Worksheet W2Thu: Path Integrals

Problem 1. Calculate the length of each of the following paths.

(a) The rectange with vertices $\pm 1 \pm 2i$.

(b) The top half of the circle $C[1+i, 1]$.

(c) $\gamma(t) = t - ie^{-it}$ for $t \in [0, 2\pi]$.

Problem 2. Calculate the integral of the indicated function over the indicated path.

(a) $f(z) = \operatorname{Re}(z)\operatorname{Im}(z)$ over $C[0, 2]$.

(b) $f(z) = \operatorname{Im}(z)$ over the line segment from 0 to $1 - i$.

(c) $f(z) = \exp(3z)$ over the line segment from 1 to i.

(d) $f(z) = \exp(3z)$ over $C[0, 3]$.

(e) $f(z) = \exp(3z)$ over the arc of the parabola $\gamma(t) = t + it^2$ for $t \in [0, 1]$.

(f) $f(z) = z^{1/2}$ over $C[0, 2]$.

Problem 3. Let γ be the semicircle centered at 0 of radius 1 from 1 to -1. Calculate

$$\int_\gamma \left(\operatorname{Re}(z)^2 - i\operatorname{Im}(z)^2\right)\, dz.$$

Problem 4. Let $I(k) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{ikt}\, dt$. Calculate the following.

(a) $I(0)$. (b) $I(k)$ for $k \in \mathbb{Z} \setminus \{0\}$. (c) $I(1/2)$.

Problem 5. Suppose $\sigma : [c, d] \to \mathbb{C}$ is a reparametrization of $\gamma : [a, b] \to \mathbb{C}$.

(a) Prove that $\int_\gamma f = \int_\sigma f$ for any continuous function f defined on an open neighborhood of γ.

(b) Prove that γ and σ have the same length.

Hint. u-substitution!

Problem 6. Let γ be the straight line path from 4 to $4i$. Calculate $\int_\gamma f$ for each of the following functions f.

(a) $f(z) = 1$. (b) $f(z) = \dfrac{1}{z}$. (c) $f(z) = \dfrac{1}{z^2 + z}$.

Problem 7. Calculate $\displaystyle\int_\gamma z^i\, dz$ for $\gamma(t) = e^{it}$ where $t \in [\pi/2, 3\pi/2]$. *Caution*. The path crosses the negative real axis!