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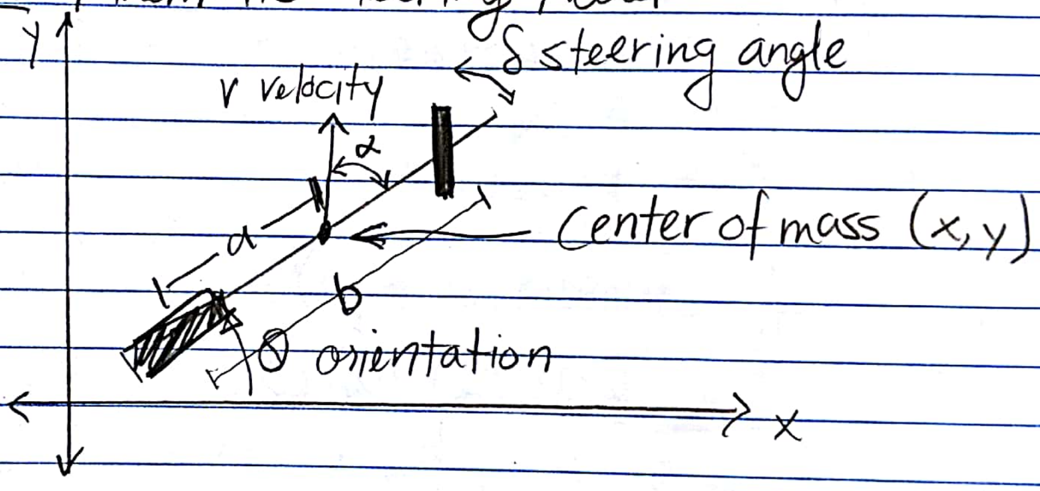
Lecture 2

Goals

1) State Space Models
2) Linearizations
3) Time Response
4) Stability
5) Control Design

I. State Space Models

Ex: Kinematic Steering Model



b = wheel base
a = distance to rear wheel from center of mass
α = angle between applied velocity and vehicle frame.
δ = front wheel steering angle
θ = orientation
v = velocity applied at center of mass

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Our objective is to control v and δ to ensure that the vehicle achieves some behavior.

To accomplish this objective

1. understand properties of models of motion
2. theory to design safe controllers
   a. linear
   b. nonlinear.

We write down models of motion using ordinary differential equations (ODE):

Ex:

$$\left.\begin{aligned}
\frac{dx}{dt} &= v\cos(\alpha+\theta) \\
\frac{dy}{dt} &= v\sin(\alpha+\theta) \\
\frac{d\theta}{dt} &= \frac{v\cos(\alpha)\tan(\delta)}{b} \\
\text{where} \quad \alpha &= \tan^{-1}\left(\frac{a\tan\delta}{b}\right)
\end{aligned}\right\}\ \text{ODE Model for Kinematic Steering model.}$$

Def: a) The state of a system is a collection of variables that summarize the past of a system for the purpose of predicting its future.

b) The state vector is the collection of these states in a vector $x \in \mathbb{R}^n$

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c) A model that describes a system using a differential equation:

$$\frac{dx}{dt} = f(t, x, u)$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth mapping, $u \in \mathbb{R}^m$ is a control (or input) variable is called a state space model

Ex: vehicle model.

Control related questions (e.g. how do I get my system from pt A to pt B) are easier for certain classes of systems which form the backbone of all of our control design techniques

Def: a) A system is called Linear Time Invariant or LTI if f does not depend on time and if f is a linear function of x and u.

b) A system is called Linear Time Varying or LTV if f is a linear function of x and u for each t.

Note: a) If the system is LTI, then f can be represented as

$$\frac{dx}{dt} = Ax + Bu$$

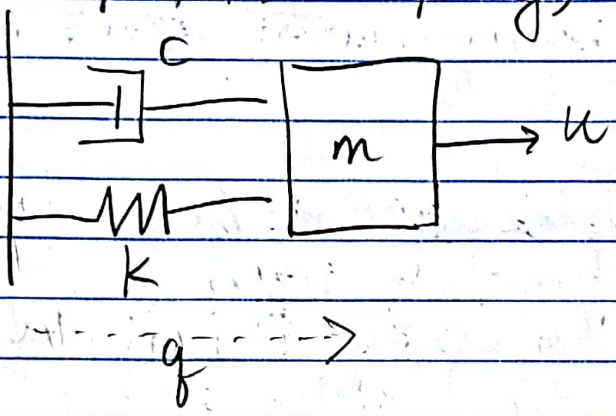
where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$

b) If f is LTV, then it can be represented as:

$$\frac{dx}{dt} = A(t)x + B(t)u$$

where for each $t \in \mathbb{R}_+$, $A(t) \in \mathbb{R}^{n \times n}$ and $B(t) \in \mathbb{R}^{n \times m}$.

Ex: (Damped Mass Spring)



Force balance to write dynamics as

$$m\ddot{q} + c\dot{q} + kq = u$$

If we let $x = (q, \dot{q})$, then

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{m} & -\frac{c}{m} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1/m \end{pmatrix}}_{B} u$$

II Linearizations

As we will see, LTI and even LTV based control design is much easier than doing control design for nonlinear systems.

A. Linearizations at a point

We focus on linearizations usually (but not always) about specific points:

Def: Suppose $\dot{x} = f(x,u)$ is time invariant. The equillibria of the dynamical system are the points x and u s.t. $\dot{x} = 0$.

Ex: Consider the kinematic steering model. For simplicity ignore x-coordinate and v is fixed then:

let $s = (y, \theta)$ and $u = \delta$

$$f(s,u) = \begin{pmatrix} v\sin(\alpha+\theta) \\ v\cos(\alpha)\tan(\delta)/b \end{pmatrix}$$

where $\alpha = \tan^{-1}\left(\frac{a\tan\delta}{b}\right)$

Suppose we are interested in an equillibria around $\theta = 0$, then

$$0 = v\sin(\alpha) \rightarrow \delta = 0$$

This implies the equilibria of this system is $\theta = 0, \delta = 0, y = \text{anything}$.

Def: The Jacobian Linearization of $\dot{x} = f(x,u)$ at an equilibria x_e, u_e is

$$\frac{dz}{dt} = Az + Bv$$

where $A = \left.\frac{df}{dx}\right|_{(x_e,u_e)}$, $B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$

where $z = (x - x_e)$ and $v = (u - u_e)$

Ex: For vehicle steering model at $s = (0,0)$

$u_e = 0$, $\alpha(u_e) \neq 0$

$$\left.\frac{df}{dx}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} 0 & v\cos(\alpha\delta + y) \\ 0 & 0 \end{bmatrix}\right|_{x_e,u_e} \rightarrow \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)} = \begin{bmatrix} av/b \\ v/b \end{bmatrix}$$

Remark: To understand the relationship between original system and Jacobian Linearization, let:

$$\delta_x(t) = x(t) - x_e, \quad \delta_u(t) = u(t) - u_e$$

then

$$\dot{\delta_x}(t) = \dot{x}(t) = f(\delta_x(t) + x_e, \delta_u(t) + u_e)$$

Apply Taylor Expansion:

$$\dot{\delta_x}(t) \cong f(x_e, u_e) + \left.\frac{df}{dx}\right|_{x_e,u_e} \delta_x(t) + \left.\frac{df}{du}\right|_{x_e,u_e} \delta_u(t)$$

$+ \text{h.o.t.}$

(higher order terms)

so we get:

$$\dot{z} = Az + Bv$$

since $f(x_e, u_e) = 0$.

B. Linearization about a Trajectory

Suppose $(x_e(t), u_e(t))$ are a trajectory that we want to linearize about for a system $\frac{dx}{dt} = f(x(t), u(t))$

Consider $\delta_x(t) = x(t) - x_e(t)$ and $\delta_u(t) = u(t) - u_e(t)$

$$\dot{x}(t) = \underbrace{f(\delta_x(t) + x_e(t), \delta_u(t) + u_e(t))}_{\text{Taylor Expand}}$$

$$\cong f(x_e(t), u_e(t)) + \frac{\partial f}{\partial x}\bigg|_{x_e(t), u_e(t)} \delta_x(t) + \frac{\partial f}{\partial u}\bigg|_{x_e, u_e} \delta_u(t) +$$

$$+ \text{h.o.t.}$$

$$\dot{x}(t) - \dot{x}_e(t) = \frac{\partial f}{\partial x}\bigg|_{x_e(t), u_e(t)} \delta_x(t) + \frac{\partial f}{\partial u}\bigg|_{x_e, u_e} \delta_u(t)$$

$$\dot{\delta}_x(t) = A(t)\,\delta_x(t) + B(t)\,\delta_u(t)$$

III Time Response

A. Simulation

Hard to solve nonlinear state space models from an i.c. given an input analytically. We instead use approximations.

Theorem: (Eulers Integration) Let $\dot{x} = f(t, x, u)$ be a differentiable function, $u : [0,T] \to \mathbb{R}^m$ be square integrable, $x_0 \in \mathbb{R}^n$ and $x : [0,T] \to \mathbb{R}^n$ a solution to f under u with $x(0) = x_0$. Let $h > 0$ with $T/h \in \mathbb{N}$ and $\{\tilde{x}(k)\}_{k=0}^{T/h}$ defined as:

$$\tilde{x}(k+1) = \tilde{x}(k) + h f(kh, \tilde{x}(k), u(kh))$$

with $\tilde{x}(0) = x_0$.

Suppose we linearly interpolate these states to get $\hat{x} : [0,T] \to \mathbb{R}^n$

$$\hat{x}(t) = \begin{cases} \tilde{x}(k) + (\tilde{x}(k+1) - \tilde{x}(k))\left(\frac{t - kh}{h}\right) & t \in [kh, (k+1)h) \\ 0 & \text{O.W} \end{cases}$$

then

$$\lim_{h \to 0} \int_0^T \| x(t) - \hat{x}(t) \|_2 \, dt = 0.$$

Remark: 1) Useful while doing optimization for trajectory design.

2) ODE 45 MATLAB function that does simulation w/ more accuracy.

B. Solutions to LTI Systems

Theorem: Any solution to an LTI system can be decomposed into a solution w/ zero input (homogeneous solution) plus a solution w/ zero i.c. (particular solution)

Lets begin by understanding how to find the homogeneous solution to an LTI system:

Ex: Suppose $\dot{x} = Ax$, what is $x(t)$?
If A is a scalar, then $x(t) = e^{At}x(0)$.
What if A is a matrix?

Def: The matrix exponential to $X \in \mathbb{R}^{n\times n}$ is the infinite series:

$$e^{X} = I + X + X^2/2 + \dots = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Lemma: This series converges for any matrix $X \in \mathbb{R}^{n\times n}$.

Ex: (a) $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, then $A^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}$

$$\text{so } e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a_1^k & 0 \\ 0 & a_2^k \end{pmatrix} = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}$$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ notice $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $e^{A} = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Note: MATLAB function expm.

Ex: $\frac{dx}{dt} = Ax$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \ldots = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

$$\frac{d}{dt}\left(e^{At}\right) = A + A^2t + \frac{1}{2}A^3t^2 + \ldots = A\sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

$$= Ae^{At}$$

so if $x(t) = e^{At}$, then $\frac{dx}{dt} = Ax$.

Ex: Consider linearization of steering model:

$$\dot{x}(t) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} x(t)$$

Recall $\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so

$$e^{At} = \begin{bmatrix} 1 & vt \\ 0 & 1 \end{bmatrix} \text{ so}$$

$$x(t) = \begin{pmatrix} 1 & vt \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) + vt\,x_2(0) \\ x_2(0) \end{pmatrix}$$

What about the particular solution?

Theorem: (Convolution Integral)

Let $\dot{x} = Ax + Bu$, then

$$x(t) = \underbrace{e^{At}x(0)}_{\text{homogeneous}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{particular}}$$

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Def: The time response of a system when $t \to \infty$ is called the <u>steady state response</u>.

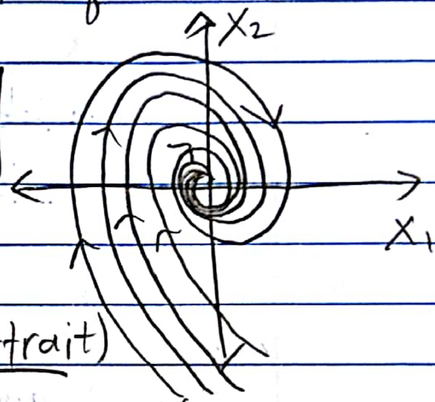
IV. Stability

Start our study of control by looking at systems w/o input:

$$\dot{x} = f(x)$$

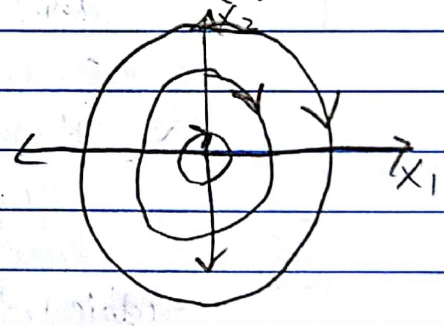
and explore when we converge to equillibria.

Ex: (a) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

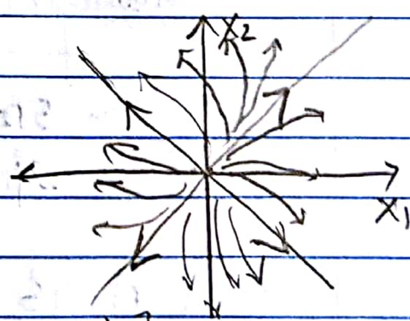


lets plot solutions of the system (called a <u>phase portrait</u>)

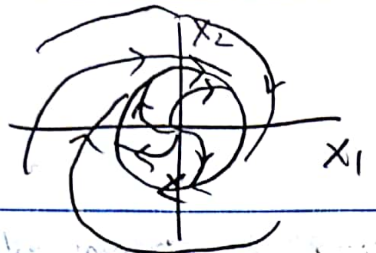
(b) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



(c) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



(d) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1(1 - x_1^2 - x_2^2) \\ -x_1 + x_2(1 - x_1^2 - x_2^2) \end{bmatrix}$



Note: all problems have equillibria at O

Def: (a) Let $x(t;a)$ be a solution to $\dot{x}(t) = f(x(t))$ from i.c. a. A solution is stable if all other solutions that start near a stay near $x(t;a)$ for all time.

(b) If all other solutions stay close to $x(t;a)$ then the system is called globally stable, otherwise it is called locally stable.

Ex: Check origin for all examples
(a) and (b) are globally stable
(c) and (d) are unstable.
If we negate the dynamics of (d) then the origin is locally stable.

Def: a) A solution $x(t;a)$ is called asymptotically stable if it is stable and $x(t;b) \to x(t;a)$ as $t \to \infty$ for b sufficiently close to a.
b) Can similarly define global and local asymptotic stability.

Ex: (a) is asymptotically stable
(b) is not.

Theorem: The system $\dot{x} = Ax$ is asymptotically stable at the origin iff all its eigenvalues of A all have strictly negative real part, and is unstable if any eigenvalue has a strictly positive real part.

Question: Why useful for control?

Ex: Suppose we have a statespace model:

$$\dot{x} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x \leftarrow \text{called an output.}$$

Suppose we want $y(t) = y_d \in \mathbb{R}$ as $t \to \infty$. We decide to use state based feedback to design a control (this imbues robustness to error), so we choose a control

$$u = -K(y - y_d) + u_d = -Kx_2 + Ky_d + u_d$$

Need to select K and u_d.
We will also assume $k_0, k_1, k_2 > 0$.
Substitute controller in:

$$\dot{x} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} (u_d + Ky_d) + \begin{pmatrix} -b_0 K x_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -k_0 - k_1 & k_1 - b_0 K \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} (u_d + Ky_d)$$

$$= Ax + B\tilde{u}$$

$$y = Cx$$

We want to arrive at yd and stay there.
Lets suppose there exists an x_e that generates yd (i.e. $y_d = Cx_e$).
Then lets make x_e a stable equillibria.

$$\frac{dx_e}{dt} = 0 = Ax_e + Bu_e \leftarrow \text{generates equillibria.}$$

$$\Rightarrow \quad x_e = -A^{-1}Bu \quad (A \text{ invertible})$$

$$y_d = -CA^{-1}Bu_e$$

$$= \frac{b_0 k_2}{k_0 k_2 + b_0 k_2 k} \underbrace{(u_d + K y_d)}_{u_e}$$

If we solve for

$$u_d = \left[\frac{k_0 + b_0 k}{b_0} - k\right] y_d$$

Now we have to ensure the system is stable.
Lets shift equillibria to origin and select K to ensure asymptotic stability.

Let $z = x - x_e$

$$\frac{dz}{dt} = \frac{dx}{dt} = Ax + Bu \quad u = u_e$$

$$= A(z + x_e) + Bu_e$$

$$= Az + A(-A^{-1}Bu_e) + Bu_e$$

$$= Az$$

$$\frac{dz}{dt} = \begin{pmatrix} \frac{-k_0 - k_1}{k_2} & \frac{k_1 - b_0 k}{k_2} \end{pmatrix} z$$

How do we ensure eigenvalues have negative real part?

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Check characteristic polynomial:

$$\det\left(sI - \begin{pmatrix} -k_0-k_1 & k_1-b_0k \\ k_2 & -k_2 \end{pmatrix}\right) = 0$$

$$s^2 + (k_0+k_1+k_2)s + (k_0k_2 + b_0k_2k) = 0$$

Want to select k to guarantee $Re(s) < 0$.

<u>Remark:</u> We do not prove it here but $k > -k_0/b_0$ ensures stability.

<u>Strategy for control design:</u>

1) Use linear state feedback control
2) ensure desired behavior is stable.

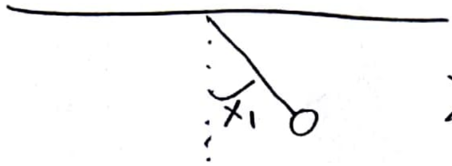
What about stability for nonlinear systems?

<u>Theorem:</u> Let $\dot{x} = f(x)$ w/ an equillibria at x_e, and f continuously differentiable. If all the eigenvalues of

$$\left.\frac{df}{dx}\right|_{x=x_e}$$

have negative real part than x_e is locally asymptotically stable. If any of the eigenvalues has positive real part than x_e is unstable.

Ex:



$\dot{x}_1 = x_2$

Consider a pendulum w/ damping
$m, g, l = 1$, then

$$\dot{x} = \begin{pmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{pmatrix}, \quad \underbrace{\gamma}_{\text{damping}} > 0$$

Lets find equillibria

$$\frac{dx}{dt} = 0 = \begin{pmatrix} x_2 \\ -\sin(x_1) - \gamma x_2 \end{pmatrix} \Rightarrow \begin{array}{l} x_{2e} = 0 \\ x_{1e} = 0, \pm\pi, \ldots \end{array}$$

a) Linearize about $x_{1e} = 0$

$$A = \begin{pmatrix} 0 & 1 \\ -\cos(x_1) & -\gamma \end{pmatrix}\Bigg|_{x_{1e}=0} = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$$

all eigenvalues have negative real part.
↳ locally asymptotically stable.

b) Linearize about $x_{1e} = \pi$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -\gamma \end{pmatrix}$$

has eigenvalue w/ positive real part $\Rightarrow$ unstable.

Note: 1) We do not know which points converge.
2) If eigenvalues of linearization have real part equal to zero, we know nothing.

V. Control Design

Strategy: For time invariant systems, linearize about desired equillibria, and ensure stability.

Theorem: (Ackerman's Formula)

Given a single input, single output LTI system:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

i) Suppose $A \in \mathbb{R}^{n \times n}$, and

$$rk\left[B \ AB \ \cdots \ A^{n-1}B\right] = n$$

(systems that satisfy this requirement are called controllable). Suppose we want y to be equal to some reference signal r as $t \to \infty$

ii) Suppose we design a feedback controller

$$u = -Kx + k_r r$$

which generates a closed loop system

$$\dot{x} = (A - BK)x + Bk_r r$$

iii) Suppose we want $(A-BK)$ to have a characteristic polynomial

$$\alpha_c(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$$

where all of its roots have negative real part. Then if:

$$K = [0 \ \cdots \ 0 \ 1]\left[B \ AB \ \cdots \ A^{n-1}B\right]^{-1}\alpha_c(A)$$

and $k_r = -(C(A-BK)^{-1}B)^{-1}$, we can get the desired characteristic polynomial and steady state behavior.

Ex: (Vehicle Steering) Linearized model w/ $a=b=2$ and $v=1$.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

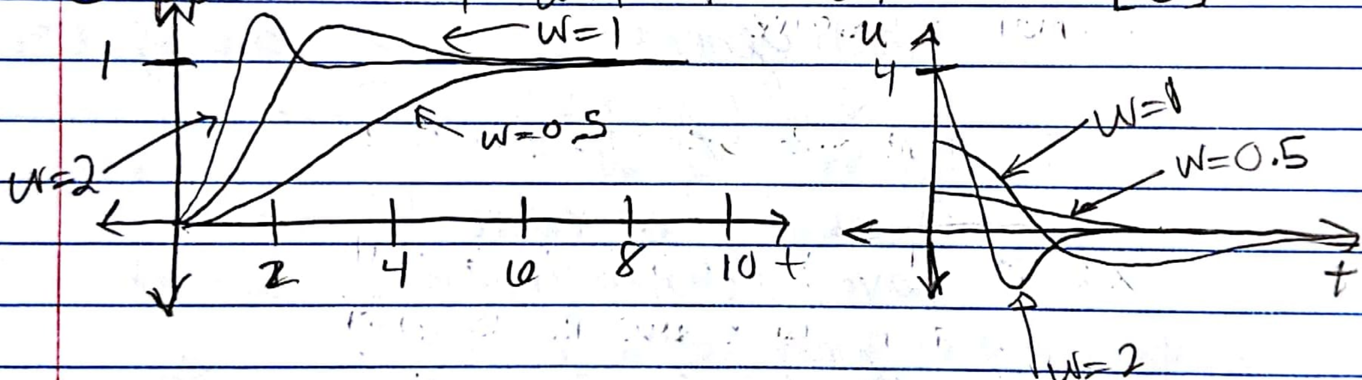
Note the system is called controllable! Suppose we want $(A-BK)$ to have a characteristic polynomial

$$\alpha_c(s) = s^2 + 2\zeta w s + w^2$$

where $\zeta = 0.7$ and $w = 0.5, 1, \& 2$ the real part of the eigenvalues go from -0.35, -0.7, to -1.4

w	$Re(s)$	K
0.5	−0.35	[0.25 0.5750]
1	−0.7	[1 0.9]
2	−1.4	[4 0.8]

Suppose $r=1$ and start at $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Note: 1) Eigenvalues $\to -\infty$, faster convergence, more overshoot, larger magnitude input
2) Does not work for T.V. systems
3) No information about how it performs in a nbhd (e.g. size).
4) no ability to bound inputs
5) no ability to enforce state constraints.