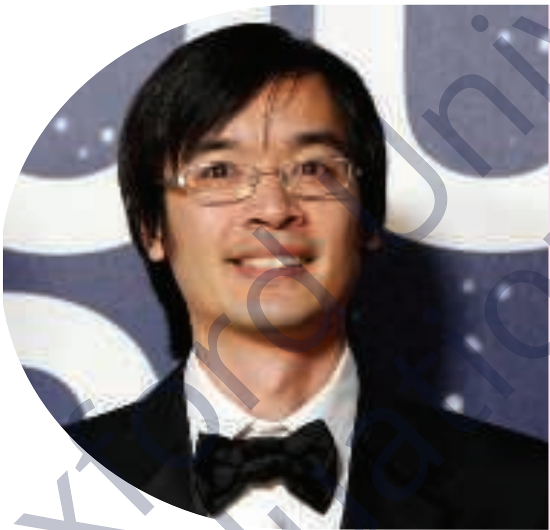
# 8 Algebra 3

**Girolamo Cardano** (1501–1576) was an Italian doctor who became a Professor of Mathematics at Milan. As well as being a distinguished academic, he was an astrologer, a physician and an enthusiastic chess player. In 1545 he published *Ars Magna*, the first Latin treatise devoted solely to algebra, in which he described methods for solving cubic and quartic equations. The treatise also contained the first calculation using complex numbers.



**Terence Tao** (1975– ) was born to ethnic Chinese immigrant parents and raised in Australia. He is generally regarded as one of the greatest living mathematicians and has been called the 'Mozart of mathematics' by his colleagues. He has authored over 300 research papers on topics such as differential equations, probability, number theory and an area of mathematics called combinatorics, for which he won the prestigious Fields Medal in 2006.

- Work with algebraic fractions including using the four rules, factorising and simplifying.
- Change the subject of a formula.
- Use and understand linear inequalities including:
  - representing on a number line
  - constructing and solving
  - graphical representation of inequalities in two variables.
- Use and understand direct and inverse proportion including expressing algebraically and finding unknown quantitates.

# 8.1 Algebraic fractions

## Simplifying fractions

As with numerical fractions, in order to simplify algebraic fractions you need to look for common factors in the numerator and denominator. You can then cancel the fraction by dividing both the numerator and denominator by these factors.

### Example

Simplify: **a)** $\frac{32x^2}{56x}$ **b)** $\frac{3a}{5a^2}$ **c)** $\frac{3y + y^2}{6y}$

**a)** $\frac{32x^2}{56x} = \frac{\cancel{8} \times 4 \times \cancel{x} \times x}{\cancel{8} \times 7 \times \cancel{x}} = \frac{4x}{7}$ **b)** $\frac{3a}{5a^2} = \frac{3 \times \cancel{a}}{5 \times a \times \cancel{a}} = \frac{3}{5a}$

**c)** $\frac{3y + y^2}{6y} = \frac{\cancel{y}(3 + y)}{6\cancel{y}} = \frac{3 + y}{6}$

### Tip

Always remember to factorise *before* simplifying when there are terms being added (or subtracted) in either the numerator or the denominator.

## Exercise 8.1A

Simplify as far as possible, where you can:

**1.** $\frac{25x^2}{35x}$ **2.** $\frac{84y^2}{96y}$ **3.** $\frac{5y^2}{y}$

**4.** $\frac{y}{2y}$ **5.** $\frac{8x^2}{2x^2}$ **6.** $\frac{2x}{4y}$

**7.** $\frac{6y}{3y}$ **8.** $\frac{5ab}{10b}$ **9.** $\frac{8ab^2}{12ab}$

**10.** $\frac{7a^2b}{35ab^2}$ **11.** $\frac{(2a)^2}{4a}$ **12.** $\frac{7yx}{8xy}$

**13.** $\frac{5x + 2x^2}{3x}$ **14.** $\frac{9x + 3}{3x}$ **15.** $\frac{25 + 7x}{25}$

**16.** $\frac{4a + 5a^2}{5a}$ **17.** $\frac{3x}{4x - x^2}$ **18.** $\frac{5ab}{15a + 10a^2}$

**19.** $\frac{5x + 4}{8x}$ **20.** $\frac{12x + 6}{6y}$ **21.** $\frac{5x + 10y}{15xy}$

**22.** $\frac{18a - 3ab}{6a^2}$ **23.** $\frac{4ab + 8a^2}{2ab}$ **24.** $\frac{(2x)^2 - 8x}{4x}$

Factorising quadratic expressions will sometimes reveal common factors that can be cancelled.

### Example

Simplify:

**a)** $\dfrac{x^2 + x - 6}{x^2 + 2x - 3}$ **b)** $\dfrac{x^2 + 3x - 10}{x^2 - 4}$ **c)** $\dfrac{3x^2 - 9x}{x^2 - 4x + 3}$

**a)** $\dfrac{x^2 + x - 6}{x^2 + 2x - 3} = \dfrac{(x-2)\cancel{(x+3)}}{\cancel{(x+3)}(x-1)} = \dfrac{x-2}{x-1}$

**b)** $\dfrac{x^2 + 3x - 10}{x^2 - 4} = \dfrac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}(x+2)} = \dfrac{x+5}{x+2}$

**c)** $\dfrac{3x^2 - 9x}{x^2 - 4x + 3} = \dfrac{3x\cancel{(x-3)}}{(x-1)\cancel{(x-3)}} = \dfrac{3x}{x-1}$

## Exercise 8.1B

Write as a fraction in its simplest form:

**1.** $\dfrac{x^2 + 2x}{x^2 - 3x}$

**2.** $\dfrac{x^2 - 3x}{x^2 - 2x - 3}$

**3.** $\dfrac{x^2 + 4x}{2x^2 - 10x}$

**4.** $\dfrac{x^2 + 6x + 5}{x^2 - x - 2}$

**5.** $\dfrac{x^2 - 4x - 21}{x^2 - 5x - 14}$

**6.** $\dfrac{x^2 + 7x + 10}{x^2 - 4}$

**7.** $\dfrac{x^2 + x - 2}{x^2 - x}$

**8.** $\dfrac{3x^2 - 6x}{x^2 + 3x - 10}$

**9.** $\dfrac{6x^2 - 2x}{12x^2 - 4x}$

**10.** $\dfrac{3x^2 + 15x}{x^2 - 25}$

**11.** $\dfrac{12x^2 - 20x}{4x^2}$

**12.** $\dfrac{x^2 + x - 6}{x^2 + 2x - 3}$

## Multiplication and division of algebraic fractions

### Example

Write as a single fraction:

**a)** $\dfrac{2x}{3} \times \dfrac{x}{4}$ **b)** $\dfrac{x-4}{6} \times \dfrac{3x}{2x-8}$ **c)** $\dfrac{2x}{9} \div \dfrac{5}{3x}$ **d)** $\dfrac{5(x-3)}{2} \div \dfrac{x+1}{4x-12}$

a) $\frac{2x}{3} \times \frac{x}{4} = \frac{2x^2}{12}$

$= \frac{x^2}{6}$

b) $\frac{x-4}{6} \times \frac{3x}{2x-8} = \frac{3x(x-4)}{6(2x-8)}$

$= \frac{\cancel{3}x\,\cancel{(x-4)}}{\cancel{3} \times 2 \times 2\,\cancel{(x-4)}}$

$= \frac{x}{4}$

c) $\frac{2x}{9} \div \frac{5}{3x} = \frac{2x}{9} \times \frac{3x}{5}$

$= \frac{6x^2}{45}$

$= \frac{\cancel{3} \times 2 \times x^2}{\cancel{3} \times 15}$

$= \frac{2x^2}{15}$

d) $\frac{5(x-3)}{2} \div \frac{x+1}{4x-12} = \frac{5(x-3)}{2} \times \frac{4x-12}{x+1}$

$= \frac{5(x-3)}{2} \times \frac{4(x-3)}{x+1}$

$= \frac{20(x-3)^2}{2(x+1)}$

$= \frac{10(x-3)^2}{x+1}$

**Tip**

Dividing by a fraction is equivalent to multiplying by its reciprocal.



## Exercise 8.1C

Simplify the following:

**1.** $\frac{2x}{5} \times \frac{x}{3}$

**2.** $\frac{5x}{2} \times \frac{4}{3x}$

**3.** $\frac{2}{x} \times \frac{5}{x}$

**4.** $\frac{2x}{5} \times \frac{5}{2x}$

**5.** $\frac{3x}{7} \times \frac{7x}{3}$

**6.** $\frac{1}{7x} \times \frac{3}{7x}$

**7.** $\frac{x+1}{5} \times \frac{2x}{x+1}$

**8.** $\frac{x-3}{8} \times \frac{2x}{2x-6}$

**9.** $\frac{4(x+2)}{10x} \times \frac{x+3}{2x+4}$

**10.** $\frac{3x}{4} \div \frac{x}{2}$

**11.** $\frac{4x}{5} \div \frac{3}{2x}$

**12.** $\frac{3}{x} \div \frac{7}{x}$

**13.** $\frac{6x}{7} \surd \frac{12}{14x}$

**14.** $\frac{-5x}{9} \div \frac{9x}{5}$

**15.** $\frac{3}{4x} \div \frac{1}{8x}$

**16.** $\frac{x+4}{3} \div \frac{x+4}{6x}$

**17.** $\frac{4x-8}{3} \div \frac{4}{x-2}$

**18.** $\frac{5(x+1)}{2} \div \frac{x+1}{2x+2}$

## Addition and subtraction of algebraic fractions

### Example

Write as a single fraction:

**a)** $\frac{2x}{3} + \frac{3x}{4}$ **b)** $\frac{2}{x} + \frac{3}{y}$ **c)** $\frac{5x}{6} - \frac{2x}{9}$ **d)** $\frac{x+1}{4} + \frac{x-2}{5}$

**a)** $\frac{2x}{3} + \frac{3x}{4}$; the LCM of 3 and 4 is 12

$$\therefore \frac{2x}{3} + \frac{3x}{4} = \frac{8x}{12} + \frac{9x}{12}$$
$$= \frac{17x}{12}$$

**b)** $\frac{2}{x} + \frac{3}{y}$; the LCM of $x$ and $y$ is $xy$

$$\therefore \frac{2}{x} + \frac{3}{y} = \frac{2y}{xy} + \frac{3x}{xy}$$
$$= \frac{2y + 3x}{xy}$$

**c)** $\frac{5x}{6} - \frac{2x}{9}$; the LCM of 6 and 9 is 18

$$\therefore \frac{5x}{6} - \frac{2x}{9} = \frac{5x \times 3}{6 \times 3} - \frac{2x \times 2}{9 \times 2}$$
$$= \frac{15x}{18} - \frac{4x}{18}$$
$$= \frac{11x}{18}$$

**d)** $\frac{x+1}{4} + \frac{x-2}{5}$; the LCM of 4 and 5 is 20

$$\therefore \frac{x+1}{4} + \frac{x-2}{5} = \frac{(x+1) \times 5}{4 \times 5} + \frac{(x-2) \times 4}{5 \times 4}$$
$$= \frac{5x+5}{20} + \frac{4x-8}{20}$$
$$= \frac{9x-3}{20}$$

## Exercise 8.1D

Simplify the following:

1. $\frac{2x}{5} + \frac{x}{5}$
2. $\frac{2}{x} + \frac{1}{x}$
3. $\frac{x}{7} + \frac{3x}{7}$
4. $\frac{1}{7x} + \frac{3}{7x}$
5. $\frac{5x}{8} + \frac{x}{4}$
6. $\frac{5}{8x} + \frac{1}{4x}$
7. $\frac{2x}{3} + \frac{x}{6}$
8. $\frac{2}{3x} + \frac{1}{6x}$
9. $\frac{3x}{4} + \frac{2x}{5}$
10. $\frac{3}{4x} + \frac{2}{5x}$
11. $\frac{3x}{4} - \frac{2x}{3}$
12. $\frac{3}{4x} - \frac{2}{3x}$
13. $\frac{x}{2} + \frac{x+1}{3}$
14. $\frac{x-1}{3} + \frac{x+2}{4}$
15. $\frac{2x-1}{5} + \frac{x+3}{2}$
16. $\frac{x+1}{3} - \frac{2x+1}{4}$
17. $\frac{x-3}{3} - \frac{x-2}{5}$
18. $\frac{2x+1}{7} - \frac{x+2}{2}$
19. $\frac{2x}{3} + \frac{3(x-5)}{2}$
20. $\frac{4x}{3} - \frac{2(x+1)}{5}$
21. $\frac{2(x+3)}{4} + \frac{4(x-2)}{7}$
22. $\frac{1}{x} + \frac{2}{x+1}$
23. $\frac{3}{x-2} + \frac{4}{x}$
24. $\frac{5}{x-2} + \frac{3}{x+3}$
25. $\frac{7}{x+1} - \frac{3}{x+2}$
26. $\frac{2}{x+3} - \frac{5}{x-1}$
27. $\frac{3}{x-2} - \frac{4}{x+1}$

## Solving quadratics involving algebraic fractions

### Example

A girl bought a certain number of golf balls for \$20. If each ball had cost 20 cents less, she could have bought five more for the same money. How many golf balls did she buy?

Let the number of balls bought be $x$.

Cost of each ball $= \frac{2000}{x}$ cents

If five more balls had been bought:

Cost of each ball now $= \frac{2000}{(x+5)}$ cents

The new price is 20 cents less than the original price.

$$\therefore \quad \frac{2000}{x} - \frac{2000}{(x+5)} = 20$$

$$x\left(\frac{2000}{x}\right) - x\left(\frac{2000}{(x+5)}\right) = 20x \qquad \text{(Multiply by } x\text{.)}$$

$$2000(x+5) - x\,\frac{2000}{(x+5)}(x+5) = 20x(x+5) \qquad \text{(Multiply by } (x+5)\text{.)}$$

$$2000x + 10000 - 2000x = 20x^2 + 100x$$

$$20x^2 + 100x - 10000 = 0$$

$$x^2 + 5x - 500 = 0$$

$$(x-20)(x+25) = 0$$

$$x = 20$$

$$\text{or} \quad x = -25$$

Reject $x = -25$ as not relevant to this context.

The number of balls bought = 20

## Exercise 8.1E

1. A number exceeds four times its reciprocal by 3. Find the number.
2. Two integers differ by 3. The sum of their reciprocals is $\frac{7}{10}$. Find the integers.
3. A cyclist travels 40 km at a speed of $x$ km/h.
   **a)** Find the time taken in terms of $x$.
   **b)** Find the time taken when his speed is reduced by 2 km/h.
   **c)** If the difference between the times is 1 hour, find the original speed $x$.
4. An increase of speed of 4 km/h on a journey of 32 km reduces the time taken by 4 hours. Find the original speed.
5. A train normally travels 240 km at a certain speed. One day, due to bad weather, the train's speed is reduced by 20 km/h so that the journey takes two hours longer. Find the normal speed.
6. The speed of a sparrow is $x$ km/h in still air. When the wind is blowing at 1 km/h, the sparrow takes 5 hours to fly 12 kilometres to her nest and 12 kilometres back again. She goes out directly into the wind and returns with the wind behind her. Find her speed in still air.

7. An aircraft flies a certain distance on a bearing of 135° and then twice the distance on a bearing of 225°. Its distance from the starting point is then 350 km. Find the length of the first part of the journey.

8. The numerator of a fraction is 1 less than the denominator. When both numerator and denominator are increased by 2, the new fraction is greater than the original fraction by $\frac{1}{12}$. Find the original fraction.

## 8.2 Changing the subject of a formula

The operations involved in solving ordinary linear equations are exactly the same as the operations required in changing the subject of a formula. Compare the two parts of the following example.

### Example 1

**a)** Solve the equation $3x + 1 = 12$

**b)** Make $x$ the subject of the formula $Mx + B = A$

**a)** $3x + 1 = 12$

$3x = 12 - 1$ (Subtract 1 from both sides.)

$x = \frac{12 - 1}{3} = \frac{11}{3}$ (Divide both sides by 3.)

**b)** $Mx + B = A$

$Mx = A - B$ (Subtract $B$ from both sides.)

$x = \frac{A - B}{M}$ (Divide both sides by $M$.)

### Example 2

**a)** Solve the equation $3(y - 2) = 5$

**b)** Make $y$ the subject of the formula $x(y - a) = e$

**a)** $3(y - 2) = 5$ (Expand the brackets.)

$3y - 6 = 5$

$3y = 11$ (Add 6 to both sides.)

$y = \frac{11}{3}$ (Divide both sides by 3.)

**b)** $x(y - a) = e$ (Expand the brackets.)

$xy - xa = e$

$xy = e + xa$ (Add $xa$ to both sides.)

$y = \frac{e + xa}{x}$ (Divide both sides by $x$.)

## Exercise 8.2A

Make $x$ the subject of each formula.

**1.** $2x = 5$
**2.** $Ax = B$
**3.** $Mx = K$
**4.** $xy = 4$
**5.** $4x = D$
**6.** $9x = T + N$
**7.** $Ax = B - R$
**8.** $Lx = N - R^2$
**9.** $R - S^2 = Nx$
**10.** $x + A = T$
**11.** $M = x + B$
**12.** $L = x + D^2$
**13.** $N^2 + x = T$
**14.** $L + x = N + M$
**15.** $Z + x = R - S$
**16.** $x - 5 = 2$
**17.** $x - R = A$
**18.** $F = x - B$
**19.** $F^2 = x - B^2$
**20.** $x - D = A + B$
**21.** $x - E = A^2$

Make $y$ the subject of the following formulae.

**22.** $L = y - B$
**23.** $Ay + C = N$
**24.** $Ny - F = H$
**25.** $Vy + m = Q$
**26.** $ty - m = n + a$
**27.** $qy + n = s - t$
**28.** $ny - s^2 = t$
**29.** $V^2y + b = c$
**30.** $r = ny - 6$
**31.** $s = my + d$
**32.** $t = my - b$
**33.** $2(y + 1) = 6$
**34.** $3(y - 1) = 5$
**35.** $A(y + B) = C$
**36.** $h(y + n) = a$
**37.** $b(y - d) = q$
**38.** $n = r(y + t)$
**39.** $t(y - 4) = b$

### Example 1

**a)** Solve the equation $\dfrac{3a + 1}{2} = 4$

**b)** Make $a$ the subject of the formula $\dfrac{na + b}{m} = n$

**a)**
$$\frac{3a + 1}{2} = 4$$
$$3a + 1 = 8$$
$$3a = 7$$
$$a = \frac{7}{3}$$

**b)**
$$\frac{na + b}{m} = n$$
$$na + b = mn$$
$$na = mn - b$$
$$a = \frac{mn - b}{n}$$

### Example 2

Make $a$ the subject of the formula $x - na = y$

Make the '$a$' term positive.
$$x = y + na$$
$$x - y = na$$
$$\frac{x - y}{n} = a$$

Rearrange so that the subject $a$ is on the left.
$$a = \frac{x - y}{n}$$

## Exercise 8.2B

Make $a$ the subject of each formula.

1. $\frac{a}{4} = 3$
2. $\frac{a}{D} = B$
3. $b = \frac{a}{m}$
4. $\frac{a-2}{4} = 6$
5. $\frac{a-A}{B} = T$
6. $\frac{a+Q}{N} = B^2$
7. $g = \frac{a-r}{e}$
8. $\frac{2a+1}{5} = 2$
9. $\frac{Aa+B}{C} = D$
10. $\frac{ra-t}{S} = v$
11. $\frac{za-m}{q} = t$
12. $\frac{m+Aa}{b} = c$
13. $A = \frac{Ba+D}{E}$
14. $n = \frac{ea-f}{h}$
15. $q = \frac{ga+b}{r}$
16. $6 - a = 2$
17. $7 - a = 9$
18. $5 = 7 - a$
19. $A - a = B$
20. $D - a = H$
21. $n - a = m$
22. $t = q - a$
23. $r = v^2 - ra$
24. $t^2 = w - na$
25. $n - qa = 2$
26. $\frac{3-4a}{2} = 1$
27. $\frac{D-Ea}{N} = B$
28. $\frac{h-fa}{b} = x$
29. $\frac{v^2-ha}{C} = d$
30. $\frac{M(a+B)}{N} = T$
31. $\frac{f(Na-e)}{m} = B$
32. $\frac{T(M-a)}{E} = F$
33. $\frac{y(x-a)}{z} = t$

### Example 1

**a)** Solve the equation $7 = \frac{4}{z}$

**b)** Make $z$ the subject of the formula $k = \frac{n}{z}$

**a)**
$$7 = \frac{4}{z}$$
$$7z = 4$$
$$z = \frac{4}{7}$$

**b)**
$$k = \frac{n}{z}$$
$$kz = n$$
$$z = \frac{n}{k}$$

## Example 2

Make $t$ the subject of the formula $\frac{x}{t} + m = a$

$$\frac{x}{t} = a - m$$

$$x = (a - m)t$$

$$\frac{x}{(a - m)} = t$$

You can rearrange this so that $t$ is on the left side:

$$t = \frac{x}{(a - m)}$$

## Exercise 8.2C

Make $a$ the subject of each formula.

**1.** $\frac{7}{a} = 14$

**2.** $\frac{B}{a} = C$

**3.** $m = \frac{n}{a}$

**4.** $\frac{B}{a} = x$

**5.** $\frac{5}{a} = \frac{3}{4}$

**6.** $\frac{N}{a} = \frac{B}{D}$

**7.** $\frac{5}{a + 1} = 2$

**8.** $\frac{7}{a - 1} = 3$

**9.** $\frac{B}{a + D} = C$

**10.** $\frac{Q}{a - C} = T$

**11.** $\frac{L}{Ma} = B$

**12.** $\frac{m}{ca} = d$

**13.** $x = \frac{z}{y - a}$

Make $x$ the subject of each formula.

**14.** $\frac{2}{x} + 1 = 3$

**15.** $\frac{5}{x} - 2 = 4$

**16.** $\frac{A}{x} + B = C$

**17.** $\frac{r}{x} - t = n$

**18.** $h = d - \frac{b}{x}$

**19.** $C - \frac{d}{x} = e$

**20.** $r - \frac{m}{x} = e^2$

**21.** $t^2 = b - \frac{n}{x}$

**22.** $\frac{d}{x} + b = mn$

**23.** $\frac{M}{x + q} - N = 0$

**24.** $\frac{Y}{x - c} - T = 0$

**25.** $3M = M + \frac{N}{P + x}$

**26.** $A = \frac{B}{c + x} - 5A$

**27.** $\frac{K}{Mx} + B = C$

**28.** $\frac{z}{xy} - z = y$

**29.** $\frac{m^2}{x} - n = -p$

**30.** $t = w - \frac{q}{x}$

### Example

Make $x$ the subject of each formula.

**a)** $\sqrt{x^2 + A} = B$ **b)** $(Ax - B)^2 = M$ **c)** $\sqrt{R - x} = T$

**a)** $\sqrt{x^2 + A} = B$

$x^2 + A = B^2$ (square both sides)

$x^2 = B^2 - A$

$x = \pm\sqrt{B^2 - A}$

**b)** $(Ax - B)^2 = M$

$Ax - B = \pm\sqrt{M}$ (square root both sides)

$Ax = B \pm \sqrt{M}$

$x = \dfrac{B \pm \sqrt{M}}{A}$

**c)** $\sqrt{R - x} = T$

$R - x = T^2$

$R = T^2 + x$

$x = R - T^2$



## Exercise 8.2D

Make $x$ the subject of each formula.

**1.** $\sqrt{x} = 2$ **2.** $\sqrt{x - 2} = 3$ **3.** $\sqrt{x + a} = B$

**4.** $\sqrt{x - E} = H$ **5.** $\sqrt{ax + b} = c$ **6.** $\sqrt{x - m} = a$

**7.** $b = \sqrt{gx - t}$ **8.** $r = \sqrt{b - x}$ **9.** $b = \sqrt{x - d}$

**10.** $\sqrt{M - Nx} = P$ **11.** $\sqrt{Ax + B} = \sqrt{D}$ **12.** $\sqrt{x - D} = A^2$

**13.** $x^2 = g$ **14.** $x^2 + 1 = 17$ **15.** $x^2 - A = M$

**16.** $b = a + x^2$ **17.** $C - x^2 = m$ **18.** $N = d - x^2$

Make $k$ the subject.

**19.** $\dfrac{kz}{a} = t$ **20.** $ak^2 - t = m$ **21.** $n = a - k^2$

**22.** $\sqrt{k^2 - A} = B$ **23.** $t = \sqrt{m + k^2}$ **24.** $2\sqrt{k + 1} = 6$

**25.** $A\sqrt{k+B} = M$

**26.** $\sqrt{\dfrac{M}{k}} = N$

**27.** $\sqrt{a-k} = b$

**28.** $\sqrt{a^2 - k^2} = t$

**29.** $\sqrt{m - k^2} = x$

**30.** $2\pi\sqrt{k+t} = 4$

**31.** $A\sqrt{k+1} = B$

**32.** $\sqrt{ak^2 - b} = C$

**33.** $a\sqrt{k^2 - x} = b$

**34.** $k^2 + b = x^2$

**35.** $\dfrac{k^2}{a} + b = c$

**36.** $\sqrt{c^2 - ak} = b$

## Rearranging when the subject appears twice

When the letter you want to make the subject appears twice in the equation, the way to make it appear only once is to collect all the terms that contain that letter, then factorise that letter out.

### Example

Make $x$ the subject of each formula.

**a)** $Ax - B = Cx + D$ **b)** $x + a = \dfrac{x+b}{c}$

**a)**

$$\begin{aligned} Ax - B &= Cx + D \\ Ax - Cx &= D + B \\ x(A - C) &= D + B \text{ (factorise)} \\ x &= \frac{D+B}{A-C} \end{aligned}$$

**b)**

$$\begin{aligned} x + a &= \frac{x+b}{c} \\ c(x + a) &= x + b \\ cx + ca &= x + b \\ cx - x &= b - ca \\ x(c - 1) &= b - ca \text{ (factorise)} \\ x &= \frac{b - ca}{c - 1} \end{aligned}$$

### Exercise 8.2E

Make $y$ the subject of each formula.

**1.** $5(y - 1) = 2(y + 3)$

**2.** $Ny + B = D - Ny$

**3.** $m(y + a) = n(y + b)$

**4.** $\dfrac{a-y}{a+y} = b$

**5.** $\dfrac{1-y}{1+y} = \dfrac{c}{d}$

**6.** $y + m = \dfrac{2y-5}{m}$

**7.** $y - n = \dfrac{y+2}{n}$

**8.** $\dfrac{ay+x}{x} = 4 - y$

**9.** $c - dy = e - ay$

**10.** $y(a - c) = by + d$

**11.** $\dfrac{y+x}{y-x} = 3$

**12.** $y(b - a) = a(y + b + c)$

**13.** $\sqrt{\dfrac{y+x}{y-x}} = 2$

**14.** $\sqrt{\dfrac{m(y+n)}{y}} = p$

**15.** $n - y = \dfrac{4y-n}{m}$

## Example

Make $w$ the subject of the formula $\sqrt{\dfrac{w}{w+a}} = c$

$$\frac{w}{w+a} = c^2$$
$$w = c^2(w+a) \quad \text{(Square both sides.)}$$
$$w = c^2w + c^2a$$
$$w - c^2w = c^2a$$
$$w(1-c^2) = c^2a$$
$$w = \frac{c^2a}{1-c^2}$$

## Exercise 8.2F

Make the letter in square brackets the subject of each formula.

**1.** $ax + by + c = 0$ [$x$]

**2.** $\sqrt{a\left(y^2 - b\right)} = e$ [$y$]

**3.** $\dfrac{\sqrt{k-m}}{n} = \dfrac{1}{m}$ [$k$]

**4.** $\dfrac{x+y}{x-y} = 2$ [$x$]

**5.** $t = 2\pi\sqrt{\dfrac{d}{g}}$ [$d$]

**6.** $\sqrt{x^2 + a} = 2x$ [$x$]

**7.** $\sqrt{\dfrac{b\left(m^2 + a\right)}{e}} = t$ [$m$]

**8.** $\sqrt{\dfrac{x+1}{x}} = a$ [$x$]

**9.** $\sqrt{a^2 + b^2} = x^2$ [$a$]

**10.** $\dfrac{a}{k} + b = \dfrac{c}{k}$ [$k$]

**11.** $a - y = \dfrac{b+y}{a}$ [$y$]

**12.** $G = 4\pi\sqrt{x^2 + T^2}$ [$x$]

**13.** $a\sqrt{\dfrac{x^2 - n}{m}} = \dfrac{a^2}{b}$ [$x$]

**14.** $\dfrac{M}{N} + E = \dfrac{P}{N}$ [$N$]

**15.** $\dfrac{Q}{P-x} = R$ [$x$]

**16.** $\sqrt{z - ax} = t$ [$a$]

**17.** $e + \sqrt{x + f} = g$ [$x$]

**18.** $\dfrac{m\left(ny - e^2\right)}{p} + n = 5n$ [$y$]

# 8.3 Proportion

## Direct proportion

In Chapter 3, section 3.1, you looked at the ideas of direct and inverse proportion, and how they were closely related to the idea of ratio. In this section, you will approach it more algebraically.

If $x$ is directly proportional to $y$, you write $x \propto y$, using the 'is proportional to' symbol $\propto$.

The '$\propto$' symbol can be replaced by '$= k$' where $k$ is a constant, which means that $x = ky$.

Suppose you are told that $x = 3$ when $y = 12$.

Then $3 = k \times 12$ and $k = \frac{1}{4}$

You can then write $x = \frac{1}{4}y$, and this allows you to find the value of $x$ for any value of $y$, and vice versa.

### Example 1

$y$ is directly proportional to $z$.

When $z = 5$, $y = 2$.

Find:

**a)** the value of $y$ when $z = 6$

**b)** the value of $z$ when $y = 5$

Because $y \propto z$, then $y = kz$ where $k$ is a constant.

$$y = 2 \text{ when } z = 5$$

$$2 = k \times 5$$

$$k = \frac{2}{5}$$

So $$y = \frac{2}{5}z$$

**a)** When $z = 6$, $y = \frac{2}{5} \times 6 = 2\frac{2}{5}$

**b)** When $y = 5$, $5 = \frac{2}{5}z$

$$z = \frac{25}{2} = 12\frac{1}{2}$$

## Example 2

The value in dollars, $V$, of a diamond is proportional to the square of its mass $M$. If a diamond with a mass of 10 grams is worth \$200, find:

**a)** the value of a diamond with a mass of 30 grams

**b)** the mass of a diamond worth \$5000.

$V \propto M^2$

or $V = kM^2$ where $k$ is a constant.

$V = 200$ when $M = 10$

$\therefore\ 200 = k \times 10^2$

$k = 2$

So $V = 2M^2$

**a)** When $M = 30$,

$V = 2 \times 30^2 = 2 \times 900$

$V = \$1800$

A diamond with a mass of 30 grams is worth \$1800

**b)** When $V = 5000$,

$5000 = 2 \times M^2$

$M^2 = \dfrac{5000}{2} = 2500$

$M = \sqrt{2500} = 50$

A diamond of value \$5000 has a mass of 50 grams.

## Exercise 8.3A

1. Rewrite the statement connecting each pair of variables using a constant $k$ instead of '$\propto$'.

   **a)** $S \propto e$ **b)** $y \propto \sqrt{x}$ **c)** $T \propto \sqrt{L}$

   **d)** $C \propto r$ **e)** $A \propto r^2$ **f)** $V \propto r^3$

2. $y$ is directly proportional to $t$. If $y = 6$ when $t = 4$, calculate:

   **a)** the value of $y$, when $t = 6$ **b)** the value of $t$, when $y = 4$

3. $z$ is directly proportional to $m$. If $z = 20$ when $m = 4$, calculate:

   **a)** the value of $z$, when $m = 7$ **b)** the value of $m$, when $z = 55$

4. $A$ is directly proportional to $r^2$. If A = 12, when $r = 2$, calculate:

   **a)** the value of $A$, when $r = 5$ **b)** the value of $r$, when $A = 48$

5. Given that $z \propto x$, copy and complete the table.

| $x$ | 1 | 3 | | $5\frac{1}{2}$ |
|---|---|---|---|---|
| $z$ | 4 | | 16 | |

6. Given that $V \propto r^3$, copy and complete the table.

| $r$ | 1 | 2 | | $1\frac{1}{2}$ |
|---|---|---|---|---|
| $V$ | 4 | | 256 | |

7. Given that $w \propto \sqrt{h}$, copy and complete the table.

| $h$ | 4 | 9 | | $2\frac{1}{4}$ |
|---|---|---|---|---|
| $w$ | 6 | | 15 | |

8. The pressure of the water $P$ at any point below the surface of the sea is directly proportional to the depth of the point below the surface $d$. If the pressure is 200 newtons/cm$^2$ at a depth of 3 m, calculate the pressure at a depth of 5 m.

9. The distance $d$ through which a stone falls from rest is directly proportional to the square of the time taken $t$. If the stone falls 45 m in 3 seconds, how far will it fall in 6 seconds? How long will it take to fall 20 m?

10. The energy $E$ stored in an elastic band is directly proportional to the square of the extension $x$. When the elastic is extended by 3 cm, the energy stored is 243 joules. What is the energy stored when the extension is 5 cm? What is the extension when the stored energy is 36 joules?

11. In the first few days of its life, the length of an earthworm $L$ is thought to be directly proportional to the square root of the number of hours $n$ which have elapsed since its birth. If a worm is 2 cm long after 1 hour, how long will it be after 4 hours? How long will it take to grow to a length of 14 cm?

12. The number of eggs which a goose lays in a week is directly proportional to the cube root of the average number of hours of sleep she has. When she has 8 hours sleep, she lays 4 eggs. How long does she sleep when she lays 5 eggs?

**13.** The resistance to motion of a car is directly proportional to the square of the speed of the car. If the resistance is 4000 newtons at a speed of 20 m/s, what is the resistance at a speed of 30 m/s?

At what speed is the resistance 6250 newtons?

**14.** A road research organisation recently claimed that the damage to road surfaces was directly proportional to the fourth power of the axle load. The axle load of a 44-tonne HGV is about 15 times that of a car. Calculate the ratio of the damage to road surfaces made by a 44-tonne HGV : damage to road surfaces made by a car.

## Inverse proportion

If $x$ is inversely proportional to $y$, you write $x \propto \frac{1}{y}$, again using the 'is proportional to' symbol $\propto$.

The '$\propto$' symbol can be replaced by '$= k$' where $k$ is a constant, which means that $x = \frac{k}{y}$

Suppose you are told that $x = 4$ when $y = 5$.

Then $4 = \frac{k}{5}$

and $k = 20$

You can then write $x = \frac{20}{y}$, and this again allows you to find the value of $x$ for any value of $y$ and vice versa.

### Example

$z$ is inversely proportional to $t^2$, and $z = 4$ when $t = 1$. Calculate:

**a)** $z$ when $t = 2$ **b)** $t$ when $z = 16$.

Write $z \propto \frac{1}{t^2}$

or $z = k \times \frac{1}{t^2}$ ($k$ is a constant)

$z = 4$ when $t = 1$,

$\therefore \quad 4 = k\left(\frac{1}{1^2}\right)$

so $k = 4$

$\therefore \quad z = 4 \times \frac{1}{t^2}$

**a)** When $t = 2$, $z = 4 \times \frac{1}{2^2} = 1$

**b)** When $z = 16$, $16 = 4 \times \frac{1}{t^2}$

$$16t^2 = 4$$

$$t^2 = \frac{1}{4}$$

$$t = \pm\frac{1}{2}$$

## Exercise 8.3B

1. Rewrite the statements connecting the variables using a constant of variation, $k$.

   **a)** $x \propto \frac{1}{y}$ **b)** $s \propto \frac{1}{t^2}$ **c)** $t \propto \frac{1}{\sqrt{q}}$

   **d)** $m$ is inversely proportional to $w$

   **e)** $z$ is inversely proportional to $t^2$.

2. $b$ is inversely proportional to $e$. If $b = 6$ when $e = 2$, calculate:

   **a)** the value of $b$ when $e = 12$ **b)** the value of $e$ when $b = 3$

3. $q$ is inversely proportional to $r$. If $q = 5$ when $r = 2$, calculate:

   **a)** the value of $q$ when $r = 4$ **b)** the value of $r$ when $q = 20$

4. $x$ is inversely proportional to $y^2$. If $x = 4$ when $y = 3$, calculate:

   **a)** the value of $x$ when $y = 1$ **b)** the value of $y$ when $x = 2\frac{1}{4}$.

5. $R$ is inversely proportional to $v^2$. If $R = 120$ when $v = 1$, calculate:

   **a)** the value of $R$ when $v = 10$ **b)** the value of $v$ when $R = 30$

6. $T$ is inversely proportional to $x^2$. If $T = 36$ when $x = 2$, calculate:

   **a)** the value of $T$ when $x = 3$ **b)** the value of $x$ when $T = 1.44$

7. $p$ is inversely proportional to $\sqrt{y}$. If $p = 1.2$ when $y = 100$, calculate:

   **a)** the value of $p$ when $y = 4$ **b)** the value of $y$ when $p = 3$

8. $y$ is inversely proportional to $z$. If $y = \frac{1}{8}$ when $z = 4$, calculate:

   **a)** the value of $y$ when $z = 1$ **b)** the value of $z$ when $y = 10$

**9.** Given that $z \propto \frac{1}{y}$, copy and complete the table.

| | | | | |
|---|---|---|---|---|
| $y$ | 2 | 4 | | $\frac{1}{4}$ |
| $z$ | 8 | | 16 | |

**10.** Given that $v \propto \frac{1}{t^2}$, copy and complete the table.

| | | | | |
|---|---|---|---|---|
| $t$ | 2 | 5 | | 10 |
| $v$ | 25 | | $\frac{1}{4}$ | |

**11.** Given that $r \propto \frac{1}{\sqrt{x}}$ copy and complete the table.

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 4 | | |
| $r$ | 12 | | $\frac{3}{4}$ | 2 |

**12.** $M$ is inversely proportional to the square of $l$.
If $M = 9$ when $l = 2$, and if $M$ and $l$ are always positive, find:

**a)** $M$ when $l = 10$ **b)** $l$ when $M = 1$

**13.** Given $z = \frac{k}{x^n}$, find $k$ and $n$, and then copy and complete the table.

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 2 | 4 | |
| $z$ | 100 | $12\frac{1}{2}$ | | $\frac{1}{10}$ |

**14.** Given $y = \frac{k}{\sqrt[n]{v}}$, find $k$ and $n$, and then copy and complete the table.

| | | | | |
|---|---|---|---|---|
| $v$ | 1 | 4 | 36 | |
| $y$ | 12 | 6 | | $\frac{3}{25}$ |

**15.** The volume $V$ of a given mass of gas is inversely proportional to the pressure $P$.
When $V = 2$ m$^3$, $P = 500$ N/m$^2$. Find the volume when the pressure is 400 N/m$^2$.
Find the pressure when the volume is 5 m$^3$.

**16.** The number of hours $N$ required to dig a certain hole is inversely proportional to the number of workers available, $x$. When 6 workers are digging, the hole takes 4 hours. Find the time taken when 8 workers are available. If it takes $\frac{1}{2}$ hour to dig the hole, how many workers are there?

**17.** The force of attraction $F$ between two magnets is inversely proportional to the square of the distance $d$ between them. When the magnets are 2 cm apart, the force of attraction is 18 newtons. How far apart are they if the attractive force is 2 newtons?

## 8.4 Indices 2

In Chapter 1 you learned how to use indices when dealing with numbers. The same rules for indices apply when working with algebra. Here are rules you need to remember.

**1.** $a^n \times a^m = a^{n+m}$ **2.** $a^n \div a^m = a^{n-m}$ **3.** $(a^n)^m = a^{nm}$

Also remember that:

- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{n}}$ means 'the $n$th root of $a$'
- $a^{\frac{m}{n}}$ means 'the $n$th root of $a$ raised to the power $m$'
- $a^0 = 1$ whenever $a \neq 0$, since $1 = \dfrac{a^n}{a^n} = a^{(n-n)} = a^0$

### Example

Simplify:

**a)** $x^7 \times x^{13}$ **b)** $x^3 \div x^7$

**c)** $(x^4)^3$ **d)** $(3x^2)^3$

**e)** $(2x^{-1})^2 \div x^{-5}$ **f)** $3y^2 \times 4y^3$

**a)** $x^7 \times x^{13} = x^{7+13} = x^{20}$

**b)** $x^3 \div x^7 = x^{3-7} = x^{-4} = \dfrac{1}{x^4}$

**c)** $(x^4)^3 = x^{12}$

**d)** $(3x^2)^3 = 3^3 \times (x^2)^3 = 27x^6$

**e)** $\left(2x^{-1}\right)^2 \div x^{-5} = 4x^{-2} \div x^{-5}$

$= 4x^{(-2--5)}$

$= 4x^3$

**f)** $3y^2 \times 4y^3 = 12y^5$

## Exercise 8.4A

Simplify:

**1.** $x^3 \times x^4$ **2.** $y^6 \times y^7$ **3.** $z^7 \div z^3$ **4.** $m^3 \div m^2$

**5.** $e^{-3} \times e^{-2}$ **6.** $y^{-2} \times y^4$ **7.** $w^4 \div w^{-2}$ **8.** $y^{\frac{1}{2}} \times y^{\frac{1}{2}}$

**9.** $(x^2)^5$ **10.** $x^{-2} \div x^{-2}$ **11.** $w^{-3} \times w^{-2}$ **12.** $w^{-7} \times w^2$

**13.** $x^3 \div x^{-4}$ **14.** $\left(k^{\frac{1}{2}}\right)^6$ **15.** $e^{-4} \times e^4$ **16.** $\left(y^4\right)^{\frac{1}{2}}$

**17.** $(x^{-3})^{-2}$ **18.** $t^{-3} \div t$ **19.** $(2x^3)^2$ **20.** $2x^2 \times 3x^2$

**21.** $5y^3 \times 2y^2$ **22.** $5a^3 \times 3a$ **23.** $(2a)^3$ **24.** $3x^3 \div x^3$

**25.** $8y^3 \div 2y$ **26.** $(2x)^2 \times (3x)^3$ **27.** $4z^4 \times z^{-7}$ **28.** $6x^{-2} \div 3x^2$

**29.** $5y^3 \div 2y^{-2}$ **30.** $(x^2)^{\frac{1}{2}} \div \left(x^{\frac{1}{3}}\right)^3$ **31.** $(2n)^4 \div 8n^0$ **32.** $x^{\frac{3}{2}} \div 2x^{\frac{1}{2}}$

**33.** $3x^{-4} \times \frac{2}{3}x^{\frac{1}{2}}$ **34.** $\frac{2}{7}x^{-\frac{1}{2}} \div 2x^{-3}$ **35.** $\left(\frac{3x^5}{4}\right)^3$ **36.** $5x^{-7} \div \frac{3}{10}x^{-\frac{1}{2}}$

**37.** $\frac{3}{5}x^{-\frac{1}{2}} \div 6x^{-4}$ **38.** $\left(\frac{3x^{\frac{1}{2}}}{2}\right)^4$ **39.** $\left(\frac{5x^{-\frac{1}{2}}}{-2}\right)^2$

### Example

Simplify:

**a)** $(2a)^3 \div (9a^2)^{\frac{1}{2}}$ **b)** $(3ac^2)^3 \times 2a^{-2}$ **c)** $(2x)^2 \div 2x^2$

**a)** $(2a)^3 \div (9a^2)^{\frac{1}{2}} = 8a^3 \div 3a$

$= \frac{8}{3}a^2$

**b)** $(3ac^2)^3 \times 2a^{-2} = 27a^3c^6 \times 2a^{-2} = 54ac^6$

**c)** $(2x)^2 \div 2x^2 = 4x^2 \div 2x^2$

$= 2$

## Exercise 8.4B

Rewrite without brackets:

**1.** $(5x^2)^2$ **2.** $(7y^3)^2$ **3.** $(10ab)^2$ **4.** $(2xy^2)^2$

**5.** $\left(4x^2\right)^{\frac{1}{2}}$ **6.** $(9y)^{-1}$ **7.** $(x^{-2})^{-1}$ **8.** $(2x^{-2})^{-1}$

**9.** $(5x^2y)^0$ **10.** $\left(\frac{1}{2}x\right)^{-1}$ **11.** $(3x)^2 \times (2x)^2$ **12.** $(5y)^2 \div y$

**13.** $\left(2x^{\frac{1}{2}}\right)^4$ **14.** $\left(3y^{\frac{1}{3}}\right)^3$ **15.** $(5x^0)^2$ **16.** $((5x)^0)^2$

**17.** $(7y^0)^2$ **18.** $((7y)^0)^2$ **19.** $(2x^2y)^3$ **20.** $(10xy^3)^2$

Simplify:

**21.** $(3x^{-1})^2 \div 6x^{-3}$ **22.** $(4x)^{\frac{1}{2}} \div x^{\frac{3}{2}}$ **23.** $x^2y^2 \times xy^3$ **24.** $4xy \times 3x^2y$

**25.** $10x^{-1}y^3 \times xy$ **26.** $(3x)^2 \times \left(\frac{1}{9}x^2\right)^{\frac{1}{2}}$ **27.** $z^3yx \times x^2yz$ **28.** $(2x)^{-2} \times 4x^3$

**29.** $(3y)^{-1} \div (9y^2)^{-1}$ **30.** $(xy)^0 \times (9x)^{\frac{3}{2}}$ **31.** $(x^2y)(2xy)(5y^3)$ **32.** $\left(4x^{\frac{1}{2}}\right) \times \left(8x^{\frac{3}{2}}\right)$

**33.** $5x^{-3} \div 2x^{-5}$ **34.** $((3x^{-1})^{-2})^{-1}$ **35.** $(2a)^{-2} \times 8a^4$ **36.** $(abc^2)^3$

Evaluate, with $x = 16$ and $y = 8$:

**37.** $2x^{\frac{1}{2}} \times y^{\frac{1}{3}}$ **38.** $x^{\frac{1}{4}} \times y^{-1}$ **39.** $(y^2)^{\frac{1}{6}} \div (9x)^{\frac{1}{2}}$ **40.** $(x^2y^3)^0$

**41.** $x + y^{-1}$ **42.** $x^{-\frac{1}{2}} + y^{-1}$ **43.** $y^{\frac{1}{3}} \div x^{\frac{3}{4}}$ **44.** $(1000y)^{\frac{1}{3}} \times x^{-\frac{5}{2}}$

**45.** $\left(x^{\frac{1}{4}} + y^{-1}\right) \div x^{\frac{1}{4}}$ **46.** $x^{\frac{1}{2}} - y^{\frac{2}{3}}$ **47.** $\left(x^{\frac{3}{4}}y\right)^{-\frac{1}{3}}$ **48.** $\left(\frac{x}{y}\right)^{-2}$

It is sometimes useful to express one number as a power of another number. To do this, use the rules of indices.

### Example 1

Write $8 \times \frac{1}{16}$ in the form $2^p$

$$8 \times \frac{1}{16} = 2^3 \times 2^{-4} = 2^{3-4} = 2^{-1}$$

You can then use your knowledge of indices to solve simple equations where the variable is in the power.

### Example 2

Solve the equations:

**a)** $4^{x-1} = 8^x$ **b)** $9^{x+2} = 3^{4x-2}$

**a)** $4^{x-1} = 8^x$

$(2^2)^{x-1} = (2^3)^x$ (Because 4 and 8 are both powers of 2.)

$2^{2(x-1)} = 2^{3x}$ (Using the rules of indices.)

$2(x-1) = 3x$

$2x - 2 = 3x$

$x = -2$

**b)** $9^{x+2} = 3^{4x-2}$

$(3^2)^{x+2} = 3^{4x-2}$ (Because 9 and 3 are both powers of 3.)

$3^{2(x+2)} = 3^{4x-2}$ (Using the rules of indices.)

$2(x+2) = 4x - 2$

$2x + 4 = 4x - 2$

$2x = 6$

$x = 3$

## Exercise 8.4C

**1.** Write in the form $2^p$ (e.g. $4 = 2^2$).

**a)** 32 **b)** 128 **c)** 64 **d)** 1

**2.** Write in the form $3^q$.

**a)** $\frac{1}{27}$ **b)** $\frac{1}{81}$ **c)** $\frac{1}{3}$ **d)** $9 \times \frac{1}{81}$

Make $x$ the subject of each equation.

**3.** $2^x = 8$ **4.** $3^x = 81$ **5.** $5^x = \frac{1}{5}$

**6.** $10^x = \frac{1}{100}$ **7.** $3^{-x} = \frac{1}{27}$ **8.** $4^x = 64$

**9.** $6^{-x} = \frac{1}{6}$ **10.** $100\,000^x = 10$ **11.** $12^x = 1$

**12.** $10^x = 0.0001$ **13.** $2^x + 3^x = 13$ **14.** $\left(\frac{1}{2}\right)^x = 32$

**15.** $5^{2x} = 25$ **16.** $1\,000\,000^{3x} = 10$

**17.** These two are more difficult. Use a calculator to find solutions correct to three significant figures.

**a)** $x^x = 100$ **b)** $x^x = 10\,000$

Make $x$ the subject of each equation.

**18.** $3^{x+1} = 9^x$ **19.** $4^{x+1} = 2^{3x-1}$ **20.** $7^{3x} = 49^{4x}$

**21.** $25^{x+2} = 5^{x+3}$ **22.** $2^{2x} = 4^x$

# 8.5 Inequalities

When solving inequalities, you follow the same procedure used for solving equations except that when you multiply or divide by a *negative* number the inequality symbol is *reversed*.

e.g. $4 > -2$

but multiplying by $-2$,

$-8 < 4$

### Example

Solve the inequalities:

**a)** $2x - 1 > 5$ **b)** $5 - 3x \leqslant 1$

**a)** $2x - 1 > 5$

$$2x > 5 + 1$$

$$x > \frac{6}{2}$$

$$x > 3$$

**b)** $5 - 3x \leqslant 1$

$$-3x \leqslant 1 - 5$$

$$-3x \leqslant -4$$

$$x \geqslant \frac{4}{3}$$

### Exercise 8.5A

Solve the inequalities:

**1.** $x - 3 > 10$ **2.** $x + 1 < 0$ **3.** $5 > x - 7$ **4.** $2x + 1 \leqslant 6$

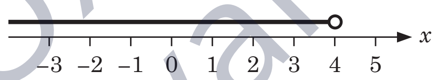
**5.** $3x - 4 > 5$ **6.** $10 \leqslant 2x - 6$ **7.** $5x < x + 1$ **8.** $2x \geqslant x - 3$

**9.** $4 + x < -4$ **10.** $3x + 1 < 2x + 5$ **11.** $2(x + 1) > x - 7$ **12.** $7 < 15 - x$

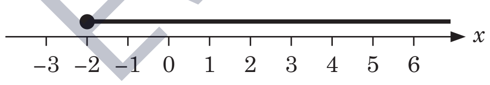
**13.** $9 > 12 - x$ **14.** $4 - 2x \leqslant 2$ **15.** $3(x - 1) < 2(1 - x)$ **16.** $7 - 3x < 0$

## Representing inequalities on a number line

The inequality $x < 4$ is represented on the number line as



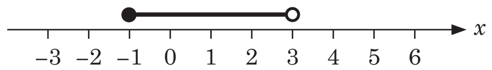

$x \geqslant -2$ is shown as




In the first case, 4 is *not* included so you use an unfilled circle ○

In the second case, $-2$ *is* included so you use a filled circle ●

$-1 \leqslant x < 3$ is shown as




Sometimes you may have to solve a three-part inequality. Sometimes you solve one of these as one inequality, but sometimes you have to split it and solve it as though it were two inequalities. Consider the following examples.

## Example

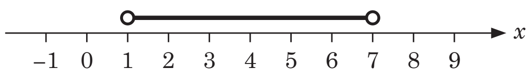
Solve the inequalities:

**a)** $-8 < 4(x - 3) < 16$ **b)** $9 \leqslant 2x - 1 \leqslant x + 9$

**a)** $-8 < 4(x - 3) < 16$ Divide each part by 4

$-2 < x - 3 < 4$ then add 3 to each part

$1 < x < 7$




**b)** If you try to solve the three parts together you will not be able to get to the point where $x$ is only in the middle part. Instead you need to solve two two-part inequalities.

The left-hand part: $9 \leqslant 2x - 1$

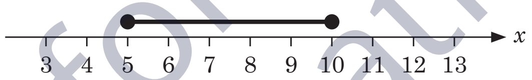
$10 \leqslant 2x$

$5 \leqslant x$

The right-hand part: $2x - 1 \leqslant x + 9$

$x \leqslant 10$

Putting these parts together gives the solution $5 \leqslant x \leqslant 10$




## Exercise 8.5B

Solve each inequality and show the result on a number line.

**1.** $2x + 1 > 11$
**2.** $3x - 4 \leqslant 5$
**3.** $2 < x - 4$
**4.** $6 \geqslant 10 - x$
**5.** $8 < 9 - x$
**6.** $8x - 1 < 5x - 10$
**7.** $2x > 0$
**8.** $1 < 3x - 11$
**9.** $4 - x > 6 - 2x$
**10.** $\frac{x}{3} < -1$
**11.** $3 < x + 2 < 6$
**12.** $-5 \leqslant x - 3 \leqslant 2$
**13.** $3 \leqslant 3x < 18$
**14.** $0 \leqslant 2x < 10$
**15.** $-3 \leqslant 3x \leqslant 21$

16. $1 < 5x < 10$
17. $\frac{x}{4} > 20$
18. $3x - 1 > x + 19$
19. $7(x + 2) < 3x + 4$
20. $1 < 2x + 1 < 9$
21. $10 \leqslant 2x \leqslant x + 9$
22. $x < 3x + 2 < 2x + 6$
23. $10 \leqslant 2x - 1 \leqslant x + 5$
24. $x < 3x - 1 < 2x + 7$
25. $x - 10 < 2(x - 1) < x$
26. $18 - x < 5x \leqslant 7 + 4x$
27. $-4 \leqslant 4x \leqslant 12 + x$
28. $3x < 4x < 2(15 - x)$
29. $2(x - 9) < 8x < 2(x - 3)$
30. $11x - 20 \leqslant 15x < 5(20 - x)$

## Exercise 8.5C

Find the solutions, subject to the given condition.

1. $3a + 1 < 20$; $a$ is a positive integer
2. $b - 1 \geqslant 6$; $b$ is a prime number less than 20
3. $2e - 3 < 21$; $e$ is a positive even number
4. $1 < z < 50$; $z$ is a square number
5. $0 < 3x < 40$; $x$ is divisible by 5
6. $2x > -10$; $x$ is a negative integer
7. $x + 1 < 2x < x + 13$; $x$ is an integer
8. $x^2 < 100$; $x$ is a positive square number
9. $0 \leqslant 2z - 3 \leqslant z + 8$; $z$ is a prime number
10. $\frac{a}{2} + 10 > a$; $a$ is a positive even number
11. State the smallest integer $n$ for which $4n > 19$.
12. Find an integer value of $x$ such that $2x - 7 < 8 < 3x - 11$.
13. Find an integer value of $y$ such that $3y - 4 < 12 < 4y - 5$.
14. Find any value of $z$ such that $9 < z + 5 < 10$.
15. Find any value of $p$ such that $9 < 2p + 1 < 11$.
16. Find a simple fraction $q$ such that $\frac{4}{9} < q < \frac{5}{9}$.
17. Find an integer value of $a$ such that $a - 3 \leqslant 11 \leqslant 2a + 10$.
18. State the largest prime number $z$ for which $3z < 66$.
19. Find the largest prime number $p$ such that $p^2 < 400$.
20. Find the integer $n$ such that $n < \sqrt{300} < n + 1$.
21. A youth club organiser is planning a day trip for the club members. The cost of the trip is \$330 and the club has already saved \$75. The price of a ticket for the trip is \$$x$ and there are 21 people going on the trip.
    a) Write down an inequality in terms of $x$ to determine the price of each ticket if the cost of the trip is to be completely funded.
    b) What is the minimum ticket price that the youth club organiser must charge?

**22.** Chailai has \$700 in her bank account. She wants to keep at least \$300. She plans to withdraw \$$y$ per week for the next 12 weeks to pay for entertainment and food.

**a)** Write down an inequality in terms of $y$ to determine the amount of money Chailai can withdraw each week.

**b)** How much can Chailai withdraw per week?

**23.** A car rental firm charges \$30 per day plus a flat fee of \$240 to rent a car. Neema has no more than \$470 to pay for the car rental.

**a)** Write down an inequality in terms of the number of days, $d$, that Neema rents the car for.

**b)** Solve the inequality to work out the maximum number of days for which Neema can rent the car.

## Revision exercise 8

**1.** Write the following as single fractions.

**a)** $\frac{x}{4} + \frac{x}{5}$ **b)** $\frac{1}{2x} + \frac{2}{3x}$

**c)** $\frac{x+2}{2} + \frac{x-4}{3}$ **d)** $\frac{7}{x-1} - \frac{2}{x+3}$

**2. a)** Factorise $x^2 - 4$

**b)** Simplify $\frac{3x-6}{x^2-4}$

**3.** Given that $s - 3t = rt$, express:

**a)** $s$ in terms of $r$ and $t$

**b)** $r$ in terms of $s$ and $t$

**c)** $t$ in terms of $s$ and $r$.

**4. a)** Given that $x - z = 5y$, express $z$ in terms of $x$ and $y$.

**b)** Given that $mk + 3m = 11$, express $m$ in terms of $k$.

**c)** For the formula $T = C\sqrt{z}$, express $z$ in terms of $T$ and $C$.

**5.** It is given that $y = \frac{k}{x}$ and that $1 \leqslant x \leqslant 10$.

**a)** If the smallest possible value of $y$ is 5, find the value of the constant $k$.

**b)** Find the largest possible value of $y$.

**6.** Given that $y$ is directly proportional to $x^2$ and that $y = 36$ when $x = 3$, find:

**a)** the value of $y$ when $x = 2$

**b)** the value of $x$ when $y = 64$

**7.** Find $x$, given that:

**a)** $3^x = 81$ **b)** $7^x = 1$

**8.** Two integers differ by 6. The sum of their reciprocals is $\frac{5}{36}$.
Find these numbers.

**9.** List the integer values of $x$ which satisfy.

**a)** $2x - 1 < 20 < 3x - 5$

**b)** $5 < 3x + 1 < 17$

**10.** Given that $t = k\sqrt{x+5}$, express $x$ in terms of $t$ and $k$.

**11.** Given that $z = \frac{3y+2}{y-1}$, express $y$ in terms of $z$.

**12.** Given that $y = \frac{k}{k+w}$

**a)** Find the value of $y$ when $k = \frac{1}{2}$ and $w = \frac{1}{3}$

**b)** Express $w$ in terms of $y$ and $k$.

**13.** Calculate the value of:

**a)** $9^{-\frac{1}{2}} + \left(\frac{1}{8}\right)^{\frac{1}{3}} + (-3)^0$

**b)** $(1000)^{-\frac{1}{3}} - (0.1)^2$

**14.** It is given that $10^x = 3$ and $10^y = 7$. Find the value of $10^{x+y}$

**15.** Make $x$ the subject of the following formulae.

**a)** $x + a = \dfrac{2x - 5}{a}$

**b)** $cz + ax + b = 0$

**c)** $a = \sqrt{\dfrac{x+1}{x-1}}$

**16.** Write the following as single fractions.

**a)** $\dfrac{5x}{10} \times \dfrac{20}{15x}$

**b)** $\dfrac{x+6}{4} \div \dfrac{2x+12}{8x}$

**c)** $\dfrac{3}{x} + \dfrac{1}{2x}$

**d)** $\dfrac{3}{a-2} + \dfrac{1}{a^2 - 4}$

**e)** $\dfrac{3}{x(x+1)} - \dfrac{2}{x(x-2)}$

**17.** $p$ is directly proportional to the square of $t$, and inversely proportional to $s$. Given that $p = 5$ when $t = 1$ and $s = 2$, find a formula for $s$ in terms of $t$.

**18.** In the diagram, the solution set $-1 \leqslant x < 2$ is shown on a number line.


Illustrate, on similar diagrams, the solution set of the following pairs of simultaneous inequalities.

**a)** $x > 2$; $x \leqslant 7$

**b)** $4 + x \geqslant 2$; $x + 4 < 10$

**c)** $2x + 1 \geqslant 3$; $x - 3 \leqslant 3$

# Examination-style exercise 8

## NON-CALCULATOR

1. $\dfrac{x-2}{5}+\dfrac{5}{x-2}=\dfrac{x^2-ax+b}{c(x-2)}$, where $a$, $b$ and $c$ are integers.

   Work out $a$, $b$ and $c$. [3]

2. Simplify $\left(64x^3\right)^{\frac{2}{3}}$ [2]

3. Work out the value of $n$ in each of the following statements.

   **a)** $16^n = 1$ [1]

   **b)** $16^n = 2$ [1]

   **c)** $16^n = 8$ [1]

4. $(125)^{-\frac{2}{3}} = 25^p$. Determine $p$. [2]

5. Write $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{b}{ab}$ as a single fraction in its simplest form. [3]

6. Solve the inequality $\dfrac{3x-4}{7}<\dfrac{x+3}{4}$ [3]

7. Rearrange the formula to make $b$ the subject: $a=\sqrt{\dfrac{b}{4}}-2$ [3]

8. **a)** Factorise $cp^3 + dp^3$ [1]

   **b)** Make $p$ the subject of the formula $cp^3 + dp^3 - a^2 = b^3$ [2]

9. The quantity $x$ is inversely proportional to the square root of $(y + 1)$.

   When $x = 20$, $y = 8$.

   Work out $y$ when $x = 12$. [3]

10. A torch is used to light up a wall.

    The brightness of the light on the wall is $b$. The distance of the light from the wall is $d$. $b$ is inversely proportional to the square of $d$.

    What happens to $b$ when $d$ is doubled? [3]

11. Solve the equation $\dfrac{x}{x+1}-\dfrac{x+1}{3x-1}=\dfrac{1}{4}$

12. Aisha takes 2 hours 35 minutes to complete a walk.

    **a)** Show that the time of 2 hours 35 minutes can be written as $\dfrac{31}{12}$ hours. [1]

    **b)** Aisha walks $(x + 3)$ kilometres at $(x + 2)$ km/h and then a further $2x$ kilometres at 3 km/h.

    Show that the total time taken is $\dfrac{2x^2+7x+9}{3x+6}$ hours. [2]

    **c)** If the total time to complete the walk is 2 hours 35 minutes, work out $x$. [2]

    **d)** Calculate Aisha's average speed, in kilometres per hour, for the whole walk, giving your answer correct to 3 s.f. [3]