> Overall I am very confused about envelopes.

> Where can we use envelopes?

This notion of envelopes has never really come up for me, which is why I suggested that this part of the reading could be skipped. On the other hand, singular points, multiplicity of intersections, etc, are quite ubiquitous in algebraic geometry and related fields. I'd suggest focusing your attention on this part of the reading.

> I might need some more examples and explanation on exercise 3.4.1.

> The calculation part of 3.4.1 is hard

Before proceeding, we make a small observation: in order for $x = 1 + ct$ and $y = 1 + dt$ to parametrize a line, either c or d must be nonzero. If both are 0, then these equations parametrize just a point!

To calculate the multiplicity of the intersection, we plug $x = 1 + ct$ and $y = 1 + dt$ into $x^3 - xy + y^2 - 1$.

$$\begin{aligned} x^3 - xy + y^2 - 1 &= (1 + ct)^3 - (1 + ct)(1 + dt) + (1 + dt)^2 - 1 \\ &= (1 + 3ct + 3c^2t^2 + c^3d^3) - (1 + (c + d)t + cdt^2) + (1 + 2dt + d^2t^2) \\ &= (2c + d)t + (3c^2 - cd + d^2)t^2 + c^3t^3 \end{aligned}$$

We now look for the multiplicity of $t = 0$ as a root of this polynomial. This is just the smallest power of t that shows up with a nonzero coefficient in this polynomial. This smallest power is usually 1, *unless* $2c + d = 0$.

If $2c + d = 0$, this means hat $c = -d/2$. Then the coefficient in front of t^2 is

$$3c^2 - cd + d^2 = 3 \cdot \left(-\frac{d}{2}\right)^2 - \left(-\frac{d}{2}\right) \cdot d + d^2 = \cdots = \frac{9d^2}{4}.$$

This equals 0 if and only if $d = 0$, but this would imply $c = -d/2 = 0$ also, which is a contradiction, since we noted above that either c or d must be nonzero. Thus if $2c + d = 0$, the smallest power of t that appears with a nonzero coefficient is t^2.

The conclusion is that the multiplicity of the intersection is 1 unless $c = -d/2$, in which case it is 2. This also tells us that any choice of c and d such that $c = -d/2$ gives us a parametrization tangent line. For simplicity, we take $d = 2$, which gives us the following parametrization of the tangent line:

$$\begin{aligned} x &= 1 - t \\ y &= 1 + 2t \end{aligned}$$