4.2.1. Show that $\sqrt{\langle x^2, y^2\rangle} = \langle x, y\rangle$.

Proof. Let $I = \langle x^2, y^2\rangle$. Since $x^2 \in I$, we have $x \in \sqrt{I}$, and similarly we have $y \in \sqrt{I}$, which shows that $\langle x, y\rangle \subseteq \sqrt{I}$.

Conversely, suppose for a contradiction that $f \in \sqrt{I}$ but $f \notin \langle x, y\rangle$. Since $f \in \sqrt{I}$, there exists a positive integer m such that $f^m \in I$. Since I is a monomial ideal, this means that every term of f^m is divisible by either x^2 or y^2 by lemma 2.4.3. On the other hand, since $f \notin \langle x, y\rangle$, we can write $f = a + \tilde{f}$ where $a \in k \setminus \{0\}$ and every term of $\tilde{f}$ is divisible by x or y. Then

$$f^m = (a + \tilde{f})^m = a^m + ma^{m-1}\tilde{f} + \cdots + \tilde{f}^m$$

by the binomial theorem. Since $\tilde{f}$ has no constant term, the same is true of $ma^{m-1}\tilde{f} + \cdots + \tilde{f}^m$, so a^m is the constant term of f^m. But this this term is not divisible by x^2 or y^2. This is a contradiction. $\square$

I encourage you to prove the generalization of this given in exercise 4.2.1.

4.2.2. Let f and g be distinct nonconstant polynomials in $k[x, y]$ and let $I = \langle f^2, g^2\rangle$. Is it necessarily true that $\sqrt{I} = \langle f, g\rangle$?

Solution. It is not true. For example, consider $f = x^2$ and $y = y$ in $k[x, y]$. Then $I = \langle f^2, g^2\rangle = \langle x^4, y^2\rangle$. Since $x^4 \in I$, we have $x \in \sqrt{I}$. But $\langle f, g\rangle = \langle x^2, y\rangle$ and $x \notin \langle x^2, y\rangle$ since $\langle x^2, y\rangle$ is a monomial ideal and x is divisible by neither x^2 nor y. Thus we have

$$x \in \sqrt{I} \setminus \langle x^2, y\rangle,$$

proving that $\sqrt{I} \neq \langle f, g\rangle$ in general.

4.2.14. Let $J = \langle xy, (x - y)x\rangle$. Describe $V(J)$ and show that $\sqrt{J} = \langle x\rangle$.

Solution. Observe that $xy = 0$ if and only if $x = 0$ or $y = 0$. Also, $(x - y)x = 0$ if and only if $x = 0$ or $(x - y) = 0$. If $x \neq 0$, then we must have $y = 0$ and $x - y = 0$, which impies that $x = 0$, and this is a contradiction. Thus $(x, y) \in V(J)$ if and only if $x = 0$. In other words,

$$V(J) = y\text{-axis}.$$

There are (at least) two ways to prove that $\sqrt{J} = \langle x\rangle$. One way is to use the nullstellensatz, but this requires that that our ground field be algebraically field. To see how this goes, suppose k is algebraically closed. Observe that $x \in I(y\text{-axis}) = I(V(J))$, so $\langle x\rangle \subseteq I(V(J))$. On the other hand, suppose $f \in I(V(J)) = I(y\text{-axis})$. By the division algorithm using lex order where $x > y$, we can write

$$f = xq + r$$

where $r \in k[y]$. Since $f \in I(y\text{-axis})$, we have

$$0 = f(0, y) = 0 \cdot q(0, y) + r(y) = r(y)$$

for all y. This means that r has infinitely many roots, so $r = 0$, which shows that $f = xq \in \langle x \rangle$. Thus $I(V(J)) = \langle x \rangle$. Finally, by applying the nullstellensatz, we have

$$\sqrt{J} = I(V(J)) = \langle x \rangle.$$

We can also prove that $\sqrt{J} = \langle x \rangle$ without the nullstellensatz, using a similar argument as the one for exercise 4.2.1 above. The nice thing about this is that this doesn't require the ground field to be algebraically field. Running Sage as follows, we find that $J = \langle x^2, xy \rangle$.

```
R.<x,y> = PolynomialRing(QQ, order='lex')
I = Ideal(x*y,(x-y)*x)
I.groebner_basis()
```

Since $x^2 \in J$, it is clear that $x \in \sqrt{J}$, so $\langle x \rangle \subseteq \sqrt{J}$. For the other inclusion, suppose for a contradiction that $f \in \sqrt{J}$ but $x \notin \langle x \rangle$. Since $f \in \sqrt{J}$, there exists some positive integer m such that $f^m \in J$. Since $J = \langle x^2, xy \rangle$ is a monomial ideal, we know that every term of f is divisible by x^2 or xy. On the other hand, since $f \notin \langle x \rangle$, we can write $f = \tilde{f} + r$ where $r \in k[y]$ is nonzero and $\tilde{f} \in langlex\rangle$. Then

$$f^m = \tilde{f}^m + \cdots + mr^{m-1}\tilde{f} + r^m.$$

Then $\tilde{f}^m + \cdots + mr^{m-1}\tilde{f} \in \langle x \rangle$ and $r^m \in k[y]$. Since r is nonzero, r^m is also nonzero. But $r^m \in k[y]$ means that none of its terms are divisible by x^2 or xy. This is a contradiction.