Homework 1

March 22, 2022

Homework #1

2022

[ ]:
```
using QuadGK, Calculus, Roots, LaTeXStrings, PlotlyJS
```

0.1 1.

0.2 2.

Let $u(x)$ and $v(x)$ two functions defined as follows

$$u(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 \leq x \leq 2 \end{cases} \quad y \quad v(x) = \sin \pi x$$

with their derivatives

$$u'(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ -1 & \text{if } 1 \leq x \leq 2 \end{cases} \quad y \quad v'(x) = \pi \cos \pi x$$

[2]:
```
function u(x)
    if 0 <= x <=1
        return x
    elseif 1 < x <=2
        return 2 - x
    else
        return 0
    end
end

function up(x)
    if 0 <= x <=1
        return 1
    elseif 1 < x <=2
        return -1
    else
        return 0
    end
end
```

```
v(x)  = sin(pi * x)
vp(x) = pi * cos(pi * x)

x = range(0, 2, step=0.1) |> collect;
```

[3]:
```
plot_u  = scatter(;x=x, y=u.(x), mode="lines+markers", name="u(x)")
    plot_v  = scatter(;x=x, y=v.(x), mode="lines+markers", name="v(x)")
    plot_uv = scatter(;x=x, y=v.(x).*u.(x), mode="lines+markers",␣
↪fill="tozeroy", name="u(x) * v(x)")

    plot_up = scatter(;x=x, y=up.(x), mode="lines+markers", name="u'(x)")
    plot_vp = scatter(;x=x, y=vp.(x), mode="lines+markers", name="v'(x)")
    plot_uvp = scatter(;x=x, y=vp.(x).*up.(x), mode="lines+markers",␣
↪fill="tozeroy", name="u'(x) * v'(x)")

    p1 = plot([plot_u, plot_v,plot_uv], Layout(title="Functions", ␣
↪xaxis_title="x"))
    p2 = plot([plot_up, plot_vp,plot_uvp], Layout(title="Function Derivative", ␣
↪xaxis_title="x"))

    p = [p1 p2]
    #relayout!(p, height=300, width=700, title_text="Functions",␣
↪legend_title_text="Legend")
    #p
```

[3]:
```
data: [
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields fill, mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields fill, mode, name, type, x, xaxis, y, and yaxis"
]

layout: "layout with fields annotations, margin, xaxis1, xaxis2, yaxis1, and
yaxis2"
```

0.2.1 Orthogonality

- $L^2(0,2)$ **Space** Norm of L^2 and its norm $||u||_{L^2}$

$$(u,v) = \int_0^2 u(x)v(x)dx = \int_0^1 x\ \sin(\pi x)dx + \int_1^2 (2-x)\ \sin(\pi x)dx$$

Integranting by steps with the sustitution $w = x$ and $dz = \sin(\pi x)\ dx$

$$(u,v)_{L^2} = \frac{1}{\pi}x\cos(\pi x)\Big|_0^1 - \frac{1}{\pi}\int_0^1 \cos(\pi x)dx + \frac{1}{\pi}(2-x)\cos(\pi x)\Big|_1^2 - \frac{1}{\pi}\int_1^2 \cos(\pi x)dx(u,v)_{L^2} = \frac{1}{\pi}\cos(\pi) - \frac{1}{\pi^2}\sin(\pi x)\Big|_0^1 -$$

The function u and v are orthogonals in $L^2(0,2)$.

```
[4]: integral, err = quadgk(x -> u(x)*v(x), 0, 2, rtol=1e-8)
```

```
[4]: (2.0816681711721685e-17, 0.0)
```

- $H^1(0,2)$ Space Now, let's check if u and v are orthogonal in $H^1(0,2)$ usign the its norm and their derivatives

$$(u,v)_{H^1} = \int_0^2 u(x)v(x)dx + \int_0^2 u'(x)v'(x)dx = 0 + \int_0^1 u'(x)v'(x)dx + \int_1^2 u'(x)v'(x)dx(u,v)_{H^1} = \pi\int_0^1 1\ \cos(\pi x)dx +$$

Then, the functions u and v are orthogonal on $H^1(0,2)$.

```
[5]: integral, err = quadgk(x -> up(x)*vp(x), 0, 2, rtol=1e-8)
```

```
[5]: (-4.105099355129983e-15, 3.7515060282571523e-19)
```

0.2.2 $||u - v||$ Difference

Let's calculate the distance between u and v in both spaces $L^2(0,2)$ and $H^1(0,2)$.

```
[6]: plot_umv2  = scatter(;x=x, y=broadcast(abs, u.(x) - v.(x)).^2,␣
      ↪mode="lines+markers", fill="tozeroy", name="|u(x)-v(x)|^2")
     plot_umv   = scatter(;x=x, y=u.(x) - v.(x), mode="lines+markers",␣
      ↪name="u(x)-v(x)")

     plot_upmvp2 = scatter(;x=x, y=broadcast(abs, up.(x) - vp.(x)).^2,␣
      ↪mode="lines+markers", fill="tozeroy", name="|u'(x) -v'(x)|^2")
     plot_upmvp  = scatter(;x=x, y= up.(x) - vp.(x), mode="lines+markers",␣
      ↪name="u'(x)-v'(x)")

     p3 = plot([plot_umv2, plot_umv], Layout(title="Distance Between Functions", ␣
      ↪xaxis_title="x"))
     p4 = plot([plot_upmvp2, plot_upmvp], Layout(title="Distance Between Derivative␣
      ↪Functions",  xaxis_title="x"))

     p = [p3 p4]
     #relayout!(p, height=300, width=700, title_text="Functions",␣
      ↪legend_title_text="Legend")
     #p
```

[6]: data: [
  "scatter with fields fill, mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields fill, mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis"
]

layout: "layout with fields annotations, margin, xaxis1, xaxis2, yaxis1, and yaxis2"

- $L^2(0,2)$ **Space**

$$||u-v||_{L^2}^2 = \int_0^2 |u-v|^2 dx = \int_0^2 (u-v)^2 dx ||u-v||_{L^2}^2 = \int_0^1 (x-\sin(\pi x))^2 dx + \int_1^2 (2-x-\sin(\pi x))^2 dx ||u-v||_{L^2}^2 = \int_0^1 (x$$

Using integral tables

$$||u-v||_{L^2}^2 = \left(\frac{1}{3}x^3 - \frac{2}{\pi^2}\sin(\pi x) + \frac{2}{\pi}x\cos(\pi x) + \frac{x}{2} - \frac{1}{4\pi}\sin(2\pi x)\right)\Big|_0^1 + \left(4x + \frac{1}{3}x^3 + \frac{x}{2} - \frac{1}{4\pi}\sin(2\pi x) - 2x^2 + \frac{4}{\pi}\text{co}\right.$$

$$||u-v||_{L^2}^2 = \left(\frac{1^3}{3} + \frac{2}{\pi}\cos(\pi) + \frac{1}{2}\right) + \left(4(2) + \frac{2^3}{3} + \frac{2}{2} - (2)2^2 + \frac{4}{\pi}\cos(2\pi) - \frac{4}{\pi}\cos(2\pi)\right) - \left(4 + \frac{1^3}{3} + \frac{1}{2} - (2)1^2 + \frac{4}{\pi}\right.$$

```
[7]: integral1, err = quadgk(x -> broadcast(abs, u(x)-v(x))^2, 0, 2, rtol=1e-8)
```

[7]: (1.6666666666666665, 8.316655142337481e-9)

- $H^1(0,2)$ **Space**

$$||u-v||_{H^1}^2 = ||u-v||_{L^2}^2 + ||(u-v)'||_{L^2}^2 = \int_0^2 |u-v|^2 dx + \int_0^2 |(u-v)'|^2 dx$$

The first integral is already done, the second is

$$||(u-v)'||_{L^2}^2 = \int_0^2 |(u-v)'|^2 dx = \int_0^1 (1-\pi\cos(\pi x))^2 dx + \int_1^2 (-1-\pi\cos(\pi x))^2 dx \int_0^2 |(u-v)'|^2 dx = \int_0^1 (1 - 2\pi\cos(\pi$$

```
[8]: integral2, err = quadgk(x -> broadcast(abs, up(x)-vp(x))^2, 0, 2, rtol=1e-8)
```

[8]: (11.869604401089356, 8.208209756332963e-8)

Finally, the distance between u and v in $H^1(0,2)$ is

$$||u-v||_{H^1}^2 = \frac{5}{3} + \pi^2 + 2 = \frac{11}{3} + \pi^2 \qquad \Longrightarrow \qquad \boxed{||u-v||_{H^1} = 3.6792}$$

[9]:
```
umvH = sqrt(integral1 + integral2)
```

[9]:
```
3.6791671704009348
```

0.3 3.

[10]:
```
plot_umumv2    = scatter(;x=x, y=broadcast(abs, u.(x).*(u.(x) - v.(x))).^2,␣
 ↪mode="lines+markers", fill="tozeroy", name="|u(u-v)|^2")
plot_upmupmvp2 = scatter(;x=x, y=broadcast(abs, up.(x).*(up.(x) - vp.(x))).^2,␣
 ↪mode="lines+markers", fill="tozeroy", name="|u'(u'-v')|^2")

p5 = plot([plot_u, plot_umv, plot_umumv2], Layout(title="Functions", ␣
 ↪xaxis_title="x"))
p6 = plot([plot_up, plot_upmvp, plot_upmupmvp2], Layout(title="Derivative␣
 ↪Functions",  xaxis_title="x"))

p = [p5 p6]
#relayout!(p, height=300, width=700, title_text="Functions",␣
 ↪legend_title_text="Legend")
#p
```

[10]:
```
data: [
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields fill, mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields mode, name, type, x, xaxis, y, and yaxis",
  "scatter with fields fill, mode, name, type, x, xaxis, y, and yaxis"
]

layout: "layout with fields annotations, margin, xaxis1, xaxis2, yaxis1, and
yaxis2"
```

Now, let's verify the Cauchy-Schwarz inequality in the space $L^2(0,2)$ for the previous functions, u and $u-v$

$$||(u, u-v)||_{L^2} \leq ||u||_{L^2} \cdot |u-v||_{L^2}$$

Notice that only the last term is calculated. First let's calculate the norm of the inner product between u and $u-v$

$$||(u,u-v)||_{L^2}^2 = \int_0^2 |u(u-v)|^2 dx = \int_0^1 (x(x-\sin(\pi x)))^2 dx + \int_1^2 ((2-x)(2-x-\sin(\pi x)))^2 dx ||(u,u-v)||_{L^2}^2 = \int_0^1 (x^4 +$$

Now, let's calculate the norm of u

$$||u||_{L^2}^2 = \int_0^2 |u|^2 dx = \int_0^1 x^2 dx + \int_1^2 (2-x)^2 dx = \frac{1}{3}x^3\Big|_0^1 - \frac{1}{3}(2-x)^3\Big|_1^2 = \frac{2}{3}$$

To check the Cauchy-Schwarz inequality let's use it to the power of two

$$||(u,u-v)||_{L^2}^2 \leq ||u||_{L^2}^2 \cdot ||u-v||_{L^2}^2 \qquad \Longleftrightarrow \qquad \boxed{0.683 \leq \frac{2}{3}(1.67) = 1.1111}$$

[11]:
```
integraluumv, err = quadgk(x -> broadcast(abs, u.(x).*(u.(x) - v.(x))).^2, 0,␣
↪2, rtol=1e-8)
```

[11]: `(0.6826727415121643, 1.262843996041596e-12)`

[12]:
```
integralu2, err = quadgk(x -> broadcast(abs, u.(x)).^2, 0, 2, rtol=1e-8)
```

[12]: `(0.6666666666666667, 5.551115123125783e-17)`

[13]:
```
sqrt(integraluumv) <= sqrt(integralu2) *  sqrt(umvH)
```

[13]: `true`

Now, let's verify the Cauchy-Schwarz inequality in the space $H^1(0,2)$ for the previous functions, u and $u-v$

$$||(u,u-v)||_{H^1} \leq ||u||_{H^1} \cdot ||u-v||_{H^1} = \left(||u||_{L^2}^2 + ||u'||_{L^2}^2\right)^{1/2} \left(||u-v||_{L^2}^2 + ||(u-v)'||_{L^2}^2\right)^{1/2}$$

Notice that only the last term is calculated. Let's start calculating the norm of the inner product between u and $u-v$

$$||(u,u-v)||_{H^1}^2 = ||(u,u-v)||_{L^2}^2 + ||(u',u'-v')||_{L^2}^2 = \int_0^2 |u(u-v)|^2 dx + \int_0^2 |u'(u'-v')|^2 dx$$

Let's start calculating the inner product between the derivatives of u and $u-v$

$$\int_0^2 |u'(u'-v')|^2 dx = \int_0^1 (1-\pi\cos(\pi x))^2 dx + \int_1^2 (1+\pi\cos(\pi x))^2 dx = \int_0^1 (1-2\pi\cos(\pi x)+\pi^2\cos^2(\pi x))dx + \int_1^2 (1+2\pi$$

Then,

$$||(u,u-v)||_{H^1}^2 = \frac{11}{15} - \frac{1}{2\pi^2} + 2 + \pi^2 = \frac{41}{15} - \frac{1}{2\pi^2} + \pi^2 \qquad \Longrightarrow \qquad \boxed{||(u,u-v)||_{H^1}^2 = 12.552}$$

Now let's calculate the u norm in H^1

$$||u||_{H^l}^2 = ||u||_{L^2}^2 + ||u'||_{L^2}^2 = \frac{2}{3} + \int_0^2 u'^2 dx = \frac{2}{3} + \int_0^1 dx + \int_1^2 dx = \frac{2}{3} + 2 = \frac{8}{3}$$

Finally, checking the Cauchy-Schwarz inequality to the power of two

$$||(u,u-v)||_{H^1}^2 \leq ||u||_{H^1}^2 \cdot ||u-v||_{H^1}^2 \qquad \Longleftrightarrow \qquad \boxed{12.552 \leq \frac{8}{3} \cdot (13.536) = 36.0.97}$$

[14]:
```
integralupupmp, err = quadgk(x -> broadcast(abs, up.(x).*(up.(x) - vp.(x))).^2,␣
↪0, 2, rtol=1e-8)
```

[14]:
```
(11.869604401089356, 8.208209756332963e-8)
```

[15]:
```
sqrt(integraluumv + integralupupmp) <= sqrt(integralu2 + 2) * umvH
```

[15]:
```
true
```

[16]:
```
plot_u2  = scatter(;x=x, y=broadcast(abs, u.(x).*u.(x)), mode="lines+markers",␣
↪fill="tozeroy", name="|u|^2")
plot_up2 = scatter(;x=x, y=broadcast(abs, up.(x).*up.(x)),␣
↪mode="lines+markers", fill="tozeroy", name="|u'|^2")

plot([plot_u2, plot_up2], Layout(title="Functions",  xaxis_title="x"))
```

[16]:
```
data: [
  "scatter with fields fill, mode, name, type, x, and y",
  "scatter with fields fill, mode, name, type, x, and y"
]

layout: "layout with fields margin, title, and xaxis"
```

0.4 4.

Let $f(x,y)$ a continous function over $\mathbb{R}^2$ such that

$$\iint_\Omega f(x,y)dxfy = 0$$

for all rectangle $\Omega \subseteq \mathbb{R}^2$, then $f(x,y) = 0$ for all (x,y).

Since f is continous it satisfy

$$\lim_{(x,y)->(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

for all (x_0, y_0) in Ω. Let's start integrating the last equation in x and y, i.e. in Ω

$$\iint_\Omega \lim_{(x,y)->(x_0,y_0)} f(x,y)dxdy = \iint_\Omega f(x_0,y_0)dxdy \lim_{(x,y)->(x_0,y_0)} \iint_\Omega f(x,y)dxdy = \iint_\Omega f(x_0,y_0)dxdy \lim_{(x,y)->(x_0,y_0)} 0$$

The integral is zero then its integrating should be zero

$$f(x_0,y_0) = 0 \quad \forall (x_0,y_0) \in \Omega \qquad \Longrightarrow \qquad \boxed{f(x,y) = 0}$$

0.5 5.