x)$

Solution. We have $f'(x) = \cos(x) - x\sin(x)$ by the product rule, so $f'(0) = 1$.

(b) $f(x) = \dfrac{x^2 - 1}{x + 2}$

Solution. We have

$$f'(x) = \frac{2x(x+2) - (x^2-1)}{(x+2)^2} = \frac{x^2+4x+1}{(x+2)^2}$$

by the quotient rule, so $f'(0) = 1/4$.

(c) $f(x) = \ln(x^2 + 2x + 2)$

Solution. We have

$$f'(x) = \frac{2x+2}{x^2+2x+2}$$

by the chain rule, so $f'(0) = 1$.

Problem 3 (2 points)**.** What is the slope of the tangent line to the curve defined by

$$e^y = \cos(xy)$$

at the point $(0, 0)$?

Solution. We differentiate implicitly and find

$$e^y \cdot \frac{dy}{dx} = -\sin(xy) \cdot \left(y + x\frac{dy}{dx}\right) \implies \frac{dy}{dx} = \frac{-y\sin(xy)}{e^y + x\sin(xy)}$$

We then plug in $x = y = 0$ and find that $\dfrac{dy}{dx} = 0$.

Problem 4 (2 points)**.** Approximate $\sqrt{101}$. Do better than $\sqrt{101} \approx 10$, and show your work. Is your approximation an overestimate or an underestimate?

Solution. Let $f(x) = \sqrt{x}$, so that $f'(x) = \dfrac{1}{2\sqrt{x}}$. Then $f(100) = 10$ and $f'(100) = \dfrac{1}{20}$, so the tangent line is given by

$$y - 10 = \frac{1}{20}(x - 100).$$

When $x = 101$, we find that

$$y = 10 + \frac{1}{20} = 10.05.$$

Thus $\sqrt{101} \approx 10.05$. The tangent line to f at $x = 100$ sits above the graph of f, so this approximation overestimates the true value of $\sqrt{101}$.

An alternative approach is to use Newton's method. Consider the function $f(x) = x^2 - 101$, which has a zero at $\sqrt{101}$. Suppose we start with the initial guess $x_0 = 10$. Then the tangent line at x_0 has slope $f'(10) = 20$ and so has equation

$$y + 1 = 20(x - 10).$$

The x-intercept of this line happens when

$$1 = 20(x - 10) \implies x = 10 + \frac{1}{20} = 10.05.$$

This too has to be an overestimate: it is clear from a graph of the function $f(x) = x^2 - 101$ that the zero of the tangent line at $x_0 = 10$ must overshoot the zero of f itself. An advantage of using Newton's method instead of just a linearization is that one could improve the estimate by setting $x_1 = 10.05$ and running another iteration, but the problem does not ask for this.

Incidentally, the actual square root is $10.0498756211\cdots$ so we got pretty darn close!

Problem 5 (3 points)**.** A car travels down a highway at a constant speed of 30 m/s. An observer stands 400 m from the highway. How fast is the distance between the observer and the car increasing 10 s after the car passes directly in front of the observer? Show your work.

Solution. The observer, the car, and the point on the highway directly in front of the observer form a right triangle. If h is the distance between the observer and the car, and x is the distance between the car and the point on the highway directly in front of the observer, then, by the Pythagorean theorem, we have

$$h^2 = 400^2 + x^2.$$

Differentiating with respect to time, we get

$$2h\frac{dh}{dt} = 2x\frac{dx}{dt}$$

so

$$\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}.$$

So we need to figure out x and h when 10 s have elapsed since the car passed directly in front of the observer. Since the car is traveling at a constant speed of 30 m/s, we see that

$$x|_{t=10} = 30 \text{ m/s} \times 10 \text{ s} = 300 \text{ m}.$$

By the Pythagorean theorem, we see that

$$h|_{t=10} = \sqrt{400^2 + 300^2} = 500.$$

Thus

$$\left.\frac{dh}{dt}\right|_{t=10} = \frac{300}{500} \cdot 30 = 18.$$

Problem 6 (Extra Credit! 2 points, no partial credit)**.** A train leaves a station, going directly east at a constant velocity v. A second train leaves the station at the same time, also going at the same constant velocity v, but it follows a track that runs $\pi/3$ radians north of east. At what rate is the distance between the two trains growing? Explain your answer.

Solution. Let x be the distance between the station and the first train. Then $dx/dt = v$. Since the second train is traveling at the same speed and it leaves the station at the same time, the distance between the station and the second train is also x.

Consider the triangle formed by the station and the two trains. We know that two of the side lengths are both x, and the angle between them is $\pi/3$. But such a triangle must be equilateral! Thus the third side (that is, the side corresponding to the distance between the two trains) is also x. Thus the distance between the two trains is changing at a rate of $dx/dt = v$.

Honor code. If you have neither given nor received any unauthorized aid on this quiz, please write either "HCU" or "Honor Code Upheld" below, and sign your name next to it.

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