Goals

1) MPC
2) Trajectory Optimization
3) Summary of the State of the Art

I. MPC → Convex QP

1) Convert LTV continuous dynamical system into a discrete LTV dynamical system.

$$\dot{x} = f(x, u)$$

$$\Big\downarrow \text{Linearize About Traj}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$\Big\downarrow \begin{array}{l}\text{LTV discrete time system}\\ \text{Euler Int}\end{array}$$

$$x(h(k+1)) = x(hk) + h \cdot \dot{x}(hk)$$

$$x(h(k+1)) = x(hk) + h\big(A(hk)x(hk) + B(hk)u(hk)\big)$$

$$\downarrow$$

$$x(\cancel{h}(k+1)) = \tilde{A}(k)\,x(\cancel{h}k) + \tilde{B}(k)\,u(\cancel{h}k)$$

$$\boxed{\tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k) + \tilde{B}(k)\,\tilde{u}(k)}$$

1) $u_{k+k'|k}$ this is the input computed at time instance k that is to be applied into the system at $k+k'$

2) $\{u_{k+k'|k}\}_{k'=0}^{N-1}$ this is the MPC time horizon

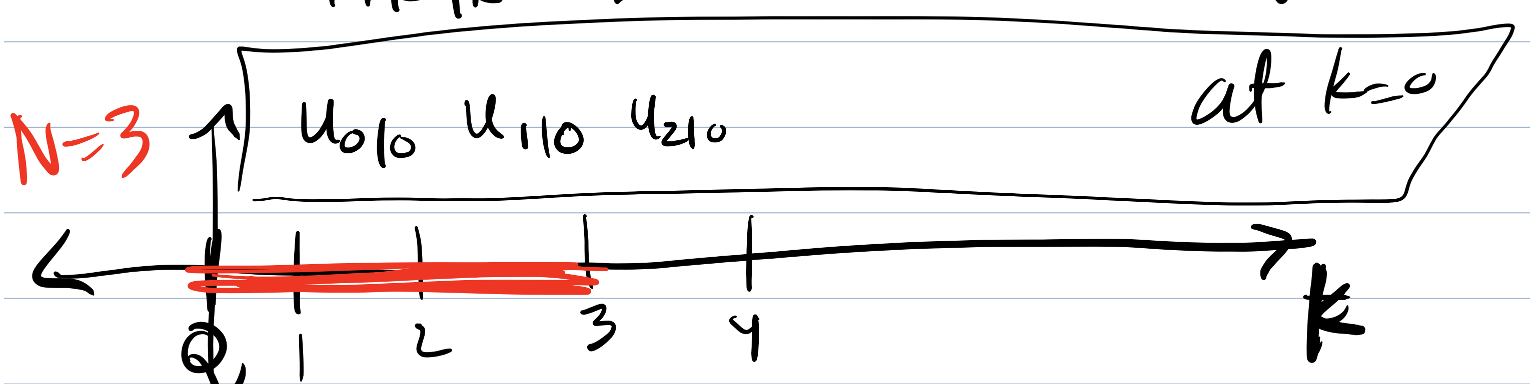
$\{x_{k+k'|k}\}_{k'=0}^{N}$

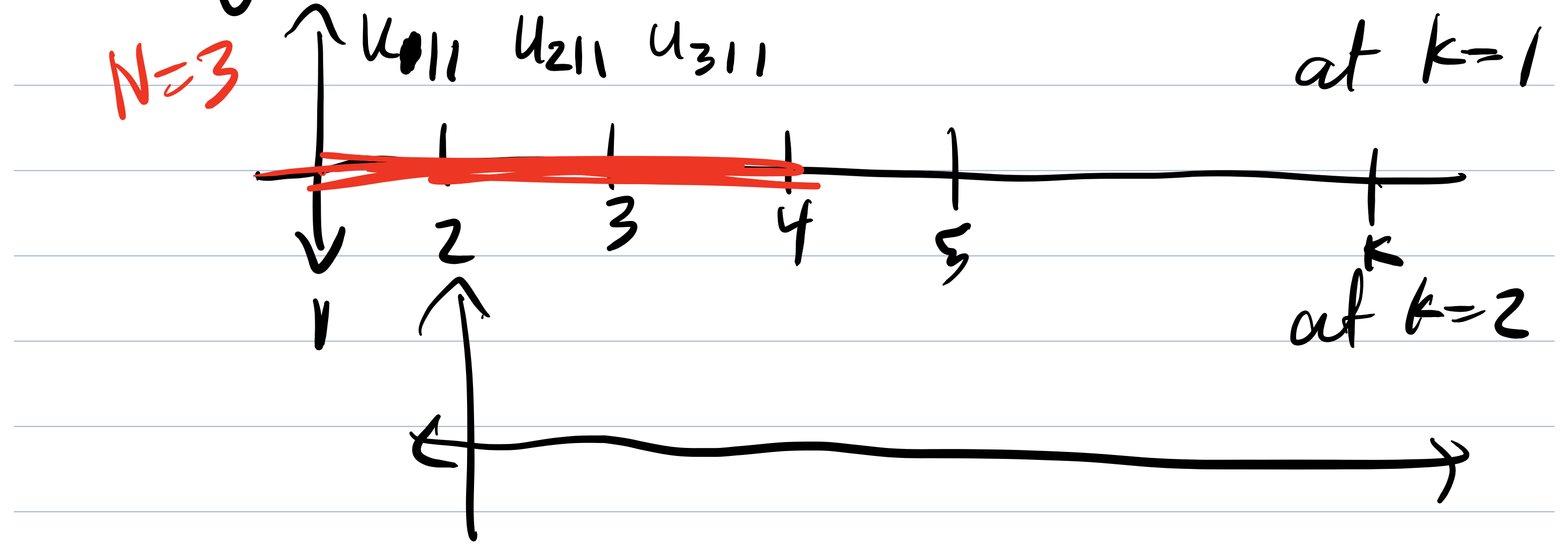
At time instance k we solve the following convex QP:

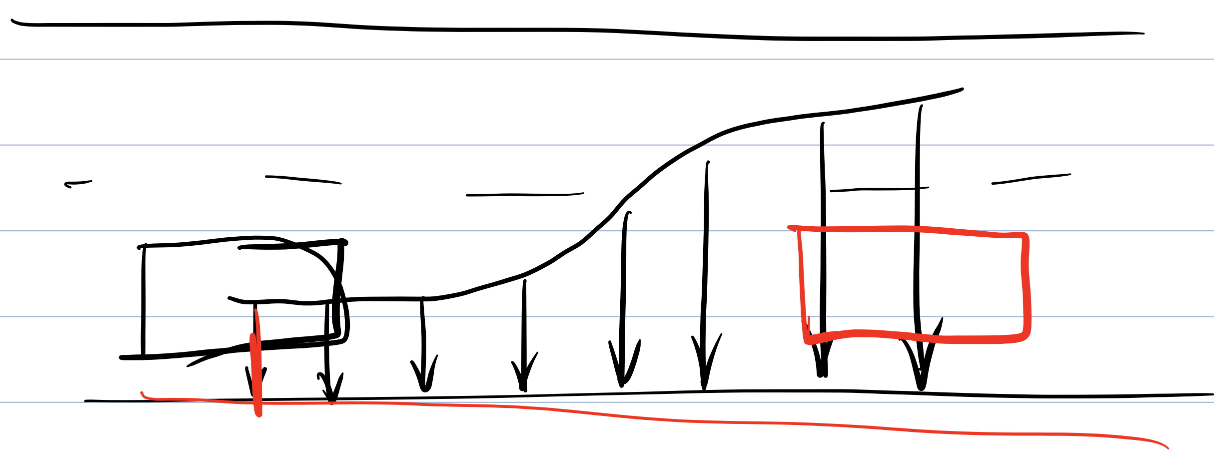
$$\min_{\{u_{k+k'|k}\}_{k'=0}^{N-1},\ \{x_{k+k'|k}\}_{k'=0}^{N}} \sum_{k'=0}^{N} x_{k+k'|k}^T Q(k') x_{k+k'|k} + \sum_{k'=0}^{N-1} u_{k+k'|k}^T R(k') u_{k+k'|k}$$

$$x_{k+k'+1|k} = A(k+k')x_{k+k'|k} + B(k+k')u_{k+k'|k}$$
$$\forall k' = \{0, \dots, N-1\}$$

$$u_{k+k'|k} \in \mathcal{U} \qquad \forall k' \in \{0, \dots, N-1\}$$
$$x_{k+k'|k} \in \mathcal{X} \qquad \forall k' \in \{0, \dots, N\}$$







II. Trajectory Optimization

Def: A trajectory optimization problem is

$$\min_{u:[0,T]\to\mathbb{R}^m} \int_0^T L(t,x(t),u(t))\,dt + \phi(x(T))$$

Subject to

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \forall t\in[0,T]$$

$$x(0) = x_0$$

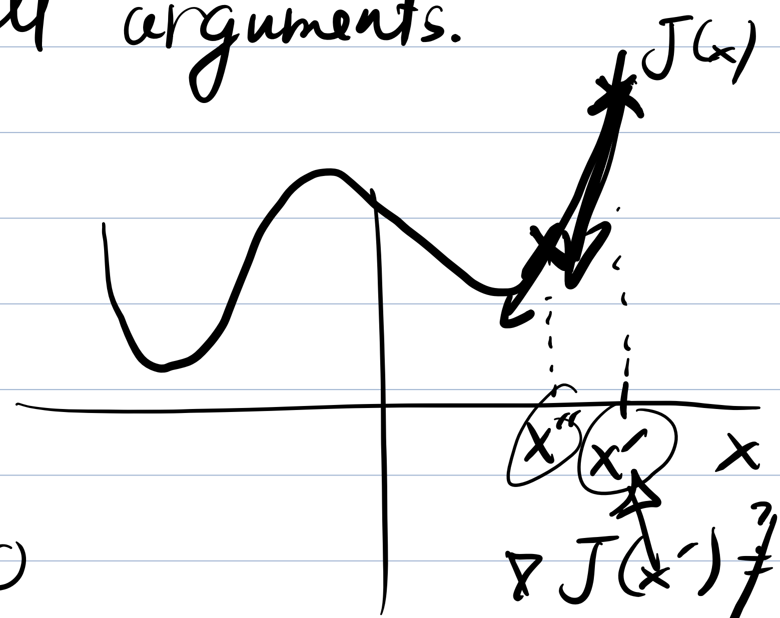
$$g_i(x(t)) \le 0 \quad \forall t\in[0,T], \ \forall i\in\{1,\dots,p_{ie}\}$$

$L: [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ is the running cost.

$\phi: \mathbb{R}^n\to\mathbb{R}$ is the terminal/final cost

$g_i: \mathbb{R}^n\to\mathbb{R}$ the inequality constraints.

L, ϕ, g_i, f are all twice continuously differentiable in all arguments.



$$\min_{x\in X} J(x)$$

$$\downarrow$$

$$\nabla J(x) \stackrel{?}{=} 0$$

$$\nabla J(x') \stackrel{?}{\neq} 0$$

$$\nabla J(x'') \stackrel{?}{=} 0$$

What do people do for traj opt solving?

Option 1: $\min_{u} J(u)$

$$\downarrow$$

$$\boxed{\nabla J(u)}$$ Indirect Methods

Option 2: $\min_{u} J(u)$

$$\Big\downarrow \text{ discretize time}$$

$$\min_{\{x(hk)\},\{u(hk)\}} J(\{u(hk)\})$$

Treat traj opt as a discrete time system via Euler Integration (assume $\frac{1}{\Delta T} \in \mathbb{N}$)

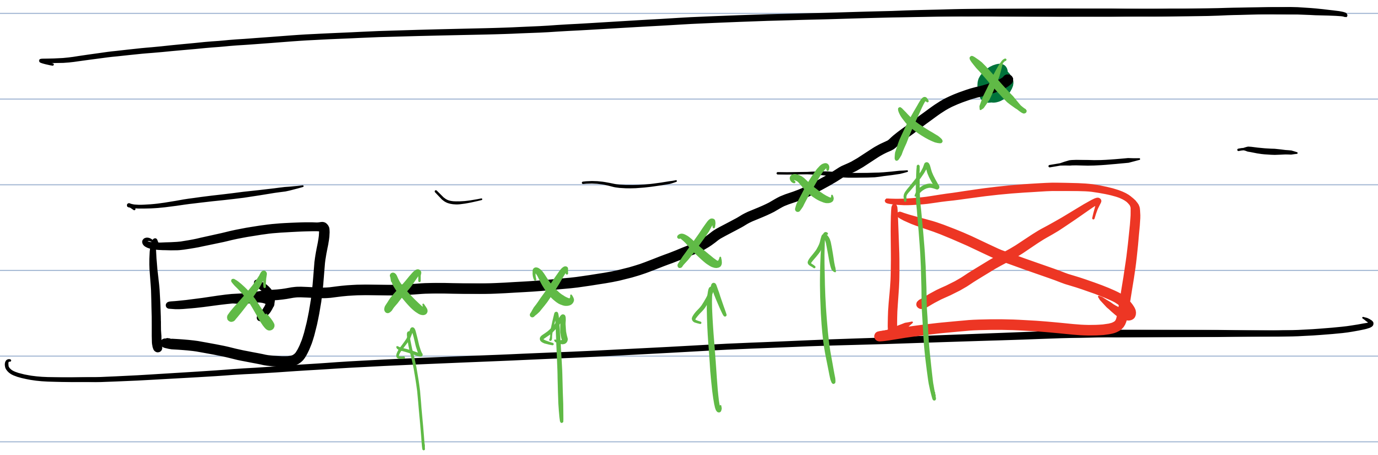
$$\min_{\substack{\{u(\Delta T k)\}_{k=0}^{\frac{1}{\Delta T}-1} \\ \{x(\Delta T k)\}_{k=0}^{\frac{1}{\Delta T}}}} \Delta T \sum_{k=0}^{\frac{1}{\Delta T}-1} L(\Delta T k, x(\Delta T k), u(\Delta T k)) + \varphi\left(x\left(\tfrac{1}{\Delta T}\right)\right)$$

subject to $x(\Delta T(k+1)) = x(\Delta T k) + \Delta T f(\Delta T k, x(\Delta T k), u(\Delta T k))$

$$\forall k \in \left\{0, \dots, \tfrac{1}{\Delta T} - 1\right\}$$

$x(0) = x_0$

$$g_i(x(\Delta T k)) \le 0 \quad \forall k \in \{0, \dots, \tfrac{1}{\Delta T}\}$$
$$\forall i \in \{1, \dots, p_{ie}\}$$



B. Solving Nonlinear Programs

Recall

$$\min_{z \in Z} \; J(z)$$
$$g_i(z) \le 0 \quad \forall i \in \{1, \dots, p_{ie}\}$$
$$h_i(z) = 0 \quad \forall i \in \{1, \dots, p_{e}\}$$

Note: 1) What is z?

$$z = [x(0), x(\Delta T \cdot 1), \dots, x(\tfrac{1}{\Delta T}), u(0), u(\Delta T) \dots u(\tfrac{1}{\Delta T} - 1)]$$

What is Z?

2) What is gi and hi?

Sidenote:

the i.c. to the optimizer

z = fmincon(cost, z_0, A, b, Aeq, beq, lb, ub, nonlcon, options)

$$\min_{z} \ cost(z)$$

s.t. $Az \le b$

$A_{eq} z = b_{eq}$

$lb \le z \le ub$

nonlcon(z)

1) $[J(z), \nabla J(z)] = cost(z)$

$$J(z) = z^3$$

$$z = 2$$

$$8 = 2^3$$

$$\nabla J(z) = 3z^2 \Big|_{z=2} = 12$$

Ex: $x(0), x(1)$

$u(0)$

$$L = 0 \qquad \phi(x(1)) = \underline{x(1)^2}$$

$$z = [x(0)\ x(1)\ u(0)]$$

$$J(z) = z_2^2$$

$$\frac{dJ}{dz}\Big|_z = 2z_2\Big|_z$$

2) $[\tilde{g}, h, \nabla g, \nabla h] = \text{nonlcon}(z)$

$$(\tilde{g}(i))(z) = g_i(z) \overset{?}{\leq} 0$$

$$(h(i))(z) = h_i(z) \overset{?}{=} 0$$

$$[\nabla g]_{ij} = \frac{\partial g_i}{\partial z_j}(z)$$

$$[\nabla h]_{ij} = \frac{dh_i}{dz_j}(z)$$

Ex: $x(1) = x(0) + 1 \cdot f(0, x(0), u(0))$

$$h_1(z) = x(1) - x(0) - f(0, x(0), u(0)) = 0$$

nonlcon(z)

$$z = [x(0) \;\; x(1) \;\; u(0)]$$

$$h_1(z) = \boxed{z_2 - z_1 - f(0, z_1, z_3)}$$

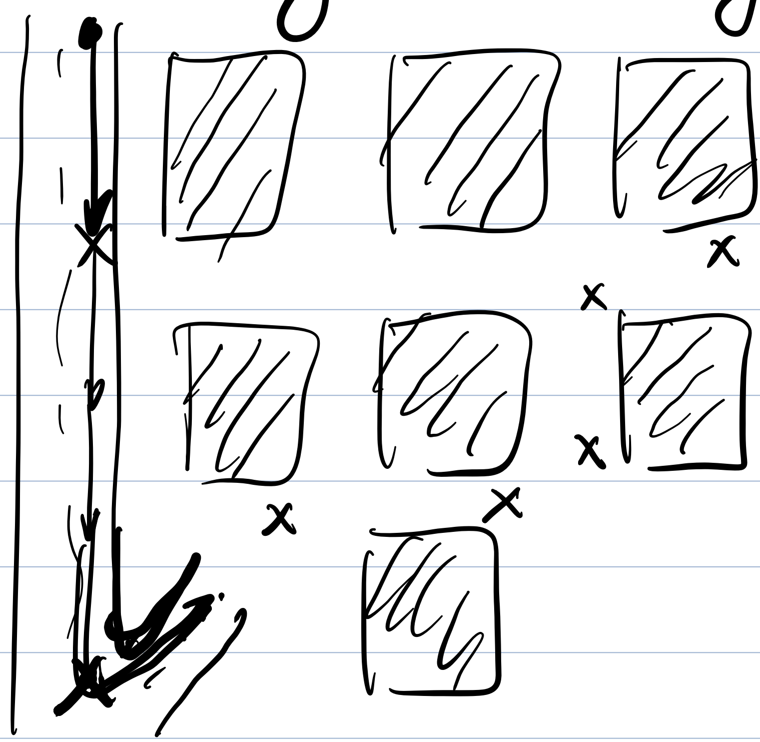
$$\frac{dh_1}{dz} = \left[\frac{dh_1}{dz_1} \quad \frac{dh_1}{dz_2} \quad \frac{dh_1}{dz_3} \right]$$

for $i = 1$: Peq

for $j = 1$: #z

3) options = optimoptions('fmincon', ...
'SpecifyConstraintGradient', true, ...
'SpecifyObjectiveGradient', true)

III. Putting Pieces Together



Assume I've computed a family of trajectories
{spdup, exit traj

IV. Frontiers

1) Solve nonlinear programs fast

2) Deal w/ discretization