Worksheet W1Tue: Complex Numbers

Instructions. Work together with your tablemates to work on the following problems. Start with the problems that are congruent to your table number mod 3. (For example, table 2 should do problems 2, 5, and 8.) If/when you finish these, keep going with any of the other problems that seem interesting to all of you. At some point, I'll ask you to put the problems congruent to your table number mod 3 (or rather, as many of them as you were able to figure out) on the board, and then we'll discuss all of them together.

Problem 1. Find the real and imaginary parts of each of the following complex numbers.

(a) $(1+2i)(2+3i)$

(b) $\overline{(1+2i)}+i$

(c) $(2+i)^{-1}$

(d) $(2+i)^3$

(e) $\overline{(2-i)^3}$

(f) $2e^{4\pi i}$

(g) $\sqrt{2}e^{i\pi/4}$

(h) $2e^{i\pi/3}$

Problem 2. Express each of the following complex numbers in polar form.

(a) $2+i$

(b) $2-i$

(c) $4+5i$

(d) $\overline{2e^{i\pi/3}}$

Problem 3. For each of the following sets, do the following: (i) Draw a picture. (ii) Identify the interior points. (iii) Identify the boundary points. (iv) Determine if it is open. (v) Determine if it is closed. (vi) Determine if it is connected. (vii) Determine if it is bounded.

(a) $\{z \in \mathbb{C} : |z+i| \leqslant 3\}$

(b) $\{z \in \mathbb{C} : \mathrm{Re}(z+2-2i) < 3\}$

(c) $\{z \in \mathbb{C} : \mathrm{Im}(z^2) = 1\}$

(d) $\{z \in \mathbb{C} : |z| = |z-1|\}$

Problem 4. Find all solutions to each of the following equations. Then draw a picture of the solution set.

(a) $z^2 + 2z + (1-i) = 0$

(b) $z^6 = -64$.

(c) $1 - z^2 + z^4 - z^6 = 0$.

Problem 5. Parametrize each of the following paths.

(a) The line segment from $-1-i$ to $2i$

(b) The top half of the circle $C[0,3]$, oriented clockwise

(c) The rectangle with vertices $\pm 1 \pm i$, oriented counter-clockwise

Problem 6. Determine whether each of the following statements is true or false. Prove your assertion formally.

(a) There is a subset of $\mathbb{C}$ which is neither open nor closed.

(b) There is a subset of $\mathbb{C}$ which is both open and closed.

(c) If B is a closed subset $\mathbb{C}$ and $A \subseteq B$, then the boundary of A is contained in B.

(d) If A is an open subset of $\mathbb{C}$ and $A \subseteq B$, then A is contained in the interior of B.

Problem 7. Suppose $\theta \in \mathbb{R}$.

(a) Show that $1 - e^{i\theta} = -2ie^{i\theta/2}\sin(\theta/2)$.

(b) If θ is not an integer multiple of 2π, show that $\displaystyle\sum_{k=0}^{n} e^{ik\theta} = e^{in\theta/2} \cdot \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}$.

Problem 8. For any positive integer n, show that $\displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} = 2^{n/2}\cos\left(\frac{n\pi}{4}\right)$. *Hint*. Consider $(1+i)^n$.