Worksheet W1Wed: Continuity and Differentiation

Problem 1. Suppose $G \subseteq \mathbb{C}$ is a region and $z_0 \in G$ and $f : G \to \mathbb{C}$ is a function. The point of this problem is to show that

$$\lim_{z \to z_0} f(z)$$

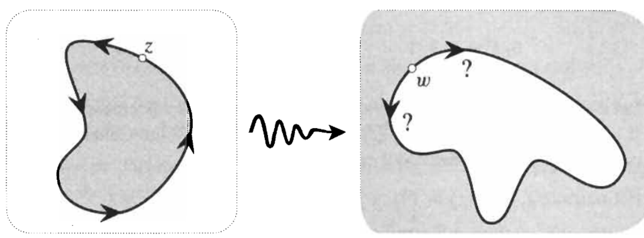
is unique if it exists. More precisely, suppose $a, b \in \mathbb{C}$ are numbers such that f approaches a as z approaches z_0 and f also approaches b as z approaches z_0. Show that $a = b$. *Hint*. Carefully apply the ϵ-δ definition of limits to show that $|a - b| < \epsilon$ for all $\epsilon > 0$, and then use this to conclude that $a = b$.

Problem 2. Let $f : \mathbb{C} \to \mathbb{R}$ be the function $f(x + iy) = \dfrac{x^2 y}{x^4 + y^2}$.

(a) Show that the limits of f at 0 along *all* straight lines through the origin exist and equal.

(b) Show that $\lim_{z \to 0} f(z)$ does not exist. *Hint*. Consider the limit along the parabola $y = x^2$.

Problem 3. The picture below depicts a holomorphic function with nonzero derivative mapping a path to another path, and the interior of the original path to the exterior of the image path. If z travels around the original path counterclockwise, which way does its image w travel around the image path?



Problem 4. (a) Show that $z \mapsto \mathrm{Re}(z)$ is continuous on $\mathbb{C}$.

(b) On what subset of $\mathbb{C}$ is $z \mapsto \mathrm{Re}(z)$ holomorphic? Justify.

(c) Suppose $G \subseteq \mathbb{C}$ is a region and $u, v : G \to \mathbb{R}$ are functions. Show that

$$\lim_{z \to z_0} (u(z) + iv(z)) = u_0 + iv_0$$

if and only if

$$\lim_{z \to z_0} u(z) = u_0 \text{ and } \lim_{z \to z_0} v(z) = v_0.$$

Problem 5. (a) Show that $f(z) = \bar{z}$ is continuous on $\mathbb{C}$.

(b) Show that $g(z) = \bar{z}/z$ is continuous on $\mathbb{C} \setminus \{0\}$.

(c) Is it possible to define $g(0)$ so that g becomes continuous on $\mathbb{C}$? Justify.

Problem 6. Let $f(z) = z^2$. Decsribe what f does to the following subsets of $\mathbb{C}$.

(a) A circle centered at the origin.

(b) A ray starting at the origin.

(c) The figure formed by the horizontal segment from 0 to 2, the circual arc from 2 to $2i$, and then the vertical segment from $2i$ to 0. *Follow-up*. What happens to the right angle at the origin? Does this contradict proposition 2.11 of BMPS?

(d) The square between $0, 2, 2 + 2i$, and $2i$. *Note*. Be careful with the vertical segments that are not connected to the origin. Their images are neither straight lines nor circles.