WORKSHEET: INVERSE MATRICES

Let $f : \mathbb{R}^3 \to \mathcal{P}_2$ the linear map given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto a + (a+b)x + (a+b+c)x^2.$$

Observe that if $f(a,b,c) = a + (a+b)x + (a+b+c)x^2 = 0$, then

$$a = a + b = a + b + c = 0,$$

which means $a = b = c = 0$. In other words, f is injective. But $\dim(\mathbb{R}^3) = \dim(\mathcal{P}_2) = 3$, so, by the rank-nullity theorem, we can conclude that f is an isomorphism, which means that there is an inverse function $f^{-1} : \mathcal{P}_2 \to \mathbb{R}^3$ which is also an isomorphism.

Let B be the standard basis of $\mathbb{R}^3$ and $C = \langle 1, x, x^2 \rangle$ the standard basis of $\mathcal{P}_2$. The point of this worksheet is to work through an example explaining how to compute $\mathrm{Rep}_{C,B}(f^{-1})$ from $\mathrm{Rep}_{B,C}(f)$.

1. Find the matrix $\mathrm{Rep}_{B,C}(f)$.
2. Since f is an isomorphism, we know that there exists a unique $(a,b,c) \in \mathbb{R}^3$ such that $f(a,b,c) = 1$. Set up a system of linear equations whose solution would be the unique $(a,b,c) \in \mathbb{R}^3$ such that $f(a,b,c) = 1$, and write down the corresponding augmented matrix. (Don't row reduce yet.) Compare what you've written down to the matrix $\mathrm{Rep}_{B,C}(f)$ from the previous problem.
3. Now row reduce the augmented matrix, *making sure to record all of the row operations you use.*
4. Since f is an isomorphism, we also know that there exists a unique $(a,b,c) \in \mathbb{R}^3$ such that $f(a,b,c) = x$. Do the same thing to find this (a,b,c). Using the information of the row operations you recorded in the previous step will make this go faster.
5. Do the same thing to find the unique $(a,b,c) \in \mathbb{R}^3$ such that $f(a,b,c) = x^2$.
6. What are $f^{-1}(1)$, $f^{-1}(x)$, and $f^{-1}(x^2)$?
7. Find the matrix $\mathrm{Rep}_{C,B}(f^{-1})$.
8. Looking at what you've done above, convince yourself that setting up the augmented matrix

   $$\left(\mathrm{Rep}_{B,C}(f) \;\; \middle| \;\; I\right),$$

   where I is the 3×3 identity matrix, and then row reducing, will eventually yield

   $$\left(I \;\; \middle| \;\; \mathrm{Rep}_{C,B}(f^{-1})\right).$$