SRQR: Basics of Number Theory (March 31, 2021)

> I am still fairly confused about how to apply the division algorithm. Could you do an example for 2.2.3?

Sure, let's do 2.2.3(a). We want to show that the square of any integer is of the form $3k$ or $3k + 1$.

Suppose a is an integer. We'll show that a^2 must be of the form $3k$ or $3k + 1$. By the division algorithm, we know that $a = 3q + r$ where $0 \leqslant r < 3$. In other words, $a = 3q$ or $a = 3q + 1$ or $a = 3q + 2$. So we have three cases to consider:

Case 1. If $a = 3q$, then

$$a^2 = (3q)^2 = 9q^2 = 3(3q^2).$$

Thus a^2 is of the form $3k$, where $k = 3q^2$.

Case 2. If $a = 3q + 1$, then

$$a^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1.$$

Thus a^2 is of the form $3k + 1$, where $k = 3q^2 + 2q$.

Case 3. If $a = 3q + 2$, then

$$a^2 = (3q + 2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1.$$

Thus a^2 is again of the form $3k + 1$, where $k = 3q^2 + 4q + 1$.

> In Example 2.1 on page 18–19, it says, "According to the Division Algorithm, every a is of the form $3q$, $3q + 1$, or $3q + 2$." I'm wondering where the 3 is from?

> For example 2.1 on page 18–19, the form of a is decided to better eliminate the 3 at the denominator, right?

I think the second question answers the first! ☺

What's happening here is that the number a is being divided by 3, and then we're considering the three possible remainders that can result from this division: 0, 1, and 2. Why is a being divided by 3? It's sort of an "inspired guess." We're trying to prove that $a(a^2 + 2)/3$ is an integer, and the thing that's preventing this expression from being an integer is the denominator of 3. So we might be led to think, if we can somehow introduce some 3's in the numerator, maybe we can cancel out that 3 in the denominator. One way you might think to introduce 3's into the numerator is using the division algorithm, and this turns out to work.

> With these proofs relating to triangles, pentagons etc, are we supposed to justify ideas fully numerically or does the geometric nature of the problems sometimes demand other types of proofs? For problems like 2.1.1, I had trouble approaching some of the ideas without using properties of shapes to justify some of my points, but I suspect there are other ways to prove these ideas.

Geometry is math at least as much numbers! That means that geometric reasoning is also mathematical reasoning, and you don't need to shy away from geometry in your arguments.