Worksheet W2Mon: Cauchy-Riemann Equations

Problem 1. Suppose $u, v : \mathbb{R} \to \mathbb{R}$ are functions and $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(x + iy) = u(x) + iv(y)$. If f is entire, show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Problem 2. Find all functions $v : \mathbb{C} \to \mathbb{R}$ such that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x + iy) = (2x^2 + x + 1 - 2y^2) + iv(x + iy)$$

is entire. Then, for each such v, find a formula for $f(z)$ directly in terms of z (ie, not in terms of $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$).

Problem 3. Show that there exists no function $v : \mathbb{C} \to \mathbb{R}$ such that $f(x + iy) = (x^2 + y^2) + iv(x + iy)$ is holomorphic.

Problem 4. Let G be a region and $\bar{G} = \{z \in \mathbb{C} : \bar{z} \in G\}$. Suppose $f : G \to \mathbb{C}$ is holomorphic and define $g : \bar{G} \to \mathbb{C}$ by

$$g(z) = \overline{f(\bar{z})}.$$

Show that g is holomorphic. Do this using the definition of the derivative, and then do it again using the Cauchy-Riemann equations.

Problem 5. Suppose G is a region and $f : G \to \mathbb{C}$ is a holomorphic function. Show that f must be constant under each of the following assumptions.

(a) f is real-valued.

(b) $\bar{f}$ is holomorphic.

(c) $f(G) \subseteq C[0, 1] \setminus \{1\}$.

Possible hints. Parts (b) and (c) can be derived from (a).

Problem 6. Suppose G is a region and $f : G \to \mathbb{C}$ is analytic. Show that, if $|f(z)| = 1$ for all $z \in G$, then f is constant on G. *Possible hints.* Set $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$. Show that $u^2 + v^2 = 1$. Differentiate this equation with respect to x and y, and then use the Cauchy-Riemann equations to get two equations whose "unknowns" are the two partials of u. Then use some linear algebra to solve these equations!