Problem Set 7

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1 (1 point each part)**.** Determine whether or not each of the following functions $\mathbb{R} \to \mathbb{R}$ is differentiable at 0.

(a) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

(b) $g(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Problem 2 (1 point)**.** If f and g are differentiable functions $\mathbb{R} \to \mathbb{R}$ such that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \le g(x)$ for all $x \ge 0$.

Problem 3 (1 point each part)**.** Prove the following facts using the mean value theorem.

(a) For all $x, y \in \mathbb{R}$, $|\cos x - \cos y| \le |x - y|$.

(b) For all positive integers n,

$$\frac{1}{2\sqrt{n+1}} \le \sqrt{n+1} - \sqrt{n} \le \frac{1}{2\sqrt{n}}.$$

Problem 4 (1 point)**.** Suppose S is an open subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ is a twice differentiable function. Show that, for any $a \in S$, we have

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

Problem 5 (3 points)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and that there exists a real number C such that $\|f^{(k)}\|_{\sup} \le C$ for all $k \in \mathbb{N}$. Suppose further that $f(1/n) = 0$ for all positive integers n. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Problem 6 (3 points)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\|f'\|_{\sup} \lneq 1$. Show that there exists an $a \in \mathbb{R}$ such that $f(a) = a$. *Hint.* Pick an arbitrary point $x_0 \in \mathbb{R}$ and then inductively define $x_{n+1} := f(x_n)$ for all $n \in \mathbb{N}$. Show that the resulting sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

Problem 7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as follows.

$$f(x) = x + \cos x \sin x$$
$$g(x) = e^{\sin x} f(x)$$

(a) (2 points) Show that

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0.$$

(b) (1 point) Show that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

does not exist, and explain why this does not contradict L'Hôpital's rule.

Problem 8 (3 points)**.** Suppose S is a bounded open interval in $\mathbb{R}$ and that $k \in \mathbb{N}$. Let $\mathcal{C}^k(S)$ denote the set of functions $f : S \to \mathbb{R}$ which are continuously differentiable at least k times on S. In other words, $f \in \mathcal{C}^k(S)$ precisely if the first k derivatives $f', f'', \ldots, f^{(k)}$ of f all exist everywhere on S, and moreover the kth derivative $f^{(k)}$ is continuous. Define a metric d on $\mathcal{C}^k(S)$ by declaring that

$$d(f, g) = \max\{\|f - g\|_{\sup}, \|f' - g'\|_{\sup}, \ldots, \|f^{(k)} - g^{(k)}\|_{\sup}\}$$

for $f, g \in \mathcal{C}^k(S)$. Show that the resulting metric space $\mathcal{C}^k(S)$ is complete.