Problem Set C – Partial Solutions

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Problem 4. Suppose $p_1, p_2, \ldots, p_d$ are distinct primes. Show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_d}$$

is never an integer.

Solution. By bringing everything to a common denominator, we have

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_d} = \frac{p_2 p_3 \cdots p_d + p_1 p_3 \cdots p_d + \cdots + p_1 p_2 \cdots p_{d-1}}{p_1 \cdots p_d},$$

so we want to show that $p_1 \cdots p_d$ does not divide

$$Q = p_2 p_3 \cdots p_d + p_1 p_3 \cdots p_d + \cdots + p_1 p_2 \cdots p_{d-1}.$$

Suppose for a contradiction that $p_1 \cdots p_d \mid Q$. That means that $p_1 \mid Q$. But p_1 also divides all of the summands in Q after the first one: in other words, p_1 divides the sum $p_1 p_3 \cdots p_d + p_1 p_2 p_4 \cdots p_d + \cdots + p_1 p_2 \cdots p_{d-1}$. This means that p_1 also divides

$$Q - p_1 p_3 \cdots p_d + p_1 p_2 p_4 \cdots p_d + \cdots + p_1 p_2 \cdots p_{d-1} = p_2 p_3 \cdots p_d.$$

Since p_1 is prime and it divides the product $p_2 p_3 \cdots p_d$, it must divide one of the factors; in other words, there must exist an $i = 2, \ldots, d$ such that $p_1 \mid p_i$. But p_i is prime and $p_1 \neq 1$, so this means that $p_1 = p_i$. This contradicts our assumption that the primes are all distinct.

Problem 5. If $n > 4$ is composite, show that $(n-1)! \equiv 0 \pmod{n}$. *Note.* This result is not true for $n = 4$, so make sure your proof uses the fact that $n > 4$ at some point.

Solution. Let p be the smallest factor of n that's greater than 1 and let $q = n/p$ so that $n = pq$. Since n is composite and p is its smallest factor that's bigger than 1, we see that $2 \leq p \leq q \leq n-1$. There are two cases:

- If $p < q$, then p and q both show up separately as factors in $(n-1)!$, so $n = pq \mid (n-1)!$.

- Suppose on the other hand that $p = q$, so that $n = p^2$. We claim that $2p \leq n - 1$. Indeed, if we had $p^2 = n \leq 2p$, we could divide both sides by 2 to conclude that $p \leq 2$, but then squaring both sides would yield $n = p^2 \leq 4$, contradicting our assumption that $n > 4$. So $2p \leq n - 1$, which means that p and $2p$ both show up separately as factors in $(n-1)!$. This means that $p \cdot 2p = 2p^2 \mid (n-1)!$, and since $n = p^2 \mid 2p^2$, this shows that $n \mid (n-1)!$.