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Quiz 4

Part I (10 points). You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

(1) $\mathcal{L}(\mathcal{P}_2(\mathbf{F}), \mathcal{P}_3(\mathbf{F}))$ is isomorphic to $\mathbf{F}^4 \times \mathcal{P}_2(\mathbf{F})'$. T **F**

(2) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p(1) = p(3) = 0\}$. Then $\mathcal{P}_4(\mathbf{R})/U$ is isomorphic to $\mathbf{R}^4$. T **F**

(3) If $T \in \mathcal{L}(\mathbf{F}^4)$ is surjective, then $\mathbf{F}^4/\operatorname{null} T$ is isomorphic to $\mathcal{P}_2(\mathbf{F})$. T **F**

(4) Let $U = \{p \in \mathcal{P}_2(\mathbf{R}) : p(1) = p'(1) = 0\}$. Then $x^2 + U = 1 + U$. T **F**

(5) Suppose $T \in \mathcal{L}(\mathbf{F}^3, \mathbf{F}^2)$ and T' is injective. Then $\dim(\mathbf{F}^2/\operatorname{range} T) = 2$. T **F**

(6) Suppose $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{F}))$ and $\dim \operatorname{null} T' = 1$. Then $\dim \operatorname{range} T = 1$. T **F**

(7) Suppose $T \in \mathcal{L}(V, W)$ and we choose bases for V and W such that $M(T)$ is a 3×5 matrix whose rows span a 3 dimensional space. Then T is injective. T **F**

(8) Let $\varphi \in \mathcal{P}_3(\mathbf{R})'$ be given by T **F**

$$\varphi(p) = \int_{-1}^{1} p(x)\, dx.$$

Then $\{p \in \mathcal{P}_3(\mathbf{R}) : \varphi \in \operatorname{span}(p)^0\} = \operatorname{span}(x, x^3)$.

(9) Suppose that U is a subspace of $\mathbf{F}^2$ such that $\mathbf{F}^2 = \operatorname{span}((1,0)) \oplus U$, and let φ_1, φ_2 be dual to the standard basis $(1,0), (0,1)$ of $\mathbf{F}^2$. Then $U^0 = \operatorname{span}(\varphi_2)$. T **F**

(10) If $T \in \mathcal{L}(V, W)$ and U is a subspace of $\operatorname{null} T$, then the map $\tilde{T} : V/U \to W$ given by $\tilde{T}(v + U) = T(v)$ is a well-defined injective linear map. T **F**

Part II (10 points).

(11) If V and W are arbitrary vector spaces, show that $V' \times W'$ is isomorphic to $(V \times W)'$.

Define $\Gamma : V' \times W' \to (V \times W)'$ by

$$\Gamma(\alpha, \beta)(v, w) = \alpha(v) + \beta(w)$$

for all $(\alpha, \beta) \in V' \times W'$ and $(v, w) \in V \times W$. There are now several steps.

(The following is an excruciatingly pedantic proof. Most of this was *not* required for full credit on this problem, but you should convince yourself that all of the steps are actually necessary and be able to prove any of them if you're told to. The key points here are the third and fourth steps.)

- First, we need to show that, for fixed $(\alpha, \beta) \in V' \times W'$, the map $\Gamma(\alpha, \beta) : V \times W \to \mathbf{F}$ Defined above is actually linear. Suppose $(v, w), (v', w') \in V \times W$ and $\lambda \in \mathbf{F}$. Then for any $(v, w) \in V \times W$, we have
$$\Gamma(\alpha, \beta)((v, w) + (v', w')) = \alpha(v + v') + \beta(w + w') = \alpha(v) + \alpha(v') + \beta(w) + \beta(w')$$
and
$$\Gamma(\alpha, \beta)(v, w) + \Gamma(\alpha, \beta)(v', w') = \alpha(v) + \beta(w) + \alpha(v') + \beta(w'),$$
so these two quantities are equal since addition is commutative, so $\Gamma(\alpha, \beta)$ commutes with addition. Also,
$$\Gamma(\alpha, \beta)(\lambda(v, w)) = \alpha(\lambda v) + \beta(\lambda w) = \lambda\alpha(v) + \lambda\beta(w)$$
and
$$\lambda\Gamma(\alpha, \beta)(v, w) = \lambda(\alpha(v) + \beta(w)) = \lambda\alpha(v) + \lambda\beta(w),$$
so these two quantities are also equal and $\Gamma(\alpha, \beta)$ commutes with scalar multiplication.

- Second, we need to show that $\Gamma : V' \times W' \to (V \times W)'$ is linear. Suppose we have $(\alpha, \beta), (\alpha', \beta') \in V' \times W'$ and $\lambda \in \mathbf{F}$. Then for any $(v, w) \in V \times W$, we have
$$\Gamma((\alpha, \beta)+(\alpha', \beta'))(v, w) = \Gamma(\alpha+\alpha', \beta+\beta')(v, w) = (\alpha+\alpha')(v)+(\beta+\beta')(w) = \alpha(v)+\alpha'(v)+\beta(w)+\beta'(w)$$
and
$$(\Gamma(\alpha, \beta) + \Gamma(\alpha', \beta'))(v, w) = \Gamma(\alpha, \beta)(v, w) + \Gamma(\alpha, \beta)(v, w) = \alpha(v) + \beta(w) + \alpha'(v) + \beta'(w),$$
and these two quantities are equal since addition is commutative. Thus Γ commutes with addition. Also,
$$\Gamma(\lambda(\alpha, \beta))(v, w) = \Gamma(\lambda\alpha, \lambda\beta)(v, w) = (\lambda\alpha)(v) + (\lambda\beta)(w) = \lambda\alpha(v) + \lambda\beta(w)$$
and
$$(\lambda\Gamma(\alpha, \beta))(v, w) = \lambda(\alpha(v) + \beta(w)) = \lambda\alpha(v) + \lambda\beta(w)$$
so these two quantities are also equal and Γ commutes with scalar multiplication.

- Third, we need to show that Γ is injective. Suppose $\Gamma(\alpha, \beta) = 0$. Then $\Gamma(\alpha, \beta)(v, w) = \alpha(v)+\beta(w) = 0$ for all $v, w \in V$. Fixing $w = 0$, we see that $\alpha(v) = 0$ for all $v \in V$, so $\alpha = 0$. Similarly, fixing $v = 0$ shows that $\beta = 0$. Thus $\operatorname{null}\Gamma = \{0\}$, so Γ is injective.

- Finally, we need to show that Γ is surjective. Suppose $\varphi \in (V \times W)'$. Define $\alpha \in V'$ by $\alpha(v) = \varphi(v, 0)$ and $\beta \in W'$ by $\beta(w) = \varphi(0, w)$. We now need to check that α and β are actually linear, but this proof is already excruciating enough, so I'll omit this. Observe then that

$$\Gamma(\alpha, \beta)(v, w) = \alpha(v) + \beta(w) = \varphi(v, 0) + \varphi(0, w) = \varphi((v, 0) + (0, w)) = \varphi(v, w)$$

  using linearity of φ, so $\Gamma(\alpha, \beta) = \varphi$.

(12) Let $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{P}_2(\mathbf{R})'$ be defined by $\varphi_1(p) = p(3)$, $\varphi_2(p) = p(-2)$, and $\varphi_3(p) = p(7)$. Is there a basis p_1, p_2, p_3 of $\mathcal{P}_2(\mathbf{R})$ whose dual basis is $\varphi_1, \varphi_2, \varphi_3$? If so, find such a basis and prove that it is actually a basis. Otherwise, explain why no such basis exists.

Observe that, if such a basis existed, p_1 must satisfy $\varphi_1(p_1) = p_1(3) = 1$, $\varphi_2(p_1) = p_1(-2) = 0$ and $\varphi_3(p_1) = p_1(7) = 0$. Since p_1 has zeroes at -2 and 7 and has degree at most 2, it must be of the form $p_1(x) = c_1(x + 2)(x - 7)$ for some scalar c_1. Then

$$1 = p_1(3) = c_1 \cdot 5 \cdot (-4) = -20c_1$$

so $c_1 = -1/20$ and $p_1(x) = (-1/20)(x+2)(x-7)$. Similar calculations show that $p_2(x) = (1/45)(x-3)(x-7)$ and $p_3(x) = (1/36)(x + 2)(x - 3)$.

To see that these three polynomials p_1, p_2, p_3 actually form a basis for $\mathcal{P}_2(\mathbf{R})$, observe that $\dim \mathcal{P}_2(\mathbf{R}) = 3$, so it is sufficient to show that these polynomials are linearly independent. Suppose we had a relation

$$a_1p_1 + a_2p_2 + a_3p_3 = 0.$$

Applying φ_1 to this relation, we find that

$$0 = \varphi_1(0) = \varphi_1(a_1p_1 + a_2p_2 + a_3p_3) = a_1 + 0 + 0 = a_1$$

so $a_1 = 0$. Similarly, applying φ_2 to the relation shows that $a_2 = 0$ and applying φ_3 shows that $a_3 = 0$. Thus p_1, p_2, p_3 is linearly independent.