Name:

Quiz 7

Part I (10 points). You will get 1 point for each correct answer, 0 points for each blank answer, and -1 point for each incorrect answer. The minimum possible score for this section is 0.

(1) The function $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2$ defines an inner product on $\mathbf{R}^2$. T **F**

(2) If u_1, u_2 is a basis for a subspace U in an inner product space V and $v \in V$, then the orthogonal projection $P_U(v)$ of v onto U is $\langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2$. T **F**

(3) If $T \in \mathcal{L}(\mathbf{C}^3)$ is a normal operator, $T(1, 2, 3) = (-1, -2, -3)$, and $(x, y, z) \in \operatorname{null} T$, then $x + 2y + 3z = 0$. **T** F

(4) If V is a finite dimensional inner product space, $T \in \mathcal{L}(V)$ is a normal operator, and u and v are linearly independent eigenvectors of T, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$. T **F**

(5) Suppose $\mathcal{P}_2(\mathbf{R})$ is regarded as an inner product space with the inner product T **F**

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\, dx$$

and the matrix of an operator T with respect to the basis $1, x, x^2$ is symmetric. Then T is self-adjoint.

(6) Suppose U is a subspace of a finite dimensional inner product space V. The orthogonal projection map P_U is self-adjoint. **T** F

(7) There exists a polynomial q of degree at most 5 such that **T** F

$$p(1) = \int_0^1 p(x)q(x)\, dx$$

for all polynomials p of degree at most 5.

(8) If U is a subspace of a finite dimensional inner product space V, there exists a subspace W such that $P_U + P_W = I$. **T** F

(9) If V is a finite dimensional real inner product space, the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$. **T** F

(10) If V is a 5 dimensional inner product space and $T \in \mathcal{L}(V)$ is an operator such that $\dim \operatorname{null} T' = 2$, then $\dim \operatorname{range} T^* = 2$. T **F**

Part II (10 points).

(11) Let V be the set of continuous functions $[-1,1] \to \mathbf{R}$ regarded as an inner product space with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$

and let $U = \operatorname{span}(1, x^2, x^4)$. Calculate the orthogonal projection $P_U(h)$ of the function $h(x) = x^3$ onto U.

Observe that h is an odd function, so for any even function f, the product fh is odd and

$$\langle f, h\rangle = \int_{-1}^{1} f(x)h(x)\,dx = 0.$$

In other words, h is orthogonal to every even function. In particular, it is orthogonal to 1, x^2 and x^4, so $h \in U^{\perp}$. Thus $P_U(h) = 0$.

(12) Suppose V is a finite dimensional complex inner product space and $P \in \mathcal{L}(V)$ is a normal operator such that $P^2 = P$. Prove that there exists a subspace U such that $P = P_U$.

Since $P^2 = P$, the minimal polynomial $p_{\min}(z)$ of P divides $z^2 - z = z(z-1)$. This means that P is diagonalizable with eigenvalues either 0 or 1. Choose orthonormal bases $e_1, \ldots, e_m$ for $E(1,P)$ and $e_{m+1}, \ldots, e_n$ for $E(0,P)$. Since P is normal, distinct eigenspaces are mutually orthogonal, so $e_1, \ldots, e_n$ is an orthonormal basis for V. Let $U = E(1,P)$ and observe that

$$Pv = P(\langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m = P_U(v)$$

for any $v \in V$. Thus $P = P_U$.