Space of Continuous Functions

1 Uniform and Pointwise Convergence

Let X be a set. For any function $f : X \to \mathbb{R}$, we define

$$\|f\|_{\text{sup}} := \sup\{|f(x)| : x \in X\}.$$

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $X \to \mathbb{R}$ and that we have another function $f : X \to \mathbb{R}$. We then make the following two definitions.

- We say that $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to f if for every $\varepsilon \gneq 0$, there exists N such that $\|f_n - f\|_{\text{sup}} \lneq \varepsilon$ for all $n \geq N$.
- We say that $(f_n)_{n\in\mathbb{N}}$ *converges pointwise* to f if, for every $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ of real numbers converges to the real number $f(x)$.

Recall that a function $f : X \to \mathbb{R}$ is *bounded* if $\|f\|_{\text{sup}} \neq \infty$. The set of bounded functions $\mathcal{B}(X)$ is then a metric space equipped with the supremum metric defined by

$$d(f, g) = \|f - g\|_{\text{sup}}.$$

On problem sets, you have seen the fact that this metric makes $\mathcal{B}(X)$ a complete metric space. Notice that convergence with respect to the supremum metric is the same thing as uniform convergence.

The set of all functions (not necessarily bounded) is denoted $\mathbb{R}^X$. This isn't exactly a metric space with the supremum metric, since metrics aren't allowed to take the value ∞. One can get around this issue in one of two ways. One way is to notice that the function

$$d(f, g) = \min\{1, \|f - g\|_{\text{sup}}\}$$

defines a metric on $\mathbb{R}^X$. The second way is to go back and redefine "metric" and allow them to take the value ∞ (in other words, this redefined "metric" would be a function $X \times X \to [0, \infty]$ which satisfies the same four axioms that we stated for metrics earlier). You can then go through and check that everything we've proved about metric spaces would still be true in this generality. Moreover, if you do decide you want to generalize the definition of metric to allow this, in the end you can check that the metric $d(f, g) = \|f - g\|_{\text{sup}}$ and the metric $d'(f, g) = \min\{1, \|f - g\|_{\text{sup}}\}$ on $\mathbb{R}^X$ are equivalent. (If I were to teach this class again, I think I would have defined metrics in this slightly generalized way.)

So, in either case, the set $\mathbb{R}^X$ becomes a metric space which is even complete (if you've already proved that $\mathcal{B}(X)$ is complete, look at your proof again and convince yourself of this). Moreover, convergence in this metric space is the same thing as uniform convergence. Finally, $\mathcal{B}(X)$ is a closed subset of $\mathbb{R}^X$. It is useful to keep in mind the fact that uniform convergence is just convergence with respect to a certain metric. This means that we can apply results we've already proved about limits of sequences in general metric spaces.

In any case, a really easy fact is that uniform convergence implies pointwise convergence.

Lemma 1.1. *Let X be a set and suppose a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $X \to \mathbb{R}$ converges uniformly to some function $f : X \to \mathbb{R}$. Then $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f.*

Proof. For any $\varepsilon \gneq 0$, there exists N such that $\|f_n - f\|_{\sup} \lneq \varepsilon$ for all $n \geq N$, which means that for any $x \in X$, we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_{\sup} \lneq \varepsilon$$

for all $n \geq N$. $\square$

The converse of lemma 1.1, however, is not true: pointwise convergence need not imply uniform convergence.

Example 1.2. Let $X := [0, 1]$ and define the function $f : X \to \mathbb{R}$ by the formula $f_n(x) = x^n$ for all $n \in \mathbb{N}$. Notice that for all $x \neq 1$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$$

and that $\lim f_n(1) = 1$. In other words, the sequence $(f_n)_{n\in\mathbb{N}}$ is converging *pointwise* to the function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \text{ and} \\ 1 & \text{if } x = 1. \end{cases}$$

However, it is not converging *uniformly* to f. Indeed, notice that for any positive integer n, we have

$$|f_n(x) - f(x)| = |x^n - f(x)| = \begin{cases} |x^n| & \text{if } x \in [0,1) \\ 0 & \text{if } x = 1. \end{cases}$$

This means that

$$\|f_n - f\|_{\sup} = \sup\left(\{0\} \cup \{|x|^n : x \in [0,1)\}\right) = \sup\{|x^n| : x \in [0,1)\} = 1.$$

Notice in the above example that $(f_n)_{n\in\mathbb{N}}$ is a sequence continuous functions, and its pointwise limit f is discontinuous. A natural follow-up question is if the *uniform* limit of a sequence of continuous functions can be discontinuous. The answer is no: uniform convergence preserves continuity. This is a theme that we will see recur over the next couple of weeks: uniform limits of functions preserve "good behavior" (at least sometimes).

2 Space of Continuous Functions

Now suppose that X is not just a set but a metric space. Then inside the set $\mathbb{R}^X$ of all functions $X \to \mathbb{R}$, we can consider the subset

$$\mathcal{C}(X) := \{\text{continuous functions } f : X \to \mathbb{R}\}.$$

By restricting the metric on $\mathbb{R}^X$, we can regard $\mathcal{C}(X)$ as a metric space.

Lemma 2.1. *Let X be a metric space and suppose $f, g \in \mathcal{C}(X)$ and $\lambda \in \mathbb{R}$. Then*

(a) $\lambda f \in \mathcal{C}(X)$, and

(b) $f + g \in \mathcal{C}(X)$.

Proof. For (a), notice that the constant function $X \to \mathbb{R}$ given by $x \mapsto \lambda$ is continuous. Thus, by one of the sample problems on the lecture on continuity, the function $X \to \mathbb{R} \times \mathbb{R}$ given by $x \mapsto (\lambda, f(x))$ is also continuous. Now post-composing this with the continuous multiplication map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(a, b) \mapsto ab$ gives us exactly the function λf. For part (b), notice that since f and g are both continuous, so is the function $X \to \mathbb{R} \times \mathbb{R}$ given by $x \mapsto (f(x), g(x))$, and post-composing this with the continuous addition map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(a, b) \mapsto a + b$ gives us a continuous map which is exactly equal to $f + g$. $\square$

Proposition 2.2. *Let X be a metric space. Then $\mathcal{C}(X)$ is a closed subset of $\mathbb{R}^X$.*

Proof. Showing that $\mathcal{C}(X)$ is closed is equivalent to showing that, for any sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}(X)$ which converges uniformly to a function $f \in \mathbb{R}^X$, the limit f is actually in $\mathcal{C}(X)$ also. In other words, we are trying to show that the uniform limit f is continuous at every point $a \in X$. Observe that for any $x \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &\leq 2\|f_n - f\|_{\sup} + |f_n(x) - f_n(a)| \,. \end{aligned}$$

Fix $\varepsilon \gneq 0$. Since $(f_n)_{n\in\mathbb{N}}$ is converging uniformly to f, there exists N such that $\|f_n - f\|_{\sup} \lneq \varepsilon/4$ for all $n \geq N$. Taking $n = N$ in particular, we see that

$$|f(x) - f(a)| \leq 2\|f_N - f\|_{\sup} + |f_N(x) - f_N(a)| \lneq \frac{\varepsilon}{2} + |f_N(x) - f_N(a)|$$

for all $x \in X$. Now since f_N is continuous, there exists $\delta \gneq 0$ such that $|f_N(x) - f_N(a)| \lneq \varepsilon/2$ for all $x \in B(a, \delta)$. Then for all $x \in B(a, \delta)$, we have

$$|f(x) - f(a)| \lneq \frac{\varepsilon}{2} + |f_N(x) - f_N(a)| \lneq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ $\square$

Corollary 2.3. *$\mathcal{C}(X)$ is a complete metric space.*

Proof. We have already noted that $\mathbb{R}^X$ is complete, and on problem sets you have seen that closed subsets of complete metric spaces are themselves complete. $\square$

Remark 2.4. Another useful fact is that, when X is a closed interval inside $\mathbb{R}$, the set of polynomial functions on X is dense inside $\mathcal{C}(X)$. This fact is called the *Weierstrass approximation theorem.* We might prove this later on in the course. If you want to see a proof now, see section 27 of Ross or theorem 7.26 in Rudin. The proof doesn't use any advanced technology you haven't already seen, but it is quite technical...

3 Series of Functions

Let X be a subset of $\mathbb{R}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathbb{R}^X$. Let $s_n := f_0 + \cdots + f_n$ for all $n \in \mathbb{N}$. If the sequence $(s_n)_{n\in\mathbb{N}}$ converges uniformly, then we say that the series of functions

$$\sum_{n=0}^{\infty} f_n$$

converges uniformly. Notice that if f_n is continuous for all $n \in \mathbb{N}$, then s_n is also continuous for all $n \in \mathbb{N}$, so if $\sum f_n$ converges uniformly, then the limit must again be continuous by proposition 2.2. This means that it is useful to be able to check uniform convergence quickly. The following is a very convenient criterion for checking uniform convergence of a series of functions.

Lemma 3.1 (Weierstrass M-test)**.** *Let X be a subset of $\mathbb{R}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $X \to \mathbb{R}$. Suppose further that $M_n \geq \|f_n\|_{\sup}$ and that the series of real numbers*

$$\sum_{n=0}^{\infty} M_n$$

converges. Then the series of functions $\sum f_n$ converges uniformly.

Proof. Let $s_n := f_0 + \cdots + f_n$. Since $\mathcal{B}(X)$ is complete, it suffices to show that the sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the supremum metric. Fix $\varepsilon \gneq 0$. Since $\sum M_n$ converges, there exists some N such that for all $n \geq m \geq N$ we have

$$|M_m + \cdots + M_n| = M_m + \cdots + M_n \lneq \varepsilon.$$

Then observe that for all $n \geq m \geq N+1$, we have $m - 1 \geq N$ so

$$\begin{aligned}
\|s_n - s_{m-1}\|_{\sup} &= \|(f_0 + \cdots + f_n) - (f_0 + \cdots + f_{m-1})\|_{\sup} \\
&= \|f_m + \cdots + f_n\|_{\sup} \\
&\leq \|f_m\|_{\sup} + \cdots + \|f_n\|_{\sup} \\
&\leq M_m + \cdots + M_n \\
&\lneq \varepsilon.
\end{aligned}$$

Thus $(s_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the supremum metric. $\square$

Corollary 3.2. *Let $\sum a_n x^n$ be a power series with a positive radius of convergence R. Then $\sum a_n x^n$ converges uniformly on every compact subset of $(-R, R)$.*

Proof. Let K be a compact subset of $(-R, R)$. Then K is contained in $[-S, S]$ for some positive real number $S \lneq R$, and uniform convergence on $[-S, S]$ implies uniform convergence on K, so we can assume that $K = [-S, S]$ without loss of generality. Let $M_n := |a_n| S^n$. Then clearly $|a_n x^n| \leq M_n$ for all $x \in K$. In other words, if we let $f_n : K \to \mathbb{R}$ be the function given by $f_n(x) = a_n x^n$, then $\|f_n\|_{\sup} \leq M_n$ for all n. Moreover, we know that $\sum a_n x^n$ converges absolutely for every $x \in (-R, R)$, so in particular it converges absolutely when $x = S$. In other words, the seris of real numbers

$$\sum_{n=0}^{\infty} |a_n S^n| = \sum_{n=0}^{\infty} M_n$$

converges. The Weierstrass M-test therefore implies that $\sum f_n$ converges uniformly on K. $\square$

Corollary 3.3. *Let $\sum a_n x^n$ be a power series with positive radius of convergence R. Then $\sum a_n x^n$ defines a continuous function on $(-R, R)$.*

Proof. To see that the function $f : (-R, R) \to \mathbb{R}$ defined by $f(x) = \sum a_n x^n$ is continuous, fix a point $a \in (-R, R)$. Let S be some number such that $|a| \lneq S \lneq R$ and then consider the compact set $K := [-S, S]$. Clearly, to have the function f be continuous at a, it is sufficient to show that f is continuous on K, since a is an interior point of K. But K is a compact set and f is the uniform limit of the series $\sum f_n$ of continuous functions $f_n : K \to \mathbb{R}$ where $f_n(x) = a_n x^n$. Thus f is continuous by proposition 2.2. $\square$

4 Sample Problems

Problem 1. Let $X := \mathbb{R}$ and for each positive integer n, define the function $f_n : X \to \mathbb{R}$ by the formula

$$f_n(x) = \frac{1 + 2\cos^2(nx)}{\sqrt{n}}$$

Show that the sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly.

Hint. First use the squeeze theorem to compute the pointwise limit. Then derive the following inequality, and explain why it implies that convergence to that pointwise limit is uniform.

$$\left| \frac{1 + 2\cos^2(nx)}{\sqrt{n}} \right| \leq \frac{3}{\sqrt{n}}.$$

Problem 2. Let $X := \mathbb{R}$ and for all positive integers n, define the function $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1/n & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that f_n is discontinuous everywhere, but that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to a continuous function. *Remark.* Thus, even though uniform limits preserve good behavior, they need not preserve bad behavior.

Problem 3. Let $X := [0, \infty)$ and define the function $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \frac{x}{n}.$$

Does the sequence $(f_n)_{n\in\mathbb{N}}$ converge uniformly? What if we had $X = [0, 1]$ instead of $X = [0, \infty)$?

Problem 4. Let $X := \mathbb{R}$ and define $f_n : X \to \mathbb{R}$ by the formula

$$f_n(x) = \frac{x^2}{(1 + x^2)^n}.$$

Show that $\sum f_n(x)$ exists for every $x \in X$, and that the function $f : X \to \mathbb{R}$ defined by $f(x) = \sum f_n(x)$ is discontinuous. *Remark.* Thus, by proposition 2.2, we can conclude that the series $\sum f_n$ does not converge uniformly.

Hint. For $x \neq 0$, the series $\sum f_n(x)$ is a convergent geometric series.

Problem 5. Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Show that this power series has radius of convergence 1, and that it converges at $x = -1$ but diverges for $x = 1$. Thus, the series of functions converges uniformly on every compact subset of $(-1, 1)$ by corollary 3.2. Does it converge uniformly on all of $(-1, 1)$? Does it converge uniformly on $(-1, 0]$? What about on $[-1, 0]$?

Solution. Checking that that this power series has radius of converges 1 and that it converges precisely on $[-1, 1)$ is straightforward. It does not converge uniformly on $(-1, 1)$. Indeed, observe that each of the partial sums s_n is a continuous functions on $(-1, 1]$. If they converged uniformly on $(-1, 1)$, then by a problem on problem set 6, they would also converge uniformly on $(-1, 1]$ since $(-1, 1)$ is dense in $(-1, 1]$. But $(s_n(1))_{n\in\mathbb{N}}$ does not converge, since

$$\lim_{n\to\infty} s_n(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

In other words, the sequence $(s_n)_{n\in\mathbb{N}}$ does not even converge pointwise on $(-1, 1]$, so it definitely cannot converge uniformly.

On the other hand, it does converge uniformly on $[-1, 0]$ and $(-1, 0]$. In fact, uniform convergence on either one implies the other: uniform convergence on $[-1, 0]$ clearly implies uniform convergence on $(-1, 0]$, and for the converse we can use the fact that $(-1, 0]$ is dense in $[-1, 0]$.

We'll prove that it converges uniformly on $[-1, 0]$ as follows. Let $s(x) = \lim s_n(x)$ and observe that

$$s(x) - s_n(x) = \left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right) - \left(x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}\right) = \sum_{k=n+1}^{\infty} \frac{x^k}{k}.$$

I will prove that

$$|s(x) - s_n(x)| \leq \left|\frac{x^{n+1}}{n+1}\right|$$

for all $x \in [-1, 0]$ and $n \in \mathbb{N}$. Once we do this, notice that

$$\sup\{|s(x) - s_n(x)| : x \in [-1, 0]\} \leq \sup\left\{\left|\frac{x^{n+1}}{n+1}\right| : x \in [-1, 0]\right\} = \frac{1}{n+1}$$

and this tends to 0 as n increases, so we have uniform convergence on $[-1, 0]$.

To prove the desired inequality, let's first take some steps to simplify the notation (and to make it more reminiscent of something you've seen before). Let me first write $y = -x$, so the inequality I am trying to prove is that

$$\left|\sum_{k=n+1}^{\infty} \frac{(-1)^k y^k}{k}\right| \leq \frac{y^{n+1}}{n+1}.$$

Now note that y^k/k is a positive number for all k, and also that the sequence of these numbers is monotonically decreasing as k increases. Let

$$a_m := \frac{y^{m+(n+1)}}{m+(n+1)}$$

for all $m \in \mathbb{N}$. In other words, what we are trying to prove is that

$$\left|\sum_{m=0}^{\infty} (-1)^{m+(n+1)} a_m\right| = \left|(-1)^{n+1} \sum_{m=0}^{\infty} (-1)^m a_m\right| = \left|\sum_{m=0}^{\infty} (-1)^m a_m\right| \leq a_0.$$

The proof of this is going to look a lot like the proof of the alternating series test. Let's look at the partial sums of the series $\sum(-1)^m a_m$.

$$\begin{aligned}
&a_0 \\
&a_0 - a_1 \\
&a_0 - (a_1 - a_2) &&= (a_0 - a_1) + a_2 \\
&a_0 - (a_1 - a_2) - a_3 &&= (a_0 - a_1) + (a_2 - a_3) \\
&a_0 - (a_1 - a_2) - (a_3 - a_4) &&= (a_0 - a_1) + (a_2 - a_3) + a_4 \\
&\quad\vdots &&\quad\vdots
\end{aligned}$$

The left hand side shows that all of these partial sums are less than a_0, since they are a_0 minus some positive number. The right hand side shows that all of these partial sums are nonnegative. Thus the limit of the partial sums has to be between 0 and a_0 also. This proves the inequality we wanted to prove. $\square$