Homework 2

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Part 1: Tutorial Example

a) Theorical and CasADi solution

In order to find the optimal value x^*, let's derive the function to minimize and equals to zero

$$f(x) = x^2 - 2x \quad \Rightarrow \quad f'(x) = 2x - 2 = 0 \quad \Rightarrow \quad x^* = 1$$

Now, we are goint to solve the problem numerically with **CasADi**

$$\min_{x \in \mathbb{R}} \quad x^2 - 2x$$

```
import casadi.*

opti = casadi.Opti();
x = opti.variable();
opti.minimize(x^2 - 2*x);

opti.solver('ipopt');

sol = opti.solve();
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
```

```
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        1

Total number of variables............................:        1
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 0.00e+000 2.00e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -1.0000000e+000 0.00e+000 0.00e+000  -1.0 1.00e+000    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............: -1.0000000000000000e+000  -1.0000000000000000e+000
Dual infeasibility......:  0.0000000000000000e+000   0.0000000000000000e+000
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  0.0000000000000000e+000   0.0000000000000000e+000


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.014
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         2
  nlp_grad_f  |        0 (       0)        0 (       0)         3
  nlp_hess_l  |        0 (       0)        0 (       0)         1
       total  |  20.00ms ( 20.00ms)  20.00ms ( 20.00ms)         1
```

```
xopt = sol.value(x);
if strcmp(sol.stats.return_status, 'Solve_Succeeded')

    disp(['Optimal solution found: x = ' num2str(xopt)]);
else
    disp('Failed problem')
end
```

```
Optimal solution found: x = 1
```

b) Add constraint

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & x^2 - 2x \\ \text{subject to} \quad & z \geq 1.5 \end{aligned}$$

```
opti = casadi.Opti();
```

```
x = opti.variable();
opti.minimize(x^2 - 2*x);
opti.subject_to( x>=1.5 );

opti.solver('ipopt');

sol = opti.solve();
```

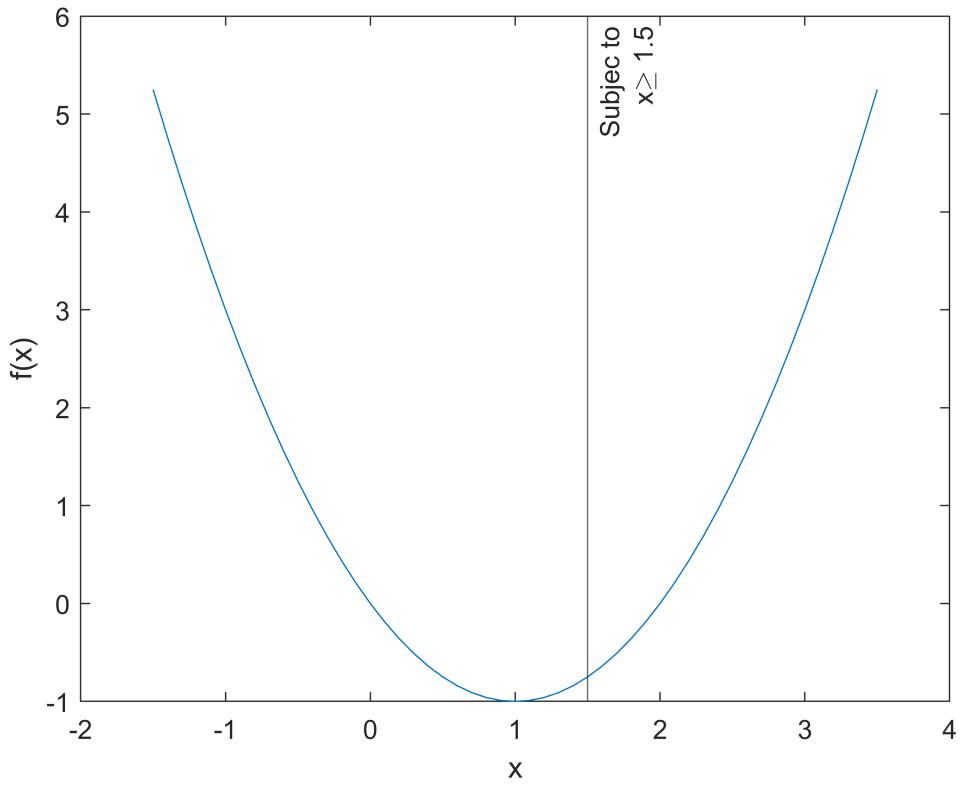
```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        1
Number of nonzeros in Lagrangian Hessian.............:        1

Total number of variables............................:        1
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        1
        inequality constraints with only lower bounds:        1
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 1.50e+000 1.50e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -6.4348670e-001 0.00e+000 1.00e-006  -1.0 1.60e+000    -  1.00e+000 1.00e+000f  1
   2 -7.3414019e-001 0.00e+000 2.83e-008  -2.5 8.15e-002    -  1.00e+000 1.00e+000f  1
   3 -7.4939903e-001 0.00e+000 1.50e-009  -3.8 1.50e-002    -  1.00e+000 1.00e+000f  1
   4 -7.4999745e-001 0.00e+000 1.84e-011  -5.7 5.98e-004    -  1.00e+000 1.00e+000f  1
   5 -7.5000001e-001 0.00e+000 2.50e-014  -8.6 2.56e-006    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............: -7.5000001248101222e-001  -7.5000001248101222e-001
Dual infeasibility......:  2.4980018054066022e-014   2.4980018054066022e-014
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  2.5189876222283079e-009   2.5189876222283079e-009
Overall NLP error.......:  2.5189876222283079e-009   2.5189876222283079e-009


Number of objective function evaluations             = 6
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 6
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 6
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      0.029
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         6
       nlp_g  |        0 (       0)        0 (       0)         6
  nlp_grad_f  |        0 (       0)        0 (       0)         7
  nlp_hess_l  |        0 (       0)        0 (       0)         5
   nlp_jac_g  |   1.00ms (142.86us)   1.00ms (142.86us)         7
       total  |  37.00ms ( 37.00ms)  37.01ms ( 37.01ms)         1
```

```
xopt1 = sol.value(x);
```

```
disp(['x optimal, x^*= ' num2str(xopt1)]);
```

```
x optimal, x^*= 1.5
```

```
x=-1.5:0.1:3.5;

figure()
plot(x,x.^2-2*x)
xlabel('x'); ylabel('f(x)');
xline(1.5,'-',{'Subjec to','x\geq 1.5'})
```



These results is according to the intuition, the minimum value now is $x^* = 1.5$.

c) Bidimensional problem

$$\begin{aligned} \min_{x,y\in\mathbb{R}} \quad & x^2 - 2x + y^2 + y \\ \text{subject to} \quad & z \geq 1.5 \\ & x + y \geq 0 \end{aligned}$$

```
opti = casadi.Opti();
x = opti.variable();
y = opti.variable();
opti.minimize(x^2-2*x+y^2+y);
```

```
opti.subject_to( x>=1.5 );
opti.subject_to( x+y>=0 );

opti.solver('ipopt');

sol = opti.solve();
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        3
Number of nonzeros in Lagrangian Hessian.............:        2

Total number of variables............................:        2
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        2
        inequality constraints with only lower bounds:        2
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 1.50e+000 1.60e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -4.0231569e-002 0.00e+000 1.89e+000  -1.0 1.57e+000    -  3.44e-001 1.00e+000f  1
   2 -7.6846321e-001 0.00e+000 1.00e-006  -1.0 1.31e+000    -  1.00e+000 1.00e+000f  1
   3 -9.8447564e-001 0.00e+000 5.65e-002  -1.7 3.64e-001    -  1.00e+000 8.85e-001f  1
   4 -9.9657324e-001 0.00e+000 2.83e-008  -2.5 1.10e-001    -  1.00e+000 1.00e+000f  1
   5 -9.9989249e-001 0.00e+000 1.50e-009  -3.8 1.31e-002    -  1.00e+000 1.00e+000f  1
   6 -9.9999819e-001 0.00e+000 1.84e-011  -5.7 3.10e-004    -  1.00e+000 1.00e+000f  1
   7 -1.0000000e+000 0.00e+000 2.51e-014  -8.6 2.81e-006    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 7

                                   (scaled)                 (unscaled)
Objective...............: -1.0000000124910460e+000  -1.0000000124910460e+000
Dual infeasibility......:  2.5091040356528538e-014   2.5091040356528538e-014
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  2.5114280819827540e-009   2.5114280819827540e-009
Overall NLP error.......:  2.5114280819827540e-009   2.5114280819827540e-009


Number of objective function evaluations             = 8
Number of objective gradient evaluations             = 8
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 8
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 8
Number of Lagrangian Hessian evaluations             = 7
Total CPU secs in IPOPT (w/o function evaluations)   =      0.086
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         8
       nlp_g  |        0 (       0)        0 (       0)         8
  nlp_grad_f  |        0 (       0)        0 (       0)         9
  nlp_hess_l  |   1.00ms (142.86us)   1.00ms (142.86us)         7
   nlp_jac_g  |        0 (       0)        0 (       0)         9
       total  |  93.00ms ( 93.00ms)  93.02ms ( 93.02ms)         1
```

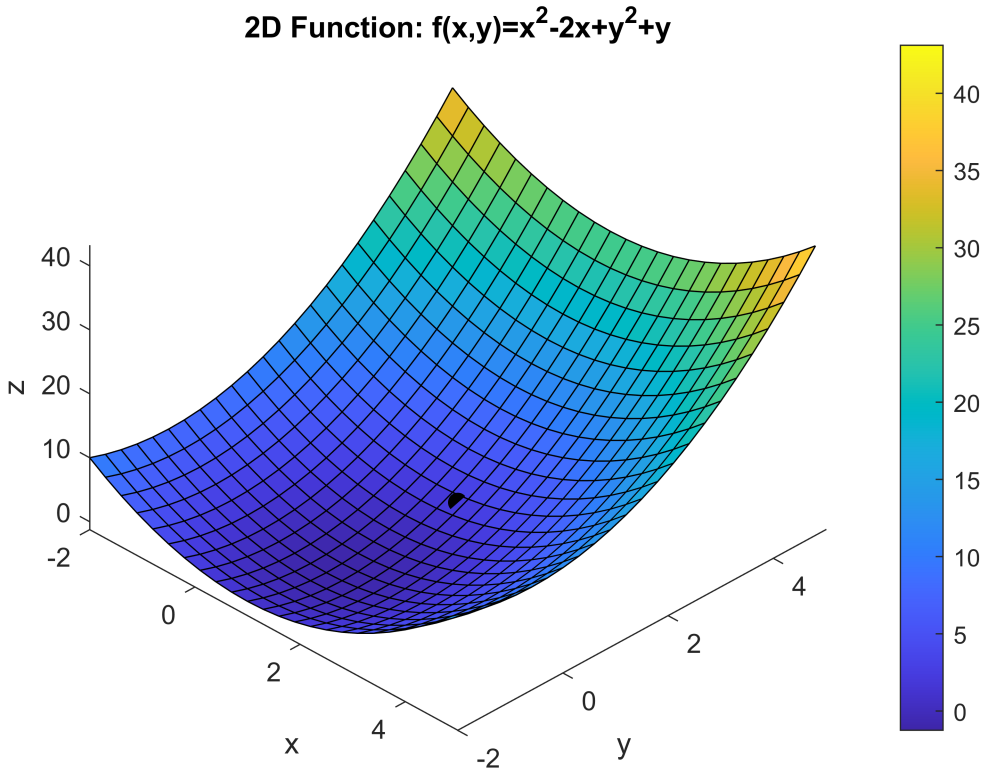
```
xopt2d = sol.value(x);
yopt2d = sol.value(x);
disp(['x optimal, x^*= ' num2str(xopt2d)]);
```

```
x optimal, x^*= 1.5
```

```
disp(['y optimal, y^*= ' num2str(xopt2d)]);
```

```
y optimal, y^*= 1.5
```

```
figure;
[X,Y] = meshgrid(-2:0.3:5,-2:0.3:5);
Z = X.^2-2*X+Y.^2+Y;

hold on;
surf(X,Y,Z)
title('2D Function: f(x,y)=x^2-2x+y^2+y')
xlabel('x'); ylabel('y'); zlabel('z')
xlim([-2 5]); ylim([-2 5]);
%view(90,90)
view(45,45)
colorbar()
plot3(xopt2d,yopt2d,xopt2d^2-2*xopt2d+yopt2d.^2+yopt2d,'.','Color','black','MarkerSize',25,...
    'MarkerFaceColor','#000000');
hold off;
```



Part 2: Equilibrium Position of the Catenary

We are going to simulate a catenary as a spring chain attached to two sports at each extrem. The chain is modeled as N masses connected by $N-1$ springs without mass, each mass m_i has a posittion (y_i, z_i) with $i = 1, \dots, N$. We are interested in finding the equilibrium position such that **minimize** the potential energy of the whole system.

The potential energy of each spring is

$$V_{el}(y_i, y_{i+1}, z_i, z_{i+1}) = \frac{1}{2} D\left((y_i - y_{i+1})^2 + (z_i - z_{i+1})^2\right)$$

for $i = 1, \dots, N-1$, and a spring constant $D \in \mathbb{R}^+$. The potential energy of each mass is

$$V_g(z_i) = mgz_i$$

for $i = 1, \dots, N$, g is the gravity, and all masses are considered equals, $m = m_1 = m_2 = \cdots = m_N$. The total potential energy is given by

$$V_{chain} = \frac{1}{2} D \sum_{i=1}^{N-1} \left((y_i - y_{i+1})^2 + (z_i - z_{i+1})^2\right) + mg \sum_{i=1}^{N} z_i$$

where $y = (y_1 \dots, y_N)$ and $z = (z_1 \dots, z_N)$. Then, the chain minimization problem is the solution to the optimization problem

$$\begin{aligned} \underset{x,y \in \mathbb{R}^N}{min} \quad & V_{\text{chain}}(y, z) \\ \text{subject to} \quad & y_1 = \bar{y}_1, \quad y_N = \bar{y}_N, \\ & z_1 = \bar{z}_1, \quad z_N = \bar{z}_N, \end{aligned}$$

where $(\bar{y_1}, \bar{z_1})$ and $(\bar{y_N}, \bar{z_N})$ are he fixed position of the outer masses.

a) Type of Problem

b) CasADi Implementation

Problem Formulation

```
opti2 = casadi.Opti();

global N m D g

N = 40;
m = N;                  % kg
D = (70/40)*N;          % N/m
```

```
g = 9.81;                % m/s^2

Y = opti2.variable(N,1);   Z = opti2.variable(N,1);
Vchain = 0.5*D*(sum(diff(Y).^2) + sum(diff(Z).^2)) +  m*g*sum(Z);

opti2.minimize(Vchain)
opti2.subject_to(Y(1)==-2); opti2.subject_to(Z(1)==1);
opti2.subject_to(Y(end)==2); opti2.subject_to(Z(end)==1);
```

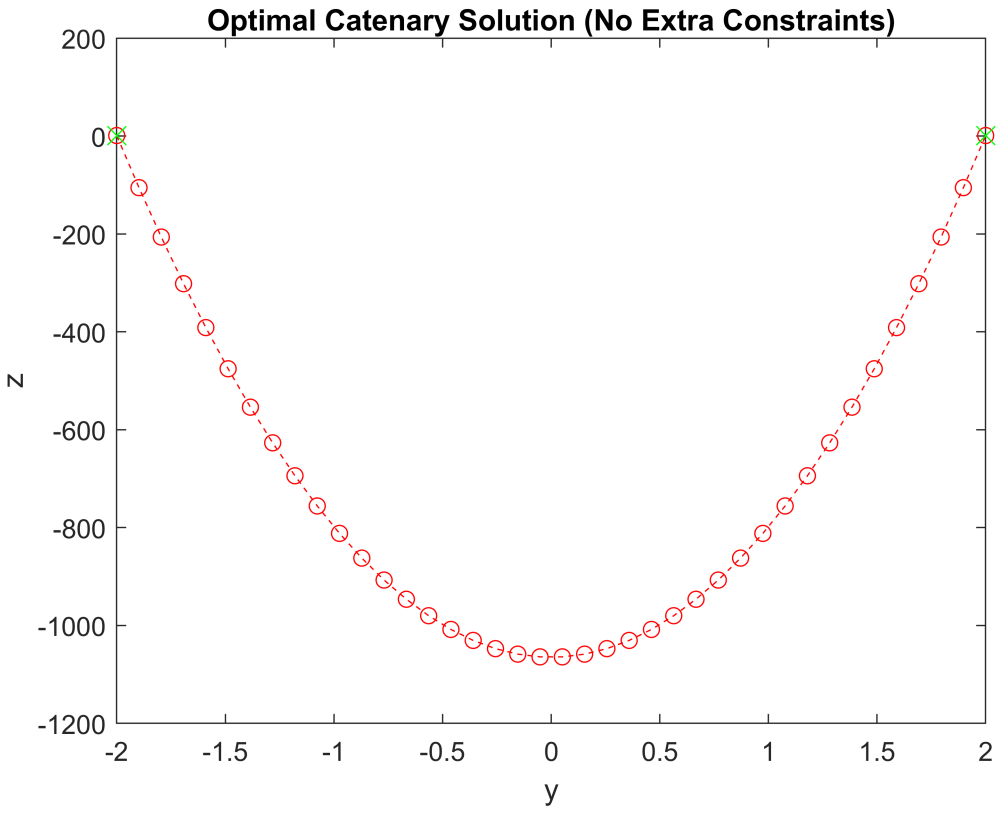
Problem Solution

```
opti2.solver('ipopt')
sol2 = opti2.solve();
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      158

Total number of variables............................:       80
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 2.00e+000 1.00e+002  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -5.4175049e+006 0.00e+000 4.29e-012  -1.0 1.06e+003    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............: -1.3806077693017153e+006  -5.4175048867399311e+006
Dual infeasibility......:  4.2878827682974440e-012   1.6825651982799172e-011
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  4.2839637236196760e-013   1.6825651982799172e-011


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 2
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.073
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         2
       nlp_g  |        0 (       0)        0 (       0)         2
  nlp_grad_f  |        0 (       0)        0 (       0)         3
  nlp_hess_l  |        0 (       0)        0 (       0)         1
```

```
   nlp_jac_g  |        0 (       0)        0 (       0)         3
       total  |  95.00ms ( 95.00ms)  95.02ms ( 95.02ms)         1
```

```
Ysol = sol2.value(Y);   Zsol = sol2.value(Z);
Plot("Optimal Catenary Solution (No Extra Constraints)", Ysol, Zsol);
```



c) Linear constraints

Adding linear constraints tho the floor. Now the complete problem is

$$\begin{aligned} \min_{x,y\in\mathbb{R}^N} \quad & V_{\text{chain}}(y,z) \\ \text{subject to} \quad & y_1 = \bar{y}_1, \quad y_N = \bar{y}_N, \\ & z_1 = \bar{z}_1, \quad z_N = \bar{z}_N, \\ & z_i \geq 0.5, z_i - 0.1 y_i \geq 0.5, \quad i = 2\ldots, N-1 \end{aligned}$$

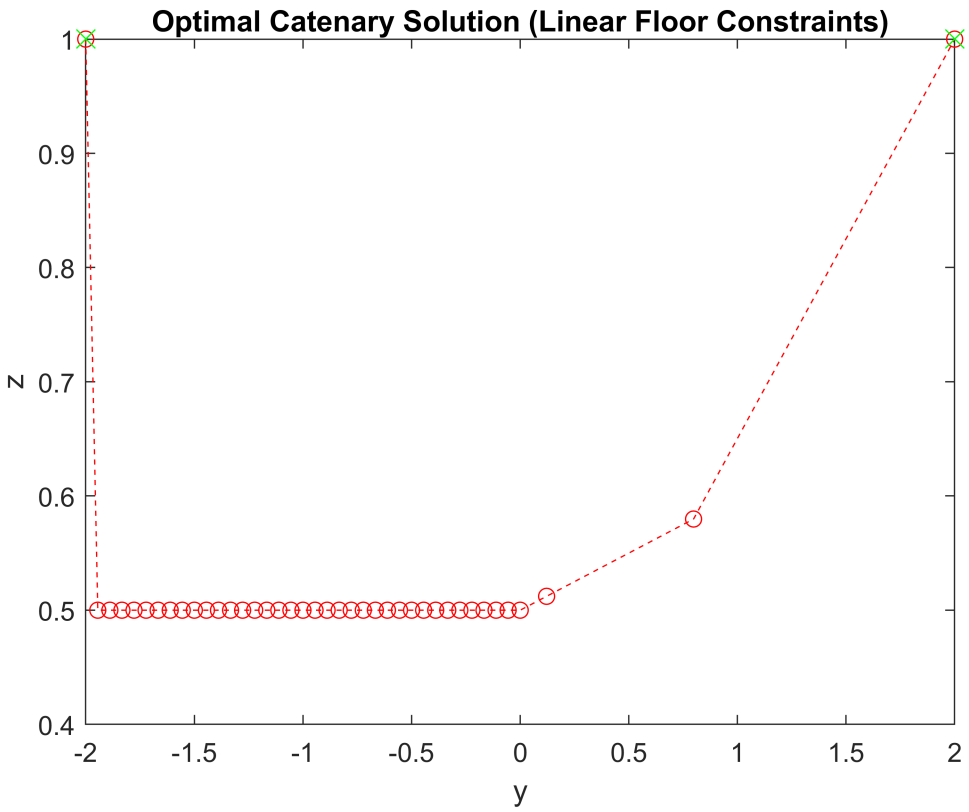
```
opti2c = casadi.Opti();

Y = opti2c.variable(N,1);  Z = opti2c.variable(N,1);
Vchain = 0.5*D*(sum(diff(Y).^2) + sum(diff(Z).^2)) +  m*g*sum(Z);

opti2c.minimize(Vchain)
opti2c.subject_to(Y(1)==-2); opti2c.subject_to(Z(1)==1);
opti2c.subject_to(Y(end)==2); opti2c.subject_to(Z(end)==1);

opti2c.subject_to(Z(2:end-1) >= 0.5);
opti2c.subject_to(Z(2:end-1)-0.1*Y(2:end-1) >= 0.5);
```

```
opti2c.solver('ipopt')
sol2c = opti2c.solve();
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:      114
Number of nonzeros in Lagrangian Hessian.............:      158

Total number of variables............................:       80
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:       76
        inequality constraints with only lower bounds:       76
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 2.00e+000 3.28e+001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -3.2147895e+001 1.98e+000 6.17e+001  -1.0 9.24e+000    -  1.00e+000 1.15e-002h  1
   2 1.9369436e+003 1.51e+000 4.69e+001  -1.0 1.98e+000    -  1.45e-001 2.36e-001h  1
   3 6.8545637e+003 3.60e-001 5.97e+001  -1.0 1.51e+000    -  2.01e-001 7.62e-001h  1
   4 8.3911272e+003 0.00e+000 3.68e+001  -1.0 7.77e-001    -  3.49e-001 1.00e+000h  1
   5 8.3794575e+003 0.00e+000 1.51e+001  -1.0 8.84e-001    -  4.55e-001 1.00e+000f  1
   6 8.3790546e+003 0.00e+000 1.00e-006  -1.0 7.38e-001    -  1.00e+000 1.00e+000f  1
   7 8.3639032e+003 0.00e+000 3.92e-001  -2.5 3.02e-001    -  1.00e+000 9.47e-001f  1
   8 8.3630223e+003 0.00e+000 2.92e-002  -2.5 1.02e-001    -  1.00e+000 9.77e-001f  1
   9 8.3630114e+003 0.00e+000 2.83e-008  -2.5 1.24e-002    -  1.00e+000 1.00e+000f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 8.3626050e+003 0.00e+000 3.34e-002  -3.8 3.27e-002    -  1.00e+000 9.63e-001f  1
  11 8.3625909e+003 0.00e+000 1.50e-009  -3.8 2.94e-003    -  1.00e+000 1.00e+000f  1
  12 8.3625681e+003 0.00e+000 1.85e-011  -5.7 2.01e-003    -  1.00e+000 1.00e+000f  1
  13 8.3625678e+003 0.00e+000 2.84e-014  -8.6 3.22e-005    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 13

                                   (scaled)                 (unscaled)
Objective...............:  2.1311334847300627e+003   8.3625677940807673e+003
Dual infeasibility......:  2.8421709430404007e-014   1.1152678780490534e-013
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  3.1319505163952296e-009   1.2289773826334882e-008
Overall NLP error.......:  3.1319505163952296e-009   1.2289773826334882e-008


Number of objective function evaluations             = 14
Number of objective gradient evaluations             = 14
Number of equality constraint evaluations            = 14
Number of inequality constraint evaluations          = 14
Number of equality constraint Jacobian evaluations   = 14
Number of inequality constraint Jacobian evaluations = 14
Number of Lagrangian Hessian evaluations             = 13
Total CPU secs in IPOPT (w/o function evaluations)   =      0.185
Total CPU secs in NLP function evaluations           =      0.002

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)        14
       nlp_g  |        0 (       0)        0 (       0)        14
  nlp_grad_f  |        0 (       0)        0 (       0)        15
  nlp_hess_l  |   1.00ms ( 76.92us)   1.00ms ( 76.92us)        13
```

```
   nlp_jac_g  |   1.00ms ( 66.67us)   1.00ms ( 66.73us)        15
       total  | 190.00ms (190.00ms) 190.04ms (190.04ms)         1
```

```
Ysolc = sol2c.value(Y);   Zsolc = sol2c.value(Z);

Plot("Optimal Catenary Solution (Linear Floor Constraints)", Ysolc, Zsolc);
```

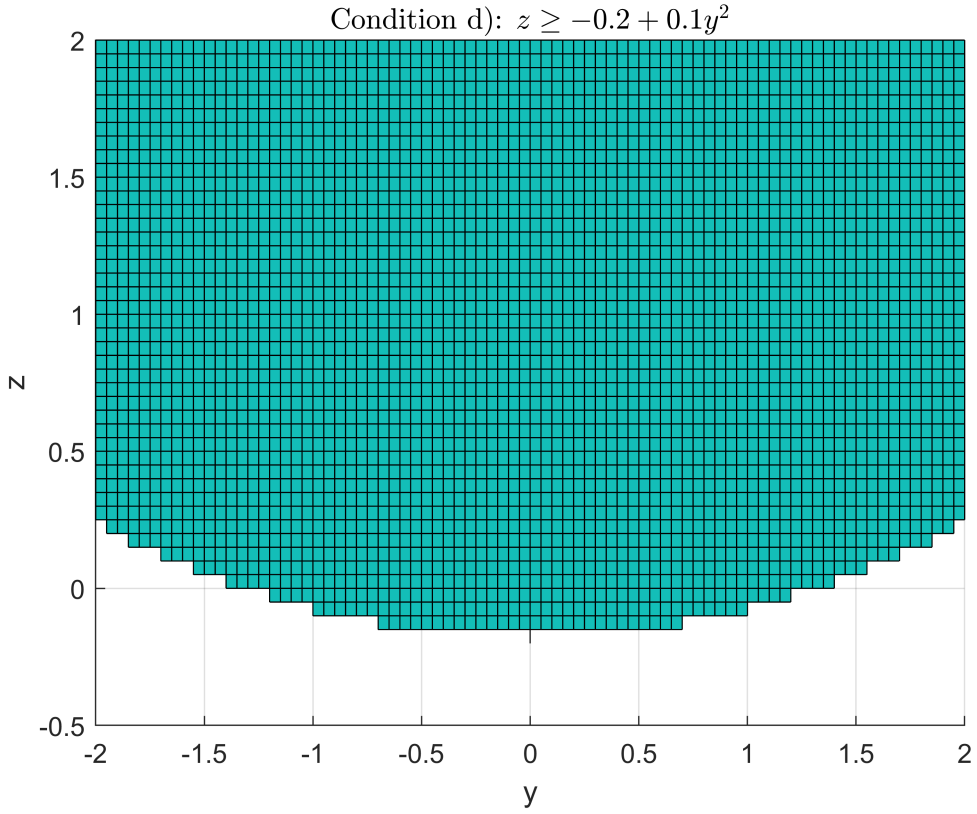


d) Nolinear constraints

Adding non linear constraints to the floor. Now the problem is

$$
\begin{aligned}
\underset{x,y\in\mathbb{R}^N}{min} \quad & V_{\text{chain}}(y,z) \\
\text{subject to} \quad & y_1 = \bar{y}_1, \quad y_N = \bar{y}_N, \\
& z_1 = \bar{z}_1, \quad z_N = \bar{z}_N, \\
& z_i \geq -0.2 + 0.1 y_i^2, \quad i = 2\ldots, N-1
\end{aligned}
$$

```
[x y] = meshgrid(-2:0.05:2);  % get 2-D mesh for x and y
cond1 = y >= -0.2+0.1*x.^2;  % check conditions for these values
cond1 = double(cond1);  % convert to double for plotting
cond1(cond1 == 0) = NaN;  % set the 0s to NaN so they are not plotted
surf(x,y,cond1)
xlabel("y"); ylabel("z");
title("Condition d): $z \geq -0.2 +0.1y^2$",'interpreter','latex');
view(0,90)    % change to top view
```

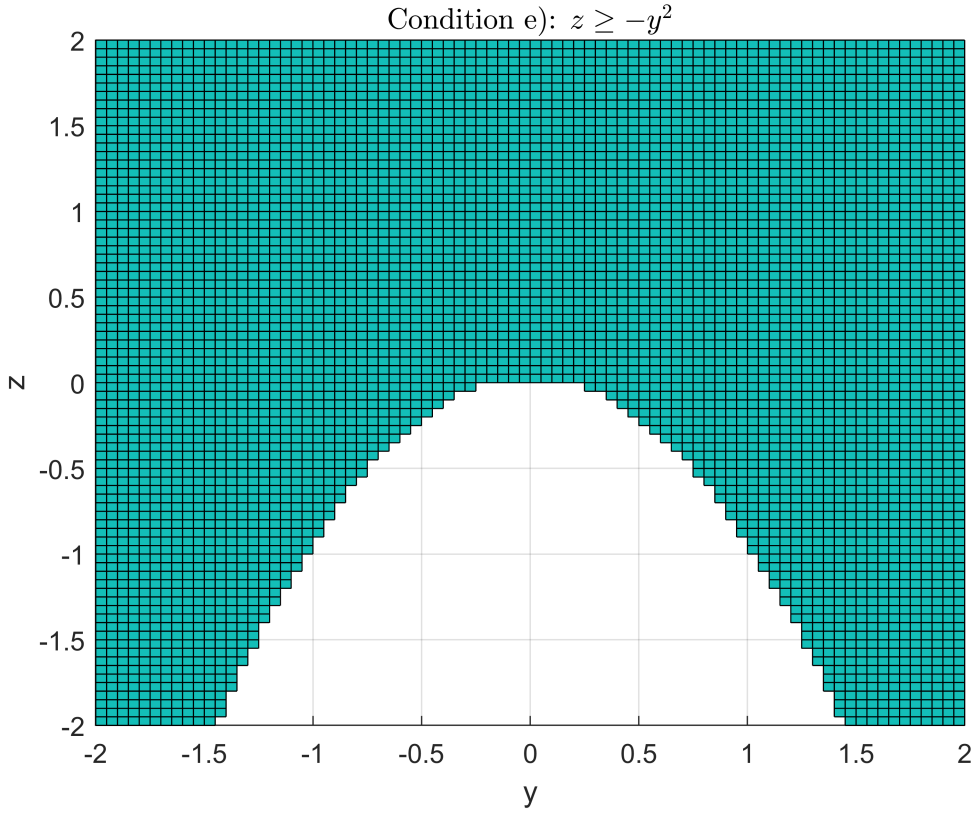


e) Nolinear constraints

Changing the non linear constraints to the floor. Now the problem is

$$\begin{aligned} \underset{x,y\in\mathbb{R}^N}{min} \quad & V_{\text{chain}}(y,z) \\ \text{subject to} \quad & y_1 = \bar{y}_1, \quad y_N = \bar{y}_N, \\ & z_1 = \bar{z}_1, \quad z_N = \bar{z}_N, \\ & z_i \geq -y_i^2, \quad i = 2\ldots, N-1 \end{aligned}$$

```
[x y] = meshgrid(-2:0.05:2);  % get 2-D mesh for x and y
cond1 = y >= -x.^2;  % check conditions for these values
cond1 = double(cond1);  % convert to double for plotting
cond1(cond1 == 0) = NaN;  % set the 0s to NaN so they are not plotted
surf(x,y,cond1)
xlabel("y"); ylabel("z");
title("Condition e): $z \geq -y^2$",'interpreter','latex');
view(0,90)    % change to top view
```



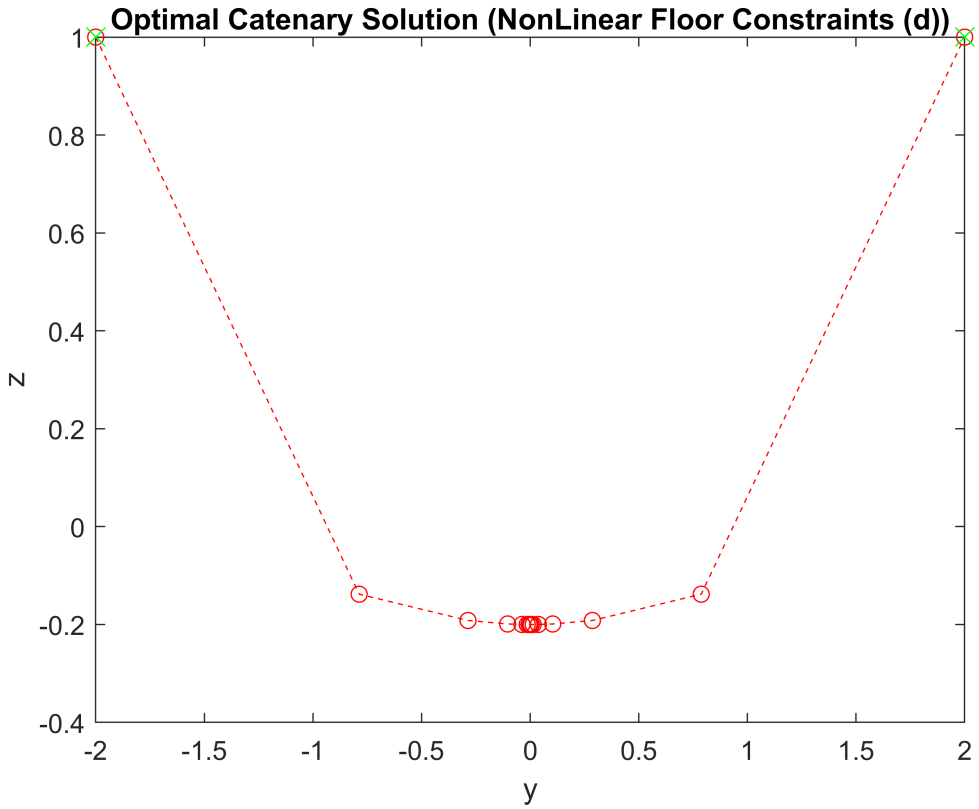
f) CasADi solution

i. Solution item d)

```
pd = CatenaryD(zeros(N,1),zeros(N,1));
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:       76
Number of nonzeros in Lagrangian Hessian.............:      158

Total number of variables............................:       80
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:       38
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       38

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 2.00e+000 4.95e+001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -2.7067643e+003 1.98e+000 4.90e+001  -1.0 1.99e+001    -  1.00e+000 9.95e-003f  1
   2 -2.4518837e+003 9.76e-001 2.47e+001  -1.0 1.98e+000    -  1.00e+000 5.07e-001h  1
   3 -1.9191513e+003 9.37e-003 1.14e+000  -1.0 9.76e-001    -  1.00e+000 1.00e+000h  1
   4 -1.9124907e+003 0.00e+000 1.28e-003  -1.0 2.84e-002    -  1.00e+000 1.00e+000h  1   5 -1.9269057e+003 0.00e+000
   6 -1.9273052e+003 0.00e+000 1.50e-009  -3.8 3.35e-005    -  1.00e+000 1.00e+000f  1
   7 -1.9273274e+003 0.00e+000 1.85e-011  -5.7 1.84e-006    -  1.00e+000 1.00e+000f  1
   8 -1.9273277e+003 0.00e+000 2.84e-014  -8.6 2.28e-008    -  1.00e+000 1.00e+000f  1
```

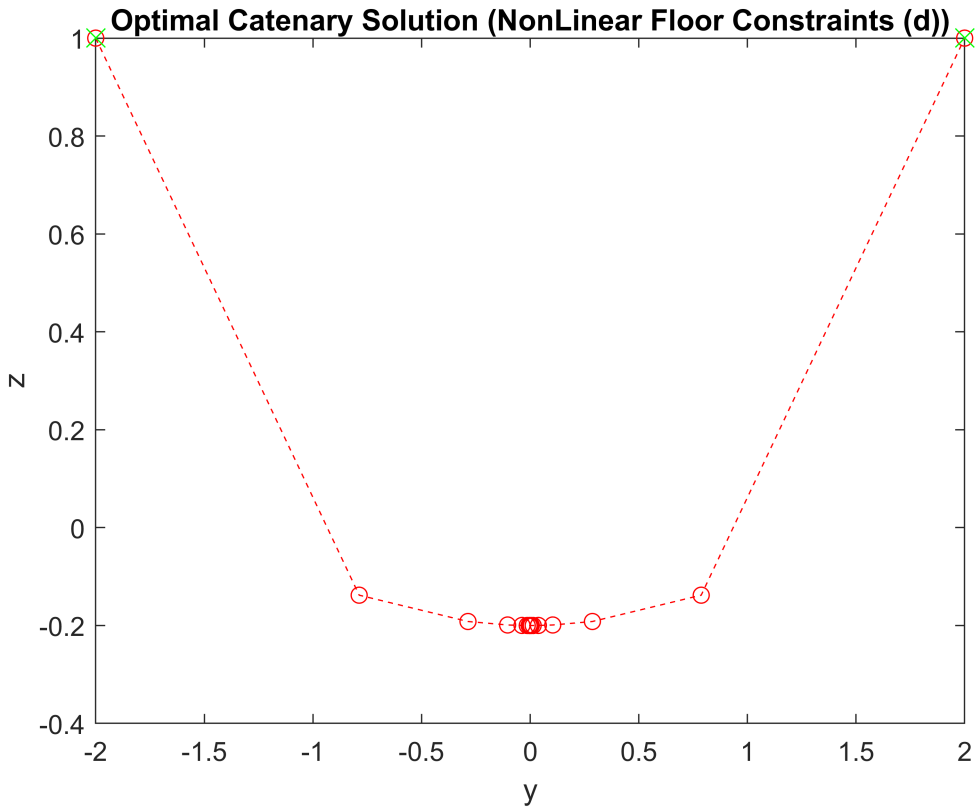
```
Number of Iterations....: 8

                                   (scaled)                 (unscaled)
Objective...............: -4.9116403117291986e+002  -1.9273276583225377e+003
Dual infeasibility......:  2.8421709430404007e-014   1.1152678780490534e-013
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  2.5059144105418960e-009   9.8332081469663999e-009
Overall NLP error.......:  2.5059144105418960e-009   9.8332081469663999e-009


Number of objective function evaluations             = 9
Number of objective gradient evaluations             = 9
Number of equality constraint evaluations            = 9
Number of inequality constraint evaluations          = 9
Number of equality constraint Jacobian evaluations   = 9
Number of inequality constraint Jacobian evaluations = 9
Number of Lagrangian Hessian evaluations             = 8
Total CPU secs in IPOPT (w/o function evaluations)   =      0.288
Total CPU secs in NLP function evaluations           =      0.003

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         9
       nlp_g  |        0 (       0)        0 (       0)         9
  nlp_grad_f  |   1.00ms (100.00us)   1.00ms (100.10us)        10
  nlp_hess_l  |   1.00ms (125.00us)   1.00ms (125.12us)         8
   nlp_jac_g  |        0 (       0)        0 (       0)        10
       total  | 319.00ms (319.00ms) 319.07ms (319.07ms)         1
```



changing both initial vector to a vector of ones

```
pd1 = CatenaryD(ones(N,1),ones(N,1));
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:       76
Number of nonzeros in Lagrangian Hessian.............:      158

Total number of variables............................:       80
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:       38
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       38

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 3.00e+000 5.05e+001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -1.4433756e+003 2.97e+000 5.00e+001  -1.0 1.03e+001    -  1.00e+000 1.00e-002f  1
   2 -4.3014839e+003 1.18e+000 2.01e+001  -1.0 2.97e+000    -  1.00e+000 6.04e-001f  1
   3 -1.9931775e+003 1.55e-002 1.45e+000  -1.0 1.18e+000    -  1.00e+000 1.00e+000h  1
   4 -1.9154336e+003 0.00e+000 2.49e-003  -1.0 5.02e-002    -  1.00e+000 1.00e+000h  1
   5 -1.9269057e+003 0.00e+000 5.68e-007  -2.5 1.21e-003    -  1.00e+000 1.00e+000f  1
   6 -1.9273052e+003 0.00e+000 1.50e-009  -3.8 3.36e-005    -  1.00e+000 1.00e+000f  1
   7 -1.9273274e+003 0.00e+000 1.85e-011  -5.7 1.84e-006    -  1.00e+000 1.00e+000f  1
   8 -1.9273277e+003 0.00e+000 2.84e-014  -8.6 2.28e-008    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 8

                                   (scaled)                 (unscaled)
Objective...............: -4.9116403117291986e+002  -1.9273276583225377e+003
Dual infeasibility......:  2.8421709430404007e-014   1.1152678780490534e-013
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  2.5059144105421628e-009   9.8332081469674471e-009
Overall NLP error.......:  2.5059144105421628e-009   9.8332081469674471e-009


Number of objective function evaluations             = 9
Number of objective gradient evaluations             = 9
Number of equality constraint evaluations            = 9
Number of inequality constraint evaluations          = 9
Number of equality constraint Jacobian evaluations   = 9
Number of inequality constraint Jacobian evaluations = 9
Number of Lagrangian Hessian evaluations             = 8
Total CPU secs in IPOPT (w/o function evaluations)   =      0.142
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         9
       nlp_g  |        0 (       0)        0 (       0)         9
  nlp_grad_f  |        0 (       0)        0 (       0)        10
  nlp_hess_l  |        0 (       0)        0 (       0)         8
   nlp_jac_g  |        0 (       0)        0 (       0)        10
       total  | 150.00ms (150.00ms) 150.03ms (150.03ms)         1
```



Note that if we change the initial vector the result does not change.

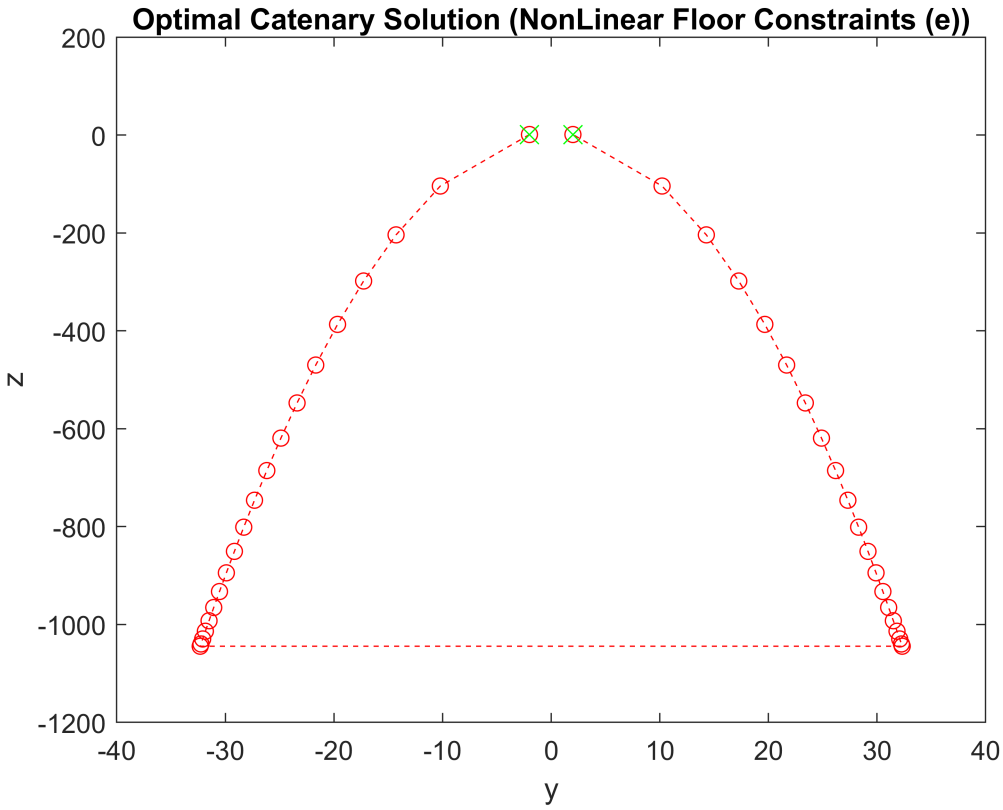
ii. Solution item e)

```
pe1 = CatenaryE(zeros(N,1),zeros(N,1));
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:       76
Number of nonzeros in Lagrangian Hessian.............:      158

Total number of variables............................:       80
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:       38
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       38

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 2.00e+000 4.95e+001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 1.1424517e+003 0.00e+000 2.00e+004  -1.0 2.00e+000   4.0 1.00e+000 1.00e+000h  1
   2 1.1524626e+003 0.00e+000 1.50e+004  -1.0 1.14e-002   3.5 1.00e+000 2.50e-001f  3
   3 1.1380924e+003 0.00e+000 4.14e+001  -1.0 3.79e-002   3.0 1.00e+000 1.00e+000f  1
   4 1.0847865e+003 0.00e+000 6.07e+001  -1.0 1.69e-001   2.6 1.00e+000 1.00e+000f  1
   5 1.0425952e+003 0.00e+000 4.81e+001  -1.0 4.87e-002   3.0 1.00e+000 1.00e+000f  1
   6 9.4539317e+002 0.00e+000 6.21e+001  -1.0 1.74e-001   2.5 1.00e+000 1.00e+000f  1
   7 8.8515543e+002 0.00e+000 5.90e+001  -1.0 6.72e-002   2.9 1.00e+000 1.00e+000f  1
   8 7.0740048e+002 0.00e+000 6.27e+001  -1.0 2.24e-001   2.5 1.00e+000 1.00e+000f  1
```

```
   9 5.7698118e+002 0.00e+000 6.50e+001  -1.0 8.33e-002   2.9 1.00e+000 1.00e+000f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 2.1347679e+002 0.00e+000 6.91e+001  -1.0 2.88e-001   2.4 1.00e+000 1.00e+000f  1
  11 -5.4841728e+001 0.00e+000 7.89e+001  -1.0 1.14e-001   2.8 1.00e+000 1.00e+000f  1
  12 -7.9577684e+002 0.00e+000 7.58e+001  -1.0 3.55e-001   2.4 1.00e+000 1.00e+000f  1
  13 -1.3449944e+003 0.00e+000 9.02e+001  -1.0 1.46e-001   2.8 1.00e+000 1.00e+000f  1
  14 -2.8626907e+003 0.00e+000 1.30e+002  -1.0 7.04e-001   2.3 1.00e+000 1.00e+000f  1
  15 -4.3171785e+003 0.00e+000 1.40e+002  -1.0 2.55e-001   2.7 6.59e-001 1.00e+000f  1
  16 -4.7002601e+003 0.00e+000 8.30e+001  -1.0 5.68e-002   3.2 1.00e+000 1.00e+000f  1
  17 -5.8832638e+003 0.00e+000 8.36e+001  -1.0 1.72e-001   2.7 1.00e+000 1.00e+000f  1
  18 -6.4005373e+003 0.00e+000 8.58e+001  -1.0 6.60e-002   3.1 1.00e+000 1.00e+000f  1
  19 -7.9483772e+003 0.00e+000 8.55e+001  -1.0 1.97e-001   2.6 1.00e+000 1.00e+000f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  20 -8.6255449e+003 0.00e+000 8.74e+001  -1.0 7.57e-002   3.1 1.00e+000 1.00e+000f  1
  21 -1.0648856e+004 0.00e+000 8.71e+001  -1.0 2.26e-001   2.6 1.00e+000 1.00e+000f  1
  22 -1.1533109e+004 0.00e+000 8.88e+001  -1.0 8.65e-002   3.0 1.00e+000 1.00e+000f  1
  23 -1.4162536e+004 0.00e+000 8.87e+001  -1.0 2.59e-001   2.5 1.00e+000 1.00e+000f  1
  24 -1.5303034e+004 0.00e+000 9.06e+001  -1.0 9.93e-002   3.0 1.00e+000 1.00e+000f  1
  25 -1.8659231e+004 0.00e+000 9.04e+001  -1.0 2.97e-001   2.5 1.00e+000 1.00e+000f  1
  26 -2.0076080e+004 0.00e+000 9.21e+001  -1.0 1.14e-001   2.9 1.00e+000 1.00e+000f  1
  27 -2.4223108e+004 0.00e+000 9.19e+001  -1.0 3.40e-001   2.4 1.00e+000 1.00e+000f  1
  28 -3.8390691e+004 0.00e+000 1.14e+002  -1.0 1.32e+000   2.0 8.54e-001 1.00e+000f  1
  29 -8.1892901e+004 0.00e+000 9.70e+001  -1.0 3.23e+000   1.5 1.00e+000 1.00e+000f  1iter    objective    inf_pr
  30 -2.0963680e+005 0.00e+000 9.87e+001  -1.0 9.86e+000   1.0 7.04e-001 1.00e+000f  1
  31 -2.2037644e+005 0.00e+000 9.86e+001  -1.0 1.06e+003    -  5.52e-002 9.98e-004f  1
  32 -1.6348646e+006 0.00e+000 5.75e+002  -1.0 1.05e+003    -  1.00e-003 1.48e-001f  1
  33 -1.8918057e+006 0.00e+000 5.54e+002  -1.0 8.72e+002    -  4.35e-001 3.65e-002f  1
  34 -2.4495318e+006 0.00e+000 5.02e+002  -1.0 8.40e+002    -  8.86e-003 8.75e-002f  1
  35 -5.2250672e+006 0.00e+000 5.27e+002  -1.0 7.73e+002    -  4.70e-004 1.00e+000f  1
  36 -5.2523992e+006 0.00e+000 1.98e+002  -1.0 6.40e+002    -  7.14e-001 6.49e-001f  1
  37 -5.2609835e+006 0.00e+000 2.70e+001  -1.0 1.43e+002    -  9.83e-001 8.81e-001f  1
  38 -5.2618522e+006 0.00e+000 1.42e-001  -1.0 1.57e+001    -  5.59e-001 1.00e+000f  1
  39 -5.2618710e+006 0.00e+000 1.78e-002  -1.7 1.36e+001    -  1.00e+000 8.73e-001f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  40 -5.2618762e+006 0.00e+000 3.09e-003  -2.5 3.06e+000    -  1.00e+000 9.12e-001f  1
  41 -5.2618770e+006 0.00e+000 3.06e-004  -2.5 1.08e+000    -  1.00e+000 1.00e+000f  1
  42 -5.2618775e+006 0.00e+000 2.19e-005  -3.8 3.56e-001    -  1.00e+000 9.99e-001f  1
  43 -5.2618775e+006 0.00e+000 2.87e-007  -5.7 3.26e-002    -  1.00e+000 1.00e+000h  1
  44 -5.2618775e+006 0.00e+000 4.29e-011  -8.6 4.29e-004    -  1.00e+000 1.00e+000h  1

Number of Iterations....: 44

                                   (scaled)                 (unscaled)
Objective...............: -1.3409473792764160e+006  -5.2618775162806567e+006
Dual infeasibility......:  4.2919223730564227e-011   1.6841503391873404e-010
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  3.8717785706145332e-009   1.5192859111091429e-008
Overall NLP error.......:  3.8717785706145332e-009   1.5192859111091429e-008


Number of objective function evaluations             = 51
Number of objective gradient evaluations             = 45
Number of equality constraint evaluations            = 51
Number of inequality constraint evaluations          = 51
Number of equality constraint Jacobian evaluations   = 45
Number of inequality constraint Jacobian evaluations = 45
Number of Lagrangian Hessian evaluations             = 44
Total CPU secs in IPOPT (w/o function evaluations)   =      0.329
Total CPU secs in NLP function evaluations           =      0.002

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)        51
       nlp_g  |        0 (       0)        0 (       0)        51
```

```
  nlp_grad_f  |        0 (       0)        0 (       0)        46
  nlp_hess_l  |        0 (       0)        0 (       0)        44
   nlp_jac_g  |        0 (       0)        0 (       0)        46
       total  | 337.00ms (337.00ms) 337.07ms (337.07ms)         1
```

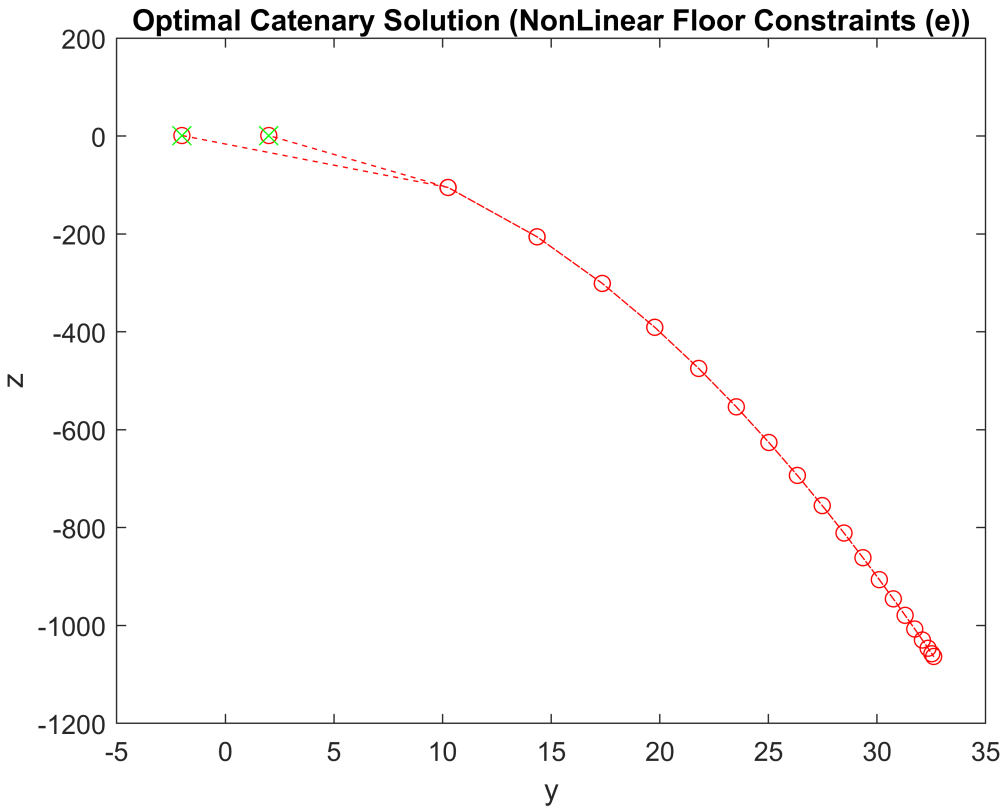


changing initial values to vector of ones

```
pe2 = CatenaryE(ones(N,1),ones(N,1));
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        4
Number of nonzeros in inequality constraint Jacobian.:       76
Number of nonzeros in Lagrangian Hessian.............:      158

Total number of variables............................:       80
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        4
Total number of inequality constraints...............:       38
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       38

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 3.00e+000 8.32e+001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 -1.1467814e+004 0.00e+000 3.00e+002  -1.0 3.00e+000   2.0 1.00e+000 1.00e+000f  1
   2 -5.2592604e+004 0.00e+000 9.70e+001  -1.0 2.91e+000   1.5 1.00e+000 1.00e+000f  1
   3 -7.4953999e+004 0.00e+000 9.72e+001  -1.0 8.84e+000   1.0 1.00e+000 1.86e-001f  1
   4 -4.2181431e+005 0.00e+000 1.02e+002  -1.0 2.75e+001   0.6 1.00e+000 1.00e+000f  1
   5 -4.5867182e+005 0.00e+000 1.01e+002  -1.0 9.49e+002    -  1.00e+000 4.01e-003f  1
```

```
   6 -2.1321924e+006 0.00e+000 8.02e+001  -1.0 9.19e+002    -  1.00e+000 2.08e-001f  1
   7 -5.4029573e+006 0.00e+000 7.98e+001  -1.0 8.43e+002    -  9.12e-001 1.00e+000f  1
   8 -5.4063330e+006 0.00e+000 8.01e+000  -1.0 6.90e+000    -  5.51e-001 1.00e+000f  1
   9 -5.4065172e+006 0.00e+000 7.79e-002  -1.0 3.38e+001    -  1.00e+000 1.00e+000f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 -5.4065282e+006 0.00e+000 1.53e-002  -1.0 2.12e+001    -  1.00e+000 1.00e+000f  1
  11 -5.4065416e+006 0.00e+000 4.02e-003  -1.7 9.74e+000    -  1.00e+000 9.16e-001f  1
  12 -5.4065456e+006 0.00e+000 8.46e-004  -2.5 3.51e+000    -  1.00e+000 1.00e+000f  1
  13 -5.4065461e+006 0.00e+000 2.91e-004  -3.8 8.71e-001    -  1.00e+000 9.12e-001f  1
  14 -5.4065461e+006 0.00e+000 4.69e-006  -3.8 1.87e-001    -  1.00e+000 1.00e+000f  1
  15 -5.4065462e+006 0.00e+000 5.84e-009  -5.7 1.06e-002    -  1.00e+000 1.00e+000f  1
  16 -5.4065462e+006 0.00e+000 2.47e-012  -8.6 8.64e-005    -  1.00e+000 1.00e+000h  1
Number of Iterations....: 16

                                   (scaled)                 (unscaled)
Objective...............: -1.3778150273094985e+006  -5.4065461671624724e+006
Dual infeasibility......:  2.4720711588877009e-012   9.7004072274753391e-012
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  2.5109652084769612e-009   9.8530274780635964e-009
Overall NLP error.......:  2.5109652084769612e-009   9.8530274780635964e-009


Number of objective function evaluations             = 17
Number of objective gradient evaluations             = 17
Number of equality constraint evaluations            = 17
Number of inequality constraint evaluations          = 17
Number of equality constraint Jacobian evaluations   = 17
Number of inequality constraint Jacobian evaluations = 17
Number of Lagrangian Hessian evaluations             = 16
Total CPU secs in IPOPT (w/o function evaluations)   =      0.136
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)        17
       nlp_g  |        0 (       0)        0 (       0)        17
  nlp_grad_f  |        0 (       0)        0 (       0)        18
  nlp_hess_l  |   1.00ms ( 62.50us)   1.00ms ( 62.62us)        16
   nlp_jac_g  |        0 (       0)        0 (       0)        18
       total  | 147.00ms (147.00ms) 147.03ms (147.03ms)         1
```



Note that when an initial vecotr is changed, moreover of the vector z, the solution change drastically.

Auxiliar Functions

```
function p=Plot(ti, Y, Z)
    figure;
    p=plot(Y,Z,'--or'); hold on;
    plot(-2,1,'xg','MarkerSize',10);
    plot(2,1,'xg','MarkerSize',10);
    xlabel('y'); ylabel('z');
    title(ti);          hold off;
end

function [p] = CatenaryD(Y0,Z0)
    global N m D g
    opti2d = casadi.Opti();

    Y = opti2d.variable(N,1);
    Z = opti2d.variable(N,1);

    opti2d.set_initial(Y,Y0);
    opti2d.set_initial(Y,Z0);

    Vchain = 0.5*D*(sum(diff(Y).^2) + sum(diff(Z).^2)) +  m*g*sum(Z);
```

```
    opti2d.minimize(Vchain)
    opti2d.subject_to(Y(1)==-2); opti2d.subject_to(Z(1)==1);
    opti2d.subject_to(Y(end)==2); opti2d.subject_to(Z(end)==1);

    opti2d.subject_to(Z(2:end-1) >= -0.2+0.1*Y(2:end-1).^2);

    opti2d.solver('ipopt')
    sol2d = opti2d.solve();

    Ysold = opti2d.value(Y);
    Zsold = opti2d.value(Z);

    p = Plot("Optimal Catenary Solution (NonLinear Floor Constraints (d))", Ysold, Zsold);
end


function [p] = CatenaryE(Y0,Z0)
    global N m D g
    opti2e = casadi.Opti();

    Y = opti2e.variable(N,1);
    Z = opti2e.variable(N,1);

    opti2e.set_initial(Y,Y0);
    opti2e.set_initial(Y,Z0);

    Vchain = 0.5*D*(sum(diff(Y).^2) + sum(diff(Z).^2)) +  m*g*sum(Z);

    opti2e.minimize(Vchain)
    opti2e.subject_to(Y(1)==-2); opti2e.subject_to(Z(1)==1);
    opti2e.subject_to(Y(end)==2); opti2e.subject_to(Z(end)==1);

    opti2e.subject_to(Z(2:end-1) >= -Y(2:end-1).^2);

    opti2e.solver('ipopt')
    sol2e = opti2e.solve();

    Ysole = opti2e.value(Y);
    Zsole = opti2e.value(Z);

    p = Plot("Optimal Catenary Solution (NonLinear Floor Constraints (e))", Ysole, Zsole);
end
```

This work was made using CasADi.