A Rapid Introduction to Sets

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It'll be important to know a few things about set theory for this class. This note is intended to be a kind of minimal introduction to sets for those who may not have seen them before.

1 Sets

A *set* is a collection of things. Perhaps the most common kind of "thing" that a set might contain is numbers, but it can really be anything (functions, points, rectangles, other sets, ...). The things that are contained in a set are called the *elements* (or *members*) of the set. Usually sets are denoted with capital letters. If S is a set and s is something (eg, a number), we write $s \in S$ to mean that s is an element of S. In other words, you can read the symbol "$\in$" as "is in" or "is an element of." You can also write $s \notin S$ to mean that s is *not* an element of S. The LaTeX commands for producing these symbols are `\in` and `\notin`.

Note. A somewhat subtle point that is useful to remember is that a set only knows whether or not something is inside it, and its elements have no order and no multiplicity. Also, there can be many different ways of describing the same set.

We'll return to this point below once we introduce some ways of describing sets.

1.1 Natural Language Sentences

The most versatile way we have of describing sets is using sentences in English or another natural language. For example, I might say something like "Let S be the set of even integers." This means that I'm defining S is the set whose elements are precisely the even numbers; nothing else besides even numbers is an element of S. A Francophone mathematician might say instead "Soit S l'ensemble des entiers pairs," and they would mean the same exactly the same thing. A Japanese mathematician might say "S を偶数の整数の集合とする," and they too would mean exactly the same thing.

In the following subsections, we'll discuss a number of symbolic ways we have of describing sets symbolically. These have the advantage of being independent of any particular natural language, and occasionally more precise. That being said, if you run into a symbolic description of a set when you're reading math, I strongly encourage you to translate it into a natural language description in your mind. Otherwise, you may sometimes find yourselves lost in the symbols when the basic idea is not very difficult.

Note. There is a "style guideline" for mathematical writing which says that, if you can give a *concise and precise* natural language description of a set (or any other mathematical object), you should do that instead of giving a symbolic description, because concise natural language descriptions tend to be easier for your readers to understand than symbolic descriptions. Stated differently, you should only give a symbolic description of a set if you cannot think of any concise and precise way of describing your set in natural language. For example, the sentence "Let S be the set of even integers" is concise and precise. On the other hand, "Let S be the set of integers whose square leaves a remainder of 1 when divided by 4" is a rather cumbersome description: this is a good candidate for something where a symbolic description would be preferrable. There are obviously some subjective judgment calls to be made here; just do your best to be reasonable!

1.2 Roster Notation

"Roster notation" for a set involves specifying what things are inside the set with a list enclosed in curly braces. For example,

$$\{1, 4, 5\}$$

denotes the set whose elements are the numbers 1, 4, and 5; nothing else is an element of this set. In other words, $4 \in \{1, 4, 5\}$ is a true statement. $6 \in \{1, 4, 5\}$ is a false statement. $6 \notin \{1, 4, 5\}$ is a true statement.

Since the elements of a set have no order and no multiplicity, there are many different ways of denoting the same set using roster notation:

$$\{1, 4, 5\} = \{1, 5, 4\} = \{1, 4, 4, 5\} = \cdots.$$

Sometimes, ellipses ($\ldots$) are also used in roster notation if there's a clear pattern to the elements. For example, one might write

$$\{2, 4, 6, 8, \ldots, 24\}$$

to denote the set of all even numbers between 2 and 24 (inclusive). This is sometimes also done with sets that have infinitely many elements. For example, one might write

$$\{2, 4, 6, \ldots\}$$

to denote the set of all even numbers. But you should be very careful when using ellipses: make sure that the pattern is very clear! If I wrote something like

$$\{2, 4, \ldots\}$$

you might have no idea if I was talking about even numbers or about powers of 2!

1.3 Special Sets

Some sets of numbers are denoted using special symbols that are very widely used.

$$\begin{aligned}
\mathbb{N} &= \text{the set of all natural numbers} &&= \{1, 2, 3, 4, ...\} \\
\mathbb{Z} &= \text{the set of all integers} &&= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathbb{Q} &= \text{the set of all rational numbers} \\
\mathbb{R} &= \text{the set of all real numbers}
\end{aligned}$$

This style of writing capital letters is called "blackboard bold" (because real boldface is hard to do when handwriting and this style was developed as a workaround to writing in boldface on the blackboard). The LaTeX commands for producing these symbols are `\mathbb{N}`, `\mathbb{Z}`, etc, where the `bb` stands for "blackboard bold."

Note. A rational number is a number that can be written as a fraction of the form an integer over another integer. For example, $3/4$ is a rational number. $\sqrt{2}$ is an irrational number, and we will prove this later. It is also true that a number is rational if its decimal representation either ends (eg, 0.75) or starts repeating at some point (eg, $0.8333\ldots$). This requires proof, and we won't really *need* this fact, but I find it useful to remember for the sake of intuition.

Note. There's some disagreement in the literature about whether 0 is a "natural number." Many mathematicians say it is, and they would say that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. But Burton excludes 0, so I propose we exclude 0 for the purposes of our class as well.

1.4 Set-Builder Notation

There's another way of describing sets that's very useful, sometimes called "set-builder notation." In fact, there are actually two different kinds of set-builder notation. Both are best explained through examples.

An example of the first type of set-builder notation is something like this:

$$S = \{n \in \mathbb{Z} : n \text{ is even}\}.$$

This means that S is the set of all integers n which have the property "n is even." In other words, S is just the set of even integers. It doesn't matter what variable name you use here: someone else might describe the same set by

$$\{m \in \mathbb{Z} : m \text{ is even}\}.$$

Also, you can put any condition you want after the colon. For example:

$$T = \{n \in \mathbb{Z} : n \text{ is even and } 0 \leq n \leq 100\}$$

is the set of all even integers between 0 and 100 (inclusive). To summarize, then, the pattern for the first type of set-builder notation is

$$\{\text{membership} : \text{properties}\}.$$

The second type of set-builder is something like this:

$$S = \{n^2 : n \in \mathbb{Z}\}.$$

This means that S is the set of things of the form n^2 where n is an integer. In other words, S is just the set of perfect squares. Again, the variable name you use doesn't matter. The same set could equally well be described by

$$\{m^2 : m \in \mathbb{Z}\}.$$

And you can put any pattern you want before the colon. For example:

$$T = \{2n + 3 : n \in \mathbb{Z}\}$$

is the set of things which can be written in the form $2n + 3$, where n is an integer. (It may require some thought to see why, but this is actually just the set of all odd integers!) To summarize, then, the pattern for the second type of set-builder notation is

$$\{\text{pattern} : \text{membership}\}.$$

Note that, on the face of it, both types of set-builder notation sort of look the same: open curly brace, some stuff, colon, some other stuff, close curly brace. You have to decide based on context which type of set-builder notation you're looking at.

Note. A common variant of of set-builder notation replaces the colon with a vertical bar (|, whose LaTeXcommand is `\mid`). So you might see something like

$$\{n \in \mathbb{Z} \mid n \text{ is even}\}$$

used to describe the set of all even integers. This is especially convenient in handwriting, I think; vertical bars are a bit easier to write by hand than colons for me!

1.5 The Empty Set

There is another set that has special notation, called the *empty set*. This is the set that has absolutely no elements. It is denoted by $\emptyset$, whose LaTeXcommand is `\emptyset`. You might think it weird that a set with absolutely no elements should get a special name, but it comes up surprisingly often as a special case.

A set is *nonempty* if it is not the empty set. Any set that has any elements whatsoever is nonempty.

2 Subsets and Set Equality

If S and T are two sets, we say that S is a *subset* of T if every element of S is also an element of T. There may or may not be elements of T that are not elements of S. The notation used for saying that S is a subset of T is $S \subseteq T$. The LaTeX command for the symbol "$\subseteq$" is `\subseteq`.

Example. Suppose $S = \{1, 4\}$ and $T = \{1, 4, 5\}$. The only elements of S are 1 and 4, and both of these are also elements of T, so it is true that S is a subset of T. On the other hand, notice that 5 is an element of T that is not an element of S, so T is *not* a subset of S.

This leads us to our next definition, which in fact we've already been thinking about a bit, though we're now making that intuitive definition precise. If S and T are sets, we say that S *equals* T if every element of S is an element of T *and* every element of T is an element of S. Equivalently, $S = T$ means precisely that $S \subseteq T$ and $T \subseteq S$.

Example. Suppose $S = \{1, 4, 5\}$ and $T = \{1, 5, 4\}$. The only elements of S are 1, 4, and 5, and all three of these are also elements of T, so we can say that $S \subseteq T$. But similarly, the only elements of T are 1, 5, and 4, and all three of these are also elements of S, so it is also true that $T \subseteq S$. Thus $S = T$.

If you'd like an analogy, this definition of set equality is a bit like saying two real numbers a and b are equal if $a \leq b$ and $b \leq a$. You might be wondering at this point why we've gone through the effort of making this somewhat bizarre definition of equality, especially since we had already noted earlier that $\{1, 4, 5\} = \{1, 5, 4\}$ using the fact that order doesn't matter. The reason is that there are arbitrarily many ways of describing a set, and sometimes the best way of showing that two descriptions are really describing the same set is by using this formal definition. Let's do an example like this: in fact, this example will even involve sets whose elements aren't numbers!

Example. Let R be the set of all rectangles whose side lengths are positive integers. We now define two subsets of R as follows:

$$S = \{r \in R : r \text{ has even length or width}\}$$
$$T = \{r \in R : r \text{ has even area}\}$$

For example, the rectangle with length 4 and width 5 is one element of S. It is also an element of T, since its area is 20. Let's prove that $S = T$.

First, to prove that $S \subseteq T$, we need to show that every element of S is also an element of T. Suppose r is a rectangle in S. Let us write a and b for the length and width of the rectangle. Since r is in S, we know that either a or b is even. But then the area of r, which is ab, must also be even since the product of an even number with any integer is always even. This means that $r \in T$. This shows that any element of S is also an element of T, so we have proved that $S \subseteq T$.

Next, let us prove that $T \subseteq S$. Suppose r is a rectangle in T. Let a and b be its length and width. Since r is in T, we know that its area, ab, is even. But the only way that ab can be even is if either a or b is even (if they were both odd, then ab would also be odd!). This means that either the length or the width of r is even, ie, that $r \in S$. This shows that every element of T is also an element of S, ie, that $T \subseteq S$. Thus we have proved that $S = T$.

Note. In the above example, we've used some facts about even and odd numbers that you're probably familiar with but maybe have never proved formally. We'll formally prove these facts later.

3 Practice Problems

(1) Give a natural language description of each of the following sets. If the set has fewer than 5 elements, give a complete roster description of the set as well.

  (a) $\{2n : n \in \mathbb{Z}\}$.

  (b) $\{x \in \mathbb{R} : x^2 = 25\}$.

  (c) $\{n^2 : n \in \mathbb{Z} \text{ and } 1 \leq n \leq 5\}$.

  (d) $\{x \in \mathbb{R} : x > 0 \text{ and } x < 0\}$.

(2) In each of the following, a roster description of an infinite set using ellipses is given. Describe the same set using set-builder notation.

  (a) $\{\dots, -9, -4, -1, 0\}$.

  (b) $\{1, 8, 27, 64, \dots\}$.

(3) Here's another set you may have seen before: $\mathbb{R}^2$ is the set of points in the plane. The point with x-coordinate a and y-coordinate b is denoted (a, b), so one could write $\mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$. Now let

$$S = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 0.\}$$

How many elements does S have?

(4) Explain in your own words why every statement of the form "$a \in \emptyset$" is false, no matter what a is.

(5) Let R be the set of all rectangles with integer side-lengths. Let S be the set of rectangles in R whose length or width is *odd*, and let T be the set of rectangles in R with *odd* area.

  (a) Prove that $T \subseteq S$.

  (b) Give an example of an element of S that is not an element of T in order to prove that $S \neq T$.

(6) Let R be the set of all *squares* with integer side-lengths. Let S be the set of squares in R with odd area, and let T be the set of squares in R with odd side-lengths. Prove that $S = T$.

(7) Consider the following two subsets of $\mathbb{R}^2$:

$$S = \{(a, b) \in \mathbb{R}^2 : a + b = 0\}$$
$$T = \{(x, -x) : x \in \mathbb{R}\}.$$

Is it true that $S = T$? If so, prove it. If not, give an example of an element of one set that is not an element of the other.

(8) Is it true that $\emptyset = \{\emptyset\}$? Explain your answer.