Goals

1) Stability
2) Control Design
3) LQR

I. Stability

Q: Why is this useful?

Ex:

$$\dot{x} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x \leftarrow \text{called an output}$$

Suppose we want $y(t) = y_d \in \mathbb{R}$ as $t \to \infty$

Assume controller is of the following form:

$$u = \underbrace{-K(y - y_d)}_{\text{feedback controller}} + \underbrace{u_d}_{\text{feedforward controller}}$$

$$u = -Kx_2 + Ky_d + u_d$$

Design $\underline{\underline{K}}$ and $\underline{\underline{u_d}}$ to satisfy control spec.

$$\dot{x} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} (u_d + Ky_d) + \begin{pmatrix} -b_0 K x_2 \\ 0 \end{pmatrix}$$

$$\dot{x} = \underline{\begin{pmatrix} -k_0 - k_1 & k_1 - b_0 K \\ k_2 & -k_2 \end{pmatrix}} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} \underline{(u_d + Ky_d)}$$

$$\dot{x} = A^K x + B\tilde{u}$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$
$$y = Cx$$

Want to arrive at y_d and stay there forever. Lets suppose there exists an x_e that generates y_d (i.e. $y_d = Cx_e$)
Lets make x_e a stable equillibrium of the system.

$$\frac{dx_e}{dt} = \underline{\underline{0}} = Ax_e + Bu_e \leftarrow \text{generates equillibrium}$$

$$\underline{\underline{x_e = -A^{-1}Bu_e}} \quad (\text{that } A \text{ is invertible})$$

$$y_d = -CA^{-1}Bu_e$$
$$y_d = -CA^{-1}B(u_d + Ky_d)$$

If we solve

$$u_d = \left[\frac{k_0 + b_0 k}{b_0} - k \right] y_d$$

Now we want to ensure the system is stable!
Lets shift x_e to origin and select K to ensure asymptotic stability

$$\text{let } z = x - x_e$$

$$\frac{dz}{dt} = \frac{dx}{dt} = Ax + Bu$$

$$\frac{d}{dt} = \frac{d}{dt} = Ax + Bu \qquad u = u_e$$

$$= A(z + x_e) + Bu_e$$

$$= Az + \cancel{A(-A^{-1}Bu_e)} + \cancel{Bu_e}$$

$$\dot{z} = Az$$

$$\frac{dz}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 - b_0 k \\ k_2 & k_2 \end{pmatrix} z$$

$$\det\left(sI - \begin{pmatrix} k_0 - k_1 & k_1 - b_0 k \\ k_2 & -k_2 \end{pmatrix}\right) = 0$$

$$s^2 + (k_0 + k_1 + k_2)s + (k_0 k_2 + b_0 k_2 k) = 0$$

Theorem: Let $\dot{x} = f(x)$ w/ an equillibrium of x_e and f continuously differentiable. If all the eigenvalues of

$$\left.\frac{\partial f}{\partial x}\right|_{x = x_e}$$

have (strictly) negative real part then x_e is locally asymptotically stable. If any eigenvalues have positive real part, then x_e is unstable.

Note: 1) We do not know which points converge to x_e even if we are locally asymptotically stable.

2) If any eigenvalues of $\frac{\partial f}{\partial x}$ have real part equal to zero, you know nothing.

II. Control Design

Theorem: (Ackermann's Formula)

Given a single input, single output LTI system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

1) Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are controllable (i.e.

$$rk([B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \cdots \, A^{n-1}B]) = n).$$

Suppose we want y to be equal to some reference signal r as $t \to \infty$.

2) Suppose we design a feedback controller

$$u = -Kx + k_r r$$

$u = -K(y - y_d)$

Ky_d

which generates a closed loop system:

$$\dot{x} = (A - BK)x + Bk_r r.$$

3) Suppose we want $(A-BK)$ to have a charcteristic polynomial

$$\alpha_c(s) = s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_0$$

Φ $\quad s \in \mathbb{C}$ $\quad \det(sI-(A-BK))$

where all of its roots have negative real part.

Then if:

$$K = [0 \; \dots \; 0 \; 1][B \;\; AB \;\; \dots \;\; A^{n-1}B]^{-1} \alpha_c(A)$$

and $k_r = -(C(A-BK)^{-1}B)^{-1}$, we get the desired characteristic polynomial and all states converge to the output where $y(t) = r$ as $t \to \infty$.

MATLAB: place

$u^T(t)\, R(t)\, u(t)$

$10 \cdot u^2(t)$

Ex: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$

$y = [1 \;\; 0]x$

Linearized vehicle model w/ $a=b=2$ and $v=1$

$$rk([B \;\; AB]) = rk\left(\begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix}\right) = 2$$

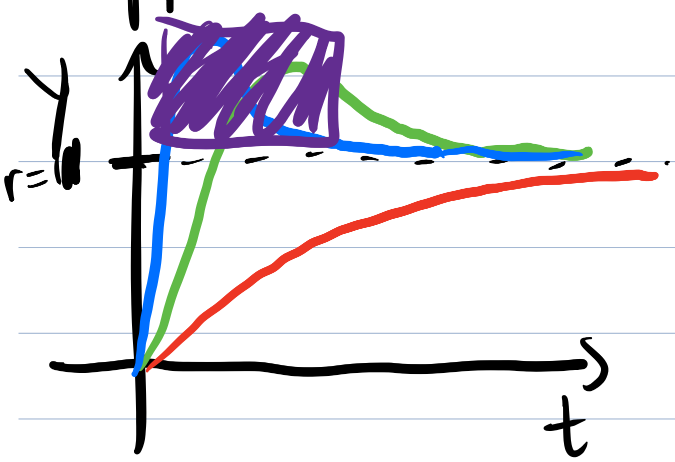
$$\boxed{\alpha_c(s) = s^2 + 2\xi \omega s + \omega^2}$$

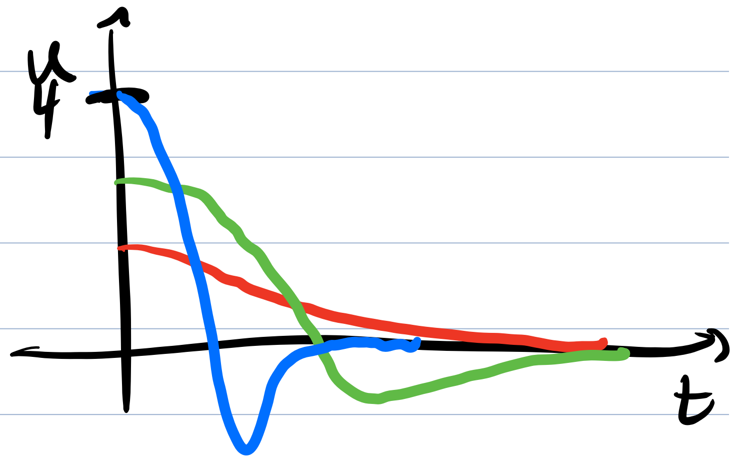
$\xi = 0.7$ and $\omega = 0.5, 1, 2$

eigenvalues of α_c will have real part that is equal to $-0.35, -0.7, -1.4$

ω	$Re(s)$	K
0.5	-0.35	$[0.25 \quad 0.5750]$
1	-0.7	$[1 \quad 0.9]$
2	-1.4	$[4 \quad 0.8]$

Suppose $r=1$ and we start at $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$





<u>Note:</u> 1) Re(Eigenvalues) $\to -\infty$, faster convergence, more overshoot, larger magnitude input.

2) Does not work for LTV systems.

3) No information about how it performs in a nbhd for a linearized nonlinear system.

4) no ability to bound inputs

5) no ability to enforce state constraints

III. Linear Quadratic Regulator (LQR)

What optimization problems are nice?

$$\min f(x)$$

$$\nabla f(x) = 0$$

A. Positive Semidefinite Matrices (p.s.d)

Def: a) A square matrix is called symmetric if it is equal to its transpose.

b) A symmetric matrix A is called positive definite (p.d) if $x^T A x > 0$ for any $x \neq 0$. Denoted as $A > 0$

c) A symmetric matrix A is called positive semidefinite if $x^T A x \geq 0$. Denoted as $A \succeq 0$.

Ex: a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x^T A x = x_1^2 + x_2^2 > 0 \quad \forall x \neq 0$

So $A > 0$.

b) $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \end{bmatrix}$

$$x^T A x = (x_1 + x_2)^2 \geq 0 \quad \forall x, \quad A \geq 0$$

$\boxed{x_1 = -x_2}$ $x^T A x = 0$

Remark: We want to use quadratic polynomials in our cost function, but to ensure they have a bounded minimum we need some "trick"

Idea: connect them to p.s.d. matrices!

Theorem: a) A quadratic function

$$f(x) = x^T D x + C^T x + C_0 \text{ where}$$

$D \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^n$, $C_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$

has a unique minimum value iff $D > 0$.

b) Similarly f has a bounded minimum value iff $D \geq 0$.

Theorem: a) A matrix is p.d. iff all its eigenvalues are real & positive.

b) A matrix is p.s.d. iff all its eigenvalues are real & non-negative.

$$\min f(x)$$

state cost

(LQR)

Def: The Linear Quadratic Regulator Problem is

$$\min_{u:[0,\tau]\to\mathbb{R}^m} \int_0^\tau \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right)dt$$

$$+ \quad x^T(\tau)Q(\tau)x(\tau)$$

input cost

subject to $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

<u>Note</u>: 1) τ is the time horizon

2) To ensure well-posedness we will require
$Q(t) = Q^T(t) \geq 0 \quad \forall t \in [0,\tau]$ and
$R(t) = R^T(t) > 0 \quad \forall t \in [0,\tau]$

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>Theorem:</u> The optimal input $u^*:[0,\tau]\to\mathbb{R}^m$ to the LQR Problem is a linear state feedback controller:

$$u^*(t) = -K(t)x(t) \quad \forall t\in[0,\tau]$$

where

$$K(t) = R^{-1}(t)B^T(t)P(t) \quad \forall t\in[0,\tau]$$

and $P(t)$ is a solution to to the Differential Ricatti Equation:

$$-\dot{P}(t) = A^T(t)P(t) + A(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) +$$
$$+ Q(t)$$

w/ final condition $P(T) = Q(T)$.