11 Sets and functions

Bertrand Russell (1872–1970) was a British mathematician and philosopher who co-wrote the book *Principia Mathematica*, in which the authors tried to reduce all mathematics to formal logic. In 1901, he discovered what became known as Russell's paradox, based on an abstract question about sets. This inspired mathematicians in the early 20th century to search for a consistent, contradiction-free, version of set theory.

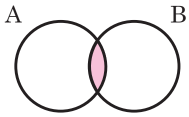
Georg Cantor (1845–1918) was a German mathematician who played a big role in the development of set theory. He is also responsible for proving several surprising mathematical ideas, one of which is that even though there are an infinite number of real numbers on the number line, and an infinite number of fractions, there are actually infinitely more real numbers than there are fractions. In proving this, Cantor showed that there are in fact an infinite number of different infinities.

- Understand and use Venn diagrams and set notation.
- Functions including notation, domain, range, inverse functions and composite functions.

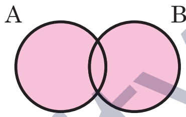
11.1 Sets

In mathematics, a **set** is a collection of well defined and distinct items that share a particular property. Members of a set are called **elements**. Sets are often displayed graphically in a **Venn diagram**. You need to understand the following language and notation for dealing with sets.

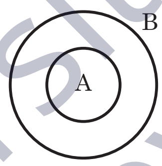
1. The symbol $\cap$ means 'intersection'.
   In this diagram, this means A **and** B.
   $A \cap B$ is shaded.



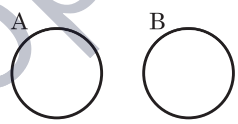
2. The symbol $\cup$ means 'union'.
   Here, this means A **or** B **or both**.
   $A \cup B$ is shaded.

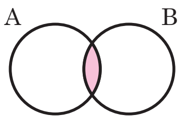


3. The symbol $\subseteq$ means 'is a subset of'.
   The symbol $\nsubseteq$ means 'is not a subset of'.
   $A \subseteq B$ means A lies completely inside B.
   A can actually equal B, which means every set is a subset of itself.

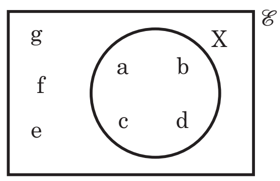


   Set A might lie completely outside B (having no elements in common) or may only partly overlap B (having only some elements in common).

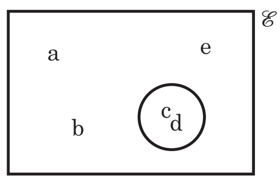




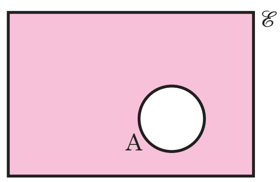
4. The symbol $\in$ means 'is an element of'.
   You can also think of it as meaning 'belongs to' or 'is a member of'.
   The symbol $\notin$ means 'is not an element of'.
   $b \in X$ (the element b is a member of set X)
   $e \notin X$ (the element e is not an element of set X)



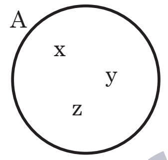
5. The symbol $\mathscr{E}$ means the 'universal set'.
   The universal set is the set of all things being considered at the time. In a Venn diagram such as this one, everything inside the rectangle makes up the universal set.
   $\mathscr{E} = \{a, b, c, d, e\}$



6. The symbol A' means 'the complement of set A'.
   This means all elements that are *not* in set A.
   A' is shaded.
   Note that $A \cup A' = \mathscr{E}$



7. The symbol n(A) means 'the number of elements in set A'.
   In the set A here, n(A) = 3, as set A has 3 elements.



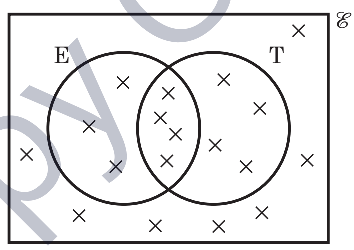
It is not always possible to list all the elements in a set; sometimes because the set contains an infinite number of elements, and sometimes because it is just not practical to do so if the set has a lot of elements.

8. Set notation
   $A = \{x: x \text{ is an integer}, 2 \leqslant x < 9\}$
   This reads, 'A is the set of all elements x, such that x is an integer, and $2 \leqslant x < 9$'.
   The set A is {2, 3, 4, 5, 6, 7, 8}.

9. The symbol $\varnothing$ means 'the empty set'. This is the set that contains no elements.
   Note that $\varnothing$ is considered to be a subset of every set.

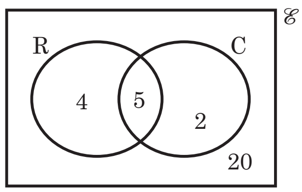
Exercise 11.1A

1. In the Venn diagram:



   ℰ = {people eating breakfast at a hotel}
   T = {people who like toast}
   E = {people who like eggs}
   a) How many people like toast?
   b) How many people like eggs but not toast?
   c) How many people like toast and eggs?
   d) How many people are in the hotel?
   e) How many people like neither toast nor eggs?

2. In the Venn diagram:



   ℰ = {students in form group 10ARO}
   R = {members of the robotics club}
   C = {members of the chess club}
   a) How many students are in the robotics club?
   b) How many students are in both clubs?
   c) How many are in the robotics club but not in the chess club?
   d) How many are in neither club?
   e) How many students are there in 10ARO?

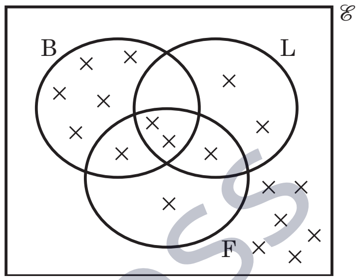
3. In the Venn diagram:

ℰ = {cars parked on a street}

B = {blue cars}

L = {cars with left-hand drive}

F = {cars with four doors}



Each cross represents one car.

a) How many cars are blue?

b) How many blue cars have four doors?

c) How many cars with left-hand drive have four doors?

d) How many blue cars have left-hand drive?

e) How many cars are in the street?

f) How many blue cars with left-hand drive do not have four doors?

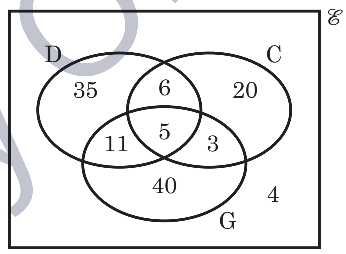
4. In the Venn diagram:

ℰ = {houses on Abbey Road}

C = {houses with a rooftop chimney}

D = {houses with a driveway}

G = {houses with a garden}



a) How many houses have a garden?

b) How many houses have a driveway and a chimney?

c) How many houses have a driveway and a chimney and a garden?

d) How many houses have a garden, but not a driveway or a chimney?

e) How many houses have a driveway and a garden, but not a chimney?

f) How many houses are there on Abbey Road?

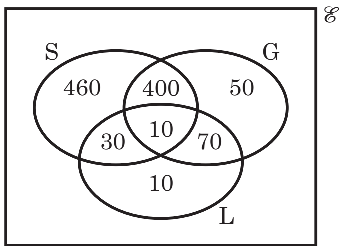
5. In the Venn diagram:

ℰ = {children in a school}

G = {girls}

S = {children who can swim}

L = {children who are left-handed}



a) How many left-handed children are there?

b) How many girls cannot swim?

c) How many boys can swim?

d) How many girls are left-handed?

e) How many boys are left-handed?

f) How many left-handed girls can swim?

g) How many boys are there in the school?

Set notation

You need to be able to use and understand set notation. Sometimes sets will be defined directly and sometimes you will be given them in a Venn diagram.

Example 1

$A = \{x: x \text{ is a natural number}\}$ and $B = \{x: x \text{ is a prime number}\}$

State whether true or false:

a) $5 \in A \cap B$ **b)** $8 \in A \cup B$

c) $B \subseteq A$ **d)** $2 \in A \cap B'$

Set A is the set of natural numbers, which are the counting numbers 1, 2, 3, 4, …

Set B are the prime numbers.

a) 5 is a natural number and it is prime, so the statement is true.

b) 8 is a natural number, so the statement is true.

c) All prime numbers are natural numbers, so the statement is true.

d) 2 is a natural number but it is prime, so the statement is false.

Example 2

$P = \{(x, y): y = 2x + 3\}$ and $Q = \{(x, y): y = -2x - 1\}$

State whether true or false:

a) $(1, 5) \in P \cup Q$ **b)** $(-1, 1) \in P \cap Q$

c) $(3, 10) \in P' \cap Q$ **d)** $\{(0, -1), (1, -3), (2, -4)\} \subseteq Q$

a) (1, 5) belongs to set P because $5 = 2 \times 1 + 3$. Therefore the statement is true.

b) (–1, 1) belongs to both sets. Therefore the statement is true.

c) (3, 10) does not belong to set P, but it also does not belong to set Q. Therefore the statement is false.

d) (0, –1) and (1, –3) are members of set Q, but (2, –4) is not. Therefore the statement is false.

Example 3

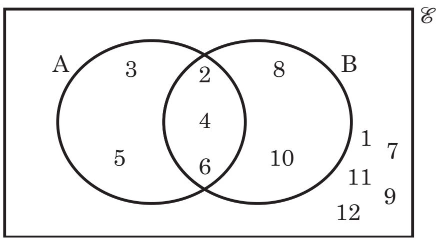
$\mathscr{E} = \{1, 2, 3, \ldots, 12\}$, $A = \{2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10\}$

a) Draw a Venn diagram to represent these sets.

b) Use your diagram to find:

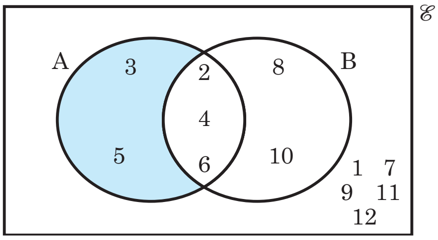
i) $A \cup B$ **ii)** $A \cap B$ **iii)** A' **iv)** $n(A \cup B) = 7$ **v)** $B' \cap A$

a)



b) i) $A \cup B = \{2, 3, 4, 5, 6, 8, 10\}$ **ii)** $A \cap B = \{2, 4, 6\}$

iii) $A' = \{1, 7, 8, 9, 10, 11, 12\}$ **iv)** $n(A \cup B) = 7$ **v)** $B' \cap A = \{3, 5\}$



Exercise 11.1B

In this exercise, be careful to use set notation only when the answer *is* a set.

1. If $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $N = \{5, 7, 9, 11, 13\}$, find:

a) $M \cap N$ **b)** $M \cup N$ **c)** $n(N)$ **d)** $n(M \cup N)$

State whether true or false:

e) $5 \in M$ **f)** $7 \in (M \cup N)$ **g)** $N \subseteq M$ **h)** $\{5, 6, 7\} \subseteq M$

2. If $A = \{2, 3, 5, 7\}$, $B = \{1, 2, 3, \ldots, 9\}$, find:

a) $A \cap B$ **b)** $A \cup B$ **c)** $n(A \cap B)$ **d)** $\{1, 4\} \cap A$

State whether true or false:

e) $A \in B$ **f)** $A \subseteq B$ **g)** $9 \subseteq B$ **h)** $3 \in (A \cap B)$

3. $A = \{x: x \text{ is a natural number}\}$ and $B = \{x: x \text{ is an even number}\}$

State whether true or false:

a) $21 \in A \cap B$ **b)** $7 \in A \cup B$ **c)** $A \subseteq B$ **d)** $6 \in A' \cap B$

4. $P = \{(x, y): y = x + 5\}$ and $Q = \{(x, y): y = 1 - x\}$

State whether true or false:

a) $(-2, 3) \in P \cup Q$ **b)** $(-2, 3) \in P \cap Q$

c) $(-1, 2) \in P' \cap Q$ **d)** $\{(2, -1), (-4, 1), (-2, 3)\} \subseteq P \cup Q$

5. If $X = \{1, 2, 3, \ldots, 10\}$, $Y = \{2, 4, 6, \ldots, 20\}$ and $Z = \{x : x \text{ is an integer}, 15 \leqslant x \leqslant 25\}$, find:

   a) $X \cap Y$ **b)** $Y \cap Z$ **c)** $X \cap Z$
   d) $n(X \cup Y)$ **e)** $n(Z)$ **f)** $n(X \cup Z)$

   State whether true or false:

   g) $5 \in Y$ **h)** $20 \in X$
   i) $n(X \cap Y) = 5$ **j)** $\{15, 20, 25\} \subseteq Z$

6. If $D = \{1, 3, 5\}$, $E = \{3, 4, 5\}$, $F = \{1, 5, 10\}$, find:

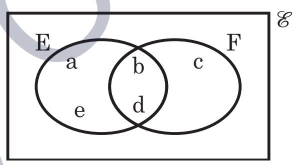
   a) $D \cup E$ **b)** $D \cap F$ **c)** $n(E \cap F)$
   d) $(D \cup E) \cap F$ **e)** $(D \cap E) \cup F$ **f)** $n(D \cup F)$

   State whether true or false:

   g) $D \subseteq (E \cup F)$ **h)** $3 \in (E \cap F)$ **i)** $4 \notin (D \cap E)$

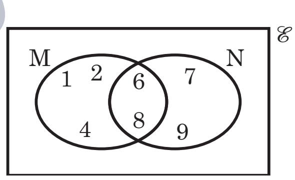
7. Find:

   a) $n(E)$ **b)** $n(F)$ **c)** $E \cap F$
   d) $E \cup F$ **e)** $n(E \cup F)$ **f)** $n(E \cap F)$



8. Find:

   a) $n(M \cap N)$ **b)** $n(N)$ **c)** $M \cup N$
   d) $M' \cap N$ **e)** $N' \cap M$ **f)** $(M \cap N)'$
   g) $M \cup N'$ **h)** $N \cup M'$ **i)** $M' \cup N'$



Shading Venn diagrams

You need to know how to shade areas on a Venn diagram that have been described using set notation.

To shade the union of two sets, for example, $A \cup B$, first shade the whole of the first set, and then shade the whole of the second set.

To shade the intersection of two sets, for example, $A \cap B$, lightly shade the whole of the first set using diagonal lines in one direction, and then lightly shade the whole of the second set using diagonal lines in the other direction. The intersection that you need to shade properly is the part that has been shaded twice.

Example

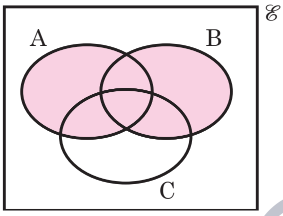
On a Venn diagram, shade the regions:

a) $A \cup B$ **b)** $A \cap C$ **c)** $(B \cap C) \cap A'$

where A, B, C are intersecting sets.

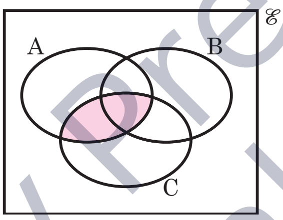
a) $A \cup B$

Shading set A then set B gives this diagram.



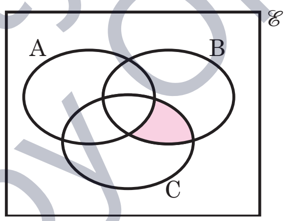
b) $A \cap C$

'A intersection C' contains only those elements common to both sets.



c) $(B \cap C) \cap A'$

First identify the intersection of sets B and C, then remove the part that is also in set A.



Exercise 11.1C

1. Draw six diagrams similar to Figure 1 and shade the following sets:

   a) $A \cap B$ **b)** $A \cup B$ **c)** A'

   d) $A' \cap B$ **e)** $B' \cap A$ **f)** $(B \cup A)'$

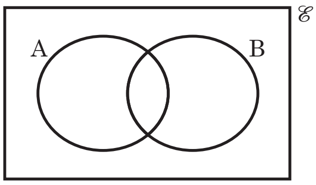


Figure 1

2. Draw four diagrams similar to Figure 2 and shade the following sets:

   a) $A \cap B$ **b)** $A \cup B$ **c)** $B' \cap A$

   d) $(B \cup A)'$

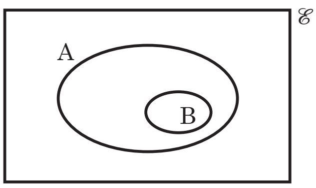


Figure 2

3. Draw four diagrams similar to Figure 3 and shade the following sets:

   a) $A \cup B$ **b)** $A \cap B$ **c)** $A \cap B'$

   d) $(B \cup A)'$

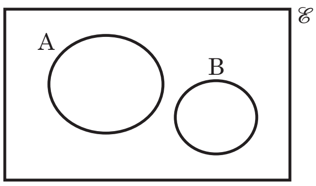


Figure 3

4. Draw nine diagrams similar to Figure 4 and shade the following sets:

a) $A \cap B$ **b)** $A \cup C$ **c)** $A \cap (B \cap C)$

d) $(A \cup B) \cap C$ **e)** $A \cap B'$ **f)** $A \cap (B \cup C)'$

g) $C' \cap (A \cap B)$ **h)** $(A \cup C) \cup B'$ **i)** $(A \cup C) \cap (B \cap C)$

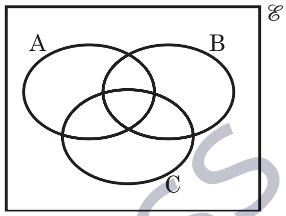


Figure 4

5. Draw nine diagrams similar to Figure 5 and shade the following sets:

a) $(A \cup B) \cap C$ **b)** $(A \cap B) \cup C$ **c)** $(A \cup B) \cup C$

d) $A \cap (B \cup C)$ **e)** $A' \cap C$ **f)** $C' \cap (A \cup B)$

g) $(A \cap B) \cap C$ **h)** $(A \cap C) \cup (B \cap C)$ **i)** $(A \cup B \cup C)'$

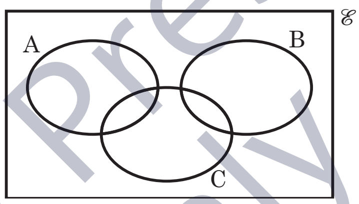
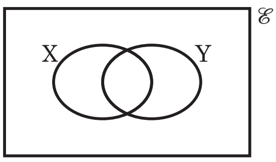
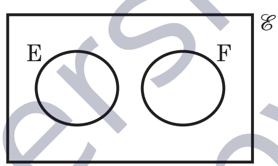


Figure 5

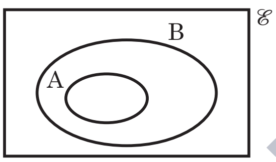
6. Copy each diagram and shade the region indicated.

a)


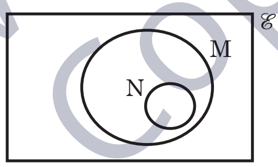
$X' \cap Y$

b)


$E \cup F$

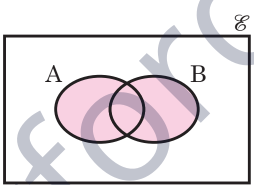
c)


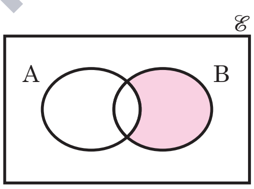
$A \cap B$

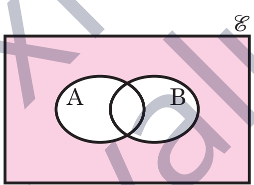
d)


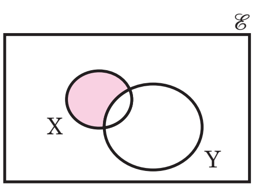
$(M \cap N)'$

7. Describe the shaded region in each Venn diagram using set notation.

a)


b)


c)


d)


11.2 Logical problems

Example 1

In a form of 30 students, 18 play basketball and 14 play hockey, while 5 students play neither sport. Find the number who play both basketball and hockey.

Let $\mathscr{E}$ = {students in the form}

B = {students who play basketball}

H = {students who play hockey}

and x = the number of students who play both basketball and hockey

> **Tip**
>
> Note that $18 + 14 + 5 > 30$, so there will be some intersection of the two sets.

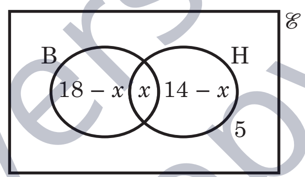
The number of students in each portion of the universal set is shown in the Venn diagram.

Since $n(\mathscr{E}) = 30$

$$18 - x + x + 14 - x + 5 = 30$$

$$37 - x = 30$$

$$x = 7$$



Seven students play both basketball and hockey.

Example 2

Sets A to D are defined as follows:

A = {sheep} C = {‘intelligent’ animals}

B = {horses} D = {animals that make good pets}

a) Express the following sentences in set language:

i) No sheep are ‘intelligent’ animals. **ii)** All horses make good pets.

iii) Some sheep make good pets.

b) Interpret the following statements:

i) $B \subseteq C$ **ii)** $B \cup C = D$

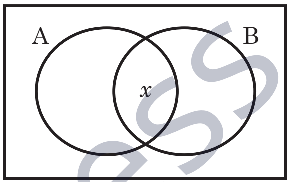
a) i) $A \cap C = \varnothing$ **ii)** $B \subseteq D$ **iii)** $A \cap D \neq \varnothing$

b) i) All horses are intelligent animals.

ii) Animals that make good pets are either horses or ‘intelligent’ animals (or both).

Exercise 11.2A

1. In the Venn diagram, $n(A) = 10$, $n(B) = 13$, $n(A \cap B) = x$ and $n(A \cup B) = 18$.



   a) Write in terms of x the number of elements in A but not in B.

   b) Write in terms of x the number of elements in B but not in A.

   c) Add together the number of elements in the three parts of the diagram to obtain the equation $10 - x + x + 13 - x = 18$.

   d) Hence find the number of elements that are in both A and B.

2. In the Venn diagram, $n(A) = 21$, $n(B) = 17$, $n(A \cap B) = x$ and $n(A \cup B) = 29$.

   a) Write down in terms of x the number of elements in each part of the diagram.

   b) Form an equation and hence find x.

3. The sets M and N intersect such that $n(M) = 31$, $n(N) = 18$ and $n(M \cup N) = 35$. Find the number of elements in both M and N.

4. The sets P and Q intersect such that $n(P) = 11$, $n(Q) = 29$ and $n(P \cup Q) = 37$. Find the number of elements in both P and Q.

5. The sets A and B intersect such that $n(A \cap B) = 7$, $n(A) = 20$ and $n(B) = 23$. Find $n(A \cup B)$.

6. Twenty students all play either pickle ball or table tennis (or both). If thirteen play pickle ball and ten play table tennis, how many play both sports?

7. Of the 53 staff at a school, 36 drink tea, 18 drink coffee and 10 drink neither tea nor coffee. How many staff members drink both tea and coffee?

8. Of the 32 students in a class, 18 play golf, 16 play the piano and 7 play both. How many students play neither sport?

9. Of the students in a class, 15 can correctly spell the word ‘parallel’, 14 can correctly spell ‘asymptote’, 5 can spell both words correctly, and 4 can spell neither. How many students are there in the class?

10. In a school, students must take at least one of these subjects: Maths, Physics or Chemistry. Here is information about a group of 50 students:
    - 7 take all three subjects
    - 9 take Physics and Chemistry only
    - 8 take Maths and Physics only
    - 5 take Maths and Chemistry only
    - x take Maths only
    - x take Physics only
    - $x + 3$ take Chemistry only.

    Draw a Venn diagram, find x, and hence find the number taking Maths.

11. All of 60 different vitamin pills contain at least one of the vitamins A, B and C.

- 12 contain A only
- 7 contain B only
- 11 contain C only
- 6 pills contain all three vitamins
- x contain A and B only
- x contain B and C only
- x contain A and C only

How many pills contain vitamin A?

12. The IGCSE results of the 30 members of a badminton team were as follows: all 30 players passed at least two subjects, 18 players passed at least three subjects, and 3 players passed four subjects or more. Calculate:

a) how many passed exactly two subjects

b) the fraction of the team who passed exactly three subjects.

13. In a group of 59 people, some are wearing hats, gloves or scarves (or a combination of these) as follows:

- 4 people are wearing all three items
- 7 people are wearing just a hat and gloves
- 3 are wearing just gloves and a scarf
- 9 are wearing just a hat and scarf
- x are wearing only a hat
- x are wearing only gloves
- $(x - 2)$ are wearing only a scarf
- $(x - 2)$ are wearing none of the three items.

Find x and hence the number of people wearing a hat.

14. In a street of 150 houses, three different newspapers are delivered: the Tribune, the Herald and the Chronicle, as follows:

- 40 receive the Tribune
- 35 receive the Herald
- 60 receive the Chronicle
- 7 receive the Tribune and the Herald
- 10 receive the Herald and the Chronicle
- 4 receive the Tribune and the Chronicle
- 34 receive no newspaper at all.

How many receive all three newspapers?

Note: If ‘7 receive Tribune and the Herald’, this information does not mean that 7 people receive the Tribune and the Herald *only*.

15. If S = {skydivers}, G = {good swimmers}, express the following statements in words:

a) $G \subseteq S$ **b)** $G \cap S = \varnothing$ **c)** $G \cap S \neq \varnothing$

(Ignore the truth or otherwise of the statements.)

16. Given that $\mathscr{E}$ = {students in a school}, G = {girls}, I = {ice skaters}, S = {surfers}, express the following possible situations in words:

a) $S \subseteq G$ **b)** $I \subseteq G'$ **c)** $S \cap I \neq \varnothing$ **d)** $G \cap I = \varnothing$

Express in set notation:

e) No girls are surfers. **f)** All students either surf or ice skate.

17. $\mathscr{E}$ = {living creatures}, S = {spiders}, F = {animals that fly}, C = {animals that are cute} Express in set notation:

a) No spiders are cute. **b)** All animals that fly are cute.

c) Some spiders can fly.

Express in words:

d) $S \cup F \cup C = \mathscr{E}$ **e)** $C \subseteq S$

11.3 Functions

The idea of a function is used in almost every branch of mathematics.
The two common forms of notation used are:

- $f(x) = x^2 + 4$
- $f : x \mapsto x^2 + 4$

In this course, however, only the first notation is used.
You interpret the second form as: 'function f such that x is mapped onto $x^2 + 4$'.

Example 1

If $f(x) = 3x - 1$ and $g(x) = 1 - x^2$ find:

a) f(2) **b)** f(−2) **c)** g(1) **d)** g(3) **e)** x if $f(x) = 1$

---

a) $f(2) = 3 \times 2 - 1 = 5$ **b)** $f(-2) = 3 \times (-2) - 1 = -7$

c) $g(1) = 1 - 1^2 = 0$ **d)** $g(3) = 1 - 3^2 = -8$

e) If $f(x) = 1$

Then $3x - 1 = 1$

$3x = 2$

$x = \frac{2}{3}$

Flow diagrams

The function f in Example 1 consisted of two simpler functions as illustrated by a flow diagram.

$x \longrightarrow \boxed{\text{multiply by } 3} \xrightarrow{3x} \boxed{\text{subtract } 1} \longrightarrow 3x - 1$

The input is x, and the output is $3x - 1$.

It is important to 'multiply by 3' and 'subtract 1' in the correct order.

Example 2

Draw flow diagrams for the functions:

a) $f(x) = (2x + 5)^2$ **b)** $g(x) = \dfrac{5 - 7x}{3}$

Find:

c) $f(2)$ **d)** $g(-4)$

a) $x \longrightarrow \boxed{\text{multiply by } 2} \xrightarrow{2x} \boxed{\text{add } 5} \xrightarrow{2x+5} \boxed{\text{square}} \longrightarrow (2x + 5)^2$

b) $x \longrightarrow \boxed{\text{multiply by } (-7)} \xrightarrow{-7x} \boxed{\text{add } 5} \xrightarrow{5-7x} \boxed{\text{divide by } 3} \longrightarrow \dfrac{5 - 7x}{3}$

c) $f(2) = (2 \times 2 + 5)^2$

$= 9^2$

$= 81$

d) $-4 \longrightarrow \boxed{\text{multiply by } (-7)} \xrightarrow{28} \boxed{\text{add } 5} \xrightarrow{33} \boxed{\text{divide by } 3} \longrightarrow 11$

Therefore, $g(-4) = 11$

Exercise 11.3A

1. Given the functions $h(x) = x^2 + 1$ and $g(x) = 10x + 1$. Find:

a) $h(2)$, $h(-3)$, $h(0)$ **b)** $g(2)$, $g(10)$, $g(-3)$

For Questions **2** to **15**, draw a flow diagram for each function.

2. $f(x) = 5x + 4$

3. $f(x) = 3(x - 4)$

4. $f(x) = (2x + 7)^2$

5. $f(x) = \dfrac{9 + 5x}{4}$

6. $f(x) = \dfrac{4 - 3x}{5}$

7. $f(x) = 2x^2 + 1$

8. $f(x) = \dfrac{3x^2}{2} + 5$

9. $f(x) = \sqrt{4x - 5}$

10. $f(x) = 4\sqrt{x^2 + 10}$

11. $f(x) = (7 - 3x)^2$

12. $f(x) = 4(3x + 1)^2 + 5$

13. $f(x) = 5 - x^2$

14. $f(x) = \dfrac{10\sqrt{x^2 + 1} + 6}{4}$

15. $f(x) = \left(\dfrac{x^3}{4} + 1\right)^2 - 6$

For Questions **16**, **17** and **18**, the functions f, g and h are defined as follows:

$\mathrm{f}(x) = 1 - 2x \qquad \mathrm{g}(x) = \dfrac{x^3}{10} \qquad \mathrm{h}(x) = \dfrac{12}{x}$

16. Find:

a) $\mathrm{f}(5), \mathrm{f}(-5), \mathrm{f}\left(\frac{1}{4}\right)$ **b)** $\mathrm{g}(2), \mathrm{g}(-3), \mathrm{g}\left(\frac{1}{2}\right)$ **c)** $\mathrm{h}(3), \mathrm{h}(10), \mathrm{h}\left(\frac{1}{3}\right)$

17. Find:

a) x if $\mathrm{f}(x) = 1$ **b)** x if $\mathrm{f}(x) = -11$ **c)** x if $\mathrm{h}(x) = 1$

18. Find:

a) y if $\mathrm{g}(y) = 100$ **b)** z if $\mathrm{h}(z) = 24$ **c)** w if $\mathrm{g}(w) = 0.8$

For Questions **19** and **20**, the functions f, g and h are defined as follows:

$\mathrm{f}(x) = \dfrac{2x^2}{3} \qquad \mathrm{g}(x) = \sqrt{(y-1)(y-2)} \qquad \mathrm{h}(x) = 10 - x^2$

19. Find:

a) $\mathrm{f}(3), \mathrm{f}(6), \mathrm{f}(-3)$ **b)** $\mathrm{g}(2), \mathrm{g}(0), \mathrm{g}(4)$ **c)** $\mathrm{h}(4), \mathrm{h}(-2), \mathrm{h}\left(\frac{1}{2}\right)$

20. Find:

a) x if $\mathrm{f}(x) = 6$ **b)** x if $\mathrm{h}(x) = 1$ **c)** y if $\mathrm{f}(y) = 2\frac{2}{3}$ **d)** p if $\mathrm{h}(p) = -26$

21. If $\mathrm{g}(x) = 2^x + 1$, find:

a) $\mathrm{g}(2)$ **b)** $\mathrm{g}(4)$ **c)** $\mathrm{g}(-1)$ **d)** the value of x if $\mathrm{g}(x) = 9$

22. The function f is defined as $\mathrm{f}(x) = ax + b$ where a and b are constants.
If $\mathrm{f}(1) = 8$ and $\mathrm{f}(4) = 17$, find the values of a and b.

23. The function g is defined as $g(x) = ax^2 + b$ where a and b are constants.
If $\mathrm{g}(2) = 3$ and $\mathrm{g}(-3) = 13$, find the values of a and b.

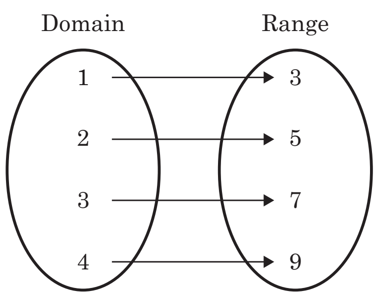
24. Functions h and k are defined as follows:
$\mathrm{h}(x) = x^2 + 1$, $k(x) = ax + b$, where a and b are constants.
If $\mathrm{h}(0) = \mathrm{k}(0)$, and $\mathrm{k}(2) = 15$, find the values of a and b.

Domain and range

The numbers that are allowed to go into a function are called its **domain**. The numbers that come out of the function are called its **range**.

A function's domain and range can be illustrated using a **mapping diagram**.

Here is a mapping diagram for the function $\mathrm{f}(x) = 2x + 1$.



In the diagram on the previous page, the domain has been defined to be the set of integers {1, 2, 3, 4}.
The range is therefore the set {3, 5, 7, 9}.

Note that it is not the function itself that has restricted the domain to just these four numbers. Normally, there would be nothing to stop you evaluating f(5), for example. In this example, restricting the domain to just the numbers 1, 2, 3 and 4 is a choice that has been made.

Example 1

Use the mapping diagram to find the values of a, b and c for the function $f(x) = 3x - 2$.

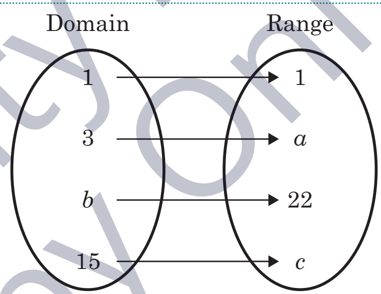
$a = 3 \times 3 - 2 = 7$

$3b - 2 = 22$

$3b = 24$

$b = 8$

$c = 3 \times 15 - 2 = 43$



So far, you have looked at mapping diagrams for functions that are one-to-one. This means that every input value in the domain has a unique output value in the range.

However, a function does not have to be one-to-one. It can also be many-to-one, which means that more than one different input value in the domain can have the same output value in the range.

Example 2

Draw a mapping diagram for the function $f(x) = x^2$ with domain {–2, –1, 0, 1, 2, 3}.

$(-2)^2 = 4$

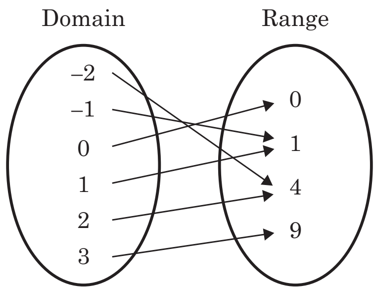
$(-1)^2 = 1$

$0^2 = 0$

$1^2 = 1$

$2^2 = 4$

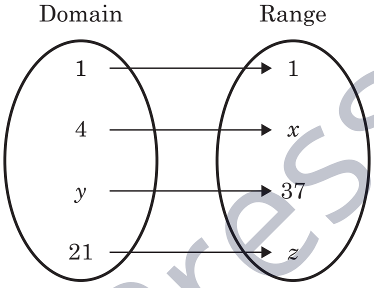
$3^2 = 9$



Tip

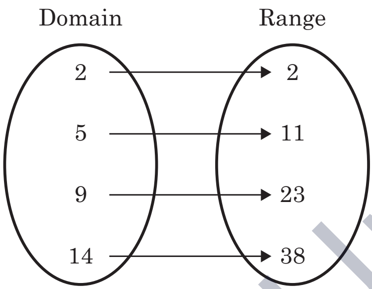
Functions are not allowed to be one-to-many (which would imply that one value in the domain would give more than one value in the range).

Exercise 11.3B

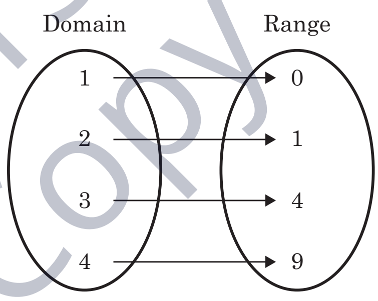


1. Copy and complete the mapping diagram shown for the function $f(x) = 5x + 2$.
2. Draw a mapping diagram for:
   a) the function $f(x) = 6x - 5$, with domain {2, 4, 6, 8}
   b) the function $f(x) = 2x + 12$, with domain {1, 3, 5, 7}.
3. Draw a mapping diagram for:
   a) the function $f(x) = x^2 + 1$, with domain {1, 2, 3, 4}
   b) the function $f(x) = x^2 - 4$, with domain {–3, –1, 1, 4}.
4. Draw a mapping diagram for:
   a) the function $f(x) = 4x + 10$, with range {14, 18, 22, 26}
   b) the function $f(x) = 3x - 9$, with range {–9, 3, 15, 21}.
5. Identify the functions represented by the following mapping diagrams.

a)



b)



6. If the domain of the function $y = 2x^2 + 7$ is {1, 2, 5, 11}, write down the range of the function.
7. If the range of the function $y = 3x^2 - 2$ is { 1, 10, 73, 241}, write down the domain of the function, if all the numbers in the domain are positive.
8. Can you think of a function, other than $y = x$, whose domain and range are the same sets?

11.4 Composite and inverse functions

Composite functions

If $f(x) = 3x + 2$ and $g(x) = x^2$ then $f[g(x)]$ is a composite function where g is performed first and then f is performed on the result of g. $f[g(x)]$ is usually abbreviated to $fg(x)$.

The function fg may be found using a flow diagram:

$$x \longrightarrow \underbrace{\boxed{\text{square}}}_{\text{g}} \xrightarrow{x^2} \underbrace{\boxed{\text{multiply by 3}} \xrightarrow{3x^2} \boxed{\text{add 2}}}_{\text{f}} \longrightarrow 3x^2 + 2$$

$\text{fg}(x) = 3x^2 + 2$

Sometimes, once you have found the composite function, it may need simplifying. Look at the following example.

Example 1

If $\text{f}(x) = 3x + 1$ and $\text{g}(x) = 2x^2 - 3$, find:

a) $\text{fg}(x)$

b) $\text{gf}(x)$

a) $3(2x^2 - 3) + 1$
$= 6x^2 - 9 + 1$
$= 6x^2 - 8$

b) $2(3x + 1)^2 - 3$
$= 2(9x^2 + 6x + 1) - 3$
$= 18x^2 + 12x + 2 - 3$
$= 18x^2 + 12x - 1$

To evaluate a composite function, you do not necessarily need to find the composite function first.

Example 2

If $\text{f}(x) = 2x^3 + 4x^2 + 7x + 1$ and $\text{g}(x) = 7x^2 - 25$, find $\text{fg}(2)$.

Evaluate the function g when $x = 2$, then use the output as the input to function f:

$\text{g}(2) = 7 \times 2^2 - 25 = 3$

$\text{f}(3) = 2 \times 3^3 + 4 \times 3^2 + 7 \times 3 + 1$
$= 2 \times 27 + 4 \times 9 + 21 + 1$
$= 54 + 36 + 21 + 1$
$= 112$

Inverse functions

If a function f maps a number n onto m, then the inverse function f^{-1} maps m onto n.
The inverse of a given function can be found in two main ways.

Example

Find the inverse of the function f, where $\text{f}(x) = \dfrac{5x - 2}{3}$

Method 1

a) Draw a flow diagram for f.

$$x \longrightarrow \boxed{\text{multiply by 5}} \xrightarrow{5x} \boxed{\text{subtract 2}} \xrightarrow{5x - 2} \boxed{\text{divide by 3}} \longrightarrow \frac{5x - 2}{3}$$

b) Draw a new flow diagram with each operation replaced by its inverse. Start with x on the right.

$$\frac{3x+2}{5} \longleftarrow \boxed{\text{divide by 5}} \longleftarrow 3x+2 \longleftarrow \boxed{\text{add 2}} \longleftarrow 3x \longleftarrow \boxed{\text{multiply by 3}} \longleftarrow x$$

So, the inverse of the function f is given by $f^{-1}(x) = \dfrac{3x+2}{5}$

Method 2

Let $y = \dfrac{5x-2}{3}$, then interchange x and y.

$$x = \frac{5y-2}{3}$$

Now rearrange this equation to make y the subject again:

$$3x = 5y - 2$$
$$3x + 2 = 5y$$
$$y = \frac{3x+2}{5}$$

So, the inverse function is $\dfrac{3x+2}{5}$

Tip

Many people prefer this algebraic method. You should use the method which you find easier.

Tip

Because a function cannot be one-to-many, a many-to-one function does not have an inverse.

Exercise 11.4A

For Questions **1** and **2**, the functions f, g and h are as follows:

$f(x) = 4x$ $\quad g(x) = x + 5$ $\quad h(x) = x^2$

1. Find the following:

a) fg **b)** gf **c)** hf **d)** fh **e)** gh

2. Find:

a) x if $hg(x) = h(x)$ **b)** x if $fh(x) = gh(x)$

For Questions **3** to **8**, find the inverse of each function.

3. $f(x) = 5x - 2$

4. $f(x) = 5(x - 2)$

5. $f(x) = 3(2x + 4)$

6. $g(x) = \dfrac{2x+1}{3}$

7. $f(x) = \dfrac{3(x-1)}{4}$

8. $g(x) = 2(3x + 4) - 6$

For Questions **9**, **10** and **11**, the functions p, q and r are as follows:

p(x) = $2x + 1$ q(x) = $3x - 1$ r(x) = x^2

9. Find:

a) pq **b)** qp **c)** pr **d)** rq

10. Find:

a) pq(2) **b)** rp(1) **c)** qp(−2) **d)** qq(2)

11. If p(x) = $2x + 1$, q(x) = $3x - 1$ and r(x) = x^2, find:

a) x if p(x) = q(x) **b)** two values of x if rp(x) = rq(x) **c)** x if pr(x) = qr(x)

12. If f(x) = $2x^2$, g(x) = $x^2 + 5x - 3$, h(x) = $\sqrt[3]{x}$, evaluate

a) fg(2) **b)** gf(–1) **c)** gh(27)

For Questions **13** to **19**, find the inverse of each function.

13. $h(x) = \frac{1}{2}\left(4 + 5x\right) + 10$ **14.** $k(x) = -7x + 3$ **15.** $j(x) = \frac{12 - 5x}{3}$

16. $n(x) = \frac{4 - x}{3} + 2$ **17.** $m(x) = \frac{\left(\frac{2x - 1}{4} - 3\right)}{5}$ **18.** $f(x) = \frac{3(10 - 2x)}{7}$

19. $g(x) = \left(\frac{\frac{x}{4} + 6}{5}\right) + 7$

20. Explain why the function $y = 3x^2 + 4$ does not have an inverse.

Revision exercise 11

1. Given that $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5\}$ and $B = \{5, 6, 7\}$, list the members of the sets:

a) $A \cap B$ **b)** $A \cup B$

c) A' **d)** $A' \cap B'$

e) $A \cup B'$

2. The sets P and Q are such that $n(P \cup Q) = 50$, $n(P \cap Q) = 9$ and $n(P) = 27$. Find the value of $n(Q)$.

3. Draw three diagrams similar to Figure 1, and shade the following:

a) $Q \cap R'$ **b)** $(P \cup Q) \cap R$

c) $(P \cap Q) \cap R'$

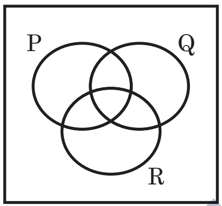


Figure 1

4. Describe the shaded regions in Figures 2 and 3 using set notation.

a)

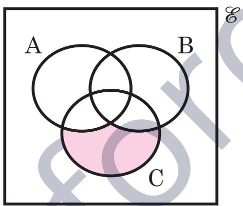


Figure 2

b)

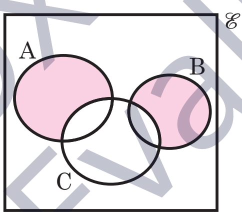


Figure 3

5. Given are: $\mathscr{E}$ = {people on a train}, M = {males}, T = {people over 25 years old} and S = {Spanish speakers}

a) Express in set notation:

i) all the Spanish speakers are over 25

ii) some Spanish speakers are women.

b) Express in words: $T \cap M' = \varnothing$

6. The figures in the diagram indicate the number of elements in each subset of E.

Find:

a) $n(P \cap R)$

b) $n(Q \cup R)'$

c) $n(P' \cap Q')$

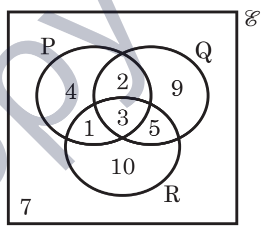


Figure 4

7. Given $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find:

a) $f(-1)$ **b)** $g(-1)$

c) $fg(-1)$ **d)** $gf(3)$

8. Given $f(x) = 3x + 4$, find the inverse function f^{-1}, and then find:

a) $f^{-1}(13)$ **b)** the value of z if $f(z) = 20$

9. Find the inverse of the function

$$f(x) = \frac{x+1}{x-1}$$

What do you notice?

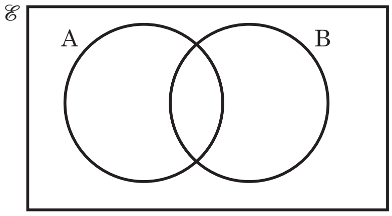
10. If $f(x) = 4x - 1$, find:

a) $f^{-1}(x)$ **b)** $ff^{-1}(10)$

Examination-style exercise 11

NON-CALCULATOR SECTION

For Question **1**, make three copies of this Venn diagram in your notebook, one for each question part:



1. a) Shade the region $A \cup B$. [1]

b) Shade the region $(A \cap B)'$. [1]

c) Shade the complement of set A. [1]

2. $\mathscr{E} = \{1, 3, 5, 7, 9, 10, 12, 14, 16\}$

$A = \{1, 5, 9, 12, 14\}$

$B = \{1, 3, 9, 16\}$

$C = \{1, 5, 9, 16\}$

a) Draw a Venn diagram to show this information. [2]

b) Write down the value of $n(B' \cap C)$. [1]

3. A and B are sets.

Write the following sets in their simplest form.

a) $A \cup A'$ [1]

b) $A \cap A'$ [1]

c) $(A \cup B) \cap (A \cup B)'$ [1]

CALCULATOR SECTION

4. $f(x) = x^3 - 2x^2 + 5x - 1$ and $g(x) = 3x + 1$

Work out:

a) $f(-2)$ [1]

b) $gf(x)$ [2]

c) $g^{-1}(x)$ [2]

5. $f(x) = 7 - 2x$

Work out:

a) $f(-2)$ [1]

b) $f^{-1}(x)$ [2]

c) $ff^{-1}(4)$ [1]

6. $f(x) = \sin x$, $g(x) = 3x + 6$

Work out:

a) $f(30)$ [1]

b) $fg(58)$ [2]

c) $g^{-1}(f(x))$ [2]

7. $f(x) = 4x - 1$, $g(x) = \frac{2}{x} + 1$, $h(x) = 3^x$

a) Work out the value of $fg(3)$ [1]

b) Write, as a single fraction, $gf(x)$ in terms of x. [3]

c) Work out $g^{-1}(x)$ [3]

d) Work out $hh(2)$ [2]

e) Calculate x when $h(x) = g\left(-\frac{9}{4}\right)$ [2]