Problem Set 2

Problem 1. (1 point) Let X be a set and let d_1 and d_2 be two metrics on X. Then d_1 and d_2 are *equivalent* if a subset E of X is open with respect to d_1 if and only if it is open with respect to d_2.

Show that on $X := \mathbb{R}^2$, the euclidean metric, the Manhattan metric, and the maximum metric are all equivalent to each other.

Proof. Let d_1 be the maximum metric, d_2 the euclidean metric and d_3 the Manhattan metric, and let

$$B_i(a,r) := \{x \in \mathbb{R}^2 : d_i(a,x) \lneq r\}$$

for $i = 1, 2, 3$. Notice that $B_1(a,r) \supseteq B_2(a,r) \supseteq B_3(a,r)$. Moreover, elementary geometry shows that $B_3(a,r) \supseteq B_1(a,r/2)$.

Now, to show that d_1, d_2 and d_3 are equivalent, it suffices to show that the following three statements are equivalent for a set $U \subseteq X$.

(i) U is open with respect to d_1.

(ii) U is open with repsect to d_2.

(iii) U is open with respect to d_3.

Suppose U is open with respect to d_1. Then for every $a \in U$, we know that $B_1(a,r) \subseteq U$ for some r. But then $B_2(a,r) \subseteq B_1(a,r) \subseteq U$ also, so U is also open with respect to d_2. The proof that (ii) implies (iii) is identical. Finally, suppose U is open with respect to d_3. Then, for any $a \in U$, there exists some positive r such that $B_3(a,r) \subseteq U$. But then $B_1(a,r/2) \subseteq B_3(a,r) \subseteq U$. Thus U is also open with respect to d_1, proving (iii) implies (i) and completing the circle. $\square$

Problem 2. (1 point) Determine the closures of the following subsets of $\mathbb{R}$ with the euclidean metric. You must prove your assertions.

(a) $E_1 = \mathbb{Z}$.

(b) $E_2 = \{1/n : n = 1, 2, 3, \ldots\}$.

(c) $E_3 = \{r \in \mathbb{Q} : r^2 \leq 2\}$.

Proof. Note that $\mathbb{R} \smallsetminus \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} B(n + 1/2, 1/2)$ is a union of open sets and is therefore open, so $\mathbb{Z}$ is closed in $\mathbb{R}$ and is therefore equal to its own closure.

The closure of E_2 is $E_2 \cup \{0\}$. There are many ways to see this. One way is to recall that we showed directly from the definition that $E_2 \cup \{0\}$ is a compact subset of $\mathbb{R}$, and therefore must be closed in $\mathbb{R}$. Moreover, any closed set containing E_2 must also contain 0, since 0 is a limit point of E_2 as we showed in class. Thus $E_2 \cup \{0\}$ is the smallest closed set containing E_2, and is therefore equal to the closure of E_2.

The closure of E_3 is the interval $[-\sqrt{2}, \sqrt{2}]$. This is because $\mathbb{Q}$ is dense in $\mathbb{R}$, which implies that $\mathbb{Q} \cap [-\sqrt{2}, \sqrt{2}] = E_3$ is dense in $[-\sqrt{2}, \sqrt{2}]$. In other words, the closure of E_3 in $[-\sqrt{2}, \sqrt{2}]$ is $[-\sqrt{2}, \sqrt{2}]$, but $[-\sqrt{2}, \sqrt{2}]$ is closed in $\mathbb{R}$ already, so the closure of E_3 in $\mathbb{R}$ is also $[-\sqrt{2}, \sqrt{2}]$. $\square$

Problem 3. (1 point) Let E be a subset of a metric space X.

(a) Must E and its closure $\bar{E}$ have the same interiors? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

(b) Must E and its interior E° have the same closures? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

Proof. (a) No. We proved in class that, inside $X := \mathbb{R}$, $E := \mathbb{Q}$ has empty interior, whose interior is again empty, and that $\bar{E} = \mathbb{R}$, whose interior is $\mathbb{R}$.

(b) No. Again, we proved in class that inside $X := \mathbb{R}$, $E := \mathbb{Q}$ has empty interior, and the closure of the empty set is the empty set again, but $\bar{E} = \mathbb{R}$ and the interior of $\mathbb{R}$ is $\mathbb{R}$.

$\square$

Problem 4. (1 point) Let E be a connected subset of a metric space.

(a) Must its closure $\bar{E}$ be connected? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

(b) Must its interior E° be connected? If so, prove it. If not, provide a counterexample and prove that it is a counterexample.

Proof. (a) Yes. Suppose that E is connected but that $\bar{E}$ is disconnected and let U be a nonempty proper open and closed subset of $\bar{E}$, and consider $V := U \cap E$. Then U is open in E by a result we proved in class. Similarly,

$$(\bar{E} \setminus U) \cap E = E \setminus U = E \setminus V$$

is also open in E, so V is open and closed in E.

Since E is connected, this means that either $V = \emptyset$ or $V = E$. Suppose first that $V = \emptyset$. This means that $U \subseteq \bar{E} \setminus E$. But every point of $\bar{E} \setminus E$ is a limit point of E and therefore has no open neighborhood contained entirely in $\bar{E} \setminus E$. In other words, $\bar{E} \setminus E$ has empty interior, so no nonempty open subset of $\bar{E}$ can be contained in $\bar{E} \setminus E$. This is a contradiction, so instead let's consider the case when $V = E$. But then notice that U is a closed subset of $\bar{E}$ containing E, but E is clearly dense in $\bar{E}$, so actually we must have $U = \bar{E}$, which is again a contradiction.

(b) No. Consider $E := \{(x, y) \in \mathbb{R}^2 : x \neq 0\} \cup \{(0, 0)\}$. In other words, we take the set of points off of the y-axis, plus the origin. This set is connected, but we will not prove this at this time. Then $(0, 0)$ is not an interior point of E, since any open ball around E contains points of the form $(0,\varepsilon)$ for ε sufficiently small, which are not points of E. On the other hand, it is easy to see that every other point of E is an interior point: in other words,

$$E^\circ = E \smallsetminus \{(0,0)\}$$

is just the set of points in $\mathbb{R}^2$ off of the y-axis. Then $U := \{(x,y) \in \mathbb{R}^2 : x \gneq 0\}$ is an example of a nonempty proper open and closed subset of E. □

Problem 5. (1 point) Does there exist a compact subset of $\mathbb{R}$ whose set of limit points is countably infinite? If so, provide an example and prove that it is an example. If not, prove that no such set exists.

Proof. Yes. Let $(x_m)_{m\in\mathbb{N}}$ be any strictly monotonically decreasing sequence converging to 0: for example, $x_m = 1/(m+1)$, and then, for each m, let $(y_{m,n})_{n\in\mathbb{N}}$ be any strictly monotonically increasing sequence converging to x_m that is entirely contained in the interval $[0, x_m]$: for example, if $x_m = 1/(m+1)$, then we can take $y_{m,n} = 1/(m+1) - 1/(m+1+n)$.

Then consider the set E which contains 0, all of the points x_m, and all of the points $y_{m,n}$. This is obviously a countably infinite set. To see that it is compact, suppose $\mathfrak{U}$ is a collection of open subsets of $\mathbb{R}$ whose union contains E. Then there exists some $U_0 \in \mathfrak{U}$ containing 0. Then there exists an open ball $B(0,r) \subseteq U_0$ and some M such that $x_m \in B(0,r) \subseteq U_0$ for all $m \gneq M$. Since $B(0,r)$ is connected, the entire interval $[0, x_m]$ is contained in $B(0,r)$, which means in particular that $y_{m,n} \in B(0,r) \subseteq U_0$ for all $m \gneq M$ and $n \in \mathbb{N}$.

Now for each $m = 0, \ldots, M$, let $U_m \in \mathfrak{U}$ be some open set containing x_m. Since U_m is open, there is some integer N_m such that, for all $n \gneq N_m$, we have $y_{m,n} \in U_m$. So then, for each $n = 0, \ldots, N_m$, let $U_{m,n} \in \mathfrak{U}$ be some open set in $\mathfrak{U}$ containing $y_{m,n}$. Now the finite set

$$\mathfrak{U}' = \{U_0, U_1, \ldots, U_m\} \cup \{U_{m,n} : m = 0, \ldots, M, n = 0, \ldots, N_m\}$$

is a subset of $\mathfrak{U}$ whose union still covers E, so E is compact.

Now note that, since E is compact, it contains all of its limit points. In particular, since E is itself countable, the set of its limit points must also be countable. But clearly each x_m is a limit point of E by construction, so in fact the set of limit points of E is also infinite. □

Problem 6. (3 points)

(a) A metric space X is *separable* if it contains a countable dense subset. Give an example of a metric space which is *not* separable.

(b) A *base* for a metric space X is a collection $\mathcal{U}$ of open subsets of X such that, for every open set $G \subseteq X$, there exists a collection of open sets $U_\alpha \in \mathcal{U}$ such that

$$G = \bigcup_\alpha U_\alpha.$$

Give an example for a base for the metric space $\mathbb{R}^2$.

(c) Show that a metric space X is separable if and only if it has a countable base.

Proof. (a) Let X be any uncountable set with the discrete metric. Then every subset of X is closed and is therefore equal to its own closure. Thus, the only dense subset of X is X itself. Since X is uncountable, we see in particular that it has no countable dense subsets.

(b) The set $\{B(a,r) : a \in X, r \gneq 0\}$ is a base for $X := \mathbb{R}^2$, and in fact for any metric space X. Indeed, given any open set U, we know that for every $a \in U$ there exists some positive r_a such that $B(a, r_a) \subseteq U$, and then it is easy to see thatc

$$U = \bigcup_{a \in U} B(a, r_a).$$

(c) Suppose that X has a countable base $\mathcal{B}$. For each nonempty $B \in \mathcal{B}$, pick a point $x_B \in B$ and then consider the set $E := \{x_B : B \in \mathcal{B}\}$. This is countable because B is countable, and we claim that it is also dense. Indeed, suppose U is any nonempty open subset of X. Then U must contain B for some $B \in \mathcal{B}$ since $\mathcal{B}$ is a base, which means it also contains x_B, so $U \cap E$ is nonempty. Thus, by a lemma we proved in class, E is dense in X.

Conversely, suppose X is separable. Let E be a countable dense subset and consider the set $\mathcal{B} := \{B(a,r) : a \in E, r \in \mathbb{Q} \text{ and } r \gneq 0\}$. Note that we have a surjective function $E \times \{r \in \mathbb{Q} : r \gneq 0\} \to \mathcal{B}$ given by $(a,r) \mapsto B(a,r)$ and the domain is countable, so $\mathcal{B}$ is countable as well. Moreover, it is a base for X. Indeed, given any open subset U of X, we claim that

$$U = \bigcup_{\substack{a \in U \cap E, \\ r \in \mathbb{Q}, r \gneq 0, \\ B(a,r) \subseteq U}} B(a,r).$$

Clearly U contains the right-hand side. Conversely, suppose that $b \in U$. Then there exists some open ball $B(b,r) \subseteq U$. Using density of the rationals, we can shrink r and assume that $r \in \mathbb{Q}$. Now consider the smaller open ball $B(b, r/2)$. Since it is a nonempty open subset of X, and E is dense in X, by a lemma we proved in class we know that there exists some $a \in B(b, r/2) \cap E$. Moreover, note that $B(a, r/2) \subseteq B(b,r)$. Indeed, for any $x \in B(a, r/2)$, we have

$$d(b,x) = d(b,a) + d(a,x) \lneq r/2 + r/2 = r.$$

Thus $B(a, r/2) \subseteq U$ also, and we have $b \in B(a, r/2)$ because we chose a so that $a \in B(b, r/2)$, so we have just shown that the point b is also an element of the union on the right-hand side. This completes the proof.

$\square$

Problem 7. (5 points) Let X be a metric space in which every infinite subset has a limit point.

(a) For any $\varepsilon \gneq 0$, show that X can be covered by finitely many open balls of radius ε.

(b) Show that X is separable.

(c) Show that X is compact.

Problem 8. (5 points) Let (X, d) be a metric space and let $\mathcal{C}$ be the set of all nonempty, bounded and closed subsets of X. Define $f : X \times \mathcal{C} \to \mathbb{R}$ by

$$f(x, B) := \inf_{b \in B} d(x, b)$$

and define $g : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ by

$$g(A, B) := \sup_{a \in A} f(a, B).$$

(a) Show that g need not be a metric on $\mathcal{C}$.

(b) Let $h : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be defined by

$$h(A, B) := \max\{g(A, B), g(B, A)\}.$$

Show that h is a metric on $\mathcal{C}$.

Proof. Let us first check that $g(A, B) \neq \infty$ so that g is actually a function into $\mathbb{R}$ as we have claimed. Suppose that $g(A, B) = \infty$. Then for any real number R, there exists some $a_R \in A$ such that $f(a_R, B) \gneq R$. In other words, we have $d(a_R, b) \gneq R$ for all $b \in B$. Fix a point $b_0 \in B$. Then let $S = d(a_R, b_0)$. Then for any $S' \geq S$, observe that $d(a_{S'}, b_0) \gneq S' \geq S = d(a_R, b_0)$. Thus, using the reverse triangle inequality,

$$S' - S \leq d(a_{S'}, b_0) - S = |d(a_{S'}, b_0) - d(a_R, b_0)| \leq d(a_{S'}, a_R).$$

Thus

$$\infty \geq \operatorname{diam}(A) \geq \sup\{d(a_{S'}, a_R) : S' \geq S\} \geq \sup\{S' - S : S' \geq S\} = \infty$$

so $\operatorname{diam}(A) = \infty$. In other words, we cannot have $g(A, B) = \infty$ when A is bounded.

Next, let A, B and C be nonempty bounded subsets of X. Fix a point $a \in A$. Then we know that, for any $b \in B$ and $c \in C$, we have

$$f(a, C) \leq d(a, c) \leq d(a, b) + d(b, c).$$

In other words, for fixed $b \in B$, the quantity $f(a, C) - d(a, b)$ is a lower bound for the set $\{d(b, c) : c \in C\}$. This gives us the first inequality in the following.

$$f(a, C) \leq d(a, b) + f(b, C) \leq d(a, b) + g(B, C) \leq d(a, b) + h(B, C).$$

In other words, $f(a, C) - h(B, C)$ is a lower bound for the set $\{d(a, b) : b \in B\}$, which means that

$$f(a, C) \leq f(a, B) + h(B, C) \leq g(A, B) + h(B, C) \leq h(A, B) + h(B, C).$$

Thus $h(A, B) + h(B, C)$ is an upper bound for $\{f(a, C) : a \in A\}$, so we conclude that

$$g(A, C) \leq h(A, B) + h(B, C).$$

Now interchanging the roles of A and C and applying the fact that h is clearly symmetric shows that $g(C, A)$ is also bounded above by the same quantity, so

$$h(A, C) \leq h(A, B) + h(B, C).$$

This, together with some other easy observations, proves that h defines a pseudometric on the set of nonempty bounded subsets of X.

Now if $h(A, B) = 0$, then $g(A, B) = g(B, A) = 0$. The fact that $g(A, B) = 0$ tells us that $f(a, B) = 0$ for all $a \in A$, which means that for every $\varepsilon \gneq 0$, there exists $b \in B$ such that $d(a, b) \lneq \varepsilon$. In other words, $a \in \bar{B}$, so $A \subseteq \bar{B}$. Similarly, $g(B, A) = 0$ tells us that $B \subseteq \bar{A}$. In other words, the equivalence relation "is distance 0 away from" induced by the pseudometric h identifies bounded subsets of X which have the same closure. In particular, h restricts to a metric on the set $\mathcal{C}$ of nonempty, closed and bounded subsets of X. $\square$