For question 3.3.1, it is not enough to check that the domains and codomains of $\pi_m \circ i$ and F are the same. To show that two functions are equal, you have to check that they have the same output on every possible input. This means basically that you have to check that the formulas for the two functions are the same, as follows.

3.3.1. Prove that $F = \pi_m \circ i$ and $i(k^m) = V$.

Solution. To show that $F = \pi_m \circ i$, we calculate that

$$\begin{aligned}
(\pi_m \circ i)(t_1, \ldots, t_m) &= \pi_m(i(t_1, \ldots, t_m)) \\
&= \pi_m(t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), f_n(t_1, \ldots, t_m)) \\
&= (f_1(t_1, \ldots, t_m), f_n(t_1, \ldots, t_m)) \\
&= F(t_1, \ldots, t_m)
\end{aligned}$$

Thus $\pi_m \circ i$ and F have the same output on every possible input, so they are equal as functions.

To show that $i(k^m) = V$, we have to show an inclusion two ways. First, let us show that $i(k^m) \subseteq V$. If $(t_1, \ldots, t_m) \in k^m$, then

$$i(t_1, \ldots, t_m) = (t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), f_n(t_1, \ldots, t_m)).$$

We have to show that the polynomials defining $V = V(x_1 - f_1, \ldots, x_n - f_n)$ vanish on $(t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), f_n(t_1, \ldots, t_m))$. Note that

$$(x_i - f_i)(t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), f_n(t_1, \ldots, t_m)) = f_i(t_1, \ldots, t_m) - f_i(t_1, \ldots, t_m) = 0$$

for all $i = 1, \ldots, n$, which shows that $i(k^m) \subseteq V$.

On the other hand, we also need to show that $V \subseteq i(k^m)$. Suppose $p = (t_1, \ldots, t_m, x_1, \ldots, x_n) \in V$. This means that $(x_i - f_i)(p) = x_i - f_i(t_1, \ldots, t_m) = 0$ for all $i = 1, \ldots, n$, which means that $x_i = f_i(t_1, \ldots, t_m)$ for all i. Thus p is of the form

$$(t_1, \ldots, t_m, f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m)) = i(t_1, \ldots, t_m)$$

which proves that $p \in i(k^m)$. This shows that $V \subseteq i(k^m)$.