Worksheet 5: Biconditionals, Existence and Uniqueness, Bézout's Theorem

Problem 1. Let a be an integer. Show that $a^2 + 4a + 5$ is odd if and only if a is even.

Problem 2. Suppose a and b are real numbers with $a \neq 0$. Show that there exists a unique real number x such that $ax + b = 0$.

Problem 3. Suppose a and b are nonzero integers. Show that any common divisor of a and b divides $\gcd(a, b)$.

Problem 4. Show that $\gcd(n, n+2) \in \{1, 2\}$ for an integer n.

Problem 5. Suppose a, b, c are integers and $\gcd(a, c) = \gcd(b, c) = 1$. Prove that $\gcd(ab, c) = 1$.

Problem 6. Suppose a, b, c are integers and $a \mid bc$. Then $a \mid \gcd(a, b) \gcd(a, c)$.

Problem 7. Excised.

Problem 8. Let a and b be integers with $b > 0$. Prove that there exist unique integers q and r such that $a = bq + r$ where $2b \leqslant r < 3b$. *Note.* You may use the existence and uniqueness statement of the division algorithm.

Problem 9. Let p be an odd prime. Show that, for any integer a, we have

$$\gcd\left(a+1, \frac{a^p+1}{a+1}\right) = 1 \text{ or } p.$$

Possible hint. Since p is odd, we have $a^p + 1 = (a+1)(a^{p-1} - a^{p-2} + a^{p-3} - \cdots - a + 1)$. This shows that $(a^p+1)/(a+1)$ is an integer, and you can also use this to calculate what $(a^p+1)/(a+1)$ is congruent to mod $a+1$.

Problem 10. *Prelude*. The point of this problem is to prove *one direction* of a fact that you may be familiar with from grade school, namely, that a number is rational if and only if its decimal expansion repeats after some point. We'll prove the other direction later. *Problem Statement*. If $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0, a_{-1}, \ldots$ are all in $\{0, 1, \cdots, 9\}$, we use the string

$$a_n \cdots a_1 a_0 . a_{-1} a_{-2} \ldots$$

to represent the number

$$A = 10^n a_n + 10^{n-1} a_{n-1} \cdots 10 a_1 + a_0 + 10^{-1} a_{-1} + 10^{-2} a_{-2} + \cdots .$$

Suppose there exists an integer $m \leqslant 0$ and an integer $k > 0$ such that $a_i = a_{i-k}$ for all $i > m$. Prove that A is rational. *Suggestion*. Start by proving that the number

$$12.7515151515151515151515151 \cdots$$

is rational. Then generalize your argument.