Problem Set B – Partial Solutions

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Problem 1. There are 63 piles of bananas with nnn bananas each, and 7 additional bananas. All of these bananas are divided evenly among 23 travelers. How many bananas can be in each pile? (Describe the set of all possible values of nnn.)

Solution. Let n be the number of bananas in each pile, and let y be the number of bananas that each traveler gets. Then $63n + 7 = 23y$, or, in other words,

$$63n - 23y = -7.$$

This is a linear diophantine equation in two variables. To solve it, note that $63 = 3 \cdot 3 \cdot 7$ is relatively prime to -23, so we can find integers a and b such that $63a - 23b = 1$. To calculate these integers, we use the Euclidean algorithm.

$$\begin{aligned} 63 &= 2 \cdot 23 + 17 \\ 23 &= 1 \cdot 17 + 6 \\ 17 &= 2 \cdot 6 + 5 \\ 6 &= 1 \cdot 5 + 1 \end{aligned}$$

Then

$$\begin{aligned} 1 &= 6 - 5 \\ &= 6 - (17 - 2 \cdot 6) \\ &= 3 \cdot 6 - 17 \\ &= 3 \cdot (23 - 17) - 17 \\ &= 3 \cdot 23 - 4 \cdot 17 \\ &= 3 \cdot 23 - 4 \cdot (63 - 2 \cdot 23) \\ &= -4 \cdot 63 + 11 \cdot 23 \\ &= 63(-4) - 23(-11). \end{aligned}$$

Multiplying through by -7 shows that

$$-7 = 63(28) - 23(77)$$

so $n_0 = 28$ and $y_0 = 77$ is one solution to this diophantine equation. Other solutions to the same diophantine equation are given by

$$n = 28 - 23t \quad y = 77 - 63t.$$

We need for both n and y to be positive, which happens when $t \leq 1$. In other words, the set of possible bananas in each pile is

$$\{28 - 23t \mid t \leq 1\}.$$

Problem 3. Prove that

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for any $n \geq m \geq 1$.

Proof. By Bézout's theorem, we know that $\gcd(m,n) = mx + ny$ for some integers x and y. Then

$$\begin{aligned}
\frac{\gcd(m,n)}{n}\binom{n}{m} &= \frac{mx+ny}{n}\binom{n}{m} \\
&= \frac{mx}{n}\binom{n}{m} + y\binom{n}{m} \\
&= \frac{mx}{n} \cdot \frac{n!}{m!(n-m)!} + y\binom{n}{m} \\
&= x\frac{(n-1)!}{(m-1)!(n-m)!} + y\binom{n}{m} \\
&= x\binom{n-1}{m-1} + y\binom{n}{m}.
\end{aligned}$$

Binomial coefficients are integers by the binomial theorem, and sums and products of integers are again integers, so this expression is an integer. $\square$