On the largest minimum distances of [n, 6] LCD codes

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Abstract

Linear complementary dual (LCD) codes can be used to against side-channel attacks and fault noninvasive attacks. Let $d_a(n,6)$ and $d_l(n,6)$ be the minimum weights of all binary optimal linear codes and LCD codes with length n and dimension 6, respectively. In this article, we aim to obtain the values of $d_l(n,6)$ for $n \geq 51$ by investigating the nonexistence and constructions of LCD codes with given parameters. Suppose that $s \geq 0$ and $0 \leq t \leq 62$ are two integers and n = 63s + t. Using the theories of defining vectors, generalized anti-codes, reduced codes and nested codes, we exactly determine $d_l(n,6)$ for $t \notin \{21,22,25,26,33,34,37,38,45,46\}$, while we show that $d_l(n,6) \in \{d_a(n,6)-1,d_a(n,6)\}$ for $t \in \{21,22,26,34,37,38,46\}$ and $d_l(n,6) \in \{d_a(n,6)-2,d_a(n,6)-1\}$ for $t \in \{25,33,45\}$.

Index terms: optimal linear code, LCD code, generalized anti-code, defining vector, reduced code

1 Introduction

Let F_2^n be the n-dimensional row vector space over binary field F_2 . A binary linear code $\mathcal{C} = [n,k]$ is a k-dimensional subspace of F_2^n . The weight w(x) of a vector $x \in F_2^n$ is the number of its nonzero coordinates. If the minimum weight of nonzero vectors in $\mathcal{C} = [n,k]$ is d, then d is called the minimum distance of \mathcal{C} and we denote $\mathcal{C} = [n,k,d]$. The dual code \mathcal{C}^{\perp} of \mathcal{C} is defined as $\mathcal{C}^{\perp} = \{x \in F_2^n \mid x \cdot y = xy^T = 0 \text{ for all } y \in \mathcal{C}\}$. The hull (or radical code) of \mathcal{C} is defined as $Hu(\mathcal{C}) = \mathcal{C}^{\perp} \cap \mathcal{C}$ and $h(\mathcal{C}) = \dim Hu(\mathcal{C})$ is the hull dimension of \mathcal{C} [1]. If a generator matrix of a linear code is denoted by G, then it is easy to know $h(\mathcal{C}) = k$ rank(G^TG). A code \mathcal{C} is self-orthogonal (SO) if $h(\mathcal{C}) = k$. And if $h(\mathcal{C}) = 0$, \mathcal{C} is a linear complementary dual (LCD) code.

LCD cyclic codes were referred to as reversible cyclic codes by Massey [2] and could provide an optimal linear coding solution for the two user binary adder channel. He also pointed out that asymptotically good LCD codes exist [3]. Soon afterwards Yang et al. provided the condition for a cyclic code having a complementary dual [4]. In Ref. [5] Sendrier verified that LCD codes can meet the Gilbert-Varshamov bound. Carlet et al. showed that LCD codes can be used to fight against fault noninvasive attacks and side-channel attacks [6]. It shed a new light on LCD codes and posed more attention to the construction of LCD codes with greatest possible minimum distance, which can improve the resistance against those two attacks. Since then, much work aimed at determining the upper bound of the minimum distances of LCD codes and constructing LCD codes with best parameters, see [7–18].

In this paper, we use $d_a(n,k)$ to denote the distance of optimal linear code for given n, k, and $d_l(n,k)$ to denote the largest distance of LCD codes for given n, k. A linear [n,k] code \mathcal{C} is optimal if \mathcal{C} has the greatest minimum weight $d_a(n,k)$ among all linear [n,k] codes. And a linear [n,k,d] code with $d=d_a(n,k)-1$ is called a near optimal code. If an LCD [n,k] code with the largest minimum weight $d_l(n,k)$ among all LCD [n,k] codes, then it is an optimal LCD code. And an LCD [n,k,d] code with $d=d_l(n,k)-1$ is called a near optimal LCD code.

Ref. [7] showed that any code over F_q is equivalent to an LCD code for $q \geq 4$. This suggests us pay more attention on LCD codes over F_q for q = 2, 3. Here we only consider binary LCD codes. In latest years, constructions of optimal LCD codes with short lengths or low dimensions are discussed, and the lower and upper bounds for $d_l(n, k)$ have been established in [8–18]. For $n \leq 24$ and $1 \leq k \leq n$, all $d_l(n, k)$ were determined. For $k \leq n \leq 40$, most of $d_l(n, k)$ were determined.

For $k \leq 5$, all $d_l(n, k)$ were obtained in [10–12, 15, 17, 18]. As for larger n and $k \geq 6$, only a few results of $d_l(n, k)$ are known, see [9–11, 11–14, 19, 20]. When k = 6, $d_l(n, 6)$ was given for $1 \leq n \leq 12$ in [9], for $13 \leq n \leq 16$ in [10], for $17 \leq n \leq 24$ in [11, 12], for $25 \leq n \leq 40$ in [11, 13], for $41 \leq n \leq 50$ in [14].

Recently, Li et al. introduced a new concept "generalized anti-code" to verify that 11 classes of binary optimal codes $[2^{k-1}s+2^k-a,k]$ are not LCD codes for given k and a in [20] (Lemma 18 in this article). The new approach is useful to study optimal LCD codes with higher dimensions. For example, they showed that some optimal linear codes are not LCD for k=6 and $52 \le n \le 62$. However, they hadn't accurately provided the upper bound of the minimum distances of optimal linear codes.

Motivated by the above results, the objective of this paper is to investigate the values of $d_l(n,6)$ for all lengths $n \geq 51$. This paper is organized as follows. In Section 2, some definitions, notations and basic results about defining vectors, generalized anti-codes and reduced codes will be are given. In Sections 3 and 4, we will determine the minimum distances of optimal [n = 63s + t, 6] LCD codes by the following two steps:

Step1: We infer that many (near) optimal linear codes with given lengths are not LCD, which will be showed in Section 3.

Step2: We construct some (near) optimal LCD codes with parameters $[63s + t, 6, d_t]$ from some nested code chains, which will be exhibited in Section 4.

In final section we will provide a table to conclude the above results and give the minimum distances of optimal [n = 63s + t, 6] LCD codes.

2 Preliminaries

In this section, some concepts, lemmas and notations about defining vectors, generalized anti-codes and reduced codes will be introduced. For clarity, they are divided into the following four subsections.

2.1 Basic knowledge

Lemma 1. (Griesmer Bound) [21] The length, dimension and minimum distance for all linear [n, k, d] codes over F_q achieve the following relation:

$$n \ge \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$$

It should be noted that all linear codes meeting the Griesmer Bound are optimal codes while not all optimal codes can achieve the Griesmer Bound.

Two binary codes C and C' are equivalent if one can be obtained from the other by permuting the coordinates, we denote them as $C \cong C'$. If two matrices G_1 and G_2 generate equivalent codes, we denote them as $G_1 \cong G_2$. Two $k \times n$ matrices $G_1 \cong G_2$ if and only if there are an invertible matrix P and a permutation Π on $\{1, 2, \dots, n\}$ such that $G_2 = PG_1\Pi$ [22].

2.2 Definitions about the defining vector

We use $\mathbf{1_n} = (1, 1, \dots, 1)_{1 \times n}$ and $\mathbf{0_n} = (0, 0, \dots, 0)_{1 \times n}$ to denote the all-one vector and the zero vector of length n, respectively. Let J_k be the $(2^k - 1) \times (2^k - 1)$ all-one matrix. And use $iG = (G, G, \dots, G)$ to denote the juxtaposition of i copies of G for given matrix G. We only consider linear codes without zero coordinates and matrices without zero columns.

Let α_i be the binary column vector representation of i for $1 \leq i \leq N = 2^k - 1$, i.e., $\alpha_1 = (1, 0, \dots, 0)^T$, $\alpha_2 = (0, 1, \dots, 0)^T$, \dots , $\alpha_N = (1, 1, \dots, 1)^T$. Then the matrix $\mathbf{S}_k = (\alpha_1, \alpha_2, \dots, \alpha_N)$ generates the k-dimensional simplex code $\mathcal{S}_k = [2^k - 1, k, 2^{k-1}]$. Recall the matrix \mathbf{S}_k can be constructed inductively [23, 24]. Let

$$\mathbf{S}_2 = \left(\begin{array}{c} 101\\011 \end{array}\right), \, \mathbf{S}_3 = \left(\begin{array}{ccc} \mathbf{S}_2 & \mathbf{0}_2^T & \mathbf{S}_2\\ \mathbf{0}_3 & 1 & \mathbf{1}_3 \end{array}\right)$$

and recursively, we have

$$\mathbf{S}_{k+1} = \left(egin{array}{ccc} \mathbf{S}_k & \mathbf{0}_k^T & \mathbf{S}_k \ \mathbf{0}_{2^k-1} & 1 & \mathbf{1}_{2^k-1} \end{array}
ight).$$

Let G be a matrix of size $k \times n$. If the columns of G contain l_i copies of α_i for $1 \le i \le N$, we denote G as $G = (l_1\alpha_1, l_2\alpha_2, \dots, l_N\alpha_N)$ for short, and call $L = (l_1, l_2, \dots, l_N)$ the **defining vector** of G [25]. Let l_{max} and l_{min} be the largest and the smallest values of all l_i for $1 \le i \le N = 2^k - 1$, respectively.

Suppose $L=(l_1,l_2,\cdots,l_N)$, let l_{j_l} $(1 \leq l \leq t)$ be different coordinates of L with $l_{j_1} < l_{j_2} < \cdots < l_{j_t}$. Defining m_l as the number of equal l_{j_l} , we say L is of type

$$[(l_{j_1})_{m_1} | (l_{j_2})_{m_2} | \cdots | (l_{j_t})_{m_t}]].$$

If C = [n, k] has a generator matrix $G = (l_1 \alpha_1, \dots, l_N \alpha_N)$, the minimum distance d of C and its codewords weight can be determined by its defining vector L and some special matrices P_k and Q_k from simplex codes.

Let P_2 be a $(2^2 - 1) \times (2^2 - 1)$ matrix whose rows are the non-zero codewords of S_2 . Construct

$$P_2 = \begin{pmatrix} 101\\011\\110 \end{pmatrix}, P_3 = \begin{pmatrix} P_2 & 0 & P_2\\\mathbf{0_3} & 1 & \mathbf{1_3}\\P_2 & \mathbf{1_3^T} & (J_2 - P_2). \end{pmatrix}$$

Then the seven rows of P_3 are just the seven nonzero vectors of the simplex code $S_3 = [7,3,4]$. For $k \geq 3$, using the recursive method, let P_k be a $(2^k - 1) \times (2^k - 1)$ matrix whose rows are just the $2^k - 1$ nonzero codewords of k-dimensional binary simplex code. Then the matrix formed by nonzero codewords of k + 1-dimensional simplex code is as

follows:

$$P_{k+1} = \begin{pmatrix} P_k & \mathbf{0_{2^k-1}^T} & P_k \\ \mathbf{0_{2^k-1}} & 1 & \mathbf{1_{2^k-1}} \\ P_k & \mathbf{1_{2^k-1}^T} & (J_k - P_k) \end{pmatrix}.$$

Let $Q_k = J_k - P_k$, Each row of Q_k has $2^{k-1} - 1$ ones and 2^{k-1} zeros. According to Refs. [23–25], the matrix P_k is invertible over the rational field and

$$P_k^{-1} = \frac{1}{2^{k-1}}(J_k - 2Q_k) = \frac{1}{2^{k-1}}(2P_k - J_k).$$

For an [n, k, d] code with defining vector $L = (l_1, l_2, \dots, l_N)$, the weights of its $2^k - 1$ codes form a vector $W = (w_1, w_2, \dots, w_N)$, called its weight vector. Set $W = (w_1, w_2, \dots, w_N) = d\mathbf{1}_{2^k - 1} + \mathbf{\Lambda}$, where $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ with $\lambda_i = w_i - d \geq 0$ and at least one $\lambda_i = 0$. Denote $\sigma = \lambda_1 + \lambda_2 + \dots + \lambda_N$. Then the weight vector W and defining vector L have the following relations:

$$W^{T} = P_{k}L^{T}, \sigma = 2^{k-1}n - d(2^{k} - 1)$$

$$L^{T} = P_{k}^{-1}W^{T} = \frac{1}{2^{k-1}}(J_{k} - 2Q_{k})(d\mathbf{1}_{2^{k}-1} + \mathbf{\Lambda})^{\mathbf{T}}$$

$$= \frac{1}{2^{k-1}}((d+\sigma)\mathbf{1}_{2^{k}-1}^{T} - 2Q_{k}\Lambda^{T})$$

For an [n, k, d] code with defining vector $L = (l_1, l_2, \dots, l_{2^k-1})$, one can infer from the above equations that

$$\lfloor \frac{1}{2^{k-1}}(d+\sigma) \rfloor \le l_i \le \lceil \frac{1}{2^{k-1}}(d-\sigma) \rceil,$$

which will be frequently used in the process of verifying the below conclusions.

2.3 Definitions about the generalized anti-code

Binary anti-codes were first introduced by Farrell in [28]. To verify the nonexistence of binary optimal LCD codes $[2^{k-1}s + 2^k - a, k]$ for given k and a, Li et al. proposed some concepts about the generalized anti-code in [20].

Definition 1. [20] Suppose C = [n, k] has a generator matrix $G = (l_1\alpha_1, \dots, l_N\alpha_N)$ and its defining vector is $L = (l_1, \dots, l_N)$. If $l_{max} = a$, then G is a sub-matrix of aS_k , $L^c = ((a - l_1), \dots, (a - l_N))$ is called the anti-vector of $L = (l_1, \dots, l_N)$ and it is the defining vector of $G^c = ((a - l_1)\alpha_1, \dots, (a - l_N)\alpha_N)$. G^c is also a sub-matrix of aS_k and called an anti-matrix of aS_k . A linear code generated by aS_k is called as a **generalized anti-code**. If the largest weight of these aS_k -vectors is aS_k , this generalized anticode is denoted as aS_k and aS_k and aS_k is called an anti-matrix of aS_k .

Parameters of C = [n, k] with generator matrix G can be derived from its generalized anti-code as follows.

Lemma 2. [20] Let $G = (l_1\alpha_1, l_2\alpha_1, \dots, l_N\alpha_N)$ be a generator matrix of $\mathcal{C} = [n, k, d]$, where $N = 2^k - 1$. Set $a = \max_{1 \leq i \leq N} \{l_i\}$, n < aN, m = aN - n and $l_i^c = a - l_i$. If $G^c = (l_1^c\alpha_1, l_2^c\alpha_1, \dots, l_N^c\alpha_N)$ generates a generalized anti-code $\mathcal{C}^a = (m, 2^k, \{\delta\})$, then $d = a2^{k-1} - \delta$ and $GG^T = G^c(G^c)^T$ for $k \geq 3$.

Combining Ref. [3] with Ref. [20], equivalent conditions for an LCD code are given as follows.

Lemma 3. [3, 20] Let $k \geq 3$, C = [n, k] be a linear code with a generator matrix G and a parity check matrix H. If the anti-matrix of G is G^c , then the following properties are equivalent:

- (1) C is LCD;
- (2) the matrix HH^T is invertible;
- (3) the matrix GG^T is invertible;
- (4) the matrix $G^c(G^c)^T$ is invertible.

The following two examples can make these previous concepts understandable.

Example 1. If a linear code has a defining vector L_1 and $L_1 = (s + 1, s - 1, s, s, s + 1, s - 1, s + 1)$ is of type $]](s - 1)_2 \mid (s)_2 \mid (s + 1)_3]]$. One can know $l_{max} = s + 1$ is the largest value of all l_i for $1 \le i \le N = 2^3 - 1 = 7$, then G is a sub-matrix of $(s + 1)S_3$ and $L_1^c = ((s + 1 - l_1), \dots, (s + 1 - l_7)) = (0, 2, 1, 1, 0, 2, 0)$ is the anti-vector of $L = (l_1, \dots, l_7)$. Here L_1^c is the defining vector of $G^c = (3\alpha_1, \alpha_2, \dots, \alpha_7)$ and of type $]](2)_2 \mid (1)_2 \mid (0)_3]]$.

Example 2. If a linear code has a defining vector $L_2 = (3, 1, 1, 3, 1, 3, 1)$, then one can know L_2 is of type $]](1)_4 \mid (3)_3]]$ and it is easy to derive that L_2^c is of type $]](2)_4 \mid (0)_3]]$ similar to the above example. From the form of L_2^c , one can infer that a code with defining vector L_2 is an SO code owing to rank $(G^c(G^c)^T)$ =rank $(G^c(G^c)^T)$ =0. Naturally, it is not LCD by lemma 3.

2.4 Some preliminary results

The hull h and d_l of a linear code with parameters [n,k] can be estimated from extended codes or codes with lower dimensions, which are given as follows.

Definition 2. [19] Let G, G_1 be generator matrices of C = [n, k, d] and $C_1 = [n-m, k-1, \ge d]$, respectively. If

$$G = \left(\begin{array}{cc} \mathbf{1}_m & u \\ \mathbf{0}_m & G_1 \end{array}\right),$$

then C_1 is called a **reduced code** of C.

Lemma 4. [19] If C_1 is a reduced code of C = [n, k, d] and $h(C_1) = r \ge 2$, then $h(C) \ge r - 1$ and C is not an LCD code.

Lemma 5. [13] If k is even and $d_l(n,k)$ is odd, then $d_l(n+1,k) \ge d_l(n,k) + 1$.

Lemma 6. [26] Let $s \geq 0$, $k \geq 6$, $1 \leq m \leq k-1$, then the code $C = \mathcal{MD}_s(k,m) = [sN + 2^k - 2^m, k, s2^{k-1} + 2^{k-1} - 2^{m-1}]$ has h(C) = k for m = 0, h(C) = k-1 for m = 1 and $h(C) = k-2 \geq 4$ for $m \geq 2$, hence C is not an LCD code.

3 On the nonexistence of $[n, 6, d_n]$ linear LCD codes

In this section, let $s \ge 1$ be an integer and we consider the nonexistence of LCD codes $[n, 6, d_n]$. For clarity, some known results about the Griesmer bounds d_g , the minimum

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\overline{d_g}$	1	2	2	3	4	4	4	5	6	6	7	8	8	8	8
d_a	1	2	2	2	3	4	4	4	5	6	6	7	8	8	8
d_l	1	2	2	2	3	4	4	4	5	6	6	6	7	8	8
\overline{n}	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$\overline{d_g}$	9	10	10	11	12	12	12	13	14	14	15	16	16	16	16
d_a	8	9	10	10	11	12	12	12	13	14	15	16	16	16	16
d_l	8	9	10	10	10	11	12	12	12	13	14	14	14	15	16
\overline{n}	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$\overline{d_g}$	16	17	18	18	19	20	20	20	21	22	22	23	24	24	24
d_a	16	17	18	18	18	19	20	20	21	22	22	23	24	24	24
d_l	16	16	17	18	18	19	20	20	20	20	21	22	22	23	24

 d_t is used to denote d_g , d_a or d_l in different lines.

distances d_a of all optimal linear codes and the minimum distances d_l of optimal LCD codes are listed in Table 1.

From Table 1 and the code tables given by Grassl in [8], one can easily know the following items:

- (1) For $n = 6, 7, 8, 11, 12, 15, 18, 19, 20, 23, 26, 27, 30 \le n \le 63 (n \ne 40, 41)$, the optimal linear $[n, 6, d_a]$ codes saturate the Griesmer bound. If $n \ge 64 (s \ge 1)$, one can infer that all optimal $[63s + t, 6, 32s + d_a(t)]$ codes also saturate the Griesmer bound.
- (2) The greatest minimum distance d_l of optimal LCD codes can not meet the corresponding d_a for n = 18, 26, 30, 31, 32, 33, 34, 37, 38, 44 49.
- (3) For n = 9, 10, 13, 14, 16, 17, 21, 22, 24, 25, 28, 29, 40, 41, the optimal linear codes $[n, 6, d_a]$ can not saturate the Griesmer bound.

Thus, one should only pay special attention to LCD codes of lengths $n = 63s + t \ge 51$ for $t = 0, 1, 2, 3, 4, 5, 9, 10, 13, 14, 16, 17, 18, 21, 22, 24, 25, 26, 28 - 34, 37, 38, 40, 41, 44 - 49, 51 - 62. In the sequel, we will investigate LCD codes with length <math>n = 63s + t \ge 51$ for given s, t.

Theorem 7. There is no [63s, 6, 32s], [63s, 6, 32s - 1], [63s + 1, 6, 32s], [63s + 1, 6, 32s - 1] or [63s + 2, 6, 32s] LCD code.

Proof. Obviously, a [63s, 6, 32s] code is the juxtaposition of s simplex codes with dimension 6 and then naturally SO.

If C is a [63s + 1, 6, 32s] linear code, then $\sigma = 32$ and its defining vector L satisfies $l_{max} = s + 1$. It follows that a [63s + 1, 6, 32s] code has a reduced code $[62s, 5, 32s] = [31 \times 2s, 5, 16 \times 2s]$, which is SO. Thus, one can deduce a code with parameters [63s + 1, 6, 32s] has $h \ge 4$ and a [63s + 1, 6, 32s] code is not LCD by Lemma 4. Further, there exists no [63s, 6, 32s - 1] LCD code considering that a [63s, 6, 32s - 1] code can be extended to a [63s + 1, 6, 32s] LCD code by Lemma 5.

A [63s + 2, 6, 32s] code has $\sigma = 64$ and then its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then \mathcal{C} has a reduced code $[62s, 5, 32s] = [31 \times 2s, 5, 16 \times 2s]$, which is SO.

If $l_{max} = s + 1$, it has a reduced code $[62s + 1, 5, 32s] = [31 \times 2s + 1, 5, 16 \times 2s]$. And this code has a reduced $[60s = 15 \times 4s, 4, 16 \times 2s]$ SO code, then a [63s + 2, 6, 32s] code has $h \ge 2$ and not LCD by Lemma 4. Similarly, we can derive that a [63s + 1, 6, 32s - 1] code is not LCD by Lemma 5.

Theorem 8. There is no [63s + 10, 6, 32s + 4] or [63s + 9, 6, 32s + 3] LCD code.

Proof. A [63s + 10, 6, 32s + 4] code has $\sigma = 68$ and then its defining vector L satisfies $s - 2 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code [62s + 8, 5, 32s + 4] violating the Griesmer bound, a contradiction.

When $l_{max} = s + 1$, a [63s + 10, 6, 32s + 4] code has a reduced code [62s + 9, 5, 32s + 4] code, and [62s + 9, 5, 32s + 4] has a reduced [60s + 8, 4, 32s + 4] SO code, thus we can deduce a [63s + 10, 6, 32s + 4] code is not LCD. It immediately follows that [63s + 9, 6, 32s + 3] code is not LCD.

Theorem 9. There is no [63s + 14, 6, 32s + 6] or [63s + 13, 6, 32s + 5] LCD code.

Proof. A [63s + 14, 6, 32s + 6] code has $\sigma = 70$ and then its defining vector L satisfies $s - 2 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code [62s + 12, 5, 32s + 6] violating the Griesmer bound, a contradiction.

When $l_{max} = s + 1$, a [63s + 14, 6, 32s + 6] code has a reduced code $\mathcal{D} = [62s + 13, 5, 32s + 6]$ code, which has $h(\mathcal{D}) \geq 3$ according to Ref. [15], thus $h(\mathcal{C}) \geq 2$. It then follows that [63s + 13, 6, 32s + 5] code is not LCD.

Theorem 10. There is no [63s+17,6,32s+8], [63s+16,6,32s+7], [63s+17,6,32s+7] or [63s+18,6,32s+8] LCD code.

Proof. A [63s + 17, 6, 32s + 8] code has $\sigma = 40$ and then its defining vector L satisfies $l_{max} = s + 1$. Then it has a reduced [62s + 16, 5, 32s + 8] SO code and one can deduce a [63s + 17, 6, 32s + 8] code is not LCD by Lemma 4. It immediately follows that a [63s + 16, 6, 32s + 7] code is not LCD.

Suppose that there is a [63s + 17, 6, 32s + 7] LCD code, then there exists a [63s + 18, 6, 32s + 8] LCD code by Lemma 5. However, a [63s + 18, 6, 32s + 8] code has a reduced [63s + 17, 5, 32s + 8] code with $h \ge 3$ by Ref. [15]. It naturally follows that a [63s + 18, 6, 32s + 8] code doesn't exist and then there is no [63s + 17, 6, 32s + 7] LCD code. \square

Theorem 11. There is no [63s + 25, 6, 32s + 12] or [63s + 24, 6, 32s + 11] LCD code.

Proof. A [63s + 25, 6, 32s + 12] code has $\sigma = 44$ and then its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code [62s + 23, 5, 32s + 12] violating the Griesmer bound, a contradiction.

When $l_{max} = s + 1$, it has a reduced [62s + 24, 5, 32s + 12] SO code. Thus we can deduce a [63s + 25, 6, 32s + 12] code is not LCD, it follows that a [63s + 24, 6, 32s + 11] code is also not LCD.

Theorem 12. There is no [63s + 29, 6, 32s + 14] or [63s + 28, 6, 32s + 13] LCD code.

Proof. A [63s + 29, 6, 32s + 14] code has $\sigma = 46$ and then its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code [62s + 27, 5, 32s + 14] violating the Griesmer bound, a contradiction.

When $l_{max} = s + 1$, Then it has a reduced code [62s + 28, 5, 32s + 14] code, which is an $\mathcal{MD}_s(5,2)$ code and has h = 3 by Lemma 6. Hence a [63s + 29, 6, 32s + 14] code has $h \geq 2$, thus there is no [63s + 29, 6, 32s + 14] LCD code according to lemma 4, which implies a [63s + 28, 6, 32s + 13] LCD code does not exist.

Theorem 13. There is no [63s + 30, 6, 32s + 14] or [63s + 29, 6, 32s + 13] LCD code.

Proof. If C is a [63s + 30, 6, 32s + 14] linear code, then $\sigma(C) = 32 \times 2 + 14 = 78$ and its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

- (1) If $l_{max} = s + 2$, then \mathcal{C} has a reduced code $\mathcal{D} = [62s + 28, 5, 32s + 14]$ with $h(\mathcal{D}) = 3$ by lemma 6, hence \mathcal{C} is not LCD according to lemma 4.
- (2) If $l_{max} = s + 1$, a [63s + 30, 6, 32s + 14] code has a reduced code $C' = [31 \times 2s + 29, 5, 16 \times 2s + 14]$.

For C', $\sigma(C') = 16 + 14 = 30$ and its defining vector L' satisfies $2s - 1 \le l'_i \le 2s + 2$.

- (2.1) If $l'_{max} = 2s + 2$, then C' has a reduced code $[30 \times 2s + 27, 5, 16 \times 2s + 14] = [15 \times (4s + 1) + 12, 4, 8 \times (4s + 1) + 6]$ violating the Griesmer bound, a contradiction.
- (2.2) If $l'_{max} = 2s + 1$ and $l'_{min} = 2s$, \mathcal{C} has a generalized anti-code $(2, 2^6, \{2\})$ by lemma 2. Owing to the maximum weight of 2^6 codewords is 2, so it is easy to know rank $(\mathcal{C}') \leq 2$. It then follows that $h(\mathcal{C}') = 5$ -rank $(\mathcal{C}'(\mathcal{C}')^T) \geq 5 2 = 3$.
- (2.3) If $l'_{max} = 2s + 1$ and $l'_{min} = 2s 1$, let m_1 , m_2 and m_3 be the numbers of equal 2s 1, 2s and 2s + 1, respectively. Then the following two equations hold:

$$m_1 + m_2 + m_3 = 31$$

 $(2s-1) \times m_1 + 2s \times m_2 + (2s+1) \times m_3 = 31 \times 2s + 29$

Solving the above equation system, we can obtain the only one integer solution:

$$m_1 = 1, m_2 = 0, m_3 = 30.$$

It follows that the defining vector L' of C' and the corresponding anti-vector L'^c have the following types:

$$L':][(2s-1)_1|(2s)_0|(2s+1)_{30}]]$$
 and $L'^c:][(2)_1|(1)_0|(0)_{30}]].$

From the form of L'^c , one can derive that $\operatorname{rank}(G^c(G^c)^T) = 0$ and C' is SO and h(C') = 5. Summarizing previous discussions of (1) and (2), we know $h(C) \geq 2$ by lemma 4 and then a [63s + 61, 6, 32s + 30] code does not exist. Hence, a [63s + 60, 6, 32s + 29] LCD code doe not exist.

Theorem 14. There is no [63s+31,6,32s+15], [63s+32,6,32s+15], [63s+32,6,32s+16] or [63s+33,6,32s+16] LCD code.

Proof. A [63s+32,6,32s+16] code is SO and then a [63s+31,6,32s+15] LCD code doesn't exist.

If C is a [63s + 33, 6, 32s + 16] linear code, then $\sigma = 32 + 16$ and its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code $[62s + 31, 5, 32s + 16] = [31 \times (2s + 1), 5, 16 \times (2s + 1)]$, which is SO.

If $l_{max} = s + 1$, a $[62s + 32, 5, 32s + 16] = [31 \times (2s + 1) + 1, 5, 16 \times (2s + 1)]$ code has a reduced $[30 \times (2s + 1), 4, 16 \times (2s + 1)]$ SO code, hence a [63s + 33, 6, 32s + 16] has $h \ge 2$, [63s + 33, 6, 32s + 16] code is not LCD. It follows that a [63s + 32, 6, 32s + 15] code is not LCD.

Theorem 15. There is no [63s + 41, 6, 32s + 20] or [63s + 40, 6, 32s + 19] LCD code.

Proof. A [63s+41, 6, 32s+20] has $\sigma = 52$ and then its defining vector L with $s-1 \le l_i \le s+2$.

If $l_{max} = s + 2$, then \mathcal{C} has a reduced code [62s + 39 = 31(2s + 1) + 8, 5, 32s + 20 = 16(2s + 1) + 4] violating the Griesmer bound, a contradiction.

When $l_{max} = s + 1$, it has a reduced [62s + 40, 5, 32s + 20] code, and [62s + 31 + 9, 5, 32s + 16 + 4] has a reduced [60s + 30 + 8, 4, 32s + 16 + 4] SO code. Thus one can deduce a [63s + 41, 6, 32s + 20] code is not LCD code, it follows that a [63s + 40, 6, 32s + 19] is not LCD.

Theorem 16. There is no [63s + 44, 6, 32s + 21] or [63s + 45, 6, 32s + 22] LCD code.

Proof. If C is a [63s + 45, 6, 32s + 22] linear code, then $\sigma = 32 + 22$ and its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code [62s + 43, 5, 32s + 22] violating the Griesmer bound, a contradiction.

If $l_{max} = s + 1$ and $l_{min} = s$, there is a unique [63s + 45, 6, 32s + 22] in [27], one can calculate $h(\mathcal{C}) = 4$.

If $l_{max} = s+1$ and $l_{min} = s-1$, then \mathcal{C} has a reduced code D = [62s+44, 5, 32s+22] = [31(2s+1)+13, 5, 16(2s+1)+6], which has $h(D) \geq 3$ according to Ref. [15], thus $h(\mathcal{C}) \geq 2$.

Theorem 17. There is no [63s+47, 6, 32s+23], [63s+48, 6, 32s+24], [63s+48, 6, 32s+24], [63s+49, 6, 32s+24] LCD code.

Proof. If C is a [63s + 49, 6, 32s + 24], then $\sigma = 56$ and its defining vector L satisfies $s - 1 \le l_i \le s + 2$.

If $l_{max} = s + 2$, then C has a reduced code [62s + 47, 5, 32s + 24] = [31(2s + 1) + 16, 5, 16(2s + 1) + 8] which is an SO code, and it easily follows that $h(C) \ge 4$.

If $l_{max} = s + 1$ and $l_{min} = s$, there are two [63s + 49, 6, 32s + 24] linear codes in [27]. Thus we can further calculate $h(C_1) = 6$ and $h(C_2) = 5$.

If $l_{max} = s+1$ and $l_{min} = s-1$, then \mathcal{C} has a reduced code D = [62s+48, 5, 32s+24] = [31(2s+1)+17, 5, 16(2s+1)+8], this reduced code has $h(D) \geq 3$ by Ref. [15], thus $h(\mathcal{C}) \geq 2$.

It is easy to know that a [63s+48, 6, 32s+24] code is SO and then a [63s+47, 6, 32s+23] code is not LCD.

Summarizing previous discussions, one can obtain that all of the [63s+47,6,32s+23], [63s+48,6,32s+24], [63s+48,6,32s+24] and [63s+49,6,32s+24] codes are not LCD. \square

Lemma 18. [20] Let $s \ge 0$, $N_k = 2^k - 1$ and $1 \le a \le 11$. If $k \ge 4$ for a = 1, 3, 4, 7, 8, $k \ge 5$ for a = 2, 6, 10 and $k \ge 7$ for a = 5, 9, 11, then the corresponding optimal $[sN_k + N_k - a, k]$ code is not LCD.

From Lemma 18, it is natural to derive the following conclusion.

Corollary 1. If $s \ge 0$, then these optimal [63s+53,6,32s+26], [63s+55,6,32s+27], [63s+56,6,32s+28], [63s+57,6,32s+28], [63s+59,6,32s+29], [63s+60,6,32s+30], [63s+61,6,32s+30], [63s+62,6,32s+31] linear codes are not LCD. Then one can infer that these [63s+52,6,32s+25], [63s+56,6,32s+27], [63s+60,6,32s+29] codes are also not LCD by Lemma 5.

4 Constructions of (near) optimal [n, 6] LCD codes for $n \ge 51$

In last section we have determined the nonexistence of many classes of $[n, 6, d_n]$ LCD codes, and then one can know it is optimal LCD if there exists an $[n, 6, d_n - 1]$ LCD code or it is near optimal LCD if there exists an $[n, 6, d_n - 2]$ LCD code. Next we will construct LCD codes of lengths greater than 50, the most of which are optimal LCD while the rest are at least near optimal LCD. According to the following lemma in Ref. [19], the optimal LCD codes can be constructed from these two linear codes [45, 6, 22] and [33, 6, 16] and simplex codes of given dimension.

Lemma 19. [19] Suppose $[n, k_1, d_1]$ is an LCD code and there are $[n, k_1, d_1] \subseteq [n, k, d_2] = \mathcal{C}$ with $h(\mathcal{C}) = k - k_1$. If there is an $[m, r, d_3]$ LCD code, then there is an [n + m, k, d] code with $d \ge \min\{d_1, d_2 + d_3\}$.

4.1 Constructions of optimal LCD codes [n, 6, d] for $51 \le n \le 64$

Theorem 20. There are optimal LCD codes with parameters [51, 6, 24], [52, 6, 24], [53, 6, 25], [54, 6, 26], [55, 6, 26], [56, 6, 26], [57, 6, 27], [58, 6, 28], [59, 6, 28], [60, 6, 28], [61, 6, 29], [62, 6, 30], [63, 6, 30] and [64, 6, 30].

Proof. Consider the three simplex matrices S_2 , S_4 and S_6 . Set

$$\mathbf{K}_{6,18} = \left(\begin{array}{cc} S_4 & \mathbf{0}_{15\times3} \\ \mathbf{0}_{3\times15} & S_2 \end{array}\right)$$

and delete the columns of $\mathbf{K}_{6,18}$ from S_6 . Thus, one can obtain

$$\mathbf{G}_{6,45} = S_6 \setminus \mathbf{K}_{6,18} = \begin{pmatrix} X \\ Y \end{pmatrix},$$

which generates a [45, 6, 22] code with hull dimension 4, where X is a 4×45 matrix and Y is a 2×45 matrix. X generates a [45, 4, 28] code and Y generates a [45, 2, 30] LCD code. Naturally, we have the nested codes [45, 2, 30] \subseteq [45, 6, 22]. Let $[m, 4, d_l(m, 4)]$ be an optimal LCD code for $6 \le m \le 19$ in [13].

By these codes [45, 2, 30], [45, 6, 22] and $[m, 4, d_l]$, using Lemma 19, one can constuct the optimal LCD codes listed in the following table.

	T_{ϵ}	able 2	2: Th	ne op	$_{ m timal}$	LCI	$\bigcap [n,$	[6,d]	code	s for	51 ≤	$n \leq$	64	
\overline{n}	51	52	53	54	55	56	57	58	59	60	61	62	63	64
d_l	24	24	25	26	26	26	27	28	28	28	29	30	30	30

4.2 Constructions of optimal LCD codes [n, 6, d] for $65 \le n \le 68$

Theorem 21. There are optimal LCD codes with parameters [65, 6, 31], [66, 6, 32], [67, 6, 32] and [68, 6, 32].

Proof. Consider the simplex matrix S_5 . Set

$$\mathbf{K}_{6,33} = \left(\begin{array}{cc} \mathbf{1_2} & \mathbf{1_{31}} \\ \mathbf{0}_{5\times2} & S_5 \end{array}\right)$$

and $\mathbf{K}_{6,33}$ generates a [33, 6, 16] code \mathcal{C} , which has a reduced SO code [32, 5, 16] and

$$h(\mathbf{K}_{6.33}) = 6 - rank(\mathbf{K}\mathbf{K}^T) = 6 - 1 = 5.$$

Obviously there is an LCD code $[33, 1, 33] \subseteq [33, 6, 16]$. Let $[m, 5, d_l(m, 5)]$ be an optimal LCD code for $32 \le m \le 35$ in [15].

By these codes [33, 1, 33], [33, 6, 16] and $[m, 5, d_l]$, using Lemma 19, one can obtain these LCD codes [65, 6, 31], [66, 6, 31], [67, 6, 32] and [68, 6, 32], respectively.

Further, one can know an optimal [66,6,32] LCD code exists by the existence of a [65,6,31] LCD code by Lemma 5. Combining with the above constructions, this lemma holds. For clarity, we list them in the following table.

4.3 Constructions of (near) optimal LCD codes [n, 6, d] for $n = 63s + t \ge 68$

When $6 \le t \le 68$, the optimal LCD codes with $[t, 6, d_l(t, 6)]$ can be obtained in Refs. [8–14] and the constructions in the above two subsections. By the juxtaposition of s simplex codes [63, 5, 32] and an optimal LCD code $[t, 6, d_a(t, 6)]$ for $6 \le t \le 68$, one can easily obtain all $[n = 63s + t, 6, 32s + d_l(n, 6)]$ (near) optimal LCD codes for $n \ge 68$. For clarity, we give some examples as follows.

Example 3. When $n = 131 = 63 \times 1 + 68$, there exits an optimal $[131, 6, 32 \times 1 + 32]$ LCD code saturating the Griesmer bound, which can be constructed by the juxtaposition of one simplex code [63, 5, 32] and an optimal LCD code [68, 6, 32]. It is obviously also an optimal code.

Example 4. When $n = 136 = 63 \times 2 + 10$, there exits a $[136, 6, 32 \times 2 + 3]$ LCD code, which can be constructed by the juxtaposition of two simplex codes [63, 5, 32] and an optimal LCD code [10, 6, 3]. By Theorem 2, we can know there is not [63s + 10, 6, 32s + 4] LCD codes. It naturally follows that the $[136, 6, 32 \times 2 + 3]$ LCD code is optimal LCD code, which is also a near optimal code in Ref. [8].

Example 5. When $n = 206 = 63 \times 3 + 17$, there exits a $[206, 6, 32 \times 3 + 6]$ LCD code, which can be constructed by the juxtaposition of three simplex codes [63, 5, 32] and an optimal LCD code [17, 6, 6]. By Theorem 3, we can know there is no [63s + 17, 6, 32s + 7] LCD code. It naturally follows that a $[206, 6, 32 \times 3 + 6]$ LCD code is optimal LCD, whose minimum distance is smaller by 2 than the corresponding optimal code in Ref. [8].

Example 6. When $n = 273 = 63 \times 4 + 21$, there exits a $[273, 6, 32 \times 4 + 8]$ LCD code, which can be constructed by the juxtaposition of four simplex codes [63, 5, 32] and an optimal LCD code [21, 6, 8]. One can infer that it is at least near optimal LCD and near optimal because the optimal code has the parameter $[273, 6, 32 \times 4 + 9]$, which can be derived from Ref. [8].

5 Conclusion

By the theories of defining vectors, generalized anti-codes, reduced codes and nested code chains, the nonexistence and constructions of LCD codes have been studied in last two sections for $n = 63s + t \ge 51$. To conclude the above results, $d_l(n,6)$ has been exactly determined for $0 \le t \le 62$ and $t \notin \{21, 22, 25, 26, 33, 34, 37, 38, 45, 46\}$. We have also showed that $d_l(n,6) \in \{d_a(n,6) - 1, d_a(n,6)\}$ for $t \in \{21, 22, 26, 34, 37, 38, 46\}$ and $d_l(n,6) \in \{d_a(n,6) - 2, d_a(n,6) - 1\}$ for $t \in \{25, 33, 45\}$.

From Ref. [8], it is easy to know all optimal [n,6] linear codes can achieve the Griesmer bound for $42 \le n \le 256$. For n > 256, the length n can be denoted as n = 63s + t, where $s \ge 3$ and $63 \le t \le 125$ are integers. By the juxtaposition of s simplex codes [63, 5, 16] and an optimal linear code $[t, 6, d_a(t, 5)]$, one can obtain all $[n, 6, d_a(n, 6)]$ optimal linear codes with $d_a(n, 6)$ achieving the Griesmer bound for n > 256. That is to say any $d_a(n, 6)$ can meet the Griesmer bound for all lengths $n \ge 42$.

Let d_a and d_l be defined as the previous sections, we list optimal linear codes and (near) optimal LCD codes in Table 4 for $n \ge 42$. Among them, d_l of (near) optimal LCD codes of lengths $n \ge 51$ are obtained in this paper.

Remark 1. From Table 4, these codes can be clearly divided into four groups:

Let $n = 63s + t \ge 42$, the following items hold:

- (i) For t=3, 4, 5, 6, 7, 8, 11, 12, 15, 19, 20, 23, 27, 35, 36, 39, 42, 43, 50, 51, 54, 58, the corresponding $[n, 6, d_l]$ optimal LCD codes are also optimal linear codes.
- (ii) For t=2, 9, 10, 13, 14, 16, 18, 21, 22, 24, 25, 26, 28, 30, 31, 34, 37, 38, 40, 41, 42, 44, 46, 47, 49, 52, 53, 55, 57, 59, 61, 62, the corresponding $[n, 6, d_l]$ LCD codes are near optimal linear codes, i.e. $d_l = d_a 1$. If $t \neq 21$, 22, 26, 34, 37, 38, 46, the $[n, 6, d_l]$ codes are optimal LCD codes.
- (iii) For t=0, 1, 17, 25, 29, 32, 33, 45, 48, 56, 60, the corresponding $[n, 6, d_l]$ LCD codes are not near optimal linear codes and $d_l = d_a 2$. If $t \neq 25, 33, 45$, the $[n, 6, d_l]$ codes are optimal LCD codes.
- (iv) For each t=21, 22, 25, 26, 33, 34, 37, 38, 45, 46, the corresponding $[n, 6, d_l]$ is at least near optimal LCD codes and the optimal LCD codes still can not determined. We sincerely look forward to the completion of the follow-up work. When the dimensions of LCD codes are higher, the approach in the paper can be also employed, one can further study optimal LCD codes with higher dimensions. It is hoped that more results about optimal LCD codes will be obtained by more scholars in future study.

Table 4:	[n, 6, 32s -	$+d_t$] coe	des with n	a = 63s + t	> 42
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$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$egin{array}{ccc} 3 & 4 \ 2 & 3 \end{array}$
	9 9
	2 3
t 11 12 13 14 15 16 17 18	19 20
d_a 4 4 5 6 6 7 8 8	8 8
d_l 4 4 4 5 6 6 6 7	8 8
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	29 30
d_a 9 10 10 11 12 12 13	14 14
d_l 8/9 9/10 10 10 10/11 11/12 12 12	12 13
t 31 32 33 34 35 36 37 38	39 40
d_a 15 16 16 16 16 16 17 18	18 19
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	/18 18 18
n 41 42 43 44 45 46 47 48	49 50
d_a 20 20 20 21 22 22 23 24	24 24
d_l 19 20 20 20 20/21 21/22 22 22	23 24
n 51 52 53 54 55 56 57 58	59 60
d_a 24 25 26 26 27 28 28 28	29 30
d_l 24 24 25 26 26 26 27 28	28 28
n 61 62	
d_a 30 31	
d_l 29 30	

 d_t denotes d_a or d_l in different lines and the left slash"/" means "or". For example, "14/15" implies the largest minimum distance of [63s + 33, 6] LCD codes is 32s + 14 or 32s + 15.

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