Differentiation

1 Differentiation

Let S be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ a function on S. If a is an interior point of S, then a function $f : S \to \mathbb{R}$ is *differentiable at* a if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, we write $f'(a)$ for the value of this limit and call this the *derivative* of f at a. If $E \subseteq S^\circ$, then we say that f is *differentiable on* E if it is differentiable at every point of E.

Lemma 1.1. *Let S be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ a function. If f is differentiable at an interior point $a \in S$, then f is also continuous at a.*

Proof. Since

$$\lim_{x \to a} \frac{f(a) - f(x)}{a - x} = f'(a)$$

is some real number, and also

$$\lim_{x \to a} (a - x) = 0,$$

we see that

$$\lim_{x \to a} (f(a) - f(x)) = \lim_{x \to a} \frac{f(a) - f(x)}{a - x} \cdot (a - x) = \lim_{x \to a} \frac{f(a) - f(x)}{a - x} \cdot \lim_{x \to a} (a - x) = 0. \qquad \square$$

Lemma 1.2. *Let S be a subset of $\mathbb{R}$ and suppose f and g are functions $S \to \mathbb{R}$ which are both differentiable at an interior point $a \in S$.*

(a) The sum $f + g$ is differentiable at a, and $(f + g)(a) = f'(a) + g'(a)$.

(b) The product fg is differentiable at a, and $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.

(c) If $g(a) \neq 0$, then a is an interior point of the set

$$S' := \{x \in S : g(x) \neq 0\},$$

the quotient function $f/g : S' \to \mathbb{R}$ is differentiable at a, and

$$(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof. Each of these statements is just the consequence of a clever rewriting of the formula defining the derivative. For part (a), we observe the following for all $x \in S \setminus \{a\}$.

$$\frac{(f+g)(a)-(f+g)(x)}{a-x} = \frac{f(a)-f(x)}{a-x} + \frac{g(a)-g(x)}{a-x}$$

Letting x tend towards a gives the result. For part (b), we do the same with the following formula.

$$\frac{(fg)(a)-(fg)(x)}{a-x} = f(a)\cdot\frac{g(a)-g(x)}{a-x} + g(x)\cdot\frac{f(a)-f(x)}{a-x}$$

Notice that this time, we have to use the fact that g is continuous at a from lemma 1.1. Finally, for part (c), observe that since g is differentiable at a, it is also continuous at a. Moreover, $g(a) \neq 0$, which means that there exists some open ball $B(a,\delta)$ such that, for all $x \in B(a,\delta)$, we have $g(x) \in B(g(a), |g(a)|)$. This condition means that $g(x) \neq 0$. In other words, the point a is still an interior point of the set $S' = \{x \in S : g(x) \neq 0\}$, as claimed. Now we observe the following formula and then let x tend towards a.

$$\frac{(f/g)(a)-(f/g)(x)}{a-x} = \frac{1}{g(a)g(x)}\left(g(x)\cdot\frac{f(a)-f(x)}{a-x} - f(x)\cdot\frac{g(a)-g(x)}{a-x}\right)$$

We again have to use the fact that f and g are continuous at a, as established in lemma 1.1. $\square$

Lemma 1.3 (Chain rule)**.** *Let S and T be subsets of $\mathbb{R}$ and suppose we have functions $f : S \to T$ and $g : T \to \mathbb{R}$ be functions such that f is differentiable at an interior point $a \in S$, $f(a)$ is an interior point of T, and g is differentiable at $f(a)$. Then the composite $g \circ f$ is differentiable at a, and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. We define a function $h : T \to \mathbb{R}$ by the formula

$$h(y) = \begin{cases} \dfrac{g(y)-g(f(a))}{y-f(a)} & \text{if } y \neq f(a), \text{ and} \\ g'(f(a)) & \text{if } y = f(a). \end{cases}$$

Then

$$\lim_{y \to f(a)} h(y) = g'(f(a)) = h(f(a))$$

so h is continuous at $f(a)$. Moreover, observe that we have

$$g(y) - g(f(a)) = h(y)(y - f(a))$$

for all $y \in f(S)$, so in particular, by taking $y = f(x)$ for $x \in S$, we have

$$g(f(x)) - g(f(a)) = h(f(x))(f(x) - f(a))$$

for all $x \in S$. Dividing by $x - a$, we see that

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = h(f(x)) \cdot \frac{f(x) - f(a)}{x - a}$$

for all $x \in S \smallsetminus \{a\}$. We now let x tend towards a, and observe that, since h is continuous at $f(a)$ and f is continuous at a, we see that $h \circ f$ is contuous at a, so

$$\lim_{x \to a} h(f(x)) = (h \circ f)(a) = h(f(a)) = g'(f(a)).$$ □

2 Mean Value Theorem

Lemma 2.1. *Let $f : S \to \mathbb{R}$ be a function which is differentiable at an interior point $a \in S$. If either $f(a) = \sup f(S)$ or $f(a) = \inf f(S)$, then $f'(a) = 0$.*

Proof. By replacing f with $-f$ if necessary, we can assume that $f(a) = \sup f(S)$ without loss of generality. Suppose that $f'(a) \neq 0$. Then either $f'(a) \gneq 0$ or $f'(a) \gneq 0$. If $f'(a) \gneq 0$, then there exists some $\delta \gneq 0$ such that for all $x \in B(a, \delta)$, we have

$$\frac{f(x) - f(a)}{x - a} \gneq 0.$$

Then for any $x \in (a, a+\delta)$, observe that $x - a \gneq 0$, so that means that $f(x) - f(a) \gneq 0$, contradicting the assumption that $f(a) = \sup f(S)$. Similarly, if $f'(a) \gneq 0$, we can find some x slightly smaller than a such that $f(x) \gneq f(a)$, again contradicting $f(a) = \sup f(S)$. □

Lemma 2.2 (Rolle's theorem)**.** *Let $I = [a, b]$ be a closed interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function which is differentiable on I° and such that $f(a) = f(b)$. Then there exists $c \in I^\circ$ such that $f'(c) = 0$.*

Proof. Since f is continuous on a compact set, it achieves its minimum and maximum values at some points a' and b', respectively. If $a' = a$ and $b' = b$, then the assumotion that $f(a) = f(b)$ tells us that f is a constant function, so lemma 2.1 guarantees that $f'(c) = 0$ for all interior points $c \in I$. On the other hand, if either $a' \neq a$ or $b' \neq b$, then let c be whichever of the values a' or b' is not an endpoint of I, and then lemma 2.1 guarantees that $f'(c) = 0$ again. □

Theorem 2.3 (Generalized Mean Value Theorem)**.** *Let f and g be continuous functions on a closed interval $I = [a, b]$ which are differentable on I°. Then there exists some $c \in I^\circ$ such that*

$$f'(c)(g(b) - g(a)) = g'(a)(f(b) - f(a)).$$

Proof. Define a function $h : I \to \mathbb{R}$ by

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

We are trying to show that h' vanishes at some point in I°. But notice that h is continuous on I, differentiable on I°, and

$$h(a) = f(a)g(b) - g(a)f(b) = h(b),$$

so Rolle's theorem guarantees this immediately. $\square$

Corollary 2.4 (Mean value theorem)**.** *Let $I = [a, b]$ be a closed interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function which is differentiable on I°. Then there exists $c \in I^\circ$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $g : I \to \mathbb{R}$ be the function $g(x) = x$. Then applying the generalized mean value theorem shows us immediately that that there exists some $c \in I^\circ$ such that $f'(c)(b - a) = f(b) - f(a)$. $\square$

Alternative proof. Let $\ell : I \to \mathbb{R}$ whose graph is the straight line connecting the points $(a, f(a))$ and $(b, f(b))$. In other words,

$$\ell(x) = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a).$$

Then it is easy to compute that

$$\ell(x) = \frac{f(b) - f(a)}{b - a}$$

for all interior points $x \in I$. Moreover, $\ell(a) = f(a)$ and $\ell(b) = f(b)$. Now consider the function $g : I \to \mathbb{R}$ defined by $g(x) = f(x) - \ell(x)$. Then g is continuous on I and differentiable at all interior points, and moreover

$$g(a) = f(a) - \ell(a) = 0 = f(b) - \ell(b) = g(b),$$

so Rolle's theorem 2.2 guarantees that there exists some interior point $c \in I$ such that

$$0 = g'(c) = f'(c) - \ell(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

$\square$

3 Uniform Limits and Differentiation

Unfortunately, it is not true that a uniform limit of differentiable functions must be differentiable, and even if it is, it need not be that the derivative of the limit is the limit of the derivatives.

Example 3.1. Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

Then it is easy to see that these functions are differentiable, since the functions $x \mapsto x^2$ and $x \mapsto \sqrt{x + (1/n)}$ are both differentiable. Moreover, observe that $|x| \leq |f_n(x)|$, and also that

$$|f_n(x)| \leq |x| + \frac{1}{\sqrt{n}}.$$

To see this, notice that

$$|f_n(x)|^2 = x^2 + \frac{1}{n} \leq x^2 + \frac{2\,|x|}{\sqrt{n}} + \frac{1}{n} = \left(|x| + \frac{1}{\sqrt{n}}\right)^2$$

so we can take the square root of both sides to get the desired inequality. Thus, if we let $f : S \to \mathbb{R}$ be the absolute value function $f(x) = |x|$, then

$$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}}$$

for all $x \in S$, so $\|f_n - f\|_{\text{sup}} \leq 1/\sqrt{n}$. This shows that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. However, clearly the limit function f is not differentiable at 0.

Example 3.2. Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{\sin(nx)}{n}.$$

This is a sequence of bounded differentiable functions which converges uniformly to 0. Indeed, for any $\varepsilon \gneq 0$, there exists N such that $1/N \lneq \varepsilon$, and then for all $n \geq N$ and $x \in \mathbb{R}$, we see that

$$|f_n(x)| = \left|\frac{\sin(nx)}{n}\right| \leq \frac{1}{n} \leq \frac{1}{N}$$

which means that $\|f_n\|_{\text{sup}} \leq 1/N \lneq \varepsilon$. Thus in this case, we have a sequence of differentiable functions converging to another differentiable function.

However, notice that $f_n'(x) = \cos(nx)$. This sequence of functions $(f_n')_{n\in\mathbb{N}}$ does not even converge pointwise. For example, at $x = \pi$, we have the sequence

$$(f_n'(\pi))_{n\in\mathbb{N}} = (\cos(\pi), \cos(2\pi), \cos(3\pi), \dots) = (-1, 1, -1, 1, \dots)$$

which is divergent.

Despite this bad behavior, we do have the following result. The statement is quite technical, and the proof is also quite difficult.

Theorem 3.3. *Let S be a bounded connected open subset of $\mathbb{R}$ and suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of differentiable functions $S \to \mathbb{R}$. Assume furthermore that $(f_n(s_0))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ for some fixed $s_0 \in S$. If the sequence of derivatives $(f_n')_{n\in\mathbb{N}}$ converges uniformly on S, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly on S to a differentiable function $f : S \to \mathbb{R}$ such that $\lim f_n' = f'$.*

Proof. First we will show that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy. Fix $\varepsilon \gneq 0$. Let $R := \operatorname{diam}(S)$. Uniform convergence of $(f_n')_{n\in\mathbb{N}}$ means in particular that this sequence is uniformly Cauchy, so there exists some integer N such that

$$\|f_m' - f_n'\|_{\text{sup}} \lneq \frac{\varepsilon}{2R}$$

for all $m, n \geq N$ also. Now for any pair of points $s \lneq t$ in S, we know that, since S is connected, the entire interval $[s,t]$ is contained in S. The function $f_m - f_n$ is differentiable on S, so in particular it is continuous on $[s,t]$ and differentiable on its interior, so by the mean value theorem, there exists some $c \in (s,t)$ such that

$$\frac{(f_m(t) - f_n(t)) - (f_m(s) - f_n(s))}{t-s} = f_m'(c) - f_n'(c).$$

This means that

$$|(f_m(t) - f_n(t)) - (f_m(s) - f_n(s))| = |f_m'(c) - f_n'(c)| \cdot |t-s| \leq \|f_m' - f_n'\|_{\sup} |t-s| \lneq \frac{\varepsilon\,|t-s|}{2R} \leq \frac{\varepsilon}{2}.$$

Now recall that we have this point s_0 such that $(f_n(s_0))_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}$. In particular Cauchy, so, by making N larger if necessary, we can assume that

$$|f_m(s_0) - f_n(s_0)| \lneq \frac{\varepsilon}{2}$$

for all $m, n \geq N$. Then for any $s \in S$, observe that

$$\begin{aligned}
|f_m(s) - f_n(s)| &= |f_m(s) - f_n(s) + f_m(s_0) - f_m(s_0) + f_n(s_0) - f_n(s_0)| \\
&\leq |(f_m(s) - f_n(s)) + (f_m(s_0) - f_n(s_0))| + |f_n(s_0) - f_m(s_0)| \\
&\lneq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

In other words, $\|f_m - f_n\|_{\sup} \leq \varepsilon$. In other words, the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy. Since $\mathbb{R}^S$ is a complete metric space, this Cauchy sequence converges. Let $f := \lim f_n$.

Now we need to show that the function f is differentiable and that $f' = \lim f_n'$. Fix a point $a \in S$. For each $n \in \mathbb{N}$, define $\varphi_n : S \to \mathbb{R}$ by

$$\varphi_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(a)}{x-a} & \text{if } x \neq a, \text{ and} \\ f_n'(a) & \text{if } x = a. \end{cases}$$

The fact that f_n is differentiable at a is then equivalent to saying that φ_n is continuous. In other words, we have a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of continuous functions $S \to \mathbb{R}$. Then, like we did above, we apply the mean value theorem to the function $f_m - f_n$ on the set of points between x and a in order to see that, for any $\varepsilon \gneq 0$, there exists N such that for all $m, n \geq N$, we have

$$|\varphi_m(x) - \varphi_n(x)| = \left| \frac{f_m(x) - f_m(a)}{x-a} - \frac{f_n(x) - f_n(a)}{x-a} \right| \lneq \frac{\varepsilon}{2R}.$$

In other words, $(\varphi_n)_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence of continuous functions on S. Since $\mathcal{C}(S)$ is complete, this sequence actually converges uniformly to a continuous function $\varphi : S \to \mathbb{R}$. Now observe that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \lim_{n\to\infty} \varphi_n(x) = \lim_{x\to a} \varphi(x) = \varphi(a)$$

which means we have

$$f'(a) = \varphi(a) = \lim_{n\to\infty} \varphi_n(a) = \lim_{n\to\infty} f_n'(a).$$

Thus $(f_n')_{n\in\mathbb{N}}$ is a uniformly convergent sequence which converges pointwise to f', so actually f' is the uniform limit of $(f_n')_{n\in\mathbb{N}}$. In other words, $f' = \lim f_n'$ in $\mathbb{R}^S$. $\square$

4 Sample Problems

Problem 1. Determine whether each of the following functions is differentiable at the specified point a. If it is not differentiable at a, is it at least continuous at a? If it is differentiable, is the derivative continuous at a?

(a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0 \end{cases}$$

at the point $a = 0$.

(b) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0 \end{cases}$$

at the point $a = 0$.

Reference. See example 5.6 in Rudin.

Problem 2. Prove that, for any positive integer n, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is differentiable at all points $x \in \mathbb{R}$ and that

$$f'(x) = nx^{n-1}.$$

Then prove that the same formula is true for all negative integers n as well.

Hint. Deal with the case $n = 1$ from the definition of the derivative. Then induct on n using the product rule for derivatives to obtain the result for all positive integers. Then use the quotient rule to get the result for all negative integers n as well.

Problem 3. Suppose S is a connected subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ is a continuous function such that $f'(x) = 0$ for all interior points $x \in S$. Show that f must be a constant function. What happens if S is not connected?

Hint. First explain why f being constant on S° implies that it is also constant on S. Thus we can reduce the case $S = S^\circ$. Now if f is not constant, there are two distinct points $a, b \in S$ where f has two different values. We can assume $a \lneq b$ without loss of generality. Since S is connected, we know that $[a, b] \subseteq S$. Now use the mean value theorem to find a contradiction.

Problem 4. Let S be a connected subset of $\mathbb{R}$ and suppose f and g be continuous functions $S \to \mathbb{R}$ which are differentiable on S° and such that

$$f'(x) = g'(x)$$

for all $x \in S^\circ$. Show that there exists a constant c such that $f(x) = g(x) + c$ for all $x \in S$.

Hint. Consider the function $h := f - g$.

Problem 5. Let S be a connected subset of $\mathbb{R}$ and suppose a $f : S \to \mathbb{R}$ is a continuous function which is differentiable on S°. If the derivative $f' : S^\circ \to \mathbb{R}$ is bounded, show that f is uniformly continuous on S.

Proof. Let M be some real number such that $|f'(x)| \leq M$ for all $x \in S^\circ$. Suppose we have two distinct points $a, b \in S$. Without loss of generality, suppose that $a \lneq b$. Since S is connected, we know that $[a, b] \in S$, and then the mean value theorem guarantees that there exists some $c \in (a, b)$ such that

$$|f(b) - f(a)| = |f'(c)| \, |b - a| \leq M \, |b - a| \, .$$

Thus, if we fix $\varepsilon \gneq 0$ and let $\delta = \varepsilon/M$, then if $|b - a| \lneq \delta$, we see that $|f(b) - f(a)| \lneq \varepsilon$. □

Problem 6. Let S be a connected open subset of $\mathbb{R}$ and let $g : S \to \mathbb{R}$ be a differentiable function such that $g'(x) \neq 0$ for all $x \in S$. Show that g is injective.

Problem 7. Let S be a connected open subset of $\mathbb{R}$ and let $g : S \to \mathbb{R}$ be a differentiable function. Show that g is increasing (in other words, that $x \leq y$ implies $g(x) \leq g(y)$) if and only if $g'(x) \geq 0$ for all $x \in S$.

Problem 8. Show that theorem 3.3 would be false if we omitted the hypothesis that $(f_n(s_0))_{n \in \mathbb{N}}$ is a convergent sequence for some $s_0 \in S$.