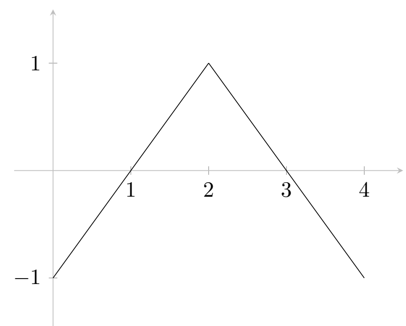
Name:

Final Exam Solutions

Instructions. The only tools you are permitted to use are pencils, pens, erasers, a handwritten sheet of notes, and your mind. No electronic devices! Remember to explain your answers unless the problem explicitly says otherwise, and also to sign the Honor Code statement at the end. You have 2.5 hours. Good luck! ☺

Problem 1 (True-false — 9 points, 1 point each)**.** No explanations are required.

(1) If f is the function on $[0,4]$ whose graph is depicted below, then $\int_0^4 f(x)\,dx = 4$. T **F**



(2) If $v(t)$ denotes the velocity function of a projectile that is launched directly upwards with some initial velocity, then the total displacement of the projectile between $t = 3$ s and $t = 5$ s is equal to the definite integral $\int_3^5 v(t)\,dt$. **T** F

(3) The equation $e^x = x^2$ has a solution. **T** F

(4) Any continuous function on the closed interval $[-1,1]$ has both an absolute minimum and an absolute maximum on the interval. **T** F

(5) The function $f(x) = e^{-x^2}$ has an antiderivative. **T** F

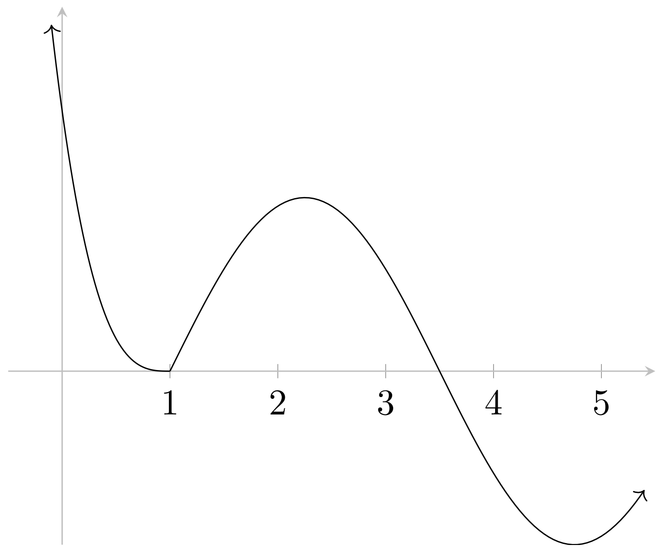
(6) $\sum_{k=1}^{2019} \left(\frac{1}{k+1} - \frac{1}{k} \right) = \frac{1}{2020} - 1$ **T** F

(7) If f is a differentiable function whose graph passes through $(2,4)$ and $(3,6)$, then there must exist some b between 2 and 3 such that $f'(b) = 2$. **T** F

(8) $\lim_{x\to 0} x\sin(1/x) = 0$. **T** F

(9) Suppose f is a function that has the property that the slope of the secant line passing through $(0, f(0))$ and $(x, f(x))$ is $2x + 2$ for all $x \neq 0$. Then it must be the case that $f'(0) = 2$. **T** F

Problem 2 (3 points, 1 point each)**.** The graph of a function f on the interval $[0, 5]$ is depicted below.



The following questions can all be answered with a single number. No explanations are required.

(a) How many critical points does f have on the closed interval $[0, 5]$?

Solution. 3

(b) At what x-value does f attain its absolute maximum on the closed interval $[0, 5]$?

Solution. 0

(c) How many inflection points does f have on the closed interval $[0, 5]$?

Solution. 2

Problem 3 (3 points, 1 point each)**.** Calculate the following limits, or state that the limit does not exist. If the limit is equal to $+\infty$ or $-\infty$, say so. Show your work.

(a) $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x - 2}$

Solution. We have

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x-3)}{x - 2} = \lim_{x \to 2} x - 3 = -1.$$

(b) $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$

Solution. We apply L'Hôpital's rule:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} \overset{\text{L'H}}{=} \lim_{x \to 0} \frac{\cos x}{\cos x - x \sin x} = 1.$$

(c) $\lim\limits_{x \to 0} \dfrac{e^x}{e^x - 1}$

Solution. The numerator tends to 1. When x is a negative number close to 0, the denominator is a small negative number, so

$$\lim_{x \to 0^-} \frac{e^x}{e^x - 1} = -\infty.$$

When x is a positive number close to 0, the denominator is a small positive number, so

$$\lim_{x \to 0^+} \frac{e^x}{e^x - 1} = +\infty.$$

Thus the limit does not exist.

Problem 4 (3 points, 1 point each)**.** In each of the following, a function f and a value a is given. Calculate $f'(a)$, or state that this derivative does not exist. Show your work.

(a) $f(x) = 3e^x \sin(2x)$ and $a = 0$.

Solution. We use the product and chain rules.

$$\begin{aligned} f'(x) &= 3e^x \sin(2x) + 6e^x \cos(2x) \\ f'(0) &= 6 \end{aligned}$$

(b) $f(x) = x^{\cos(x)}$ and $a = \pi$.

Solution. We use logarithmic differentiation.

$$\begin{aligned} \ln f(x) &= \cos x \ln x \\ \frac{f'(x)}{f(x)} &= \frac{\cos x}{x} - \sin x \ln x \\ f'(x) &= x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right) \\ f'(\pi) &= \pi^{-1}\left(\frac{-1}{\pi} - 0\right) = -\frac{1}{\pi^2}. \end{aligned}$$

(c) $f(x) = \displaystyle\int_0^{x^2} \sqrt{t + e^t}\, dt$ and $a = 1$

Solution. Let G be an antiderivative of the function $g(x) = \sqrt{x + e^x}$. Then

$$\int_0^{x^2} \sqrt{t + e^t}\, dt = G(x^2) - G(0),$$

so

$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_0^{x^2} \sqrt{t + e^t}\, dt \\ &= \frac{d}{dx}\left(G(x^2) - G(0)\right) \\ &= 2x\sqrt{x^2 + e^{x^2}} \end{aligned}$$

where we have used the chain rule and the fact that derivatives of constants vanish for the last step. Thus

$$f'(1) = 2\sqrt{1 + e}.$$

Problem 5 (3 points, 1 point each)**.** Calculate the following definite integrals. Show your work.

(a) $\displaystyle\int_0^\pi (2x + \sin x)\, dx$

Solution. We have

$$\int_0^\pi (2x + \sin x)\, dx = x^2 - \cos(x)\big|_0^\pi = (\pi^2 + 1) - (0 - 1) = \pi^2 + 2.$$

(b) $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$

Solution. Let $u = \ln x$, so that $du = dx/x$. Then

$$\int_1^e \frac{\ln x}{x}\, dx = \int_0^1 u\, du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}.$$

(c) $\displaystyle\int_{-1}^0 x(x+1)^5\, dx$

Solution. Let $u = x + 1$, so that $x = u - 1$ and $du = dx$. Then

$$\int_{-1}^0 x(x+1)^5\, dx = \int_0^1 (u-1)u^5\, du = \int_0^1 (u^6 - u^5)\, du = \left.\frac{u^7}{7} - \frac{u^6}{6}\right|_0^1 = \frac{1}{7} - \frac{1}{6} = \frac{-1}{42}.$$

Problem 6 (2 points)**.** What is the slope of the tangent line to the curve defined by

$$e^y = \sin(xy) + 1$$

at the point $(0, 0)$?

Solution. We differentiate implicitly and find

$$e^y \cdot \frac{dy}{dx} = \cos(xy) \cdot \left(y + x\frac{dy}{dx}\right) \implies \frac{dy}{dx} = \frac{y\cos(xy)}{e^y - x\cos(xy)}$$

We then plug in $x = y = 0$ and find that $\dfrac{dy}{dx} = 0$.

Problem 7 (3 points)**.** Approximate $\sqrt{99}$. Do better than $\sqrt{99} \approx 10$, and explain your answer. Is your approximation an overestimate or an underestimate?

Solution. Let $f(x) = \sqrt{x}$, so that $f'(x) = \dfrac{1}{2\sqrt{x}}$. Then $f(100) = 10$ and $f'(10) = \dfrac{1}{20}$, so the tangent line is given by

$$y - 10 = \frac{1}{20}(x - 100).$$

When $x = 99$, we find that

$$y = 10 - \frac{1}{20} = 9.95.$$

Thus $\sqrt{99} \approx 9.95$. The tangent line to f at $x = 100$ sits above the graph of f, so this approximation overestimates the true value of $\sqrt{99}$.

It's also possible to do this using Newton's method: use the function $f(x) = x^2 - 99$ and start with $x_0 = 10$. Then $x_1 = 9.95$ again.

Problem 8 (3 points)**.** A jewelry box with a square base is to be built with copper plated sides, nickel plated bottom and top, and a volume of 250 cm^3. Copper plating costs \$1 cm^2 and nickel plating costs \$2 cm^2. Find the side lengths of the base of the box that minimize cost.

Solution. Let x denote the length of the sides of the base, and let y denote the height of the box. We are trying to optimize the cost

$$C = 2 \cdot 2 \cdot x^2 + 4 \cdot xy = 4x^2 + 4xy.$$

The constraint is that the volume $x^2 y$ must equal 250 cm^3, which means that $y = 250/x^2$. Thus

$$C = 4x^2 + \frac{4 \cdot 250x}{x^2} = 4x^2 + \frac{1000}{x}.$$

Since x is a side length, we are trying to optimize C over the open interval $(0, \infty)$.

To find critical points, we differentiate, set equal to zero, and solve for x.

$$\begin{aligned} \frac{dC}{dx} = 8x - \frac{1000}{x^2} &= 0 \\ 8x &= \frac{1000}{x^2} \\ x^3 &= 125 \\ x &= 5 \end{aligned}$$

The final step is to verify that the critical point $x = 5$ does in fact minimize cost, and there are many ways to do this. One possibility is to compute a second derivative. Notice that

$$\frac{d^2C}{dx^2} = 8 + \frac{2000}{x^3}$$

is always positive for x in $(0, \infty)$. In other words, the function is always concave up, so the critical point $x = 5$ must be a global minimum.

Problem 9 (3 points)**.** The process of *carbon dating* involves a radioactive isotope of carbon, called carbon-14, that is formed when cosmic rays strike the Earth's atmosphere. This carbon-14 forms carbon dioxide and is incorporated into plants during photosynthesis, and subsequently into animals when they consume plants. When an organism dies, it stops incorporating carbon-14 from the atmosphere, and the carbon-14 left in the organism's body undergoes radioactive decay. The half-life of carbon-14 is 5730 years. By comparing the quantity of carbon-14 that is left over in the dead organism to the carbon-14 found in a similar living organism, we can determine how long ago the organism died.[1]

If a dead tree is found that contains 75% of the carbon-14 that similar living trees contain, how long ago did the tree die? You can leave your answer in terms of logarithms.

Solution. Let $Q(t)$ denote the quantity of carbon-14 in the tree at year t after the death of the tree. Then

$$Q(t) = Q_0 \cdot 0.5^{t/5730},$$

where Q_0 is the amount of carbon-14 that was in the tree at time of death. This is equal to the amount of carbon-14 that similar living trees contain, so at the moment when the dead tree is found, we have

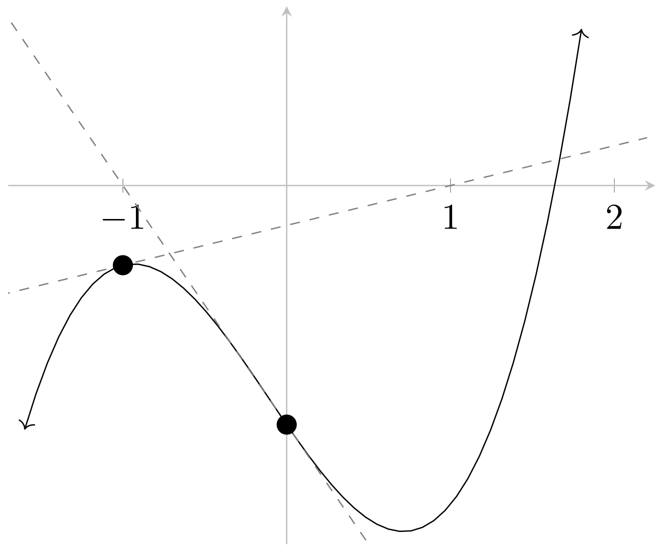
$$\begin{aligned} Q_0 \cdot 0.5^{t/5730} &= 0.75 \cdot Q_0 \\ \frac{t}{5730} &= \log_{0.5}(0.75) \\ t &= 5730 \cdot \log_{0.5}(0.75) \end{aligned}$$

Other expressions for this include...

$$t = 5730 \cdot \frac{\ln(3/4)}{\ln(1/2)} = 5730 \cdot \frac{\ln(4/3)}{\ln(2)} = 5730 \cdot \log_2(4/3) = \cdots$$

[1]Carbon dating was discovered by Willard Libby, who was born into a farming family in 1908 in the Grand Valley in western Colorado. This work was revolutionary, and Libby was awarded the Nobel Prize in Chemistry for it in 1960.

Problem 10 (2 points)**.** The graph of a function f is depicted below in the solid black line, and the two dashed gray lines are tangent to the graph of f.



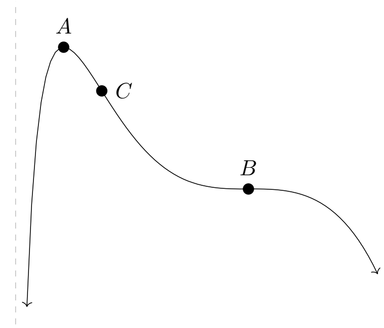
If we run 2 iterations of Newton's method to approximate the root of f, starting with the initial guess $x_0 = 0$, what will the new estimate of the root of f be? Explain briefly.

Solution. Since the tangent line at $x_0 = 0$ intersects the x-axis at -1, we have $x_1 = -1$. Then since the tangent line at $x_1 = -1$ intersects the x-axis at 1, we have $x_2 = 1$.

Problem 11 (4 points)**.** Below is a picture of the graph of the function

$$f(x) = \ln(x^3 + 2) - x^3,$$

but the x- and y-axes are missing. The dashed gray line is a vertical asymptote of the function.



(a) (1 point) What is the equation of the vertical asymptote?

Solution. Note that

$$\lim_{x \to a^+} f(x) = -\infty$$

where a is the value such that $a^3 + 2 = 0$. In other words, we need to have

$$a = -\sqrt[3]{2},$$

so the equation of the asymptote is $x = -\sqrt[3]{2}$.

(b) (2 points) Points A and B are critical points. What are the x-coordinates of A and B?

Solution. Observe that

$$f'(x) = \frac{3x^2}{x^3 + 2} - 3x^2.$$

We now set this equal to 0 and solve for x.

$$\begin{aligned}
\frac{3x^2}{x^3+2} - 3x^2 &= 0 \\
3x^2(x^3 + 2) &= 3x^2 \\
x^3 + 2 &= 1 \qquad \text{or} \qquad x = 0 \\
x^3 &= -1 \qquad \text{or} \qquad x = 0 \\
x &= -1 \qquad \text{or} \qquad x = 0
\end{aligned}$$

Thus the x-coordinate of A is -1 and the x-coordinate of B is 0.

(c) (1 point) Point C is an inflection point. What is its x-coordinate?

Solution. We calculate the second derivative.

$$f''(x) = \frac{6x(x^3+2) - 9x^4}{(x^3+2)^2} - 6x = -\frac{3x(2x^6+9x^3+4)}{(x^3+2)^2}.$$

This vanishes when the numerator vanishes, which happens when $x = 0$, which happens at point B, or when $2x^6 + 9x^3 + 4 = 0$.

$$\begin{aligned} 2x^6 + 9x^3 + 4 &= 0 \\ (x^3+4)(2x^3+1) &= 0 \\ x^3 &= -4 \qquad \text{or} \quad \frac{-1}{2} \\ x &= -\sqrt[3]{4} \quad \text{or} \quad \frac{-1}{\sqrt[3]{2}} \end{aligned}$$

Now $-\sqrt[3]{4} < -\sqrt[3]{2}$ so f is undefined at $x = -\sqrt[3]{4}$. Thus the x-coordinate of C must be $\frac{-1}{\sqrt[3]{2}}$.

Problem 12 (Extra credit! 2 points, no partial credit)**.** The minute hand on a watch is 8 mm long and the hour hand is 4 mm long, and both hands move continuously. How fast is the distance between the ends of the two hands changing at 3 o'clock?

Solution. Let t denote the number of minutes that have elapsed since 12 o'clock. Set up a coordinate system such that the center of the clock is at the origin and the position of the minute hand at $t = 0$ is $(0, 8)$. Then the position of the minute hand at time t can be described as

$$P_m = \left(8\sin\left(\frac{\pi t}{30}\right), 8\cos\left(\frac{\pi t}{30}\right)\right).$$

Similarly, the position of the hour hand can be described as

$$P_h = \left(4\sin\left(\frac{\pi t}{360}\right), 4\cos\left(\frac{\pi t}{360}\right)\right).$$

The distance D between these points satisfies

$$D^2 = \left(8\sin\left(\frac{\pi t}{30}\right) - 4\sin\left(\frac{\pi t}{360}\right)\right)^2 + \left(8\cos\left(\frac{\pi t}{30}\right) - 4\cos\left(\frac{\pi t}{360}\right)\right)^2.$$

Differentiating implicitly, we find

$$\begin{aligned} 2D \cdot \frac{dD}{dt} = \quad & 2\left(8\sin\left(\frac{\pi t}{30}\right) - 4\sin\left(\frac{\pi t}{360}\right)\right)\left(\frac{8\pi}{30}\cos\left(\frac{\pi t}{30}\right) - \frac{4\pi}{360}\cos\left(\frac{\pi t}{360}\right)\right) \\ & + 2\left(8\cos\left(\frac{\pi t}{30}\right) - 4\cos\left(\frac{\pi t}{360}\right)\right)\left(\frac{-8\pi}{30}\sin\left(\frac{\pi t}{30}\right) + \frac{4\pi}{360}\sin\left(\frac{\pi t}{360}\right)\right). \end{aligned}$$

We now need to plug in $t = 60 \cdot 3 = 180$ and calculate dD/dt.

First, observe that we can calculate D when $t = 180$ using the Pythagorean theorem since the two hands are at right angles. The hands form a right triangle whose side lengths are 8 and 4, so the distance between the tips of the hands is $\sqrt{8^2 + 4^2} = 4\sqrt{5}$.

Then note the following.

$$\sin\left(\frac{180\pi}{30}\right) = 0 \qquad \cos\left(\frac{180\pi}{30}\right) = 1$$
$$\sin\left(\frac{180\pi}{360}\right) = 1 \qquad \cos\left(\frac{180\pi}{360}\right) = 0$$

Plugging all of this in, we find that

$$\begin{aligned} 4\sqrt{5} \cdot \frac{dD}{dt} &= (8 \cdot 0 - 4 \cdot 1)\left(\frac{8\pi}{30} \cdot 1 - \frac{4\pi}{360} \cdot 0\right) + (8 \cdot 1 - 4 \cdot 0)\left(\frac{-8\pi}{30} \cdot 0 + \frac{4\pi}{360} \cdot 1\right) = -\frac{44\pi}{45} \\ \frac{dD}{dt} &= -\frac{11\pi}{45\sqrt{5}}. \end{aligned}$$

Thus the hands of the clock are moving closer at a rate of $\dfrac{11\pi}{45\sqrt{5}}$ mm/min.

There is another, slightly easier solution to this problem using the law of cosines, but I don't have the law of cosines memorized... If you're interested, try to figure out this alternative solution goes! It's kind of fun.

Honor code. If you have neither given nor received any unauthorized aid on this quiz, please write either "HCU" or "Honor Code Upheld" below, and sign your name next to it.

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