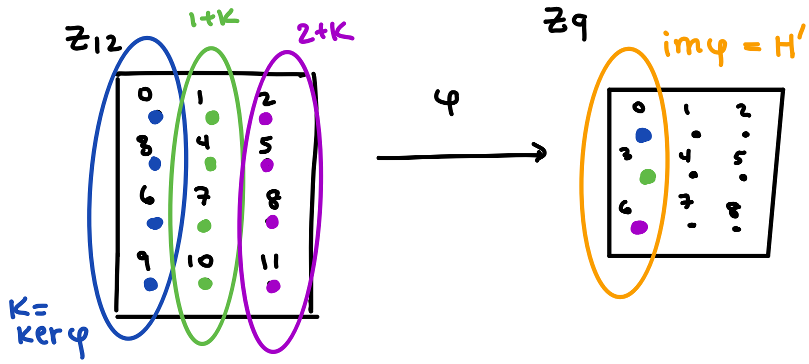
First Isomorphism Theorem

Reminder: CAPE!
(see Zulip for EC policy ☺)

Suppose $\varphi: G \to H$ is a homomorphism with kernel K and image H'.
Then $\bar{\varphi}(aK) = \varphi(a)$ is a well-defined isomorphism $\bar{\varphi}: G/K \to H'$.
In particular, $G/K \approx H'$.

Ex. $\varphi: \mathbb{Z}_{12} \to \mathbb{Z}_9$ given by $\varphi(x) = 3x \mod 9$



$G/K = \{\text{cosets of } K\}$

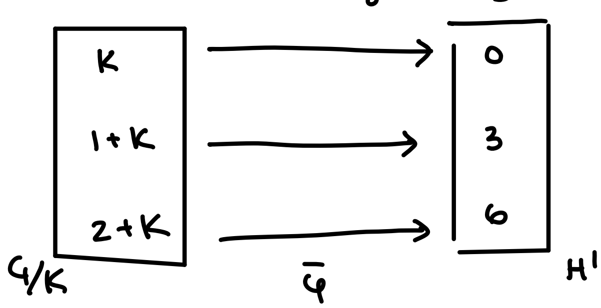
$= \{ \underbrace{\{0,3,6,9\}}_{\substack{K = 0+K \\ = 3+K \\ = 6+K \\ = 9+K}}, \underbrace{\{1,4,7,10\}}_{\substack{1+K \\ = 4+K \\ = 7+K \\ = 10+K}}, \underbrace{\{2,5,8,11\}}_{\substack{2+K \\ = 5+K \\ = 8+K \\ = 11+K}} \}$

$= \{K, 1+K, 2+K\}$

K is a normal subgroup, so we have a group operation on G/K.

- $(1+K) + (1+K) = (1+1) + K = 2+K$
- $(1+K) + (2+K) = (1+2) + K = 3+K = K$.

Have a function $\bar{\varphi}: G/K \to H'$ given by $\bar{\varphi}(a+K) = \varphi(a)$.



- This is well-defined: $1+K = 7+K$. Is it true that $\bar{\varphi}(1+K) = \bar{\varphi}(7+K)$?
  Is it true that $\varphi(1) = \varphi(7)$? Yes! [Recall: $\varphi(a) = \varphi(b)$ iff $aK = bK$]

$\bar{\varphi}$ is bijective.

$\bar{\varphi}$ is an isomorphism, ie, it is operation-preserving as well.

$$\left.\begin{aligned} \bar{\varphi}(1+K) + \bar{\varphi}(1+K) &= \varphi(1) + \varphi(1) = 3 + 3 = 6 \\ \bar{\varphi}((1+K) + (1+K)) &= \bar{\varphi}(2+K) = \varphi(2) = 6 \end{aligned}\right]$$

$\bar{\varphi}(1+K)+\bar{\varphi}(2+K)=\varphi(1)+\varphi(2)=3+6=0$
$\bar{\varphi}((1+K)+(2+K))=\bar{\varphi}(K)=\varphi(0)=0$]

1. $\varphi: \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ $\varphi(x)=2x$.

$\ker\varphi = \{0,5\}$
$\operatorname{im}\varphi = \{0,2,4,6,8\}$

$\left|\mathbb{Z}_{10}/\ker\varphi\right| = \dfrac{|\mathbb{Z}_{10}|}{|\ker\varphi|} = \dfrac{10}{2} = 5$] same!
$|\operatorname{im}\varphi| = 5$

2. $\mathbb{Z}_{27}\oplus\mathbb{Z}_3 \to \mathbb{Z}_9\oplus\mathbb{Z}_9$ surjective homomorphism?

Suppose there did exist a surjective homomorphism $\varphi: \mathbb{Z}_{27}\oplus\mathbb{Z}_3 \to \mathbb{Z}_9\oplus\mathbb{Z}_9$.
Then φ must be an isomorphism:

- $|\mathbb{Z}_9\oplus\mathbb{Z}_9| = 81$, and $|\mathbb{Z}_{27}\oplus\mathbb{Z}_3| = 81$, so φ must be injective!
- Let $K=\ker\varphi$. Then we know that $\mathbb{Z}_{27}\oplus\mathbb{Z}_3/K \cong \mathbb{Z}_9\oplus\mathbb{Z}_9$, so $|\mathbb{Z}_{27}\oplus\mathbb{Z}_3/K| = |\mathbb{Z}_9\oplus\mathbb{Z}_9|$. But $|\mathbb{Z}_{27}\oplus\mathbb{Z}_3/K| = \dfrac{|\mathbb{Z}_{27}\oplus\mathbb{Z}_3|}{|K|} = \dfrac{81}{|K|}$ and $|\mathbb{Z}_9\oplus\mathbb{Z}_9| = 81$, so we must have $|K|=1$, ie, K is trivial, so φ is injective.

In $\mathbb{Z}_{27}\oplus\mathbb{Z}_3$, we have elements of order 27 — eg, $(1,0)$.
But no element of $\mathbb{Z}_9\oplus\mathbb{Z}_9$ can have order 27 — (a,b) will have order 1, 3, or 9 (since $\operatorname{lcm}([1,3,\text{or }9],[1,3,\text{or }9]) = [1,3,\text{or }9]$).

3. $U(24)/U_{12}(24)$

$U_{12}(24) = \{x \in U(24) \mid x \bmod 12 = 1\}$ subgroup of $U(24)$.
"quotient group" = factor group

Consider $\varphi: U(24) \to U(12)$ given by $\varphi(x) = x \bmod 12$.
Then φ is surjective & $\ker\varphi = U_{12}(24)$. so $U(24)/U_{12}(24) \cong U(12)$.

because every number rel. prime to 12 is also rel. prime to 24.

$U(12) = \{1, 5, 7, 11\}$ $U_{12}(24) = \{1, 13\}$ $|U(24)| = 8$