2.1.(b) n is triangular iff $8n+1$ is a perfect square

2 things to show

n is triangular $\Longrightarrow$ $8n+1$ is a perfect square.

$\longrightarrow n = \frac{a(a+1)}{2}$ for some a.

$$8n+1 = 8\cdot\frac{a(a+1)}{2} + 1 = 4a(a+1)+1$$
$$= 4a^2+4a+1$$
$$= (2a+1)^2$$

$8n+1$ is a perfect square $\Longrightarrow$ n is triangular.

$\underline{8n+1} = b^2$ for some b.

Since $8n+1$ is odd, b must be odd also

(If it was even, b^2 would also be even!)

so b is of the form $2a+1$ for some a.

$$8n+1 = (2a+1)^2$$
$$= 4a^2+4a+1$$
$$= 8\cdot\frac{a(a+1)}{2}+1$$

$\Longrightarrow n = \frac{a(a+1)}{2}$

So n is triangular.

4^2
6^2
8^2

$8n+1$

$8n+1=16$
$8n=15$
$n=15/8$

$8n+1=36$
$8n=35/8$

two things to prove!

2.2.1 Prove that, if a, b with $b>0$, then there exist unique integers q and r such that $a = qb+r$ where $2b \le r < 3b$.

Looks similar to the statement of division algorithm, which says that: there exist unique integers $\tilde{q}$ and $\tilde{r}$ such that $a = \tilde{q}b + \tilde{r}$ and $0 \le \tilde{r} < b$.

$\tilde{q}, \tilde{r}$ are the actual quotient & remainder. q & r are not quite ... "modified" quotient & remainder.

$a = 15$ $\quad$ $b = 4$

$$15 = \underbrace{3}_{\tilde{q}}\cdot 4 + \underbrace{3}_{\tilde{r}}$$

$$\left[\begin{array}{l} 3\cdot 15 = 3\cdot 3\cdot 4 + 3\cdot 3 \\ 45 = 9\cdot 4 + 9 \end{array}\right]$$

$0 \le \tilde{r} < 4$

$2\cdot 4 \le r < 3\cdot 4 \leadsto 8 \le r < 12$

$$23 = 4\cdot 5 + 3$$
$$= 2\cdot 5 + 13$$
$$= (4-2)\cdot 5 + 2\cdot 5 + 3$$

$0 \le 3 < 5$

$10 \le ? < 15$

$$15 = (3-2)\cdot 4 + 2\cdot 4 + 3$$
$$= 1\cdot 4 + 11$$

Proof of existence. By division algorithm, there exist $\tilde{q}$, $\tilde{r}$ such that

$$a = \tilde{q}b + \tilde{r} \qquad 0 \le \tilde{r} < b$$

Notice that

$$a = \tilde{q}b + \tilde{r} = \underbrace{(\tilde{q} - 2)}_{q} b + \underbrace{(2)b + \tilde{r}}_{r}$$

$$= qb + r$$

and $0 \le \tilde{r} < b \implies 2b \le \tilde{r} + 2b < b + 2b$

$$2b \le r < 3b.$$

Proof of uniqueness. Suppose we have q, r and q', r' such that

$$a = bq + r \qquad a = bq' + r'$$
$$2b \le r < 3b \qquad 2b \le r' < 3b.$$

We would like to show that $q = q'$ and $r = r'$.

$$a = bq + r = bq' + r'$$
$$bq - bq' = r' - r$$
$$b(q - q') = r' - r \quad (*)$$

$r' - r$ is a multiple of b. On the other hand, since $2b \le r, r' < 3b$, $|r' - r| < b$. But the only multiple of b whose absolute value is $< b$ is 0. So $r' - r = 0 \implies r = r'$.

By $(*)$, $r' - r = 0$ so $b(q - q') = 0$. But $b > 0$, so $q - q' = 0 \implies q = q'$.

$\underline{13}\ \underline{13}\ \underline{13} \ldots$

$\}\ 13^{13}$

$\}\ 13^{12}$

$\vdots$

13