Closed Subsets

1 Closed Subsets

Let X be a metric space. A subset E of X is *closed* if its complement $X \setminus E$ is open.

Example 1.1. In any metric space X, the sets $\emptyset$ and X are always closed, because their complements are X and $\emptyset$, respectively, and we have already noted that these are always open. In particular, both the sets $\emptyset$ and X are both open *and* closed!

Example 1.2. Let $E := \{1/n : n = 1, 2, \ldots\}$ as a subset of $X := \mathbb{R}$ with the euclidean metric. This set is not closed. Indeed, note that its complement includes the point 0, and 0 is not an interior point of the complement. Indeed, for every open ball $B(0, r)$, there exists some positive integer n such that $1/n \lneq r$ using the archimedean property of the reals, which means that $1/n$ is a point of $B(0, r)$, so $B(0, r)$ is not entirely contained in the complement of E.

Lemma 1.3. *Let X be a metric space.*

(a) If a set I indexes a collection $(E_i)_{i \in I}$ of closed subsets of X, then $E := \bigcap_{i \in I} E_i$ is also closed.

(b) If $E_1, \ldots, E_n$ are all closed subsets of X, then $E_1 \cup \cdots \cup E_n$ is also closed.

Proof. This follows immediately from the analogous result for open sets and De Morgan's Laws. I leave it as an exercise for you to fill in the details. $\square$

Closely related to the notion of closedness is the notion of limit points. Let X be a metric space. A point $a \in X$ is a *limit point* of E if, for every open set U containing a, there exists some point $x \in E \cap U$ which is distinct from a. Note that a limit point of E need not be a point of E itself.

Example 1.4. Let $X := \mathbb{R}$ and $E := \{1/n : n = 1, 2, \ldots\}$. The only limit point of E is 0. To see that 0 is a limit point, we basically repeat the same argument as above. Given any open set U containing 0, there exists an open ball $B(0, r)$ contained entirely inside U since U is open. Then we can find some positive integer n such that $1/n \lneq r$ using the archimedean property, which means that $1/n \in E \cap B(0, r) \subseteq E \cap U$ and $1/n$ is distinct from 0. I will leave it as an exercise for you to convince yourself that 0 is the only limit point.

Example 1.5. Inside $X := \mathbb{R}$ with the euclidean metric, consider the set $E := (-\infty, 0]$. This set is closed, because its complement $(0, \infty)$ is open as we saw last time. Also, the set of limit points of E is precisely equal to E itself. Indeed, suppose first that $a \in E$. Then for any open set U

containing a, there exists an open ball $B(a,r) \subseteq U$. Notice that $a-(r/2) \in E \cap B(a,r) \subseteq E \cap U$ and $a-(r/2)$ is distinct from a itself, so a is a limit point. Conversely, suppose that $a \notin E$. Then a is positive, and the open ball $B(a,a)$ is entirely disjoint from E, so a cannot be a limit point of E.

Lemma 1.6. *Let X be a metric space. Then a subset E of X is closed if and only if every limit point of E is contained in E.*

Proof. Suppose E is closed. Let us show that if $a \notin E$, then a is not a limit point of E. Since $a \notin E$, a is a point in the open set $X \setminus E$. But $X \setminus E$ is disjoint from E, so it contains no points of E, so a is not a limit point of E.

Conversely, suppose every limit point of E is contained in E. We want to show that E is closed, or, in other words, that $X \setminus E$ is open. Suppose $a \in X \setminus E$. Since a is not in E, in particular it is not a limit point of E. That means that there exists an open set U containing a which contains no points of E. In other words, $U \subseteq X \setminus E$. But then since U is open, there exists an open ball $B(a,r) \subseteq U$, so $B(a,r) \subseteq X \setminus E$ also. Thus $X \setminus E$ is open, so E is closed. ∎

Example 1.7. It need not be that every point of a closed set is a limit point of that set. For example, consider $E := \{0\}$ inside $X := \mathbb{R}$. Then $X \setminus E = (-\infty, 0) \cup (0, \infty)$ is open, but 0 is not a limit point of E since no open subset of $\mathbb{R}$ containing 0 also contains a point of E which is distinct from E.

2 Closure

If X is a metric space and E is any subset, we define the *closure* of E, denoted $\bar{E}$, to be the intersection of all closed subsets of X which contain E. Lemma 1.3(a) guarantees immediately that $\bar{E}$ is closed. Thus, $\bar{E}$ is the smallest closed subset of X containing E.

Proposition 2.1. *Let X be a metric space and let E be a subset. If E' is the set of limit points of E, then $\bar{E} = E \cup E'$.*

Proof. Suppose F is any closed subset of X containing E. If $a \in E'$, then a is also a limit point of F clearly, but F is closed and contains all of its limit points by lemma 1.6, so $a \in F$. Thus $E' \subseteq F$, so $E \cup E' \subseteq F$. Thus every closed set containing E contains $E \cup E'$, so $E \cup E' \subseteq \bar{E}$.

To see that $E \cup E' = \bar{E}$, it suffices to show that $E \cup E'$ is itself closed. We do this using lemma 1.6 by showing that $E \cup E'$ contains all of its limit points. Suppose that a is a limit point of $E \cup E'$ and suppose that $a \notin E$. We want to show that then a must be a limit point of E also. Suppose that it is not a limit point of E, and let U be an open set containing a which contains no points of E. Since a is a limit point of $E \cup E'$, we know that U contains some point $b \in E \cup E'$ distinct from a. But U contains no points of E by assumption, so actually we must have that b is a point of E' distinct from a. But then U is an open set containing b, so since b is a limit point of E, U must contain some point of E. This is a contradiction, since U was supposed to contain no points of E. ∎

3 Density

Let X be a metric space. Then a subset $E \subseteq X$ is *dense* inside X if $\bar{E} = X$.

Lemma 3.1. *Let X be a metric space and E a subset. Then E is dense inside X if and only if $E \cap U$ is nonempty for every nonempty open subset U of X.*

Proof. Suppose E is dense inside X and suppose for a contradiction that there exists some nonempty open subset U of X such that $E \cap U = \emptyset$. Then $F := X \setminus U$ is a closed set containing E, so $\bar{E} \subseteq F$. In particular, $\bar{E}$ does not contain any points of U, but this is a contradiction since U is nonempty and we were supposed to have $\bar{E} = X$.

Conversely, suppose $E \cap U$ is nonempty whenever U is a nonempty open subset of X. Let F be a closed set containing E. Then $U := X \setminus F$ is an open set and clearly $E \cap U = \emptyset$ since E is contained inside F. This means that U must already be empty, since if it was nonempty it would have nonempty intersection with E by assumption. Now U being empty means that $F = X$. Since any closed set containing E is equal to X, we conclude that $\bar{E} = X$. $\square$

4 Sample Problems

Problem 1. Let E be a subset of a metric space X and let a be a limit point of E. Show that every open subset of X containing a also contains infinitely many points of E.

Solution. Let U be an open subset of X containing a and suppose for that U contains only finitely many points of E. In other words, we are assuming that $E \cap U$ is finite. Let $x_1, \ldots, x_n$ be the elements of $E \cap U$ distinct from a. Since U is open, there exists an open ball $B(a, r) \subseteq U$. Let

$$\varepsilon := \min\{r, d(a, x_1), \ldots, d(a, x_n)\}.$$

Then $\varepsilon \gneq 0$ since all of the elements of the set are also positive, so $B(a, \varepsilon)$ is an open ball. It is contained inside U since $\varepsilon \leq r$, so $B(a, \varepsilon) \subseteq B(a, r) \subseteq U$, and moreover $x_i \notin B(a, \varepsilon)$ for all i since $d(a, x_i) \geq \varepsilon$ for all i. Thus the open ball $B(a, \varepsilon)$ contains no points of E other than possibly a itself, so a is not a limit point for E. $\square$

Problem 2. Let $X := \mathbb{R}$ and $E := (0, \infty)$. What is the closure of E when $\mathbb{R}$ is given the euclidean metric? What is the closure of E when $\mathbb{R}$ is given the discrete metric?

Solution. The closure of E when $\mathbb{R}$ has the euclidean metric is $[0, \infty)$. Indeed, 0 is the only limit point of E which is not already contained in E. To see this, notice that for any $a \lneq 0$, the set $(-\infty, 0)$ is an open subset of $\mathbb{R}$ containing a but containing no points of E, so a cannot be a limit point of E. Next, when $\mathbb{R}$ is given the discrete metric, all subsets are open, so all subsets are also closed. Thus $(0, \infty)$ is already closed, so it is its own closure. $\square$

Problem 3. Let E be a nonempty subset of $\mathbb{R}$ which is bounded above, and let $\alpha := \sup E$. Show that $\alpha \in \bar{E}$.

Solution. If $\alpha \in E$, then we are done immediately, so suppose $\alpha \notin E$. Since $\bar{E} = E \cup E'$ where E' is the set of limit points of E, in order to have $\alpha \in \bar{E}$ even though $\alpha \notin E$, we must have $\alpha \in E'$, so let us show this. Let U be any open subset containing α. Then there exists some open ball $B(\alpha, r) \subseteq U$. Since $\alpha - r \lneq \alpha$, we know that $\alpha - r$ is not an upper bound for α, so there exists some $x \in E$ such that $\alpha - r \lneq x \lneq \alpha$. In other words, we have $|\alpha - x| \lneq r$, so $x \in B(a, r) \cap E \subseteq U \cap E$. Thus α is a limit point of E. □

Problem 4. Show that $\mathbb{Q}$ is dense inside $\mathbb{R}$ with the euclidean metric.

Proof. Let U be a nonempty open subset of $\mathbb{R}$. Then there exists some $a \in U$ and since U is open there exists some open ball $B(a, r)$ inside U. But by the result we've been calling "density of the rationals," we know that there exists a rational number x such that $a \lneq x \lneq a + r$, which means that $x \in B(a, r) \subseteq U$, so $\mathbb{Q} \cap U$ is nonempty. Thus $\mathbb{Q}$ is dense using lemma 3.1. □

Problem 5. Consider $\mathbb{Z}$ with the discrete metric. Describe all dense subsets of $\mathbb{Z}$, and prove that you have described all of them.

Solution. Suppose E is a dense subset of $\mathbb{Z}$. Pick any $a \in \mathbb{Z}$. Then $B(a, 1/2) = \{a\}$ is an open ball, so $E \cap B(a, 1/2)$ must be nonempty by lemma 3.1. But the only element of $B(a, 1/2)$ is a itself, so $a \in E$. Thus $E = \mathbb{Z}$. Thus the only dense subset of $\mathbb{Z}$ is $\mathbb{Z}$ itself. □