HOMEWORK 1

Numerical Analysis II - 2022

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17]:
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using QuadGK, Calculus, Roots, LaTeXStrings, PlotlyJS, Images
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1. T

Show that the space $X = \{u \in C^1([-1,1]) : u(-1) = u(1) = 0\}$, equipped with the norm

$$||u||_X^2 = \int_0^1 (u(x)^2 + u'(x)^2)\, dx$$

is not a Banach space. (Hint: Prove that given the sequence $u_n(x) = \sqrt{1+\frac{1}{n}} - \sqrt{x^2+\frac{1}{n}}$, $x \in [-1,1]$, converges to a function $u \notin X$

$X = \{u \in C^1[-1,1] : u(-1) = u(1) = 0\}$

$$||u||_x^2 = \int_0^1 (u(x))^2 + (u'(x))^2 dx$$

We are going to show that the limit of the sequence $u_n(x) = \sqrt{1+\frac{1}{n}} - \sqrt{x^2+\frac{1}{n}}$ is not in X $\Big($that is, it does not converge in $X\Big)$, even though the sequence is Cauchy in X.

Solution

Let $\epsilon > 0$ be, we are going to find $N \in \mathbb{N}$ such that for every $n, m > N \quad \Rightarrow \quad ||u_n(x) - u_m(x)||_X < \epsilon$

Suppose $m > n \Rightarrow \frac{1}{m} < \frac{1}{n} \Rightarrow x^2 + \frac{1}{m} < x^2 + \frac{1}{n} \Rightarrow \sqrt{x^2+\frac{1}{m}} < \sqrt{x^2+\frac{1}{n}}$ and similarly $\sqrt{1+\frac{1}{m}} < \sqrt{1+\frac{1}{n}}$ then :

$$||u_n(x) - u_m(x)||_x < \left\|\left(\sqrt{1+\frac{1}{n}} - \sqrt{x^2+\frac{1}{n}}\right) - \left(\sqrt{1+\frac{1}{m}} - \sqrt{x^2+\frac{1}{m}}\right)\right\|_X$$

$$< \left\|\sqrt{1+\frac{1}{n}} - \sqrt{x^2+\frac{1}{n}}\right\|_X < \left\|\sqrt{1+\frac{1}{n}}\right\|_X$$

since $x \in [-1,1]$

$$\left\|\sqrt{1+\frac{1}{n}}\right\|_X = \left(\int_0^1 \left(\sqrt{1+\frac{1}{n}}\right)^2 + (0)^2) dx\right)^{1/2} = \left(\int_0^1 \left(\sqrt{1+\frac{1}{n}}\right)^2 dx\right)^{1/2}$$

$$= \left(\int_0^1 \left(1+\frac{1}{n}\right) dx\right)^{1/2} = \left(1+\frac{1}{n}\right)^{1/2}$$

Then,

$$\left(1+\frac{1}{n}\right)^{1/2} < \left(1+\frac{1}{N}\right)^{1/2} < \epsilon$$

$$1 + \frac{1}{N} < \epsilon^2 \quad \leftrightarrow \quad \frac{1}{N} < \epsilon^2 - 1 \quad \implies \quad N > \frac{1}{\epsilon^2 - 1}$$

Taking $N > \frac{1}{\epsilon^2-1}$ we have proved that $u_n(x)$ is the cauchy and convergent and $\lim_{n\to\infty}(u_n(x)) = 1 - |x| = u(x)$ but $u(x)$ does not converge in X because its firts derivate is not continuous in $[-1,1]$.

Hence, X is not complete and therefore X is not a Banach space.

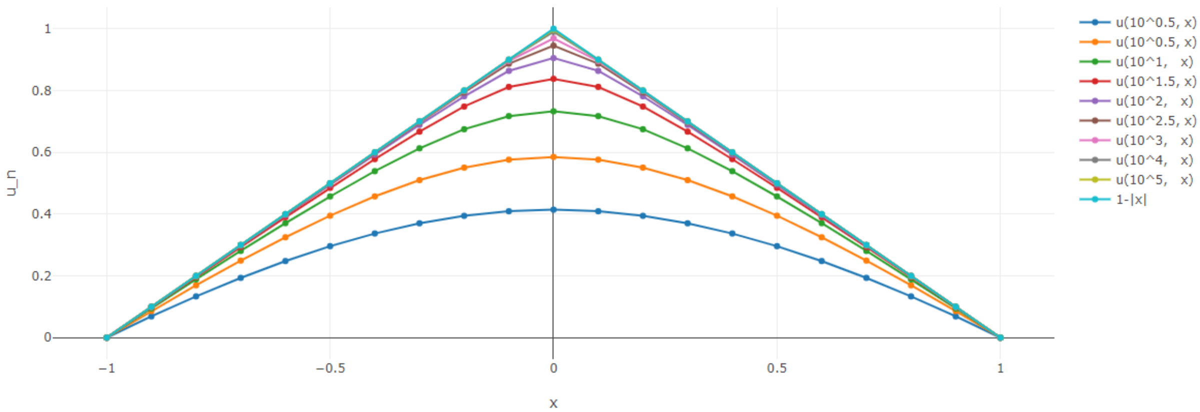
[69]:
```
x = range(-1, 1, step=0.1) |> collect
u(n,x) = sqrt(1+1/n) -sqrt(x^2+1/n)

pp0 = scatter(;x=x, y=u.(10^0,  x),  mode="lines+markers", name="u(10^0.5, x)")
pp1 = scatter(;x=x, y=u.(10^0.5,x),  mode="lines+markers", name="u(10^0.5, x)")
pp2 = scatter(;x=x, y=u.(10^1,  x),  mode="lines+markers", name="u(10^1,   x)")
pp3 = scatter(;x=x, y=u.(10^1.5,x),  mode="lines+markers", name="u(10^1.5, x)")
pp4 = scatter(;x=x, y=u.(10^2,  x),  mode="lines+markers", name="u(10^2,   x)")
pp5 = scatter(;x=x, y=u.(10^2.5,x),  mode="lines+markers", name="u(10^2.5, x)")
pp6 = scatter(;x=x, y=u.(10^3,  x),  mode="lines+markers", name="u(10^3,   x)")
pp7 = scatter(;x=x, y=u.(10^4,  x),  mode="lines+markers", name="u(10^4,   x)")
pp8 = scatter(;x=x, y=u.(10^5,  x),  mode="lines+markers", name="u(10^5,   x)")
ppx = scatter(;x=x, y=-abs.(x).+1,  mode="lines+markers", name="1-|x|")

plots = [pp0, pp1, pp2, pp3, pp4, pp5, pp6, pp7, pp8, ppx]
pp1 = plot(plots, Layout(title="Un Sequence",  xaxis_title="x",  yaxis_title="u_n"))
```

[69]:

Un Sequence



2. T

Let $u(x)$ and $v(x)$ two functions defined as follows

$$u(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 \leq x \leq 2 \end{cases} \quad y \quad v(x) = \sin \pi x$$

with their derivatives

$$u'(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ -1 & \text{if } 1 \leq x \leq 2 \end{cases} \quad y \quad v'(x) = \pi \cos \pi x$$

[2]:

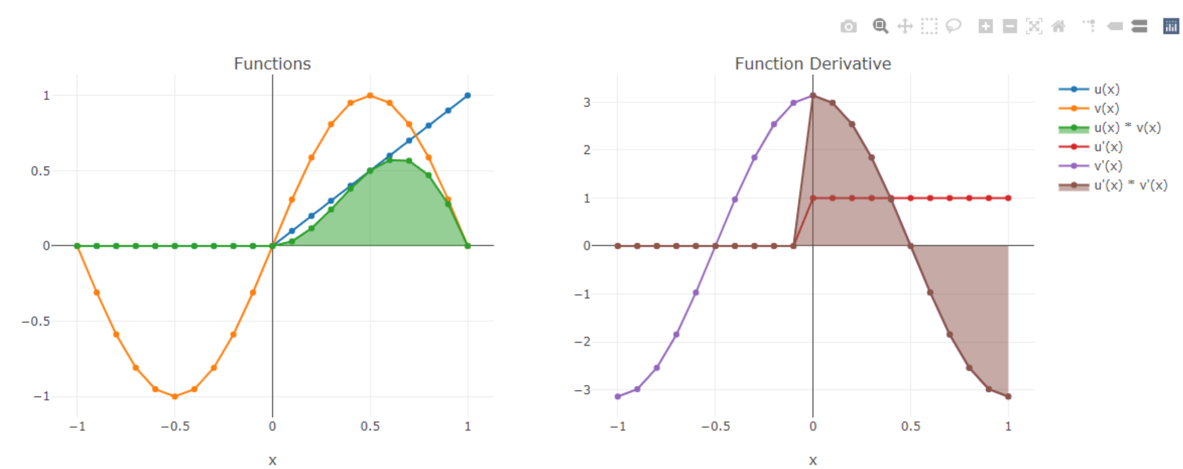
```
function u(x)
    if 0 <= x <=1
        return x
    elseif 1 < x <=2
        return 2 - x
    else
        return 0
    end
end

function up(x)
    if 0 <= x <=1
        return 1
    elseif 1 < x <=2
        return -1
    else
        return 0
    end
end

v(x)  = sin(pi * x)
vp(x) = pi * cos(pi * x)

x = range(0, 2, step=0.1) |> collect;
```

[55]:



Orthogonality

- $L^2(0,2)$ Space

Norm of L^2 and its norm $||u||_{L^2}$

$$(u,v) = \int_0^2 u(x)v(x)dx = \int_0^1 x\ \sin(\pi x)dx + \int_1^2 (2-x)\ \sin(\pi x)dx$$

Integranting by steps with the sustitution $w = x$ and $dz = \sin(\pi x)\ dx$

$$(u,v)_{L^2} = \frac{1}{\pi}x\cos(\pi x)\Big|_0^1 - \frac{1}{\pi}\int_0^1 \cos(\pi x)dx + \frac{1}{\pi}(2-x)\cos(\pi x)\Big|_1^2 - \frac{1}{\pi}\int_1^2 \cos(\pi x)dx$$
$$(u,v)_{L^2} = \frac{1}{\pi}\cos(\pi) - \frac{1}{\pi^2}\sin(\pi x)\Big|_0^1 - \frac{1}{\pi}\cos(\pi) - \frac{1}{\pi^2}\sin(\pi x)\Big|_0^1$$
$$(u,v)_{L^2} = \frac{1}{\pi}\cos(\pi) - \frac{1}{\pi}\cos(\pi) - \frac{1}{\pi^2}(\sin(\pi) - \sin(0)) - \frac{1}{\pi^2}(\sin(2\pi) - \sin(\pi)) = \frac{1}{\pi} - \frac{1}{\pi} - (0-0) - (0-0)$$
$$\implies \quad \boxed{(u,v)_{L^2} = 0}$$

The function u and v are orthogonals in $L^2(0,2)$.

[4]:
```
integral, err = quadgk(x -> u(x)*v(x), 0, 2, rtol=1e-8)
```

[4]:
```
(2.0816681711721685e-17, 0.0)
```

- $H^1(0,2)$ Space

Now, let's check if u and v are orthogonal in $H^1(0,2)$ usign the its norm and their derivatives

$$(u,v)_{H^1} = \int_0^2 u(x)v(x)dx + \int_0^2 u'(x)v'(x)dx = 0 + \int_0^1 u'(x)v'(x)dx + \int_1^2 u'(x)v'(x)dx$$
$$(u,v)_{H^1} = \pi\int_0^1 1\ \cos(\pi x)dx + \pi\int_1^2 (-1)\ \cos(\pi x)dx = \sin(\pi x)\Big|_0^1 - \sin(\pi x)\Big|_1^2$$
$$\implies \quad \boxed{(u,v)_{H^1} = 0}$$

Then, the functions u and v are orthogonal on $H^1(0,2)$.

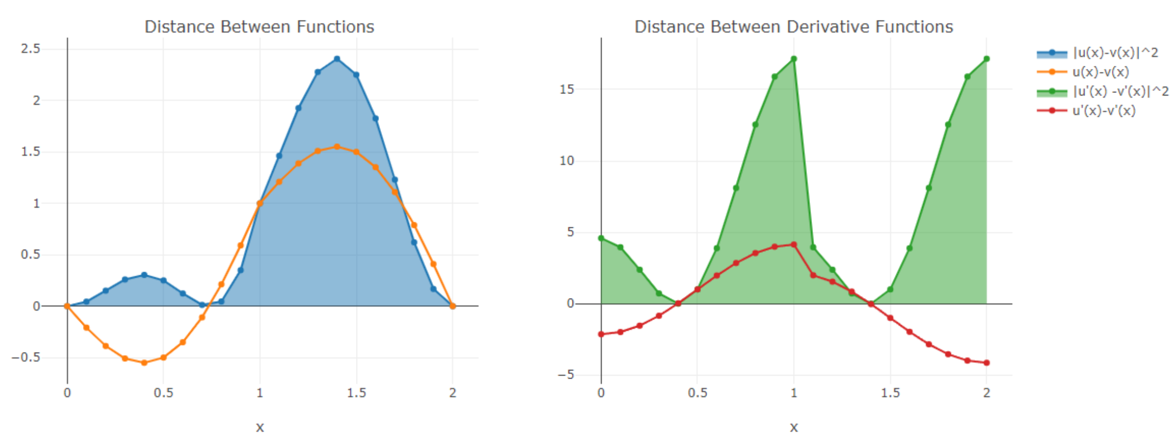
[5]:
```
integral, err = quadgk(x -> up(x)*vp(x), 0, 2, rtol=1e-8)
```

[5]:
```
(-4.105099355129983e-15, 3.7515060282571523e-19)
```

Distance Between u and v

Let's calculate the distance between u and v in both spaces $L^2(0,2)$ and $H^1(0,2)$.

[6]:



[55]:
```
plot_u  = scatter(;x=x, y=u.(x), mode="lines+markers", name="u(x)")
plot_v  = scatter(;x=x, y=v.(x), mode="lines+markers", name="v(x)")
plot_uv = scatter(;x=x, y=v.(x).*u.(x), mode="lines+markers", fill="tozeroy", name="u(x) * v(x)")

plot_up = scatter(;x=x, y=up.(x), mode="lines+markers", name="u'(x)")
plot_vp = scatter(;x=x, y=vp.(x), mode="lines+markers", name="v'(x)")
plot_uvp = scatter(;x=x, y=vp.(x).*up.(x), mode="lines+markers", fill="tozeroy", name="u'(x) * v'(x)")

p1 = plot([plot_u, plot_v,plot_uv], Layout(title="Functions",  xaxis_title="x"))
p2 = plot([plot_up, plot_vp,plot_uvp], Layout(title="Function Derivative",  xaxis_title="x"))

p12 = [p1 p2]
#relayout!(p, height=300, width=700, title_text="Functions", legend_title_text="Legend")
#p
```

- $L^2(0,2)$ Space

$$||u-v||_{L^2}^2 = \int_0^2 |u-v|^2 dx = \int_0^2 (u-v)^2 dx$$

$$||u-v||_{L^2}^2 = \int_0^1 (x-\sin(\pi x))^2 dx + \int_1^2 (2-x-\sin(\pi x))^2 dx$$

$$||u-v||_{L^2}^2 = \int_0^1 \left(x^2 - 2x\sin(\pi x) + \sin^2(\pi x)\right) dx + \int_1^2 \left(4 + x^2 + \sin^2(\pi x) - 4x - 4\sin(\pi x) + 2x\sin(\pi x)\right) dx$$

Using integral tables

$$||u-v||_{L^2}^2 = \left(\frac{1}{3}x^3 - \frac{2}{\pi^2}\sin(\pi x) + \frac{2}{\pi}x\cos(\pi x) + \frac{x}{2} - \frac{1}{4\pi}\sin(2\pi x)\right)\Big|_0^1 + \left(4x + \frac{1}{3}x^3 + \frac{x}{2} - \frac{1}{4\pi}\sin(2\pi x) - 2x^2 + \frac{4}{\pi}\cos(\pi x) + \frac{2}{\pi^2}\sin(\pi x) - \frac{2}{\pi}x\cos(\pi x)\right)\Big|_1^2$$

$$||u-v||_{L^2}^2 = \left(\frac{1^3}{3} + \frac{2}{\pi}\cos(\pi) + \frac{1}{2}\right) + \left(4(2) + \frac{2^3}{3} + \frac{2}{2} - (2)2^2 + \frac{4}{\pi}\cos(2\pi) - \frac{4}{\pi}\cos(2\pi)\right) - \left(4 + \frac{1^3}{3} + \frac{1}{2} - (2)1^2 + \frac{4}{\pi}\cos(\pi) - \frac{2}{\pi}\cos(\pi)\right)$$

$$||u-v||_{L^2}^2 = \left(\frac{1}{3} - \frac{2}{\pi} + \frac{1}{2}\right) + \left(8 + \frac{2^3}{3} + 1 - 8\right) - \left(4 + \frac{1}{3} + \frac{1}{2} - 2 - \frac{4}{\pi} + \frac{2}{\pi}\right)$$

$$||u-v||_{L^2}^2 = \left(\frac{5}{6} - \frac{2}{\pi}\right) + \left(\frac{5}{6} - \frac{2}{\pi}\right) = \frac{5}{3} \qquad \Longrightarrow \qquad \boxed{||u-v||_{L^2}^2 = 1.67}$$

[7]:
```
integral1, err = quadgk(x -> broadcast(abs, u(x)-v(x))^2, 0, 2, rtol=1e-8)
```

[7]:
```
(1.6666666666666665, 8.316655142337481e-9)
```

- $H^1(0,2)$ Space

$$||u-v||_{H^1}^2 = ||u-v||_{L^2}^2 + ||(u-v)'||_{L^2}^2 = \int_0^2 |u-v|^2 dx + \int_0^2 |(u-v)'|^2 dx$$

The first integral is already done, the second is

$$||(u-v)'||_{L^2}^2 = \int_0^2 |(u-v)'|^2 dx = \int_0^1 (1-\pi\cos(\pi x))^2 dx + \int_1^2 (-1-\pi\cos(\pi x))^2 dx$$

$$\int_0^2 |(u-v)'|^2 dx = \int_0^1 \left(1 - 2\pi\cos(\pi x) + \pi^2\cos^2(\pi x)\right) dx + \int_1^2 \left(1 + 2\pi\cos(\pi x) + \pi^2\cos^2(\pi x)\right) dx$$

$$||(u-v)'||_{L^2}^2 = \int_0^2 |(u-v)'|^2 dx = \left(x - 2\sin(\pi x) + \frac{\pi^2}{2}x + \frac{\pi}{4}\sin(2\pi x)\right)\Big|_0^1 + \left(x + 2\sin(\pi x) + \frac{\pi^2}{2}x + \frac{\pi}{4}\sin(2\pi x)\right)\Big|_1^2$$

$$||(u-v)'||_{L^2}^2 = \int_0^2 |(u-v)'|^2 dx = \left(1 + \frac{\pi^2}{2}\right) + \left(1 + \frac{\pi^2}{2}\right) = \pi^2 + 2 \qquad \Longrightarrow \qquad \boxed{||(u-v)'||_{L^2}^2 = 11.8696}$$

[8]:
```
integral2, err = quadgk(x -> broadcast(abs, up(x)-vp(x))^2, 0, 2, rtol=1e-8)
```

[8]:
```
(11.869604401089356, 8.208209756332963e-8)
```

Finally, the distance between u and v in $H^1(0,2)$ is

$$||u-v||_{H^1}^2 = \frac{5}{3} + \pi^2 + 2 = \frac{11}{3} + \pi^2 \qquad \Longrightarrow \qquad \boxed{||u-v||_{H^1} = 3.6792}$$

[9]:
```
umvH = sqrt(integral1 + integral2)
```

[9]:
```
3.6791671704009348
```

3. T

Using the functions u and $u-v$ from the previous item, verify the Cauchy-Schwarz inequality for $L^2(0,2)$ and $H^1(0,2)$.

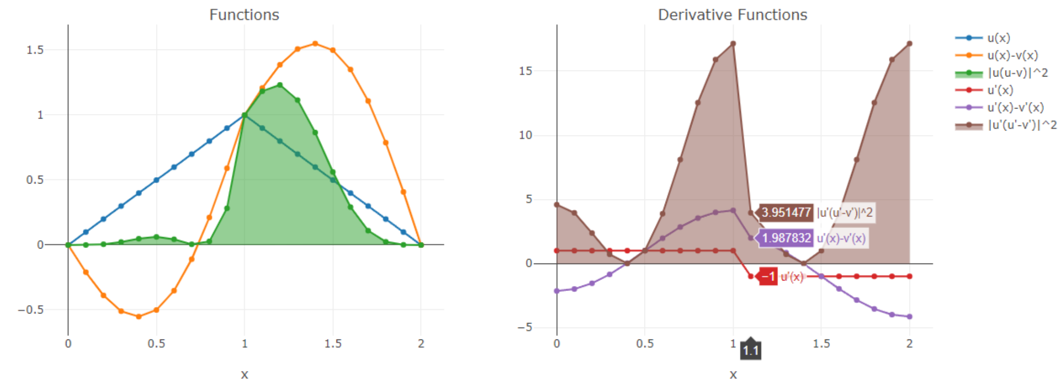
[10]:
```
plot_umumv2    = scatter(;x=x, y=broadcast(abs, u.(x).*(u.(x) - v.(x))).^2, mode="lines+markers", fill="tozeroy", name="|u(u-v)|^2")
plot_upmupmvp2 = scatter(;x=x, y=broadcast(abs, up.(x).*(up.(x) - vp.(x))).^2, mode="lines+markers", fill="tozeroy", name="|u'(u'-v')|^2")

p5 = plot([plot_u, plot_umv, plot_umumv2], Layout(title="Functions",  xaxis_title="x"))
p6 = plot([plot_up, plot_upmvp, plot_upmupmvp2], Layout(title="Derivative Functions",  xaxis_title="x"))

p = [p5 p6]
#relayout!(p, height=300, width=700, title_text="Functions", legend_title_text="Legend")
#p
```

[10]:

[10]:



Solution

Now, let's verify the Cauchy-Schwarz inequality in the space $L^2(0,2)$ for the previous functions, u and $u-v$

$$||(u, u-v)||_{L^2} \leq ||u||_{L^2} \cdot |u-v||_{L^2}$$

Notice that only the last term is calculated. First let's calculate the norm of the inner product between u and $u-v$

$$||(u, u-v)||_{L^2}^2 = \int_0^2 |u(u-v)|^2 dx = \int_0^1 (x(x-\sin(\pi x)))^2 dx + \int_1^2 ((2-x)(2-x-\sin(\pi x)))^2 dx$$

$$||(u, u-v)||_{L^2}^2 = \int_0^1 \left(x^4 + x^2\sin^2(\pi x) - 2x^3\sin(\pi x)\right) dx + \int_1^2 \left((2-x)^4 + (2-x)^2\sin^2(\pi x) - 2(2-x)^3\sin(\pi x)\right) dx$$

$$||(u, u-v)||_{L^2}^2 = \int_0^1 x^4 dx + \frac{1}{2}\int_0^1 x^2\left(1-\cos(2\pi x)\right) dx - 2\int_0^1 x^3\sin(\pi x)dx + \int_1^2 (2-x)^4 dx + \frac{1}{2}\int_1^2 (2-x)^2\left(1-\cos(2\pi x)\right) dx - 2\int_1^2 (2-x)^3\sin(\pi x)dx$$

$$||(u, u-v)||_{L^2}^2 = \int_0^1 x^4 dx + \frac{1}{2}\int_0^1 x^2 dx - \frac{1}{2}\int_0^1 x^2\cos(2\pi x)dx - 2\int_0^1 x^3\sin(\pi x)dx + \int_1^2 (2-x)^4 dx + \frac{1}{2}\int_1^2 (2-x)^2 dx - \frac{1}{2}\int_1^2 (2-x)^2\cos(2\pi x)dx - 2$$
$$\int_1^2 (2-x)^3\sin(\pi x)dx$$

$$||(u, u-v)||_{L^2}^2 = \frac{1}{5} + \frac{1}{6} - \frac{1}{2}\frac{1}{2\pi^2} - \frac{2}{\pi} + \frac{12}{\pi^3} + \frac{1}{5} + \frac{1}{6} - \frac{1}{2}\frac{1}{2\pi^2} - \frac{12}{\pi^3} + \frac{2}{\pi} = \frac{2}{5} + \frac{1}{3} - \frac{1}{2\pi^2} = \frac{11}{15} - \frac{1}{2\pi^2} \quad \Longrightarrow \quad \boxed{||(u, u-v)||_{L^2}^2 = 0.683}$$

Now, let's calculate the norm of u

$$||u||_{L^2}^2 = \int_0^2 |u|^2 dx = \int_0^1 x^2 dx + \int_1^2 (2-x)^2 dx = \frac{1}{3}x^3\Big|_0^1 - \frac{1}{3}(2-x)^3\Big|_1^2 = \frac{2}{3}$$

To check the Cauchy-Schwarz inequality let's use it to the power of two

$$||(u, u-v)||_{L^2}^2 \leq ||u||_{L^2}^2 \cdot ||u-v||_{L^2}^2 \quad \Longleftrightarrow \quad \boxed{0.683 \leq \frac{2}{3}(1.67) = 1.1111}$$

[11]:
```
integraluumv, err = quadgk(x -> broadcast(abs, u.(x).*(u.(x) - v.(x))).^2, 0, 2, rtol=1e-8)
```

[11]: (0.6826727415121643, 1.262843996041596e-12)

[12]:
```
integralu2, err = quadgk(x -> broadcast(abs, u.(x)).^2, 0, 2, rtol=1e-8)
```

[12]: (0.6666666666666667, 5.551115123125783e-17)

[13]:
```
sqrt(integraluumv) <= sqrt(integralu2) *  sqrt(umvH)
```

[13]: true

Now, let's verify the Cauchy-Schwarz inequality in the space $H^1(0,2)$ for the previous functions, u and $u-v$

$$||(u,u-v)||_{H^1} \leq ||u||_{H^1} \cdot ||u-v||_{H^1} = \left(||u||_{L^2}^2 + ||u'||_{L^2}^2\right)^{1/2}\left(||u-v||_{L^2}^2 + ||(u-v)'||_{L^2}^2\right)^{1/2}$$

Notice that only the last term is calculated. Let's start calculating the norm of the inner product between u and $u-v$

$$||(u,u-v)||_{H^1}^2 = ||(u,u-v)||_{L^2}^2 + ||(u',u'-v')||_{L^2}^2 = \int_0^2 |u(u-v)|^2 dx + \int_0^2 |u'(u'-v')|^2 dx$$

Let's start calculating the inner product between the derivatives of u and $u-v$

$$\int_0^2 |u'(u'-v')|^2 dx = \int_0^1 (1-\pi\cos(\pi x))^2 dx + \int_1^2 (1+\pi\cos(\pi x))^2 dx = \int_0^1 (1-2\pi\cos(\pi x)+\pi^2\cos^2(\pi x))dx + \int_1^2 (1+2\pi\cos(\pi x)+\pi^2\cos^2(\pi x))dx$$

$$\int_0^2 |u'(u'-v')|^2 dx = \int_0^1 dx - 2\pi\int_0^1 \cos(\pi x)dx + \frac{\pi^2}{2}\int_0^1 (1+\cos(2\pi x))dx + \int_1^2 dx + 2\pi\int_1^2 \cos(\pi x)dx + \frac{\pi^2}{2}\int_1^2 (1+\cos(2\pi x))dx$$

$$\int_0^2 |u'(u'-v')|^2 dx = \int_0^1 dx - 2\pi\int_0^1 \cos(\pi x)dx + \frac{\pi^2}{2}\int_0^1 dx + \frac{\pi^2}{2}\int_0^1 \cos(2\pi x)dx + \int_1^2 dx + 2\pi\int_1^2 \cos(\pi x)dx + \frac{\pi^2}{2}\int_1^2 dx + \frac{\pi^2}{2}\int_1^2 \cos(2\pi x)dx$$

$$\int_0^2 |u'(u'-v')|^2 dx = 1 + \frac{\pi^2}{2} + 1 + \frac{\pi^2}{2} = 2 + \pi^2 \qquad \Longrightarrow \qquad \boxed{\int_0^2 |u'(u'-v')|^2 dx = 11.87}$$

Then,

$$||(u,u-v)||_{H^1}^2 = \frac{11}{15} - \frac{1}{2\pi^2} + 2 + \pi^2 = \frac{41}{15} - \frac{1}{2\pi^2} + \pi^2 \qquad \Longrightarrow \qquad \boxed{||(u,u-v)||_{H^1}^2 = 12.552}$$

Now let's calculate the u norm in H^1

$$||u||_{H^1}^2 = ||u||_{L^2}^2 + ||u'||_{L^2}^2 = \frac{2}{3} + \int_0^2 u'^2 dx = \frac{2}{3} + \int_0^1 dx + \int_1^2 dx = \frac{2}{3} + 2 = \frac{8}{3}$$

Finally, checking the Cauchy-Schwarz inequality to the power of two

$$||(u,u-v)||_{H^1}^2 \leq ||u||_{H^1}^2 \cdot ||u-v||_{H^1}^2 \qquad \Longleftrightarrow \qquad \boxed{12.552 \leq \frac{8}{3} \cdot (13.536) = 36.0.97}$$

[14]:
```
integralupupmp, err = quadgk(x -> broadcast(abs, up.(x).*(up.(x) - vp.(x))).^2, 0, 2, rtol=1e-8)
```

[14]: (11.869604401089356, 8.208209756332963e-8)

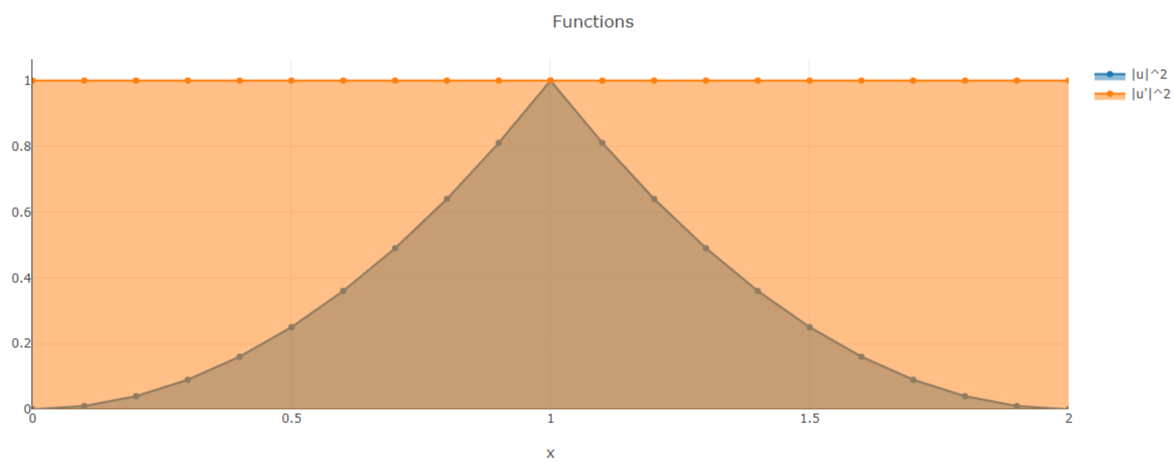
[15]:
```
sqrt(integraluumv + integralupupmp) <= sqrt(integralu2 + 2) * umvH
```

[15]: true

[16]:
```
plot_u2  = scatter(;x=x, y=broadcast(abs, u.(x).*u.(x)), mode="lines+markers", fill="tozeroy", name="|u|^2")
plot_up2 = scatter(;x=x, y=broadcast(abs, up.(x).*up.(x)), mode="lines+markers", fill="tozeroy", name="|u'|^2")

plot([plot_u2, plot_up2], Layout(title="Functions",  xaxis_title="x"))
```

[16]:



4. T

Show that, if $f(x, y)$ is a continous function defined in $\mathbb{R}^2$ such that

$$\iint_\Omega f(x, y)\ dxdy = 0$$

for all rectangle $\Omega \in \mathbb{R}^2$, the $f(x, y) = 0$ for all (x, y).

Solution

Let's suppose a continous function f such that $f(x_0, y_0) = L > 0$ for some $(x_0, y_0) \in \Omega$ and $\iint_\Omega f(x, y)dxdy = 0$.

Since f is continuous, for all $\epsilon > 0$, there exist δ such that if $|(x, y) - (x_0, y_0)| < \delta$ then $|f(x, y) - L| < \epsilon$. It is valid for any $\epsilon > 0$. In particular, taking $\epsilon = L/2$ the $f(x, y)$ inequality takes the following form

$$-\epsilon < f(x, y) - L < \epsilon$$
$$-\frac{L}{2} < f(x, y) - L < \frac{L}{2}$$
$$-\frac{L}{2} + L < f(x, y) < \frac{L}{2} + L$$
$$\frac{L}{2} < f(x, y) < \frac{3L}{2}$$

integrating the last inequality over Ω

$$\iint_\Omega \frac{L}{2}\ dxdy < \iint_\Omega f(x, y)\ dxdy < \iint_\Omega \frac{3L}{2}\ dxdy$$

then

$$\iint_\Omega f(x, y)\ dxdy > \iint_\Omega \frac{L}{2}\ dxdy \geq \int_{y_0-\delta}^{y_0+\delta} \int_{x_0-\delta}^{x_0+\delta} \frac{L}{2}\ dxdy = \frac{L}{2}(2\delta)(2\delta) = 2L\ \cdot\ \delta^2 > 0$$

which is a contradiction. Similarly, the case when $f(x_0, y_0) < 0$ is can discussed.

Hence, $f(x, y)$ must be 0 ($f(x, y) = 0$) for all $(x, y) \in \Omega$.

5. T

Let X and Y Banach spaces and $A :\ X\ \rightarrow\ Y$ a linear operator. Prove that the following statement are equivalent

- (i) A is enclosed, i.e., exits a constat $C > 0$ such that
  $||Au||_Y \leq C||u||_X, \qquad \forall u \in X$
- (ii) A is continous, i.e.
  $\text{Si } u_n \rightarrow u \text{ cuando } n \rightarrow \infty, \text{ entonces } Au_n \rightarrow Au \text{cuando } n \rightarrow \infty$

Solution

$(i) \rightarrow (ii)$

Let $(x_n) \subseteq X$ be a convergent sequence with limit $x \in X$, then $x_n \rightarrow x$ when $n \rightarrow \infty$

$$||Ax_n - Ax||_Y = ||A(x_n - x)||_Y = ||A||||x_n - x||_X \rightarrow 0, \text{ because } x_n \rightarrow x \quad \Rightarrow \quad Ax_n \rightarrow Ax, \text{ when } n \rightarrow \infty$$

$(ii) \rightarrow (i)$

A is continuous in all point, it implies A is continous in 0 .

$(*)$: For all sequences $(x_n) \subseteq X$ with $x_n \rightarrow 0$ when $n \rightarrow \infty$, we have $Ax_n \rightarrow 0$ when $n \rightarrow \infty$

$(**)$: Then, there is some $\delta > 0$ such that $||Ax||_Y < 1$ for all $x \in X$ with $||x||_X < \delta$

Claim: $(*) \Rightarrow (**)$

Proof of the claim:

$\neg(**) \Rightarrow$ for all $n \in N$ we find $x_n \in X$ with norm $||x_n||_X < \frac{1}{n}$ and $||Ax_n||_Y \geq 1 \Rightarrow \neg(*)$

$$\frac{||Ax||_Y}{||x||_X} = \frac{||Ax||_Y \frac{\delta}{2||x||_X}}{||x||_X \frac{\delta}{2||x||_X}} = \frac{\left\|A\left(\frac{\delta x}{2||x||_X}\right)\right\|_Y}{\left\|\frac{\delta x}{2||x||_X}\right\|_X} \leqslant \frac{2}{\delta} \quad \Rightarrow \quad ||A|| = sup\left\{\frac{||Ax||_Y}{||x||_X} : x \in X, x \neq 0\right\} \leqslant \frac{2}{\delta} < \infty$$

Hence, A is bounded.

You can find this homework in the github repository saguileran/NAII.