Continuity and Compactness

1 Images of Compact Spaces

Lemma 1.1. *Let X and Y be metric spaces and let $f : X \to Y$ be a continuous function. If X is compact, then the image $f(X)$ is also compact.*

First proof. Let $\mathcal{U}$ be a collection of open subsets of Y whose union contains $f(X)$. Then let us define

$$f^{-1}\mathcal{U} := \{f^{-1}(U) : U \in \mathcal{U}\}.$$

Since f is continuous, this is a collection of open subsets of X. Moreover, for any $x \in X$, we know that $f(x) \in U$ for some $U \in \mathcal{U}$, so $x \in f^{-1}(U)$, so this is actually an open cover of X. Since X is compact, it has a finite subcover $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$. Then $\mathcal{U}' := \{U_1, \dots, U_n\}$ is a finite subset of $\mathcal{U}$ whose union still covers $f(X)$. Indeed, given any $y \in f(X)$, there exists some $x \in X$ such that $f(x) = y$, and then $x \in f^{-1}(U_i)$ for some $i = 1, \dots, n$, which means that $y = f(x) \in U_i$. $\square$

Second proof. Let us show that $f(X)$ is sequentially compact using the fact that X is sequentially compact. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $f(X)$. Then for each y_n, there exists some $x_n \in X$ such that $f(x_n) = y_n$. Since X is sequentially compact, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $a = \lim x_{n_k}$. Since f is continuous, we know that

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(a)$$

so $(y_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. $\square$

Corollary 1.2. *Let X be a compact metric space and $f : X \to \mathbb{R}$ a continuous function. Then there exists some $a, b \in X$ such that*

$$f(a) = \inf f(X) \text{ and } f(b) = \sup f(X).$$

Proof. By lemma 1.1, we know that $f(X)$ is compact. This means that it is a closed and bounded subset of $\mathbb{R}$, so $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$. In other words, there exists $a \in X$ such that $f(a) = \inf f(X)$ and there exists $b \in X$ such that $f(b) = \sup f(X)$. $\square$

Corollary 1.3. *Let X and Y be metric spaces and suppose $f : X \to Y$ is a continuous bijection. If X is compact, then the inverse function f^{-1} is continuous as well.*

Proof. We need to show that $f^{-1} : Y \to X$ is continuous, which means we need to show that for every open $U \subseteq X$, its preimage under f^{-1} is open. But the preimage of U under f^{-1} is equal to $f(U)$, so we are trying to show that $f(U)$ is open in Y. Since U is open, its complement $F := X \setminus U$ is closed. Since X is compact, this means that F is also compact, so lemma 1.1 tells us that $f(F)$ is compact as well. This means that $f(F)$ is closed in Y. But notice that

$$f(F) = f(X \setminus U) = Y \setminus f(U)$$

so since $f(F)$ is closed, we see that $f(U)$ is open. □

Remark 1.4. If X and Y are metric spaces, a continuous function $f : X \to Y$ is called a *homeomorphism* if it is bijective and if f^{-1} is also continuous. In general, it need not be that any continuous bijection is a homeomorphism: see examples 3.2 and 3.3 below. The preceding corollary tells us that, if X is compact, then it does follow that any continuous bijection $X \to Y$ is automatically a homeomorphism.

2 Compactness and Uniform Continuity

Lemma 2.1. *Let X and Y be metric spaces and $f : X \to Y$ a continuous function. If X is compact, then f is uniformly continuous.*

First proof. Fix some $\varepsilon \gneq 0$. Continuity guarantees that for each point $x \in X$, there exists a $\delta_x \gneq 0$ such that the open ball $B_X(x, \delta_x)$ is mapped into $B_Y(f(x), \varepsilon/2)$. Consider the open cover

$$\mathcal{U} := \{B_X(x, \delta_x/2) : x \in X\}.$$

Comapctness of X guarantees that there exists a finite subcover. In other words, there exist finitely many points $x_1, \ldots, x_n$ such that X equals the union of the open balls $B_X(x_i, \delta_{x_i}/2)$ for $i = 1, \ldots, n$. Let

$$\delta := \frac{1}{2} \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}.$$

Now suppose that x and x' are any two points of X such that $d_X(x, x') \lneq \delta$. Then there exists some i such that $x \in B(x_i, \delta_{x_i}/2)$. Then we clearly have $f(x) \in B_Y(f(x_i), \varepsilon/2)$. Moreover, observe that

$$d(x_i, x') \leq d(x_i, x) + d(x, x') \lneq \delta_{x_i}/2 + \delta/2 \lneq \delta_{x_i}$$

which means that $f(x') \in B_Y(f(x_i), \varepsilon/2)$, so

$$d_Y(f(x), f(x')) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(x')) \lneq \varepsilon.$$ □

Second proof. Suppose X is not uniformly continuous. Then there exists an $\varepsilon \gneq 0$ such that for any $\delta \gneq 0$, there exist points x, x' which are less than δ apart but for which $f(x)$ and $f(x')$ are at least ε apart. Taking $\delta = 1/(n+1)$ for each $n \in \mathbb{N}$, we find a pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ such that $d_X(x_n, x'_n) \lneq 1/(n+1)$ but $d_Y(f(x_n), f(x'_n)) \geq \varepsilon$.

Since X is sequentially compact, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Now since $(x'_{n_k})_{k\in\mathbb{N}}$ is a sequence in X also, it too has a convergent subsequence $(x'_{n_{k_j}})_{j\in\mathbb{N}}$. Then $(x_{n_{k_j}})_{j\in\mathbb{N}}$ is a subsequence of $(x_{n_k})_{k\in\mathbb{N}}$, so it converges as well. Now we know that

$$d_X(x_{n_{k_j}}, x'_{n_{k_j}}) \lneq \frac{1}{n_{k_j}+1}$$

for all j, so actually both of these sequences muust converge to the same point $a \in X$. But f is continuous, so we must have

$$\lim_{j\to\infty} f(x_{n_{k_j}}) = f(a) = \lim_{j\to\infty} f(x_{n_{k_j}}).$$

This means that there exists some J such that for all $j \geq J$, we have $f(x_{n_{k_j}}) \in B(f(a), \varepsilon/2)$ and $f(x'_{n_{k_j}}) \in B(f(a), \varepsilon/2)$ also, so then

$$d(f(x_{n_{k_j}}), f(x'_{n_{k_j}})) \leq d(f(x_{n_{k_j}}), f(a)) + d(f(a), f(x'_{n_{k_j}})) \lneq \varepsilon,$$

which is a contradiction. $\square$

3 Counterexamples

We now discuss several examples which demonstrate that the compactness assumptions in the above results are essential.

Example 3.1. Let $X := (0, 1]$ and $Y := \mathbb{R}$ and consider the function $f : X \to Y$ given by $f(x) = 1/x$. Then f is continuous, but $f(X) = (0, \infty)$ is not compact. Notice that $X = (0, 1]$ is not compact.

Example 3.2. Let $X := [0, 1]$ with the discrete metric and let $Y := [0, 1]$ with the euclidean metric. Then the identity function $f : X \to Y$ given by $f(x) = x$ is continuous, since for any open set U, the preimage $f^{-1}(U)$ is open in X, since all subsets of X are open. Moreover f is also a bijection. However, the inverse map $f^{-1} : Y \to X$ is not continuous. Indeed, the subset $U := \{0\}$ is an open subset of X since all subsets of X are open, but $f^{-1}(U) = \{0\}$ is not an open subset of Y. Notice that X is not compact.

Example 3.3. Let $X := [0, 2\pi)$ and let $Y := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$ be the set of points on the unit circle in the plane. Consider the function $f : X \to Y$ given by

$$f(\theta) = (\cos\theta, \sin\theta).$$

This function is bijective: this follows from various properties of cos and sin which you are familiar with, but we haven't proved in this class... Anyway, from the picture, this should be fairly obvious. This function f is also continuous. This is "obvious" because cos and sin are both continuous functions, but I want to explain how we can use continuity of cos and sin to prove this more formally.

To prove that f is continuous, we will express f as a composite of continuous functions. First, let $\Delta : X \to X \times X$ is the function given by $\Delta(x) = (x, x)$. This is continuous by sample problem 7 on the lecture on continuity (applied with $f = g$ both being the identity map $x \mapsto x$ on X). Furthermore, the function $\cos \times \sin : X \times X \to Y$ given by

$$(\cos \times \sin)(\theta, \theta') = (\cos \theta, \sin \theta')$$

is continuous, using the fact that cos and sin are both continuous and another problem which will be assigned on problem set 6. Now notice that $f = (\cos \times \sin) \circ \Delta$.

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\cos \times \sin} Y$$

Since Δ and $\cos \times \sin$ are both continuous, and composites of continuous functions are continuous, we conclude that f is continuous.

Finally, we note that f^{-1} is not continuous. Indeed, consider the open ball $B_X(0, \varepsilon)$ for some small positive number ε. The preimage of this open ball under f^{-1} is a small bit of the circle going counterclockwise from the point $(1, 0) \in Y$, including the point $(1, 0)$, but the point $(1, 0)$ is included in that preimage, and is clearly not an interior point of the preimage. In other words, f^{-1} fails to be continuous at the point $(1, 0)$.

Example 3.4. Let $X = (0, 1]$ and $Y = \mathbb{R}$. Recall that last time, we saw that the function $f : X \to Y$ given by $f(x) = 1/x$ is not uniformly continuous: for example, it turned the Cauchy sequence $(1/n)_{n\in\mathbb{N}}$ into a non-Cauchy sequence in Y. Notice again that X is not compact.

4 Sample Problems

Problem 1. Show that a function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if there exists a continuous function $\widetilde{f} : [a, b] \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for all $x \in (a, b)$.

Problem 2. Let $I := [0, 1]$ and let $f : I \to \mathbb{R}$ be a nonconstant continuous function ("nonconstant" means that its image is not just a single point). Prove that $f(I)$ is a closed interval.

Proof. Since I is compact, the image $f(I)$ is compact also, so if $a := \inf f(I)$ and $b := \sup f(I)$, then $a, b \in f(I)$. We automatically have $a \leq b$. If $a = b$, then $f(I) = \{a\}$ so f is a constant function which maps all of I onto the point a, which is a contradiction. Thus we must have $a \lneq b$. Since I is connected, the image $f(I)$ is also connected, so $[a, b] \subseteq f(I)$. In fact we must have $[a, b] = f(I)$, since if not, there would exist some $c \in f(I)$ that was either smaller than a or larger than b, which contradicts the definition of a and b. $\square$