Proof Writing Advice*

Writing proofs is as much an art form as it is a scientific process. There is no algorithm that will *generate* proofs for any possible mathematical statement (though there are algorithms that can *check* a proof for correctness). Each statement you want to prove may require you to think a little bit differently. Different people may come up with different proofs for the same statement. What's more, even once you know how to prove a statement, there is an art to writing down your proof in a clear and logical way so that a human reader can understand your argument.

Scratchwork Phase

When you first see a statement that you want to prove, do some scratchwork. Maybe play with a few examples until you feel like you have some intuition. Maybe list off some facts that you can immediately deduce from the assumptions. Maybe list off some facts that would allow you to deduce the conclusion. Your scratchwork will inevitably look like a complete mess, and that's okay — that scratchwork is for no eyes but your own.

As you do this scratchwork, at some point something will click. It may take a lot of scratchwork to reach that point, but be patient with yourself.

When you do reach that point, grab a new sheet of paper or fire up LaTeX and start writing up a formal proof. You'll probably find yourself picking out pieces of your scratchwork to go into your formal proof. Make sure that you only pick out the pieces that are actually relevant (there may be many dead-ends in your scratchwork that need not go in your proof at all), and that you pick them out in the correct order for a proof (which is not necessarily the same as the order in which you had the ideas when you were doing your scratchwork).

I cannot stress the importance of scratchwork enough. When you read a proof, all you're seeing is the finished product. You're not seeing all of the scratchwork that went into that finished product, in part because that scratchwork would not be useful for anyone to look at except the peron who did it. This can make proofs seem mysterious sometimes; you may

[0] This document is adapted from a similar handout prepared by Stefan Erickson.

frequently find yourself thinking, "how did anyone ever think of his?!" The bursts of insight that lead to proofs are a little mysterious. But those bursts of insight only happen when you do the scratchwork. Keep playing until you figure it out.

Writing Phase

You want to find a balance of mathematical rigor and poetic language when writing up your proof. A well-written proof will have a structure and flow to it which can be quite aesthetically pleasing. The best way to learn how to write proofs is by reading carefully through others' proofs (eg, in the book) and trying to mimic that style and structure when writing your own proofs. Below are a few guidelines to keep in mind.

- Remember that a proof is ultimately an argument that is meant to convince a human reader of its veracity. You want to try to be as clear and concise as possible so that you accomplish this goal.

- Start your proofs with clearly stating what you are assuming, and write a concluding sentence stating what you've proven. For example, suppose you want to prove the propositional statement:

  If P, then Q.

  You have the following options.

  1. (Direct Proof) *Assume that P is true. ... Therefore, Q is true.*
  2. (Contrapositive) *Assume that Q is false. ... Therefore, P is false. By the contrapositive, P implies Q.*
  3. (Proof by Contradiction) *Assume that P is true and Q is false. ... The statement A is both true and false. This is a contradiction. Therefore, Q is true.*

  For example, the statement

  If n is even, then n^2 is even

  can be proven in the following ways:

  1. (Direct Proof) *Assume that n is even. ... Therefore, n^2 is even.*

2. (Contrapositive) *Assume that* n^2 *is odd. ... Therefore,* n *is odd. By the contrapositive,* n *is even implies* n^2 *is even.*
3. (Proof by Contradiction) *Assume that* n *is even and* n^2 *is odd. ... We have a contradiction. Therefore,* n^2 *is even.*

(In this case, proof by contradiction isn't really appropriate because the statement you end up contradicting is "n *is even or* n^2 *is odd.*")

- If your theorem is stated as

  $$P \textit{ if and only if } Q$$

  you must prove two statements:

  $$\textit{If } P\textit{, then } Q \qquad \text{AND} \qquad \textit{If } Q\textit{, then } P.$$

  As an example, the statement

  $$n \textit{ is even if and only if } n^2 \textit{ is even.}$$

  you need to prove two separate statements:

  $$n \textit{ is even implies } n^2 \textit{ is even} \qquad \text{AND} \qquad n^2 \textit{ is even implies } n \textit{ is even.}$$

- Be sure to justify each step of your proof. Every statement should be explained by a definition, an axiom, a previously proven theorem, or a previous statement in your proof. Always use definitions exactly as they appear in the book. If a theorem is used, check all the conditions for the theorem to be applied.

- Sometimes it is better to use the *properties* of some term rather than its definition. For example, if you know that a is odd and $a \cdot b$ is even, then you can conclude that b is even using properties of even and odd integers without having to write down $a = 2k+1$ for some integer k.

- Obey the laws of logic. Make sure your statements are in the correct order.

- Use symbols only when they will help your reader understand your argument. If you can think of a clear and concise way of saying something in words rather than symbols, that may be the better way to go.

Checking Phase

It's very normal to feel uncertain about your proofs when you first start writing proofs. Here are some strategies you might use to check your proofs (and to build confidence about your proofs).

- Find some representative examples and check every step of your proof is true for these examples. For example, if you have just come up with a proof of the statement "*If* n *is even, then* n^2 *is even*," take $n = 2$ and go through every line in your proof and make sure that it is true for $n = 2$. Then repeat this with $n = 4$. And then maybe repeat it again, with $n = -18$.
- Go through your proof and identify exactly where you used each assumption. If you find that there's an assumption that you think you haven't used, that might point to a problem. At that point, you should look for an example which doesn't satisfy that assumption and work through your proof using that example.

At this point, you can be pretty sure your proof is correct! Read it one more time and make sure that it's as clear and concise as it can possibly be, and then call it a day ☺