HOMEWORK # 1

Numerical Opmitization - 2022

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1.

Let $A \subset \mathbb{R}$, such that $\inf\ A$ is finite or $\inf\ A = -\infty$. Prove that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of elements in A such that $a_n \to \inf\ A$ when $n \to \infty$.

2.

Suppose that Ω is a not empty **closed** set and f is a **continious** and **cohercive** function in Ω. Prove that given $R > 0$ the set

$$\Omega_R := \{x \in \Omega | f(x) \leq R\}$$

is closed and bounded.

In order to prove that Ω_R is closed, let's prove that its complement is open Ω_R^C, by defition

$$\Omega_R^C := \{x \in \Omega | f(x) > R\}$$

now using the continuity of f at a point c of its domain Ω, for any neighborhood over te image of c, $N_1(f(c))$, there is a neighborhood $N_2(c) \in \Omega$ such that $f(x) \in N_1(f(c))$ whenever $x \in N_2(c)$, it is equivalent to say that for any $\epsilon > 0$ and $c \in \Omega$ there is $B_\epsilon(f(c)) \subseteq \Omega_R^C$ such that $B_\epsilon(c) \subseteq \Omega$. Since every point in Ω_R^C is the center of an open ball contained in Ω_R^C then it is an open set and then Ω_R is a closed set.

Again, using the definition of continuty there is some pre-image $\widetilde{x}$ of R ($f(\widetilde{x}) = R$]) such that allow to us rewrite the defition of Ω_R as follows

$$\Omega_R := \{x \in \Omega | f(x) \leq f(\widetilde{x})\}$$

this also implies that $f(\widetilde{x})$ is the suprem of Ω_R. Now, using the defition of cohercivity, $\lim_{|x|| \to \infty} f(x) = +\infty$, for any x far way from origin there is always an image and then the supremum and infimum always exists that implies that Ω_R is bounded.

Then, it has been demostrated that Ω_R is closed and bounded. □

and continuty, there exists some positive real number $\delta > 0$ such that for all x in the domain of f with $x_0 - \delta < x < x_0 + \delta$ the value of $f(x)$ satisfies $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$, for any $x \in \Omega_R$ thgere is some $\delta > 0$

3.

Prove that the quadratic optimization problem

$$\min_{x \in \Omega} x^T Q x + c^T x$$

where $\Omega \subset \mathbb{R}^n$ is a closed set, $x, c \in \mathbb{R}^n$ and the matrix $Q \in \mathbb{R}^{n \times n}$, admit at least one global solution.

4.

Given a set of n points in the plane. Find the circle of minimum radius containing all these points.

a)

Consider this problem as an optimization problem restricted with an objective lineal function and quadratic restrictions.

b)

Prove that the problem has unique solution.

5.

Let $S = \{1, 2, ..., M\}$ a set of M objects, with p_i the weight of the i-th object and v_i its value $i = 1, 2, ..., M$. It is a matter of carrying in a bag objects from S, such that the total value is maximized, but without the total weight of all the objects to be included in the bag exceeding a given weight P. Formulate the optimization problem that allow to solve the problem.

6.

Study which of the following subsets are convex

a) $S = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \geq x_1\}$

In order to check if S is convex let's rewrite the set as follows

$$S = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 - x_1 = A \cdot x \geq 0\}$$

where $A = (-1, 1)$ and $x = (x_1, x_2)$ are two vectors.

Let $x, y \in S$, i.e., $A \cdot x \geq 0,\ A \cdot y \geq 0$ with $x=(x_1,x_2)$ and $y = (y_1, y_2)$. Now, consider any point on the line joining points x and y

$$z = (1-t)x + ty, \quad t \in [0, 1]$$

and operating$A$ to left side

$$Az = (1-t)Ax + tAy$$

since all the terms are greater or equal to zero, from the definition of t and the definition of an element of S, then

$$Az = (1-t)Ax + tAy \geq 0$$

which is the condition to be in S, $z \in S$. Then, the set S is a convex set.

b) $S = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 < 4\}$

Analogous to the previous point,

$$S = \{(x_1, x_2) \in \mathbb{R}^2 |\ ||x||^2 < 4\} = \{x \in \mathbb{R}^2 |\ ||x|| < 2\}$$

Let $x, y \in S$, i.e., $||x|| < 2, ||y|| < 2$, and consider any point on the line joining points

$$z = (1-t)x + ty, \quad t \in [0, 1]$$

In order to prove that S is convex, consider the norm of the vector

$$||z||^2 = ((1-t)x + ty)^T((1-t)x + ty) = ((1-t)x^T + ty^T)((1-t)x + ty)$$
$$||z||^2 = (1-t)^2 x^T x + t^2 y^T y + 2(1-t)tx^T y = (1-t)^2||x||^2 + t^2||y||^2 + 2(1-t)tx^T y$$

using the Cauchy–Schwarz inequality

$$||z-2||^2 \leq (1-t)^2||x||^2 + t^2||y||^2 + 2(1-t)t||x||\ ||y||$$

using the pertenence condition of S (x and y statement conditions)

$$||z-2||^2 \leq (1-t)^2(4) + t^2(4) + 2(1-t)t(4) = 4(1 - 2t + t^2 + t^2 + 2t - 2t^2)$$

Finally simplifying,

$\Rightarrow ||z-2||^2 \leq 4$

This result means that z is in S, $z \in S$>. Then, the set S is convex.

c) $S = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 = \sin x_1\}$

In order to prove that S is not convex, let $x = (0,0)$ and $y = (\pi, 0)$ two points in S and consider any point on the line joining points

$$z = (1-t)x + ty = (1-t)(0,0) + t(\pi,0), \quad t \in [0,1]$$
$$z = t(\pi, 0)$$

In particular taking $t = 1/2$, $z = (\pi/2, 0)$, the point z does not live in S since $\sin \pi/2 \neq 0$. Then, the set $S$is not convex since $z \notin S$ although$x$ and y if ther are in S.

d) $S = \{y \in \mathbb{R}^m \ : \ y = Ax, \ x \in C\}$with $C \subseteq \mathbb{R}^n$ convex and A a matrix $m \times m$

Let $y_1, y_2 \in S$, i.e., $y_1 = Ax_1$ and $y_2 = Ax_2$. and consider any point on the line joining points

$$z = (1-t)y_1 + ty_2, \quad t \in [0,1]$$

$$z = (1-t)Ax_1 + tAx_2 = A[(1-t)x_1 + tx_2] = A\widetilde{x}$$

where $\widetilde{x}$ is any point on the line joining points x_1 and x_2 that live in C, since it is convex $\widetilde{x} \in C$and then z satisfies the condition to pertenence to S, $z = A\widetilde{x}$, with $\widetilde{x} \in C$.