Homework 3

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1. Limite on an Optimization Problem

Given a matrix $J \in \mathbb{R}^{m\times n}$ and a matrix $Q \in \mathbb{R}^{n\times n}$ symmetric defined positive $Q \succ 0$, a vector of measures $\eta \in \mathbb{R}^m$ and a point $\overline{x} \in \mathbb{R}$, calculate the limite

$$\lim_{\alpha\to 0^+} \arg\min_x \frac{1}{2}\|\eta - Jx\|_2^2 + \frac{\alpha}{2}(x-\bar{x})^T Q(x-\bar{x})$$

In order to calculate this limit, the function $f(x)$ is defined to find its minimum value using the derivative

$$f(x) = \frac{1}{2}||\eta - Jx||_2^2 + \frac{\alpha}{2}(x-\overline{x})^T Q(x-\overline{x}) = \frac{1}{2}(\eta - Jx)^T(\eta - Jx) + \frac{\alpha}{2}(x-\overline{x})^T Q(x-\overline{x})$$

$$= \frac{1}{2}(\eta^T\eta - x^T J^T\eta - \eta^T Jx + x^T J^T Jx) + \frac{\alpha}{2}(x-\overline{x})^T Q(x-\overline{x})$$

$$= \frac{1}{2}(\eta^T\eta - 2J^T\eta x + x^T J^T Jx) + \frac{\alpha}{2}(x-\overline{x})^T Q(x-\overline{x})$$

$$\nabla f(x) = -J^T\eta + J^T Jx + \alpha Q(x-\overline{x})$$

$$\nabla^2 f(x) = J^T J + \alpha Q \succ 0$$

then, to find the minimum we have to solve $\nabla f(x) = 0$

$$
\begin{aligned}
&\nabla f(x) = -J^T\eta + J^T Jx + \alpha Q(x-\overline{x}) = -J^T\eta + J^T Jx + \alpha Qx - \alpha\overline{x}Q\overline{1} = 0\\
&(J^T J + \alpha Q)x = J^T\eta + \alpha\overline{x}Q\overline{1}\\
&\Rightarrow \quad x(\alpha) = (J^T J + \alpha Q)^{-1}(J^T\eta + \alpha\overline{x}Q\overline{1})
\end{aligned}
$$

here $\overline{1} \in \mathbb{R}^n$ is a vector of ones. Using SVD decomposition $J = U\Sigma V^T$,with $U \in \mathbb{R}^{m\times m}, \Sigma \in \mathbb{R}^{m\times n}, V^T \in \mathbb{R}^{n\times n}$

$$
J^T J + \alpha Q = V\Sigma^T U^T U\Sigma V^T + \alpha Q = V\Sigma^T\Sigma V^T + \alpha Q
$$

descomposing Q in terms of V, $Q = U'\Sigma' V^T$ since Q is square then $U', \Sigma', V^T \in \mathbb{R}^{n\times n}$

$$
\begin{aligned}
&J^T J + \alpha Q = V\Sigma^T\Sigma V^T + \alpha U'\Sigma' V^T \ = (V\Sigma^T\Sigma + \alpha U'\Sigma')V^T = (V\Sigma^T\Sigma + \alpha VV^T U'\Sigma')V^T\\
&J^T J + \alpha Q = V(\Sigma^T\Sigma + \alpha V^T U'\Sigma')V^T
\end{aligned}
$$

calculating the inverse

$$
\begin{aligned}
&(J^T J + \alpha Q)^{-1} = (V(\Sigma^T\Sigma + \alpha V^T U'\Sigma')V^T))^{-1} = (V^T)^{-1}(\Sigma^T\Sigma + \alpha V^T U'\Sigma')^{-1}V^{-1}\\
&= V(\Sigma^T\Sigma + \alpha V^T U'\Sigma')^{-1}V^T = V(\Sigma^T\Sigma + \alpha W\Sigma')^{-1}V^T
\end{aligned}
$$

writing in matricial form and defining $W = V^T U', W \in \mathbb{R}^{n\times n}$

$$
(J^T J + \alpha Q)^{-1} = V \begin{pmatrix}
\sigma_1^2 + \alpha W_{1,1}\sigma_1' & 0 & \cdots & & & 0\\
0 & \sigma_2^2 + \alpha W_{2,2}\sigma_2' & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \sigma_r^2 + \alpha W_{r,r}\sigma_r' & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & \alpha W_{n-1,n-1}\sigma_{n-1}' & 0\\
0 & 0 & \cdots & \cdots & \cdots & \alpha W_{n,n}\sigma_n'
\end{pmatrix}^{-1} V^T
$$

$$
(J^T J + \alpha Q)^{-1} = V \begin{pmatrix}
\frac{1}{\sigma_1^2 + \alpha W_{1,1}\sigma_1'} & 0 & \cdots & & & 0\\
0 & \frac{1}{\sigma_2^2 + \alpha W_{2,2}\sigma_2'} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \frac{1}{\sigma_r^2 + \alpha W_{r,r}\sigma_r'} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & \frac{1}{\alpha W_{n-1,n-1}\sigma_{n-1}'} & 0\\
0 & 0 & \cdots & \cdots & \cdots & \frac{1}{\alpha W_{n,n}\sigma_n'}
\end{pmatrix} V^T
$$

now let's calculate the product $(J^TJ+\alpha Q)^{-1}(J^T\eta+\alpha\bar{x}Q)$

$$
\begin{aligned}
&J^T\eta+\alpha\bar{x}Q\bar{1}=V\Sigma^TU^T\eta+\alpha\bar{x}U'\Sigma'V^T\ \bar{1}=V\Sigma^TU^T\eta+\alpha\bar{x}VV^TU'\Sigma'V^T\ \bar{1}\\
&=V(\Sigma^TU^T\eta+\alpha\bar{x}W\Sigma'V^T\ \bar{1})
\end{aligned}
$$

$$
\begin{aligned}
&(J^TJ+\alpha Q)^{-1}(J^T\eta+\alpha\bar{x}Q)=V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}V^T\ V(\Sigma^TU^T\eta+\alpha\bar{x}W\Sigma'V^T)\\
&=V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}(\Sigma^TU^T\eta+\alpha\bar{x}W\Sigma'V^T)\\
&=V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}(\Sigma^T)U^T\eta+V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}(\alpha\bar{x}W\Sigma')V^T
\end{aligned}
$$

calculating each matrix by separeted

- $V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}\Sigma^TU^T\eta$

$$
V\begin{pmatrix}
\frac{1}{\sigma_1^2+\alpha W_{1,1}\sigma_1'} & 0 & \cdots & & & 0\\
0 & \frac{1}{\sigma_2^2+\alpha W_{2,2}\sigma_2'} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \frac{1}{\sigma_r^2+\alpha W_{r,r}\sigma_r'} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & \frac{1}{\alpha W_{n-1,n-1}\sigma_{n-1}'} & 0\\
0 & 0 & \cdots & \cdots & \cdots & \frac{1}{\alpha W_{n,n}\sigma_n'}
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & 0 & \cdots & & & & \\
0 & \sigma_2 & 0 & \cdots & & & \\
 & & \ddots & \vdots & \cdots & & \\
 & & \cdots & \sigma_r & \cdots & & \\
 & & & \vdots & 0 & \cdots & \\
 & & & & & \ddots & \vdots\\
 & & & & & \cdots & 0
\end{pmatrix}U^T\eta
$$

$$
V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}\Sigma^TU^T\eta=V\begin{pmatrix}
\frac{\sigma_1}{\sigma_1^2+\alpha W_{1,1}\sigma_1'} & 0 & \cdots & & & 0\\
0 & \frac{\sigma_2}{\sigma_2^2+\alpha W_{2,2}\sigma_2'} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \frac{\sigma_r}{\sigma_r^2+\alpha W_{r,r}\sigma_r'} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & 0 & 0\\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}V^T \tag{6}
$$

- $V(\Sigma^T\Sigma+\alpha W\Sigma')^{-1}(\alpha\bar{x}W\Sigma')V^T$

$$
V\begin{pmatrix}
\frac{1}{\sigma_1^2+\alpha W_{1,1}\sigma_1^{'}} & 0 & \cdots & & & 0\\
0 & \frac{1}{\sigma_2^2+\alpha W_{2,2}\sigma_2^{'}} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \frac{1}{\sigma_r^2+\alpha W_{r,r}\sigma_r^{'}} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & \frac{1}{\alpha W_{n-1,n-1}\sigma_{n-1}^{'}} & 0\\
0 & 0 & \cdots & \cdots & \cdots & \frac{1}{\alpha W_{n,n}\sigma_n^{'}}
\end{pmatrix}
\begin{pmatrix}
\alpha\bar{x}W_{1,1}\sigma_1^{'} & 0 & \cdots & & & & \\
0 & \alpha\bar{x}W_{2,2}\sigma_2^{'} & 0 & \cdots & & & \\
 & & \ddots & \vdots & \cdots & & \\
 & & \cdots & \alpha\bar{x}W_{r,r}\sigma_r^{'} & \cdots & & \\
 & & & \vdots & 0 & \cdots & \\
 & & & & & \ddots & \vdots\\
 & & & & & \cdots & 0
\end{pmatrix}V^T
$$

$$
V(\Sigma^T\Sigma+\alpha W\Sigma^{'})^{-1}(\alpha\bar{x}W\Sigma^{'})V^T = V\begin{pmatrix}
\alpha\frac{\bar{x}W_{1,1}\sigma_1^{'}}{\sigma_1^2+\alpha W_{1,1}\sigma_1^{'}} & 0 & \cdots & & & 0\\
0 & \alpha\frac{\bar{x}W_{2,2}\sigma_2^{'}}{\sigma_2^2+\alpha W_{2,2}\sigma_2^{'}} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \alpha\frac{\bar{x}W_{r,r}\sigma_r^{'}}{\sigma_r^2+\alpha W_{r,r}\sigma_r^{'}} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & 0 & 0\\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}V^T
\tag{7}
$$

Now let's calculate the limite of both matrixes

$$
\lim_{\alpha\to 0^+}(J^TJ+\alpha Q)^{-1}J^T = \lim_{\alpha\to 0^+}V(\Sigma^T\Sigma+\alpha W\Sigma^{'})^{-1}\Sigma^T
$$

$$
= \lim_{\alpha\to 0^+}V\begin{pmatrix}
\frac{\sigma_1}{\sigma_1^2+\alpha W_{1,1}\sigma_1^{'}} & 0 & \cdots & & & 0\\
0 & \frac{\sigma_2}{\sigma_2^2+\alpha W_{2,2}\sigma_2^{'}} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \frac{\sigma_r}{\sigma_r^2+\alpha W_{r,r}\sigma_r^{'}} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & 0 & 0\\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}V^T = V\begin{pmatrix}
\sigma_1^{-1} & 0 & \cdots & & & 0\\
0 & \sigma_2^{-1} & 0 & \cdots & \cdots & 0\\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots\\
0 & \cdots & \cdots & \sigma_r^{-1} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & 0 & 0\\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}V^T = V\Sigma^+V^T
$$

$$
\lim_{\alpha\to 0^+}(J^TJ+\alpha Q)^{-1}\alpha\bar{x}Q = \lim_{\alpha\to 0^+} V(\Sigma^T\Sigma+\alpha W\Sigma^{'})^{-1}(\alpha\bar{x}W\Sigma^{'})V^T
$$

$$
= \lim_{\alpha\to 0^+} V \begin{pmatrix}
\alpha\dfrac{\bar{x}W_{1,1}\sigma_1^{'}}{\sigma_1^2+\alpha W_{1,1}\sigma_1^{'}} & 0 & \cdots & & & 0 \\
0 & \alpha\dfrac{\bar{x}W_{2,2}\sigma_2^{'}}{\sigma_2^2+\alpha W_{2,2}\sigma_2^{'}} & 0 & \cdots & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \alpha\dfrac{\bar{x}W_{r,r}\sigma_r^{'}}{\sigma_r^2+\alpha W_{r,r}\sigma_r^{'}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix} V^T = V\,\bar{\bar{0}}\,V^T
$$

where $\bar{\bar{0}} \in \mathbb{R}^{n\times n}$ is a zero matrix.

Then, the limit is

$$
\lim_{\alpha\to 0}(J^TJ+\alpha Q)^{-1}(J^T\eta+\alpha\bar{x}Q) = V\Sigma^+V^T\eta + V\,\bar{\bar{0}}\,V^T = J^+\eta
$$

which gives the same result found for the laest square problem without regularization. □

2. Linear fitting in L_2

Suppose there is a set of N noisy measures $(x_i, y_i) \in \mathbb{R}^2$ that we want to fit a line $y = ax + b$. The previous can be expressed as the following optimization problem

$$
\min_{a\,b}\sum_{i=1}^{N}(ax_i+b-y_i)^2 = \min_{a\,b}\left\|J\begin{pmatrix}a\\b\end{pmatrix}-y\right\|_2^2
$$

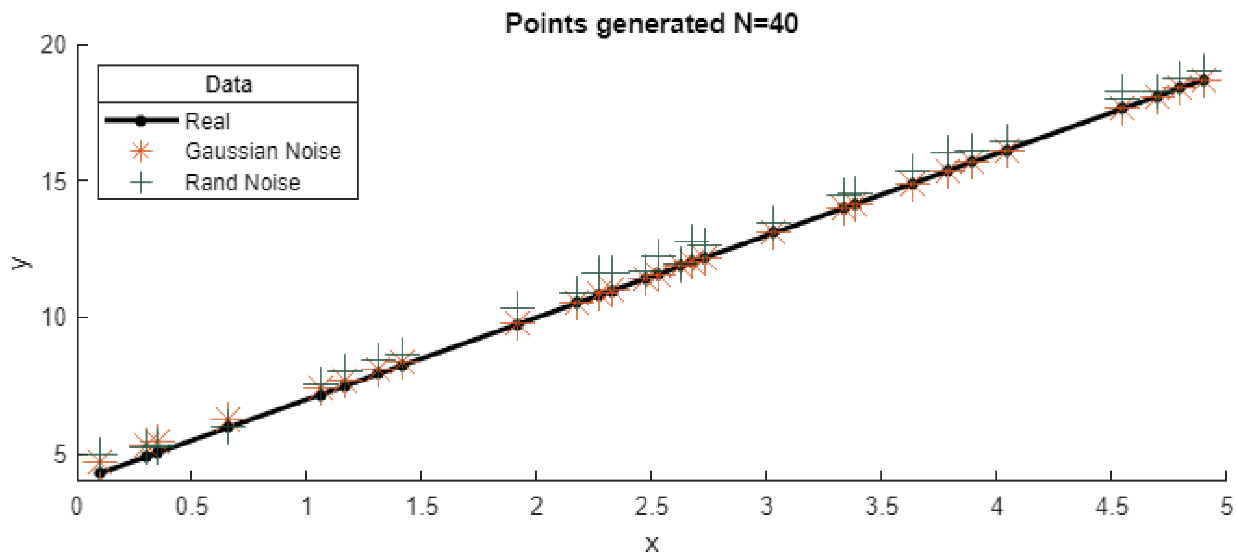
Like was discussed in class, the optimal solution to the previous problem can be calculated explicity by solving the normal Gauss equations

$$
J^TJ\begin{pmatrix}a\\b\end{pmatrix} = J^Ty
$$

1. Data generation

Generate the problem data. Take $N = 30$ points in the interval $[0, 5]$ and generate the true outputs $y_i = 3x_i + 4$. Add gaussian noise with zero mean and standard deviation 1 to get the noisy data and plot them. *Advice:* check the commands **`linspace`** and **`randn`**. If you want a "random" reproducible serie, use **`rng`**.

```
N = 30; rng(7);

xrand = zeros(N,1);
x     = linspace(0,5,100);
xi    = randi(100,1,N);

for i=1:1:N; xrand(i) = x(xi(i)); end

y       = 3*xrand + 4;
ynoise  = y + (1/sqrt(2*pi)).*exp(-xrand.^2/2);
ynoise1 = y + 0.9*rand(N,1);

clf
figure(Position=[100 100 800 300]);
hold on
plot(xrand,y, '.-', MarkerSize=14, Color='black', LineWidth=2)
plot(xrand,ynoise, '*', MarkerSize=14)
plot(xrand,ynoise1, '+', MarkerSize=14, Color=[21/250 71/250 52/250])
leg = legend('Real', 'Gaussian Noise','Rand Noise','Location','northwest');
title(leg,'Data');
xlabel('x'); ylabel('y'); title('Points generated N=40')
hold off
```



2. Problem solution

Write the matrix J. Calculate the coefficients a, b using the last equation and plot the measures and line getted in the same plane.

The equation (1) can be rewrite using residual vector r

$$r(x) = (ax_1 + b - y_1, ax_2 + b - y_2, \dots, ax_N + b - y_N)$$

then

$$\min_{a,b\in\mathbb{R}} \sum_{i=0}^{N} r_i^2(a,b) = \min_{a,b\in\mathbb{R}} ||r||_2^2$$

Jacobian

$$J = \left[\frac{\partial r_i}{\partial x_j}\right]_{ij}$$

with $i = 1, \ldots, N$, $j = 1, 2$, and $x_1 = a$, $x_2 = b$. Then the Jacobian is

$$J = \begin{pmatrix} \frac{\partial r_1}{\partial a} & \frac{\partial r_1}{\partial b} \\ \frac{\partial r_2}{\partial a} & \frac{\partial r_2}{\partial b} \\ \vdots & \vdots \\ \frac{\partial r_N}{\partial a} & \frac{\partial r_N}{\partial b} \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_{N-1} & 1 \\ x_N & 1 \end{pmatrix}$$

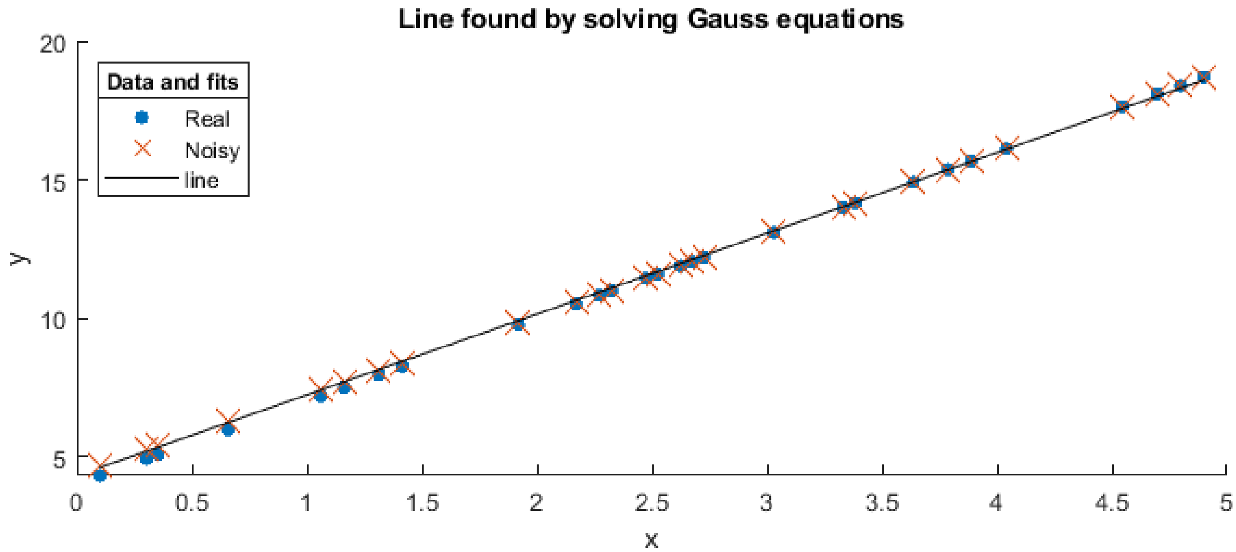
```
J = [xrand  ones(N,1)];
ab = J \ ynoise;             % A \ b %Ax = b  %'dividir a izquierda'
disp(['Optimal solution found by solving Gauss equations: a = ' num2str(ab(1)) ' b = ' num2str(
```

```
Optimal solution found by solving Gauss equations: a = 2.9181 b = 4.3045
```

```
disp(['The residual error by solgin the Gauss equation is r = ' num2str(sum(ab(1)*xrand+ab(2)-y
```

```
The residual error by solgin the Gauss equation is r = 1.3145e-13
```

```
clf
figure(Position=[100 100 800 300]);
hold on
plot(xrand, y, '.', MarkerSize=14)
plot(xrand, ynoise, 'x', MarkerSize=14)
plot(xrand,ab(1)*xrand+ab(2), '-',Color='black')
leg = legend('Real', 'Noisy','line','Location','northwest');
title(leg,'Data and fits');
xlabel('x'); ylabel('y'); title('Line found by solving Gauss equations')
hold off
```



Let's verifyt the coefficients using CasADI.

```
import casadi.*

opti = casadi.Opti();
a_opt = opti.variable();
b_opt = opti.variable();
opti.minimize(norm(a_opt*xrand+b_opt-ynoise,2));

opti.solver('ipopt');

sol = opti.solve();
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        3

Total number of variables............................:        2
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 6.8661733e+001 0.00e+000 1.59e+001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 6.6373392e+001 0.00e+000 1.59e+001  -1.0 1.39e+005    -  1.00e+000 6.10e-005f 15
   2 5.5605779e+001 0.00e+000 1.59e+001  -1.0 1.25e+005    -  1.00e+000 6.10e-005f 15
   3 1.6121968e+001 0.00e+000 1.59e+001  -1.0 7.37e+004    -  1.00e+000 6.10e-005f 15
   4 1.1849837e+001 0.00e+000 1.59e+001  -1.0 1.80e+003    -  1.00e+000 9.77e-004f 11
   5 1.0363910e+001 0.00e+000 1.59e+001  -1.0 7.13e+002    -  1.00e+000 1.95e-003f 10
   6 4.5094273e+000 0.00e+000 1.59e+001  -1.0 4.77e+002    -  1.00e+000 1.95e-003f 10
   7 5.4311814e-001 0.00e+000 1.13e+001  -1.0 3.92e+001    -  1.00e+000 7.81e-003f  8
   8 3.8251916e-001 0.00e+000 1.29e-001  -1.0 4.87e-002    -  1.00e+000 5.00e-001f  2
```

```
   9 3.8250658e-001 0.00e+000 8.50e-006  -2.5 1.95e-004    -  1.00e+000 1.00e+000f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 3.8250658e-001 0.00e+000 5.34e-014  -8.6 1.28e-008    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 10

                                   (scaled)                 (unscaled)
Objective...............:   3.8250658000710780e-001    3.8250658000710780e-001
Dual infeasibility......:   5.3401727484470030e-014    5.3401727484470030e-014
Constraint violation....:   0.0000000000000000e+000    0.0000000000000000e+000
Complementarity.........:   0.0000000000000000e+000    0.0000000000000000e+000
Overall NLP error.......:   5.3401727484470030e-014    5.3401727484470030e-014


Number of objective function evaluations             = 121
Number of objective gradient evaluations             = 11
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 10
Total CPU secs in IPOPT (w/o function evaluations)   =      0.178
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)       121
  nlp_grad_f  |        0 (       0)        0 (       0)        12
  nlp_hess_l  |        0 (       0)        0 (       0)        10
       total  | 180.00ms (180.00ms) 180.26ms (180.26ms)         1
```

```
aopt = sol.value(a_opt);
bopt = sol.value(b_opt);
disp(['Optimal solution found be CasADI: a = ' num2str(aopt) ' b = ' num2str(bopt)]);
```

```
Optimal solution found be CasADI: a = 2.9181 b = 4.3045
```

```
disp(['The residual error is by CasADI r = ' num2str(sum(aopt*xrand+bopt-ynoise))])
```

```
The residual error is by CasADI r = 7.9936e-15
```

```
disp(['CasADI - Gauss: a = ' num2str(aopt-ab(1)) ' b = ' num2str(bopt-ab(2))])
```

```
CasADI - Gauss: a = -4.4409e-16 b = -2.6645e-15
```

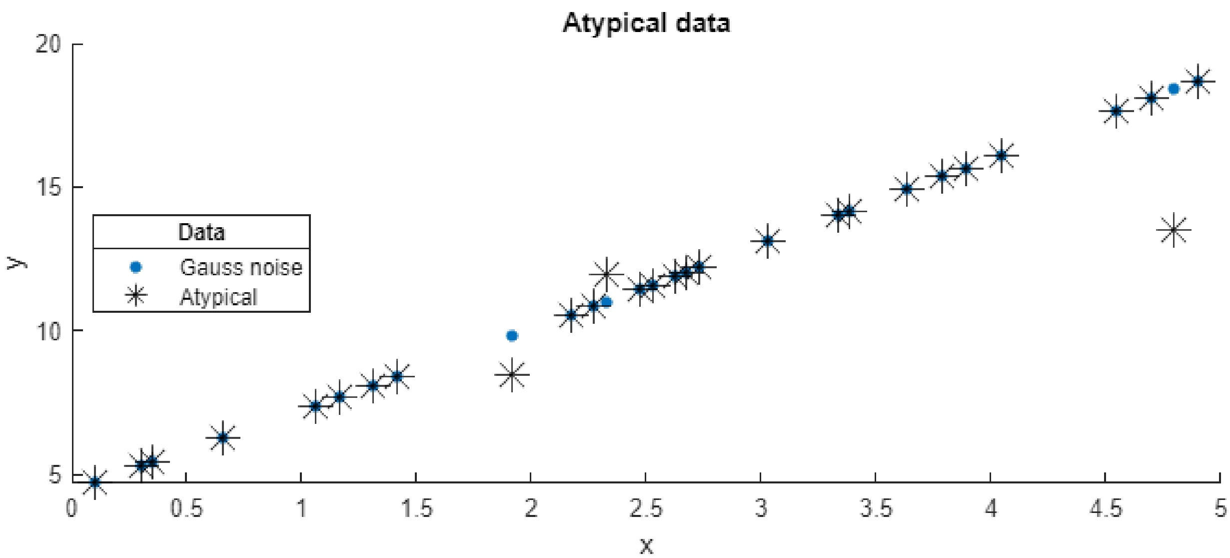
Althouh both solutions have the same value of slope and y-intercept (a, b) with difference in the 15-th decimal, the solution found by solving the Gauss equations has a smaller residual error then it is better.

3. Atypical data

Introduce 3 atypical data in the measures y and plot the new fitted line in the same plane.

```
rng(123);
ynoise_atypical = ynoise;
ynoise_atypical(randi(N,3,1)) = ynoise_atypical(randi(N,3,1))+0.7*rand(1);

clf;
figure(Position=[100 100 800 300]);
hold on
```

```
plot(xrand, ynoise, '.', MarkerSize=14)
plot(xrand, ynoise_atypical, '*', MarkerSize=14 , Color='black')
leg = legend('Gauss noise', 'Atypical','Location','west');
title(leg,'Data');
xlabel('x'); ylabel('y'); title('Atypical data')
hold off
```



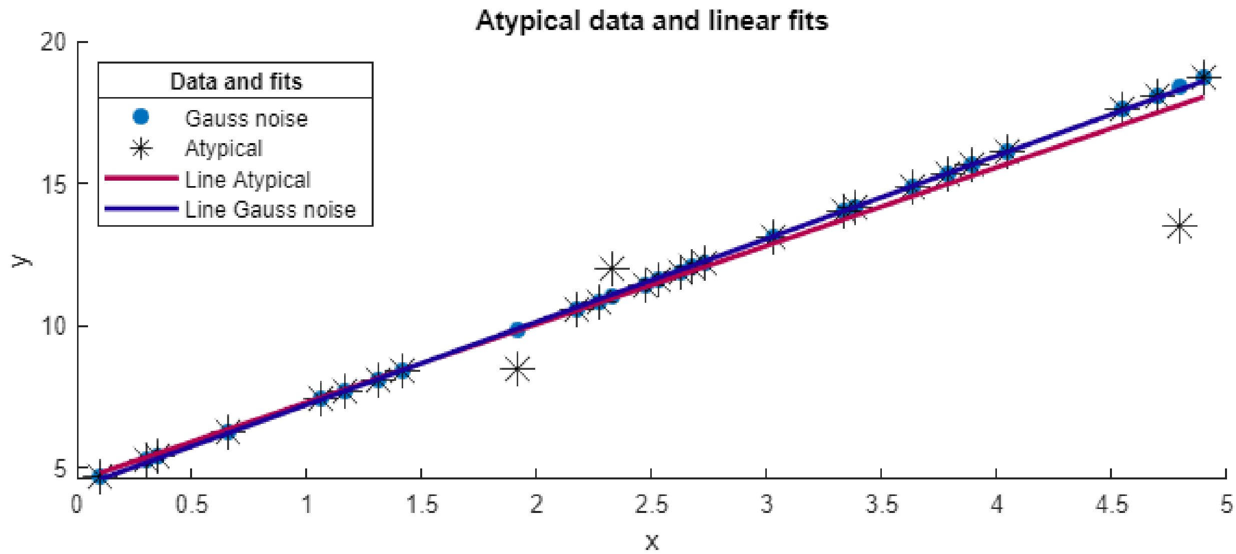

```
ab_atypical = J \ ynoise_atypical;            % A \ b %Ax = b  %'dividir a izquierda'
disp(['Optimal solution found by solving Gauss equations: a = ' num2str(ab_atypical(1)) ' b = '
```

```
Optimal solution found by solving Gauss equations: a = 2.755 b = 4.548
```

```
disp(['The residual error by solgin the Gauss equation is r = ' num2str(sum(ab_atypical(1)*xran
```

```
The residual error by solgin the Gauss equation is r = 9.9476e-14
```

```
clf;
figure(Position=[100 100 800 300]);
hold on
plot(xrand, ynoise, '.', MarkerSize=20, DisplayName='Gauss noise')
plot(xrand, ynoise_atypical, '*', MarkerSize=14 , Color='black', DisplayName='Atypical')
plot(xrand, ab_atypical(1)*xrand+ab_atypical(2), '-', LineWidth=2, Color=[.7 .0 .3], DisplayNar
plot(xrand,ab(1)*xrand+ab(2), '-', LineWidth=2, Color=[.1 0 .6], DisplayName='Line Gauss noise
leg = legend('Location','northwest'); title(leg,'Data and fits');
xlabel('x'); ylabel('y'); title('Atypical data and linear fits')
hold off
```



The measures y (with and without the atypical data) and the matrix $J$are necessary to the following point.

3. Linear Fitting in L_1

Now we are interesting to fit a line to the same measure set, but now using the following objective function

$$\min_{a\,b} \sum_{i=1}^{N} |ax_i + b - y_i|$$

The objective function is not differentiable, then we are going to use width variables to get the following equivalent linear programming problem

$$\begin{aligned} &\min_{a,b,s} \sum s_i \\ &\text{s.a} - s_i \leq ax_i + b - y_i \leq s_i \quad i = 1, \ldots, N \\ &\quad - si \leq 0 \qquad\qquad\qquad\qquad i = 1, \ldots, N \end{aligned} \tag{3}$$

1. Problem reformulation, matrix A

In order to solve the previous linear programming problem use the MATLAB command `linprog`, to do this is necessary to write the problem as follows

$$
\begin{aligned}
\min_{z} \quad & f^T z \\
\text{s.a} \quad & Az \leq b \\
& Cz = d \\
& l_z \leq z \leq u_z
\end{aligned}
$$

Write the matrix A and the vectors f, b. Organize he variables as $z^T = [a, b, s_1, \ldots, s_N]$ Use the matrix J from the previous exercise to define A.

From the equation (3), note that the vairable $s_i = ax_i + b - y_i$ and the product $f^T z$ must be a scalar. Then,

$$
f = [0, 0, 1, \ldots, 1], \quad \rightarrow \quad f^T z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \cdot [a, b, s_1, \ldots, s_N] = s_1 + s_2 + \cdots + s_N
$$

note that the i-th component of the residual vector can be write in terms of J as

$$
ax_i + b - y_i = \begin{pmatrix} x_i & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - y_i = J_i (a \ b)^T - y_i = (J(a \ b)^T - y)_i
$$

respecto to the constraints, it can be write as follos using the matrix J found in the previous point and the previous sustitution

$$
-z' \leq J \begin{pmatrix} a \\ b \end{pmatrix} - y \leq z', \quad \leftrightarrow \quad -z' - J \begin{pmatrix} a \\ b \end{pmatrix} \leq -y \leq z' - J \begin{pmatrix} a \\ b \end{pmatrix}
$$

where $z = [a, b, s]$, $s = [s_1, s_2, \ldots, s_N]$. For the components, since we are working in linear programming this constraint must be splitted in two

$$
\begin{aligned}
-s - J \begin{pmatrix} a \\ b \end{pmatrix} \leq -y \quad & \leftrightarrow \quad s + J \begin{pmatrix} a \\ b \end{pmatrix} \geq y \\
s - J \begin{pmatrix} a \\ b \end{pmatrix} \geq -y \quad & \leftrightarrow \quad -s + J \begin{pmatrix} a \\ b \end{pmatrix} \leq y \\
-s \leq 0 \quad & \leftrightarrow \quad s > 0
\end{aligned} \tag{4}
$$

where the last 0 is a vector of $N + 2$ components. Now let's contruct the matrix A takink in aocunt the previous inequalities

$$\begin{aligned} s_1 + x_1 a + b &= z_3 + x_1 z_1 + z_2 \geq y_1 \\ s_2 + x_2 a + b &= z_4 + x_2 z_1 + z_2 \geq y_2 \\ s_3 + x_3 a + b &= z_5 + x_3 z_1 + z_2 \geq y_3 \\ &\vdots \\ s_N + x_N a + b &= z_{N+2} + x_N z_1 + z_2 \geq y_N \end{aligned} \quad \leftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & 1 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N & 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \\ s_N \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

that can be write as $A_1 z \geq b$ with

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{N+2} \end{pmatrix} = \begin{pmatrix} a \\ b \\ s_1 \\ \vdots \\ s_N \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & 1 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N & 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

similarly the second constraint of (3) can be write as $A_2 z \leq b$

$$\begin{aligned} -s_1 + x_1 a + b &= -z_3 + x_1 z_0 + z_1 \leq y_1 \\ -s_2 + x_2 a + b &= -z_4 + x_2 z_0 + z_1 \leq y_2 \\ -s_3 + x_3 a + b &= -z_5 + x_3 z_0 + z_1 \leq y_3 \\ &\vdots \\ -s_N + x_N a + b &= -z_{N+2} + x_N z_0 + z_1 \leq y_N \end{aligned} \quad \leftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_1 & 1 & -1 & 0 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & -1 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N & 1 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \vdots \\ z_{N+1} \\ z_{N+2} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

with

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x_1 & 1 & -1 & 0 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & -1 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N & 1 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

and the final condition, all variables s bigger than one can be write as $-s = -Iz[2 : end] \leq 0$ where the zero is the vector of zeros.

Now implementing the matrixes A_1, A_2 and the vectors b, f in the code

```
clf;
f = [0; 0; ones(N,1)];
b = [0; 0; ynoise]; b_atypical = [0; 0; ynoise_atypical];

A1 = zeros(N+2); A1(3:end,1:2) = J; A1(3:end,3:end) = eye(N);
A2 = zeros(N+2); A2(3:end,1:2) = J; A2(3:end,3:end) = -eye(N);
```

2. Problem solution

Solve the problem using the y measures from the previous exercise (with and without the atypical data) and plot the results over the L_2 results. Which norm has a better result?

Without atypical data

```
%A1 \ b

opti3 = casadi.Opti();
z_opt3 = opti3.variable(N+2);

opti3.minimize(f'*z_opt3);

opti3.subject_to(A1*z_opt3 >= b);
opti3.subject_to(A2*z_opt3 <= b);
opti3.subject_to(-z_opt3(3:end) <= zeros(N,1));

opti3.solver('ipopt');
sol3 = opti3.solve();
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:     2078
Number of nonzeros in Lagrangian Hessian.............:        0

Total number of variables............................:       32
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:       94
        inequality constraints with only lower bounds:       32
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       62

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 1.87e+001 5.00e-001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 3.1030639e+000 2.59e+000 5.37e+000  -1.0 1.84e+001    -  3.25e-001 8.63e-001h  1
   2 1.3735394e+001 0.00e+000 3.90e+001  -1.0 1.56e+000    -  4.55e-001 9.90e-001h  1
   3 1.1344994e+001 0.00e+000 6.71e+001  -1.0 3.73e-001    -  1.00e+000 9.91e-001f  1
   4 9.3040192e+000 0.00e+000 1.00e-006  -1.0 4.44e-001    -  1.00e+000 1.00e+000f  1
   5 2.4837846e+000 0.00e+000 1.58e+005  -5.7 3.19e-001    -  1.00e+000 7.84e-001f  1
   6 2.0124862e+000 0.00e+000 7.33e+004  -5.7 7.06e-002    -  9.17e-001 5.37e-001f  1
   7 1.8889895e+000 0.00e+000 3.74e+004  -5.7 5.89e-002    -  7.57e-001 4.89e-001f  1
   8 1.8482983e+000 0.00e+000 1.77e+004  -5.7 2.25e-002    -  7.69e-001 5.27e-001f  1
   9 1.8326806e+000 0.00e+000 6.43e+003  -5.7 2.60e-002    -  5.91e-001 6.37e-001f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
```

```
  10 1.8256614e+000 0.00e+000 1.80e+003  -5.7 2.79e-002    -  8.04e-001 7.20e-001f  1
  11 1.8238984e+000 0.00e+000 2.80e+002  -5.7 6.50e-003    -  9.48e-001 8.44e-001f  1
  12 1.8237706e+000 0.00e+000 1.85e-011  -5.7 7.17e-005    -  1.00e+000 1.00e+000f  1
  13 1.8237082e+000 0.00e+000 1.96e+000  -8.6 1.68e-005    -  1.00e+000 9.89e-001f  1
  14 1.8237077e+000 0.00e+000 5.65e-001  -8.6 1.06e-007    -  8.51e-001 7.11e-001f  1
  15 1.8237076e+000 0.00e+000 2.55e-004  -8.6 8.52e-008    -  9.99e-001 1.00e+000f  1
  16 1.8237076e+000 0.00e+000 2.51e-014  -8.6 2.91e-009    -  1.00e+000 1.00e+000h  1

Number of Iterations....: 16

                                   (scaled)                 (unscaled)
Objective...............:  1.8237075577790940e+000   1.8237075577790940e+000
Dual infeasibility......:  2.5091040356528538e-014   2.5091040356528538e-014
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  2.5745917457967955e-009   2.5745917457967955e-009
Overall NLP error.......:  2.5745917457967955e-009   2.5745917457967955e-009


Number of objective function evaluations             = 17
Number of objective gradient evaluations             = 17
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 17
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 17
Number of Lagrangian Hessian evaluations             = 16
Total CPU secs in IPOPT (w/o function evaluations)   =      0.204
Total CPU secs in NLP function evaluations           =      0.003

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)        17
       nlp_g  |   2.00ms (117.65us)   2.00ms (117.76us)        17
  nlp_grad_f  |        0 (       0)        0 (       0)        18
  nlp_hess_l  |        0 (       0)        0 (       0)        16
   nlp_jac_g  |   1.00ms ( 55.56us)   1.00ms ( 55.61us)        18
       total  | 211.00ms (211.00ms) 211.05ms (211.05ms)         1
```

```
zopt3 = sol3.value(z_opt3);
zopt3'
```

```
ans = 1×32
    2.9321    4.2577    0.1411    0.0068    0.0724    0.0101    0.0752    0.0647 ⋯
```

```
%plot(zopt3(3:end),'-o')
%xlabel('i'); ylabel('s')
disp(['Optimal solution found: a = ' num2str(zopt3(1)) ' b = ' num2str(zopt3(2))]);
```

```
Optimal solution found: a = 2.9321 b = 4.2577
```

```
clf;
figure(Position=[100 100 900 400]);

subplot(1,2,1)
hold on
plot(xrand, ynoise, '.', MarkerSize=25, DisplayName='Gauss noise', Color='black')
plot(xrand, ab(1)*xrand+ab(2), '-', LineWidth=2, Color=[.7 .0 .3], DisplayName=['||_2: a=' num2
plot(xrand,zopt3(1)*xrand+zopt3(2), '-', LineWidth=1, Color=[.1 0 .6], DisplayName=['||_1: a='
leg = legend('Location','northwest');
title(leg,'Data and Fits')
```

```
xlabel('x'); ylabel('y'); title('without atypical data', 'Interpreter','latex')
hold off
disp(['Residual sum: Σ si = ' num2str(sum(zopt3(3:end)))]);
```

```
Residual sum: Σ si = 1.8237
```

Atypical data

```
opti3_aty = casadi.Opti();
z_opt3_aty = opti3_aty.variable(N+2);

opti3_aty.minimize(f'*z_opt3_aty);

opti3_aty.subject_to(A1*z_opt3_aty >= b_atypical);
opti3_aty.subject_to(A2*z_opt3_aty <= b_atypical);
opti3_aty.subject_to(-z_opt3_aty(3:end) <= zeros(N,1));

opti3_aty.solver('ipopt');
sol3_aty = opti3_aty.solve();
```

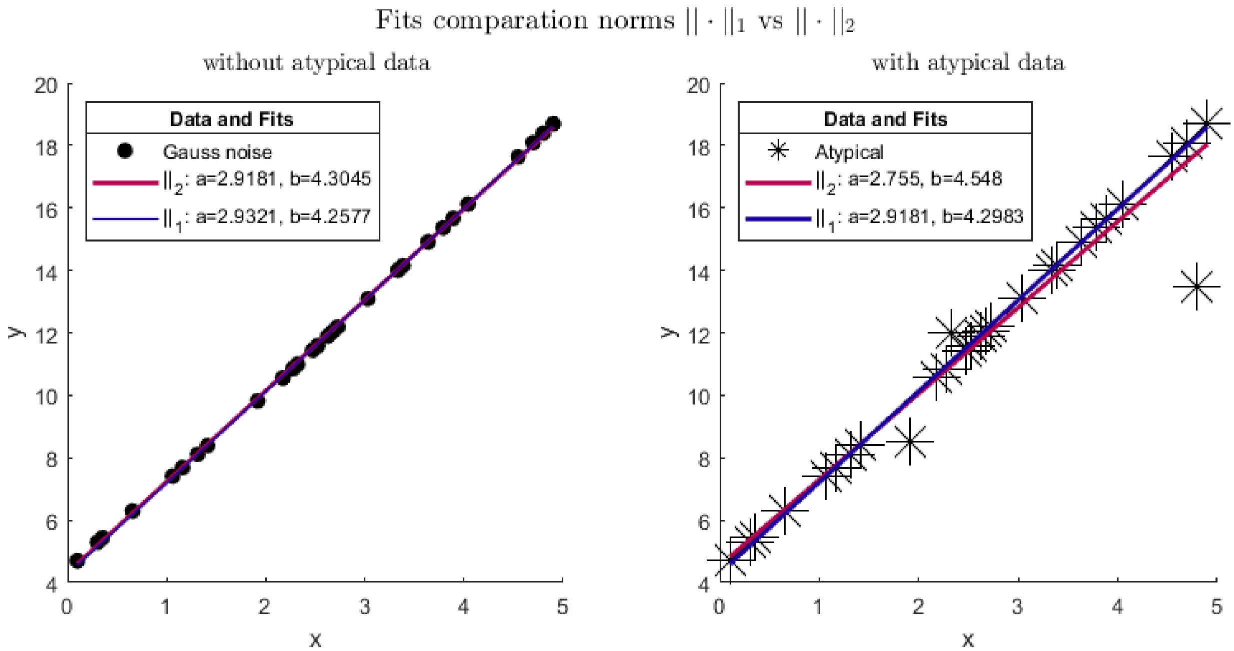
```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:     2078
Number of nonzeros in Lagrangian Hessian.............:        0

Total number of variables............................:       32
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:       94
        inequality constraints with only lower bounds:       32
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:       62

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 0.0000000e+000 1.87e+001 5.00e-001  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 2.0015184e-001 1.77e+001 1.10e+000  -1.0 1.78e+001    -  3.40e-002 5.56e-002h  1
   2 7.9248264e-001 1.53e+001 1.51e+000  -1.0 1.74e+001    -  5.61e-002 1.36e-001f  1
   3 5.0000921e+000 4.84e+000 5.42e+000  -1.0 1.48e+001    -  2.67e-001 6.83e-001h  1
   4 6.6812194e+000 4.54e+000 5.23e+000  -1.0 2.41e+000    -  5.71e-002 6.09e-002h  1
   5 2.0516499e+001 1.54e+000 1.13e+001  -1.0 3.84e+000    -  4.15e-001 6.45e-001h  1
   6 2.5562739e+001 0.00e+000 8.91e+001  -1.0 2.00e+000    -  2.22e-001 9.90e-001h  1
   7 1.9362092e+001 0.00e+000 1.02e+002  -1.0 9.41e-001    -  1.00e+000 9.91e-001f  1
   8 1.5794812e+001 0.00e+000 1.00e-006  -1.0 6.46e-001    -  1.00e+000 1.00e+000f  1
   9 9.2959021e+000 0.00e+000 2.31e+005  -5.7 4.07e-001    -  9.92e-001 7.94e-001f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10 8.8876161e+000 0.00e+000 1.12e+005  -5.7 6.40e-002    -  9.71e-001 5.33e-001f  1
  11 8.7500900e+000 0.00e+000 4.04e+004  -5.7 3.98e-002    -  8.15e-001 6.40e-001f  1
  12 8.7315483e+000 0.00e+000 2.09e+004  -5.7 1.19e-002    -  8.92e-001 4.85e-001f  1
  13 8.7153919e+000 0.00e+000 3.41e+003  -5.7 3.38e-002    -  6.30e-001 8.36e-001f  1
  14 8.7129899e+000 0.00e+000 2.07e+002  -5.7 4.42e-003    -  9.19e-001 9.39e-001f  1
  15 8.7128782e+000 0.00e+000 1.84e-011  -5.7 9.25e-005    -  1.00e+000 1.00e+000f  1
  16 8.7128161e+000 0.00e+000 2.42e+000  -8.6 1.51e-005    -  1.00e+000 9.87e-001f  1
  17 8.7128155e+000 0.00e+000 5.08e-001  -8.6 1.38e-007    -  7.97e-001 7.90e-001f  1
  18 8.7128153e+000 0.00e+000 4.86e-003  -8.6 9.81e-008    -  9.79e-001 1.00e+000f  1
  19 8.7128153e+000 0.00e+000 2.51e-014  -8.6 2.71e-009    -  1.00e+000 1.00e+000h  1
```

```
Number of Iterations....: 19

                                   (scaled)                 (unscaled)
Objective...............:   8.7128152997301669e+000    8.7128152997301669e+000
Dual infeasibility......:   2.5091040356528538e-014    2.5091040356528538e-014
Constraint violation....:   0.0000000000000000e+000    0.0000000000000000e+000
Complementarity.........:   2.6011777452244518e-009    2.6011777452244518e-009
Overall NLP error.......:   2.6011777452244518e-009    2.6011777452244518e-009


Number of objective function evaluations             = 20
Number of objective gradient evaluations             = 20
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 20
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 20
Number of Lagrangian Hessian evaluations             = 19
Total CPU secs in IPOPT (w/o function evaluations)   =      0.125
Total CPU secs in NLP function evaluations           =      0.003

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)        20
       nlp_g  |        0 (       0)        0 (       0)        20
  nlp_grad_f  |        0 (       0)        0 (       0)        21
  nlp_hess_l  |        0 (       0)        0 (       0)        19
   nlp_jac_g  |   3.00ms (142.86us)   3.00ms (142.81us)        21
       total  | 130.00ms (130.00ms) 130.03ms (130.03ms)         1
```

```
zopt3_aty = sol3_aty.value(z_opt3_aty);
zopt3_aty'
```

```
ans = 1×32
    2.9181    4.2983    0.1054    0.0203    0.0827   -0.0000    0.1029    0.0680 ⋯
```

```
subplot(1,2,2)
hold on
%plot(xrand, ynoise, '.', MarkerSize=20, DisplayName='Gauss noise', Color='blue')
plot(xrand, ynoise_atypical, '*', MarkerSize=20, DisplayName='Atypical', Color='black')
plot(xrand, ab_atypical(1)*xrand+ab_atypical(2), '-', LineWidth=2, Color=[.7 .0 .3], DisplayNar
plot(xrand,zopt3_aty (1)*xrand+zopt3_aty(2), '-', LineWidth=2, Color=[.1 0 .6], DisplayName=['
leg = legend('Location','northwest');
title(leg,'Data and Fits')
xlabel('x'); ylabel('y'); title('with atypical data', 'Interpreter','latex')
hold off
sgtitle('Fits comparation norms $||\cdot||_1$ vs $||\cdot||_2$',Interpreter='Latex');
```




```
disp(['RESIDUAL ERRORS' newline ...
    '- without atypical data:' newline ...
    '      Residual sum in ||_1: Σ si = ' num2str(sum(zopt3(3:end))) newline  ...
    '      residual error in ||_2 r = ' num2str(sum(ab(1)*xrand+ab(2)-ynoise)) newline ...
    '- with atypical data' newline ...
    '      Residual sum  in ||_1: Σ si = ' num2str(sum(zopt3_aty(3:end))) newline  ...
    '      residual error in ||_2  r = ' num2str(sum(ab_atypical(1)*xrand+ab_atypical(2)-ynoise
```

```
RESIDUAL ERRORS
- without atypical data:
      Residual sum in ||_1: Σ si = 1.8237
      residual error in ||_2 r = 1.3145e-13
- with atypical data
      Residual sum  in ||_1: Σ si = 8.7128
      residual error in ||_2  r = 9.9476e-14
```

3.

Solve the problem with CasADi and compare the results.

```
%A1 \ b_atypical %ab = J \ ynoise;
```

4. Integral Equation

Consider the following integral equation

$$Ax(t) := \int_0^1 e^{st}\, x(t)\, dt \;=\; \frac{e^{s+1}-1}{s+1} \;=:\; y(s),\quad 0 \le s \le 1$$

a) Exact solution

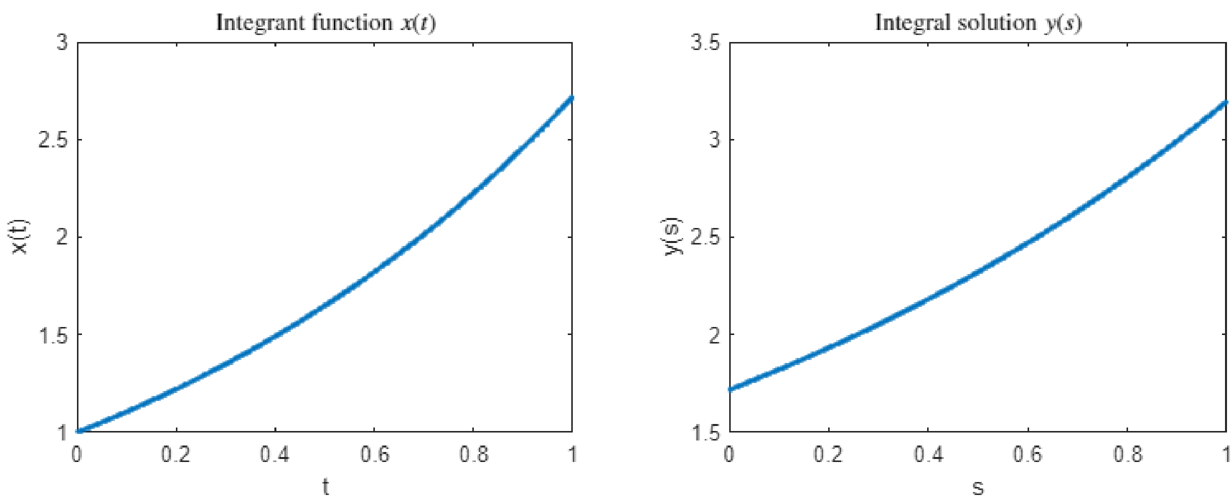
Find the exact solution (*Advice:* Interprete the operator $A : L^2(0,1) \to L^2(0,1)$ as the Laplace transform).

Let's take the solution to be an exponential function $x(t) = e^{at}$

$$\int_0^1 e^{st}e^{at}dt = \int_0^1 e^{(s+a)t}dt = \frac{1}{s+a}e^{(s+a)t}\Big|_0^1 = \frac{e^{s+a}-1}{s+a} = \frac{e^{s+1}-1}{s+1}$$

then $a = 1$ and the solution is $x(x) = e^t$.

```
t = 0:0.001:1;

clf
figure(Position=[10 10 900 300])
subplot(1,2,1)
plot(t,exp(t),'.');
xlabel('t'); ylabel('x(t)');
xlim([0 1]); %ylim([-14 5]);
title('Integrant function $x(t)$', Interpreter='latex');

subplot(1,2,2)
plot(t, (exp(t+1)-1)./(t+1), '.')
xlabel('s'); ylabel('y(s)');
title('Integral solution $y(s)$', Interpreter='latex');
```



b) Problem Discretization

Use the Trapezoidal rule with an integer step $h = 1/n$ and $s = ih$ and $i = 1, \dots, n$ to get a linear system of equations.

Remembering the trapezoidal rule. Let $\{x_k\}$ be a partition of $[0,1]$ such that $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ and $\Delta t_k = \Delta t = 1/n$ be the length of the k-th subinterval

$$\int_a^b f(t)dx \approx \sum_{k=1}^{n-1} \frac{f(t_{k+1}) + f(t_k)}{2} \Delta t_k$$

and applying it to the integral equation

$$\int_0^1 e^{st}x(t)dt \approx \sum_{k=1}^{n-1} \frac{e^{st_{k+1}}x_{k+1} + e^{st_k}x_k}{2} \cdot \frac{1}{n} = \frac{1}{2n}\sum_{k=1}^{n-1}(e^{st_{k+1}}x_{k+1} + e^{st_k}x_k)$$

$$= \frac{1}{2n}\left[e^{st_2}x_2 + e^{st_1}x_1 + e^{st_3}x_3 + e^{st_2}x_2 + e^{st_4}x_4 + e^{st_3}x_3 + \cdots + e^{st_n}x_n + e^{skt_{n-1}}x_{n-1}\right]$$

$$= \frac{1}{2n}\left[e^{st_1}x_1 + 2e^{st_2}x_2 + 2e^{st_3}x_3 + 2e^{st_4}x_4 + \cdots + 2e^{skt_{n-1}}x_{n-1} + e^{st_n}x_n\right] = \frac{e^{s+1}-1}{s+1}$$

with $h = 1/n$ and $s = ih$. Rewriting to the following system for any s. Then

$$y(s) = \int_0^1 e^{st}x(t)dt \approx \frac{1}{2n}\begin{pmatrix} e^{st_1} & 2e^{st_2} & 2e^{st_3} & \cdots & 2e^{skt_{N-1}} & e^{st_N} \end{pmatrix} \cdot \begin{pmatrix} x1 & x2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} = \frac{e^{s+1}-1}{s+1}$$

To build the system now let's evalute the previous expression to each i

$$s = 0: \quad \frac{1}{2n}(1 \quad 2 \quad 2 \quad \cdots \quad 2 \quad 1) \cdot \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} = e^1 - 1$$

$$s = h: \quad \frac{1}{2n}\begin{pmatrix} e^{ht_1} & 2e^{ht_2} & 2e^{ht_3} & \cdots & 2e^{ht_{n-1}} & e^{ht_n} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} = \frac{e^{h+1}-1}{h+1}$$

$$s = 2h: \quad \frac{1}{2n}\begin{pmatrix} e^{2ht_1} & 2e^{2ht_2} & 2e^{2ht_3} & \cdots & 2e^{2ht_{n-1}} & e^{2ht_n} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} = \frac{e^{2h+1}-1}{2h+1}$$

$$\vdots$$

$$s = (n-1)h: \quad \frac{1}{2n}\begin{pmatrix} e^{(n-1)ht_1} & 2e^{(n-1)ht_2} & 2e^{(n-1)ht_3} & \cdots & 2e^{(n-1)ht_{n-1}} & e^{(n-1)ht_n} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} = \frac{e^{(n-1)h+1}-1}{(n-1)h+1}$$

$$s = nh: \quad \frac{1}{2n}\begin{pmatrix} e^{nht_1} & 2e^{nht_2} & 2e^{nht_3} & \cdots & 2e^{nht_{n-1}} & e^{nht_n} \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} = \frac{e^{nh+1}-1}{nh+1}$$

that can be write as $Ax = b$ where

$$A_{(n\times n)} = \frac{1}{2n}\begin{pmatrix} e^{ht_1} & 2e^{ht_2} & 2e^{ht_3} & \cdots & 2e^{ht_{n-1}} & e^{ht_n} \\ e^{2ht_1} & 2e^{2ht_2} & 2e^{2ht_3} & \cdots & 2e^{2ht_{n-1}} & e^{2ht_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e^{(n-1)ht_1} & 2e^{(n-1)ht_2} & 2e^{(n-1)ht_3} & \cdots & 2e^{(n-1)ht_{n-1}} & e^{(n-1)ht_n} \\ e^{nht_1} & 2e^{nht_2} & 2e^{nht_3} & \cdots & 2e^{nht_{n-1}} & e^{nht_n} \end{pmatrix}$$

$$x_{(n\times 1)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \qquad b_{(n\times 1)} = \begin{pmatrix} \dfrac{e^{h+1}-1}{h+1} \\ \dfrac{e^{2h+1}-1}{2h+1} \\ \\ \dfrac{e^{(n-1)h+1}-1}{(n-1)h+1} \\ \dfrac{e^{nh+1}-1}{nh+1} \end{pmatrix}$$

c) Code Implementation

Find the solution to the linear system for $n = 4, 8, 16, 32$ and plot them. Point out the number condition of each linear system using the command **cond** on MATLAB.

Solution by Gauss eq

First, let's find the solution by solving the Gauss equations with the function **System(n,α)**

```
[time1, sol1, co1, M1, b1] = System(4,0);
[time2, sol2, co2, M2, b2] = System(8,0);
[time3, sol3, co3, M3, b3] = System(16,0);
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =  8.993789e-19.
```

```
[time4, sol4, co4, M4, b4] = System(32,0);
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =  7.263197e-20.
```

```
[time100, sol100, co100, M100, b100] = System(128,0);
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =  4.634772e-21.
```

```
disp(['cond n=4:    ' num2str(round(co1,2)) newline 'cond n=8:    ' num2str(co2) newline ...
      'cond n=16:   ' num2str(co3) newline 'cond n=32:   ' num2str(co4) newline 'cond n=128:  ' nu
```

```
cond n=4:   23063.77
cond n=8:   9058044760132.311
cond n=16:  3.542464016391407e+17
cond n=32:  3.190396294578456e+18
cond n=128: 2.742725726910894e+19
```

where this variable **cond** is the conndition number for inversion, it measures the sensitivity of the solution of a system of linear equations to errors in the data. By default, Matlab calculates the 2-norm

$$\text{cond} = \kappa(A) = ||A||_2 \; ||A^{-1}||_2$$

Solution by CasADI

Now let solve the problem with CasADI as the following optimization problem

$$\min_{x \in \mathbb{R}^n} \quad ||Ax - b||_2^2$$

the function **`SolCasADI(n,A,b)`** is used here.

```
x1 = SolCasADI(4, M1, b1);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:       10

Total number of variables............................:        4
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 2.6943424e+001 0.00e+000 8.15e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 4.6276502e-024 0.00e+000 8.34e-016  -1.0 3.52e+000    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  4.6276501576568773e-024   4.6276501576568773e-024
Dual infeasibility......:  8.3385863701727979e-016   8.3385863701727979e-016
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  8.3385863701727979e-016   8.3385863701727979e-016


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.148
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         2
  nlp_grad_f  |        0 (       0)        0 (       0)         3
  nlp_hess_l  |        0 (       0)        0 (       0)         1
       total  | 154.00ms (154.00ms) 154.03ms (154.03ms)         1
```

```
x2 = SolCasADI(8, M2, b2);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
```

```
Number of nonzeros in Lagrangian Hessian.............:       36

Total number of variables............................:        8
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 4.9950145e+001 0.00e+000 8.57e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 3.6378098e-006 0.00e+000 3.08e-004  -1.0 3.08e+000  -4.0 1.00e+000 1.00e+000f  1
   2 1.4262148e-006 0.00e+000 3.79e-006  -5.7 1.14e-001  -4.5 1.00e+000 1.00e+000f  1
   3 2.0457279e-007 0.00e+000 1.41e-006  -8.6 1.27e-001  -5.0 1.00e+000 1.00e+000f  1
   4 5.8550796e-009 0.00e+000 2.39e-007  -8.6 6.45e-002  -5.4 1.00e+000 1.00e+000f  1
   5 4.4693305e-011 0.00e+000 1.55e-008  -8.6 1.25e-002  -5.9 1.00e+000 1.00e+000f  1
   6 2.0353632e-011 0.00e+000 6.50e-010  -9.0 1.58e-003  -6.4 1.00e+000 1.00e+000f  1

Number of Iterations....: 6

                                   (scaled)                 (unscaled)
Objective...............:  2.0353632495740268e-011   2.0353632495740268e-011
Dual infeasibility......:  6.4957062359547385e-010   6.4957062359547385e-010
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  6.4957062359547385e-010   6.4957062359547385e-010


Number of objective function evaluations             = 7
Number of objective gradient evaluations             = 7
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 6
Total CPU secs in IPOPT (w/o function evaluations)   =      0.008
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         7
  nlp_grad_f  |        0 (       0)        0 (       0)         8
  nlp_hess_l  |        0 (       0)        0 (       0)         6
       total  |   9.00ms (  9.00ms)   9.00ms (  9.00ms)         1
```

```
x3 = SolCasADI(16, M3, b3);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      136

Total number of variables............................:       16
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
```

```
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 9.6118472e+001 0.00e+000 8.60e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 2.6605883e-006 0.00e+000 2.93e-004  -1.0 2.93e+000  -4.0 1.00e+000 1.00e+000f  1
   2 8.8600597e-007 0.00e+000 2.49e-006  -5.7 7.48e-002  -4.5 1.00e+000 1.00e+000f  1
   3 1.0617175e-007 0.00e+000 8.82e-007  -8.6 7.94e-002  -5.0 1.00e+000 1.00e+000f  1
   4 2.6596095e-009 0.00e+000 1.32e-007  -8.6 3.55e-002  -5.4 1.00e+000 1.00e+000f  1
   5 2.8730167e-010 0.00e+000 7.25e-009  -8.6 5.87e-003  -5.9 1.00e+000 1.00e+000f  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............:  2.8730166532400370e-010   2.8730166532400370e-010
Dual infeasibility......:  7.2518550442247229e-009   7.2518550442247229e-009
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  7.2518550442247229e-009   7.2518550442247229e-009


Number of objective function evaluations             = 6
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      0.121
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         6
  nlp_grad_f  |        0 (       0)        0 (       0)         7
  nlp_hess_l  |   1.00ms (200.00us)   1.00ms (200.20us)         5
       total  | 155.00ms (155.00ms) 154.90ms (154.90ms)         1
```

```
x4 = SolCasADI(32, M4, b4);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      528

Total number of variables............................:       32
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 1.8853286e+002 0.00e+000 8.59e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 1.8820221e-006 0.00e+000 2.82e-004  -1.0 2.82e+000  -4.0 1.00e+000 1.00e+000f  1
   2 4.8321288e-007 0.00e+000 1.70e-006  -5.7 5.10e-002  -4.5 1.00e+000 1.00e+000f  1
   3 5.1288904e-008 0.00e+000 5.33e-007  -8.6 4.80e-002  -5.0 1.00e+000 1.00e+000f  1
```

```
   4 1.1985394e-009 0.00e+000 7.44e-008  -8.6 2.01e-002  -5.4 1.00e+000 1.00e+000f  1
   5 2.1944447e-010 0.00e+000 4.45e-009  -8.6 3.60e-003  -5.9 1.00e+000 1.00e+000f  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............:  2.1944447377641908e-010   2.1944447377641908e-010
Dual infeasibility......:  4.4451118540402028e-009   4.4451118540402028e-009
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  4.4451118540402028e-009   4.4451118540402028e-009


Number of objective function evaluations             = 6
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      0.006
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         6
  nlp_grad_f  |        0 (       0)        0 (       0)         7
  nlp_hess_l  |        0 (       0)        0 (       0)         5
       total  |   8.00ms (  8.00ms)   8.00ms (  8.00ms)         1
```

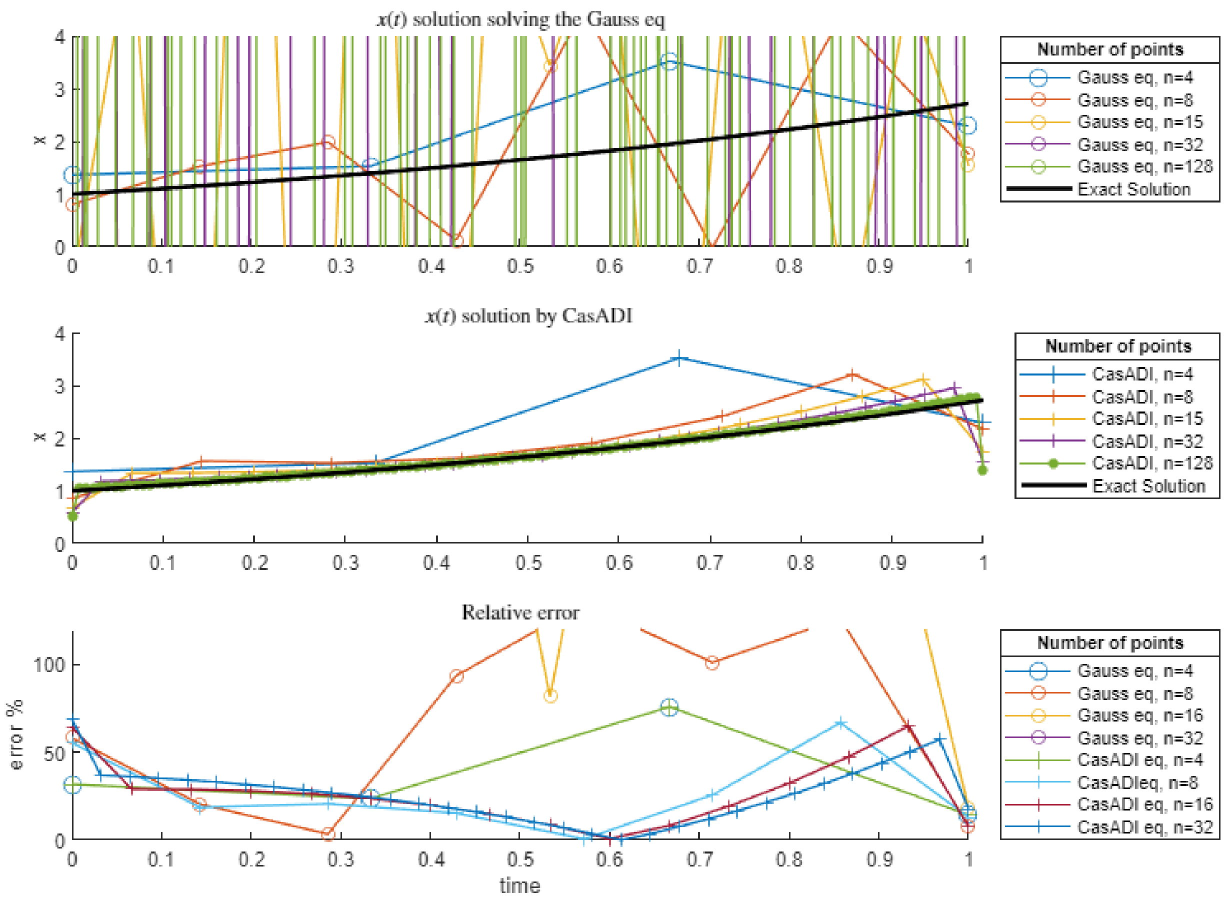
```
x100 = SolCasADI(128, M100, b100);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:     8256

Total number of variables............................:      128
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 7.4315539e+002 0.00e+000 8.56e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 2.3519607e-006 0.00e+000 2.73e-004  -1.0 2.73e+000  -4.0 1.00e+000 1.00e+000f  1
   2 2.0594929e-007 0.00e+000 6.94e-007  -5.7 2.08e-002  -4.5 1.00e+000 1.00e+000f  1
   3 1.9813707e-008 0.00e+000 1.97e-007  -8.6 1.78e-002  -5.0 1.00e+000 1.00e+000f  1
   4 4.0196681e-010 0.00e+000 2.59e-008  -8.6 6.99e-003  -5.4 1.00e+000 1.00e+000f  1
   5 6.7343212e-011 0.00e+000 1.49e-009  -8.6 1.21e-003  -5.9 1.00e+000 1.00e+000f  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............:  6.7343212389345426e-011   6.7343212389345426e-011
Dual infeasibility......:  1.4908838812067784e-009   1.4908838812067784e-009
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
```

```
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  1.4908838812067784e-009   1.4908838812067784e-009


Number of objective function evaluations             = 6
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      0.017
Total CPU secs in NLP function evaluations           =      0.010

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (        0)        0 (        0)         6
  nlp_grad_f  |        0 (        0)        0 (        0)         7
  nlp_hess_l  |  10.00ms (   2.00ms)  10.00ms (   2.00ms)         5
       total  |  28.00ms (  28.00ms)  28.01ms (  28.01ms)         1
```

```
clf
figure(Position=[10 10 900 600])
subplot(3,1,1)
hold on
plot(time1, sol1','o-', DisplayName='Gauss eq, n=4',MarkerSize=8);
plot(time2, sol2','o-', DisplayName='Gauss eq, n=8');
plot(time3, sol3','o-', DisplayName='Gauss eq, n=15');
plot(time4, sol4','o-', DisplayName='Gauss eq, n=32');
plot(time100, sol100','o-', DisplayName='Gauss eq, n=128');
title('$x(t)$ solution solving the Gauss eq',Interpreter='latex')
plot(t, exp(t), Color='black',DisplayName='Exact Solution', LineWidth=2)
leg = legend('Location','northeastoutside');
title(leg,'Number of points')
ylim([0 4]);ylabel('x')
hold off


subplot(3,1,2)
hold on
plot(time1 , x1','+-', DisplayName='CasADI, n=4',MarkerSize=8);
plot(time2, x2','+-', DisplayName='CasADI, n=8');
plot(time3, x3','+-', DisplayName='CasADI, n=15');
plot(time4, x4','+-', DisplayName='CasADI, n=32');
plot(time100, x100','.-', DisplayName='CasADI, n=128', MarkerSize=14);
plot(t, exp(t), Color='black',DisplayName='Exact Solution', LineWidth=2)
xlim([0 1]); ylabel('x')

title('$x(t)$ solution by CasADI',Interpreter='latex')
leg = legend('Location','northeastoutside');
title(leg,'Number of points')
hold off
subplot(3,1,3)
hold on
```

```
plot(time1', 100*abs((sol1-exp( 0:1/3:1))/exp( 0:1/3:1)),'o-', DisplayName='Gauss eq, n=4',Mark
plot(time2', 100*abs((sol2-exp( 0:1/7:1))/exp( 0:1/7:1)),'o-', DisplayName='Gauss eq, n=8');
plot(time3', 100*abs((sol3-exp( 0:1/15:1))/exp( 0:1/15:1)),'o-', DisplayName='Gauss eq, n=16')
plot(time4', 100*abs((sol4-exp( 0:1/31:1))/exp( 0:1/31:1)),'o-', DisplayName='Gauss eq, n=32')

plot(time1', 100*abs((x1-exp( 0:1/3:1))/exp( 0:1/3:1)),'+-', DisplayName='CasADI eq, n=4',Marke
plot(time2', 100*abs((x2-exp( 0:1/7:1))/exp( 0:1/7:1)),'+-', DisplayName='CasADIeq, n=8');
plot(time3', 100*abs((x3-exp( 0:1/15:1))/exp( 0:1/15:1)),'+-', DisplayName='CasADI eq, n=16');
plot(time4', 100*abs((x4-exp( 0:1/31:1))/exp( 0:1/31:1)),'+-', DisplayName='CasADI eq, n=32');

title('Relative error ',Interpreter='latex')
leg = legend('Location','northeastoutside');
title(leg,'Number of points')
ylim([0 120]); xlabel('time'); ylabel('error %')
hold off
```




```
disp(['cond n=4:    ' num2str(round(co1,2)) newline 'cond n=8:    ' num2str(co2) newline ...
      'cond n=16:   ' num2str(co3) newline 'cond n=32:   ' num2str(co4) newline 'cond n=128:  ' nu
```

```
cond n=4:    23063.77
cond n=8:    9058044760132.311
cond n=16:   3.542464016391407e+17
cond n=32:   3.190396294578456e+18
cond n=128:  2.742725726910894e+19
```

Note that the first and last point are atypical data, this can be solved using the following matrix

$$A_{(n\times n)} = \frac{1}{n}\begin{pmatrix} e^{ht_1} & e^{ht_2} & e^{ht_3} & \cdots & e^{ht_{n-1}} & e^{ht_n} \\ e^{2ht_1} & e^{2ht_2} & e^{2ht_3} & \cdots & e^{2ht_{n-1}} & e^{2ht_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ e^{(n-1)ht_1} & e^{(n-1)ht_2} & e^{(n-1)ht_3} & \cdots & e^{(n-1)ht_{n-1}} & e^{(n-1)ht_n} \\ e^{nht_1} & e^{nht_2} & e^{nht_3} & \cdots & e^{nht_{n-1}} & e^{nht_n} \end{pmatrix}$$

It is also important to mention that the result given by solving the Gauss equations is counterintuitive, when the number of points n increse the function starts to behave chaotically, the solution starts to oscillate drastically, then the bets solution in this case is when there are a few number of points.

In contrast, the solution with CasADI is different, the solution improves when n increases then the best solution for this point is use CasADI library with high number of points. Furthermore, the relative error is plotted for both solution methods, here the result shows that CasADI has lower relative errors than by solving Gauss eq.

d) Add Regularization

Solve the integral equation using the regularization method for each of the following parameters: i) $n = 100,\ \alpha = 0.2$, i) $n = 100,\ \alpha = 0.1$, and i) $n = 100,\ \alpha = 10^{-3}$. Plot the solutions.

Solution by Gauss eq

In this point, a regularization problem is added. The problem solution is found by solving the gauss equations

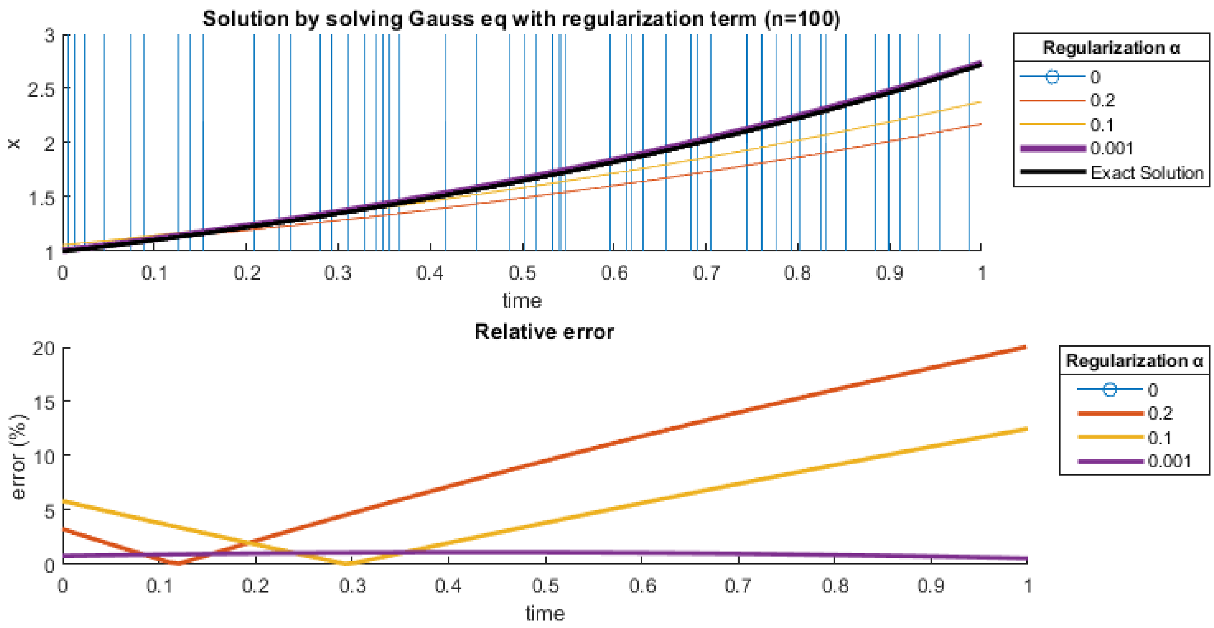
$$x = (A + \alpha I)^{-1} b = (A + \alpha I) \setminus b$$

```
[time_0,   sol_0,   co_0,   M_0,   b_0]   = System(100, 0);
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =  6.209026e-21.
```

```
[time_i,   sol_i,   co_i,   M_i,   b_i]   = System(100, 0.2);
[time_ii,   sol_ii, co_ii,  M_ii,  b_ii]  = System(100, 0.1);
[time_iii, sol_iii, co_iii, M_iii, b_iii] = System(100, 0.001);

clf;
figure(Position=[10 10 900 400])
subplot(2,1,1)
title('Solution by solving Gauss eq with regularization term (n=100)')
hold on
plot(time_0,   sol_0, '-o',DisplayName='0')
plot(time_i,   sol_i, DisplayName='0.2')
plot(time_ii,  sol_ii, DisplayName='0.1')
plot(time_iii, sol_iii, DisplayName='0.001', LineWidth=3)
plot(t,        exp(t), Color='black',DisplayName='Exact Solution', LineWidth=2)
leg = legend('Location','northeastoutside');
title(leg,'Regularization α')
xlabel('time'); ylabel('x'); ylim([1 3])
hold off
```

```
subplot(2,1,2)
title('Relative error')
hold on
plot(time_0,  100*abs((sol_0-exp(time_0)')./exp(time_0)'), '-o',DisplayName='0')
plot(time_i,  100*abs((sol_i-exp(time_i)')./exp(time_i)'), '-',  DisplayName='0.2', LineWidth=2
plot(time_ii, 100*abs((sol_ii-exp(time_ii)')./exp(time_ii)'), '-', DisplayName='0.1', LineWidth
plot(time_iii, 100*abs((sol_iii-exp(time_iii)')./exp(time_iii)'), '-', DisplayName='0.001', Lin
leg = legend('Location','northeastoutside');
title(leg,'Regularization α')
xlabel('time'); ylabel('error (%)'); ylim([0 20])
hold off
```




```
disp(['The higher relative error for α=0.2 is   ' num2str(max(100*abs((sol_i-exp(time_i)')./exp
       'The higher relative error for α=0.1 is   ' num2str(max(100*abs((sol_ii-exp(time_ii)')./e
       'The higher relative error for α=0.001 is ' num2str(max(100*abs((sol_iii-exp(time_iii)').
```

```
The higher relative error for α=0.2 is   20.0283
The higher relative error for α=0.1 is   12.4936
The higher relative error for α=0.001 is 1.1288
```

```
disp(['cond n=100, α=0:     ' num2str(round(co_0,2)) newline 'cond n=100, α=0.2:    ' num2str(co
       'cond n=100, α=0.01:  ' num2str(co_ii) newline 'cond n=100, α=0.001: ' num2str(co_iii)  ]
```

```
cond n=100, α=0:     1.855424207205099e+19
cond n=100, α=0.2:   7.7493
cond n=100, α=0.01:  14.5093
cond n=100, α=0.001: 1361.5892
```

In order to fix the condition of the matrix A a regularization term is added. The result shows that as the regularization term decreases the solution improves, without the regularizaion term the solution is still unstable, and the best solution is when $\alpha = 0.001$ because the relative error is always smaller than $1.5\ \%$.

Solution by CasADI

Now let's solve the problem, least squares, but modifying the matrix with a regularization parameter, $A = A + \alpha I$

```
x_casadi_0 = SolCasADI(100, M_0, b_0);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:     5050

Total number of variables............................:      100
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 5.8138663e+002 0.00e+000 8.56e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 2.0533839e-006 0.00e+000 2.74e-004  -1.0 2.74e+000  -4.0 1.00e+000 1.00e+000f  1
   2 2.2727703e-007 0.00e+000 8.03e-007  -5.7 2.41e-002  -4.5 1.00e+000 1.00e+000f  1
   3 2.2075292e-008 0.00e+000 2.32e-007  -8.6 2.09e-002  -5.0 1.00e+000 1.00e+000f  1
   4 4.6139277e-010 0.00e+000 3.07e-008  -8.6 8.28e-003  -5.4 1.00e+000 1.00e+000f  1
   5 8.4276125e-011 0.00e+000 1.79e-009  -8.6 1.45e-003  -5.9 1.00e+000 1.00e+000f  1

Number of Iterations....: 5

                                   (scaled)                 (unscaled)
Objective...............:  8.4276125324613342e-011   8.4276125324613342e-011
Dual infeasibility......:  1.7862071497248228e-009   1.7862071497248228e-009
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  1.7862071497248228e-009   1.7862071497248228e-009


Number of objective function evaluations             = 6
Number of objective gradient evaluations             = 6
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 5
Total CPU secs in IPOPT (w/o function evaluations)   =      1.679
Total CPU secs in NLP function evaluations           =      0.007

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         6
  nlp_grad_f  |        0 (       0)        0 (       0)         7
  nlp_hess_l  |   7.00ms (  1.40ms)   7.00ms (  1.40ms)         5
       total  |   1.71 s (  1.71 s)   1.71 s (  1.71 s)         1
```

```
x_casadi_i = SolCasADI(100, M_i, b_i);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
```

```
Number of nonzeros in Lagrangian Hessian.............:     5050

Total number of variables............................:      100
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 5.8138663e+002 0.00e+000 9.83e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 3.1715660e-027 0.00e+000 6.58e-015  -1.0 2.17e+000    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  3.1715659656345017e-027   3.1715659656345017e-027
Dual infeasibility......:  6.5825710206764181e-015   6.5825710206764181e-015
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  6.5825710206764181e-015   6.5825710206764181e-015


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.031
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         2
  nlp_grad_f  |        0 (       0)        0 (       0)         3
  nlp_hess_l  |   1.00ms (  1.00ms)   1.00ms (  1.00ms)         1
       total  |  54.00ms ( 54.00ms)  54.01ms ( 54.01ms)         1
```

```
x_casadi_ii = SolCasADI(100, M_ii, b_ii);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:     5050

Total number of variables............................:      100
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
```

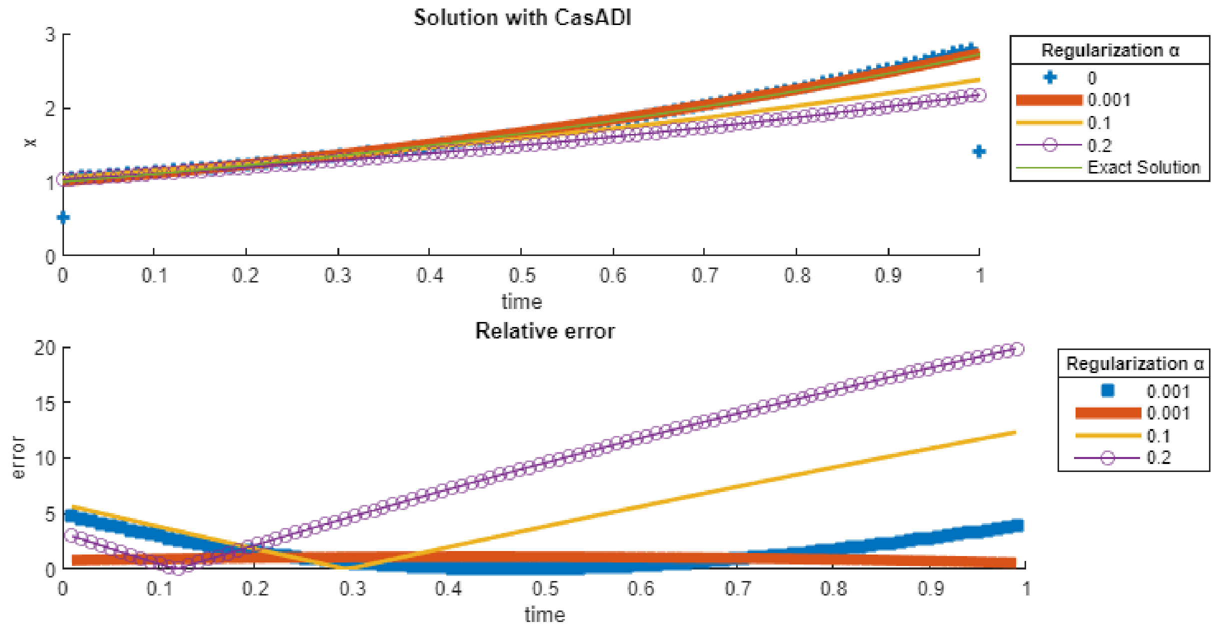
```
   0 5.8138663e+002 0.00e+000 9.20e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 1.0402265e-026 0.00e+000 5.39e-015  -1.0 2.38e+000    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  1.0402265022890296e-026   1.0402265022890296e-026
Dual infeasibility......:  5.3862704223432385e-015   5.3862704223432385e-015
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  5.3862704223432385e-015   5.3862704223432385e-015


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.025
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         2
  nlp_grad_f  |        0 (       0)        0 (       0)         3
  nlp_hess_l  |   1.00ms (  1.00ms)   1.00ms (  1.00ms)         1
       total  |  47.00ms ( 47.00ms)  47.01ms ( 47.01ms)         1
```

```
x_casadi_iii = SolCasADI(100, M_iii, b_iii);
```

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:     5050

Total number of variables............................:      100
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0 5.8138663e+002 0.00e+000 8.57e+000  -1.0 0.00e+000    -  0.00e+000 0.00e+000   0
   1 9.4746989e-023 0.00e+000 6.75e-015  -1.0 2.73e+000    -  1.00e+000 1.00e+000f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...............:  9.4746989019194999e-023   9.4746989019194999e-023
Dual infeasibility......:  6.7519697818798492e-015   6.7519697818798492e-015
Constraint violation....:  0.0000000000000000e+000   0.0000000000000000e+000
Complementarity.........:  0.0000000000000000e+000   0.0000000000000000e+000
Overall NLP error.......:  6.7519697818798492e-015   6.7519697818798492e-015
```

```
Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.035
Total CPU secs in NLP function evaluations           =      0.001

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (       0)        0 (       0)         2
  nlp_grad_f  |        0 (       0)        0 (       0)         3
  nlp_hess_l  |   1.00ms (  1.00ms)   1.00ms (  1.00ms)         1
       total  |  38.00ms ( 38.00ms)  37.40ms ( 37.40ms)         1
```

```
clf;
figure(Position=[10 10 900 400])
subplot(211)
hold on
plot(time_0, x_casadi_0, '+',DisplayName='0', LineWidth=2)
plot(time_iii, x_casadi_iii, DisplayName='0.001', LineWidth=5)
plot(time_ii, x_casadi_ii, DisplayName='0.1', LineWidth=2)
plot(time_i, x_casadi_i, '-o', DisplayName='0.2')
plot(time_i, exp(time_i), DisplayName='Exact Solution')
leg = legend('Location','northeastoutside');
title(leg,'Regularization α')
ylabel('x'); xlabel('time'); title('Solution with CasADI')
hold off
subplot(212)
hold on
plot(time_0(2:end-1), 100*abs((x_casadi_0(2:end-1)-exp(time_0(2:end-1))')./exp(time_0(2:end-1)
plot(time_iii(2:end-1), 100*abs((x_casadi_iii(2:end-1)-exp(time_iii(2:end-1))')./exp(time_iii(
plot(time_ii(2:end-1), 100*abs((x_casadi_ii(2:end-1)-exp(time_ii(2:end-1))')./exp(time_ii(2:en
plot(time_i(2:end-1), 100*abs((x_casadi_i(2:end-1)-exp(time_i(2:end-1))')./exp(time_i(2:end-1)
leg = legend('Location','northeastoutside');
title(leg,'Regularization α')
ylabel('error'); xlabel('time'); title('Relative error')
hold off
```




```
disp(['Maximum relative error ' num2str(max(100*abs((x_casadi_i(2:end-1)-exp(time_i(2:end-1))')
```

```
Maximum relative error 19.8358
```

The relative error is ploted without the first and last point since CasADI calculate these points wrong, the value getted is the half of the real value and can be fixed by multiplying by 2 the first and last columns of the matrix A. Note that the solutions are the same since CasADI solve the problem correctly for 100 points, then the regularization term is not necessary when CasADI is used.

Auxiliar Functions

```
function y=yfunc(s)
    y = (exp(s+1)-1)./(s+1);
end

function [time, sol, co, M, b]=System(n, alfa)
    time = 0:1/(n-1):1;
    M = zeros(n,n);
    for i=1:n
        s = i*(1/n);

        v = exp(s*time);
        v(2:end-1) = 2*v(2:end-1);

        M(i,:)= (1/(2*n))*v;
    end
    b   = yfunc((1:n)*(1/n))';
    M = M+alfa*eye(size(M));
    sol =  M \ b;
    co  = cond(M);
```

```
end

function x_int_opt=SolCasADI(n, M, b)
    opti_int = casadi.Opti();
    x_int = opti_int.variable(n);

    opti_int.minimize(norm(M*x_int-b,2)^2);

    opti_int.solver('ipopt');

    sol_int = opti_int.solve();
    x_int_opt = sol_int.value(x_int);
end
```