Problem Set 1 Solutions

Problem 1. (1 point) Prove that there exists no rational number whose square is 75.

Proof. Suppose there did exist such a rational number r. Then $r^2 = 75 = 25 \cdot 3$, so $(r/5)^2 = 3$. Notice that $s := r/5$ is rational as well, so we can write $s = a/b$ for some integers $a, b \in \mathbb{Z}$ with b nonzero and a and b having no common factors. Then $a^2 = 3b^2$. Then a^2 is a multiple of 3, so a is also a multiple of 3 since 3 is prime, so $a = 3c$. Then $3b^2 = a^2 = 9c^2$ so $b^2 = 3c^2$, so b^2 is a multiple of 3, so b is also a multiple of 3. This contradicts the assumption that a and b have no common factors. $\square$

Problem 2. (1 point) Let E be a nonempty subset of an ordered set X. Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Since E is nonempty, it contains some $x \in E$. Since α is a lower bound, we have $\alpha \leq x$, and since β is an upper bound, we have $x \leq \beta$, so by transitivity of orders, we have $\alpha \leq \beta$. $\square$

Problem 3. (1 point) Let X be an ordered set with the supremum property, and suppose S and T are nonempty bounded subsets of X such that T is bounded and $S \subseteq T$. Show that S is bounded, and that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

Proof. Since T is bounded, it has an upper bound β and a lower bound α. Then β is also an upper bound for S and α is also a lower bound for S, so S is also bounded.

To see that $\inf T \leq \inf S$, notice that $\inf T$ is a lower bound for T, so it is also a lower bound for S, but $\inf S$ is the greatest lower bound, so $\inf T \leq \inf S$. The proof that $\sup S \leq \sup T$ is analogous. The fact $\inf S \leq \sup S$ follows from problem 2. $\square$

Problem 4. (1 point) Are there any nonempty subsets S of $\mathbb{R}$ such that $\inf S = \sup S$? If so, describe all of them and prove that you have described all of them. If not, prove that there are none.

Proof. Yes, there are, and every such subset is a singleton set of the form $S = \{s\}$. Indeed, suppose that S is a nonempty subset such that $\inf S = \sup S$, and fix an arbitrary element $s \in S$. Then, $s \leq \sup S = \inf S$, but we also have $\inf S \leq s$, so by antisymmetry we have $s = \inf S$. Thus, given any $s, s' \in S$, we have $s = \inf S = s'$, so S has just one element. $\square$

Problem 5. (1 point) Let A be a nonempty subset of $\mathbb{R}$ and let

$$-A := \{-x : x \in A\}.$$

Prove that $\inf(A) = -\sup(-A)$. *Hint.* There are no boundedness assumptions on A in this statement. So, first consider the case when A is not bounded below, in which case $\inf(A) = -\infty$, and then consider the case when A is bounded below.

Proof. Suppose A is not bounded below. Then $-A$ is not bounded above: indeed, for any real number R, there exists some $a \in A$ such that $a \leq -R$, which means that $-a$ is an element of A such that $-a \geq R$. Thus $\sup(-A) = \infty$, so $\inf(A) = -\infty = -\sup(-A)$, as desired.

Next, suppose A is bounded below. If R is a lower bound for A, then $-R$ is an upper bound for $-A$, so $-A$ is bounded above and $\sup(-A)$ exists. For any $a \in A$, we know that $-a \leq \sup(-A)$, so $a \geq -\sup(-A)$, so $-\sup(-A)$ is a lower bound for A. Thus $-\sup(-A) \leq \inf(A)$.

Conversely, to see that $-\sup(-A) \geq \inf(A)$, it is equivalent to show that $\sup(-A) \leq -\inf(A)$, so it is sufficient to show that $-\inf(A)$ is an upper bound for $-A$. Given any $a \in -A$, we know that $a = -b$ for some $b \in A$, and we have $b \geq \inf(A)$, so $a = -b \leq -\inf(A)$. Thus $-\inf(A)$ is an upper bound for $-A$, as desired. $\square$