What exactly is multideg(f)?

multideg(f) is the tuple of natural numbers that records all of the exponents that appear leading monomial. For example, suppose we're considering the polynomial

$$f = xy + y^2z \in k[x, y, z].$$

If we're using lex order, the leading monomial is $xy = x^1y^1z^0$ and multideg$(f) = (1, 1, 0)$. If we're using grevlex order, the leading monomial is $y^2z = x^0y^2z$, so multideg$(f) = (0, 2, 1)$.

I cannot figure out what the method is for calculating x^γ [in definition 4]...

To compute the lcm of two monomials, we write down the monomial in which each variable shows up to the bigger of the two powers in the monomials we started with. For example, let's say we want to calculate lcm(xy, y^2z). Since x shows up to the power 1 in the first monomial and 0 in the second monomial, it'll show up to the power $\max\{1, 0\} = 1$ in the lcm. Since y shows up to the power 1 in the first monomial and 2 in the second monomial, it'll show up to the power $\max\{1, 2\} = 2$ in the lcm. And finally, since z shows up to the power 0 in the first monomial and 1 in the second monomial, it'll show up to the power $\max\{0, 1\} = 1$ in the lcm. Thus

$$\text{lcm}(xy, y^2z) = xy^2z.$$

What is the importance of the S-polynomial?

The importance is Buchberger's criterion. In general, it is quite difficult to check that a particular set G is a Gröbner basis directly from the definition. But Buchberger's criterion makes this very easy: we just have to calculate the S-polynomial for all pairs of elements of S, divide those S-polynomials by G, and make sure the remainder is 0.

For example, yesterday I asserted that $G = \{x + y^2, y^3\}$ is a Gröbner basis for the ideal $I = \langle x + y^2, y^3\rangle$ with respect to lex order, but I did not prove this (because it is not very easy to prove this directly from the definition!). With Buchberger's criterion, we can can prove this as follows. We calculate the S-polynomial

$$S(x + y^2, y^3) = y^3(x + y^2) - x(y^3) = y^5$$

and then we divide by G. This division has just one step: we put a y^2 in the quotient corresponding to y^3, and we get a remainder of 0.

$$S(x + y^2, y^3) = y^5 = \underbrace{0}_{q_1} \cdot (x + y^2) + \underbrace{y^2}_{q_2} \cdot (y^3) + \underbrace{0}_{r}$$

Thus G is a Gröbner basis for I for Buchberger's criterion.

In general, if G is more than 2 elements, you'll have to do this with more than 1 S-polynomial. More specifically, if G has t elements, you'll have $\binom{t}{2}$ pairs of elements of G, and each of those pairs has an S-polynomial, and then you'll have to divide each of those by G and find a remainder.

Do we always have $S(f, g) = S(g, f)$?

No: what we actually have is $S(f, g) = -S(g, f)$. For example, if $f = x + y$ and $g = y$ and we're using lex order where $x > y$, we have

$$S(x + y, y) = y(x + y) - x(y) = y^2$$
$$S(y, x + y) = x(y) - y(x + y) = -y^2.$$