6 Algebra 2

Srinivasa Ramanujan (1887–1920) was a self-taught, Indian mathematician who made a huge contribution to the subject in the early 20th century. In 1914, he travelled to England to work with G. H. Hardy at Trinity College in Cambridge, but he was plagued by ill-health and returned to India five years later. During his time at Cambridge, he made contributions to many areas of mathematics, including analysis and number theory, and solved many problems that were at that time considered unsolvable. He was made a Fellow of the Royal Society in 1918, two years before his death at the age of 32.

Ingrid Daubechies (1954–) is a Belgian physicist and mathematician, best known for her work on image compression technology. Her work has also enabled scientists to extract information from samples of bones and teeth. The image processing methods she has helped to develop can be used to establish the age and authenticity of works of art and have been used on paintings by artists such as Vincent van Gogh and Rembrandt.

- Factorising and completing the square.
- Solve simultaneous equations, where one is linear and one is nonlinear.
- Solve quadratic equations including by factorising, completing the square and using the quadratic formula.

6.1 Factorising

In Chapter 2 you expanded expressions such as $x(3x - 1)$ to give $3x^2 - x$.

The reverse of this process is called **factorising**. When you factorise an expression, you look for the *highest* common factor of all terms.

Example

Factorise: **a)** $4x + 4y$ **b)** $x^2 + 7x$ **c)** $3y^2 - 12y$ **d)** $6a^2b - 10ab^2$ **e)** $12ax^2 + 4ax + 8a^2x$

a) 4 is common to $4x$ and $4y$

$4x + 4y = 4(x + y)$

b) x is common to x^2 and $7x$

$x^2 + 7x = x(x + 7)$

c) $3y$ is common to both terms

$3y^2 - 12y = 3y(y - 4)$

d) $2ab$ is common to both terms

$6a^2b - 10ab^2 = 2ab(3a - 5b)$

e) $4ax$ is common to all three terms

$12ax^2 + 4ax + 8a^2x = 4ax(3x + 1 + 2a)$

When a question asks you to factorise, you must factorise fully. This means all common factors will be written outside of the brackets. In part **(c)** of the example above, if you had just written $3(y^2 - 4y)$ this would not be correct as it is only partially factorised.

Exercise 6.1A

Factorise:

1. $5a + 5b$ **2.** $7x + 7y$ **3.** $7x + x^2$ **4.** $y^2 + 8y$

5. $2y^2 + 3y$ **6.** $6y^2 - 4y$ **7.** $3x^2 - 21x$ **8.** $16a - 2a^2$

9. $6c^2 - 21c$ **10.** $15x - 9x^2$ **11.** $56y - 21y^2$ **12.** $ax + bx + 2cx$

13. $x^2 + xy + 3xz$ **14.** $x^2y + y^3 + z^2y$ **15.** $3a^2b + 2ab^2$ **16.** $x^2y + xy^2$

17. $6a^2 + 4ab + 2ac$ **18.** $ma + 2bm + m^2$ **19.** $2kx + 6ky + 4kz$ **20.** $ax^2 + ay + 2ab$

21. $x^2k + xk^2$ **22.** $a^3b + 2ab^2$ **23.** $abc - 3b^2c$ **24.** $2a^2e - 5ae^2$

25. $a^3b + ab^3$ **26.** $x^3y + x^2y^2$ **27.** $6xy^2 - 4x^2y$ **28.** $3ab^3 - 3a^3b$

29. $2a^3b + 5a^2b^2$ **30.** $ax^2y - 2ax^2z$ **31.** $2abx + 2ab^2 + 2a^2b$ **32.** $ayx + yx^3 - 2y^2x^2$

Example 1

Factorise $ah + ak + bh + bk$.

Divide into pairs: $(ah + ak) + (bh + bk)$

a is common to the first pair.

b is common to the second pair.

$a(h + k) + b(h + k)$

$(h + k)$ is common to both terms.

Therefore, the answer is $(h + k)(a + b)$

Example 2

Factorise $6mx - ny - 3nx + 2my$.

You may need to rearrange the expression first. In this case, rearrange to $6mx - 3nx + 2my - ny$ so that there are terms next to each other with common factors.

$$\begin{aligned}(6mx - 3nx) + (2my - ny) &= 3x(2m - n) + y(2m - n)\\ &= (2m - n)(3x + y)\end{aligned}$$

Exercise 6.1B

Factorise:

1. $ax + ay + bx + by$
2. $xm + xn + m + n$
3. $ah - ak + bh - bk$
4. $am - bn - bm + an$
5. $hs + ht + ks + kt$
6. $ax + by - ay - bx$
7. $xs - xt - ys + yt$
8. $hx - hy - bx + by$
9. $am - bm - an + bn$
10. $xk - xm - kz + mz$
11. $2ax + 6ay + bx + 3by$
12. $2ax + 2ay + bx + by$
13. $2mh - 2mk - h + k$
14. $2mh + 3mk - 2nh - 3nk$
15. $6ax + 2bx + 3ay + by$
16. $2ax - 2ay - bx + by$
17. $x^2a + x^2b + ya + yb$
18. $ms + 2mt^2 - ns - 2nt^2$

Quadratic expressions

When factorising a quadratic expression, first look at the constant term and identify pairs of numbers that multiply to make that term.

For example, if you were trying to factorise $x^2 + 6x + 8$, you would be looking for pairs of numbers that multiply to give +8.

These would be: 1 and 8, 2 and 4, −1 and −8, −2 and −4.

Then find out which of these pairs add together to make the coefficient of x, which in this case is 6.

In this case, the pair is 2 and 4.

Those are the two numbers that you put into your brackets, therefore:

$x^2 + 6x + 8 = (x + 2)(x + 4)$

Example

Factorise: **a)** $x^2 + 2x - 15$
b) $x^2 - 6x + 8$
c) $x^2 - 25$

a) Two numbers that multiply to give −15 and add to give +2 are −3 and 5.
$x^2 + 2x - 15 = (x - 3)(x + 5)$

b) Two numbers that multiply to give +8 and add to give −6 are −2 and −4.
$x^2 - 6x + 8 = (x - 2)(x - 4)$

c) Think of this as $x^2 + 0x - 25$
Two numbers that multiply to give −25 and add to give 0 are −5 and 5.
$x^2 - 25 = (x - 5)(x + 5)$

Exercise 6.1C

Factorise:

1. $x^2 + 7x + 10$
2. $x^2 + 7x + 12$
3. $x^2 + 8x + 15$
4. $x^2 + 10x + 21$
5. $x^2 + 8x + 12$
6. $y^2 + 12y + 35$
7. $y^2 + 11y + 24$
8. $y^2 + 10y + 25$
9. $y^2 + 15y + 36$
10. $a^2 - 3a - 10$
11. $a^2 - a - 12$
12. $z^2 + z - 6$
13. $x^2 - 2x - 35$
14. $x^2 - 5x - 24$
15. $x^2 - 6x + 8$
16. $y^2 - 5y + 6$
17. $x^2 - 8x + 15$
18. $a^2 - a - 6$
19. $a^2 + 14a + 45$
20. $b^2 - 4b - 21$
21. $x^2 - 8x + 16$
22. $y^2 + 2y + 1$
23. $y^2 - 3y - 28$
24. $x^2 - x - 20$
25. $x^2 - 8x - 240$
26. $x^2 - 26x + 165$
27. $y^2 + 3y - 108$
28. $x^2 - 49$
29. $x^2 - 9$
30. $x^2 - 16$

If the coefficient of x^2 is not 1, there are two main methods you can use to factorise a quadratic. The first is by inspection.

Example 1

Factorise $2x^2 + 7x + 3$.

Since 2 is a prime number, you know that the brackets must start with $2x$ and x.

This means that $2x^2 + 7x + 3 = (2x + a)(x + b)$, for some a and b.

Since the constant term is 3, you know that a and b must be either 1 and 3 (in some order) or -1 and -3 (in some order).

If you expand $(2x + a)(x + b)$ you get
$2x^2 + 2bx + ax + ab = 2x^2 + (a + 2b)x + ab$.

This means that $a + 2b = 7$

The pair of values for which this is true is $a = 1$ and $b = 3$.

The factorisation of $2x^2 + 7x + 3$ is therefore $(2x + 1)(x + 3)$.

Sometimes it may be too difficult to factorise a quadratic expression by inspection. This could be because the coefficient of x^2 or the constant term (or both) has a lot of factors, which means there are too many possible options to check. In these instances, there is a second method that can be used.

If the quadratic expression to be factorised is of the form $ax^2 + bx + c$, this method starts by looking for two numbers whose product is ac and whose sum is b.

Example 2

Factorise $3x^2 + 13x + 4$.

Here, $a = 3$, $b = 13$ and $c = 4$. Thus $ac = 3 \times 4 = 12$

Find two numbers that multiply to give 12 (ac) and add to 13 (b).

In this case the numbers you want are 1 and 12.

First, you split the '$13x$' term into x and $12x$:
$3x^2 + x + 12x + 4$

Then you factorise the first and second terms as a pair followed by the third and fourth as a pair:
$x(3x + 1) + 4(3x + 1)$

Note that at this stage the brackets in both factorisations will always be the same.

Finally, factorise one more time: $(x + 4)(3x + 1)$

Exercise 6.1D

Factorise:

1. $2x^2 + 5x + 3$
2. $2x^2 + 7x + 3$
3. $3x^2 + 7x + 2$
4. $2x^2 + 11x + 12$
5. $3x^2 + 8x + 4$
6. $2x^2 + 7x + 5$
7. $3x^2 - 5x - 2$
8. $2x^2 - x - 15$
9. $2x^2 + x - 21$
10. $3x^2 - 17x - 28$
11. $6x^2 + 7x + 2$
12. $12x^2 + 23x + 10$
13. $3x^2 - 11x + 6$
14. $3y^2 - 11y + 10$
15. $4y^2 - 23y + 15$
16. $6y^2 + 7y - 3$
17. $6x^2 - 27x + 30$
18. $10x^2 + 9x + 2$
19. $6x^2 - 19x + 3$
20. $8x^2 - 10x - 3$
21. $12x^2 + 4x - 5$
22. $16x^2 + 19x + 3$
23. $4a^2 - 4a + 1$
24. $12x^2 + 17x - 14$
25. $15x^2 + 44x - 3$
26. $48x^2 + 46x + 5$
27. $64y^2 + 4y - 3$
28. $120x^2 + 67x - 5$
29. $9x^2 - 1$
30. $4a^2 - 9$

The difference of two squares

When an expression can be seen as the difference of two perfect squares, for example $x^2 - y^2$, then it can be factorised as $(x - y)(x + y)$.

$$x^2 - y^2 = (x - y)(x + y)$$

Remember this result.

Example

Factorise: **a)** $4a^2 - b^2$ **b)** $25m^2 - 81n^2$ **c)** $3x^2 - 27y^2$

a) $4a^2 - b^2 = (2a)^2 - b^2$
$= (2a - b)(2a + b)$

b) $25m^2 - 81n^2 = (5m)^2 - (9n)^2$
$= (5m - 9n)(5m + 9n)$

c) $3x^2 - 27y^2 = 3(x^2 - 9y^2)$
$= 3(x^2 - (3y)^2)$
$= 3(x - 3y)(x + 3y)$

Exercise 6.1E

Factorise the following:

1. $y^2 - a^2$
2. $m^2 - n^2$
3. $x^2 - t^2$
4. $y^2 - 1$
5. $x^2 - 9$
6. $a^2 - 25$
7. $x^2 - \frac{1}{4}$
8. $x^2 - \frac{1}{9}$
9. $4x^2 - y^2$
10. $a^2 - 4b^2$
11. $25x^2 - 4y^2$
12. $9x^2 - 16y^2$
13. $x^2 - \frac{y^2}{4}$
14. $9m^2 - \frac{4}{9}n^2$
15. $16t^2 - \frac{4}{25}s^2$
16. $4x^2 - \frac{z^2}{100}$
17. $x^3 - x$
18. $a^3 - ab^2$
19. $4x^3 - x$
20. $8x^3 - 2xy^2$
21. $12x^3 - 3xy^2$
22. $18m^3 - 8mn^2$
23. $5x^2 - 1\frac{1}{4}$
24. $50a^3 - 18ab^2$
25. $12x^2y - 3yz^2$
26. $36a^3b - 4ab^3$
27. $50a^5 - 8a^3b^2$
28. $36x^3y - 225xy^3$

Evaluate:

29. $81^2 - 80^2$
30. $102^2 - 100^2$
31. $225^2 - 215^2$
32. $1211^2 - 1210^2$
33. $723^2 - 720^2$
34. $3.8^2 - 3.7^2$
35. $5.24^2 - 4.76^2$
36. $1234^2 - 1235^2$
37. $3.81^2 - 3.8^2$
38. $540^2 - 550^2$
39. $7.68^2 - 2.32^2$
40. $0.003^2 - 0.002^2$

Cubic expressions

The techniques you have learned so far mean that you can now also factorise many simple cubic expressions of the form $ax^3 + bx^2 + cx$. Because there is no constant term and every term contains x, you can simply take x, or a multiple of x, out as a factor, leaving you then with only a quadratic to factorise.

Example

Factorise: **a)** $x^3 + 7x^2 + 10x$ **b)** $2x^3 + 10x^2 - 12x$

a) $x^3 + 7x^2 + 10x = x(x^2 + 7x + 10) = x(x + 2)(x + 5)$

b) $2x^3 + 10x^2 - 12x = 2x(x^2 + 5x - 6) = 2x(x - 1)(x + 6)$

Exercise 6.1F

Factorise the following:

1. $x^3 + 3x^2 + 2x$
2. $x^3 - x^2 - 6x$
3. $x^3 + 3x^2 - 4x$
4. $x^3 + 8x^2 + 15x$
5. $x^3 + 4x^2 - 60x$
6. $x^3 + 3x^2 - 28x$
7. $x^3 - 8x^2 + 15x$
8. $x^3 - 15x^2 + 44x$
9. $2x^3 - 3x^2 - 2x$

10. $3x^3 + 5x^2 - 2x$ **11.** $2x^3 + x^2 - 3x$ **12.** $4x^3 + x^2 - 3x$

13. $5x^3 + 17x^2 - 12x$ **14.** $6x^3 + x^2 - x$ **15.** $8x^3 - 2x^2 - x$

16. $15x^3 + 13x^2 + 2x$ **17.** $12x^3 + 40x^2 - 7x$ **18.** $6x^3 + x^2 - 40x$

19. $2x^3 - 2x$ **20.** $3x^3 - 3x^2 - 6x$ **21.** $15x^3 + 50x^2 - 40x$

22. $12x^3 - 20x^2 - 48x$ **23.** $-2x^3 - 32x^2 - 126x$ **24.** $-30x^3 + 5x^2 + 75x$

6.2 Quadratic equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$. Unlike linear equations, which only have one solution, quadratic equations can have zero, one, or two solutions.

For an equation $ax^2 + bx + c = 0$ to be quadratic, a cannot be zero, but b and c are allowed to be zero. There must be an x^2 term, but there cannot be any terms containing a higher power of x than 2.

Example

Which of the following equations are quadratic? Explain your reasoning.

a) $x^2 - 7x + 12 = 0$ **b)** $3x^2 + 14 = 0$ **c)** $-8x^2 = 0$

d) $-13x + 4 = 0$ **e)** $x^3 - x^2 + 4x - 9 = 0$

a) Yes. It is of the form $ax^2 + bx + c = 0$, and a is not zero.

b) Yes. It is of the form $ax^2 + bx + c = 0$. The value of b is zero, but this is allowed.

c) Yes. It is of the form $ax^2 + bx + c = 0$. The values of b and c are zero, but this is allowed.

d) No. It has no x^2 term.

e) No. It has a term with a power of x greater than 2.

Solution by factorising

If the product of two numbers is zero, then one of those two numbers must be zero.

You can use this fact to help you solve some quadratic equations.

Example 1

Solve the equation $x^2 + x - 12 = 0$.

Factorising, this becomes $(x - 3)(x + 4) = 0$

The product of the two brackets can only be zero if one of the brackets equals zero.

This could be because $x - 3 = 0$, which is the case if $x = 3$.

Alternatively, it could be because $x + 4 = 0$, which would be the case if $x = -4$.

The solutions to this equation are therefore $x = 3$ or $x = -4$.

Example 2

Solve the equation $6x^2 + x - 2 = 0$.

Factorising, this becomes $(2x - 1)(3x + 2) = 0$

Either $2x - 1 = 0$, which is the case if $x = \frac{1}{2}$

Or $3x + 2 = 0$, which is the case if $x = -\frac{2}{3}$

The solutions to this equation are therefore $x = \frac{1}{2}$ or $x = -\frac{2}{3}$

Example 3

Solve the equation $3x^2 - 2x = 14x - 5$.

First, rearrange to get all the terms on one side of the equals sign.

$3x^2 - 16x + 5 = 0$

Then solve in the same way as Example 2.

Factorising gives $(3x - 1)(x - 5) = 0$.

Therefore $x = \frac{1}{3}$ or $x = 5$.

Exercise 6.2A

Solve the following equations:

1. $x^2 + 7x + 12 = 0$
2. $x^2 + 7x + 10 = 0$
3. $x^2 + 2x - 15 = 0$
4. $x^2 + x - 6 = 0$
5. $x^2 - 8x + 12 = 0$
6. $x^2 + 10x + 21 = 0$
7. $x^2 - 5x + 6 = 0$
8. $x^2 = 4x + 5$
9. $x^2 + 5x - 14 = 0$
10. $2 = 2x^2 - 3x$
11. $3x^2 + 10x - 8 = 0$
12. $2x^2 + 7x - 15 = 0$
13. $6x^2 - 13x + 6 = 0$
14. $4x^2 - 29x + 7 = 0$
15. $10x^2 - 2 = 1 + x$
16. $y^2 - 15y + 56 = 0$
17. $12y^2 - 16y + 5 = 0$
18. $y^2 + 2y - 63 = 0$
19. $x^2 = -1 - 2x$
20. $x^2 - 3x = 3x - 9$
21. $x^2 + 10x + 25 = 0$
22. $x^2 - 7x + 5 = 7x - 44$
23. $6a^2 - a - 1 = 0$
24. $6a^2 - 3a - 11 = 2a^2 - 1$
25. $z^2 - 8z - 65 = 0$
26. $x^2 + 17x + 6 = 9 - 5x^2$
27. $10k^2 + 19k - 2 = 0$
28. $y^2 - 2y + 1 = 0$
29. $36x^2 + x - 2 = 0$
30. $14x^2 + x - 3 = 8x - 6x^2$

Example 1

Solve the equation $x^2 - 7x = 0$.

Factorising, $x(x - 7) = 0$

Either $x = 0$ or $x - 7 = 0$

$x = 7$

The solutions are $x = 0$ and $x = 7$.

Example 2

Solve the equation $4x^2 - 9 = 0$.

a) Factorising, $(2x - 3)(2x + 3) = 0$

Either $2x - 3 = 0$ or $2x + 3 = 0$

$2x = 3$ $\quad$ $2x = -3$

$x = \frac{3}{2}$ $\quad$ $x = -\frac{3}{2}$

b) Alternative method:

$4x^2 - 9 = 0$

$4x^2 = 9$

$x^2 = \frac{9}{4}$

$x = +\frac{3}{2}$ or $-\frac{3}{2}$

Tip

You must give both the solutions. A common error is to only give the positive square root.

Exercise 6.2B

Solve the following equations:

1. $x^2 - 3x = 0$ **2.** $x^2 + 7x = 0$ **3.** $2x^2 - 2x = 0$

4. $3x^2 - x = 0$ **5.** $x^2 - 16 = 0$ **6.** $x^2 - 49 = 0$

7. $4x^2 - 1 = 0$ **8.** $9x^2 - 4 = 0$ **9.** $6y^2 + 9y = 0$

10. $6a^2 - 9a = 0$ **11.** $10x^2 - 55x = 0$ **12.** $16x^2 - 1 = 0$

13. $y^2 - \frac{1}{4} = 0$ **14.** $56x^2 - 35x = 0$ **15.** $36x^2 - 3x = 0$

16. $x^2 = 6x$ **17.** $x^2 = 11x$ **18.** $2x^2 = 3x$

19. $x^2 = x$ **20.** $4x = x^2$ **21.** $3x - x^2 = 0$

22. $4x^2 = 1$ **23.** $9x^2 = 16$ **24.** $x^2 = 12 - x$

25. $12x = 5x^2$ **26.** $1 - 9x^2 = 0$ **27.** $x^2 = \frac{x}{4}$

28. $2x^2 = \frac{x}{3}$ **29.** $4x^2 = \frac{1}{4}$ **30.** $\frac{x}{5} - x^2 = 0$

Solution by the quadratic formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula, one of the most well-known formulae in mathematics:

$$x = \frac{-b \pm \sqrt{\left(b^2 - 4ac\right)}}{2a}$$

You need to learn this formula, but you should only use it if you are unable to solve the equation by factorisation. Note that if you use the formula and discover that the number under the square root is a square number, then you could have solved the equation by factorising. The factorisation, however, may not have been easy to spot.

Example

Solve the equation $2x^2 - 3x - 4 = 0$, giving your answers:

a) in surd form

b) accurate to 2 decimal places.

a) Comparing with the general form $ax^2 + bx + c = 0$, you have $a = 2$, $b = -3$, $c = -4$

Using the quadratic formula gives

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - (4 \times 2 \times -4)}}{2 \times 2} = \frac{3 \pm \sqrt{9 + 32}}{4} = \frac{3 \pm \sqrt{41}}{4}$$

The solutions in surd form are $x = \frac{3 + \sqrt{41}}{4}$ or $x = \frac{3 - \sqrt{41}}{4}$

b) Accurate to 2 decimal places:

$$x = \frac{3 + \sqrt{41}}{4} = 2.35 \quad \text{or} \quad x = \frac{3 - \sqrt{41}}{4} = -0.85$$

Exercise 6.2C

For Questions **1** to **27**, solve the equations and give your answers in surd form where necessary.

1. $2x^2 + 11x + 5 = 0$
2. $3x^2 + 11x + 6 = 0$
3. $6x^2 + 7x + 2 = 0$
4. $3x^2 - 10x + 3 = 0$
5. $5x^2 - 7x + 2 = 0$
6. $6x^2 - 11x + 3 = 0$
7. $2x^2 + 6x + 3 = 0$
8. $x^2 + 4x + 1 = 0$
9. $5x^2 - 5x + 1 = 0$
10. $x^2 - 7x + 2 = 0$
11. $2x^2 + 5x - 1 = 0$
12. $3x^2 + x - 3 = 0$
13. $3x^2 + 8x - 6 = 0$
14. $3x^2 - 7x - 20 = 0$
15. $2x^2 - 7x - 15 = 0$
16. $x^2 - 3x - 2 = 0$
17. $6x^2 - 11x - 7 = 0$
18. $3x^2 + 25x + 8 = 0$
19. $3y^2 - 2y - 5 = 0$
20. $2 - x - 6x^2 = 0$
21. $20x^2 + 17x - 63 = 0$
22. $x^2 + 2.5x - 6 = 0$
23. $0.3y^2 + 0.4y - 1.5 = 0$
24. $10 - x - 3x^2 = 0$
25. $x^2 + 3.3x - 0.7 = 0$
26. $12 - 5x^2 - 11x = 0$
27. $5x - 2x^2 + 187 = 0$

For Questions **28** to **36**, solve the equations and give your answers correct to two decimal places.

28. $2x^2 + 6x - 1 = 0$
29. $2y^2 - 5y + 1 = 0$
30. $\frac{1}{2}y^2 + 3y + 1 = 0$
31. $3 + 4x - 2x^2 = 0$
32. $1 - 5x - 2x^2 = 0$
33. $3x^2 - 1 + 4x = 0$
34. $5x - x^2 + 2 = 0$
35. $24x^2 - 22x - 35 = 0$
36. $36x^2 - 17x - 35 = 0$

The solution to a problem may involve an equation which does not at first appear to be quadratic. The terms in the equation may need to be rearranged as shown in the next example.

Example

Solve: $2x(x - 1) = (x + 1)^2 - 5$

$2x^2 - 2x = x^2 + 2x + 1 - 5$

$2x^2 - 2x - x^2 - 2x - 1 + 5 = 0$

$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$x = 2$

In this example the quadratic has a repeated solution of $x = 2$.

Exercise 6.2D

Solve the following, giving answers to two decimal places where necessary:

1. $x^2 = 6 - x$
2. $x(x + 10) = -21$
3. $3x + 2 = 2x^2$
4. $x^2 + 4 = 5x$
5. $6x(x + 1) = 5 - x$
6. $(2x)^2 = x(x - 14) - 5$
7. $(x - 3)^2 = 10$
8. $(x + 1)^2 - 10 = 2x(x - 2)$
9. $(2x - 1)^2 = (x - 1)^2 + 8$
10. $3x(x + 2) - x(x - 2) + 6 = 0$
11. $2x^2 = 7x$
12. $16 = \dfrac{1}{x^2}$

Solution by completing the square

Another way to solve a quadratic equation of the form $x^2 + bx + c = 0$ $(a = 1)$ is to rearrange the equation to make x the subject. This, however, appears to be tricky at first because x appears twice in the equation and in one of those instances it is squared.

The key here is to notice that $(x + n)^2 = x^2 + 2nx + n^2$, where n is a number.

Rearranging this gives $x^2 + 2nx = (x + n)^2 - n^2$

This means that if you have an expression of the form $x^2 + 2nx$ you can swap it for an expression of the form $(x + n)^2 - n^2$. This turns out to be very useful.

Suppose you want to solve the equation $x^2 + 6x + 4 = 0$.

Note that $x^2 + 6x$ is an expression of the form $x^2 + 2nx$ where $n = 3$. This means you can swap $x^2 + 6x$ for $(x + 3)^2 - 3^2$. Note that the 3 is just half of the 6.

This turns $x^2 + 6x + 4 = 0$ into $(x + 3)^2 - 3^2 + 4 = 0$

Rearranging this gives us $(x + 3)^2 = 5$

Note that on the left-hand side of the equation you now have a perfect square. For this reason, this method of solving quadratic equations is called ‘completing the square’.

Now you can finish solving the equation:
$$x + 3 = \pm\sqrt{5}$$
$$x = -3 \pm \sqrt{5}$$

Example 1

Solve the quadratic equation $x^2 - 12x = 0$ by completing the square.

$x^2 - 12x = (x - 6)^2 - 36 = 0$

$(x - 6)^2 = 36$

$x - 6 = \pm 6$

$x = 6 + 6 = 12$ or $x = 6 - 6 = 0$

Example 2

Solve the quadratic equation $x^2 - 10x - 17 = 0$ by completing the square, giving your answers in surd form.

$x^2 - 10x - 17 = 0$

$(x - 5)^2 - 5^2 - 17 = 0$

$(x - 5)^2 = 25 + 17 = 42$

$x - 5 = \pm\sqrt{42}$

$x = 5 + \sqrt{42}$ or $x = 5 - \sqrt{42}$

Example 3

Solve the quadratic equation $x^2 + 3x - 11 = 0$ by completing the square, giving your answers correct to 2 decimal places.

$x^2 + 3x - 11 = 0$

$(x - 1.5)^2 - 1.5^2 - 11 = 0$

$(x - 1.5)^2 = 2.25 + 11 = 13.25$

$x - 1.5 = \pm\sqrt{13.25}$

$x = 1.5 + \sqrt{13.25} = 5.14$ (2 d.p.) or

$x = 1.5 - \sqrt{13.25} = -2.14$ (2 d.p.)

If in $ax^2 + bx + c = 0$, $a \neq 1$, the method can be adapted, as shown in Example 4.

Example 4

Solve the quadratic equation $2x^2 - 12x + 7 = 0$ by completing the square, giving your answers correct to 2 decimal places.

$$2x^2 - 12x + 7 = 0$$
$$2(x^2 - 6x) + 7 = 0$$
$$2[(x - 3)^2 - 9] + 7 = 0$$
$$2(x - 3)^2 - 18 + 7 = 0$$
$$2(x - 3)^2 = 18 - 7 = 11$$
$$(x - 3)^2 = 5.5$$
$$x - 3 = \pm\sqrt{5.5}$$
$$x = 3 + \sqrt{5.5} = 5.35 \text{ (2 d.p.) or}$$
$$x = 3 - \sqrt{5.5} = 0.65 \text{ (2 d.p.)}$$

Example 5

Given that $y = x^2 - 8x + 18$, show that $y \geqslant 2$ for all values of x.

Completing the square, $y = (x - 4)^2 - 16 + 18$
$= (x - 4)^2 + 2$

Now $(x - 4)^2$ is always greater than or equal to zero because it is 'something squared'.

Therefore, $y \geqslant 2$

Exercise 6.2E

For Questions **1** to **10**, complete the square for each expression by writing them in the form $(x + a)^2 + b$, where a and b can be positive or negative.

1. $x^2 + 8x$ **2.** $x^2 - 12x$ **3.** $x^2 + x$ **4.** $x^2 + 4x + 1$

5. $x^2 - 6x + 9$ **6.** $x^2 + 2x - 15$ **7.** $2x^2 + 16x + 5$ **8.** $2x^2 - 10x$

9. $6 + 4x - x^2$ **10.** $3 - 2x - x^2$

11. Solve these equations by completing the square.

a) $x^2 + 4x - 3 = 0$ **b)** $x^2 - 3x - 2 = 0$ **c)** $x^2 + 12x = 1$

12. Try to solve the equation $x^2 + 6x + 10 = 0$ by completing the square. Explain why you can find no solutions.

13. Given $y = x^2 + 6x + 12$, show that $y \geqslant 3$ for all values of x.

14. Given $y = x^2 - 7x + \frac{1}{4}$, show that the least possible value of y is -12.

15. If $y = x^2 + 4x + 7$ find:

a) the smallest possible value of y

b) the value of x for which this smallest value occurs

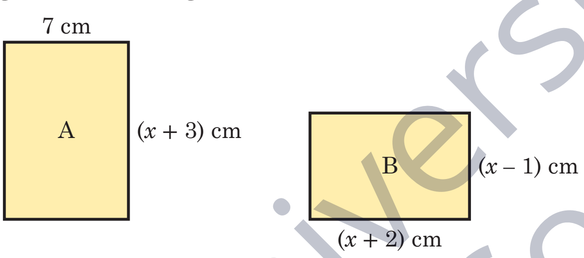
c) the greatest possible value of $\dfrac{1}{(x^2 + 4x + 7)}$

6.3 Solving problems using quadratic equations

Example

The area of rectangle A is 16 cm² greater than the area of rectangle B.

Find the height of rectangle A.



Area of rectangle A = $7(x + 3)$

Area of rectangle B = $(x + 2)(x - 1)$

You are given $(x + 2)(x - 1) + 16 = 7(x + 3)$

Solve this equation $x^2 + 2x - x - 2 + 16 = 7x + 21$

$$x^2 + x + 14 = 7x + 21$$
$$x^2 - 6x - 7 = 0$$
$$(x - 7)(x + 1) = 0$$
$$x = 7 \text{ or } x = -1$$

However, x is a length so must be positive; therefore $x = 7$.

The height of rectangle A, $x + 3$, is therefore $7 + 3 = 10$ cm.

Exercise 6.3A

Solve each problem by forming a quadratic equation.

1. Two positive numbers that differ by 3, have a product of 88. Find these numbers.

2. The product of two positive consecutive odd numbers is 143. Find the numbers.

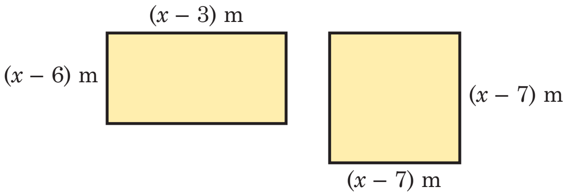
Tip

In Question **2**, let x = the first odd number, and then express the second odd number in terms of x.

3. The length of a rectangle exceeds the width by 7 cm. If the area is 60 cm^2, find the length of the rectangle.

4. The length of a rectangle exceeds the width by 2 cm. If the diagonal is 10 cm long, find the width of the rectangle.

5. The area of the rectangle exceeds the area of the square by 24 m^2. Find the value of x.



6. The perimeter of a rectangle is 68 cm. If the diagonal is 26 cm, find the dimensions of the rectangle.

7. Sang Jae walks a certain distance due north and then the same distance plus a further 7 km due east. If the final distance from the starting point is 17 km, find the distances he walks north and east.

8. A farmer makes a profit of x cents on each of the $(x + 5)$ eggs her hen lays. If her total profit was 84 cents, find the number of eggs the hen lays.

9. Sirak buys x eggs at $(x - 8)$ cents each and $(x - 2)$ bread rolls at $(x - 3)$ cents each. If the total bill is \$1.75, how many eggs does he buy?

10. In Figure 1, ABCD is a rectangle with AB = 12 cm and BC = 7 cm. AK = BL = CM = DN = x cm. If the area of KLMN is 54 cm^2, find the value of x.

11. In Figure 1, AB = 14 cm, BC = 11 cm and AK = BL = CM = DN = x cm. If the area of KLMN is now 97 cm^2, find the possible values of x.

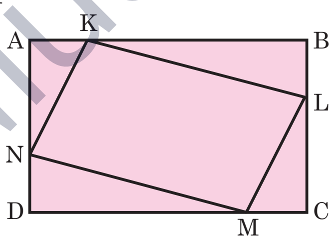


Figure 1

12. The perimeters of a square and a rectangle are equal. One side of the rectangle is 11 cm and the area of the square is 4 cm^2 more than the area of the rectangle. Find the possible lengths of the side of the square.

Tip

Questions **4**, **6** and **7** use Pythagoras' theorem.

6.4 Nonlinear simultaneous equations

Sometimes you may be given a pair of simultaneous equations where one equation is linear and one is nonlinear. When this happens, you will not be able to use the elimination method that you sometimes used in Chapter 2 when both equations were linear. In these instances, you have no choice but to use substitution.

Example 1

Solve the simultaneous equations $y = x + 1$ and $y = x^2 + 3x - 2$.

$y = x + 1$ (1)

$y = x^2 + 3x - 2$ (2)

Substitute (1) into (2):

$x + 1 = x^2 + 3x - 2$

$0 = x^2 + 2x - 3$

Solve the resulting quadratic equation by factorising:

$0 = (x + 3)(x - 1)$

$x = -3$ or $x = 1$

Substitute the x-values into (1).

When $x = -3$, $y = -2$

When $x = 1$, $y = 2$

Example 2

Solve the simultaneous equations $2x - y = 3$ and $y = 2x^2 + 9x - 1$.

$2x - y = 3$ (1)

$y = 2x^2 + 9x - 1$ (2)

Rearrange (1) to make y the subject:

$2x = y + 3$

$2x - 3 = y$

Substitute into (2):

$2x - 3 = 2x^2 + 9x - 1$

$0 = 2x^2 + 7x + 2$

Solve the resulting quadratic equation using the formula:

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(2)}}{2(2)} = \frac{-7 \pm \sqrt{33}}{4}$$

Tip

If you have to solve the quadratic equation by using the formula, use the exact x-values when finding y, then round both answers at the end.

Either $x = \frac{-7 + 5.74456...}{4} = -0.31386...$

$\Rightarrow y = -3.6277...$

or $x = \frac{-7 - 5.74456...}{4} = -3.18614...$

$\Rightarrow y = -9.3722...$

The solutions are:

$x = -0.31,\ y = -3.63$ and $x = -3.19,\ y = -9.37$ (all to 2 d.p.)

Exercise 6.4A

Solve the following pairs of simultaneous equations. Give your answers to two decimal places where necessary.

1. $y = 20 - 2x$
   $y = x^2 - 16x + 68$
2. $y = 6x - 8$
   $y = x^2 + 2x - 5$
3. $y = 2 - 2x$
   $y = x^2 - 4x + 3$
4. $y + 2x = 9$
   $y = x^2 - 6x + 12$
5. $y + 10x + 31 = 0$
   $y + 6 = x^2$
6. $y + 12x = x^2 + 40$
   $y + 8x = 38$
7. $y - 7 = x^2 + 2x$
   $y - 9 = 4x$
8. $y - x^2 - 14x = 54$
   $43 = y - 6x$
9. $y + 2x + 7 = 4$
   $y - 8 = x^2 + 6x + 2$
10. $2y - 3x = 1$
    $y = x^2 + 3x - 7$
11. $3y + 4x = 15$
    $y = 2x^2 - 3x + 5$
12. $3y - 2x + 5 = 0$
    $y = 7 - 2x - 3x^2$
13. $y = x + 1$
    $x^2 + y^2 = 6$
14. $y = x - 3$
    $y = \frac{2}{x}$
15. $x^2 + y^2 = 6$
    $4x + 3y = 2$

Revision exercise 6

1. Factorise these expressions.
   a) $4x^2 - y^2$
   b) $2x^2 + 8x + 6$
   c) $6m + 4n - 9km - 6kn$
   d) $2x^2 - 5x - 3$
   e) $4x^3 + 10x^2 + 4x$
2. Solve these equations.
   a) $2x^2 - 7x = 0$
   b) $x^2 + 5x + 6 = 0$
   c) $\frac{1}{x} + \frac{1}{4} = \frac{1}{3}$
3. Factorise these expressions.
   a) $z^3 - 16z$
   b) $x^2y^2 + x^2 + y^2 + 1$
   c) $2x^2 + 11x + 12$

4. Solve these simultaneous equations.

a) $y = x + 4$

$y = x^2 + 4x + 4$

b) $2y - 3x - 1 = 0$

$y = 2x^2 - 4x + 3$

5. Solve these equations.

a) $4(y+1) = \frac{3}{1-y}$

b) $x^2 = 5x$

6. Solve these equations, giving your answers correct to two decimal places.

a) $2x^2 - 3x - 1 = 0$

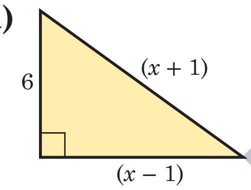
b) $x^2 - x - 1 = 0$

c) $3x^2 + 2x - 4 = 0$

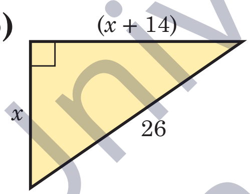
d) $x + 3 = \frac{7}{x}$

7. In each triangle, find the value of x by forming a suitable equation.

a)

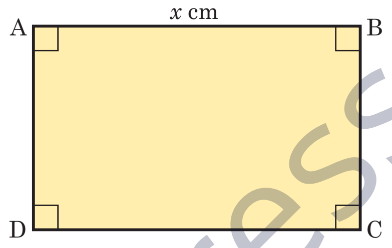


b)



8. A car travels for x hours at a speed of $(x + 2)$ km/h. If the distance travelled is 15 km, write down an equation for x and solve it to find the speed of the car.

9. ABCD is a rectangle, where AB = x cm and BC is 1.5 cm less than AB.



If the area of the rectangle is 52 cm^2, form an equation in x and solve it to find the dimensions of the rectangle.

10. Solve these equations.

a) $(2x + 1)^2 = (x + 5)^2$

b) $x^2 - 7x + 5 = 0$, giving the answers correct to two decimal places.

11. The sides of a right-angled triangle have lengths $(x - 3)$ cm, $(x + 11)$ cm and $2x$ cm, where $2x$ is the hypotenuse. Find the value of x.

12. When each edge of a cube is decreased by 1 cm, its volume is decreased by 91 cm^3. Find the length of a side of the original cube.

13. Solve these simultaneous equations, giving your answers correct to 3 d.p.

$x^2 + y^2 = 16$

$y = x + 1$

Examination-style exercise 6

NON-CALCULATOR SECTION

1. $x^2 + 6x - 12$ can be written in the form $(x + p)^2 + q$.

Work out the value of p and q. [3]

2. $a^4 - 81b^4$ can be written as $(a^2 - kb^2)(a^2 + kb^2)$.

a) Write down the value of k. [1]

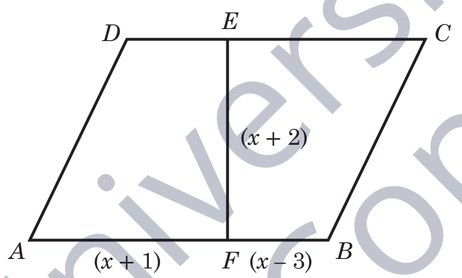
b) Fully factorise the expression $a^4b - 81b^5$. [2]

3. Solve the equation $x^2 + 4x - 11 = 0$ by completing the square.

Leave your answers in surd form. [3]

4. In parallelogram ABCD, the line AD is parallel to BC, *and* the line CD is parallel to AB.

The line EF is perpendicular to lines AB and CD.



The area of parallelogram ABCD is 176 cm^2.

a) Show that $x^2 + x - 90 = 0$. [3]

b) Solve the equation $x^2 + x - 90 = 0$. [2]

c) Calculate the length of AB. [2]

5. Solve these simultaneous equations.

$x^2 + y^2 = 25$

$x + y = -1$ [3]

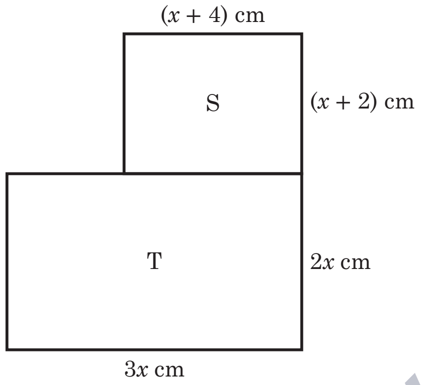
CALCULATOR SECTION

6. a) i) Write down an expression for the area of rectangle S, in the diagram below, in the form $ax^2 + bx + c$. [1]

ii) Show that the total area of rectangles S and T is $(7x^2 + 6x + 8)$ cm^2. [1]

b) The total area of rectangles S and T is 221 cm^2.

Calculate the value of x correct to 1 decimal place. [4]



7. Solve these simultaneous equations, giving your answers accurate to 3 d.p.

$$y = \frac{1}{x}$$

$$y = x + 1$$ [3]

8. One solution of the equation $2x^2 - 7x + k = 0$ is $x = -\frac{1}{2}$

a) Work out the value of k. [2]

b) Work out the other value of x. [2]