Problem Set 3

Note. You must provide a proof for all assertions you make in your solutions, whether the problem explicitly asks for it or not.

Problem 1. Let X be a metric space. For any $a \in X$ and positive real number r, let

$$B^+(a,r) := \{x \in X : d(a,x) \le r\}.$$

(a) Show that $B^+(a,r)$ is bounded.

(b) Show that $B^+(a,r)$ is closed in X.

(c) Must $B^+(a,r)$ be equal to the closure of $B(a,r)$?

Problem 2. (1 point) Let X be a metric space, E a subset of X, and S a subset of E. Consider the following statements.

(a) If E is open in X, and S is open in E, then S is open in X.

(b) If E is open in X, and S is closed in E, then S is closed in X.

(c) If E is closed in X, and S is open in E, then S is open in X.

(d) If E is closed in X, and S is closed in E, then S is closed in X.

Determine whether each of the above statements is true or false. *Remark.* There are another four statements of this form that one could make. You might optionally try listing those four statements as well and thinking about whether those are true or not, but you need not submit this.

Problem 3. (1 point) Let X be a metric space and suppose that $K_1, \ldots, K_n$ are compact subsets of X. Show that

$$K := K_1 \cup \cdots \cup K_n$$

is also compact.

Problem 4. (1 point) Let X be a compact metric space. Show that X is bounded.

Problem 5 (1 point)**.** Let X be a metric space and suppose a nonempty set I indexes a collection $(K_i)_{i \in I}$ of compact subsets $K_i \subseteq X$. Show that $K := \bigcap_{i \in I} K_i$ is also compact.

Problem 6 (1 point)**.** Let E be a subset of the closed interval $[0, 1]$ and let E' denote its set of limit points. We showed in class that when E is the Cantor set, E is closed, that $E \subseteq E'$ and that $E^\circ = \emptyset$. The point of this problem is for you to come up with other examples of sets which satisfy any two of these properties but not all three of them.

In other words, give an example of a subset E inside $[0, 1]$ such that...

(a) E is closed and $E \subseteq E'$, but $E^\circ \neq \emptyset$.

(b) E is closed and $E^\circ = \emptyset$, but $E \not\subseteq E'$.

(c) $E \subseteq E'$ and $E^\circ = \emptyset$, but E is not closed.

Problem 7 (1 point)**.** A metric space X is *totally disconnected* if the only connected subsets of X are the empty set and singleton sets.

Show that the Cantor set is totally disconnected. *Hint.* You can do this using ternary expansions if you want, but there is a much simpler proof that doesn't use ternary expansions: recall that we know precisely when subsets of $\mathbb{R}$ are connected.

Problem 8 (3 points)**.** In class, we saw that discrete metric spaces furnish counterexamples to the Heine-Borel theorem: they can have bounded and closed subsets which are not compact. This problem is about a much more important example of this phenomenon. Note that this problem requires you to be familiar with the notation and terminology introduced in the example involving the supremum metric in https://math.berkeley.edu/~sagrawal/su15_math104/metric_examples.pdf.

Let S be a set and let X be the set of bounded functions $f : S \to \mathbb{R}$ regarded as a metric space with the supremum metric. Then let

$$F := \{f \in X : \|f\|_{\sup} \leq 1\}.$$

In other words, using the notation from problem 1, we have $F = B^+(\mathbf{0}, 1)$, where $\mathbf{0}$ denotes the bounded function $S \to \mathbb{R}$ that maps every $s \in S$ to the real number 0. It therefore follows from problem 1 that F is bounded and that it is also closed in X.

Show that, if S is infinite, then F is *not* compact.

Problem 9 (3 points)**.** Let X be a compact metric space and let $\mathfrak{U}$ be an open cover of X. Show that there exists a positive real number δ such that, for every subset $E \subseteq X$ such that $\operatorname{diam}(E) \lneq \delta$, we have $E \subseteq U$ for some $U \in \mathfrak{U}$.

Problem 10 (5 points)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We define a metric d on $X \times Y$ by declaring

$$d((x, y), (x', y')) := \max\{d_X(x, x'), d_Y(y, y')\}.$$

You should check for yourself that d is indeed a metric on $X \times Y$, but you don't need to submit this proof (the proof is very similar to the proof we did in class that the maximum metric on $\mathbb{R}^2$ is actually a metric). Using this metric, we regard $X \times Y$ as a metric space.

(a) Show that, if X and Y are compact, then $X \times Y$ is also compact.

(b) Explain why part (a), together with the Heine-Borel theorem for $n = 1$, allows us to conclude the Heine-Borel theorem for all $n \geq 1$.