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Quiz 3

Part I (10 points). You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

(1) Let $V = \{a : a \in \mathbf{R}\}$ be the set of all real numbers. Define addition on V to be the usual addition of real numbers, and define a scalar multiplication operation @ on V by — T **F**

$$\lambda @ a = \begin{cases} 0 & \text{if } \lambda = 0 \\ a/\lambda & \text{if } \lambda \neq 0 \end{cases}$$

for all $a \in V$ and $\lambda \in \mathbf{R}$. Then V is a vector space over $\mathbf{R}$.

(2) The complex conjugation map $T : \mathbf{C} \to \mathbf{C}$ given by $T(a + bi) = a - bi$ is linear as a map of complex vector spaces. — T **F**

(3) Suppose U is a subspace of a vector space V and define a map $T : V \to V$ by — T **F**

$$T(v) = \begin{cases} v & \text{if } v \in U \\ 0 & \text{if } v \notin U. \end{cases}$$

Then T is linear.

(4) If V and W are vector spaces, $T \in \mathcal{L}(V, W)$, and $v_1, \ldots, v_n$ is a list in V such that $Tv_1, \ldots, Tv_n$ is linearly independent, then $v_1, \ldots, v_n$ is linearly independent. — **T** F

(5) Suppose $T \in \mathcal{L}(\mathbf{C}^4, \mathbf{C}^2)$ is such that — **T** F

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{C}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Then T is surjective.

(6) Suppose $T \in \mathcal{L}(\mathbf{R}^2, \mathcal{P}_3(\mathbf{R}))$ is such that — **T** F

$$\operatorname{range} T = \{p \in \mathcal{P}_3(\mathbf{R}) : p(1) = p(-1) = 0\}.$$

Then T is injective.

(7) Suppose V is a vector space such that $\dim V \geq 2$ and let — T **F**

$$U = \{T \in \mathcal{L}(V, V) : T \text{ is not surjective}\}.$$

Then U is a subspace of $\mathcal{L}(V, V)$.

(8) Suppose that U, V and W are vector spaces and that $S : U \to V$ and $T : V \to W$ are both surjective linear maps. Then TS is also surjective. — **T** F

(9) Suppose $S, T \in \mathcal{L}(\mathcal{P}_3(\mathbf{F}), \mathbf{F}^4)$ are such that $\dim \operatorname{range} S = \dim \operatorname{range} T = 1$. Then $(\operatorname{null} S) \cap (\operatorname{null} T)$ is 1, 2 or 3 dimensional. — T **F**

(10) Suppose V and W are 3 dimensional vector spaces and $T \in \mathcal{L}(V, W)$ has $\dim \operatorname{null} T = 1$. Then there exist bases v_1, v_2, v_3 for V and w_1, w_2, w_3 for W such that the matrix of T with respect to these bases is — **T** F

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Part II (10 points). Let $T : \mathcal{P}_2(\mathbf{R}) \to \mathcal{P}_3(\mathbf{R})$ be the linear map defined by

$$T(f)(z) = zf(z) + f'(z).$$

For example, if $f(z) = z^2$, then $T(f)$ is the polynomial $z^3 + 2z$.

(11) What is the matrix of T with respect to the basis $1, z, z^2$ for $\mathcal{P}_2(\mathbf{R})$ and the basis $1, z, z^2 + 1, z^3$ for $\mathcal{P}_3(\mathbf{R})$?

Observe that $T(1) = z$, $T(z) = z^2 + 1$ and $T(z^2) = z^3 + 2z$. Thus

$$M(T) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(12) Is T surjective? Is it injective? Justify your answers.

It is injective. Suppose $f \in \operatorname{null} T$ is of degree n. If $n \geq 0$, then clearly $\deg T(f) = n + 1 \neq -\infty$, so $T(f) \neq 0$. Thus it must be that $f = 0$, so $\operatorname{null} T = \{0\}$. Thus T is injective. It cannot be surjective since the dimension of the domain is strictly less than the dimension of the codomain.

For an alternative proof, observe that range $T = \operatorname{span}(z, z^2 + 1, z^3 + 2z)$ and that these polynomials form a basis since they are all of different degrees, so $\dim \operatorname{range} T = 3$. In particular, this shows that T cannot be surjective. Then

$$\dim \operatorname{null} T = \dim \mathcal{P}_2(\mathbf{R}) - \dim \operatorname{range} T = 3 - 3 = 0$$

so $\operatorname{null} T = \{0\}$, so T is injective.