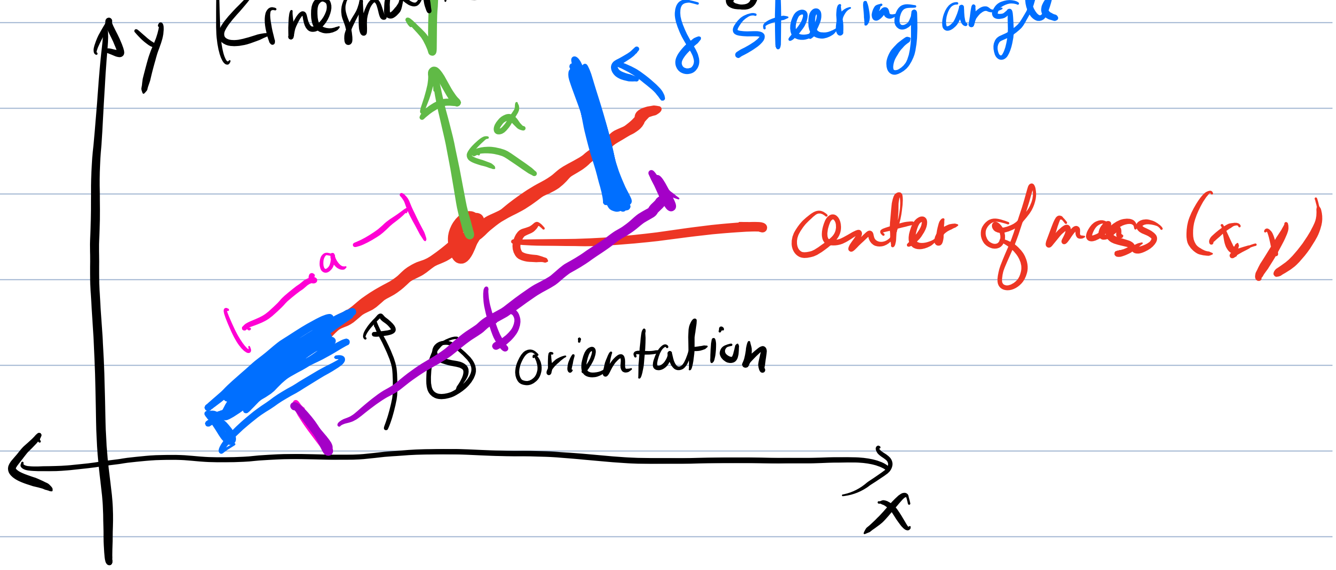
Goals

1) State Space Models
2) Linearizations
3) Time Response
4) Stability
5) Control Design

I S.S. Models

Kinematic Steering Model

Ex:



b = wheel base
a = distance to rear wheel center from center of mass
α = angle between wheel frame and the applied velocity V's direction
δ = front wheel steering angle
θ = orientation
V = velocity applied at center of mass

Ex: ordinary differential equation (ODE)

$$\cancel{\frac{dx}{dt} = v\cos(\alpha+\theta)}$$

$$\frac{dy}{dt} = v\sin(\alpha+\theta)$$

$$\frac{d\theta}{dt} = \frac{v\cos(\alpha)\tan(\delta)}{b}$$

ODE model for kinematic steering model

Find equillibria when $\theta = 0$.

$$\alpha = \tan^{-1}\left(\frac{a\cdot\tan\delta}{b}\right)$$

Def: a) The state of a system is a collection of variables that summarize the past of a system for the purpose of predicting its future.

b) The state vector is the collection of these states in a vector $x \in \mathbb{R}^n$.

c) A model that describes a system using a differential equation

$$\frac{dx}{dt} = f(t,x,u)$$

where $f: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is

$\parallel$

$[0,\infty)$

a smooth mapping, $u \in \mathbb{R}^m$ is a control (or input) variable, is called a state space model.

<u>Def</u>: a) A system is called Linear Time Invariant or LTI if f does not depend on time and if f is a linear function of x and u.

b) A system is called Linear Time Varying (LTV) if f is a linear function of x and u at each time $t \in \mathbb{R}_+$.

<u>Note</u>: 1) If a system is LTI, the f can be represented as

$$\frac{dx}{dt} = Ax + Bu$$

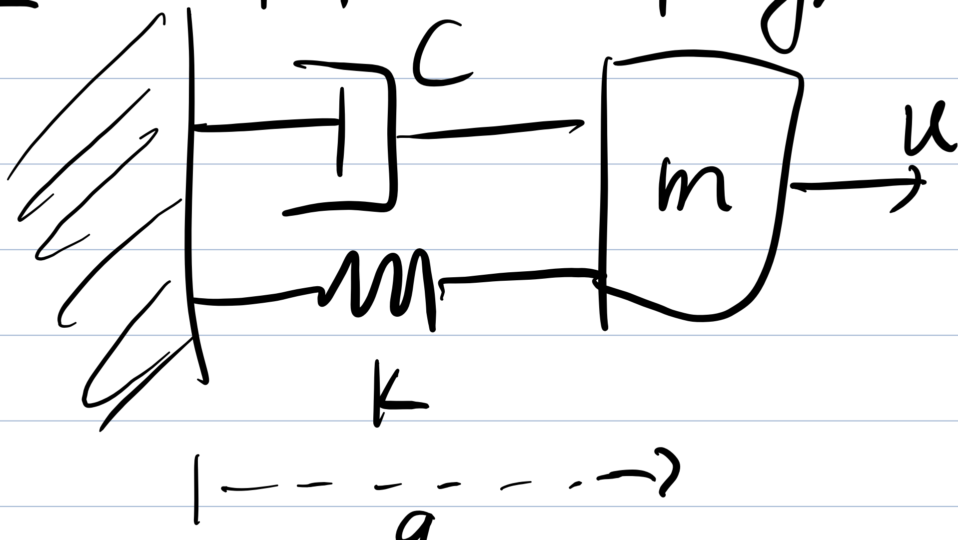
where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$

2) If a system is LTV, the f can be represented as

$$\frac{dx}{dt} = A(t)x + B(t)u$$

where for each $t \in \mathbb{R}_+$, $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$

Ex: (Damped Mass Spring)



$$m\ddot{q} + c\dot{q} + kq = u$$

select $x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1/m \end{pmatrix}}_{B} u$$

$$\dot{x} = A\,x + B\,u$$

B. Linearizations

about a point $\rightarrow$ LTI
about a traj $\rightarrow$ LTV

A. Linearizations about a point

Def: Suppose $\dot{x} = f(x,u)$ is time invariant. The equillibria of the dynamical system are points x and u s.t. $\dot{x} = 0$.

Ex:

$$\frac{dy}{dt} = v\sin(\alpha + \theta) = f_1 \qquad \frac{\partial f_1}{\partial x_1} \neq 0$$

$y \to x_1$, $\theta \to x_2$

$$\frac{d\theta}{dt} = \frac{v\cos(\alpha)\tan(\delta)}{b} = f_2 \qquad \frac{\partial f_2}{\partial x_1} = 0$$

$$\alpha = \tan^{-1}\left(\frac{a \cdot \tan\delta}{b}\right) \qquad \frac{\partial f_2}{\partial x_2} = 0$$

$\theta = 0$, $\quad \frac{dy}{dt} = 0 = v\sin(\alpha)$

$\alpha = 0$

$$0 = \tan^{-1}\left(\frac{a \cdot \tan\delta}{b}\right) \to \delta = 0$$

$\theta = 0$, $\delta = 0$, y = anything, v = anything

Def: The Jacobian Linearization of $\dot{x} = f(x,u)$ at an equillibria (x_e, u_e) is

$$\frac{dz}{dt} = Az + Bv$$

where $A = \left.\frac{\partial f}{\partial x}\right|_{(x_e, u_e)}$ $\quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e, u_e)}$

where $z = (x - x_e)$ and $v = (u - u_e)$

<u>Ex:</u> For vehicle steering model $\begin{pmatrix} y_e \\ \theta_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\delta_e = 0, \quad \alpha(\delta_e) = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{x_e, u_e} = \left.\begin{bmatrix} 0 & \underline{v\cos(\alpha + \theta)} \\ 0 & 0 \end{bmatrix}\right|_{x_e, u_e} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

$$\left[\frac{\partial f}{\partial x}\right]_{ij} = \frac{\partial f_i}{\partial x_j}$$

A

$$\left.\frac{\partial f}{\partial u}\right|_{x_e, u_e} \in \mathbb{R}^{2 \times 1}$$

$$\left.\frac{\partial f}{\partial u}\right|_{x_e, u_e} = \begin{bmatrix} av/b \\ v/b \end{bmatrix}$$

B

$$\begin{bmatrix} (y \dot{-} y_e) \\ (\theta \dot{-} \theta_e) \end{bmatrix} = A \begin{pmatrix} y - y_e \\ \theta - \theta_e \end{pmatrix} + B(\delta - \delta_e)$$

Remark: $\delta_x(t) \equiv x(t) - x_e$

$\delta_u(t) \equiv u(t) - u_e$

$$\dot{\delta}_x(t) = \dot{x}(t) = f(x(t), u(t))$$
$$= f(\delta_x(t) + x_e, \delta_u(t) + u_e)$$

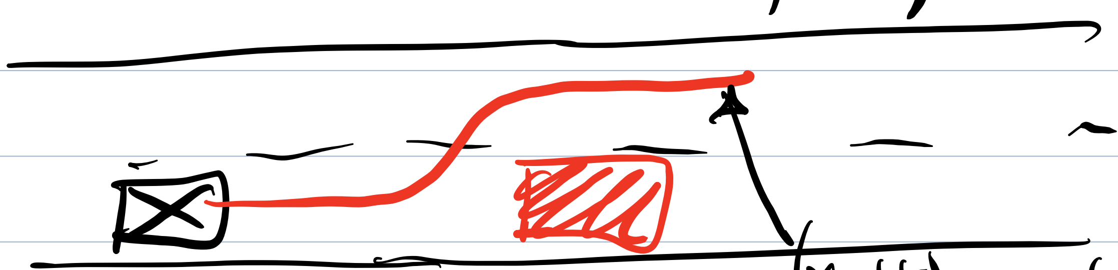
Apply Taylor Expansion

$$\dot{\delta}_x(t) \approx \underbrace{f(x_e, u_e)}_{=0} + \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \delta_x(t) + \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)} \delta_u(t) + \cancel{\text{H.O.T.}}$$

~~(higher order terms)~~

$$\dot{z} = Az + Bv$$

B. Linearizations about a Trajectory



$(x_e(t), u_e(t))$

$$\frac{dx}{dt} = f(x(t), u(t))$$

$\delta_x(t) = x(t) - x_e(t)$, $\delta_u(t) = u(t) - u_e(t)$

$$\dot{x}(t) = f(\delta_x(t) + x_e(t), \delta_u(t) + u_e(t))$$

$$\approx f(x_e(t), u_e(t)) + \frac{\partial f}{\partial x}\Big|_{x_e(t),u_e(t)} \delta x(t) + \frac{\partial f}{\partial u}\Big|_{x_e(t),u_e(t)} \delta u(t)$$

$$+ \cancel{h.o.t.}$$

$$\dot{x}(t) - f(x_e(t), u_e(t)) = \frac{\partial f}{\partial x}\Big|_{x_e(t),u_e(t)} \delta x(t) + \frac{\partial f}{\partial u}\Big|_{x_e(t),u_e(t)} \delta u(t)$$

$$\dot{x}(t) - \dot{x}_e(t) =$$

$$\dot{\delta x}(t) = A(t)\delta x(t) + B(t)\delta u(t)$$

$$\frac{\partial f}{\partial x}$$

III. Time Response

$$\int_0^T \|u(t)\|_2^2 \, dt < \infty$$

square integrable

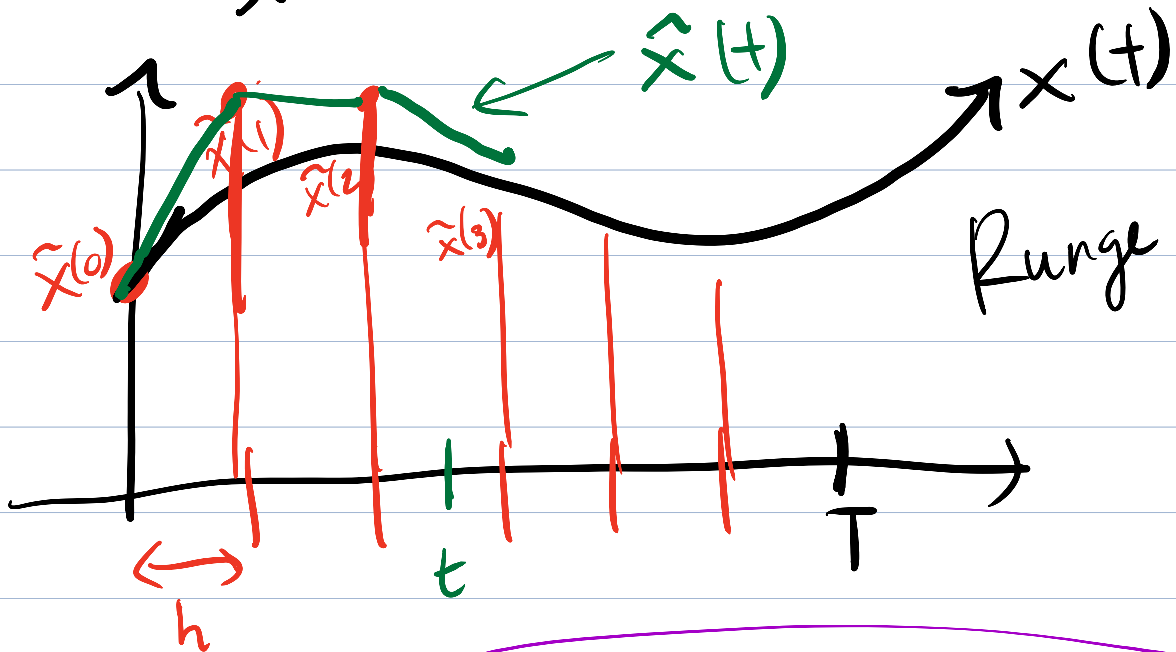
A. Simulation

<u>Theorem:</u> (Euler Integration) Let $\dot{x} = f(t, x, u)$ be a differentiable function, $u : [0,T] \to \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$ and $x : [0,T] \to \mathbb{R}^n$ a solution to f under u with $x(0) = x_0$.
Let $h > 0$ with $T/h \in \mathbb{N}$ and

$\{\tilde{x}(k)\}_{k=0}^{1/h}$ defined as:

$$\tilde{x}(k+1) = \tilde{x}(k) + h\,\underline{f(kh, \tilde{x}(k), u(kh))}$$

with $\hat{x}(0) = x_0$.



Runge Kutta

$g(z)$ $$\underline{g'(z)} \approx \frac{g(z+h) - g(z)}{h}$$

$$g'(z)h = g(z+h) - g(z)$$

$$g'(z)h + g(z) = g(z+h)$$

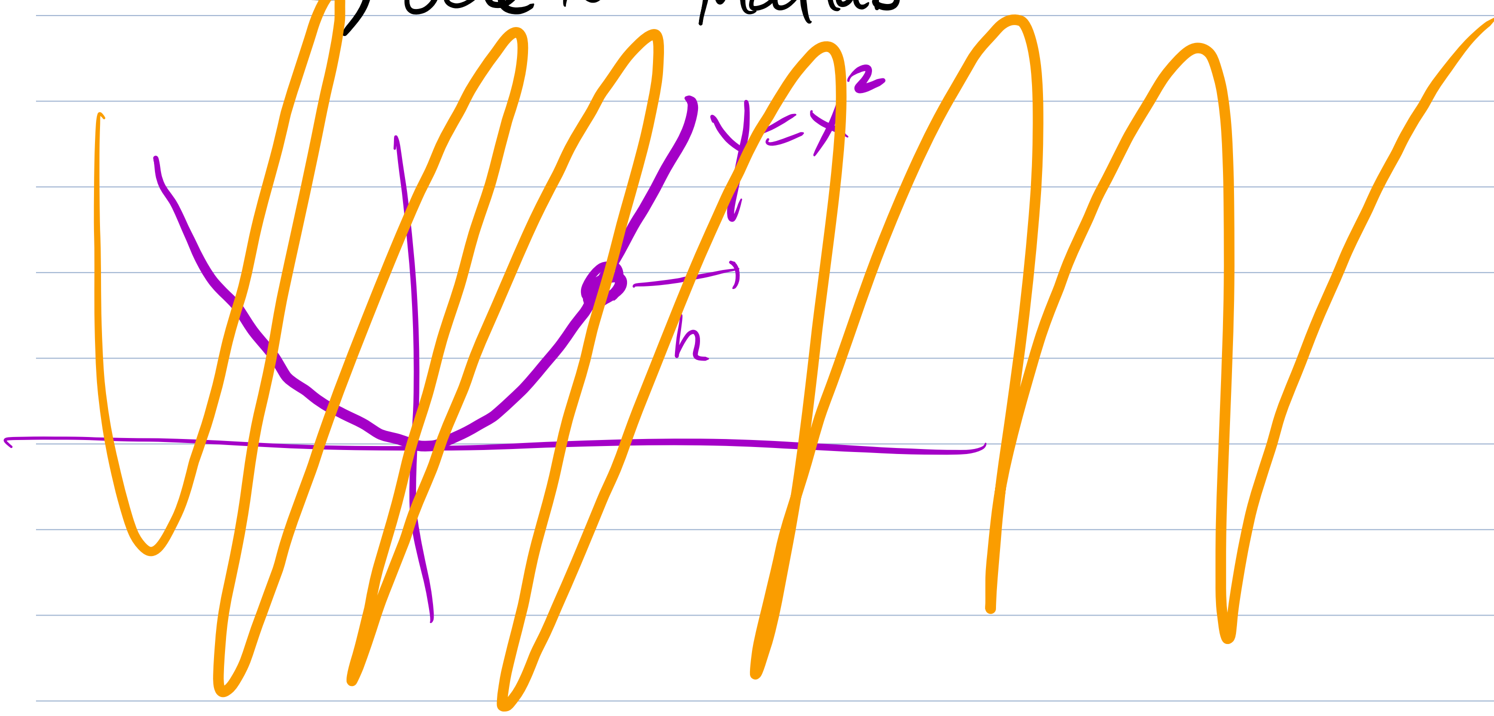
Suppose we linearly interpolate these states to get $\hat{x} : [0,T] \to \mathbb{R}^n$

f

then $$\lim_{h \to 0} \int_0^T \|x(t) - \hat{x}(t)\|_2 \, dt \le h e^{KhT}$$

Remark: 1) Useful while doing optimization for trajectory design.

2) ode45 Matlab



B. Solutions to LTI Systems

Theorem: Any solution to an LTI system can be decomposed into a solution w/ zero input (homogeneous solution) plus a solution w/ zero i.c. (particular solution).

$$\dot{x} = Ax + Bu$$

homog.

$$\begin{matrix} \dot{x} = Ax \\ x(0) = x_0 \end{matrix} \quad + \quad \begin{matrix} \dot{x} = Ax + Bu \\ x(0) = 0 \end{matrix}$$

Ex: $\dot{x} = Ax$ what is $x(t)$ when $x(0) = x_0$?

$A \in \mathbb{R}$ $\quad x(t) = e^{At} x(0)$

$$\dot{x} = 5x \qquad x(t) = e^{5t}$$

$$\frac{dx}{dt} = 5x \qquad \frac{dx}{x} = 5t$$

what if $A \in \mathbb{R}^{n \times n}$ $\quad x(t) = e^{At} x(0)$

Def: The matrix exponential to $X \in \mathbb{R}^{n \times n}$ is the infinite series

$$e^X = I + X + \frac{X^2}{2} + \dots = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Lemma: This series converges for any matrix $X \in \mathbb{R}^{n \times n}$.

Ex: (a) $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ what is e^A

then $A^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a_1^k & 0 \\ 0 & a_2^k \end{pmatrix} = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}$$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \cancel{\begin{pmatrix} 0 & e^1 \\ 0 & 0 \end{pmatrix}}$

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$e^A = \underbrace{I + A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$A^n = A^{n-2} A^2$

MATLAB use expm

$\neq$

exp

Ex: $\frac{dx}{dt} = Ax$

$e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \dots, = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

$\frac{d}{dt}(e^{At}) = A + A^2 t + \frac{1}{2} A^3 t^2 + \dots$

$= A \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = Ae^{At}$

$x(t) = e^{At} x(0)$

$\frac{dx}{dt} = Ax$

Ex: $\dot{x}(t) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} x(t)$

$x(t) = ?$ $\qquad x(t) = e^{At} x(0)$

$$\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = I + A = \begin{bmatrix} 1 & vt \\ 0 & 1 \end{bmatrix}$$

$$x(t) = \begin{pmatrix} 1 & vt \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) + vt x_2(0) \\ x_2(0) \end{pmatrix}$$

Theorem: (Convolution Integral)
Let $\dot{x} = Ax + Bu$, then

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau$$

homogeneous $\qquad$ particular

Def: The time response of a system when $t \to \infty$ is called the steady state response.

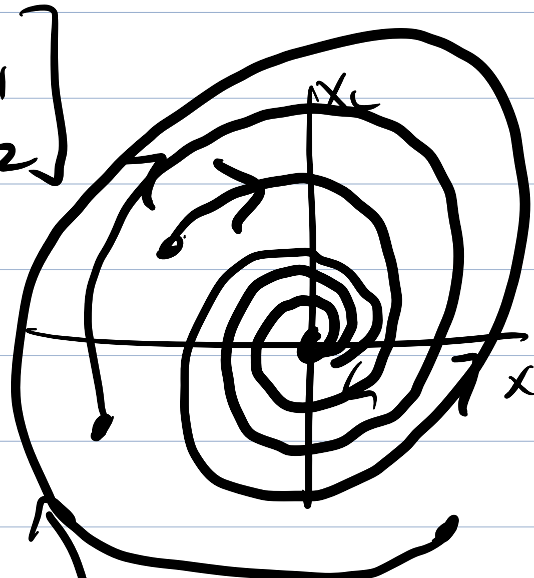
IV. Stability

$$\dot{x} = f(x)$$

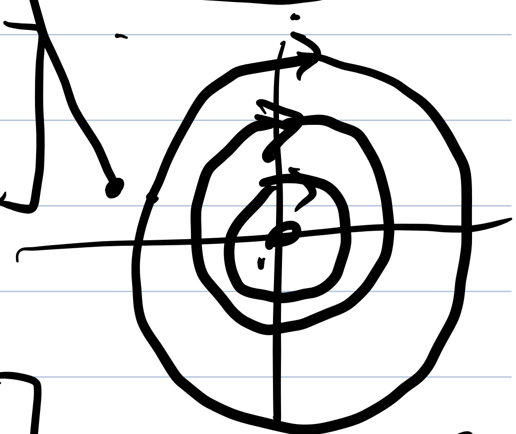
Understand when we converge to equilibria

Ex: (a) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x(t) = e^{At} x(0)$$

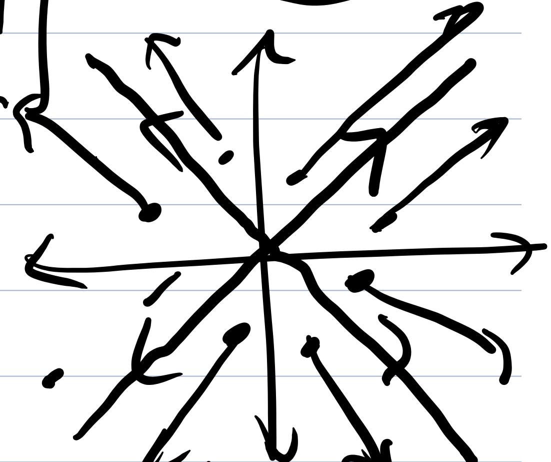


(b) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



(c) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x(t) = e^{At} x(0)$$



(d) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1(1-x_1^2-x_2^2) \\ -x_1 + x_2(1-x_1^2-x_2^2) \end{bmatrix}$

initial condition

Def: (a) Let $x(t;a)$ be a solution of $\dot{x}(t) = f(x(t))$ from i.c. a. A solution is stable if all other solutions that start near a stay near $x(t;a)$ for all time.
(b) If all other solutions stay close to $x(t;a)$, then the system is called globally stable, otherwise it is called locally stable.

Def: a) A solution $x(t;a)$ is called asymptotically stable if it is stable and $x(t;b) \to x(t;a)$ as $t \to \infty$ for b sufficiently close to a.

Theorem: The system $\dot{x} = Ax$ is globally asymptotically stable iff all eigenvalues of A have strictly negative real part. If any eigenvalue has a strictly positive real part, then it is unstable.

$\to \operatorname{Re}(i) = 0$

$\sigma(A) = \{1, 1+i, -1+i, i\}$

$Re(1) = 1 \quad Re(1+i) = 1$ $\quad Re(-1+i) = -1$

Lyapunov

$\leq$ $\quad <$

$\neq$