Problem Set 4

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1. (1 point) Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ and $|x_n| \leq y_n$ for all n. If $\lim y_n = 0$, show that $\lim x_n = 0$ also.

Proof. For any $\varepsilon \gneq 0$, there exists some N such that $y_n \lneq \varepsilon$ for all $n \geq N$. Then we have

$$d(x_n, 0) = |x_n| \leq y_n \lneq \varepsilon$$

for all $n \geq N$, so $\lim x_n = 0$. □

Problem 2. (1 point) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space X and, for any $\varepsilon \gneq 0$ and $a \in X$, define

$$J_\varepsilon(a) := \{n \in \mathbb{N} : x_n \in B(a, \varepsilon), x_n \neq a\}.$$

(a) Show that, if $J_\varepsilon(a)$ is nonempty for all $\varepsilon \gneq 0$, then a is a subsequential limit of $(x_n)_{n\in\mathbb{N}}$.

(b) Give an example to show that the result of part (a) may no longer be true if we omit the condition "$x_n \neq a$" from the definition of $J_\varepsilon(a)$.

Proof. Suppose for a contradiction that $J_\varepsilon(a)$ were finite for some $\varepsilon \gneq 0$. Then choose some ε' so that

$$0 \lneq \varepsilon' \leq \min\{d(a, x_n) : n \in J_\varepsilon(a)\}.$$

In particular, we have that $\varepsilon \lneq \varepsilon'$, which means that $J_{\varepsilon'}(a) \subseteq J_\varepsilon(a)$. Since $J_{\varepsilon'}(a)$ is nonempty, it contains some number m. But then we also have $m \in J_\varepsilon(a)$, so

$$d(a, x_m) \lneq \varepsilon' \lneq \min\{d(a, x_n) : n \in J_\varepsilon(a)\} \leq d(a, x_m),$$

which is a contradiction. Thus $J_\varepsilon(a)$ must be infinite for all $\varepsilon \gneq 0$. Now for any open set U containing a and any position N, let $B(a, \varepsilon)$ be some open ball contained in U. Since $J_\varepsilon(a)$ is infinite, it must contain some $n \geq N$, and then we have $x_n \in B(a, \varepsilon) \subseteq U$. By the lemma we proved about subsequential limits in class, we are done.

For part (b), consider the sequence $(1, 1/2, 1/3, \ldots)$. If we omit the condition "$x_n \neq a$" from the definition of $J_\varepsilon(a)$, then we see that $J_\varepsilon(1)$ is nonempty for all $\varepsilon \gneq 0$ since we always have $0 \in J_\varepsilon(1)$. But this sequence converges to 0, so 0 is its only subsequential limit, so in particular 1 is not a subsequential limit even though $J_\varepsilon(1)$ is always nonempty. □

Problem 3. (1 point) Let X be a complete metric space and F a closed subset. Show that F is complete.

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in F. Then it is also a Cauchy sequence in X, so since X is complete, it has a limit in X. But F is closed, so the limit of every sequence in F is still in F, so $(x_n)_{n\in\mathbb{N}}$ converges in F as well. $\square$

Problem 4. (1 point) Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

(a) Show that $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ has further subsequence which converges to x.

(b) If $X = \mathbb{R}$, show that $\lim x_n = \infty$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a further subsequence whose limit is ∞.

(c) Give an example to demonstrate that $(x_n)_{n\in\mathbb{N}}$ need not be convergent even if every subsequence has a further convergent subsequence.

Proof. If $x = \lim x_n$, then the limit of every subsequence is also x so the "only if" direction is easy. Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ does not converge to x. Then there exists an open set U containing x such that $x_n \notin U$ for infinitely many n. Thus we can take a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ for which $x_{n_k} \notin U$ for all $k \in \mathbb{N}$, and this subsequence clearly cannot have a further subsequence converging to x.

The proof of (b) is essentially identical. One direction is easy. Conversely, if $\lim x_n \neq \infty$, there exists some R such that $x_n \notin (R, \infty)$ for infinitely many n. Then the subsequence of such terms notin in (R, ∞) clearly has no subsequence which converges to ∞.

For (c), take any non-convergent sequence in a compact metric space X. Then any subsequence has a further convergent subsequence since X is compact and therefore sequentially compact, but the original sequence is non-convergent! $\square$

Problem 5. (1 point) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let

$$E := \left\{ a \in \mathbb{R} \cup \{\infty, -\infty\} : \lim_{k\to\infty} x_{n_k} = a \text{ for some subsequence } (x_{n_k})_{k\in\mathbb{N}} \right\}.$$

Show that E is a singleton set $\{a\}$ if and only if $\lim x_n = a$. *Hint.* You might use problem 4.

Proof. The "if" direction is clear. Conversely, suppose $E = \{a\}$. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Then, as we proved in class, this subsequence has a monotonic subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$. Since $E = \{a\}$, we must have $\lim x_{n_{k_j}} = a$. Thus every subsequence has a further subsequence whose limit is a, so by problem 4 parts (a) and (b), and the statement analogous to part (b) with $-\infty$ in place of ∞, we conclude that $\lim x_n = a$. $\square$

Problem 6. (1 point) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$.

(a) Show that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) Give an example to show that the inequality in part (a) can be strict.

Proof. Let $(x_{n_k}+y_{n_k})_{n\in\mathbb{N}}$ be a subsequence of $(x_n+y_n)_{n\in\mathbb{N}}$ such that $\limsup(x_n+y_n) = \lim(x_{n_k}+y_{n_k})$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, so it has a monotonic subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$. This subsequence is bounded since $(x_n)_{n\in\mathbb{N}}$ is bounded, so it is convergent. Now note that the sequence $(y_{n_{k_j}})_{j\in\mathbb{N}}$ is also a sequence in $\mathbb{R}$, so it has a monotonic subsequence $(y_{n_{k_{j_i}}})_{i\in\mathbb{N}}$. This too is bounded, so it also is convergent. Moreover $(x_{n_{k_{j_i}}})_{i\in\mathbb{N}}$ is a subsequence of the convergent subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$, so it is also convergent. Also note that $(x_{n_{k_{j_i}}}+y_{n_{k_{j_i}}})_{i\in\mathbb{N}}$ is a subsequence of the convergent sequence $(x_{n_k}+y_{n_k})_{k\in\mathbb{N}}$, so putting everything together, we have

$$\begin{aligned}\limsup_{n\to\infty}(x_n+y_n) &= \lim_{k\to\infty}(x_{n_k}+y_{n_k})\\ &= \lim_{i\to\infty}\left(x_{n_{k_{j_i}}}+y_{n_{k_{j_i}}}\right)\\ &= \lim_{i\to\infty} x_{n_{k_{j_i}}} + \lim_{i\to\infty} y_{n_{k_{j_i}}}\\ &\le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\end{aligned}$$

where the last inequality follows from the definition we gave in class of lim sup as the supremum of the set of subsequential limits.

For part (b), consider the sequences $(x_n)_{n\in\mathbb{N}} = (1,-1,1,-1,\dots)$ and $(y_n)_{n\in\mathbb{N}} = (-1,1,-1,1,\dots)$. Then $\limsup(x_n+y_n) = \limsup(0,0,0,\dots) = 0$ and $\limsup x_n = \limsup y_n = 1$. ☐

Problem 7. (1 point for each part) This problem describes an algorithm for approximating square roots and investigates its rate of convergence. For some positive real number a, suppose that $x_0 \gneq \sqrt{a}$ and then define

$$x_{n+1} = \frac{x_n^2+a}{2x_n}.$$

(a) Prove that $(x_n)_{n\in\mathbb{N}}$ is monotonically decreasing and that $\lim x_n = \sqrt{a}$.

(b) Let $\varepsilon_n := x_n - \sqrt{a}$. Show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}.$$

(c) Describe happens if I accidentally choose x_0 so that $0 \lneq x_0 \lneq \sqrt{a}$. Does the sequence still converge?

Takeaway. Notice that $x_n \gneq \sqrt{a}$, so if we let $\beta := 2\sqrt{a}$, and induct on part (b), we find that $\varepsilon_n \lneq \beta(\varepsilon_0/\beta)^{2^n}$. In other words, if we start with a reasonable estimate x_0 for $\sqrt{a}$ (more precisely, one for which the initial error ε_0 is less than the constant β, so that $\varepsilon_0/\beta \lneq 1$), the terms x_n of this sequence get close to $\sqrt{a}$ very very quickly! (In fact, even if if my initial estimate is bad and not within β of $\sqrt{a}$, the fact that $\lim x_n = \sqrt{a}$ tells us that *eventually* I'll wind up with some x_n such that $\varepsilon_n \lneq \beta$, and then after that I'll still have rapid convergence.)

Proof. We claim that $x_n^2 \gneq a$ for all n. The base case of the induction follows from the assumption that $x_0 \gneq \sqrt{a}$. Inductively, note that

$$x_{n+1}^2 - a = \left(\frac{x_n^2 + a}{2x_n}\right)^2 - a = \frac{x_n^4 + 2ax_n^2 + a^2}{4x_n^2} - a = \frac{x_n^4 - 2ax_n^2 + a^2}{4x_n^2} = \left(\frac{x_n^2 - a}{2x_n}\right)^2 \gneq 0$$

which shows that $x_{n+1}^2 \gneq a$ as well. This completes the induction. Notice, in particular, that this means that $x_n \gneq 0$ for all n. Moreover, we have

$$x_n - x_{n+1} = x_n - \frac{x_n^2 + a}{2x_n} = \frac{x_n^2 - a}{2x_n} \gneq 0$$

since, as we have just observed, the denominator and numerator are both positive. Thus $x_n \gneq x_{n+1}$ for all n. Thus the sequence is monotonically decreasing, and

$$x := \lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\} \geq \sqrt{a}$$

since we proved above that $x_n \gneq \sqrt{a}$ for all n. Now note that taking limits in the recurrence, we have

$$x = \frac{x^2 + a}{2x}$$

which means that $2x^2 = x^2 + a$, which means that $x^2 = a$, which means that $x = \pm\sqrt{a}$. But x must be positive as we have just observed, so $x = \sqrt{a}$.

For part (b), we just compute directly.

$$\begin{aligned}
\varepsilon_{n+1} = x_{n+1} - \sqrt{a} &= \frac{x_n^2 + a}{2x_n} - \sqrt{a} \\
&= \frac{(\varepsilon_n + \sqrt{a})^2 + a - 2x_n\sqrt{a}}{2x_n} \\
&= \frac{\varepsilon_n^2 + 2\varepsilon_n\sqrt{a} + 2a - 2(\varepsilon_n + \sqrt{a})\sqrt{a}}{2x_n} = \frac{\varepsilon_n^2}{2x_n}.
\end{aligned}$$

For part (c), we claim that if $x_0 \lneq \sqrt{a}$, then $x_1 \gneq \sqrt{a}$, so we still have $\lim x_n = \sqrt{a}$. This is because

$$x_1 - \sqrt{a} = \frac{x_0^2 + a}{2x_0} - \sqrt{a} = \frac{x_0^2 + a - 2x_0\sqrt{a}}{2x_0} = \frac{(x_0 - \sqrt{a})^2}{2x_0} \gneq 0$$

since the numerator is positive and the denominator is as well. □

Problem 8. Let X be a complete metric space.

(a) (1 point) Let $E_0 \supseteq E_1 \supseteq \cdots$ be a nested infinite chain of closed and bounded subsets of X such that $\lim \operatorname{diam}(E_n) = 0$. Show that $\bigcap_{n\in\mathbb{N}} E_n$ contains just a single point. *Hint.* Recall the proof we gave in class of the fact that complete and totally bounded implies compact.

(b) (2 points) Show that if $G_0, G_1, \ldots$ is a countable collection of dense open subsets of X, then $\bigcap_{n\in\mathbb{N}} G_n$ is also dense in X. *Hint.* Use part (a).

Proof sketch. Choose $x_n \in E_n$. Then a standard argument shows that this sequence is Cauchy, so it converges to some point a. Since E_n contains infinitely many terms of the sequence, it contains

a, so a is in the intersection. If $b \neq a$ is in the intersection, then eventually $\operatorname{diam}(E_n)$ gets smaller than $d(a, b)$, which is a contradiction.

If U is any nonempty open subset, we know that $U \cap G_0$ is nonempty and open, so there exists some closed ball $B^+(a_0, r_0) \subseteq U \cap G_0$. Now G_1 is dense, so there exists some closed ball $B^+(a_1, r_1) \subseteq B(a, r_0) \cap G_0$. We can further insist that $r_1 \lneq 1/4$. Proceeding in this way, we find a nested sequence of closed balls $B^+(a_n, r_n)$ of diameter 2^{-n}. Then the point in the intersection is a point of U. □

Problem 9. (3 points) Let S be a set and let X be the set of bounded functions $S \to \mathbb{R}$ regarded as a metric space with the supremum metric. Show that X is complete. (Thus, using problem 3, we can deduce that the closed subset $F := \{f \in X : \|f\|_{\sup} \leq 1\}$ is complete. Recall that we showed on problem set 3 that, when S is infinite, then F is *not* compact. It follows that F must not be totally bounded.)

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. We claim that $(f_n(s))_{n\in\mathbb{N}}$ is then a Cauchy sequence in $\mathbb{R}$. To see this, suppose $\varepsilon \gneq 0$. Then there exists N such that for all $m, n \geq N$, we have $\|f_m - f_n\|_{\sup} \lneq \varepsilon$. This means that

$$|f_m(s) - f_n(s)| \leq \sup_{s\in S} |f_m(s) - f_n(s)| = \|f_m - f_n\|_{\sup} \lneq \varepsilon,$$

which shows that $(f_n(s))_{n\in\mathbb{N}}$ is Cauchy. But $\mathbb{R}$ is complete, so this Cauchy sequence is convergent in $\mathbb{R}$. Define $f(s) := \lim f_n(s)$. This defines a function $f : S \to \mathbb{R}$. We claim that f is bounded and that $\lim f_n = f$.

To see that f is bounded, note that since $(f_n)_{n\in\mathbb{N}}$ is Cauchy in X, it is bounded, so there exists some large closed ball $E := B^+(\mathbf{0}, R)$ such that $f_n \in E$ for all n, where $\mathbf{0}$ denotes the zero function. Then for all $s \in S$, we have $f_n(s) \in [-R, R]$, but $[-R, R]$ is closed, so we must have $f(s) = \lim f_n(s) \in [-R, R]$ also. Thus $|f(s)| \leq R$ for all $s \in S$, so f is bounded.

To see that $f = \lim f_n$, choose some $\varepsilon \gneq 0$. Since the sequence is Cauchy, there exists some $m, n \geq N$ such that

$$\|f_m - f_n\|_{\sup} \lneq \varepsilon/2.$$

Then for all $s \in S$, we have $|f_m(s) - f_n(s)| \lneq \varepsilon/2$, so taking the limit over all $n \geq N$ shows that

$$|f_m(s) - f(s)| \leq \varepsilon/2$$

for all $s \in S$. This means that

$$\|f_m - f\|_{\sup} \leq \varepsilon/2 \lneq \varepsilon$$

for all $m \geq N$, completing the proof. □

Problem 10. (5 points) Let S be a set. Let us say that a metric d on S is *bounded* if there exists a real number R such that $d(x, y) \leq R$ for all $x, y \in S$. Let X be the set of all bounded functions $S \times S \to \mathbb{R}$ regarded as a metric space with the supremum metric. Notice that the set M of bounded metrics on S is a subset of the metric space X. Is M closed in X? If so, prove it. If not, describe all of the elements of the closure $\bar{M}$.

Proof sketch. Let M_i be the set of functions satisfying axiom (Mi) for a metric. It's easy to verify that M_i is closed for $i = 1, 2, 3$ so the space of pseudometrics is a closed set containing M.

Now note that if d and d' are pseudometrics, then one can prove that $\lambda d + (1 - \lambda)d'$ is a pseudometric for all $\lambda \in [0, 1]$. This implies that every pseudometric is a limit of a sequence in M. Indeed, let d' be a pseudometric and let d be any metric (for example, the discrete metric). Then $\lambda d + (1 - \lambda)d'$ is a metric for every $\lambda \in (0, 1]$. Then $d_n = (1/n)d + (1 - 1/n)d'$ is a sequence of metrics converging to the pseudometric d'.

In particular, as long as S has more than one element, the set M is not closed in X, and its closure is the set of bounded pseudometrics. $\square$

Further problems. If you're familiar with the concept of equivalence relations and of quotients of sets by equivalence relations, I highly recommend that you work through exercises 23, 24, and 25 in chapter 3 of Rudin at some point when you have time.