Goal

1) Motivation for Intelligence
2) Convexity
3) Convex Optimization
4) QP based MPC

I. Motivation for Intelligence

	Dynamics	Input/State Constraints	Real-Time	Feedback
Ackermann's Equation	LTI	no	yes	yes
LQR	LTV	no *	yes	yes
QP MPC	LTV	yes *	yes	yes *
traj. optimization	Nonlinear	yes	no	no

<u>Def</u>: An optimization problem is:

$$\min_{z \in Z} J(z) \qquad (P)$$

subject to $g_i(z) \leq 0 \quad \forall i \in \{1, \dots, p_{ie}\}$

$h_i(z) = 0 \quad \forall i \in \{1, \dots, p_e\}$

where Z is the decision space/set,

$J: Z \to \mathbb{R}$ is the cost/objective function

which is minimized over the decision variable $z \in Z$ that satisfy the inequality constraints $g_i : Z \to \mathbb{R}$ and equality constraints $h_i : Z \to \mathbb{R}$.
The set of points that satisfy the constraints ($\{z \in Z \mid g_i(z) \le 0 \ \forall i, h_i(z) = 0, \forall i\}$) are called the feasible set.

Ex: Consider the kinematic steering model:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} = f(\underbrace{x}_{p_x,\ p_y,\ \theta}, \underbrace{u}_{v,\ w})$$

1 $\quad p_x$

$p_x = 0$

$p_y = 0$

$\theta = 0$

$t = 1$

$v \in [-1, 1]$

$w \in [-1, 1]$

Step 1: discretize the dynamics

Use Euler Integration, choose stepsize $h < 1$

$1/h \in \mathbb{N}$, then

$$x(h(k+1)) = x(hk) + h f(x(hk), u(hk))$$

for each $k \in \{0, \dots, 1/h - 1\}$

Step 2: decision variable: $\{\underbrace{u(kh)}_{\in \mathbb{R}^2}\}_{k=0}^{\frac{1}{h}-1}$

$$\{\underbrace{x(kh)}_{\in \mathbb{R}^3}\}_{k=0}^{\frac{1}{h}}$$

Question: What is Z?

$$Z = (\mathbb{R}^2)^{1/h - 1} \times (\mathbb{R}^3)^{\frac{1}{h}}$$

Step 3:

a) Equality Constraints:

i) represent the dynamics discretized via Euler Integration

ii) initial condition

$$x(0 \cdot h) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v(0)$

$w(0)$

$-1 \le v(0) \le 1 \rightarrow v(0) - 1 \le 0$

$-1 - v(0) \le 0$

b) Inequality Constraints

i) ensuring input constraints are satisfied

$$-1 \le u(kh) \le 1 \quad \forall k \in \{0, \ldots, \tfrac{1}{h} - 1\}$$

ii) check obstacle constraint

$$(p_x(kh) - 0.5)^2 + (p_y(kh) - 0.5)^2 - (0.1)^2 \ge 0$$

for each $k \in \{1, \ldots, \frac{1}{h}\}$

Step 4: Cost Function

$$(p_x(1) - 1)^2 + (p_y(1) - 1)^2 + \sum u^2(kh)h$$

$J : Z \to \mathbb{R}$

$\cancel{J\begin{pmatrix} p_x(1) \\ p_y(1) \end{pmatrix} \in \mathbb{R}}$

$$\begin{pmatrix} u(0) \\ u(h) \\ u(2h) \\ \vdots \\ x(0) \\ x(h) \\ \vdots \\ x(1) \end{pmatrix}$$

J

$$Z = \begin{pmatrix} u(0) \in \mathbb{R}^2 \\ x(0) \in \mathbb{R}^3 \\ x(1) \in \mathbb{R}^3 \end{pmatrix} \leftarrow h = 1$$

$$\begin{pmatrix} u(0) = \begin{matrix} v(0) \\ w(0) \end{matrix} \\ x(0) = \begin{matrix} p_x(0) \\ p_y(0) \\ \theta(0) \end{matrix} \\ x(1) = \begin{matrix} p_x(1) \\ p_y(1) \\ \theta(1) \end{matrix} \end{pmatrix}$$

$J(z)$

$$(z(6) - 1)^2 + (z(7) - 1)^2$$

When is it easy to solve?

Our approach is to use derivatives to solve this problem.

<u>Def</u>: a) Let J^* be the optimal value of the optimization problem (P). A global optimizer is a feasible z^* s.t.

$J(z^*) = J^*$.

b) A feasible $\bar{z}$ is a local optimizer to (P) if $\exists R \geq 0$ s.t.

$$J(\bar{z}) = \min_{z \in Z} J(z)$$
$$\text{s.t.} \quad g_i(z) \leq 0 \quad \forall i$$
$$h_i(z) = 0 \quad \forall i$$
$$\|z - \bar{z}\| \leq R$$

When does local = global? $\Rightarrow$ QP-MPC

objectives for today:
1) when is local = global?
2) how to formulate problems where local = global?

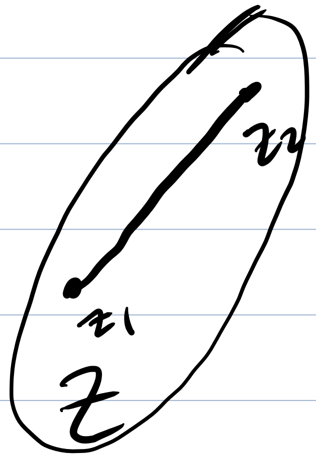
II Convexity

Def: 1) A set $Z \subset \mathbb{R}^n$ is convex if
$$\lambda z_1 + (1-\lambda) z_2 \in Z$$
$$\forall z_1, z_2 \in Z, \lambda \in [0,1].$$
2) A function $f: Z \to \mathbb{R}$ is convex if Z is convex and

$$f(\lambda z_1 + (1-\lambda) z_2) \le \lambda f(z_1) + (1-\lambda) f(z_2)$$

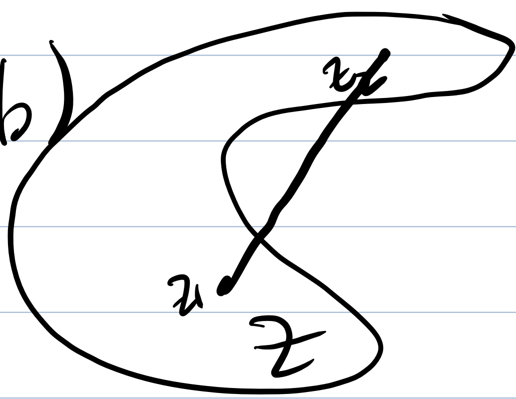
$\forall z_1, z_2 \in Z$ and $\lambda \in [0, 1]$.

Ex: a)



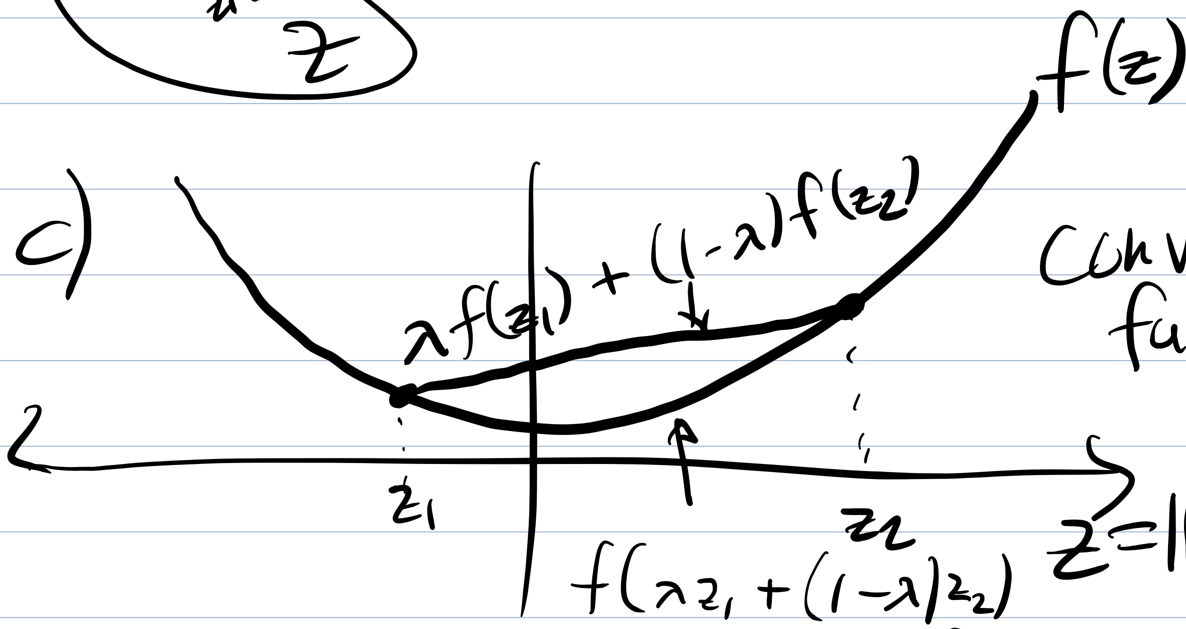
Z is convex

b)



Z is not convex

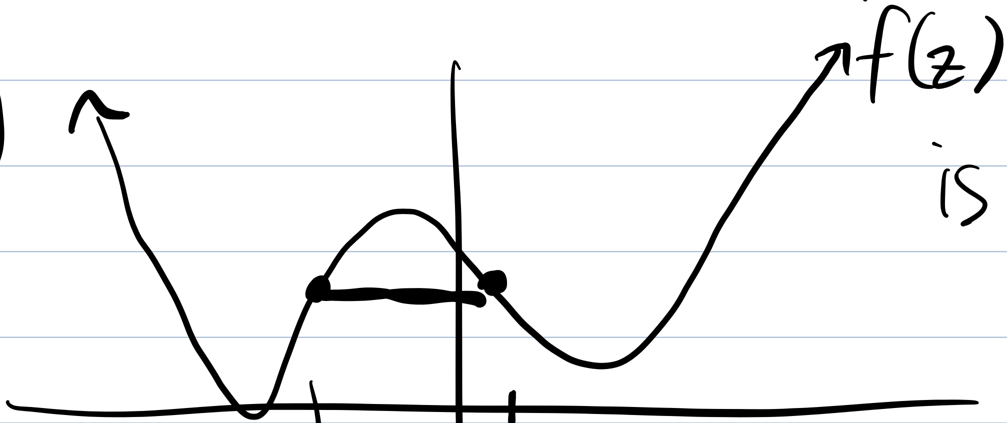
c)



Convex function!

$Z = \mathbb{R}$

d)



is not convex

z_1 | z_2 $\qquad Z=\mathbb{R}$

e) An affine function $f(z)=c'z+d$ is convex

f) A quadratic function $f(z)=z'Qz+2sz+r$ is convex iff $Q\succeq 0$

Theorem: 1) The intersection of an arbitrary number of convex sets is convex.

2) If $f_1,\dots,f_N$ are convex functions, then $\sum_i \alpha_i f_i$ is a convex function for all $\alpha_i \geq 0$.

Def: A convex polytope $P\subset\mathbb{R}^n$ is a set defined as the intersection of a finite number of half spaces:

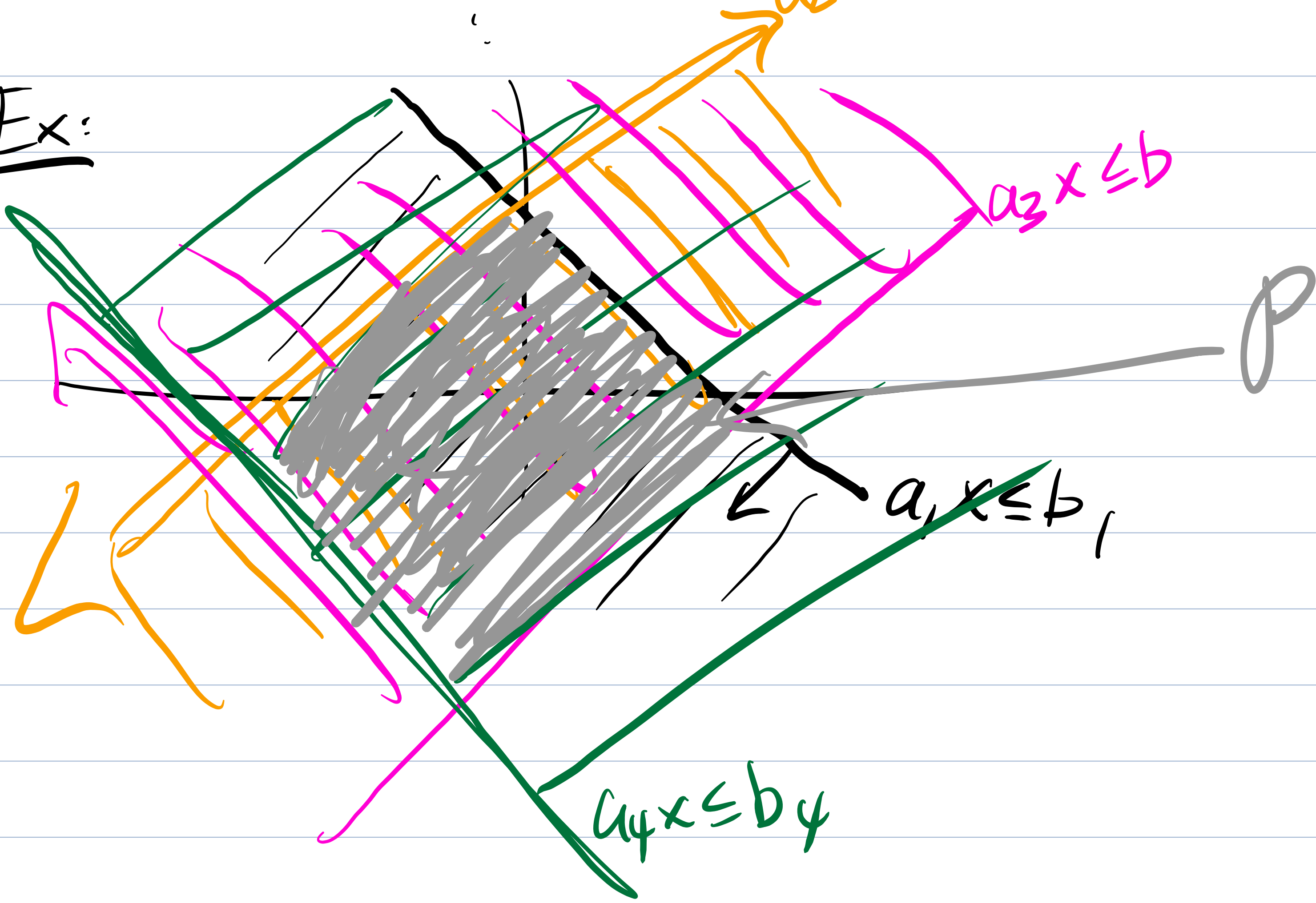
$$P=\{x\in\mathbb{R}^n \mid Ax\leq b\}$$

where $Ax\leq b$ is $\underline{a_i x\leq b_i}$

$$\begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \\ & \vdots & \end{bmatrix}$$

$a_2 x \leq b_2$

Ex:



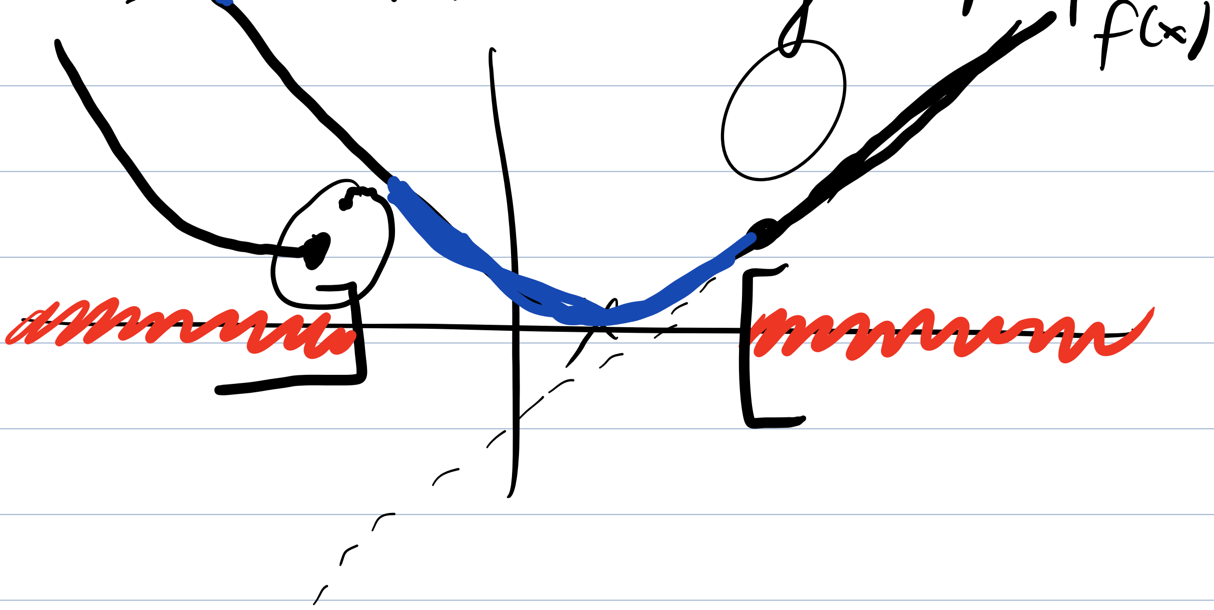
Note: Convex polytopes are convex!

$\mathbb{R}^n$

III. Convex Optimization

Def: An optimization problem is convex if the objective function is convex and the feasible set is convex.

Thm: In convex optimization problems local minima are global optimizers.



$$f(x,y) = (x+y)^2$$

$$x = -y$$

A. Quadratic Programming

Def: A quadratic program ($\nearrow$ QP) is an optimization problem that can be written as:

$$\min_{z \in \mathbb{R}^n} \frac{1}{2} z' Q z + c' z$$

$$\text{s.t. } Az \leq b$$

where Q is symmetric.

Thm: If $Q \succeq 0$ then the quadratic program is convex.

MATLAB: Quadprog

CPLEX, Gurobi, MOSEK

IV QP based Model Predictive Control.

Basic Idea:

1) Rely on fast speed of QP solvers.

2) At each sampling time k, starting at current state, compute an optimal controller at a sequence of time steps over a finite horizon, $[k, k+K]$

3) Use the solution for the first time instance $[k, k+1]$

4) Repeat step 2 at the new location $k = k+1$

Note: 1) We have LTV system discretized:

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

2) Let $u_{k+k'|k}$ be the input at time step $k+k'$ computed at time step k.

3) Given $\{u_{k+k'|k}\}_{k'=0}^{K-1}$ let $x_{k+k'|k}$ be the state at time step $k+k'$.

Suppose I have dynamics in (1) and I have constraints such that $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}$ $\forall k$ where $\mathcal{X}, \mathcal{U}$ are convex polytopes

$$\min_{\{u_{k+k'|k}\}_{k'=0}^{K-1},\ \{x_{k+k'|k}\}_{k'=0}^{K}} \quad (QP)$$

$$\sum_{k'=0}^{K} x_{k+k'|k}^T Q(k') x_{k+k'|k} + \sum_{k'=0}^{K-1} u_{k+k'|k}^T R(k') u_{k+k'|k}$$

$$x_{k+k'+1|k} = A(k+k') x_{k+k'|k} + B(k+k') u_{k+k'|k}$$

$\forall k' \in \{0, \dots, \bar{K}-1\}$

$A_u u_{k+k'|k} \le b_u$

$u_{k+k'|k} \in \mathcal{U} \quad \forall k'$

$x_{k+k'|k} \in \mathcal{X} \quad \forall k'$

Algorithm for MPC

1) Measure the state $x_{k|k}$ at time k
2) Solve the QP
3) Apply $u_{k|k}$ at time step k
4) Wait for new sampling time $k+1$, go to step 1.