Problem Set 5

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1. (1 point) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Hint. Notice that the nth term of the series can be rewritten as $1/n - 1/(n+1)$.

Proof. The sum telescopes: the nth partial sum is equal to

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

This tends to 1 as n increases, so the sum of the series is 1. $\square$

Problem 2. (1 point) Determine whether or not each of the following series converges.

$$\sum_{n=2}^{\infty} \frac{1}{n+(-1)^n}$$
$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

Proof. Notice that $n + (-1)^n \geq n - 1$ for all n, and the series

$$\sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Thus, by the comparison test, we see that

$$\sum_{n=2}^{\infty} \frac{1}{n+(-1)^n}$$

must diverge also. For the second series, we use the ratio test. Observe that

$$\frac{(n+1)^2/(n+1)!}{n^2/n!} = \frac{(n+1)^2 n!}{n^2(n+1)!} = \frac{(n+1)^2}{n^2(n+1)} = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}.$$

This sequence converges to 0 as n increases. Thus

$$\beta := \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 \lneq 1$$

so the ratio test guarantees convergence. $\square$

Problem 3. (1 point)

(a) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

(b) Does there exist an *absolutely* convergent series $\sum a_n$ for which $\sum a_n^2$ diverges?

Proof. For (a), the series $\sum a_n$ where $a_n = (-1)^n/\sqrt{n}$ converges by the alternating series test, but $a_n^2 = 1/n$ so the series $\sum a_n^2$ is the harmonic series, which diverges. For (b), notice that if $\sum a_n$ is absolutely convergent, in particular it is convergent so $\lim a_n = 0$, which means that there exists an N such that $|a_n| \leq 1$ for all $n \geq N$. Then

$$\left|a_n^2\right| = a_n^2 \leq |a_n|$$

for all $n \geq N$. The series

$$\sum_{n=N}^{\infty} |a_n|$$

is convergent since $\sum a_n$ is absolutely convergent, so

$$\sum_{n=N}^{\infty} a_n^2$$

is also convergent by the comparison test. Then

$$\sum_{n=0}^{\infty} a_n^2 = a_0^2 + \cdots + a_{N-1}^2 + \sum_{n=N}^{\infty} a_n^2$$

so $\sum a_n^2$ also converges. Thus there cannot exist such a series. $\square$

Problem 4. (1 point) Find the set of all $x \in \mathbb{R}$ for which the following power series converges.

$$\sum_{n=0}^{\infty} \left(\frac{4+2(-1)^n}{5}\right)^n x^n$$

Proof. We first compute

$$\alpha = \limsup_{n\to\infty} \left(\frac{4+2(-1)^n}{5}\right) = \frac{6}{5}.$$

Thus the radius of convergence is $R := 1/\alpha = 5/6$, so the power series definitely converges absolutely on $(-5/6, 5/6)$. At $x = \pm 5/6$, we observe that the nth term of the resulting series is

$$\left(\frac{4+2(-1)^n}{5}\right)^n \left(\pm\frac{5}{6}\right)^n = \left(\pm\frac{4+2(-1)^n}{6}\right)^n$$

and that this is equal to 1 whenever n is even. In particular, the terms of the series are not tending to 0, so this must diverge. Thus the set of points where this power series converges is exactly the open interval

$$(-5/6, 5/6).$$

$\square$

Problem 5. (1 point) Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if the series

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1})$$

converges.

Proof. Fix $\varepsilon \gneq 0$. Since the series $\sum d(x_n, x_{n+1})$ converges, the Cauchy criterion implies that there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$ we have

$$d(x_m, x_{m+1}) + \cdots + d(x_n, x_{n+1}) \lneq \varepsilon.$$

Then for all $n \geq m \geq N$ we see by repeated application of the triangle inequality that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_n) \leq \cdots \\ &\leq d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n) \\ &\leq d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \\ &\lneq \varepsilon. \end{aligned}$$

Thus the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. $\square$

Problem 6. (1 point) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\liminf |a_n| = 0$. Show that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that the series

$$\sum_{k=0}^{\infty} a_{n_k}$$

converges absolutely.

Proof. Without loss of generality, by passing to a subsequence if necessary, we can assume that $\lim |a_n| = 0$ (since a subsequence of a subsequence is also itself a subsequence). Now let $n_0 := 0$ and inductively, for each k, let $n_{k+1} \gneq n_k$ be such that $\left|a_{n_{k+1}}\right| \lneq 2^{-k}$. Then $\sum a_{n_k}$ converges absolutely by comparison to the convergent series $\sum 2^{-k}$. $\square$

Problem 7. (3 points) Show that if $(a_n)_{n\in\mathbb{N}}$ is a monontonically decreasing sequence in $\mathbb{R}$ and $\sum a_n$ converges, then $\lim n a_n = 0$. *Remark.* Notice that this result gives another proof that the harmonic series diverges.

Proof. Since $\sum a_n$ converges and $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing, we know that

$$0 = \lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

In particular, $a_n \geq 0$ for all n. Let $\varepsilon \gneq 0$. By the Cauchy criterion for convergence of series, there exist some N such that

$$(n - m + 1)a_n \leq a_m + \cdots + a_n = |a_m + \cdots + a_n| \lneq \varepsilon/2$$

for all $n \geq m \geq N$. Taking $m = N+1$ in particular we see that $(n-N)a_n \lneq \varepsilon/2$. Now notice that

$$na_n = Na_n + (n-N)a_n \lneq Na_m + \frac{\varepsilon}{2}$$

for all $n \geq m \geq N$. Since $\lim a_n = 0$, there exists an $M \geq N$ such that $a_M \lneq \varepsilon/2N$. Then for any $n \geq M$ we have

$$na_n \lneq Na_M + \frac{\varepsilon}{2} \lneq \varepsilon$$

and this completes the proof. □

Problem 8. (3 points) Is it true that, for any closed subset $E \subseteq \mathbb{R}$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ whose set of subsequential limits in $\mathbb{R}$ is exactly equal to E?

Proof sketch. $\mathbb{R}$ is separable, so it has a countable base, so E has a countable base, so it is separable. Take a countable dense subset of E and then take a sequence in which each element of this countable dense subset is repeated infinitely often. □

Problem 9. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection and suppose further that there exists a strictly increasing sequence $N_0 \lneq N_1 \lneq \cdots$ of natural numbers such that $\sigma(\{0, \ldots, N_i\}) \subseteq \{0, \ldots N_i\}$ for all i. Further, let $\sum a_n$ be a convergent series.

(a) (2 points) Show that, if the rearrangement $\sum a_{\sigma(n)}$ also converges, then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\sigma(n)}.$$

(b) (2 points) Give an example to show that the rearrangement $\sum a_{\sigma(n)}$ need not converge.

(c) (2 points) Show that, if there exists a constant M such that $N_{i+1} - N_i \leq M$ for all i, then the rearrangement $\sum a_{\sigma(n)}$ must converge.

Proof. Let s_n be the nth partial sum of $\sum a_n$ and t_n the nth partial sum of the rearrangement $\sum a_{\sigma(n)}$. Then observe that for any i, σ restricts to an injective map from the finite set $\{0, \ldots, N_i\}$ into itself, so this restriction must actually be bijective. Thus we have

$$t_{N_i} = a_{\sigma(0)} + \cdots + a_{\sigma(N_i)} = a_0 + \cdots + a_{N_i} = s_{N_i}.$$

In other words, the subsequences of $(s_n)_{n\in\mathbb{N}}$ and of $(t_n)_{n\in\mathbb{N}}$ determined by the strictly increasing sequence $N_0 \lneq N_1 \lneq \cdots$ are identical. If $\sum a_{\sigma(n)}$ converges, then $(t_n)_{n\in\mathbb{N}}$ converges and

$$\sum_{n=0}^{\infty} a_{\sigma(n)} = \lim_{n\to\infty} t_n = \lim_{i\to\infty} t_{N_i} = \lim_{i\to\infty} s_{N_i} = \lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} a_n.$$

For (b), consider the series

$$\underbrace{1-1}_{\text{2 terms}} + \underbrace{\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}}_{\text{4 terms}} + \underbrace{\frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4}}_{\text{8 terms}} + \underbrace{\frac{1}{8} - \frac{1}{8} + \cdots - \frac{1}{8}}_{\text{16 terms}} + \cdots$$

which we showed in class converges to 0, and consider the rearrangement

$$1 - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \cdots .$$

The rearrangement bijection σ here satisfies the condition needed with $N_0 = 0$, $N_1 = 1$, $N_2 = 5$, $N_3 = 13$, and, inductively, $N_i = N_{i-1} + 2^i$. But it is easy to see that there is a subsequence of the sequence of partial sums of this rearrangement which is constantly 1. If this rearrangement converged, it would have to converge to 0 by part (a), so it could not have a subsequence which is constantly 1. Thus this rearrangement does not converge.

Finally, for (c), let s_n and t_n the the nth partial sum for $\sum a_n$ and $\sum a_{\sigma(n)}$, respectively. As we noted in the proof of part (a), we know that $s_{N_i} = t_{N_i}$ for all i. For any $n \in \mathbb{N}$, let N_i be such that $N_i \leq n \lneq N_{i+1}$ and observe that, by applying the triangle inequality and using the fact that $s_{N_i} = t_{N_i}$, we have

$$|s_n - t_n| \leq |s_n - s_{N_i}| + |t_{N_i} - t_n| .$$

Note that $s_n - s_{N_i}$ and $t_n - t_{N_i}$ are both terms of terms of the form a_k for $N_i \lneq k \lneq N_{i+1}$. There are at most M such terms. Thus we have

$$|s_n - t_n| \leq 2M \max\{|a_k| : N_i \lneq k \lneq N_{i+1}\}.$$

Now for any $\varepsilon \gneq 0$, since $\sum a_n$ converges we know that there exists an N such that $|a_n| \lneq \varepsilon/2M$ for all $n \geq N$. By making N larger if necessary, we can assume that $N = N_j$ for some j. Then for any $n \geq N$, let N_i be such that $N_i \leq n \lneq N_{i+1}$ as above. Then $j \leq i$, so $k \gneq N_i$ means that $k \geq N$, which means that $|a_k| \lneq \varepsilon/2M$. Thus

$$|s_n - t_n| \leq 2M \max\{|a_k| : N_i \lneq k \lneq N_{i+1}\} \lneq \varepsilon.$$

Thus $\lim |s_n - t_n| = 0$. $\square$