Continuous Functions

1 Notation

There will now be many metric spaces running around. So, when there can be ambiguity, we will write d_X for the metric on a metric space X and $B_X(a, r)$ for the open ball in X of radius r centered at $a \in X$. Also, while it is not strictly necessary, I find it is useful to be familiar with the word "neighborhood" at this point.

Let X be a metric space and $a \in X$ a point. A *neighborhood* of a is a subset U of X in which a is an interior point. Note that U need not itself be open in X. If it does happen to be open in X, we say that U is an *open neighborhood* of a.

Example 1.1. $(-1, 1), [-1, 1], (-0.5, 72]$ and $(-1, 1.2] \cup (72, 100]$ are all examples of neighborhoods of the point $a = 0$ in the metric space $X = \mathbb{R}$ with the euclidean metric. Only the first one of these is an open neighborhood.

Remark 1.2. Rudin uses the word "neighborhood" for the concept we have been calling "open ball." Rudin's usage is not standard: the definition of "neighborhood" given above is what most people mean when they say "neighborhood."

2 Limits of Functions

Let X and Y be metric spaces and, for some fixed point $a \in X$, let $f : X \setminus \{a\} \to Y$ be a function. We say that

$$\lim_{x \to a} f(x) = b$$

if, whenever U is a neighborhood of b in Y, then $f^{-1}(U) \cup \{a\}$ is also a neighborhood of a in X.

Example 2.1. Let $X = Y = \mathbb{R}$ and $a = 0$ and consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by the formula

$$f(x) = x^2.$$

Let's show that

$$\lim_{x \to 0} f(x) = 0.$$

Let U be a neighborhood of $b := 0$ in Y. In other words, b is an interior point of U, so there exists an open ball $B_Y(b, \varepsilon) \subseteq U$. Now consider the open ball $V := B_X(a, \sqrt{\varepsilon})$. Then for any

$x \in V \cap E \setminus \{a\} = V \setminus \{0\}$, we have

$$d(b, f(x)) = |f(x)| = \left|x^2\right| \lneq \left(\sqrt{\varepsilon}\right)^2 = \varepsilon$$

which means that $f(x) \in B_Y(b, \varepsilon) \subseteq U$. In other words, the entire open ball V is contained in $f^{-1}(U) \cup \{a\}$, so a is an interior point of $f^{-1}(U) \cup \{a\}$. In other words, $f^{-1}(U) \cup \{a\}$ is a neighborhood of a.

Example 2.2. Let X and Y be metric spaces, $a \in X$ a point and $f : X \setminus \{a\} \to Y$ a function such that

$$\lim_{x \to a} f(x) = b.$$

By definition, we know that if U is a neighborhood of b, then $f^{-1}(U) \cup \{a\}$ is a neighborhood of a. But, even if U is an *open* neighborhood, it need not be that $f^{-1}(U) \cup \{a\}$ is an *open* neighborhood of a. To see this, let $X = Y = \mathbb{R}$ and $a = 0$ again, and consider the function $f : X \setminus \{a\} \to Y$ given by

$$f(x) = \begin{cases} 1/2 & \text{if } x \in [2,3] \\ x^2 & \text{if } x \notin [2,3]. \end{cases}$$

It's easy to see that this function still has

$$\lim_{x \to a} f(x) = 0$$

just like in the previous example. But, consider the open neighborhood $U := B_X(a, 1)$. Then

$$f^{-1}(U) \cup \{a\} = (-1, 1) \cup [2, 3]$$

which is clearly not open. This is the reason for introducing the terminology "neighborhood of a" instead of just sticking with "open set containing a."

If we have metric spaces X and Y, a point $a \in X$ and a function $f : X \to Y$, we can "forget" about the value of f at a and restrict f to a function $X \setminus \{a\} \to Y$. This means that it makes sense to write

$$\lim_{x \to a} f(x) = b$$

even if f is defined at a. Of course, this also means that the fact that

$$\lim_{x \to a} f(x) = b$$

is totally independent of the value of f at a.

Example 2.3. Keep the same notation as in example 2.1, but now suppose that $f : X \to Y$ is actually defined everywhere by the formula

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

The same proof as in example 2.1 shows that we still have

$$\lim_{x \to 0} f(x) = 0$$

even though $f(x) \neq 0$. In fact, the above would be true no matter what the value of $f(x)$ was. It could be 0, or 1, or 100...

Lemma 2.4. *Let X and Y be metric spaces, $a \in X$ a point and $f : X \smallsetminus \{a\} \to Y$ a function. If there exist points b and b' in B such that*

$$\lim_{x \to a} f(x) = b \text{ and } \lim_{x \to a} f(x) = b'$$

then $b = b'$.

The proof of this lemma is left as an exercise: see problem 2.4.

Lemma 2.5. *Let X and Y be metric spaces, $a \in X$ a point, and $f : X \smallsetminus \{a\} \to Y$ a function. Then*

$$\lim_{x \to a} f(x) = b$$

for some $b \in B$ if and only if

$$\lim_{n \to \infty} f(x_n) = b$$

for every sequence $(x_n)_{n \in \mathbb{N}}$ in $X \smallsetminus \{a\}$ which converges to a.

Proof. For the "only if" direction, let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $X \smallsetminus \{a\}$ converging to a and let U be a neighborhood of b. Then $f^{-1}(U) \cup \{a\}$ is a neighborhood of a, which means that there exists an open ball V such that $f(V \smallsetminus \{a\}) \subseteq U$. Since $\lim x_n = a$, there exists an $N \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq N$. Then $x_n \in V \smallsetminus \{a\}$, so $f(x_n) \in U$ for all $n \in \mathbb{N}$. This shows that $\lim f(x_n) = b$.

For the "if" direction, suppose that

$$\lim_{x \to a} f(x) \neq b.$$

This means that there exists a neighborhood U containing b for which $f^{-1}(U) \cup \{a\}$ is *not* a neighborhood of a. In other words, there does not exist any open ball around a which is mapped by f entirely into U. Consider the open ball $V_n := B_X(a, 1/n)$. Then f does not map $V_n \smallsetminus \{a\}$ into U, so there exists some $x_n \in V_n \smallsetminus \{a\}$ such that $f(x_n) \notin U$. Now notice that $\lim x_n = a$, since for any $\varepsilon \gneq 0$, there exists N such that $1/N \lneq \varepsilon$, and then, since $x_n \in V_n := B_X(a, 1/n)$, we have that

$$d(a, x_n) \lneq 1/n \leq 1/N \lneq \varepsilon.$$

But we cannot have $\lim f(x_n) \neq b$. Indeed, since U is a neighborhood of b, there exists some open ball $B_Y(b, r) \subseteq U$, and we know that $f(x_n) \notin U$ for all n, so $f(x_n) \notin B_Y(b, r)$ for all n. In other words, $B_Y(b, r)$ is an example of an open set containing b which contains none of the points of the sequence $(f(x_n))_{n \in \mathbb{N}}$. $\square$

Example 2.6. Let $X = Y = \mathbb{R}$, suppose $a \in X$ and consider theh function $f : X \to Y$ given by $f(x) = 2x^2 + 1$. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $X \smallsetminus \{a\}$ converging to a. Then

$$\lim_{x\to a} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2x_n^2 + 1 = 2a^2 + 1 = f(a).$$

Functions like this, where the limit at every point is equal to the value of the function itself, are called "continuous."

3 Continuity

Let X and Y be metric spaces and let $f : X \to Y$ be a function. Then f is *continuous at* $a \in X$ if

$$\lim_{x\to a} f(x) = f(a).$$

Also we say that f is *continuous* if it is continuous at every $a \in X$.

Lemma 3.1. *Let X and Y be metric spaces. A function $f : X \to Y$ is continuous at a point $a \in X$ if and only if, for every neighborhood U of $f(a)$ in Y, the preimage $f^{-1}(U)$ is a neighborhood of a in X.*

Proof. By definition, we have that f is continuous at a if and only if, for every neighborhood U of $f(a)$, the preimage $f^{-1}(U \smallsetminus \{a\}) \cup \{a\}$ is a neighborhood of a. But notice that

$$f^{-1}(U \smallsetminus \{a\}) \cup \{a\} = f^{-1}(U). \qquad \square$$

Example 3.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(a, b) = a + b$. We give $\mathbb{R}$ and $\mathbb{R}^2$ both the euclidean metric. We want to show that f is continuous. Notice that f being continuous depends only on what subsets of $\mathbb{R}^2$ are open and not on the specific metric we put on $\mathbb{R}^2$, so actually it is more convenient to put the maximum metric on $\mathbb{R}^2$ instead of the euclidean metric.

Fix a point $(a, b) \in \mathbb{R}^2$ and let U be a neighborhood of $f(a, b) = a + b$. Then there exists some $\varepsilon \gneq 0$ such that $B_{\mathbb{R}}(f(a, b), \varepsilon) \subseteq U$. Now notice that

$$B_{\mathbb{R}^2}((a, b), \varepsilon/2) = B_{\mathbb{R}}(a, \varepsilon/2) \times B_{\mathbb{R}}(b, \varepsilon/2)$$

so for any (x, y) in this open ball, we have

$$d(f(a, b), f(x, y)) = |(a + b) - (x + y)| \le |a - x| + |b - y| \lneq \varepsilon.$$

In other words, the open ball $B_{\mathbb{R}^2}((a, b), \varepsilon/2)$ is entirely contained in $f^{-1}(U)$, so $f^{-1}(U)$ is a neighborhood of (a, b). This completes the proof of continuity.

Now if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ converging to x and y, respectively, then $((a_n, b_n))_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^2$ converging to (a, b), which means that

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} f(a_n, b_n) = \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) = a + b.$$

Here, we used lemma 2.5 for the second equality and the fact that f is continuous for the third. This proves the proposition concerning sums of limits we had left unproved before. For the other parts of that proposition, see problem 3.

Corollary 3.3. *Let X, Y and Z be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be functions. Suppose f is continuous at $a \in X$ and g is continuous at $f(a)$. Then the composite $h := g \circ f$ is continuous at a.*

Proof. Let U be a neighborhood of $h(a) = g(f(a))$. Since g is continuous at $f(a)$, we know that $g^{-1}(U)$ is a neighborhood of $f(a)$, and then since f is continuous at a, we also know that $f^{-1}(g^{-1}(U))$ is a neighborhood of a. But notice that

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) = h^{-1}(U). \qquad \square$$

Corollary 3.4. *Let X and Y be metric spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every open subset U of Y.*

Proof. Suppose first that f is continuous and let U be an open subset of Y. We want to show that every point $a \in f^{-1}(U)$ is an interior point of $f^{-1}(U)$. But notice that $a \in f^{-1}(U)$ means that $f(a) \in U$, so U is a neighborhood of $f(a)$, so $f^{-1}(U)$ is a neighborhood of a. In other words, a is an interior point of $f^{-1}(U)$.

Conversely, suppose that $f^{-1}(U)$ is open for every open subset $U \subseteq X$. We want to show that f is continuous at every point $a \in X$, so let U be any neighborhood of $f(a)$. Then there exists an open set U' containing $f(a)$ such that $U' \subseteq U$, and by assumption we know that $f^{-1}(U')$ is open. Then there exists an open ball $B_X(a, r) \subseteq f^{-1}(U') \subseteq f^{-1}(U)$. Thus a is an interior point of $f^{-1}(U)$, so $f^{-1}(U)$ is a neighborhood of a and we are done. $\square$

Example 3.5. Let $X = Y = \mathbb{R}$ with the euclidean metric and consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then for any irrational number a, notice that $B_Y(0, 1/2)$ is a neighborhood of $f(a) = 0$, but its preimage is the set of all irrational numbers, which is not a neighborhood of a. Similarly, for every rational number a, notice that $B_Y(1, 1/2)$ is a neighborhood of $f(a) = 1$, but its preimage is the set of all rational numbers, which is again not a neighborhood of a. Thus, this function f is discontinuous everywhere!

4 Continuity and Connectedness

Lemma 4.1. *Let X and Y be metric spaces and let $f : X \to Y$ be a continuous function. If X is connected, then its image $f(X)$ is also connected.*

Proof. Suppose that U is a nonempty proper open and closed subset of $f(X)$. Since U is open in $f(X)$, there exists an open set V in Y such that $U = V \cap f(X)$. Since f is continuous, we know that $f^{-1}(V)$ is open in X, but observe that $f^{-1}(V) = f^{-1}(U)$ since $U = V \cap f(X)$. Similarly, since $f(X) \setminus U$ is open in $f(X)$, we conclude that

$$f^{-1}(f(X) \setminus U) = X \setminus f^{-1}(U)$$

is also open in X. Thus $f^{-1}(U)$ is an open and closed subset of X, so, since X is connected, we have either $f^{-1}(U) = \emptyset$ or $f^{-1}(U) = X$. But notice that since U is a subset of $f(X)$, having $f^{-1}(U) = \emptyset$ forces $U = \emptyset$, which is a contradiction. On the other hand, having $f^{-1}(U) = X$ forces $U = f(X)$, which again is a contradiction. $\square$

Theorem 4.2 (Intermediate value theorem)**.** *Suppose $a \lneq b$ are real numbers and $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) \lneq f(b)$. If $f(a) \leq y \leq f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof. We know that $[a, b]$ is connected, so its continuous image $f([a, b])$ is also connected by lemma 4.1. Since $f(a)$ and $f(b)$ are elements of $f([a, b])$ and this is a connected subset of $\mathbb{R}$, we know that the entire interval $[f(a), f(b)]$ is contained in $f([a, b])$. In particular, since $y \in [f(a), f(b)]$, we have that $y \in f([a, b])$ also, so there exists $c \in [a, b]$ such that $f(c) = y$. $\square$

5 Sample Problems

Problem 1. Prove lemma 2.4.

Hint. The proof is very similar to the proof that limits of sequences are unique.

Problem 2. Let X and Y be metric spaces and let $f : X \to Y$ be a function. Show that f is continuous at $a \in X$ if and only if, for all $\varepsilon \gneq 0$, there exists $\delta \gneq 0$ such that $d(a, x) \lneq \delta$ implies $d(f(a), f(x)) \lneq \varepsilon$.

Problem 3. Mimic the proof of example 3.2 in order to prove that each of the following functions is continuous, where $\mathbb{R}$ and $\mathbb{R}^2$ have the euclidean metric.

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$.

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$.

(c) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$.

Problem 4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous at 0. *Remark.* We haven't formally defined sin in this class yet, but in any case, you can use the fact that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$.

Problem 5. Let X be a set regarded as a metric space with the discrete metric.

(a) Show that any function $f : X \to Y$ into a metric space Y is continuous.

(b) Is it also true that any function $f : Y \to X$ from any metric space Y is also continuous?

Hint for (b). Let $X = \mathbb{R}$ with the discrete metric and let $Y = \mathbb{R}$ with the euclidean metric, and consider the function $f : X \to Y$ given by $f(x) = x$. Is this continuous?

Problem 6. Determine all points $a \in \mathbb{R}$ where the function $f : \mathbb{R} \to \mathbb{R}$ defined by the following formula is continuous.

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$