Name:

QUIZ 1 SOLUTIONS

Instructions. The only tools you are permitted to use are pencils, pens, erasers, a sheet of handwritten notes, and your mind. No electronic devices! You have 1 hour. Good luck!

Problem 1 (True-false, 9 points)**.** You will get 1 point for a correct answer and 0 points for blank and incorrect answers.[1] No explanations are required.

(1) If

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

then $\lim_{x\to 0} f(x) = f(0)$. T **F**

(2) $\lim_{x\to 2^-} \frac{1}{x-2} = \infty$. T **F**

(3) $\lim_{x\to 0} \frac{2^{2x}-1}{2^x-1} = 1$. T **F**

(4) There exist functions f and g such that $\lim_{x\to 0} (f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exist. **T** F

(5) The function $f(x) = \sin(x^2+2x)$ is continuous at all real numbers. **T** F

(6) The function

$$f(x) = \begin{cases} x^2-1 & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0 \end{cases}$$

is right continuous at 0. **T** F

(7) $\lim_{x\to 0^+} \sqrt{x}\cos(1/x) = 0$. **T** F

(8) The equation $\cos x = x$ has a solution. **T** F

(9) The equation $e^x = -x^2$ has a solution. T **F**

[1]During quiz revisions, you'll have the chance to meet with me one-on-one and convince me that you understand and can fully explain *up to 2 of the true-false questions that you left blank* to get full credit on those questions.

Problem 2 (3 points)**.** Calculate each of the following limits, or state that the limit does not exist. If it does not exist, be sure to specify whether or not it equals $\pm\infty$. Show your work.

(a) $\lim_{x \to 2} \dfrac{x^2 + 2x + 1}{x + 1}$

Solution. The function is continuous at $x = 2$, so we can just plug in $x = 2$. We find that

$$\lim_{x \to 2} \frac{(x+1)^2}{x+1} = \lim_{x \to 2} (x + 1) = 3.$$

(b) $\lim_{x \to \infty} \dfrac{x^3 - 7x}{2x^3 - 5}$.

Solution. Note that

$$\lim_{x \to \infty} \frac{x^3 - 7x}{2x^3 - 5} = \lim_{x \to \infty} \frac{1 - \frac{7}{x^2}}{2 - \frac{5}{x^3}} = \frac{1}{2}.$$

(c) $\lim_{x \to -\infty} \dfrac{x^3 - 7x}{x^2 + 2}$.

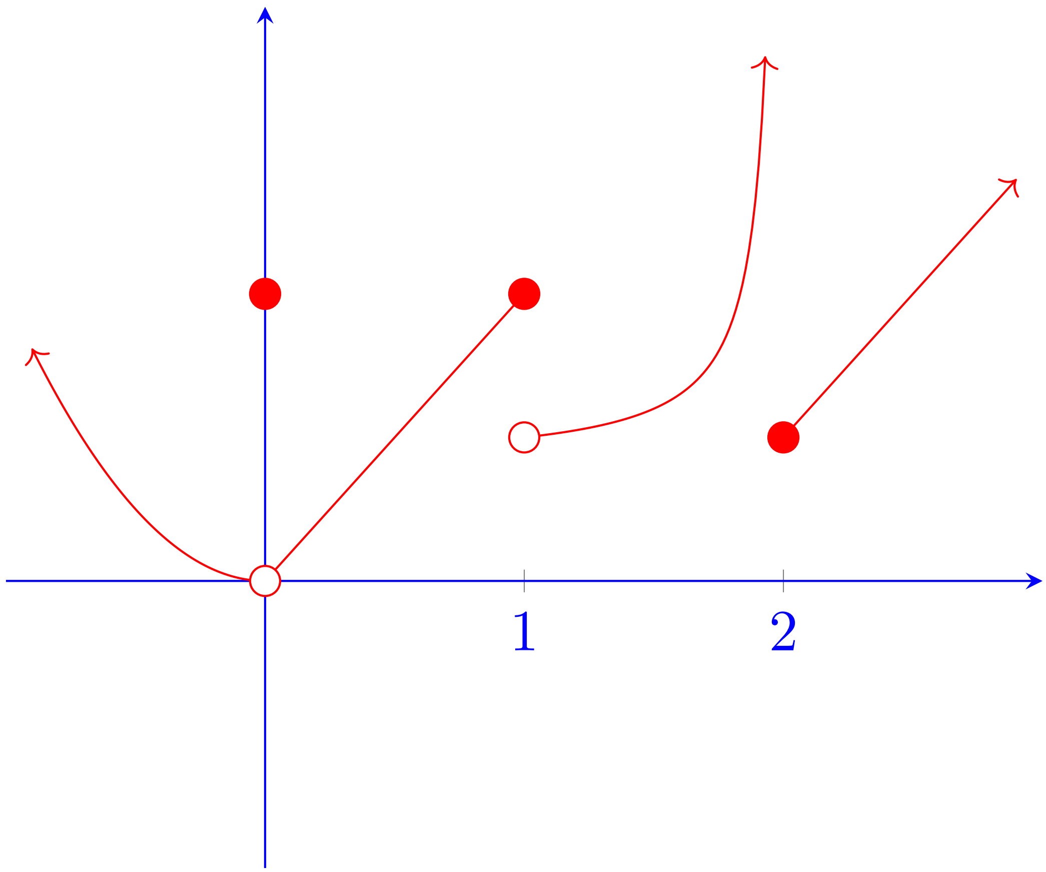
Solution. We have

$$\lim_{x \to -\infty} \frac{x^3 - 7x}{x^2 + 2} = \lim_{x \to -\infty} \frac{1 - \frac{7}{x^2}}{\frac{1}{x} + \frac{2}{x^3}} = -\infty$$

because, when x is a large negative number, the denominator of the fraction (ie, $\frac{2}{x} + \frac{2}{x^3}$) is a small negative number. This is a limit that does not exist.

Problem 3 (3 points)**.** Sketch the graph of a function f that is defined for all real numbers, that has a removable discontinuity at $x = 0$, a jump discontinuity at $x = 1$, an infinite discontinuity at $x = 2$, and that is continuous at all other real numbers.

Solution. Something like this would work.



Note in particular that, since the domain of f is supposed to be all real numbers, it must have some finite value at $x = 2$.

Problem 4 (3 points)**.** Find all values of c such that the function f defined by the formula

$$f(x) = \frac{x^2 + 3x - 4}{x - c}$$

has a removable discontinuity at $x = c$. For each such value of c, what is $\lim_{x \to c} f(x)$?

Solution. The numerator factors as $(x + 4)(x - 1)$, so if f is to have a removable discontinuity, it must be that $c = -4$ or $c = 1$. For all other values of c, the function has an infinite discontinuity at $x = c$.

For $c = -4$, we have

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} x - 1 = -5$$

and for $c = 1$, we have

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x + 4 = 5.$$

Problem 5 (Extra credit! 2 points, no partial credit)**.** You are a mountain climber. At midnight on Monday, you are asleep at base camp. At some point during the day on Monday, you wake up and climb the mountain. At midnight on Tuesday, you are asleep at the top of the mountain. At some point during the day on Tuesday, you wake up and descend, and you are back at base camp before midnight. Must there be some time t_0 such that you are at the same elevation on both Monday at time t_0 and Tuesday at time t_0?[2] Explain your answer.

Solution. Yes, there must be such a time!

Let t be the time elapsed in hours since midnight. Let $f(t)$ be your elevation from base camp on Monday. We don't know anything about f, except that $f(0) = 0$ and $f(24) = L$ where L is the elevation change between base camp and the top of the mountain. Also, we know that f has to be continuous (you can't teleport and suddenly change elevation). Similarly, let $g(t)$ be your elevation from base camp on Tuesday, so that $g(0) = L$ and $g(24) = 0$. The function g is also continuous.

Let $h(t) = f(t) - g(t)$. Since f and g are continuous, so is h. Furthermore,

$$h(0) = f(0) - g(0) = 0 - L = -L < 0$$

and

$$h(24) = f(24) - g(24) = L - 0 = L > 0,$$

so by the intermediate value theorem, there exists some t_0 such that $h(t_0) = 0$. In other words, $f(t_0) = g(t_0)$, which means precisely that you have the same elevation at time t_0 on both Monday and Tuesday.

Honor code. If you have neither given nor received any unauthorized aid on this quiz, please write either "HCU" or "Honor Code Upheld" below, and sign your name next to it.

[2]For example, if you were at an elevation of 5000 ft above base camp on Monday at 2:37pm, and you were also at an elevation of 5000 ft above base camp again on Tuesday at 2:37pm, then $t_0 =$ 2:37pm would be such a time.

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