TEXT BOOKS:

1) Edward Angel: Interactive Computer Graphics A

TEXT BOOKS:

1) Frederick S. Hillier and Gerald J. Lieberman:

Introduction to operations Research: Concepts & cases, 8^{th} Edition, Tata Mc GrawHill, 2005. (chapters: 1, 2, 3.1 to 3.4, 4.1 to 4.8, 5, 6.1 to 6.7, 7.1 to 7.3, 8, 13, 14, 15.1 to 15.4)

REFERENCE BOOKS:

1) Wayne L. Winston: Operations Research Applications and algorithms, 4^{th} Edition, Cengage Learning, 2003.

2) Hamdy A Taha: Operations Research: An Introduction, 8^{th} Edition, pearson Education, 2007

Subject code: 10CS661/10IS661

S.D. Sharma — VTU — Book.
Kalavathy,

UNIT - I

Introduction, Linear programming - I: Introduction: The origin, nature and impact of OR; Defining the problem and gathering data; formulating a Mathematical model; Deriving solutions from the model; Testing the model; Preparing to apply the model; Implementation. Introduction to Linear programming (LPP): prototype example; The (LP) model. — 6 Hours

Introduction to OR:

The origin of OR:-

→ The term 'operations Research' was coined in 1940 by Mc Closky and Trefthen in a small town of Bawdsey in England. It is a science that came into existence in Military context.

→ In order to understand 'What is OR', we must know something of its history and evolution. The main origin of operations Research was during the Second World-War.

→ At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air & land defence of the country. Since they were having very limited military & resources, it was necessary to decide upon the most effective utilization of them eg. the efficient ocean transport, effective bombing etc.

→ During world war II, the military commands of UK & USA engaged inter-disciplinary teams of scientists to undertake scientific research into strategic & tactical military operations.

→ Their mission was to formulate specific proposals and plans for aiding the military commands to arrive at the decisions on optimal utilization of scarce military resources & efforts, & also to implement the decisions effectively.

→ The OR teams were not actually engaged in military operations & in fighting the war. But, they were only advisors and significantly instrumental in winning the war to the extent that the scientific & systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence, OR can be associated with

"an art of winning the war without actually fighting it".

→ As the name implies, 'Operations Research' (OR) was apparently invented because the team was dealing with research on (military) operations. The work of this team of scientists was named as Operational Research in England.

→ The work of OR team was given various names in the United States

- Operational Analysis
- Operations Evaluation
- Operations Research,
- Systems Analysis
- Systems Evaluation
- Systems Research
- Management science
- ~~Systems Analysis~~
- ~~Systems Evaluation~~,
- Systems

→ The success of military teams attracted the attention of industrial managers who were seeking solutions to their complex executive-type problems.

Most common problem was-
What methods should be adopted so that the total cost is minimum or total profit maximum?

The first mathematical technique in this field (the simplex method of linear programming) was developed in 1947 by American mathematician, George B. Dantzig.

→ Today, the impact of OR can be felt in many areas like transportation system, libraries, hospitals, city planning, financial institutions etc.

→ In business & other organizations, OR scientists and specialists always remain engaged in the background. But, they help the top management officials & other line managers in doing their 'fighting' job better.

→ While making use of techniques of OR,
Step1: a mathematical model of the problem is formulated.
~~Step~~1. This model is actually a simplified representation of the problems in which only the most important features are considered for reasons of simplicity.
Step2: Then, an optimal or most favourable solution is found. Since the model is an idealized instead of exact representation of real problem, the optimal solution thus obtained may not prove to be the best solution to the actual problem. Although, extremely accurate but highly complex mathematical models can be developed.

The Nature & Meaning of 'OR'

(1) OR is a Scientific method of providing executive departments with a quantitative basis for decision regarding the operation under their control.

— Morse & Kimbal (1946)

(2) OR is a scientific method of providing executive departments with an analytical & objective basis for decisions

— P.M.S. Blackett (1948)

(3) OR can be considered to be an attempt to study those operations of modern society which involved organizations of men or of men and machines.

— P.M. Morse (1948)

(4) OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problem

— Churchman, Acoff, Arnoff (1957)

(5) OR is the art of giving bad answers to problems to which otherwise worse answers are given

— T.L. Saaty (1958)

(6) OR is a management activity pursued in two complementary ways - one half by the free & bold exercise of commonsense untrammelled by any routine, & other half by the application of a repertoire of well established precreated methods & techniques.

(7) OR is the attack of modern methods on complex problems arising in the direction & management to large systems of men, machines, materials, & money in industry, business & defence.

(8) OR is actually winning war without actually fighting it.

(9) OR is an applied decision theory. It uses any scientific mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a through going 'rationality' in dealing with his decision problems.

— Miller & Starr

(10) OR is a Scientific approach to problem solving for executive management.

— H.M. Wagner.

(11) OR is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the Scientific method of analysis.

— C. Kittel

(12) OR is the Systematic method oriented study of the basic structure, characteristics, functions & relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making

— E.L. Arnoff & M.J. Netzorg

(13) OR is the application of scientific methods to problems arising from operations involving integrated systems of provide the managers of such systems with optimum operating solutions.

— Fabrycky & Torgersen

14) OR is an experimental & applied science devoted to observing, understanding & predicting the behaviour of purposeful man-machine systems & OR workers are actively engaged ~~by~~ in applying this knowledge to practical problems in business, government & society.

— OR Society of America

15) OR is the application of scientific method by inter-disciplinary teams to problems involving the control of organized (man-machine) systems so as to provide solutions which best serve the purpose of the organization as a whole.

— Ackoff & Sasieni, (1968)

16) OR utilizes the planned approach (updated scientific method) & an inter-disciplinary team in order to represent complex functional relationships as mathematical models for purpose of providing a quantitative basis ~~of~~ for decision making & uncovering new problems for quantitative analysis

— Thieart & Klekamp (1975)

(17) OR is a "scientific approach to decision making, which seeks to determine how best to design and operate a system, under conditions requiring the allocation of ~~scarce~~ scarce resources"

Comments on Definitions of OR:

A decision, which taking into account all the present circumstances can be considered the best one, is called an optimal decision.

The three main ~~reas~~ reasons for why most of the decisions of OR are not satisfactory.

(1) OR is not a science like any well-defined physical, biological, social phenomena.

While chemists know about atoms and molecules & have theories about their interactions; & biologists know about living organisms & have theories about vital process, OR do not claim to know or have theories about operations.

∴ OR is not a scientific research into the control of operations. It is essentially a collection of mathematical techniques & tools which is conjunction with a system approach are applied to solve practical decision problems of an economic or engineering nature. Thus, it is very difficult to define OR precisely.

(2) OR is inherently inter-disciplinary in nature with applications not only in military & business but also in medicine, engineering, physics & so on.

Operations research makes use of experience & expertise of people from different disciplines for developing new methods & procedures. Thus, inter-disciplinary approach are applied to solve practical decision problems of an economic or engineering nature.

Thus it is very difficult to define operations research precisely.

(3) Most of the definitions of OR have been offered at different times of development of 'OR' & hence are bound to emphasise its only one or the other aspect.

Management Applications of operations research

Some of areas of management decision making, where the tools & techniques of OR is applied

1) Finance - Budgeting & Investments

(i) cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.

(ii) credit policies, Credit risks, & delinquent account procedures.

(iii) claim & complaint ~~to~~ procedures.

(2) production Management

(i) Physical Distribution

(a) Location & size of warehouses, distribution centres & retail outlets.

(b) Distribution policy.

(ii) Facility planning

(a) Numbers & location of factories, warehouses, hospitals, etc.

(b) loading and unloading facilities for railroads and trucks determining the transport schedule.

(iii) Manufacturing

(a) production scheduling & sequencing.

(b) Stabilization of production & employment training, layoffs & optimum product ~~m~~ mix.

(iv) Maintenance & project Scheduling

(a) Maintenance policies and preventive maintenance.

(b) Maintenance crew sizes.

(c) project scheduling & allocating of resources.

Research and Development

(i) Determination of the areas of concentration of research and development.

(ii) project selection

(iii) Determination of time cost trade-off and control of development projects.

(iv) Reliability & alternative design.

From all above areas of applications, we may conclude that OR can be widely used in taking timely management decisions and also used as a corrective measure.

Impact of operation research

→ OR has made a significant contribution to increasing the productivity of economics of various countries.

The Nature of operations research

As name implies, operations research involves "research on operations",

Thus operations research is applied to problems that concern how to conduct and coordinate the operations within an organization.

→ The research part of the name means that operations research uses an approach that resembles the way research is conducted in established scientific fields. To a considerable extent, the scientific method is used to investigate the problem of concern.

In particular, the process begins by

~~Step~~ * observing & formulating problem, including gathering all relevant data.

* Next to construct a scientific model that attempts to abstract the essence of real problem.
* It is then hypothesized that this model is a sufficiently precise representation of the essential features of the situation that the solutions obtained from the models are also valid for the real problem.

* Next, suitable experiments are conducted to test this hypothesis, modify it as needed, & eventually verify some form of the ~~hyp~~ hypothesis.

Thus, in a certain sense, operations research involves creative scientific research into the fundamental properties of operations.

1) → OR is concerned with practical management of organization.
2) OR also provide positive, understandable conclusions to the decision makers when they are needed.
3) Another characteristics of OR is its broad viewpoint. OR adopts an organizational point of view. Thus, it attempts to resolve the conflicts of interest among the components of the organization in a way that is the best for the organization as a whole. This does not imply that the study of each problem must give explicit consideration to all aspects of the organization; rather, the objectives being sought must be consistent with those of overall organization.
4) additional characteristics is that OR frequently attempts to search for a best solution rather than simply improving the status quo. "Search for optimality".

(5) All these characteristics lead quite naturally i.e., It is evident that no single individual should be expected to be an expert on all the many aspects of OR work or the problems typically considered; this would require a group of individuals having diverse backgrounds & skills.

∴ Full-fledged OR study of a new problem is undertaken, it is usually necessary to use to team approach.

Impact of OR

OR has made a significant contribution to increasing the productivity of economics of various countries.

There, now are a few dozen member countries in the International federation of OR Societies (IFORS) with each country having a national OR societies.

Both Europe & Asia have federations of OR Societies to co-ordinate holding international conference & publishing international journals in those continents. The Institute for Oprn Research & the management science (INFORMS) is an international OR society.

Wide application of OR, we list some actual award winning appl^n

Organizations	Nature of appl^n	Year of pub	Annual savings
1) Netherlands	~~Developmental~~ national water management policy	1985	$15M
2) United ~~Airal~~ Airlines	Reservation offices & airports to meet customer needs with min costs	1986	$6M

Main characteristics (Features) of OR

The main characteristics of OR are as follows:

(1) Inter-disciplinary team approach: In OR, the maximum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering physics etc.

Eg: An OR team required for a big organization may include a statistician, an economist, a mathematician, one or more engineers, a psychologist, and some supporting staff like computer programmers etc.

A mathematician or a probabilist can apply his tools in a plant problem only if he gets to understand some of the physician implications of the plant from an engineer. Otherwise, he may give such solution which may not be possible to apply.

(2) Wholistic approach to the system: The most of the problems tackled by OR have the characteristic that OR tries to find best (optimum) decisions relative to largest possible portion of the total organization.

Eg: in attempting to solve a maintenance problem in a factory, OR tries to consider how ~~it~~ this affects the production department and the business as a whole. It may even try to go further & investigate how the effect on this particular business organization in turn affects the industry as a whole, etc.

Thus, OR attempts to consider inter-actions or chain of effects as far out as these effects are significant.

(3) Imperfectness of solutions: By OR Techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.

(4) Use of Scientific research: OR uses techniques of scientific research to reach the optimum solution.

(5) To optimize the total ~~Outcom~~ Output: OR tries to optimize total return by maximizing the profit & minimizing the cost or loss.

Main phases of operations research study

The procedure for an OR study generally involves the following major phases.

Phase 1. Formulating the problem: Before preceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model. To do so, the following information will be required.

(i) Who has to take the decision?
(ii) What are the objectives?
(iii) What are the ranges of controlled variables?
(iv) What are the uncontrolled variables that may affect the possible solutions?
(v) What are the restrictions or constraints on the variables?

+ wrong formulation cannot yield a right decision, one must be considerably careful while execution this phase.

Phase 2: Constructing a mathematical model.

The second phase of the investigations is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis.

The most suitable form for this purpose is to construct a mathematical model representing the system under study.

It requires the identification of both static & dynamic structural elements.

A mathematical model should include the following three important basic factors:

(i) Decision variables and parameters
(ii) Constraints or restrictions
(iii) Objective function

Phase 3: Deriving the solutions from the model.

This phase is devoted to the computation of those values of decision variables that maximize (or minimize) the objective function. Such solution is called an optimal solution which is always in the best interest of the problem under consideration.

The general techniques for deriving the solution of OR model are discussed in the following sections & further details are given in the text.

Phase 4: Testing & model its solution (updating the model)

After completing the model, it is once again tested as a whole for the errors if any.

A model may be said to be valid if it can provide a reliable prediction of the system's performance.

A good practitioner of OR realises the model be applicable for a longer time & thus he updates the model time to time by taking into account the ... problem

Phase 5. Controlling the Solution: This phase establishes controls over the solution with any ~~dre~~ degree of satisfaction. The model requires immediate modification as soon as the controlled variables change significantly, otherwise the model goes out of control.

As the conditions are constantly changing in the world, the model & the solution may not ~~return~~ remain valid for a long time.

Phase 6. Implementing the Solution:

Finally tested results of the model are implemented to work. This phase is primarily executed with the cooperation of OR experts & those who are responsible for managing & operating the systems.

Scope of operations research

OR has entered successfully many different areas of research for military, government & industry.

The basic problem in most of the developing countries in Asia & Africa is to remove the poverty & hunger as quickly as possible. So there is a great scope for economists, statisticians, administrators, politicians & the technicians working in a team to solve this problem by OR approach.

Besides OR is useful in various important fields.

(1)

(i) In Agriculture: With the explosion of population and consequent shortage of food, every country is facing the problem of —

(i) optimum allocation of land to various crops in accordance with the climatic condition

(ii) optimum distribution of water from various resources like canal for irrigation purposes.

Thus, there is a need of determining best policies under the prescribed restrictions. ~~Beside this, OR is useful in the following~~ Hence a good amount of work can be done in this direction.

(ii) In Production management: A production manager can use OR techniques:

(i) to find out the number & size of the items to be produced.

(ii) in scheduling and sequencing the production run by proper allocation of machines.

(iii) in calculating the optimum product mix; &

(iv) to select, locate, & design the sites for the production plants.

(iii) In LIC: OR approach is also applicable to enable the LIC offices to decide:

(i) what should ~~to~~ be the premium rates for various modes of policies

(ii) how best the ~~app~~ profits could be ~~distr~~ distributed in the cases of with profit policies? etc

Role of Operations research in ~~decision making~~

The six important steps in OR study, but in all each & every step does not necessarily follow logical order as below:

Step 1: Observing the problem Environment

The activities in this step are visits, conferences, observations, research etc.; with such activities analyst gets sufficient information & support to define the problem.

Step 2: Analysing & defining the problem

In this step the problem is defined, & objectives and limitations of the study are stated in its context.

Step 3: Developing a Model

is to construct a model. A model is representation of some real or abstract situations.

OR models are basically mathematical models representing systems, process or environment in the form of equations, relationships or formulae.

The activities in this step are ~~analysing~~ defining interrelationships among variables, formulating equations, using known OR Models or searching suitable alternate models.

The proposed model may be practically tested & modified in order to work under given environmental constraints.

Step 4: Selecting Appropriate data Input

No model will work appropriately if data input is not appropriate.

Important activities in this step are analysing internal - external data and facts, collecting opinions and using computer data banks.

Step 5: Providing a solution with the help of model & data input. Such a solution is not implemented immediately. First it is tested & to have sufficient input to operate & test the models.

The end result is solution that supports current organization objective.

Step 6: Implementing the Solution

Implementation of the solution is the last step of OR
→ In O.R. the decision making is scientific, but implementation of decision involves many behavioural issues.

∴ The implementing authority has to resolve the behavioural issues.

$\boxed{\text{Observation}}$
↓
$\boxed{\text{Analysing \& Defining the Problem}}$
↓
$\boxed{\text{Constructing a Model}}$
↓
$\boxed{\text{Selecting data input}}$
↓
$\boxed{\text{Obtain Solution \& its testing}}$
↓
$\boxed{\text{Implementing of Solution}}$

Uses and Limitations of OR

Uses:-

(1) It provides a logical & Systematic approach to the problem.

(2) It allows modification of mathematical solutions before they are put of use.

(3) Suggests all the alternative courses of action for the same management.

(4) Helps in finding avenues for new research and improvement in system.

(5) Facilitates improved quality of decision.

(6) Leads to optimum use of manager's production factor.

(7) It makes the overall structure of the production problem more comprehensible and helps in dealing with the problem as a whole.

(8) Aids in preparation of future managers by improving their knowledge & skill.

(9) Indicates the scope as well as limitations of problems.

Limitations:

(1) Models are only idealized representations of reality and cannot be regarded as absolute in any case.

(2) The validity of a model, for a particular situation can be ascertained only by conducting experiments on it.

(3) Mathematical models are applicable to only specific categories of problems as they do not take ~~quan~~ qualitative factor into account.

All influencing factors, which cannot be quantified, find no place in mathematical model.

(4) OR requires huge calculations which cannot be handled manually and require computer, resulting in heavy costs.

(5) As it is a new field, there is a resistance from the employees to the new proposals.

(6) The implementation of O.R mainly depends on the person (or input) who provides the solution, & the person who uses the solution.

Basic points ~~of~~ to Remember

(1) problem solving: The process of identifying a difference between some actual and some desired state of affairs & then taking action to resolve the difference.

(2) Decision making: The process of defining the problem, identifying the alternatives, determining the criteria, evaluating the alternatives, & choosing an alternative.

(3) Constraint: Restrictions or limitations imposed on a problem.

(4) Objective function: A mathematical expression used to represent the criterion for evaluating solutions to a problem.

(5) Controllable input: The decision alternatives or inputs that can be specialised by the decision maker.

(6) Decision Variable: Another term for controllable input.

(7) Uncontrollable input: The environmental factors or inputs that cannot be controlled by the decision maker.

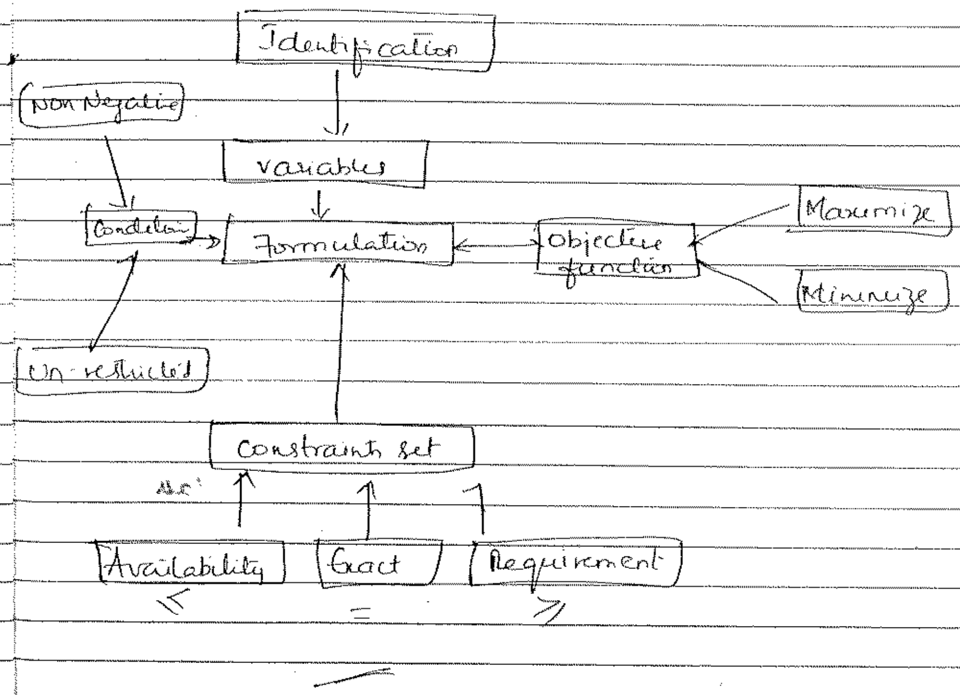
(8)

Introduction to Linear programming

Linear programming:

→ was first conceived by George B Dantzig around 1947. End of II World war.

→ Very historically work on Russian mathematician in 1939.
→ The work of Kantorovich (1939) was published in 1959.
→ Dantzig's first paper was titled "programming in linear structure"
→ Koopmans coined the term "Linear programming" in 1948.
→ Simplex Method was published in 1949 by Dantzig.



(1) Linear programming involves the planning activity to obtain an optimal result, i.e., a result that reaches the specified goal best among all the feasible alternatives.

→ It is one of the most important optimization technique developed in the field of OR. A linear form is meant a mathematical expression of degree 1.
i.e., $c_1x_1 + c_2x_2 \ldots + c_nx_n$
where $c_1, c_2 \ldots \rightarrow$ real nos
x_1, x_2 — are unknown variables.
The term programming refers to the process of determining the solⁿ to the problem.

(2) Linear programming problem consists of optimizing [maximizing or minimizing] a linear function of variables called objective function subjected to a set of linear equations or inequality called constraints.
Eg:- Maximize $Z = 5x_1 - 3x_2 + 4.5x_3 \rightarrow$ objective fuⁿ
subjected to constraints
$2x_1 - 3x_2 + x_3 \leq 1$ ---

Prototype Example

The WYNDOR Glass Co. produces high quality glass products, including windows & glass doors.
It has 3 plants
plant 1: Aluminium frames & hardware
Plant 2: wood frames are made
plant 3: produces the glass & assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

product 1: An 8-foot glass door with aluminium framing.
product 2: A 4 x 6 foot double-hung wood-framed window.

Product 1 requires some of the production capacity in plants 1 & 3, but none in plant 2.

~~Plant~~ Product 2 needs only Plants 2 & 3
The marketing division has concluded that the company could sell as much of either product as could be produced by these plants.
However, because both products would be competing for the same production capacity in plant 3, it is not clear which mix of the two products would be most profitable. ∴ an OR team has been formed to study this question.
The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem.

* Determine what the production rates should be for the two products in order to maximize their total profit, subject to the restrictions imposed by the limited production capacities available in the three plants.
(Each product will be produced in batches of 20, so the production rate is defined as the number of batches

produced per week). Any combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

The OR team also identified the data that needed to be gathered:

(1) Number of hours of production time available per week in each ~~b bad~~ plant for these new products.

(2) Number of hours of production time used in each plant for each batch produced of each new product.

(3) Profit per batch produced of each new product.

Obtaining reasonable estimates of these quantities required enlisting the help of key personnel in various units of company.

→ Staff in the manufacturing division provided the data in the first category.

→ Developing estimates for the second category of data required some analysis by manufacturing engineers involved in designing the production process ~~of~~ for the new products.

→ accounting department developed estimates for third category.

Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours — Product 1	Product 2	Production Time Available per week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

Formulation as linear programming ~~model~~ problem

To formulate the mathematical (linear programming) model for this problem, let

x_1 = number of batches of product 1 produced per week
x_2 = " " " " 2 " " "
Z = total profit per week from producing these two products (in dollars)

∴ x_1 & x_2 are the decision variables for the model.

~~Max~~ $Z = 3x_1 + 5x_2$

i.e. Max $Z = 3x_1 + 5x_2$, subject to the restrictions imposed on their values by the limited production capacities available in the three plants.

→ Table indicates that each batch of product 1 produced per week uses 1 hour of production time per week in plant-1, whereas only 4 hours per week are available. This ~~expression~~ restriction is expressed mathematically by the inequality -

$$x_1 \leq 4$$

III^{ly} plant-2 imposes the restriction

$$2x_2 \leq 12$$

III^{ly} plant-3

$$3x_1 + 2x_2 \leq 18$$

Finally, since production rates cannot be negative, it is necessary to restrict the decision variables to the nonnegative: $x_1 \geq 0$ & $x_2 \geq 0$

∴ To summarize, in the mathematical language of linear programming, the problem is to choose values of x_1 & x_2 as to

$$\text{Maximize } Z = 3x_1 + 5x_2,$$

Subject to restriction

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

&

$$x_1 \geq 0, \ \& \ x_2 \geq 0$$

Example 1: A manufacturer produces two type of models M_1 and M_2. Each model of the type M_1 requires 4 hour of grinding & 2 hours of polishing, whereas each model of the type M_2 requires 2 hours of grinding & 5 hours of polishing. The Manufacturer has 2 grinders & 3 polishers. Each grinder works 40 hours a week & each polisher works for 60 hours a week. Profit on M_1 model is Rs. 3.00 & on model M_2 is Rs. 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate to his production capacity to the two types of models, so that he may make the maximum profit a week?

Soln: Decision Variables:

Let x_1 & x_2 be the no of units of M_1 & M_2 models.

Objective function:- Since the profit on both the models are given, we have to maximize the profit.

	M_1	M_2	Avail	hours a week
G	4	2	2	40
P	2	5	3	60
Profit	3	4		

The profit on Model M_1 is $3x_1$

$$\text{Max } Z = 3x_1 + 4x_2$$

Constraints:- There are two constraints one for grinding and the other for polishing.

- No of hrs available on each grinder for one week is 40

∴ 2 grinders — $2 \times 40 = 80$

Hence, Manufacturer does not have more than $2 \times 40 = 80$ hrs of grinding

M_1 requires 4 hrs of grinding & M_2 requires 2 hours of grinding

$\therefore 4x_1 + 2x_2 \leq 80$

Since 3 polishes, available time for polishing in a week is given by $3 \times 60 = 180$ hrs of polishing.

M_1 require 2 hrs of polishing
M_2 require 5 hrs of polishing

$\therefore 2x_1 + 5x_2 \leq 180$

Finally, we have : Max $Z = 3x_1 + 4x_2$

$4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

You cannot produce -ve quantities for both x_1 & x_2

$x_1, x_2 \geq 0$

Example 2: A firm manufactures two type of products A & B and sells them at a profit of Rs. 2 on type A & Rs. 3 on type B. Each product is processed on two machines G & H. Type A requires one minute of processing time on G and two minutes on H. Type B req one min on G & one min on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hrs during any working day.

Soln: decision variable :

Let x_1 be the no of products of type A.
Let x_2 " " " " type B.

Machine	Time of products (min) Products A (x_1 unit)	B (x_2 unit)	Available time (min)
G	1	1	6 hrs 40 minutes 400 $60 \times 60 = 360$ $+ 40$ $= 400$
H	2	1	10 hrs 600 $10 \times 60 = 600$
Profit	2	3	

Since the profit on type A is Rs. 2 per product

$2x_1$ will be the profit on selling x_1 units of type A.
$|||^{ly}$ $3x_2$ " " " " x_2 " type B.

$\therefore$ Total profit on selling x_1 units of A and x_2 units of B given by

$Z_{max} = 2x_1 + 3x_2$ (Objective function)

Since machine 'G' takes 1 min on type A & 1 min type B
∴ total no of minutes required on M/c G is given by $x_1 + x_2$

Ily the total no. of min required on m/c H is given by $2x_1 + x_2$

But m/c 'G' is not available for more than 6 hour 40 minutes : 400 min

$\therefore x_1 + x_2 \le 400$ (first constraint)

Ily M/c 'H' is not available for 10 hours only

$\therefore 2x_1 + x_2 \le 600$ (2^{nd} Constraint)

Since, it is not possible to produce negative quantities.

$x_1 \ge 0$ & $x_2 \ge 0$ (non-negativity restriction)

Hence allocation problem of the firm can be finally put in the form,
find x_1 & x_2 such that the profit

Max $Z = 2x_1 + 3x_2$ is
subject to the condition:

$x_1 + x_2 \le 400$
$2x_1 + x_2 \le 600$
$x_1 \ge 0,\ x_2 \ge 0$

Terminology :

→ The problem variables x_1 & x_2 are called decision variables & they represent the Solution or the ~~Outp~~ Output decision from the problem (x_j where $j = 1, 2 \dots n$)

→ The profit function that the manufacturer wishes to increase, represents the objective of making the decisions on the production quantities & is called objective function.

$Z = f(x_1, x_2 \dots x_n)$

→ The conditions matching the resource availability and resource requirement are called constraints

Subject to $(x_1, x_2 \dots x_n) \le, =, \ge$
constraints $x_1, x_2 \dots x_n \ge 0$ are called the non negative restrictions

→ We have also explicitly stated that the decision variable should take non-negative values.
- This is true for all linear programming problem
- " " called non-negative restriction.

→ The problem that we have written down in algebraic form represents the mathematical model of the given System & is called problem formulation

→ The problem formulation has the following steps
→ Identifying the decision variables
→ writing the objective function
→ writing the constraints.
→ writing the non negativity restriction.

→ In the above formulation, the objective function and the constraints are linear.
∴ the model that we formulated is a linear programming problem.

→ A linear programming problem has
→ a linear objective function,
→ Linear constraints and
→ the non negativity constraints on all the decision variables.

(3) The following table gives the data for a problem. Formulate the problem as a LP model.

Raw Materials	Requirement /unit I	II	III	Availability
A	2	3	5	4000
B	4	2	7	6000
Min. demand	200	200	150	
Profit /unit	30	20	50	

Sol^n: Decision variable:
x_1, x_2, x_3

Objective function: $Z: 30x_1 + 20x_2 + 50x_3$

Constraints: $2x_1 + 3x_2 + 5x_3 \leq 4000$
$4x_1 + 2x_2 + 7x_3 \leq 6000$

$x_1 \geq 200, \quad x_2 \geq 200, \quad x_3 \geq 150$

Example (4): A firm manufactures 3 products A, B & C. June/July 2011
The profit are Rs 3/-, Rs. 2/-, Rs. 4/- respectively. The firm has 2 machines G & H and below is the required processing time in minutes for each machine on each product & total M/c available minutes on each machine are given below.

Machine	M/c Minutes required A	B	C	Total M/c minutes available
G	4	3	5	2000
H	2	2	4	2500
profit	3	2	4	

The firm must manufacture atleast 100 A's & 200 B's and 50 C's but not more than 150 A.
Setup LP Model to solve maximize the profit.

Sol^n. Decision Variables.

x_1, x_2, x_3

$100 \le x_1 \le 150$

~~$100 \le x_1$~~ $x_1 \ge 100$

$x_1 \le 150$

Objective function:

$$Z = 3x_1 + 2x_2 + 4x_3$$

Constraints:
$$4x_1 + 3x_2 + 5x_3 \le 2000$$
$$2x_1 + 2x_2 + 4x_3 \le 2500$$

and $100 \le x_1 \le 150$, $x_2 \ge 200$, $x_3 \ge 50$

Example 5: A firm can produce 3 types of cloth say A, B & C. 3 kinds of wool are required for it say Red wool, green wool & blue wool. One unit length of type A cloth needs 3 yards of red wool, 3 yards of blue wool and one unit length of type 'B' cloth needs 3 yards of Red, 2 yards of green & 2 yards of blue wool & type C cloth of one unit needs 5 yards of green, 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, 10 yards of green wool & 15 yards of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs 3/- type 'B' cloth is Rs 5 and type C cloth is Rs 4/-. Form the LPP to max the income.

	Type A	Type B	Type C	Stock available
Red wool	3	3	0	8
green wool	0	2	5	10
blue wool	3	2	4	15
Income per unit	3	5	4	

∴ LPP is max $Z = 3x_1 + 5x_2 + 4x_3$

Subjected to

$$3x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$

& $x_1, x_2, x_3 \ge 0$

(6) June/July 2009

The prime Insurance company is introducing 2 new product lines: special risk insurance & mortgages. The expected profit is \$5/unit on special risk & \$2/unit on mortgage. Mgmt wishes to establish share quotas for the new product lines to maximize total profit. The work requirements are as follows:

Department	work hour / unit Special risk	Mortgage	work hours available
under writing	3	2	2400
Administration	0	1	800
claims	2	0	1200
Profit	5	2	

Solⁿ: Formulate the problem into LPP

Let x_1 be number of units of Special risk
x_2 " " " Mortgages

The work requirement on under writing dept is
$3x_1 + 2x_2 \leq 2400$

The work req. on administration dept is
$x_2 \leq 800$

The work req on claims dept is
$2x_1 \leq 1200$

Objective function is max $Z = 5x_1 + 2x_2$

$\therefore$ LPP is to max $Z = 5x_1 + 2x_2$
Subjected to

$$3x_1 + 2x_2 \leq 2400$$
$$x_2 \leq 800$$
$$2x_1 \leq 1200$$

& $x_1, x_2 \geq 0$

Example 7: A farmer has to plant two kinds of trees p & Q in a land of 4000 sq.m area. Each p tree requires atleast 25 sqm and Q tree requires atleast 40 sq m of land. The annual water requirements of p tree is 30 units & of Q tree is 15 units per tree, while at most 3000 units of water is available. It is also estimated that the ratio of the number Q trees to the number of p trees should not be less than 6/19 & should not be ~~less~~ more than 17/8. The return per tree from p is expected to be one and half times as much as from Q tree. Formulate the problem as a LP model.

Solⁿ: x_1 & x_2 be no of type 'p' & 'Q' trees

	Type P	Type Q	Avail
Area	25	40	4000 (sq m)
water	30	15	3000 (units)
Profit	1.5	1	

Profit -

$Z_{max} = 1.5x_1 + x_2$ (as the profit on type p tree is 1.5 times more than on type Q tree's profit)

Subjected to,

$25x_1 + 40x_2 \leq 4000 \rightarrow$ area constraint
$30x_1 + 15x_2 \leq 3000 \rightarrow$ water "

P.T.O

It is given that ratio of no of trees of type 'a' to that of type 'p' should be in the range of 6/19 to 17/8 (not less than 6/19 but not more than 17/8)

i.e., $17/8 \leq \frac{x_1}{x_2} \geq 6/19$ or

$\frac{x_2}{x_1} \geq \frac{6}{19}$, $\frac{x_2}{x_1} \leq \frac{17}{18}$

Example 8: A farmer has 100 acre farm. He can sell all tomatoes, Lettuce & radishes he can raise. The price he can obtain is ₹1.00 per kg for tomatoes, ₹0.75 a head for lettuce and ₹2.00 per kg for radishes. The average yield per acre is 2000 kg of tomatoes, 3000 heads of lettuce and 1000 kg of radishes. Fertilizer is available at ₹0.50 per kg and the amount required per acre is 100 kg each for the tomatoes & lettuce and 50 kg for radishes. Labour required for sowing, cultivating & harvesting per acre is 5 man-days for tomatoes and radishes & 6 man-days for lettuce. A total of 400 man days of labour are available at ₹ ~~2.00~~ 20.0 per man-day. Formulate this problem as a linear programming model to maximize the farmer's total profit.

Solⁿ:

	Tomatoes	Lettuce	radish	
Farmer				
Prof	1.00	0.75	2.00	
Average yield	2000 kg	3000	1000	
Fertilizer	100 kg	100 kg	50 kg	400 man-days – ₹20
	5 man	6	5	

decision variables

x_1, x_2, x_3.

$2000x_1, 3000x_2, 1000x_3$

∴ The total sales of the farmer will be

Rs. $(1 \times 2000x_1 + 0.75 \times 3000x_2 + 2 \times 1000x_3)$

∴ Fertilizer expenditure will be

Rs. $[0.50\,[100(x_1+x_2) + 50x_3]]$

∴ Labour expenditure will be

$=$ Rs $[20 \times (5x_1 + 6x_2 + 5x_3)]$

∴ farmer's net profit will be

$P =$ total sale (in Rs) – total expenditure (in Rs)

$P = [2000x_1 + 0.75 \times 3000x_2 + 2 \times 1000x_3] -$

$0.50 \times [100(x_1+x_2) + 50x_3] - 20 \times [5x_1 + 6x_2 + 5x_3]$

$P = 1850x_1 + 2080x_2 + 1875x_3$.

Since total area of the farm is restricted to 100 acre,

$x_1 + x_2 + x_3 \leq 100$

Also, the total man-days labour is restricted to 400 man days,

$\therefore 5x_1 + 6x_2 + 5x_3 \leq 400$

P.T.O.

Hence the farmers allocation problem can be find the of values of x_1, x_2, x_3 so as to

max: $p = 1800x_1 + 2080x_2 + 1875x_3$

Subjected to conditions:

$$x_1 + x_2 + x_3 \leq 100$$
$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$x_1, x_2, x_3 \geq 0$$

Example 9: Old hens can be bought at Rs 50/- each but young ones cost Rs 100/- each. The old hens lay 3 eggs/week and young hens 5 eggs/week. Each egg costs Rs 2/-. A hen costs Rs 5/- per week to feed. If a person has only Rs 2,000/- to spend for hens, formulate the problem to decide how many of each kind of hen should be buy? Assume that he cannot house more than 40 hens.

Sol^n:

$x_1, x_2 \Rightarrow$ be the no of old & young hens

No of eggs laid by old hens: 3,

No of eggs laid by young hens: 5.

Total income from the eggs

$(3x_1 + 5x_2) \times 2 = \cancel{6x_1} + 10$

$6x_1 + 10x_2$

$\therefore$ (No of eggs) × Selling price

Feeding cost $(x_1 + x_2) \times 5$

$= 5x_1 + 5x_2$

Income − feeding cost = profit

$\therefore$ profit $= x_1 + 5x_2$ is to be max

$Z_{max} = x_1 + 5x_2$

Sub to the constraints,

$$50x_1 + 100x_2 \leq 2000 \text{ (Budget constraint)}$$
$$x_1 + x_2 \leq 40 \text{ (Housing constraint)}$$
$$x_1, x_2 \geq 0$$

Example 10: A toy company manufactures two types of ~~doo~~ dolls, a basic vector - doll A & a deluxe version - doll B. Each doll of type B takes twice as long to produce as one of type A and the company would have time to make maximum of 2,000 per day. The supply of plastic is sufficient to produce 1,500 doll per day (both A & B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of ₹10/- & ₹18/- per doll on doll A & B respectively, then how many of each doll should be produced per day in order to max. the total profit. Formulate the problem as LPP.

Sol^n: $x_1, x_2 \longrightarrow A, B$

Let the doll A req t hours

doll B req $2t$ hours.

$\therefore$ total time to manufacture x_1 & x_2 dolls should not exceed 2,000t hours.

i.e., $\cancel{t}x_1 + 2tx_2 \leq 2000t$

$t(x_1 + 2x_2) \leq 2000t$

The LPP is

$Z_{max} = 10x_1 + 18x_2$

Subject to constraints

$x_1 + 2x_2 \leq 2000$ (time constraint)

$x_1 + x_2 \leq 1,500$ (plastic supply const)

$x_2 \leq 600$ (fancy dress const)

Graphical procedure: To solve LPP model:

(i) Graphical method
(ii) Simplex method.

1) Consider each inequality constraints as eqn.
2) Plot each equation on the graph, as each one will geometrically represent a st. line.
3) Shade the feasible region, every point on the line will satisfy the eqn of the line.

If the inequality constraint corresponding to that line is '≤' then the region below the line lying in the quadrant is shaded. ~~The points lying in~~

The points lying in with '≥' sign constraint, the region above the line in the first quadrant is shaded.

The points lying in common region will satisfy all the const simultaneously. The common region thus obtained is called the 'feasible region'.

(4) Choose the convenient value of Z (say Z=0) & plot the objective function line.

(5) Pull the objective function line until the extreme points of the feasible region.

In the max. case this line will stop farthest from the origin & passing through at least one ~~corner~~ corner of the feasible region.

In the min. case this line will stop nearest to the origin & passing through at least one corner of the feasible region.

(6) Read the co-ordinates of the extreme points selected in step 5 & find the max or min value of Z.

Graphical method:- provides pictorial representation of the problems & its solutions & which gives the basic for concepts used in solving general LPP.

The redundant constraints are automatically eliminated from the system. Multiple solns, unbounded solutions & infeasible soln get highlighted very clearly in graphical method.

Ex1. Solve the following LPP by graphical method

$Z_{max} = 3x_1 + 4x_2$

subj. to $x_1 + x_2 \le 450$,

$2x_1 + x_2 \le 600$

$x_1, x_2 \ge 0$

Soln: Converting the inequalities into equations we get,

$x_1 + x_2 = 450$

$2x_1 + x_2 = 600$

$x_1 + x_2 = 450$

if $x_2 = 0$

$x_1 = 450$ $(450, 0)$

if $x_1 = 0$

$x_2 = 450$ $(0, 450)$

IIly $2x_1 + x_2 = 600$

if $x_2 = 0$

$2x_1 = 600$

$x_1 = 300$ $(300, 0)$

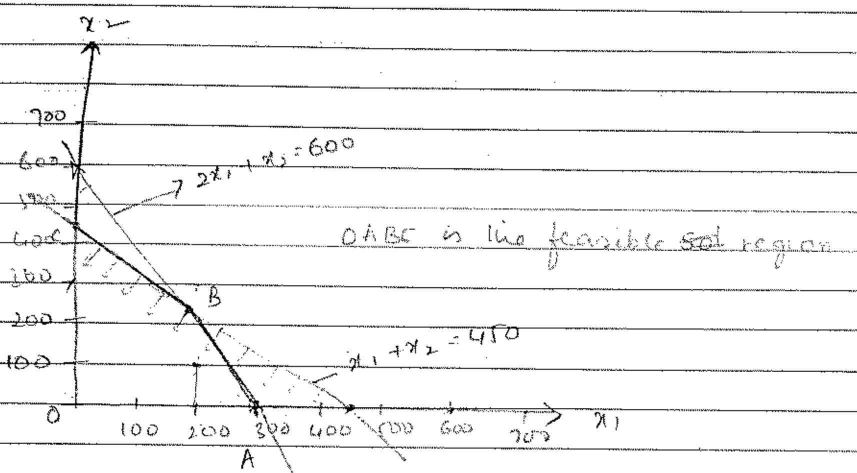
if $x_1 = 0$

or $x_2 = 600$

$x_2 = 600$ $(0, 600)$

plot the co-ordinates on the graph sheet

Mathematically/ graphically any set of values of x_1 & x_2 lying on or below the eq^n or line will satisfy the constraint $\leq$, in other words represent the areas downwards of the constraint is $\leq$ & upwards if the constraint is $\geq$.



Now, the common area shared by considering all these constraints is the feasible region or the solution space (OABC) that is the most common area between upward & downward arrows in the graph.

Using the method of corner points or vertices (which are the bounded points of the region), we can find the value of Z

$Z_{max} = 3x_1 + 4x_2$

$Z_O = 3(0) + 4(0) = 0$

$Z_A = 3(300) + 4(0) = 900$

$Z_B = 3(150) + 4(300) = 450 + 1200 = 1650$

$= 3(0) + 4(450) = 1800$

$(x_1 + x_2 = 450) \times 2$

$2x_1 + x_2 = 600$

$x_1 = 900$

$2x_1 + 2x_2 = 600$

$2x_1 + x_2 = 300$

$x_2 = 300$

$+(x_1 + x_2 = 450) \times 2$

$2x_1 + x_2 = 600$

$x_2 = 300$

$x_1 + x_2 = 450$

$\boxed{x_1 = 150}$

Z_{max} occurs at C,

$\therefore$ the value $= 1800$

Hence the corresponding co-ordinates of x_1, x_2

$x_1 = 0,\ x_2 = 450$

* $Z = 3x_1 + 4x_2$

Here the problem is to find the point or points in the feasible region which max the objective fun

$\therefore$ For some fixed value of Z, $Z = 3x_1 + 4x_2$ is a straight line and any point on it gives the same value of Z.

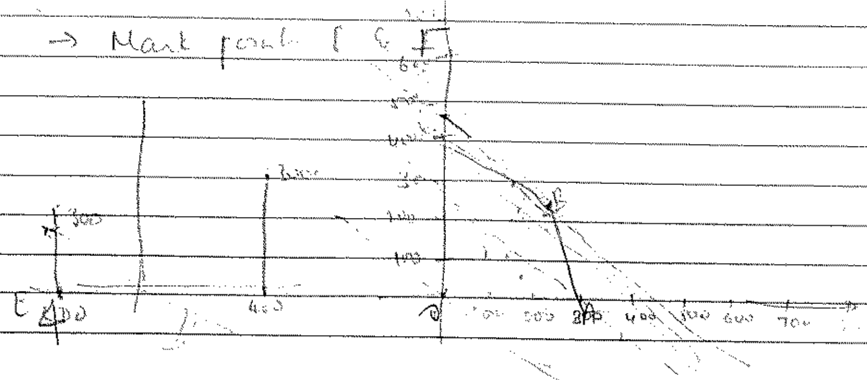
Also, it should be noted that the lines corresponding to different values of Z are parallel $\because$ the gradient ~~(-3/4)~~ of $(-4/3)$ of the line $Z = 3x_1 + 4x_2$ remains the same throught

For $Z = 0$ i.e. $0 = 3x_1 + 4x_2$ means a line which passes through the origin

→ To draw the line $3x_1 + 4x_2$ … determine the value x_1: -4, -400

x_2: 3, 300

→ Mark point (& F)



→ Joining the point F & O, draw the line $3x_1+4x_2=0$

→ Now go on drawing the lines parallel to this until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the max value of Z is attained.

→ It is also possible that such a line may coincide with one of the edge of feasible region.

Note

Note If the no of vertices of feasible region is small find the coordinate ... by solving the pairs of lines whose intersections are their points respectively.

Ex 2: Solve the following LPP by graphical method.

Minimize $Z = 20x_1 + 10x_2$

Subjected to $x_1 + 2x_2 \le 40$

$3x_1 + x_2 \ge 30$

$4x_1 + 3x_2 \ge 60$

$x_1, x_2 \ge 0$

Sol^n: Replace all inequalities of the constraints by eq^n

$x_1 + 2x_2 = 40$

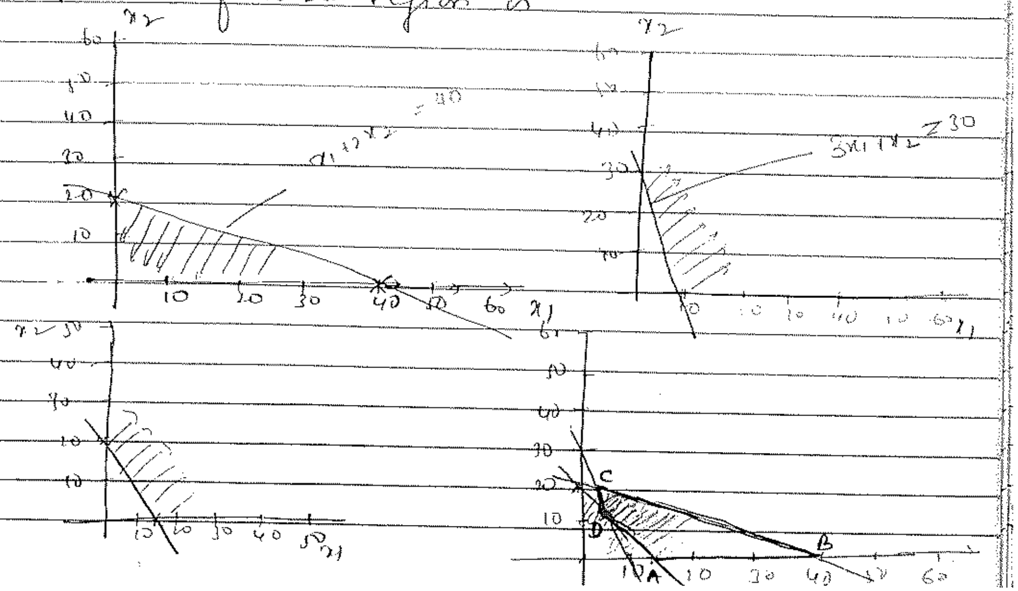
if $x_1 = 0$, $x_2 = 20$ if $x_2 = 0$, $x_1 = 40$

∴ $x_1+2x_2 = 40$ passes through $(0,20)$ $(40,0) = (40,20)$

$3x_1 + x_2 = 30$ " " $(0,30)$ $(10,0) = (10,30)$

$4x_1 + 3x_2 = 60$ " " $(0,20)$ $(15,0) = (15,20)$

∴ The feasible region is

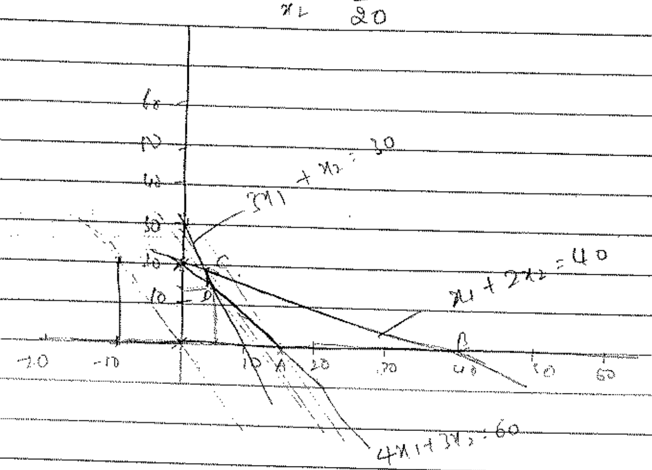


$Z = 20x_1 + 10x_2$

$Z = 0$

$20x_1 + 10x_2 = 0$ $\quad 20x_1 = -10x_2$

$\frac{x_1}{x_2} = \frac{-10}{20}$



To solve $x_1 + 2x_2 = 40$

$3x_1 + x_2 = 30$

$4x_1 + 3x_2 = 60$

Co-ordinates of A (15, 0)

B (40, 0)

C (4, 18)

D (6, 12)

To find coordinates of C & D

At point 'C' $(x_1 + 2x_2 = 40) \times 3$

$3x_1 + x_2 = 30$

$3x_1 + 6x_2 = 120$

$\cancel{3x_1} + x_2 = 30$

$5x_2 = 90$

$\therefore \boxed{x_2 = 18}$ $\quad$ (4, 18)

$x_1 + 2x_2 = 40$

At point 'D'

$4x_1 + 3x_2 = 60$

$(3x_1 + x_2 = 30) \times 3$

$9x_1 + 3x_2 = 90$

$4x_1 + \cancel{3x_2} = 60$

$5x_1 = 30$

$\boxed{x_1 = 6}$

Corner points	Value of Z
	$Z = 20x_1 + 10x_2$
A(15, 0)	$Z = 20\times15 + 10\times0$, $Z = 300$
B(40, 0)	$Z = 20\times40 + 10\times0$, $Z = 800$
C(4, 18)	$Z = 20\times4 + 10\times18$, $Z = 260$
D(6, 12)	$Z = 20\times6 + 10\times12$, $\boxed{Z = 240}$ → Min value

$\therefore$ The minimum value of 'Z' occurs at D(6, 12).

Hence the optimal soln is

$\boxed{x_1 = 6,\ x_2 = 12}$

Example (3) Graphical Method to solve the LPP

Max $Z = 6x_1 + 4x_2$

Subject to $-2x_1 + x_2 \le 2$

$x_1 - x_2 \ge 2$

$3x_1 + 2x_2 \le 9$

$x_1, x_2 \ge 0$

$-2x_1 + x_2 = 2$

if $x_1 = 0$ $x_2 = 2$ (0, 2)

$x_2 = 0$ $x_1 = -1$ (−1, 0)

(0, 2) (−1, 2)

$x_1 - x_2 = 2$

if $x_1 = 0$ (0, −2)

$x_2 = -2$

if $x_2 = 0$ (2, 0)

$x_1 = 2$ $\quad$ (−2, 2)

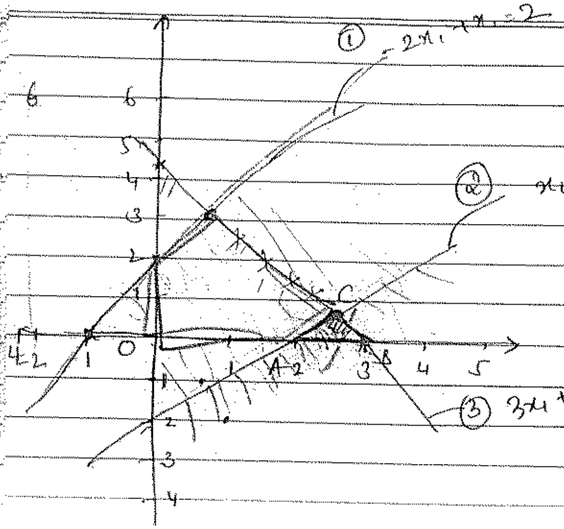
$3x_1 + 2x_2 = 9$

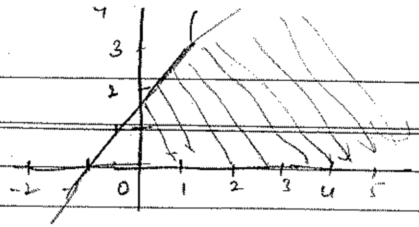
if $x_1 = 0$

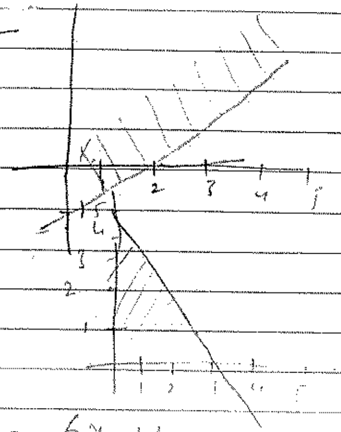
$x_2 = 9/2$ (0, 9/2)

if $x_2 = 0$

$x_1 = 9/3 = 3$ (3, 0)







$\textcircled{0}$. $6x_1 + 4x_2$

$6x_1 = -4x_2$

$\frac{x_1}{x_2} = \frac{-4}{6}$

Feasible region is given by ABC

We know the corner point A & B

A (2, 0)
B (3, 0)

At point 'C'

$3x_1 + 2x_2 = 9$ $\Rightarrow$ $3x_1 + 2x_2 = 9$

$(x_1 - x_2 = 2)\,2$ $\quad$ $2x_1 - 2x_2 = 4$

$5x_1 = 13$

$x_1 = 13/5 = 2.6$

$2.6 - x_2 = 2$

$-x_2 = 2 - 2.6$

$x_2 = 0.6$

C (2.6, 0.6)

Value of Z

$Z = 6x_1 + 4x_2$

at A (2, 0) $Z = 6(2) + 4(0)$
$= 12$

B (3, 0) $- Z = 6(3) + 4(0)$
$= 18$

C (2.6, 0.6) $, Z = 6(2.6) + 4(0.6)$
$= 15.6 + 2.4$
$= 18$

2.6×6
15.6

Example ④ ~~Minimization~~ unbound problem

8. Use graphical method to solve the LPP

Minimize $Z = 3x_1 + 2x_2$

Subject to

$5x_1 + x_2 \geq 10$

$x_1 + x_2 \geq 6$

$x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

① $5x_1 + x_2 = 10$

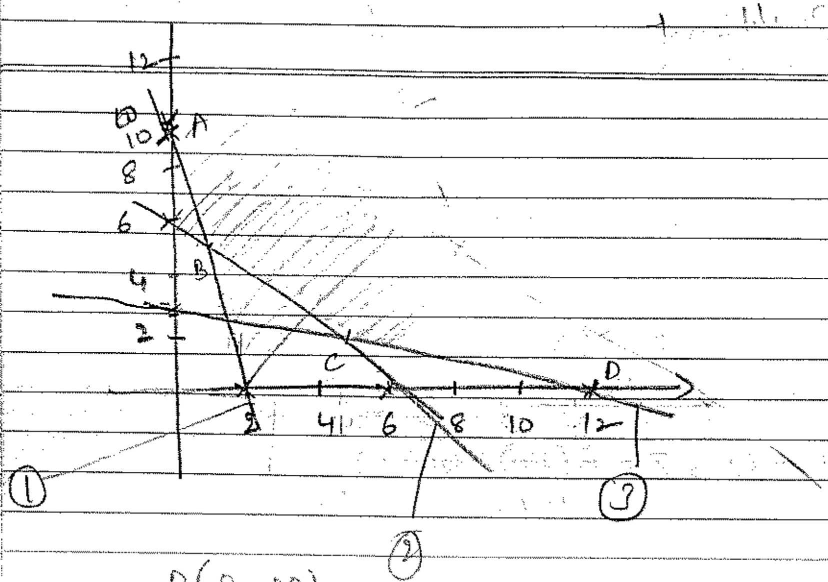
$x_1 = 0$ $\quad$ $x_2 = 0$
$x_2 = 10$ $\quad$ $x_1 = 2$
(0, 10) $\quad$ (2, 0)
(2, 10)

② $x_1 + x_2 = 6$

if $x_1 = 0$ (0, 6)
$x_2 = 6$
$x_2 = 0$ (6, 0)
$x_1 = 6$
(6, 6)

③ $x_1 + 4x_2 = 12$

if $x_1 = 0$
$x_2 = 3$
if $x_2 = 0$
$x_1 = 12$
(12, 3)



D(0, 10)
C(0, 12)

at point 'B'

(1) & (2)

$+5x_1 + x_2 = 10$

$+x_1 + x_2 = 6$

(−) (−) (−)

$4x_1 = 4$

$\boxed{x_1 = 1}$

$\boxed{x_2 = 5}$

at point 'C'

(2) & (3)

$x_1 + 4x_2 = 12$

$x_1 + x_2 = 6$

$3x_2 = 6$

$\boxed{x_2 = 2}$

$\boxed{x_1 = 4}$

Value of $Z = 3x_1 + 2x_2$

B(1,5) $Z = 3(1) + 2(5) = 3 + 10 = 13$ → Minimum

C(4,2) $Z = 3(4) + 2(2) = 12 + 4 = 16$

D(12,0) $Z = 3(12) + 2(0) = 36 + 0 = 36$

A(0,10) $Z = 3(0) + 2(10) = 0 + 20 = 20$

Corner points	value of $Z = 3x_1 + 2x_2$
A(0,10)	20
B(1,5)	13 (Minimum value)
C(4,2)	16
D(12,0)	36

Since the min value is attained at B(1,5), the optimal solution is

$$\boxed{x_1 = 1,\ x_2 = 5}$$

NOTE :- In the above problem if the objective function is maximization then solution is unbounded, as max value of Z occurs at infinity.

(5) Solution for Prototype Example problem using Graphical method

Solution

Max $Z = 3x_1 + 5x_2$

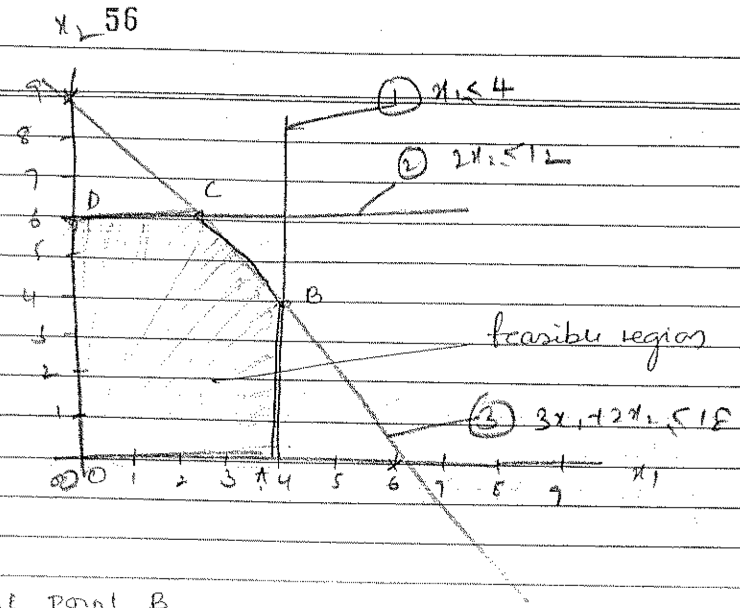
$x_1 \le 4$ —— (1)

$2x_2 \le 12$ —— (2)

$3x_1 + 2x_2 \le 18$ —— (3)

$x_1, x_2 \ge 0$

$x_1 = 4$	$2x_2 = 12$	$3x_1 + 2x_2 = 18$
$\frac{x_1}{4} = 1$	$x_2 = 6$	$x_1 = 0$ $x_2 = 0$
	$\frac{x_2}{6} = 1$	$x_2 = 9$ $x_1 = 6$
		(0,9) (6,0)



At point B

$3x_1 + 2x_2 \le 18$ $\qquad$ $3x_1 + 2x_2 = 18$

$x_1 \le 4$ $\qquad$ $x_1 = 4$ $\quad 2x_2 = 18 - 12$

$x_2 = 6/2 = 3$

$(4,3)$

At point C

$3x_1 + 2x_2 = 18 \Rightarrow 3x_1 + 12 = 18$

$2x_2 = 12$ $\qquad$ $x_1 = 6/3 = 2$

$x_2 = 6$

$(2,6)$

Value of Z $\quad Z = 3x_1 + 5x_2$

At point A $(4,0)$; $3(4) + 0 = 12$

B $(4,3)$; $3(4) + 5(3) = 12 + 15 = 27$

C $(2,6)$; $3(2) + 5(6) = 6 + 30 = \boxed{36}$

D $(0,6)$; $3(0) + 5(6) = 30$

Z is max at C $(2,6)$

Example 6: $Z_{max} = 3x_1 + 4x_2$

Subjected to $5x_1 + 4x_2 \le 200$

$3x_1 + 5x_2 \le 150$

$5x_1 + 4x_2 \ge 100$

$8x_1 + 4x_2 \ge 80$

$x_1, x_2 \ge 0$

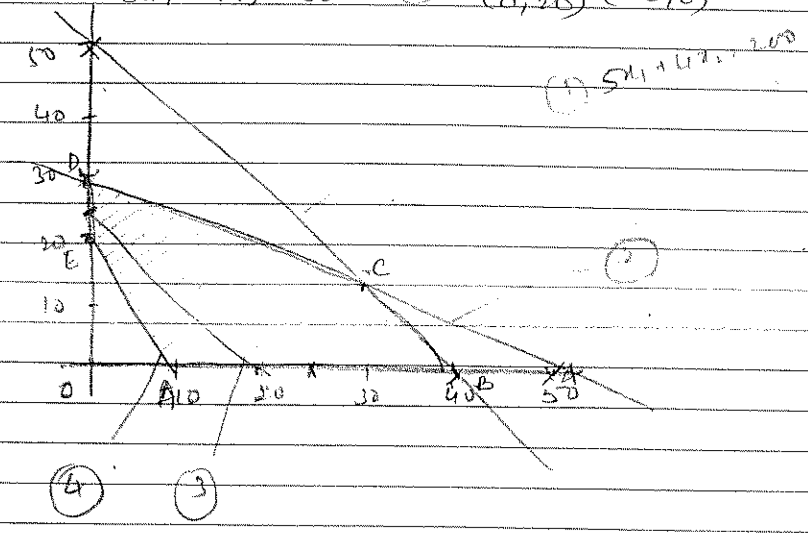
Sol^n. Converting the inequalities into eq^n

$5x_1 + 4x_2 = 200$ — ① $\quad (0,50)\,(40,0)$

$3x_1 + 5x_2 = 150$ — ② $\quad (0,30)\,(50,0)$

$5x_1 + 4x_2 = 100$ — ③ $\quad (0,25)\,(20,0)$

$8x_1 + 4x_2 = 80$ — ④ $\quad (0,20)\,(10,0)$

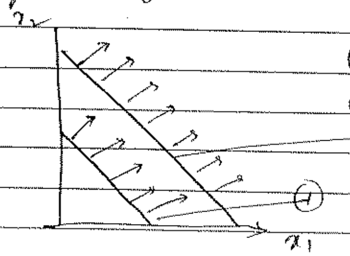


Graphical Solution in Some Exceptional cases

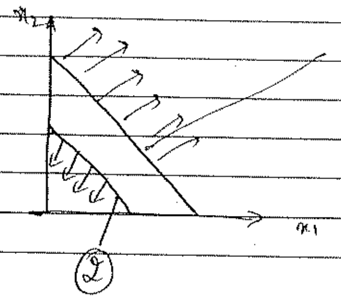
(1) A unique optimal solⁿ
(2) Multiple optimal solⁿ
(3) Unbounded solⁿ
(4) Infeasible solⁿ
(5) Redundancy

(2) Multiple optimal solⁿ. In certain cases a given LP problem may have more than one optimal solution yielding the same objective function value.

(3) Unbounded solⁿ. If an LPP has no limit on constraints then it is stated as unbounded LPP. The common feasible region is not bounded by the given constraint as represented below.



(4) Infeasible solution: If it is not possible to find a feasible solⁿ satisfying all the constraints, then LPP is said to have an infeasible solⁿ as represented below.



① problem having unbounded solution for Max

Max $Z = 3x_1 + 2x_2$

Subject to

$x_1 - x_2 \leq 1$

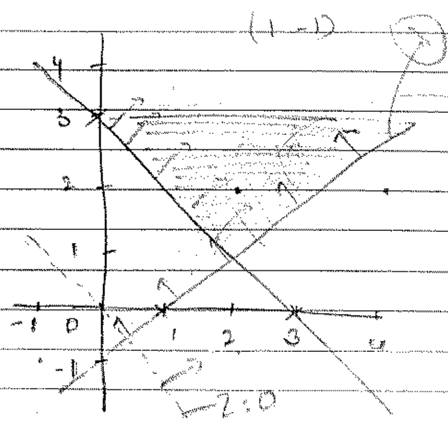
$x_1 + x_2 \geq 3$

& $x_1, x_2 \geq 0$

Solⁿ: $x_1 - x_2 = 1$	$x_1 + x_2 = 3$
if $x_1 = 0$, $x_2 = -1$ $(0,-1)$	if $x_1 = 0$, $x_2 = 3$ $(0,3)$
if $x_2 = 0$, $x_1 = 1$ $(1,0)$	if $x_2 = 0$, $x_1 = 3$ $(3,0)$

(1,-1)

(3,3)



feasible Region

No bounding

The region of feasible solution is the shaded area in the figure.

It is clear from the figure that the line representing the objective function can be moved far even parallel to itself in the direction of increasing Z, & still have some points in the region of feasible solutions.

Hence Z can be made arbitrarily large, & the problem has no finite maximum value of Z. Such problems are said to have unbounded solutions.

Infinite profit in practical problem of linear programming cannot be expected. If LP problem has been formulated

② Problem which is not completely normal

Max $Z = -x_1 + 2x_2$

Subject to

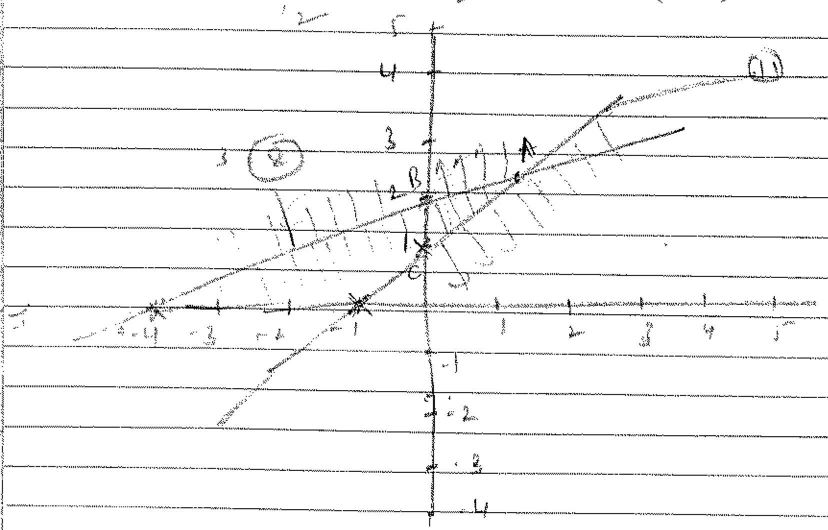
$-x_1 + x_2 \leq 1$

$-x_1 + 2x_2 \leq 4$

& $x_1, x_2 \geq 0$

Sol^n.

$-x_1 + x_2 = 1$	$-x_1 + 2x_2 = 4$
if $x_1 = 0$, $x_2 = 1$ (0,1)	if $x_1 = 0$, $x_2 = 2$ (0,2)
$x_2 = 0$, $x_1 = -1$ (-1,0)	if $x_2 = 0$, $x_1 = -4$ (-4,0)
$(-1,1)$	$(4,2)$



$-x_1 + x_2 = 1$

$-x_1 + 2x_2 = 4$

$(+)\ (-)\ \ (-)$

$-x_2 = -3$

$x_2 = 3$

x_1: $-x_1 + x_2 = 1$

$-x_1 + 3 = 1$

$-x_1 = -2$

$x_1 = 2$

$Z = -x_1 + 2x_2$

$= -2 + 2(3)$

...

Z Max

$C(-1,1)$: $Z = -x_1 + 2x_2 = -(-1) + 2(1) = 3$

$A(4,2)$: $Z = -(-4) + 2(2) = 4 + 4 = \boxed{8}$

③ Problem with inconsistent System of constraints (infeasible Sol^n)

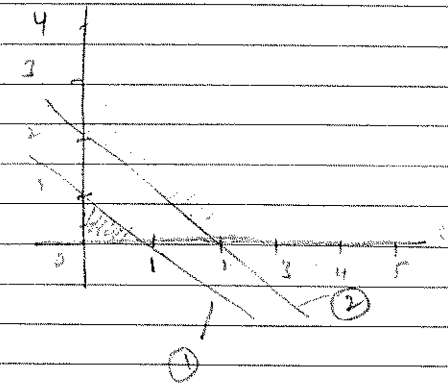
Max $Z = 3x_1 - 2x_2$

Subject to

$x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \geq 4$

$x_1 + x_2 = 1$	$2x_1 + 2x_2 = 4$
if $x_1 = 0$, $x_2 = 1$ (0,1)	if $x_1 = 0$, $x_2 = 2$ (0,2)
if $x_2 = 0$, $x_1 = 1$ (1,0)	$x_2 = 0$, $x_1 = 2$ (2,0)



The problem is represented graphically in the fig. The figure shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence, the problem has no solution because the constraints are inconsistent.

(4) problem in which constraints are equations rather than inequalities

Max $Z = 5x_1 + 3x_2$

Subject to

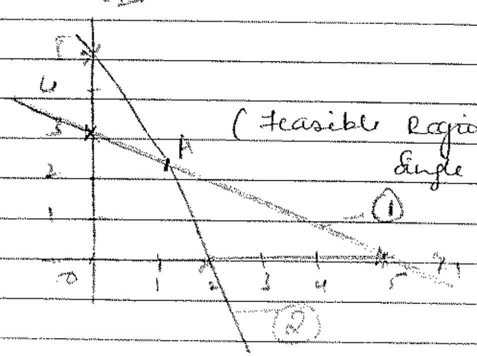
$3x_1 + 5x_2 = 15$ — (1)

$5x_1 + 2x_2 = 10$ — (2)

& $x_1 \geq 0,\ x_2 \geq 0$

Sol^n:

(1) $3x_1 + 5x_2 = 15$	$5x_1 + 2x_2 = 10$
if $x_1 = 0$, $x_2 = 3$ $(0,3)$	if $x_1 = 0$, $x_2 = 5$ $(0,5)$
if $x_2 = 0$, $x_1 = 5$ $(5,0)$	if $x_2 = 0$, $x_1 = 2$ $(2,0)$



(Feasible Region consists of Single point A)

$3x_1 + 5x_2 = 15)\times 2$ $10x_1 + 4x_2 = 20$

$(5x_1 + 2x_2 = 10)\times 2$ $3x$

$3x_1 + 5x_2 = 15 \quad \times 2$ $6x_1 + 10x_2 = 30$

$5x_1 + 2x_2 = 10 \quad \times 5$ $25x_1 + 10x_2 = 50$

$\frac{25}{-6}{19}$

$-19x_1 = -20$

$x_1 = \frac{20}{19}$ $\therefore x_2 = \frac{45}{19}$

Figure shows the graphical solution. Since there is only a single solution point $A(20/19,\ 45/19)$, there is nothing to be maximized. Hence a problem of this kind is of no importance. Such problems can arise only when the number of equations in the constraints is atleast equal to number of variables. If the solution is feasible, it is optimal. If it is not feasible, the problem has no solution.

(5) problem on Multiple Solution:

Max $3x_1 + 3x_2$

Subject to

$4x_1 + 6x_2 \leq 48$

$4x_1 + 2x_2 \leq 12$

$3x_2 \geq 3$

$2x_1 \geq 2$

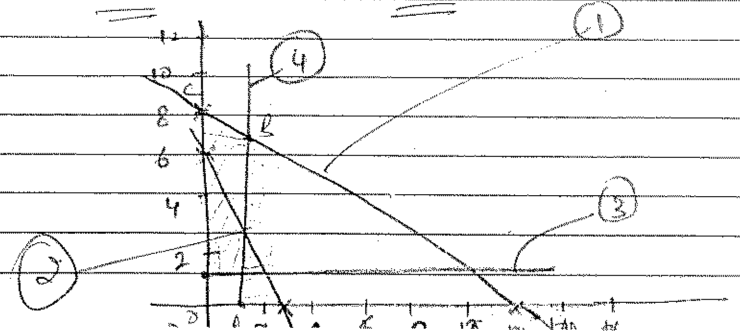
& $x_1, x_2 \geq 0$

(1) $4x_1 + 6x_2 = 48$	(2) $4x_1 + 2x_2 = 12$
if $x_1 = 0$, $x_2 = 8$ $(0,8)$	if $x_1 = 0$, $x_2 = 6$ $(0,6)$
if $x_2 = 0$, $x_1 = 12$ $(12,0)$	if $x_2 = 0$, $x_1 = 3$ $(3,0)$

3,6

(3) $3x_2 = 3$ $\therefore x_2 = 1$

(4) $2x_1 = 2$ $x_1 = 1$



~~Multiple Solution~~

⑥ Solve the LPP by graphical method

Max $Z = 100x_1 + 40x_2$

Subject to constraint

$5x_1 + 2x_2 \le 1000$ — ①

$3x_1 + 2x_2 \le 900$ — ②

$x_1 + 2x_2 \le 500$ — ③

& $x_1 \ge 0,\ x_2 \ge 0$ — ④

TB

①	②	③
$x_1 = 200$ $x_2 = 500$	$x_1 = 300$ $x_2 = 450$	$x_1 = 500$ $x_2 = 250$

Z_{max}

$A(200, 0) = 20{,}000$

~~$B(0, 210) = 10{,}000$~~

$C(0, 210) = 10{,}000$

$D(\quad)$

By solving ② & ③

~~$3x_1 + 2x_2 = 900$~~

~~$(-)\ x_1 + 2x_2 = 500$~~

~~$2x_1 = 400$~~ $x_1 =$

~~$x_1 = 200$~~

~~$x_1 + 2x_2 = 500$~~

~~$200 + 2x_2 = 500$~~

~~$x_2 = \frac{500 - 200}{2} = \frac{300}{2} = 150$~~

~~$Z_{max} = 100x_1 + 40x_2$~~

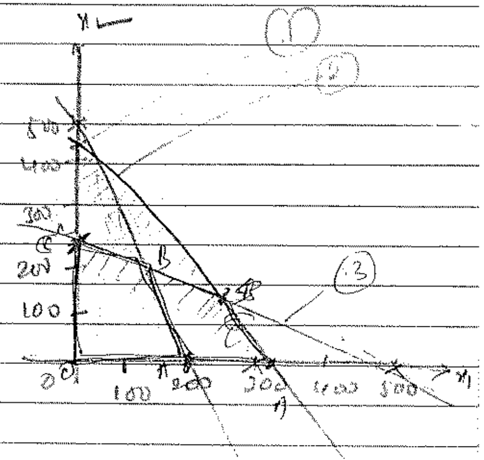
~~$= 100(200) + 40(150)$~~

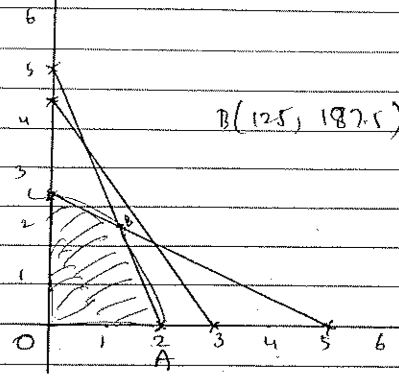
~~$= 20000 + 6000$~~

~~$Z_A = 100x_1 + 40x_2$~~

~~$= 100(300) + 40(0)$~~

~~$= 30000$~~





$B(125, 187.5)$

$O(0,0) = 0$

$A(200, 0) = 20{,}000$

$B(125, 187.5) = 12500 + 7500 = 20{,}000$

$C(250, 0) = 10{,}000$

Alternate optimal solⁿ

(7) XYZ corporation manufactures two electric products air conditioners & large fans. The assembly process for each is similar in that both require a certain amount of wiring and drilling. Each air conditioner takes 3 hours of wiring & 2 hour of drilling. Each fan must go through 2 hours of wiring and 1 hr of drilling. During the next production period, 240 hours of wiring time are available and upto 140 hours of drilling time may be used. Each air conditioner sold yields a profit of Rs. 25. Each fan assembled may be sold for a Rs. 15 profit. Formulate & solve this LP problem production mix situation to find the best combination of air conditioners & fans that yields the ~~heighest~~ highest profit. Use the corner point graphical approach.

Solution:

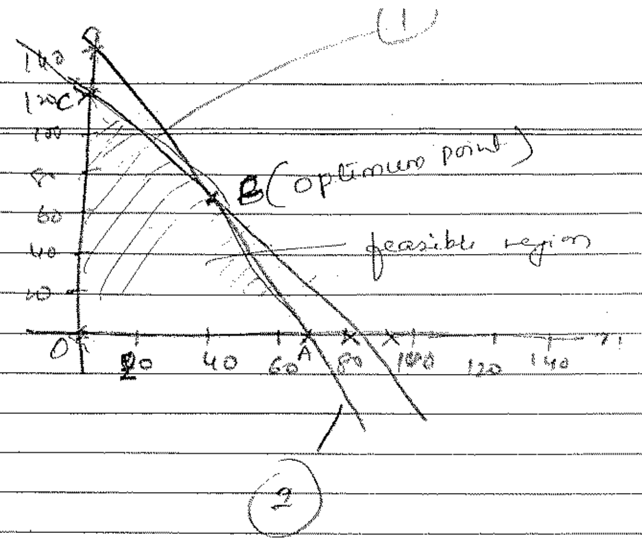
	Air conditioner ~~wiring time~~	large fans ~~Drilling time~~	Total hrs
wiring ~~Air conditioner~~	3	2	~~280~~ 240
drilling ~~large fans~~	2	1	140
~~Total~~			
Profit	25	15	

Max profit $= Z = 25x_1 + 15x_2$

$3x_1 + 2x_2 \leq 240$ — (1)

$2x_1 + x_2 \leq 140$ — (2)

(1) $3x_1 + 2x_2 = 240$		$2x_1 + x_2 = 140$	
if $x_1 = 0$, $x_2 = 120$	~~(0,140)~~	if $x_1 = 0$, $x_2 = 140$	(0,140)
if $x_2 = 0$, $x_1 = 80$	(80,0)	if $x_2 = 0$, $x_1 = 70$	(70,0)



From corner point method:

		Z value $25x_1 + 15x_2$	
~~A(0,70)~~ A(70,0)	=	25(70) + 15(0)	= 1750
C(0,120)	=	25(0) + ~~120~~ 15(120)	= 1800
~~C~~ B(40,60)	=	25(40) + 15(60) = 1000 + 900	= $\boxed{1900}$

at 'B' $3x_1 + 2x_2 = 240$

$(2x_1 + x_2 = 140) \times 2$

$4x_1 + \cancel{2x_2} = 280$

$3x_1 + \cancel{2x_2} = 240$

$x_1 = 40$

$2x_1 + x_2 = 140$

$2(40) + x_2 = 140$

$80 + x_2 = 140$

$x_2 = 140 - 80 = 60$

$\boxed{B(40,60)}$

(8) A dean of the School of Business must plan the schools course offerings for the semester. Student demands deem it necessary to offer atleast 30 under graduates and 20 graduate courses at in the term. Faculty contracts also dictate that atleast 60 courses be offered in total. Each undergraduate course taught costs the college an average of Rs. 2500 in faculty wages, while each graduate course costs Rs. 3000. How many undergraduate and graduate courses should be taught in the semester so that total faculty salaries are kept to a minimum!

Sol:

$x_1 \to$ graduate course

$x_2 \to$ under " "

(1) $x_1 \geq 20$, (2) $x_2 \geq 30$

(3) $x_1 + x_2 \geq 60$

LPP: Min cost = $3000x_1 + 2500x_2$

G.M

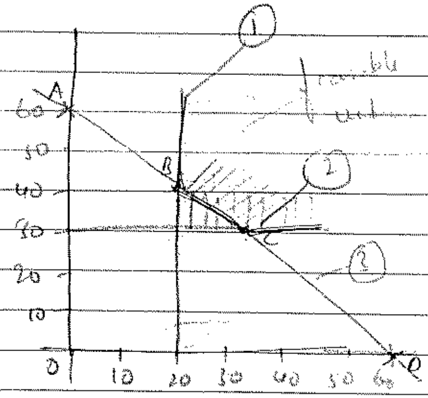
(1) $x_1 = 20$

(2) $x_2 = 30$

(3) $x_1 + x_2 = 60$

if $x_1 = 0$, $x_2 = 60$ $(0, 60)$

if $x_2 = 0$, $x_1 = 60$ $(60, 0)$



Z.

At A(0,60)

B( )

C( )

D(60,0)

At 'B'

$x_1 = 20$

$x_1 + x_2 = 60$

$20 + x_2 = 60$

$x_2 = 60 - 20$

$x_2 = 40$

B(20, 40)

At 'C'

$x_2 = 30$

$x_1 + x_2 = 60$

$x_1 + 30 = 60$

$x_1 = 30$

∴ C(30, 30)

Min value $3000x_1 + 2500(x_2)$

A(0,60) = 0 + 2500(60) = 150,000

B(20,40) = 3000(20) + 2500(40)
= 60000 + 10,0000 = 160000

C(30,30) = 3000(30) + 2500(30)
= 90000 + 75000 = 165,000

D(60,0) = 3000(60) + 0 = 180,000

Minimum cost is at

B(20,40) = 1,60,000

Problem on Redundancy Constraint

Redundancy: The presence of redundant constraints is another common situation that occurs in large linear programming formulations.

Redundancy causes no major difficulties in solving LP programs graphically, but you should be able to identify its occurance.

A redundant constraint is simply one that does not affect the feasible solution region.

In other words, one constraint may be more binding or restrictive than another & there by negate its need to be considered.

Eg: LP problem with three constraints

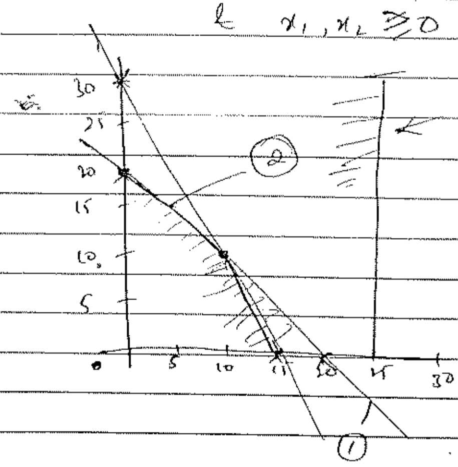
Max profit $= Rs.\ 1 \cdot x_1 + 2x_2$

subject to $x_1 + x_2 \leq 20$ — (1) (20, 20)

$2x_1 + x_2 \leq 30$ — (2) (15, 30)

$x_1 \leq 25$ — (3)

& $x_1, x_2 \geq 0$



Redundant constraint $x_1 \leq 25$

The 3rd constraint is $x_1 \leq 25$ is redundant & unnecessary in the formulation & solution of the problem because it has no effect on the feasible region set from the first two more restrictive constraints.

Slack and Surplus Variables

The inequalities encountered in linear programming problems are of the 'less-than-or equal to' or 'greater-than-or-equal-to' type.

For convenience, the constraints are to formulated that all the inequalities are of the $\leq$ type. If, in the original formulation, any inequality of the '$\geq$' type, that constraints can be multiplied by -1 to convert it into '$\leq$' type.

If the original constraint was of the ~~less~~ '$\leq$' type, a slack variable is added on the left hand side so as to convert the inequality into an equality.

Slack variables: If a constraint has $\leq$ sign, then in order to make it an equality, we have to add something positive to ~~on~~ the left hand side.

∴ The nonnegative variable which is added to the left hand side of the constraint to convert it into equation is called slack variable.

Ex: $x_1 + x_2 \leq 2$

$2x_1 + 4x_2 \leq 5$

$x_1, x_2 \geq 0$

We add the slack variables $S_3 \geq 0$, $S_4 \geq 0$ on the left hand sides of above inequalities respectively

$x_1 + x_2 + \boxed{S_3} = 2$

$2x_1 + 4x_2 + \boxed{S_4} = 5$

∴ $x_1, x_2, S_3, S_4 \geq 0$

Surplus variables. If a constraint has $\geq$ sign, then in order to make it an equality, we have to subtract something non-negative from its left hand side.

Thus, the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called surplus variable.

Eg: $x_1 + x_2 \geq 2$

$2x_1 + 4x_2 \geq 5$

$x_1, x_2 \geq 0$

we subtract the surplus variable $S_3 \geq 0$, $S_4 \geq 0$ from the left hand sides of above inequalities respectively to obtain

$x_1 + x_2 - S_3 = 2$

$2x_1 + 4x_2 - S_4 = 5$

$x_1, x_2, S_3, S_4 \geq 0$ //

In the above equation the variable x_3 shows by how much $x_1 + x_2$ is less than 2. Since $x_1 + x_2$ can be at the most be equal to 2, it follows that S_3 can never be negative.

Basic variables: are the variables which does not participate in the ~~objective function~~ optimal solution:

↓ Only used to solve problem — in objective fun its co-efficient is zero

Non Basic variables are decision variables

Simplex Method

~~Step~~ problem: Max $Z = 3x_1 + 5x_2$

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$, $x_1, x_2 \geq 0$

Step 1: Transform inequality to equality

$x_1 \leq 4 \Rightarrow x_1 + S_3 = 4$ where $S_3 \to$ slack variable

$2x_2 \leq 12 \Rightarrow 2x_2 + S_4 = 12$ S_4, S_5 /

$3x_1 + 2x_2 \leq 18 \Rightarrow 3x_1 + 2x_2 + S_5 = 18$

write the equation containing all terms

$x_1 + 0 \cdot x_2 + S_3 = 4$ — (1)

$0x_1 + 2x_2 + S_4 = 12$ — (2)

$3x_1 + 2x_2 + S_5 = 18$ — (3)

where $x_1, x_2, S_3, S_4, S_5 \geq 0$

Hence $Z = 3x_1 + 5x_2 + 0 \cdot S_3 + 0 \cdot S_4 + 0 \cdot S_5$

$\therefore Z_0 - 3x_1 - 5x_2 - 0 \cdot S_3 - 0 \cdot S_4 - 0 \cdot S_5 = 0$ — (4)

Initial simplex table

incoming variable

pivot column

Basic variable	eq^n	Z	x_1	x_2	S_3	S_4	S_5	RHS	Min ratio
Z	4	1	-3	-5	0	0	0	0	-
S_3	1	0	1	0	1	0	0	4	(4/0) ×
→ S_4	2	0	0	2	0	1	0	12	12/2 = 6
outgoing variable S_5	3	0	3	2	0	0	1	18	18/2 = 9

pivot row

x_1, x_2 are non basic variables.

Step 2: Find entering and leaving variable.

1) In Z^{th} row find minimum value (most -ve value).

$\therefore \min(-3, -5) = -5$

Hence x_2 is entering variable & column is pivot column.

2) min Ratio = $\frac{RHS}{\text{incoming variable column } (>0)}$

$\therefore \min\{4/0, 12/2, 18/2\} =$

~~Hence~~ In this min Ratio is 6 ∴ Hence S_4 is the leaving variable & pivot row.

Step 3: The intersection of pivot row & pivot column is pivot element i.e. 2

~~It has to~~ make that element to 1.

∴ BV	eqⁿ	Z	x_1	x_2	S_3	S_4	S_5	RHS	Min Ratio
Z	4	1	-3	-5	0	0	0	0	
S_3	1	0	1	0	1	0	0	4	4/0
S_4	2	0	0	$2/2 = 1$	0	1/2	0	$12/2 = 6$	$12/2 = 6$
S_5	3	0	3	2	0	0	1	18	$18/2 = 9$

$R_2 / 2$

* Make the other element in pivot column is 0.

	BV	eqⁿ	Z	x_1	x_2	S_3	S_4	S_5	RHS	Min ratio
$R_0 = R_0 + 5R_2$	Z	4	1	-3	0	0	5/2	0	30	
R_1	S_3	1	0	1	0	1	0	0	4	4/1 = 4
R_2	x_2	2	0	0	1	0	1/2	0	6	6/0
$R_3 = R_3 - 2R_2$	S_5	3	0	3	0	0	-1	1	6	

$R_0 = R_0 + 5R_2$: x_1: $-3 + 5(0) = -3$; x_2: $-5 + 5(1) = 0$; S_3: $0 + 5(0)$; S_4: $0 + 5(1/2)$; S_5: 0; RHS: $0 + 5(6)$

$R_3 = R_3 - 2R_2$: x_1: $3 - 2(0)$; x_2: $2 - 2(1)$; 0; S_4: $0 - 2/2 = -1$; S_5: $1 - 2(0)$; RHS: $18 - 2(6) = 6$

Now again find entering & leaving variable.

1) In Z^{th} row find min value i.e. -3 ∴ x_1 is ~~to~~ entering variable & column is pivot column.

(2) min Ratio = $\frac{RHS}{\text{incoming variable column}}$

∴ min ratio = $4/1, 6/0, 6/3$

Pivot column

BV	Bv	eqⁿ	Z	x_1	x_2	S_3	S_4	S_5	RHS	Min ratio
$R_0 = R_0 + 3(R_3)$	Z	4	1	-3	0	0	5/2	0	30	
	S_3	1	0	1	0	1	0	0	4	4/1 = 4
	x_2	2	0	0	1	0	1/2	0	6	6/0 = x
$R_3 = R_3 / 3$	S_5	3	0	3	0	0	-1	1	6	6/3 = 2 — Pivot row

pivot

In min Ratio, 2 is minimum so it is pivot column.

∴ Intersection of pivot row & pivot column is pivot element i.e. 3

⇒ make that element to 1

BV	Bv	eqⁿ	Z	x_1	x_2	S_3	S_4	S_5	RHS	Min ratio
$R_0 = R_0 + 3(R_3)$	Z	4	1	-3	0	0	5/2	0	30	
$R_1 = R_1 - (R_3)$	S_3	1	0	1	0	1	0	0	4	
	x_2	2	0	0	1	0	1/2	0	6	
$R_3 / 3$	x_1	3	0	1	0/3 = 0	0	-1/3	1/3	6/3 = 2	

→

R_0

$-3 + 3(1) = 0$

PTO

make the other elements in the pivot column 0.

	Bv	eqn	Z	x_1	x_2	S_3	S_4	S_5	RHS	Min Ratio
R_0+3R_3	Z	4	1	0	0	0	3/2	1	36	
R_1-R_3	S_3	1	0	0	0	1	1/3	-1/3	2	
R_2	x_2	2	0	0	1	0	1/2	0	6	
R_3	x_1	3	0	1	0	0	-1/3	1/3	2	

R_0+3R_3 x_1: $-3+3(1) = 0$; x_2: $0+0$; S_3: 0; S_4: $5/2+3(-1/3) = \frac{5}{2}+1 = 3/2$; S_5: $0+3/3 = 1$; RHS: $30+3(2)$

$\frac{5}{2}-\frac{1}{1} = \frac{5-2}{2} = 3/2$

$R_1 - R_3$

∴ we stop the process when all values in Z row are non negative

$\boxed{\therefore \text{Max } Z = 36 \text{ at } x_1 = 2,\ x_2 = 6,\ S_3 = 2}$

Augmented form: The original form has been augmented by some supplementary variables needed to apply the simplex method.

∴ An augmented solution is a solution for the original variables that has been augmented by the corresponding values of slack variables.

* A basic solution is an augmented corner-point ~~solution~~ feasible solution.
* The only difference between basic solution & CPF solution is whether the values of slack variables are included.

For any basic solution, the corresponding corner-point solution is obtained simply by deleting the slack variables.

Note: if ... of min-ratio value (/) if denominator is -ve value then reject that value

Example 2: Max $Z = 3x_1 + 2x_2$

Subject to

$x_1 - x_2 \le 2$

$x_1 + x_2 \le 4$

$x_1, x_2 \ge 0$

Step 1: $x_1 + x_2 + S_1 = 4$ —— (1)

$x_1 - x_2 + S_2 = 2$ —— (2)

$Z = 3x_1 + 2x_2 + S_1 + S_2$

$Z - 3x_1 - 2x_2 - 0 \cdot S_1 - 0 \cdot S_2 = 0$ —— (3)

Augmented form

Step 2: Initial simplex table — Entering variable

Bv	eqn	Z	x_1	x_2	S_1	S_2	RHS	Min Ratio
Z	3	1	-3	-2	0	0	0	
S_1	1	0	1	1	1	0	4	4/1 = 4
S_2	2	0	1	-1	0	1	2	2/1 = 2

leaving variable

Min ratio = 4/1 = 4

2/1 = 2 Entering

	Bv	eqn	Z	x_1	x_2	S_1	S_2	RHS	Min Ratio
$R_0: R_0+3(R_2)$	Z'	3	1	0	-5	0	3	6	6/-
$R_1: R_1-R_2$	S_1	1	0	0	2	1	-1	2	2/2 = 1
R_2	x_1	2	0	1	-1	0	1	2	2/-1 = -2 ×
$R_0 = R_0+5(R_1)$	Z	3	1	0	0	5/2	1/2	11	
$R_1 = R_1/2$	x_2	1	0	0	1	1/2	-1/2	1	
$R_2 = R_2+R_1$	x_1	2	0	1	0	1/2	1/2	3	

$Z_{max} = 11$ at $x_1 = 3,\ x_2 = 1,\ S_1 = 0,\ S_2 = 0$

$\frac{1}{2}+1,\ \frac{1+2}{2} = 3/2$ $\quad 1-\frac{1}{2} = 1/2$

Example 3:

Solve the following LPP by simplex method

Max $Z = 6x_1 + 8x_2$

Subject to

$2x_1 + 8x_2 \le 16$

$2x_1 + 4x_2 \le 8$

$x_1, x_2 \ge 0$

$2x_1 + 8x_2 = 16$; if $x_1 = 0$, $x_2 = 2$ $(0,2)$; if $x_2 = 0$, $x_1 = 8$

$2x_1 + 4x_2 = 8$; if $x_1 = 0$, $x_2 = 2$; if $x_2 = 0$, $x_1 = 4$ $(4,0)$

Sol^n: $2x_1 + 8x_2 + S_1 = 16$ —(1)

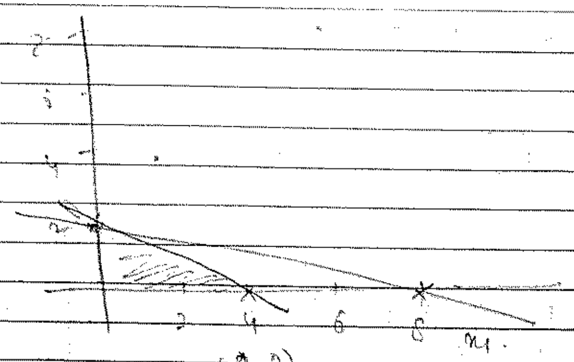
$2x_1 + 4x_2 + S_2 = 8$ —(2)

$Z - 6x_1 - 8x_2 - 0 \cdot S_1 - 0 \cdot S_2 = 0$ —(3)

$\underbrace{x_1, x_2}_{N.B.V}$, $\underbrace{S_1, S_2}_{B.V} \ge 0$

Enter variable / PC

Basic	eq^n	Z	x_1	x_2	S_1	S_2	RHS	Min
Z	3	1	-6	-8	0	0	0	0
S_1	1	0	2	8	1	0	16	$\frac{16}{8} = 2$
S_2	2	0	2	4	0	1	8	$8/4 = 2$



A $(4,0)$

B $(0,2)$

$6x_1 + 8x_2$

$6(4) + 0$

$= 24$

$6(0) + 8(2)$

$= 16$

New pblm

Example 4: Work through the simplex method step by step to solve the following pblm:

Minimize $Z = x_1 - 3x_2 + 3x_3$

Subject to $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 + 4x_2 \ge -12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

Step 1: Converting minimization to maximization problem

(Maximize $-Z$ = Minimize Z)

Max $-Z = -x_1 + 3x_2 - 3x_3$

Subject to $3x_1 - x_2 + 2x_3 \le 7$

$-2x_1 - 4x_2 \le +12$

$-4x_1 + 3x_2 + 8x_3 \le 10$, $x_1, x_2, x_3 \ge 0$

$\Downarrow$

Max $-Z + x_1 - 3x_2 + 3x_3 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3 = 0$ —(1)

$3x_1 - x_2 + 2x_3 + S_1 = 7$ —(2)

$-2x_1 - 4x_2 + S_2 = 12$ —(3)

$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$ —(4)

$x_i \ge 0$, $i = 1$ to 3, $S_1, S_2, S_3 \ge 0$

Initial simplex table

entering variable / PC

BV	eq^n	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS	Min ratio
Z	1	-1	1	-3	3	0	0	0	0	
S_1	2	0	3	-1	2	1	0	0	7	$7/-1$ ×
S_2	3	0	-2	-4	0	0	1	0	12	$12/-4$ ×
S_3	4	0	-4	3	8	0	0	1	10	$10/3$

leaving

PR

P element

	BV	eq	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS	Min ratio
$R_0 = R_0 + 3R_3$	Z	1	-1	-3	0	11	0	0	1	10	
$R_1 = R_1 + R_3$	S_1	2	0	5/3	0	14/3	1	0	1/3	31/3	31/5
$R_2 = R_2 + 4(R_3)$	S_2	3	0	-22/3	0	32/3	0	1	4/3	76/3	76/-22 X
$R_3/3 \rightarrow$	x_2	4	0	-4/3	1	8/3	0	0	1/3	10/3	10/-4 X

	BV	eq	Z	x_1	x_2	x_3	S_1	S_2	S_3	RHS
2	Z	1	-1	0	0	97/5	9/5	0	8/5	143/5
	x_1	2	0	1	0	14/5	3/5	0	1/5	31/5
	S_2	3	0	0	0	468/5	22/5	1	42/5	1862/15
	x_2	4	0	0	1	96/15	4/5	0	9/15	58/5

$(174/15)$

Optimal soln is $x_1 = \frac{31}{5}$, $x_2 = \frac{58}{5}\left(\frac{174}{15}\right)$

Min $Z = \frac{-143}{5} = -28.6$, $Z_{max} = \frac{143}{5} = 28.6$

Ex 5: A furniture manufacturing company plans to make two products, chairs & Tables from its available resources which consists of 400 board feet timber and 450 man hours. It knows that to make a chair it requires 5 board feet & 10 man hours and yields a profit of 45/- while each table uses 20 B feet & 15 man hours & yields a profit of 80/-. The problem is to determine how many chairs & tables the company can make keeping within its Resource constraints, so that it maximises it profit. Formulate a LPP model for this pblm & find soln using both.

	chair	Table	availa resource
Timber	5	20	400
Man hrs	10	15	500
profit	45	80	

Max $Z = 45x_1 + 80x_2$

Subject: $5x_1 + 20x_2 \le 400$ — (1)

$10x_1 + 15x_2 \le 450$ — (2)

$x_1, x_2 \geq 0$

$10x_1 + 15x_2 = 450$
$10x_1 + 15 \times 14 = 450$
$210 = 450$
$x_1 = \frac{450-210}{10}$
$= 24$

$5x_1 + 20x_2 = 400$
if $x_1 = 0$
$x_2 = 20$
if $x_2 = 0$
$x_1 = 80$ (80, 20)

$10x_1 + 15x_2 = 450$
if $x_1 = 0$
$x_2 = 30$
if $x_2 = 0$
$x_1 = 45$ (45, 30)

450/10

$45x_1 + 80x_2$

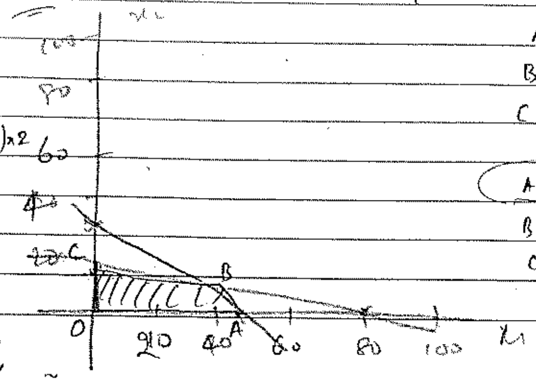
A (45, 0) = 45(45) + 0
B (24, 14) = 45(24) + 80(14)
C (0, 20) = 0 + 80(20)

A = 2025
B = 1174 2200
C = 1600

$(5x_1 + 20x_2 = 400) \times 2$
$10x_1 + 15x_2 = 450$

$10x_1 + 40x_2 = 800$
$10x_1 + 15x_2 = 450$

$25x_2 = 350$
$x_2 = 14$



(6) Simplex method:

max $Z = 45x_1 + 80x_2$

Subject to

$5x_1 + 20x_2 \le 400$

$10x_1 + 15x_2 \le 450$

$x_1, x_2 \ge 0$

$5x_1 + 20x_2 + S_1 = 400$ — (1)

$10x_1 + 15x_2 + S_2 = 450$ — (2)

$Z - 45x_1 - 80x_2 - 0 \cdot S_1 - 0 \cdot S_2 = 0$ — (3)

ent pc

	BV	eqn	Z	x_1	x_2	S_1	S_2	RHS	Min val.
R_0	Z	3	1	-45	-80	0	0	0	
R_1 leaving var	S_1	1	0	5	20	1	0	400	$\frac{400}{20} = 20$
	S_2	2	0	10	15	0	1	450	$\frac{450}{15} = 30$
	Z	3	1	-45	-80	0	0	0	
$R_1 = R_1/20$	x_2	1	0	$5/20$	1	$1/20$	0	$400/20 = 20$	
	S_2	2	0	10	15	0	1	450	
$R_0 = R_0 + 80(R_1)$	Z	3	1	-25	0	4	0	1600	$\frac{1600}{-25}$
R_1	x_2	1	0	$5/20$	1	$1/20$	0	20	$\frac{20}{5/20} = \frac{400}{5} = 80$
$R_2 = R_2 - 15(R_1)$ leav	S_2	2	0	$25/4$	0	$-3/4$	1	150	$\frac{150}{25/4} = \frac{600}{25} = 24$
	Z	3	1	-25	0	4	0	1600	
	x_2	1	0	$5/20$	1	$1/20$	0	20	
$R_2 = \frac{R_2}{25/4}$	x_1	2	0	1	0	$-3/25$	$4/25$	24	
$R_0 = R_0 + 25(R_2)$	Z	3	1	0	0	$97/25$	$4/25$	1624	
$R_1 = R_1 - \frac{5}{20}(R_2)$	x_2	1	0	0	1	$-14/25$	$20/625$		
	x_1	2	0	1	0	$-3/25$	$4/25$	24	

$1600 + 24$

$1 - \frac{5}{20}(0)$

$\frac{1}{20} - \frac{5}{20}\left(\frac{3}{25}\right)$

$4 + \frac{3}{25} = \frac{100+3}{25} = \frac{97}{25}$

$450 - 15(20)$

$= 450 - 300$

$-45 + 80\left(\frac{5}{20}\right)$

$= -45 + 20$

$= -25$

$0 - 15\left(\frac{1}{20}\right)$

$= -\frac{3}{4}$

$\frac{80}{20}$

$0 + 80(20) = 1600$

$10 - 15\left(\frac{5}{20}\right)$

$= 10 - \frac{15}{4} = \frac{40 - 15}{4}$

$-\frac{3}{4} \times \frac{4}{25} = \frac{25}{4}$

$\frac{150}{25/4} = \frac{150 \times 4}{25}$

Ex 6:

$\frac{1}{20} - \frac{3}{5} = \frac{1 - 15}{25} = -\frac{14}{25}$

$\frac{5}{25} \times \frac{4}{25} = \frac{20}{625}$

$20 - \frac{1}{4}\left(\frac{12}{24}\right)$

$= 20 - 12$

(7) Using Graphical method solve the LPP.

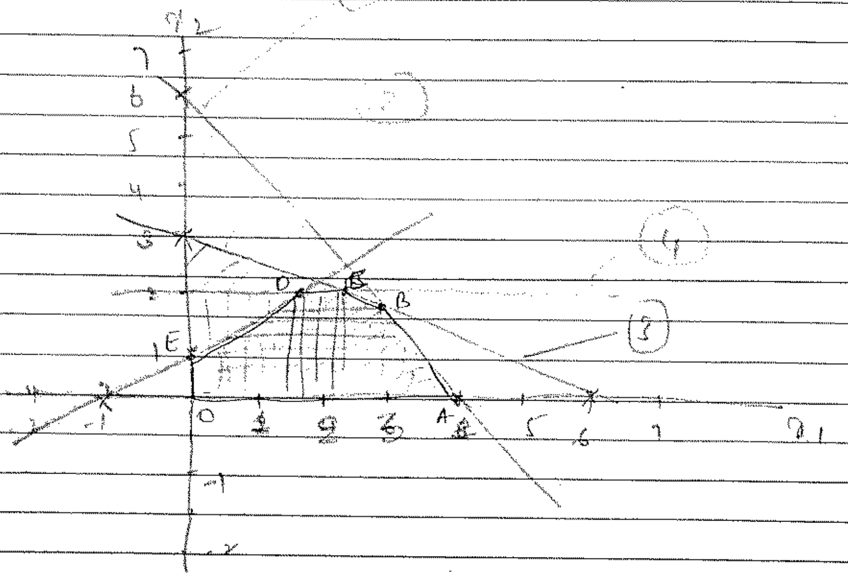
Max $Z = 5x_1 + 4x_2$

subject to
$6x_1 + 4x_2 \le 24$ — (1)
$x_1 + 2x_2 \le 6$ — (2)
$-x_1 + x_2 \le 1$ — (3)
$x_2 \le 2$ — (4)
$x_1, x_2 \ge 0$

Graphical sol^n:

Sol^n: ~~Max Z~~:

(1) $6x_1+4x_2=24$	$x_1+2x_2=6$	$-x_1+x_2=1$	$x_2=2$
if $x_1=0$, $x_2=6$	if $x_1=0$, $x_2=3$	if $x_1=0$, $x_2=1$	
if $x_2=0$, $x_1=4$	if $x_2=0$, $x_1=6$	if $x_2=0$, $x=-1$	
(4,6)	(6,3)	(-1,1)	



Points are at
A (4,0)
E (0,1)
B ( )

At 'B' Eq^r (1) & (3)

$6x_1 + 4x_2 = 24$
$x_1 + 2x_2 = 6 \quad \times 6$

$6x_1 + 12x_2 = 36$
$6x_1 + 4x_2 = 24$
$(-) \quad (-) \quad (-)$

$8x_2 = 12$
$x_2 = \frac{12}{8} = \frac{3}{2}$

$\therefore x_1 = 3 \quad B(3, 3/2)$

At 'C'
$\therefore C(2,2) \quad D(1,2)$

$Z = 5x_1 + 4x_2$

A(4,0)	= 20
B(3,3/2)	= 21
C(2,2)	= 18
D(1,2)	= 13
E(0,1)	= 4

$Z_{max} = 21$ at $(3, 3/2)$

Simplex method

$6x_1 + 4x_2 + S_1 = 24$ — (2)

$x_1 + 2x_2 + S_2 = 6$ — (3)

$-x_1 + x_2 + S_3 = 1$ — (4)

$x_2 + S_4 = 2$ — (5)

$Z - 5x_1 - 4x_2 - 0.S_1 - 0.S_2 - 0.S_3 - 0.S_4 = 0$ — (1)

BV	eq^n	x_1	x_2	S_1	S_2	S_3	S_4	RHS	Min Ratio
Z	1	-5	-4	0	0	0	0	0	
L.V S_1	2	6	4	1	0	0	0	24	$\frac{24}{6} = 4$
S_2	3	1	2	0	1	0	0	6	$\frac{6}{1}$ 6
S_3	4	-1	1	0	0	1	0	1	$\frac{1}{-1}$ x
S_4	5	0	1	0	0	0	1	2	$\frac{2}{0} = \infty$ x

R_0	Z	1	-5	-4	0	0	0	0	0	
$R_1 = R_1/6$	x_1	2	1	2/3	1/6	0	0	0	$\frac{24}{6} = 4$	
	S_2	3	1	2	0	1	0	0	6	
	S_3	4	-1	1	0	0	1	0	1	
	S_4	5	0	1	0	0	0	1	2	

$R_0 = R_0 + 5(R_1)$	Z	1	0	-2/3	5/6	0	0	0	20	-
	x_1	2	1	2/3	1/6	0	0	0	4	6
$R_2 = R_2 - R_1$	S_2	3	0	4/3	-1/6	1	0	0	2	2/(4/3): 3/2
$R_3 = R_3 + R_1$	S_3	4	0	5/3	1/6	0	1	0	5	3
$R_4 =$	S_4	5	0	1	0	0	0	1	2	2

	BV	eq^n								
R_0	Z	1	0	0						
R_1	x_1	2	1							
$R_2 = R_2 \times \frac{3}{4}$	x_2	3	0	1	-1/8	3/4	0	0	3/2	
R_3	S_3	4	0							
	S_4	5	0							

.. at $x_1 = 3$ & $x_2 = 3/2$

$x_1 + x_2 = 4$

$x_1 = 0$

$x_2 = 4$

$S_4 =$ $x_1 = 4$

$2x_2 = 12$

$3x_1 + 2x_2 = 18$

(4, 6

$3(4) + 2(6)$

$12 + 12$

$\frac{24 - 18}{6}$

$-4 + 8\left(\frac{2}{3}\right)$

$\frac{-4+2}{3}$

$\frac{-12+2}{3} = \frac{-10}{3}$

$\frac{-4+2}{3}$

$\frac{-12+2}{3} = \frac{-10}{3}$

$= -2/3$

$\frac{5}{6}$

$2 - \frac{2}{3} = \frac{4}{3}$

$1 + \frac{2}{3} = \frac{5}{3}$

$\frac{20}{1}$ $\frac{60}{-2}$

$\frac{1}{-\frac{2}{3}}$

$4/(2/3) = \frac{4 \times 3}{2} = \frac{12}{6}$

$\frac{-1}{6} \times \frac{3}{4} = \frac{-1}{8}$

$\frac{5}{1}$

$\frac{1}{5/3}$

$\frac{8}{1} \times \frac{3}{8} =$

$\frac{3}{3}$ $\frac{2}{4/3}$

$\frac{6}{4/2} = 3/2$

$\frac{2 \times 3}{4/2} = 3/2$

$\frac{5}{3} \times \frac{28}{3}$

5

Definitions for

1) Feasible Solution:- The values which satisfy all the constraints of the Model.

2) Infeasible solution: The values which will not satisfy atleast, one constraint of the Model.

3) CPF solution: Corner point Feasible Solution
The points which lies at the corner of the feasible region.

(4) optimal Solution:
The Most ~~Var~~ favarable value for the given model.
→ If the Model is of Maximization type the Optimal solution will be the largest value
→ If it is of Minimization type the optimal solⁿ will be the smallest value.

Assumptions of linear programming:

(1) Addictivity: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraints) is the sum of the individual contributions of the respective activities.
i.e, The sum of the (individual activities will give) resources used by different activity must be equal to the total quantity of the resources used by each activity for all the resources collectively.

(2) Divisibility Assumption: Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional & nonnegativity constraints. Thus these variables are not restricted to just integer values.

(3) Certainity assumption: this concerns the parameters of the model, namely the co-efficients in the objective function c_j, the co-efficients in the functional constraints a_{ij}, and right hand sides of the functional constraints b_i.

The value assigned to each parameter of a linear programming model is assumed to be a known constant.

i.e. the co-efficients in the objective function & constraints are completely known & do not change during the period under study in all the problems.
[(Sensitivity Analysis) ∴ it is important to conduct after a solution is found that is optimal under the assumed parameter values].

(4) Proportionality assumption: $\overset{c_1}{10{,}000/-}\ \overset{x_1}{42\text{ LED}} \rightarrow 12{,}000/-$

The contribution of each variable in the objective function is directly proportional to the value of the variable that means, if some resources availability increases by some percentage then the O/p should also increase by the same percentage.

i.e. $Z \propto c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ c_nx_n$

Z^{16} (profit) $\rightarrow c_1^{\uparrow 4}\ x_1$ | $x_1 = 7.5$ book, $3/2$

$\uparrow 25$ $\uparrow 5$

Standard form of LPP:

proceeding as for the Wyndor Glass Co problem, we can now formulate the mathematical model for this general problem of allocating resources to activities

$Z - c_1x_1 - c_2x_2 - c_3x_3 = $

∴ Max $Z = c_1x_1 + c_2x_2 + c_3x_3 \ \ldots\ + c_nx_n$

Subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

and

$$x_1 \ge 0,\ x_2 \ge 0\ \ldots\ x_n \ge 0$$

where

Z = value of overall measure of performance
x_j = level of activity j (for $j = 1, 2 \ldots n$)
c_j = increase in Z that would result from each unit increase in level of activity j.
b_i = amount of resource i that is available for allocation to activities (for $i = 1, 2 \ldots m$).
a_{ij} = amount of resource i consumed by each unit of activity j.

The model poses the problem in terms of making decisions about the levels of the activities, so $x_1, x_2 \ldots x_n$ are called decision variables.

Table form: Data needed for a linear pgming model

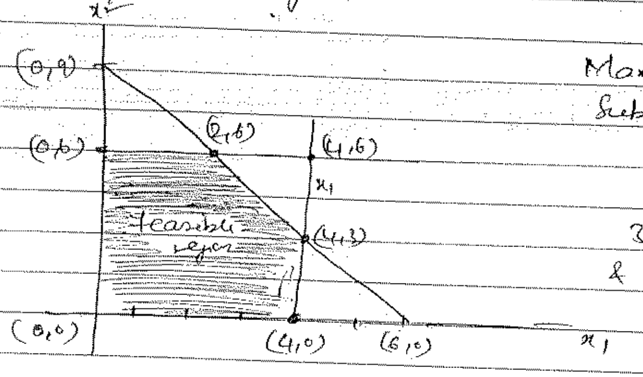
Resource	Resource usage per unit of Activity — Activity 1	2	…	n	Amount of resource available
1	a_{11}	a_{12}	…	a_{1n}	b_1
2	a_{21}	a_{22}	…	a_{2n}	b_2
⋮	⋮				⋮
m	a_{m1}	a_{m2}	…	a_{mn}	b_m
Contribution to Z per unit	c_1	c_2	…	c_n	

1

The Essence of the Simplex Method

The simplex method is an algebraic procedure. However, its underlying concepts are geometric. Understanding these geometric concepts provides a strong intuitive feeling for how the simplex method operates & what make it so efficient.

Prototype ex. Wyndor Glass Co.



Max $Z = 3x_1 + 5x_2$
Subject to
$x_1 \le 4$
$2x_2 \le 12$
$3x_1 + 2x_2 \le 18$
&
$x_1 \ge 0,\ x_2 \ge 0$

The points of intersection are the corner-point solution of the problem. The five are
(0,0), (0,6), (2,6), (4,3) & (4,0) → CPF
& Other (0,9), (4,6), (6,0) → Corner point Infeasible Solution.

if ~~$n=2$~~ n

In this example, each corner-point solution lies at the intersection of two constraint boundaries. (For a LPP with n decision variables, each of its corner-point solutions lies at the intersection of n constraint boundaries)

Since $n=2$ in the example, two of its CPF solutions are adjacent if they share one constraint boundary; Eg (0,0) & (0,6) are adjacent ∵ they share $x_1 = 0$ constraint boundary.

Adjacent CPF solutions for each CPF

CPF solution	Adjacent CPF solution
(0,0)	(0,6) & (4,0)
(0,6)	(2,6) & (0,0)
(2,6)	(4,3) & (0,6)
(4,3)	(4,0) & (2,6)
(4,0)	(0,0) & (4,3)

Optimality test: Consider any linear programming problem that possesses atleast one optimal solution. If a CPF solution has no adjacent CPF solutions that are better (as measured by Z), then it must be an optimal solution.

Thus, for the example (2,6) must be optimal simply because its Z=36 is larger than Z=30 for (0,6) & Z=27 for (4,3).

Outline of what the simplex method does is

Initialization: Choose (0,0) as the initial CPF solution to examine. (This is a convenient choice ∵ no calculations are required to identify this CPF Solution)

Optimality test: Conclude that (0,0) is not an optimal solution (Adjacent CPF solutions are better)

Iteration 1:

Move to a better adjacent CPF solution (0,6) by performing following 3 steps.

1. Considering the two edges of the feasible region that emanate from (0,0), choose to move along the edge that leads up the x_2 axis.

(2) Stop at the first new constraint boundary
$2x_2 = 12$

(3) Solve for the intersection of the new set of constraint boundaries (0,6). (The equations for these constraint boundaries $x_1 = 0$ & $2x_2 = 12$ immediately yield this solution).

Optimality test: Conclude that (0,6) is not an optimal solⁿ.

∴ Iteration 2: Move to a better adjacent CPF solution (2,6) by performing the same 3 steps.
(1) (2) (3)

∴ Next optimality test: conclude that (2,6) is optimal solution & stop.

Six Key Solution Concepts

~~Solution~~

* The first solution concept is based directly on the relationship between optimal solutions and CPF solutions.

Solution concept 1: The Simplex Method focuses solely on the CPF solutions. For any problem with atleast one optimal solution, finding one requires only finding a best CPF solution.

* Next Solution Concept defines the flow of the Simplex method.

Solution concept 2: The Simplex method is an iterative algorithm with the following structure.

Initialization → optimality test → if yes: stop; if no → Iteration → (back to optimality test)

* We next focus on how to get started.

Solution Concept 3: Whenever possible, the initialization of the Simplex method chooses the origin (all decision variables equal to zero) to the initial CPF solⁿ. When there are too many decision variables to find an initial CPF solution graphically, this choice eliminates the need to use algebraic procedures to find & solve for an initial CPF solⁿ.

* ~~Next~~ Solution concept concerns the better CPF solution at each iteration.

Solution Concept 4: Given a CPF solution, it is much quicker computationally to gather information about its adjacent CPF solutions than about other CPF solutions. ∴ each time the SM ~~perf~~ performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.

* Next focus is on which adjacent CPF solution to choose at each iteration.

Solution Concept 5: It simply identifies the rate of improvement in Z that would be obtained by moving along the edge. Among the edges with a positive rate of improvement in Z, it then chooses to move along the one with the largest rate of improvement in Z.

✶ Final Solution concept clarifies how the optimality test is performed efficiently.

Solution concept 6: optimality test consists simply of of checking whether any of the edges give a positive rate of improvement in Z. If none do, then the current CPF solution is optimal.

NBV → will have value "zero"

SETTING UP THE SIMPLEX METHOD

Definition of slack, augmented solution, basic solution, BF solⁿ

✶A basic solⁿ is an augmented corner point feasible solution (also BFS)

A basic Solution has the following properties:

(1) Each variable is designated as either a nonbasic variable or a basic variable.

(2) The number of basic variables equals the no of functional constraints (now eqns).

∴ The no of NBV = Total no of variables minus the number of functional constraints

$5 - 3 = 2$

(3) The nonbasic Variables are set equal to zero.

(4) The values of basic variables are obtained as the simultaneous solution of the system equations

(5) If the basic variables satisfy the nonnegativity constraints, the basic solution is BF solution.

Problems:

(1) Find all the basic solution to the following problem

$$\text{Max } Z = x_1 + 3x_2 + 3x_3$$

s.t.

$$x_1 + 2x_2 + 3x_3 = 4$$

$$2x_1 + 3x_2 + 5x_3 = 7$$

$\underbrace{x_1\ x_2\ x_3}_{N.B.V}\ \underbrace{x_4\ x_5}_{BV}$

Also, find which of the Basic Solutions are

(i) Basic feasible

(ii) Non degenerated Basic feasible

(iii) optimal basic feasible

BV	NBV	Value of BV	Z	Type of solution
x_1, x_2	$x_3 = 0$	$x_1 = 2, x_2 = 1$	5	(2,1,0) Non-degenerated
x_1, x_3	$x_2 = 0$	$x_1 = 1, x_3 = 1$	4	(1,0,1) — " —
x_2, x_3	$x_1 = 0$	$x_2 = -1, x_3 = 2$	3	(0,-1,2) (Since -ve value (Non-degenerated

$(x_1 + 2x_2 + 3x_3 = 4)\,2 \rightarrow 2x_1 + 4x_2 = 8$

$(2x_1 + 3x_2 + 5x_3 = 7)\,1 \quad 2x_1 + 3x_2 = 7$

$x_2 = 1$

$x_1 + 2x_2 = 4$

$x_1 = 4 - 2 = 2$

$(x_1 + 3x_3 = 4)\,2 \quad 2x_1 + 6x_3 = 8$

$2x_1 + 5x_3 = 7 \quad 2x_1 + 5x_3 = 7$

$x_3 = 1$

$x_1 = 1$

(i) Basic feasible → (2,1,0) (1,0,1)

(ii) Non Degenerated Bf → (2,1,0) (1,0,1)

(iii) optimal Bf → (2,1,0) Z = 5

② Find all the basic solution of following s/m for equations, identify in each case Basic & non basic variable.

$2x_1 + x_2 + 4x_3 = 11$
$3x_1 + x_2 + 5x_3 = 14$

BV	NBV	value of BV	feasible
$x_1\ x_2$	$x_3 = 0$	$x_1 = 3$, $x_2 = 5$, $X =$	$(3,5,0)$
$x_1\ x_3$	$x_2 = 0$	$x_1 = 0.5$, $x_3 = 2.5$, $X =$	$(0.5, 0, 2.5)$
x_2, x_3	$x_1 = 0$	$x_2 = -1$, $x_3 = 3$, $X =$	$(0, -1, 3)$

∴ The feasible solution are $(3,5,0)$ & $(0.5, 0, 2.5)$

③ Obtain all the basic solution to the following s/m of Linear equation & also find out whether the solution is non-degenerated or not.

$2x_1 + 6x_2 + 2x_3 + x_4 = 3$
$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$

The system is having n variables & m 2 equation

$n = 4$
$m = 2$ ∴ $n - m = 2$

∴ So set of two variable value to zero

BV	NBV	value of BV	feasible
x_1, x_2	$x_3, x_4 = 0$	$x_1 = 0$, $x_2 = 0.5$	$(0, 0.5, 0, 0)$
x_2, x_3	$x_1, x_4 = 0$	$x_2 = 0.5$, $x_3 = 0$	$(0, 0.5, 0, 0)$
x_3, x_4	$x_1, x_2 = 0$	$x_3 = 2$, $x_4 = -1$	$(0, 0, 2, -1)$
x_1, x_3	$x_2, x_4 = 0$	$x_1 = -2$, $x_3 = 3.5$	$(-2, 0, 3.5, 0)$
x_1, x_4	$x_2, x_3 = 0$	$x_1 = 2.6$, $x_4 = -2.23$	$(2.66, 0, 0, -2.33)$
x_2, x_4	$x_1, x_3 = 0$	$x_2 = 0.5$, $x_4 = 0$	$(0, 0.5, 0, 0)$

$(0, 0.5, 0, 0)$ are feasible Yes

& ~~(0,0,2,-1), (-2,0,3.5,0), (2.66,0,0,-2.33) are non degenerated solution~~

★ If any of the basic variable takes the value zero the basic solution of the system, that basic solution is termed as a degenerate basic solution otherwise it is called a non-degenerate basic solution.

Degen → which $= 0$
N. Degen → which > 0
Feasible → which ≥ 0

(4)

~~The~~ Basic Solution for

Max $Z = 3x_1 + 5x_2$

$x_1 + x_3 = 4$ — (1)

$2x_2 + x_4 = 12$ — (2)

$3x_1 + 2x_2 + x_5 = 18$ — (3)

$3(4) + 2(6) = 18$

$12 + 12$

$x_5 = -6$

$x_4 = 0$

$x_3 = 0$

$2x_2 = 12$

$x_2 = 6$

$x_1 = \frac{18-12}{4}$

∴ 5 variables & 3 equations

$5 - 3 = 2$

∴ So set of two variable values to zero

BV	NBV	value of BV	Feasible
x_1, x_2, x_5	$x_3, x_4 = 0$		(4, 6, 0, 0, -6)

Solve the following lpp by simplex method

Max $Z = 6x_1 + 8x_2$

S.T

$2x_1 + 8x_2 \leq 16$

$2x_1 + 4x_2 \leq 8$ and $x_1, x_2 \geq 0$

I $2 \to 1$ $3 \to 1$ ≤ 6

$4 \to 1$ $6 \to 1$ ≤ 12

8 12

sol^n.

~~$Z - 6x_1 - 8x_2 = 0$~~ — (0)

$2x_1 + 8x_2 + S_1 = 16$ — (1)

$2x_1 + 4x_2 + S_2 = 8$ — (2)

$Z - 6x_1 - 8x_2 - 0 \cdot S_1 - 0 \cdot S_2 = 0$ — (3)

E.V

BV	eq^n	Z	x_1	x_2	S_1	S_2	RHS	Min ratio
Z	3	1	-6	-8	0	0	0	$0/_{-8}$ ×
S_1	1	0	2	8	1	0	16	$16/8 = 2$
L.V S_2	2	0	2	4	0	1	8	$8/4 = 2$

The ~~Algebra~~ of the SIMPLEX METHOD

Make pivot element to 1 & other element to zero

BQ

	BV	eq^n	Z	x_1	x_2	S_1	S_2	RHS	Min Ratio
R_0	Z	3	1	-6	-8	0	0	0	
R_1	S_1	1	0	2	8	1	0	16	$16/8 = 2$
$R_2 = R_2/4$	x_2	2	0	1/2	1	0	1/4	2	$8/4 = 2$
$R_0 = R_0 + 8(R_2)$	Z	3	1	-2	0	0	2	16	$\frac{16}{-2}$ ×
$R_1 = R_1 - 8(R_2)$	S_1	1	0	-2	0	1	2	0	$0/_{-2}$ ×
	x_2	2	0	1/2	1	0	1/4	2	4
$R_0 = R_0 + 2(R_2)$	Z	3	1	0	4	0	3	24	
$R_1 = R_1 + 2(R_2)$	S_1	1	0	0	0	1	3	8	
$R_2 = R_2 \times 2$	x_1	2	0	1	2	0	1/2	4	$16 + 8(2)$

$x_1 = 4$, $x_2 = 0$

$Z_{max} = 24$

$-6 + 8(\frac{1}{2})$

$2 - 8(\frac{1}{2})$

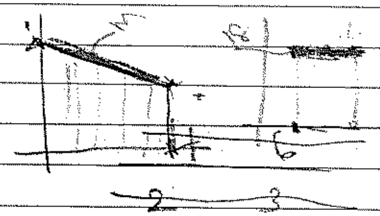
$16 + 8$

$2 + 4(\frac{1}{2})$

24

24

Z = 18



2 3

4 6

Tie Breaking in Simplex method

(1) Unbounded Solutions: In some cases if the value of a variable is increased indefinitely, the constraints are not violated. This indicates that the feasible region is unbounded atleast in one direction ∴ the objective function value can be increased indefinitely. This means that the pblm has been poorly formulated. ~~or to~~

Eg: Max $Z = 4x_1 + 3x_2$

S.T $x_1 \le 5$

$x_1 - x_2 \le 8$

$x_1, x_2 \ge 0$

Soln: Introduce slack variable

$x_1 + S_1 = 5$ — (1)

$x_1 - x_2 + S_2 = 8$ — (2)

$Z = 4x_1 + 3x_1 + 0 \cdot S_1 + 0 \cdot S_2$

$Z - 4x_1 - 3x_2 - 0 \cdot S_1 - 0 \cdot S_2$ — (3)

E.V

	BV	eqn	Z	x_1	x_2	S_1	S_2	RHS	Min ratio
	Z	3	1	-4	-3	0	0	0	0/-4 ×
LV	S_1	1	0	1	0	1	0	5	5
	S_2	2	0	1	-1	0	1	8	8
$R_0 = R_0 + 4(R_1)$	Z	3	1	0	-3	4	0	20	20/-3 ×
	x_1	1	0	1	0	1	0	5	5/0 ×
$R_2 = R_3 - R_1$	S_2	2	0	0	-1	-1	1	3	3/-1 ×

∴ No variable is ready to leave the basis

From the table it is clear that x_2 is entering variable into the basis (Pivot Column variable is still negative) but no variable is ready to leave since the ratio of solution value to key column value is infinity for R_1 & negative for R_2 both of which are to be ignored.

Thus we cannot proceed further, & this problem yields no finite solution or in other words an unbounded solution.

Multiple optimal Solution: when the objective function is parallel to one of the constraints, the multiple optimal ~~sol~~ solution may exist. As we have seen from Graphical solutions, that the optimal solution exists at the extreme point on the feasible region, the multiple optimal solutions will be noticed on atleast two points of the binding constraint parallel to that of objective function. Thus in simplex method also ~~atle~~ atleast two solution can be found.

The alternate optima is identified in SM by using following principle.

After reaching the optimality, if atleast one of the non-basic (decision) variables posses a zero value, then the multiple optimal solution exists.

Ex: Max $Z = 3x_1 + 6x_2$

S.T $x_1 + x_2 \le 5$

$x_1 + 2x_2 \le 6$

$x_1, x_2 \ge 0$

Step 1: Converting inequalities into equations we get

$x_1 + x_2 + S_1 = 5$ — (1)

$x_1 + 2x_2 + S_2 = 6$ — (2)

$x_1, x_2 \ge 0$

$Z - 3x_1 - 6x_2 - 0 \cdot S_1 - 0 \cdot S_2 = 0$ — (3)

I.1

E.V

	BV	eq	Z	x_1	x_2	S_1	S_2	RHS	Min ratio
	Z	3	1	-3	-6	0	0	0	0/-6 x
	S_1	1	0	1	1	1	0	5	5/1
L.V	S_2	2	0	1	2	0	1	6	6/2 = 3
$R_0 = R_0 + 6(R_2)$	Z	3	1	-1	0	0	6	36	36/-1 x
$R_1 = R_1 - 2(R_2)$	S_1	1	0	-1	0	1	-1	-1	-1/-1 x
$R_2 = R_2/2$	x_2	2	0	1/2	1	0	1/2	3	6
$R_0 = R_0 + R_2$	Z	3	1	0	2	0	7	42	
$R_1 = R_1 + R_2$	-	1	0	0	2	1	-1/2	2	
$R_2 = R_2 \times 2$	x_1	2	0	1	2	0	1	6	

$3 - 3(0)$

$-3 + 6(1/2)$

$0 - 2(1)$

$5 - (2(3))$

$\frac{6+1}{2} = 7/2$

36, 42, $-1 + \frac{1}{2}$

Incomplete

~~$Z_{max} = 42$~~

~~$x_1 =$~~

Ans: $Z_{max} = 18$

$x_1 = 0$, $x_2 = 0$

$x_1 = 4$

$x_2 = 1$

Change objective function $Z = 3x_1 + 3x_2 \rightarrow 3x_1 + 2x_2$

Max $Z = 3x_1 + 2x_2$

S.T $x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

$\Rightarrow$

$x_1 + S_1 = 4$ — (1)

$2x_2 + S_2 = 12$ — (2)

$3x_1 + 2x_2 + S_3 = 18$ — (3)

$Z - 3x_1 - 2x_2 - 0 \cdot S_1 - 0 \cdot S_2 - 0 \cdot S_3 = 0$ — (4)

Initial Basic feasible solution $x_1, x_2 = 0$. $S_1 = 4$, $S_2 = 12$, $S_3 = 18$

E.V E.V

	BV	eq	Z	x_1	x_2	S_1	S_2	S_3	RHS	Min ratio
	Z	4	1	-3	-2	0	0	0	0	0/-3 x
L.V	S_1	1	0	1	0	1	0	0	4	4/1 = 4
	S_2	2	0	0	2	0	1	0	12	12/0 x
	S_3	3	0	3	2	0	0	1	18	18/3 = 6
$R_0 = R_0 + 3R_1$	Z	4	1	0	-2	3	0	0	12	12/-2 x
$R_1 = R_1$	x_1	1	0	1	0	1	0	0	4	4/0 x
$R_2 = R_2$	S_2	2	0	0	2	0	1	0	12	12/2 = 6
$R_3 = R_3 - 3(R_1)$ L.V	S_3	3	0	0	2	-3	0	1	6	6/2 = 2
$R_0 = R_0 + R_3$	Z	4	1	0	0	0	0	1	18	
$R_1 = R_1$	x_1	1	0	1	0	1	0	0	4	
$R_2 = R_2 - 2(R_3)$	S_2	2	0	0	0	3	1	-1	6	6/3 = 2
$R_3 = R_3/2$	x_2	3	0	0	1	-3/2	0	1/2	3	
	Z	4	1	0	0	0	0	1	18	
$R_1 = R_1 - R_2$	x_1	1	0	1	0	0	-1/3	1/3	2	
$R_2 = R_2/3$	S_1	2	0	0	0	1	1/3	-1/3	2	
	x_2	3	0	0	1	0	1/2	0	6	

$x_1 = 4$, $x_2 = 3$

$x_1 = 2$, $x_2 = 6$

$Z_{max} = 18$

Infeasible solution:

There may not exist any solution to certain LPP, this in LPP jargon is said to be infeasible solution. In this type of solution, there exists no feasible region we do not get any infeasible solution with all constraints as 'less than or equal to type'.

Eg. Max $Z = 2x_1 + 3x_2$

S.T $x_1 \leq 5$

$x_1 - x_2 \geq 10$

$x_1 \geq 0, \; x_2 \geq 0$

Step. Max $Z = 2x_1 + 3x_2$

$-x_1 + x_2 \leq -10$

$x_1 \leq 5$

$\Longrightarrow$

$x_1 + s_1 = 5$ — (1)

$-x_1 + x_2 + s_2 = -10$ — (2)

$Z - 2x_1 - 3x_2 - 0 \cdot s_1 - 0 \cdot s_2 = 0$ — (3)

$\left.\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}\right\}$ Intial Basic feasible solution Non Basic

$\left.\begin{matrix} s_1 = 5 \\ s_2 = -10 \end{matrix}\right\}$ Basic

From the condition $s_2 \geq 0$, s_2 cannot take value -10,

$\therefore$ no IBFS can exist with such less than or equal to type constraints.

So, this method of converting '$\geq$' type constraint to '$\leq$' type with a negative sign cannot yield any result & not suitable in such cases, as said earlier.

Setting up characteristics of the two forms of simplex Method

Characteristics of the canonical form

(1) The objective function is of maximization type.

(2) All constraints are of ($\leq$) type.

(3) All variables x_i are non-negative.

Characteristics of the standard form:

(1) The objective function is of max type.

(2) All constraints are expressed as equations.

(3) Right hand side of each constraint is non-negative.

(4) All variables are non-negative.

Degeneracy

Here we present certain complications encountered in the application of the simplex method and how they are resolved. These are called variants of the simplex method.

(1) Minimization

(2) Inequality with $\geq$ or $=$ sign

(3) Degeneracy

(4) Unbounded solution

(5) Multiple solutions

(6) Non-existing feasible solution

(7) Unrestricted variables

(1) Minimization: We have problems in which the objective function has to be minimized instead of maximizing. This situation can be tackled easily in ~~either of~~

$$\text{Min } Z = -\text{Max}(-Z)$$

(2) Inequality with $\geq$ or $=$ sign:

If atleast one constraint is of $\geq$ or $=$ type with positive right hand side value, then slack or surplus variable fails to give the initial basic feasible solution. To resolve this, we have to introduce the artificial variables to get the initial basic feasible solution & then adopt Big-M Method or two phase method to find a feasible optimal solution.

(3) Degeneracy: The concept of obtaining a degenerate basic feasible solution in a LPP known as Degeneracy. Degeneracy in a LPP may arise.

(i) At the initial stage when atleast one basic variable is zero in the initial basic feasible solution.

(ii) At any subsequent iteration when more than one basic variable is eligible to leave the basic & hence one or more variables becoming zero in the next iteration and the problem is said to degenerate.

There is no assurance that the value of the objective function will improve, since the new solutions may remain degenerate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solutions. This concept is known as cycling or circling.

Perbutation Rule to Avoid cycling:

(a) Divide each element in the tied rows by the positive co-efficients of the key column in that row.

(b) Compare the resulting ratios, column by column, first in the identity & then in the body, from left to right.

(c) The row which first contains the smallest algebraic ratio & contains the leaving variable.

(1) Solve the following LPP by Simplex Method

Max $Z = x_1 + 2x_2 + x_3$

S.T $2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6$

$4x_1 + x_2 + x_3 \leq 6$, $x_1, x_2, x_3 \geq 0$

Solⁿ Given data:

Max $Z = x_1 + 2x_2 + x_3$

S.form $2x_1 + x_2 - x_3 + S_1 = 2$

$2x_1 - x_2 + 5x_3 + S_2 = 6$

$4x_1 + x_2 + x_3 + S_3 = 6$, $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

The initial basic feasible solution is

$S_1 = 2$, $S_2 = 6$, $S_3 = 6$ when ($x_1 = x_2 = x_3 = 0$, non basic)

Incomplete

~~Iteration~~:

Tie for the leaving Basic variable - Degeneracy

→ Suppose two or more basic variables tie for the being the leaving basic variable Step2 of an iteration. Does it matter which one is choosen?

Yes & in a very critical way, ∵ of following sequence of events that could occur.

1) The one or ones not chosen to be the leaving basic variable also will have a value of zero in the new BF solution. (NOTE that basic variable with a value of zero are called degenerate & is the same term is applied to the corresponding BF solution)

② if one of these degenerate basic variables retain its value of zero until it is chosen at a subsequent iteration to be a leaving basic variable, the corresponding entering basic variable also must remain zero (since it cannot be increased without making the leaving basic variable negative), so the value of Z must remain unchanged.

③ If Z may remain the same rather than increase at each iteration, the simplex method may then go around in a loop, repeating the same sequence of solutions periodically rather than eventually increasing Z toward an optimal solution.

It rarely ~~also~~ occur in practical problem, ~~& a~~ & special rules have been constructed for breaking ties so that such loops are always avoided.

[" New Finite pivoting Rules for the simplex Method"]

Pblm from page 15

BV	eqn	Z	x_1	x_2	S_1	S_2	RHS	Min ratio
Z	3	1	-6	-8	0	0	0	
S_1	1	0	2	8	1	0	16	2
S_2	2	0	2	4	0	1	8	2

tie

Apply the degeneracy procedure (tie)

First arrange the column x_1, x_2, S_1 & S_2 is such a way that the initial identity (basic) matrix appear first

BV	S1	S2	x_1	x_2	RHS	Min ratio	
Z	0	0	-6	-8	0	0/-8 x	0/-8 x
S_1	1	0	2	8	16	16/8 = 2	1/8
S_2	0	1	2	4	8	8/4 = 2	0/4 = 0

Min ratio $\left[\begin{array}{l}\text{elements of } 1^{st} \text{ column } (S_1) \\ \text{corresponding elements of keycolumn } (x_2)\end{array}\right]$

$$\min\left[\cancel{\tfrac{0}{-8}},\ \tfrac{1}{8},\ \tfrac{0}{4}\right] = 0$$

$-6+2(8)$

BV	eqn	Z	S_1	S_2	x_1	x_2	RHS
$R_0 = R_0 + 8(R_1)$ Z	1	1	8	0	10	0	128
$R_1 = R_1/8$ x_2	1	0	1/8	0	1/4	1	2
$R_2 = R_2 - 4(R_1)$ S_2	2	0	1/2	1/2	-6	0	

BV		S_1	S_2	x_1	x_2	RHS	Min ratio
$R_0 = R_0 + 8(R_1)$	Z	1	0	-2	0	16	x
$R_1 = R_1/8$	x_2	1/8	0	1/4	1	2	8
$R_2 = R_2 - 4(R_1)$	S_2	1/2	2	1	0	0	y
$R_0 = R_0 + 2(R_1)$	Z	2	0	0	8	32	
$R_1 = R_1 \times 4$	x_1	1/2	0	1	4	8	
$R_2 = R_2 - R_1$	S_2	0	2	0	-4	-8	

$Z_{max} = 32$
$x_1 = 8$
$x_2 = 0$

$\frac{16}{2}$, $\frac{2}{1} = 8$, 0

$-6+2(8)$, 16, 32, $2-4(2)$, -6, $2-4(\frac{1}{4})$, $1-4(\frac{1}{8})$, $2-4(\frac{1}{8})$, $-6+8\frac{1}{4}$, $4-\frac{1}{2}$, $0+8(2)$, $1-4(\frac{1}{8})$, $8-4(2)$, $1-\frac{1}{2}$

BV	Eq	Z	S_1	S_2	x_1	x_2	RHS	Min ratio
Z	3	1	0	0	-6	-8	0	0/-8
S_1	1	0	1	0	2	8	16	16/8
S_2	2	0	0	1	2	4	8	8/4

	BV	eq	Z	S_1	S_2	x_1	x_2		
	Z	3	1	0	0	-6	-8		
$R_1 = \frac{R_1}{8}$	x_2	1	0	1/8	0	2/8	1	16/8	
	S_2	2	0	0	1	2	4	8	
$R_0 = R_0 + 8(R_1)$	Z	3	1	1	0	-4	0	16	-4
	x_2	1	0	1/8	0	1/4	1	2	8
$R_2 = R_2 - 4(R_1)$	S_2	2	0	-1/2	-4	1	0	0	0

$-6+8(\frac{2}{8})$

$0+8(2) = -16$

$8-4(\frac{2}{2})$

$2-4(\frac{1}{4})$

Min ratio —

	BV	eq	Z	S_1	S_2	x_1	x_2	
$R_0 = R_0 + 4(R_2)$	Z	3	1	2	0	0	4	
$R_1 = R_1 - R_2/4$	x_1	1	0	1/4	0	1	4	8
$R_2 = R_2 - R_1$	S_2	2	0	-3/4	-4	0	-4	

$16+4(8)$

$= 16+32$

$-\frac{1}{2} - \frac{1}{4}$

The Algebra of the SIMPLEX METHOD

Prototype Example

Original form of the Model

Max $Z = 3x_1 + 5x_2$

S.T. $x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

& $x_1 \ge 0,\ x_2 \ge 0$

Augment form of Model

Max $Z = 3x_1 + 5x_2$

S.T

$x_1 + S_1 = 4$

$2x_2 + S_2 = 12$

$3x_1 + 2x_2 + S_3 = 18$

& $x_j \ge 0$, $S_1, S_2, S_3 \ge 0$

Initialization:

The choice of x_1 & x_2 to be the nonbasic variables for the IBF is based on concept 3.

This choice eliminates the work required to solve for the basic variable (S_1, S_2, S_3)

$x_1 = x_2 = 0$

$\therefore S_1 = 4$

$S_2 = 12$ $\quad\therefore$ IBFS $(0, 0, 4, 12, 18)$

$S_3 = 18$

Optimality test:

The O.F is $Z = 3x_1 + 5x_2$,

so $Z = 0$ for the initial Basic feasible solution.

$\because$ the basic variables has zero co-efficients in the objective function.

→ The coefficient of each nonbasic variable (x_1, x_2) gives the rate of improvement $(3, 5)$ in Z if that variable were to be increased from zero.

$\therefore$ These rates of improvement (3 & 5) are +ve.

$\therefore$ based on solution concept 6 we conclude that $(0, 0, 4, 12, 18)$ is not optimal.

Step 1: Determining the direction of Movement (1st iteration)

Increasing one basic variable from zero corresponds to moving along one edge emanating from the current CPF solution. Based on solution concept 4 and 5 the choice of which nonbasic variable to increase is as follows:

$Z = 3x_1 + 5x_2$

increase x_1? Rate of improvement in $Z = 3$

increase x_2? " " " $Z = 5$

$\because 5 > 3$ $\therefore$ choose x_2 to increase.

$\therefore x_2$ is called entering basic variable for iteration 1.

[Increasing the nonbasic variable from zero will convert it to a basic variable for the next BF solution]

Step 2: Determining where to stop

Step 2 addresses the question of how far to increase the entering basic variable x_2 before stopping.

$\therefore$ So we want to go as far as possible without leaving the feasible region.

$\therefore$ The requirement to satisfy the functional constraints in augmented form means that increasing x_2 (keeping $x_1 = 0$) changes the value of some basic variable

$x_1 = 0$ so

(1) $x_1 + s_1 = 4$ $\therefore s_1 = 4$

(2) $2x_2 + s_2 = 12$ $s_2 = 12 - 2x_2$

(3) $3x_1 + 2x_2 + s_3$ $s_3 = 18 - 2x_2$

→ The other requirement for feasibility is that all the variables be non negative. The nonbasic variables (even entering basic variable) are non-negative but we need to check how far x_2 can be increased without violating the non-negativity constraints of basic variables.

$s_1 = 4 \geq 0$ no upper bound on x_2

$s_2 = 12 - 2x_2 \geq 0 \Rightarrow x_2 \leq \frac{12}{2} = 6 \Leftarrow$ Minimum

$s_3 = 18 - 2x_2 \geq 0 \Rightarrow x_2 \leq \frac{18}{2} = 9$

Thus x_2 can be increased just to 6, at which point x_4 has dropped to 0.

Increasing x_2 beyond 6 would cause x_4 to become -ve. This calculations are referred to as minimum ratio test. The objective of this test is to determine which basic variable drops to zero first as the entering basic variable is increased.

→ We can immediately rule out the basic variable in any equation where the co-efficient of the entering variable is zero or negative.

[At any iteration of the Simplex method, Step 2 uses the minimum ratio test to determine which basic variable drops to zero first as the entering basic variable is increased. Decreasing the basic variable to zero will convert it to a non-basic variable for the next BF solution $\therefore$ this variable is called the leaving basic variable for the current iteration.]

$\therefore x_4$ is leaving basic variable.

Step 3. Solving for the New Basic feasible Solution

Increasing x_2 from $0 \to 6$ moves from IBF to NBBF

	Initial BF Solution	New BF Solution
Nonbasic variables	$x_1 = 0,\ x_2 = 0$	$x_1 = 0,\ x_2 = 6$
Basic "	$S_1 = 4,\ S_2 = 12,\ S_3 = 18$	$S_1 = ?,\ x_2 = 6,\ S_3 = ?$

The purpose of Step 3 is to convert the system of equations to a more convenient form for conducting the optimality test & the next iteration with this NBFS.

Here we also identify values of S_1 & S_3 for the new soluti

$$Z - 3x_1 - 5x_2 = 0 \quad \text{— (0)}$$
$$x_1 + S_1 = 4 \quad \text{— (1)}$$
$$2x_2 + S_2 = 12 \quad \text{— (2)}$$
$$3x_1 + 2x_2 + S_3 = 18 \quad \text{— (3)}$$

& x_2 has replaced S_2

eqn (2) $\quad 2x_2 + S_2 = 12$

$$x_2 + \frac{1}{2} S_2 = 6$$

UNIT - 6

The Transportation and Assignment problems

Here we discuss two particularly important types of linear programming problems.

(1) Transportation problem — ~~received~~ determining how to optimally transport goods.

(2) Assignment problem — assigning people to task.

(i) Transportation problem is a special class of linear programming problems.

It deals with the transportation of a commodity (Single product) from 'm' Source to (origins or supply or capacity centres) to 'n' destinations (Sinks or demand or requirement centres). The data of the transportation model includes,

(i) level of supply to each source and the amount of demand at each destination.

(ii) The unit transportation cost of commodity from each destination.

The objective is to determine the amount to be shifted from each source to each destination such that the total transportation cost is minimum.

Transportation or distribution methods of solving linear programming problems aim at minimizing costs of sending goods from dispatch stations to receiving ends.

Generally, specific quantities of goods are shipped from several dispatch stations to a number of receiving centres.

Linear programming approach in such situations tends to find, how many goods may be transported from which dispatch station to which receiving end in order to make the shipment cost economical (least costly).

Three methods for solving transportation problems

(a) North-west corner level (Stepping stone method)
(b) Least cost method (Matrix minimum method)
(c) Vogel's approximation method (unit cost penalty

Formulation of a Transportation problem

Let us assume that there are m sources & n destinations.

~~Let a_i be the~~

Let

$a_i \rightarrow$ supply (capacity) at source i

$b_j \rightarrow$ demand at destination j,

$C_{ij} \rightarrow$ unit transportation cost from source i to destination j

$x_{ij} \rightarrow$ no of units shifted from source i to j.

Then the transportation problem can be expressed mathematically as

Obj Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}$

Subject to the constraints

$\sum_{j=1}^{n} x_{ij} = a_i$, $i = 1, 2, 3 \dots m$

$$\begin{cases} x_{11} + x_{12} + x_{13} + \dots x_{1n} \\ x_{21} + x_{22} + x_{23} + \dots x_{2n} \\ \vdots \\ x_{m1} + x_{m2} + \dots + x_{mn} \end{cases}$$

$\sum_{i=1}^{m} x_{ij} = b_j$, $j = 1, 2, 3 \dots n$ & $x_{ij} \geq 0$ for all i & j

NOTES.

NOTE1: The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) (total demand)

which is the condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called balanced transportation problem.

NOTE2: if $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$

then the transportation problem is said to be unbalanced.

NOTE 3: For any transportation problem, the co-efficients of all x_{ij} in the constraints are unity.

NOTE 4: The objective function and the constraints being all linear, the problem can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we look other ~~met~~ technique.

(1) Standard Transportation Table: Transportation problem is explicitly represented by the transportation table.

Destination

		D_1	D_2	D_3	...	D_j	...	D_n	supply
	S_1	C_{11}	C_{12}	C_{13}	...	C_{1j}	...	C_{1n}	a_1
Source	S_2	C_{21}	C_{22}	C_{23}	...	C_{2j}	...	C_{2n}	a_2
	S_i	C_{i1}	C_{i2}	C_{i3}		C_{ij}		C_{in}	a_i
	S_m	C_{m1}	C_{m2}			C_{mj}		C_{mn}	a_m

The mn squares are called cells. The unit transportation cost C_{ij} from the i^{th} source to the j^{th} destination is displayed in the upper left side of the $(i,j)^{th}$ cell. Any feasible solution is shown in the table by entering the value of x_{ij} in the centre of the $(i,j)^{th}$ cell.

The various a_i's & b_j's are called rim requirements.

The feasibility of a solution can be verified by summing the values of x_{ij} along the rows & down the columns.

Definitions:

(i) A set of non-negative values x_{ij},
$i = 1,2,3 \ldots m$
$j = 1,2,3 \ldots n$,
that satisfies the constraints (rim conditions & also the non-negative restrictions) is called a feasible solution to the transportation problem.

Note: A balanced transportation problem will always have a feasible solution.

(2) A feasible solution to $(m \times n)$ transportation problem that contains no more than $m+n-1$ non-negative allocations is called a basic feasible solution to the T.P.

(3) A basic feasible solution to a $(m \times n)$ transportation problem is said to be non-degenerate basic feasible solution if it exactly $m+n-1$ non negative allocations independent positions.

(4) A basic feasible solution that contains less than $m+n-1$ non-negative allocations is said to be a degenerate basic feasible solution.

(5) A feasible solution is said to be an optimal solution if it minimizes the total transportation cost.

Methods for finding Initial Basic feasible solution.

The transportation problem has a solution if and only if the problem is balanced.

∴ before starting to find the initial basic feasible solution check whether the given transportation problem is balanced. If not one has to balance the transportation problem first.

~~these~~ Now, Consider all are balanced.

Method -1. North-west corner Rule:

Step1.

case(i): If $\min\{a_1, b_1\} = a_1$, then put $x_{11} = a_1$; decrease b_1 by a_1 & move vertically to the 2nd row (i,e) to the cell (2,1) cross out the first row.

case(ii): If $\min\{a_1, b_1\} = b_1$ then put $x_{11} = b_1$ & decrease a_1 by b_1 and move horizontally right (i,e,) to the cell (1,2) cross out the first column.

case(iii): If $\min\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row & the first column

Step2: Repeat ~~step(1) for the resulting method~~
Repeat the procedure until the requirements are satisfied.

OR

~~No~~ North-west corner method.

(a) Starting with the cell at the upper left most corner (north-west) of the transportation matrix, we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied.

i,e, $x_{11} = \min(a_1, b_1)$

(b) If b_1 is greater than a_1 $(b_1 > a_1)$, we move down vertically to the second row and make the second allocation of magnitude

$$\therefore x_{21} = \min(a_2,\ b_1 - x_{11})$$

in the cell (2,1). And if $(b_1 < a_1)$ move right horizontally to the 2nd col & make the second allocation of magnitude

$$x_{12} = \min(a_1 - x_{11},\ b_1) \text{ in the cell } (1,2)$$

(c) If $(b_1 = a_1)$, there is a tie ~~of~~ for the 2nd allocation of magnitude

$$x_{12} = \min(a_1 - a_1, b_2) = 0 \text{ in the cell } (1,2)$$

or

$$x_{21} = \min(a_2,\ b_1 - b_1) = 0 \text{ in the cell } (2,1)$$

(d) Repeat step 1 & step 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

(1) Find the initial basic Feasible solution by North-west Corner rule.

			Require or Supply
2	7	4	5 ← a_i
3	3	1	8
5	4	7	7
1	6	2	14
Demand 7	9	18	

↑ b_i

5+8+7+14 = 34

7+9+18 = 34

$\Sigma a_i = \Sigma b_i = 34$

∴ the given problem is balanced.

∴ There exists a feasible solution to the Transportation problem.

$\boxed{5}$ 2	~~7~~	4	5/0
$\boxed{2}$ 3	$\boxed{6}$ 3	1	8/6/0
5	$\boxed{3}$ 4	$\boxed{4}$ 7	7/4/0
1	6	$\boxed{14}$ 2	14
7/2/0	9/3/0	18/14	

m = no of column

n = no of rows

m+n-1

3+4-1

= 6

∴ atleast 6 solution

Following the Northwest corner rule, the first allocation is made in the cell (1,1)

Here $x_{11} = \min(a_1 b_1)$

$= \min(5, 7)$

$= 5$.

Allocate 5 to the cell (1,1) and decrease 7 by 5 i.e., 7-5=2. As the first row is satisfied, we crossout the first row and the resulting reduced transportation

∴ Hence the northwest corner is (2,1)

so allocate

$x_{21} = \min\{2, 8\} = 2$

to this cell (2,1) and allocate 2

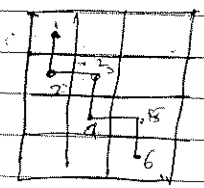
& decrease 8 by 2 8-2=6

& 2-2=0

As the first column satisfy, we crossout the first column

so repeat the same.

transportation



∴ all solutions are in independent position

i.e., no closed loops.

∴ From the table, we see that the number of positive independent allocation is equal to

m+n-1 = 3+4-1 = 6

∴ This ensures that the solution is non-degenerate basic feasible.

∴ Transportation Cost

= Rs. 5×2 + 2×3 + 6×3 + 3×4 + 4×7 + 14×2

= Rs 102

Ex (2) Determine the basic feasible solution to the following transportation problem using North west Corner rule.

	A	B	C	D	E	Supply
P	2	11	10	3	7	4
Q	1	4	7	2	1	8
R	3	9	4	8	12	9
demand	3	3	4	5	6	

Sol^n: Since $\Sigma a_i = \Sigma b_j = 21$, the given problem is balanced.

$\therefore$ $\Sigma a_i = 4+8+9 = 21$

$\Sigma b_j = 3+3+4+5+6 = 21$

$\therefore$ There exists a feasible solution to the transportation pblm

						a_i
P	2	11	10	3	7	4
Q	1	4	7	2	1	8
R	3	9	4	8	12	9
dem b_j	3	3	4	5	6	

$\Rightarrow$

3 2	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following north-west corner rule, the first allocation is made in cell (1,1)

Here $x_{11} = \min(a_1, b_1)$

$= \min(4, 3)$

$= 3$

$\therefore$ Allocate 3 to the cell (1,1) and decrease by 4 by 3 i.e., $4-3=1$. As the first column is satisfied, we cross out the first column & the resulting reduced transportation table is

1 11	10	3	7	1
4	7	2	1	8/6
9	4	8	12	9
3/2	4	5	6	

Hence the north-west corner cell is (1,2)

So allocate

$x_{12} = \min\{1, 3\} = 1$ $(3-1=2)$

to this cell (1,2) & move vertically to cell (2,2).

The resulting reduced transportation table is

2 4	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min(8, 2) = 2$

to the cell (2,2) and move horizontally to the cell (2,3). The resulting transportation table is

4 7	2	1	6
4	8	12	9
4	5	6	

Allocate $x_{23} = \min(6, 4) = 4$

and move horizontally to the cell (2,4). The resulting reduced transportation table is

2 2	1	2
8	12	9
5	6	

Allocate $x_{24} = \min(2, 5) = 2$.

& move vertically to the cell (3,4). The resulting transportation table is

3 8	12	9
3	6	

Allocate $x_{34} = \min(9,3) = 3$

and move horizontally to the cell (3,5) which is

$\boxed{12 \quad 6}$ 6

6

Allocate $x = \min\{6,6\} = 6$

Finally, the initial basic feasible solution is

Eg

2 (3)	11 (1)	10	3	7	4
1	4 (2)	7 (4)	2 (2)	1	8
3	9	4	8 (3)	12 (6)	9
3	3	4	5	6	

from the table we see that the no. of positive independent allocations is equal to

$m+n-1 = 3+5-1 = 7$

This ensures that the solution is non-degenerate basic feasible.

$\therefore$ Transportation cost $= 2\times3 + 11\times1 + 4\times2 + 7\times4 + 2\times2 + 8\times3 + 12\times6$

$= 153$

(2)

19	30	50	10	7
70	30	40	60	9
40	8	70	20	18
5	8	7	14	

$5+3$

$8-1 = 7$

$\boxed{3}$ 2	$\boxed{1}$ 11	10	3	7	4/1/0
1	$\boxed{2}$ 4	$\boxed{4}$ 7	$\boxed{5}$ 2	$\boxed{1}$ 1	8/6/1/0
3	9	4	8	4 12	9/5
3/0	3/2/0	4/0	5/0	6/5/0	

$\boxed{3}$ 2	$\boxed{1}$ 11	10	3	7	4/1/0
1	$\boxed{2}$ 4	$\boxed{4}$ 7	$\boxed{2}$ 2	1	8/6/2/0
3	9	4	$\boxed{3}$ 8	$\boxed{6}$ 12	9/6/0
3/0	3/2/0	4/0	5/3/0	6/0	

Total units transported from i to j

$= 3\times2 + 1\times11 + 2\times4 + 4\times7 + 2\times2 + 3\times8 + 6\times12$

$= 6 + 11 + 8 + 28 + 4 + 24 + 72$

$= 153$

Cost for 1 unit to transport from i to j

Example 3:

19	30	50	10	7
70	30	40	60	9
40	8	70	20	18
5	8	7	14	

$\Sigma a_i = 7+9+18 = 34$

$\Sigma b_j = 5+8+7+14 = 34$

$\Sigma a_i = \Sigma b_j \implies \therefore$ exists feasible solution.

$\boxed{5}$ 19	$\boxed{2}$ 30	50	10	7/2/0
70	$\boxed{6}$ 30	$\boxed{3}$ 40	60	9/3/0
40	8	$\boxed{4}$ 70	$\boxed{14}$ 20	18/14
5/0	8/6	7/4/0	14	

$4+3-1$

$= 6$

Total cost =

$5\times19 + 2\times30 + 6\times30 + 3\times40 + 4\times70 + 14\times20$

$= Rs\ 1015$

Example 4:

7	6	4	5	9	40
8	5	6	7	8	30
6	8	7	6	5	20
5	2	7	8	6	10
30	30	15	20	5	

					a_i
30 7	10 6	4	5	9	40/10
8	20 5	10 6	7	8	30/10
6	8	5 7	15 6	5	20/15
5	2	7	5 8	5 6	10/5
b_i 30/0	10/20	15/5	20/5/0	5	

$30\times7 + 10\times6 + 20\times5 + 10\times6 + 5\times7 + 15\times6 + 5\times8$
$+ 5\times6$
$= 625$

Example 5:

2	3	11	7	6
1	0	6	1	1
5	8	15	9	10
7	5	3	2	

Ans = 116

Example 6.

4	6	8	8	40
6	8	6	7	60
5	7	6	8	50
20	30	50	50	

Ans = 900

Example 7.

11	13	17	14	250
16	18	14	10	300
21	24	13	10	400
200	225	275	250	

Rs. 12,200

Least cost or Matrix Minima method

Step 1: Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}, allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j).

Step 2: If $\min(a_i, b_j) = a_i$ then put $x_{ij} = a_i$, cross off the i^{th} row of the transportation table and decrease b_j by a_i. Then go to Step 3.

If $x_{ij} = b_j$ cross off the j^{th} column of the transportation table and decrease a_i by b_j. Go to step 3.

Step 3: Repeat Steps 1 & Steps 2 for the resulting reduced transportation table until all the rim requirements satisfied. Whenever the minimum cost is not unique. Make an arbitrary choice among the minima.

$\boxed{\text{or}}$

Step 1: Identify the cell with smallest cost and allocate $x_{ij} = \min(a_i, b_j)$

Case 1: If $\min(a_i, b_j) = a_i$, then put $x_{ij} = a_i$ cross out the i^{th} row & decrease b_j by a_i. Go to Step 2

Case 2: If $\min(a_i, b_j) = b_j$, then put $x_{ij} = b_j$ cross out the j^{th} column & decrease a_i by b_j. Go to step 2

Case 3: If $\min(a_i, b_j) = a_i = b_j$, then put $x_{ij} = a_i = b_i$ Cross out either i^{th} row or j^{th} column but not Go to step 2

Step 2: Repeat step (1) for the resulting reduced transportation table until all the requirements are satisfied.

① Obtain the Initial feasible solution to the following Transportation problem using Matrix Minima method.

1	2	3	4	6
4	3	2	0	8
0	2	2	1	10
4	6	8	6	

1	$\boxed{6}$ 2	3	4	6/0
4	3	$\boxed{2}$ 2	$\boxed{6}$ 0	8/2/0
$\boxed{4}$ 0	2	$\boxed{6}$ 2	1	10/6/0
4/0	6/0	8/6/0	6/0	

$\therefore 6\times2 + 4\times0 + 2\times2 + 6\times0 + 6\times2 = \text{Rs. } 28$

Eg(2): Find the initial basic feasible solution for the following transportation problem by least cost method.

From \ To					Supply
	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	20	10	

Sol^n: Since $\sum a_i = \sum b_j = 100$, then given TPP is balance

$\therefore$ There exists a feasible solution to the transportation pbm.

By least cost method $C_{ij} = C_{11} = C_{13} = C_{24} = 1$. Since more than one cell having the same minimum C_{ij}, break the tie. Let us choose the cell $(1,1)$ and allocate.

$x_{11} = \min(a_1, b_1)$
$= \min(30, 20)$
$= 20$

1 $\boxed{20}$	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	20	10	

and cross out the satisfied column.

The resulting reduced transportation table is

2	10	4	10
3	2	1	50
2	5	9	20
40	30	10	

Here min $C_{ij} = C_{13} = C_{24} = 1$

Choose the cell $(1,3)$ and allocate 10

$x_{13} = \min(a_1, b_3)$

$= \min(10, 30)$

$= 10$

and cross out the satisfied row

The resulting reduced transportation table is

3	2	10 ~~1~~	50
2	5	9	20
40	20	10	

Here min $C_{ij} = C_{24} = 1$

$\therefore$ Allocate $x_{24} = \min(a_2, b_4)$

$= \min(50, 10)$

$= 10$

and cross out the satisfied column.

The resulting transportation table is

3	2 20	40
2	5	20
40	20	

Here min $C_{ij} = C_{13} = C_{22} = 2$

Choose the cell $(2,3)$ and allocate

$x_{23} = \min(a_2, b_3)$

$= \min(40, 20)$

$= 20$

& cross out the satisfied column.

The resulting reduced T.T is

3	20
2	20
40	

Here min $C_{ij} = C_{32} = 2$.

Choose the cell $(3,3)$ & allocate

$x_{32} = \min(a_3, b_3)$

$= \min(20, 40)$

$= 20$

& cross out the satisfied row.

$\therefore$ The resulting reduced transportation table is

3 20	20
20	

$\therefore$ Finally the initial basic feasible solution is as shown

1 20	2	1 10	4
3	3 20	2 20	1 10
4	2 20	5	9

We see from the table no of positive independent allocations equal to $m+n-1$

$= 3+4-1 = 6$

$\therefore$ This ensures that the solution is non-degenerate basic feasible.

$\therefore$ Transportation cost

$=$ Rs. $1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 1 \times 10 + 2 \times 20$

$=$ Rs 180/-

(3)

6	4	1	5	14
8	9	2	7	16
4	3	6	7	5
6	10	15	4	35

6	4	[14]	[0]	14/0
[6] 8	[9] 9	[1] 2	7	16/15/9/0
4	[1] 3	6	[4] 2	5/1/0
6/0	10/9/0	15/1/0	4/0	

$14\times1 + 6\times8 + 9\times9 + 2\times1 + 3\times1 + 4\times2 = 156$ Rs

(4)

19	30	50	10	7
70	30	40	60	9
40	8	70	20	18
5	8	7	14	34

19	30	50	[7] 10	7/0
[2] 70	30	[7] 40	60	9/2
[3] 40	[8] 8	70	[7] 20	18/10/3/0
5/2/0	8/0	7/0	14/7/0	

$2\times70 + 3\times40 + 8\times8 + 8\times40 + 7\times10 + 7\times20 = 814$

(5)

2	7	4	5
3	3	1	8
5	4	7	7
1	6	2	14
8	8	18	

2	[1] 7	[4] 4	5/1/0
3	3	[8] 1	8/0
5	[7] 4	7	7/0
[8] 1	6	[6] 2	14/6/0
8/0	8	18	

$1\times7 + 4\times4 + 8\times1 + 7\times4 + 8\times1 + 6\times2 =$ Rs 79

Row Minima Method

19	30	50	[7] 10	7/0
70	[8] 30	[1] 40	60	9/1
[5] 40	8	[6] 70	[7] 20	18/11/6
5/0	8	7/6	14/7	

$7\times10 + 8\times30 + 1\times40 + 5\times40 + 6\times70 + 7\times20$

= Rs 1110

Column Minima Method

[5] 19	30	50	[2] 10	7/2/0
70	30	[7] 40	[2] 60	9/2/0
40	[8] 8	70	[10] 20	18/10/0
5/0	8/0	7/0	14/12/2/0	

$5\times19 + 2\times10 + 7\times40 + 2\times60 + 8\times8 + 10\times20$

= 779

Vogel's approximation method or UNIT COST PENALTY METHOD

(1) ~~First~~ Step 1 Find the difference (penalty) between the smallest and next smallest cost in each row (column) and write them in brackets against the corresponding row (column).

Step 2. Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in the selected row or column and allocate as much as possible to this cell and crossout the satisfied row or column and goto step 3.

Step 3 Again compute the column & row penalties for the reduced transportation table & then goto step (2). Repeat the procedure until all the requirements are satisfied.

(1) Find the initial basic feasible solution

Step 1

				Available	penalties	
19	30	50	10	7	(9)	
70	30	40	60	9	(10)	(50-40)
40	$\boxed{8}$ 8	70	20	18/10	(12)	(20-8)
Require 5	8	7	14			
penalties (21)	(22)	(10)	(10)			

Step 2: Crossout column (row) in which the requirement has been satisfied

			Avail	Penalties
$\boxed{5}$ 19	50	10	7/2	(9)
70	40	60	9	(20)
40	70	20	10	(20)
5/0	7	14		
penalties (21)	(10)	(10)		

Step 3:

		Avail	Penalties
50	10	2	40
40	60	9	20
70	$\boxed{10}$ 20	10	50
Req 7	14/4		
Penalty 10	10		

Step 4

		Av	Pen
50	$\boxed{2}$ 10	2/0	40
$\boxed{7}$ 40	$\boxed{2}$ 60	9/7/0	20
7/0	4/2/0		
Penalties 10	50		

40	$\boxed{2}$ 60	9/7 ~~7/5~~ 20
7	2	
10	60	

$\boxed{7}$ 40 ~~5~~ 0/40

(7)

$\boxed{7}$ 40 7

7

[5] 19	30	50	[2] 10	7
70	30	[7] 40	[2] 60	9
40	[8] 8	70	[10] 20	18
5	8	7	14	

IBFS
$= M + N - 1$
$= 3 + 4 - 1$
$= \underline{6}$

$8\times8 + 5\times19 + 2\times10 + 7\times40 + 2\times60 + 10\times20 +$
$64 + 95 + 20 + 280 + 120 + 200$
$= 779$

---

(2)

1	5	3	3	34
3	3	1	2	15
0	2	2	3	12
2	7	2	4	19
21	25	17	17	

Step 1. identifying the penalty

				Avai	Penalty
[21] 1	5	3	3	34/13	(2)
3	3	1	2	15	(1)
0	2	2	3	12	(2) ←
2	7	2	4	19	(2)
21/0	25	17	17		
(1)	(1)	(1)	(1)		

Choose which is min cost

Select the row/column with Max penalty & assign value to Min cost.

				Penalty
5	3	3	13	(2)
3	1	2	15	(1)
[12] 2	2	3	12/0	(1)
7	2	4	19	(2)
25/13	17	17		

5	3	3	13	(2)
3	1	2	15	(1)
7	2	4	19	(2) ←
13	17	17		
(2)	(1)	(1)		

				Penalty
5	3	[13] 3	13/0	(2) ←
3	1	2	15	(1)
2	2	3	12	(1)
7	2	4	19	(2)
25	17	17/4		
(1)	(1)	(1)		

				Penalty
3	1	2	15	(1)
2	2	3	12	(1)
7	[17] 2	4	19/2	(2) ←
25	17/0	4		
(1)	(1)	(1)		

3	2	15	(1)
2	3	12	(1)
7	[2] 4	2/0	(3) ←
25	4/2		
(1)	(1)		

3	[2] 2	15/13	(1) ←
2	3	12	(1)
25	2/0		
(1)	(1)		

⇒

[13] 3	13/0	(3) ←
2	12	(2)
25/12		
(1)		

[12] 2	12	(2)
12		
25/13		
(2)		

3	13	
[13] 2	12/0	

3	11
13	

$= 1\times2 + 13\times3 + 17\times2 + 2\times4 + 2\times2 + 13\times3 + 12\times2$

$= 169$

195

(3) Find the IBFS for the following using Vogel's Approximation

			Avail	penalty
1	2	6	7	(1)
[0] 0	4	[2] 2	12/2	(2)
3	1	5	11	(2)
10	10	10	30	

(1) (1) (3)

			Avail	penalty
[7] 2	6		7/0	(4)
4	2		2	(2)
1	5		1	(4)
10/3	10			
(1)	(3)			

4	2	2	(2)
[3] 1	5	11/9	(4)
3/0	10		
(3)	(3)		

2	2	(2)
[9]	9/1	(5)
10/0		
(5)		

				I	II	III	
[7] 1	2	6	7/0	(1)	(1)	(1)	—
[2] 0	4	[10] 2	12/2/0	(2)	(4)	—	—
[1] 3	[10] 1	5	11/10/0	(2)	(2)	(1)	(2)
10/8/1/0	10/0	10/0	30				

I	(1)	(1)	(3)
II	(1)	(1)	—
III	(2)	(1)	—
IV	(3)	(1)	—

$1\times7 + 2\times0 + 10\times2 + 1\times3 + 10\times1$

$= 7 + 0 + 20 + 3 + 10$

$\boxed{= 40}$

(4)

			Supply	Penalties I[st]	II	III	IV
[9] 13	[8] 15	16	17/9	(2)	(2)	(2)	(2)
[5] 7	11	[7] 2	12/5/0	(5)	(5)	4	—
19	20	[16] 9	16/0	(10)	—	—	—
14/9	8/0	23/7	45				

40

Penalties			
1[st]	(6)	(4)	(7)
II	(6)	(4)	(14)
III	(6)	(4)	
IV	(13)	(15)	

$9\times13 + 8\times15 + 5\times7 + 7\times2 + 16\times9$

$= 430$

(5)

				Penalty: 1st	2nd	3rd	4th
[20] 4	6	8	8	40/20/0 (2)	(2)	(2)	(8)
6	[30] 8	[90] 6	[10] 7	60/10/0 (1)	(1)	(1)	(7)
15	[10] 7	[50] 6	[40] 8	50/40 (1)	(1)	(4)	(1)
20/0	30/10/0	50/0	50/40/0				

I	(1)	(1)	(2)	(1)
II	—	(1)	(2)	(1)
III	—	(8)	(0)	(7)
IV	—	(0)	—	(1)
	—	—	—	(1)

$20\times4 + 30\times8 + 7\times7 + 50\times6$

$20\times9 + 10\times6 + 50\times6 + 10\times7 + 10\times7\times40\times8$

$= 960$

(6)

				Penalties I	II	III	IV	V
[25] 2	[5] 6	[20] 6	11	50/25/20 (4)	(4)	(5)	(5)	—
10	8	[70] 7	5	70/0 (2)	(2)	(2)	(2)	(2)
13	[30] 3	9	12	30/0 (6)	—	—	—	—
4	6	[15] 8	[35] 3	50/15/0 (1)	(1)	(3)	(5)	(5)
25/0	35/5/0	105/85/15/0	35/0					

I	(2)	(3)	(1)	(2)
II	(2)	(2)	(1)	(2)
III	—	(2)	(1)	(2)
IV	—	—	(1)	(2)
V	—	—	(1)	(2)

$25\times2 + 5\times6 + 20\times6 + 70\times7 + 30\times3$
$+ 15\times8 + 35\times3$

1005

Ans - 995

(7)

21	16	15	13	11
17	18	14	23	13
32	27	18	41	19
6	10	12	15	

Ans 686

557

(8)

						I	II	III	IV
[3] 2	[1] 11	10	3	7	4/1/0	(1)	(1)	(1)	(9)
1	[2] 4	7	2	[6] 1	8/2/0	(1)	(1)	—	—
3	9	[4] 4	[5] 8	12	9/5/0	(1)	(1)	(1)	(5)
3/0	3/1/0	4/0	5/0	6/0					

I	(1)	(5)	(3)	(1)	(6)
II	(1)	(5)	(3)	(1)	—
III	(1)	(2)	(6)	(5)	—
IV	(1)	(2)	—	(5)	—

Ans 68

$3\times2 + 1\times11 + 2\times4 + 6\times1 + 4\times4 + 5\times8$
$= 6 + 11 + 8 + 6 + 16 + 40$

92

In case of tie among the highest penalties, select row or column having minimum cost. In case of tie in the minimum cost also, select the cell which can have maximum allocation. If there is tie among maximum allocation cells also, select the cell arbitrarily for allocation. Following these rules yields the best possible initial basic feasible solution and reduces the number of iterations required to reach the optimal solution.

Transportation Algorithm (or) MODI Method

Modified Distribution Method

Step 1: Find the initial basic feasible solution of the given problem by North-west Corner or LCM or VAM.

Step 2: Check the number of occupied cells. If there are less than $m+n-1$, there exists degeneracy and we introduce a very small positive assignment of $\epsilon\ (\approx 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$.

Step 3: Find the set of values u_i, v_j ($i = 1,2,3 \dots m$, $j = 1,2,3, \dots n$) from the relation $C_{ij} = u_i + v_j$ for each occupied cell (i,j) by starting initially with $u_i = 0$ or $v_j = 0$ for which the corresponding row or column have maximum number of individual allocations.

Step 4: Find the cell evaluations $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the ~~cor~~ corresponding cell (i,j).

Step 5: Find the cell evaluations $\boxed{d_{ij} = C_{ij} - (u_i + v_j)}$ (d_{ij} = upperleft − upper right) for each occupied cell (i,j) and enter at the lower right corner of the corresponding cell (i,j).

(i) if all $d_{ij} > 0$, then the solution under the test is optimal and unique.

(ii) if all $d_{ij} \geq 0$, with atleast one $d_{ij} = 0$, then the solution under the test is optimal and an alternate optimal solution exists.

(iii) if atleast one $d_{ij} < 0$, then the solution is not optimal. Go to the next step.

Step 6: Form a new B.F.S by giving maximum allocation to the cell for which d_{ij} is most negative by making an occupied cell empty.

For that, draw a closed path consisting of horizontal and vertical lines beginning and ending at the cell which d_{ij} is most negative and having its other corner at some allocated cells. Along ~~its~~ this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners.

Choose minimum of θ from the cells having $-\theta$. Add this minimum of θ to the cells with $+\theta$ & subtract this minimum of θ from the allocation to the cells with $-\theta$.

Step 7: Repeat step 2 (2) to (6) to ~~the~~ test the optimality of this new basic feasible solution.

Step 8: Continue the above procedure till the optimum solution is attained.

					I	II	III	IV
20 4	6	8	8	40/20	(2)	(2)		
6	8	6	7	60	(1)	(1)		
5	7	6	8	50	(1)	(1)		
20/0	30	50	50	150				
I (1)	(1)	(2)	(1)					
II –	(1)	(2)	(1)					
III –								
IV –								

Example: Find the optimal transportation cost of the following matrix.

4	1	2	6	9	100
6	4	3	5	7	120
5	2	6	4	8	120
40	50	70	90	90	340

Solⁿ: Since $\Sigma a_i = \Sigma b_j = 340$, the given transportation problem is balanced.

∴ There exist a basic feasible solution to this problem.

By using "Least Cost method", the initial solⁿ is

4	50 1	50 2	6	9	100/50/0
10 6	4	20 3	5	90 7	120/100/90/0
30 5	2	6	90 4	8	120/30/0
40/10/0	50/0	70/20/0	90/0	90/0	

Initial transportation cost $= 1\times50 + 2\times50 + 10\times6 + 20\times3 + 90\times7 + 30\times5 + 90\times4$

$=$ Rs. 1410/-

For optimality: Since the number of non-negative independent allocations is $3+5-1=7$, we apply MODI Method.

Hence the solution is non-degenerated.

Finding the cell evaluation u_i+v_j for each unoccupied cell(s)

c_{11} 4	50 c_{12} 1	50 c_{13} 2	c_{14} 6	c_{15} 9	$u_1=-1$
10 6	4	20 3	5	90 7	$u_2=0$
c_{31} 30 5	c_{32} 2	c_{33} 6	c_{34} 90 4	c_{35} 8	$u_3=-1$
$v_1=6$	$v_2=2$	$v_3=3$	v_4	$v_5=7$	

$u_2=0$

We have

$c_{21} = u_2+v_1$

$6 = 0+v_1$

$\boxed{v_1=6}$

$c_{23} = u_2+v_3$

$3 = 0+v_3$

$\boxed{v_3=3}$

$\boxed{v_5=7}$

⇒ ~~WKT~~ WKT,

$c_{31} = u_3+v_1$,

$5 = u_3+6$

$\boxed{u_3=-1}$

WKT

$c_{13} = u_1+v_3$

$2 = u_1+3$

$\boxed{u_1=-1}$

$c_{12} = u_1+v_2$

$1 = -1+v_2$

$\boxed{v_2=2}$

Now, finding the cell evaluations $d_{ij} = c_{ij} - (u_i+v_j)$

d_{ij} = upperleft − upper right

i.e. c_{ij} & new c_{ij}

4 5 / -1	50 1	2 50	6 4 / 2	9 6 / 3	$u_1=-1$
6 / 10	4 2 / 2	3 / 20	5 5 / 0	7 / 90	$u_2=0$
5 / 30	2 1 / 1	6 2 / 4	4 / 90	8 6 / 2	$u_3=-1$

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum to the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell $(1,1)$ has negative value $d_{11} = -1$.

We draw a closed loop consisting a horizontal and vertical lines beginning and ending at this cell $(1,1)$ and having its other corner at some occupied cells. Along this closed loop indicate $+\theta$ & $-\theta$ alternating at the corner,

4 $+\theta$	1	2 $-\theta$ 50	6	9
6 10 $-\theta$	4	3 $+\theta$ 20	5	7 90
5 30	2	6	4 90	8

From the two cells $(1,3)$, $(2,1)$ having $-\theta$, we find that θ will be the minimum of 50, 10. i.e $\boxed{\theta = 10}$

Add this θ (10) to cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

Hence, the new basic feasible solution is

4 10	1 50	2 40	6	9
6 0	4	3 30	5	7
5 30	2	6	4 90	8

We see that table satisfies the rim conditions with $(m+n-1)$ non negative allocation at independent positions. So, we apply MODI method

4 10	1 50	2 40	6 3 (3)	9 6 (3)	$u_1 = 0$
6 5 (1)	4 2 (2)	3 30	5 4 (1)	7 90	$u_2 = 1$
5 30	2 2 (0)	6 3 (3)	4 90	8 7 (1)	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

$C_{11} = u_1 + v_1$ ||| $\boxed{v_2 = 1}$ $\boxed{v_3 = 2}$

$4 = 0 + v_1$

$\boxed{v_1 = 4}$

W.K.T, $C_{23} = u_2 + v_3$ $\quad$ $C_{34} = u_3 + v_4$ $\quad$ $C_{31} = u_3 + v_1$ $\quad$ $C_{25} = u_2 + v_5$

$3 = u_2 + 2$ $\quad$ $4 = 1 + v_4$ $\quad$ $5 = u_3 + 4$ $\quad$ $7 = 1 + v_5$

$\boxed{u_2 = 1}$ $\quad$ $\boxed{v_4 = 3}$ $\quad$ $\boxed{u_3 = 1}$ $\quad$ $\boxed{v_5 = 6}$

Since all d_{ij} with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimal ~~solution~~ allocation schedule is given by $x_{11} = 10$, $x_{12} = 50$, $x_{13} = 40$, $x_{23} = 30$, $x_{25} = 90$, $x_{31} = 30$, $x_{34} = 90$

& the optimum (Minimum) Transportation cost (OPT)

$OPT = ~~4.~~\ 4 \times 10 + 1 \times 50 + 2 \times 40 + 3 \times 30 + 7 \times 90 + 5 \times 30 + 4 \times 90$

$= 1,400/-$

or Other method is

Find optimal solution using MODI method for TP:

~~Vogel's method~~

				Avail	P_1	P_2	P_3	P_4	P_5
[5] 19	30	50	[2] 10	7/2/0	9	9	40	40	–
70	30	[7] 40	[2] 60	9/1/0	10	10	20	20	20
40	[8] 8	70	[10] 20	18/10/0	12	12	52	–	–
5/0	8/0	7/0	14/4/2/0	34					

P_1	21	22↑	10	10
P_2	21↑	–	10	10
P_3	–	–	10	10
P_4	–	–	10	50↑
P_5	–	–	10	50↑

$= 5\times19 + 10\times2 + 7\times40 + 2\times60$
$+ 8\times8 + 10\times20$
$= \boxed{779}$ — IBFS – 11

Step 2: MODI method: Test for optimality
(i) Write the basic solution
(ii) Test for optimality.

$m+n-1 = 6$
$3+4-1 = 6$ ∴ all sol^n are independent Position (no closed loops)

Step 3: Find U_i & V_j such that $C_{ij} = U_i + V_j$ for the basic cells.

19			10	$U_1 = -10$
		40	60	$U_2 = 40$
	8		20	$U_3 = 0$
$V_1 = 29$	$V_2 = 8$	$V_3 = 0$	$V_4 = 20$	

Consider $U_3 = 0$

$C_{ij} = U_i + V_j$
$C_{34} = U_3 + V_4$
$20 = 0 + V_4$
$\boxed{V_4 = 20}$

$C_{24} = U_2 + V_4$
$60 = U_2 + 20$
$\boxed{U_2 = 40}$

$C_{14} = U_1 + V_4$
$10 = U_1 + 20$
$\boxed{U_1 = -10}$

$|||^{ly}$ V_1

$C_{11} = U_1 + V_1$
$19 = -10 + V_1$
$\boxed{V_1 = 29}$

$C_{32} = U_3 + V_2$
$8 = 0 + V_2$
$\boxed{V_2 = 8}$

$C_{23} = U_2 + V_3$
$40 = 40 + V_3$
$\boxed{V_3 = 0}$

Step 5: Find the new $C_{ij} = U_i + V_j$ for non basic cells (Extend the distribution)

×	-2	-10	×	-10
69	48	×	×	40
29	×	0	×	0
29	8	0	20	

$U_1 + V_1$
$C_{12} = -10 + 8$
$= -2$
$C_{13} = -10 + 0 = -10$
$C_{21} = 40 + 29$

Step 6: Find the difference between old C_{ij} – new c_{ij} for the non basic cells.

If you don't find the -ve value, so stop the problem.

×	32	60	×
1	-18	×	×
11	×	70	×

old C_{ij} – new C_{ij}
$(30) - (-2) = 32$
$(50) - (-10) = 60$
$70 - 69 = 1$
$30 - 48 = -18$
$40 - 29 = 11$
$70 - 0 = 70$

67

Step5: If all $d_{ij} \geq 0$, then the soln is optimal
∴ we have $d_{22} < 0$ then the soln is not optimal

Iteration toward optimality

Min = -18 Cell C_{22}

5			2
	*	7	2
	8		10

closed loop

$+\theta$ & $-\theta$ alternatively for the closed loops

	$+\theta$		$-\theta$
	$-\theta$		$+\theta$

E.V

outgoing variable

Start with entering variable draw horizontal & vertical skipping allocations, with allocations at corners forming a circle.

$\min\{(2-\theta)(8-\theta)\}$

$\boxed{\theta = 2}$

* 5			* 2	7
	* 2	* 7	0 * (2-2)	9
	* 6		* 12	18
5	8	7	14	

∴ Total cost = $5 \times 19 + 2 \times 10 + 2 \times 30 + 7 \times 40 + 6 \times 8 + 12 \times 20$
$= 95 + 20 + 60 + 280 + 48 + 240$
$= 743$ — IBFS - 2 :

VAM = 779 $(-18 \times 2) = 36$
$= -18 \times 2$ (2 units shipped)
743

Repeat from Step 2

Again Apply MODI Method

(i) write the basic solution
(ii) Test for optimality.

$m+n-1 = 6$ Solutions

Find u_i & v_j such that $C_{ij} = u_i + v_j$ for the basic cells.

5			2	$u_1 = -10$
	2	7	*	$u_2 = -12$
	6		12	$u_3 = 0$
$v_1 = 15$	$v_2 = 6$	$v_3 = 19$	$v_4 = 12$	

Consider $u_3 = 0$
$C_{ij} = u_i + v_j$
$C_{34} = u_3 + v_4$
$12 = 0 + v_4$
$\boxed{\therefore v_4 = 12}$

$C_{24} = u_2 + v_4$
$0 = u_2 + 12$
$\boxed{\therefore u_2 = -12}$

$C_{14} = u_1 + v_4$
$2 = u_1 + 12$
$\boxed{\therefore u_1 = -10}$

IIIly v_1:
$C_{11} = u_1 + v_1$
$5 = -10 + v_1$
$\boxed{\therefore v_1 = 15}$

$u_3 = 0$
$C_{32} = u_3 + v_2$
$6 = 0 + v_2$
$\boxed{\therefore v_2 = 6}$

$C_{23} = u_2 + v_3$
$7 = -12 + v_3$
$\boxed{v_3 = 19}$

Step6: Find the new $C_{ij} = u_i + v_j$ for the non basic cells

x	-4	9	x
3	x	x	x
15	x	19	x

$C_{12} = -10 + 6$
$= -4$
$C_{13} = -10 + 19$
$= 9$

~~Find the difference between old C_{ij} & new C_{ij}~~

x	36	51	x
-2	x	x	x
-4	x	51	x

$32-(-4)$
$80-9$
$1-3$
$11-15$
$70-19$

Again apply MODI method

(1) write the basic solution

(2) check for optimality $(m+n-1)=\underline{6}$

Find u_i and v_j such that $C_{ij} = u_i+v_j$ for the basic cells.

5			2	$u_1 = 0$
	2	7		$u_2 = 6$
	6		12	$u_3 = 10$
$v_1 = 5$	$v_2 = -4$	$v_3 = 1$	$v_4 = 2$	

$u_1+v_1=5$ Let $u_1=0$, $v_1=5$, $v_4=2$

$u_1+v_4=2$

$u_2+v_2=2$

$u_2+v_3=7$

$u_3+v_2=6$

$u_3+v_4=12$

$u_3+v_4=12$
$u_3=12-v_4$
$=12-2$
$\boxed{u_3=10}$

$u_3+v_2=6$
$v_2=6-u_3$
$=6-10$
$\boxed{v_2=-4}$

$u_2+v_2=2$
$u_2=2-v_2$

$u_2+v_3=7$
$v_3=7-u_2$

(4) Find new C_{ij} for non basic cells $C_{ij} = u_i+v_j$

x	-4	1	x	0
11	x	x	8	6
15	x	11	x	10
5	-4	1	2	

(5) Find the difference between old C_{ij} & new C_{ij}

	34	49	
59			52
25		59	

Original cost Matrix
original C_{ij} − new C_{ij}

In the above matrix all the values are greater than or equal to zero (No negative values) so the previously obtained solution $\boxed{\text{Min } Z = 743}$ is the optimal solⁿ

(2) Solve the following ~~the~~ Transportation problem in which cell entries represent unicost by vogel's method.

				P1	P2	P3	P4
[5] 2	7	4	5/0	(2) ←	—	—	—
3	3	[8] 1	8/0	(2)	(2) ←		
5	[7] 4	7	7/0	(1)	(1)	(1)	(1)
[2] 1	[2] 6	[10] 2	14/4/2	(1)	(1)	(1)	(5) ←
7/2/0	9/2/0	18/10/0	34				

P1	(1)	(1)	(1)
P2	(1)	(1)	(1)
P3	(4)	(2)	(5) ↑
P4	(4)	(2) ↑	—

Ans:

$5\times2 + 8\times1 + 7\times4 + 2\times1 + 2\times6 + 10\times2$

$= \boxed{80}$

Step 2: To test optimality, Apply MODI Method

$m+n-1 = 6$

$4+3-1 = 6$

6

Step 3: First write only basic cells, find u_i & v_j

i.e., $C_{ij} = u_i + v_j$

2	x	x	$u_1 = 1$
x	x	1	$u_2 = -1$
x	4	x	$u_3 = -2$
1	6	2	$u_4 = 0$
$v_1 = 1$	$v_2 = 6$	$v_3 = 2$	

Step 4. Find new $C_{ij} = u_i + v_j$ for non-basic cells.

x	7	3	1
0	5	x	-1
-1	x	0	-2
x	x	x	0
1	6	2	

Step 5: Find the difference between old C_{ij} - new C_{ij} for the non basic cells.

x	0	1
3	-2	x
6	x	7
x	x	x

— Entering variable

Step 6. ~~If~~ It is observed that the cell evaluation $d_{22} = -2$, is negative ∴ The solution under test is not optimal.

The solution can be improved as shown below.

[5] 2		
	$+\theta$	[8] 1 $-\theta$
	[7] 4	
[2] 1	[2] 6 $-\theta$	[10] 2 $+\theta$

⟹

[5] 2		
	[2] 3	[6] 1
	[7] 4	
[2] 1	[0] 6	[12] 2

$\min(8-\theta)(2-\theta)$

$\boxed{\theta = 2}$

$= 5\times2 + 2\times3 + 6\times1 + 7\times4 + 2\times1 + \cancel{0\times6} + 12\times2$

$= 10 + 6 + 6 + 28 + 2 + 0 + 24$

$= 76$

New feasib sol^n

$\boxed{5}$ 2			5
	$\boxed{2}$ 3	$\boxed{6}$ 1	8
	$\boxed{7}$ 4		7
$\boxed{2}$ 1		$\boxed{2}$ 2	14
7	9	18	

$\boxed{Ans = 76}$

Again Apply Modi Method

$M+N-1=6$

Step 3: first write only basic cells, find u_i & v_j

2			$u_1=1$
	3	1	$u_2=-1$
	4		$u_3=0$
1		2	$u_4=0$
$v_1=1$	$v_2=4$	$v_3=2$	

$C_{11}=u_1+v_1$

$2=u_1+1$

$\boxed{u_1=1}$

$u_1+v_1=2$

$u_2+v_2=3$

$u_2+v_3=1$

$u_3+v_2=4$

$u_4+v_1=1$

$u_4+v_3=2$

$u_2+v_3=1$

$u_2=1-v_3$

$u_2=1-2$

$=-1$

$u_2+v_2=3$

$-1+v_2=3$

$v_2=4$

$\boxed{u_3=0}$

Step 4: find new c_{ij} for non basic cells

x	5	3	1
0	x	x	-1
5	x	2	0
x	4	x	0
1	4	2	

Step 5: find the difference between old c_{ij} - new c_{ij}

old c_{ij}

·	7	4
3	x	x
5	x	7
x	6	x

−

·	5	3
0	x	x
5	x	2
x	4	x

	2	1
3		
0		5
	2	

In the above matrix all the values are greater than or equal to zero. So the previously obtained solution $\boxed{\text{Min } Z = 76}$ is the optimal sol^n

Eg (3) Solve the Transportation problem

Vogel's Method

Penalty

1	2	1	4	30	(0)
3	3	2	[10] 1	50	(1)
4	2	5	9	20	(2)
20	40	30	10	100	
(2)	(1)	(1)	(3)		

1	2	1	30	(0)
3	3	2	40	(1)
4	[20] 2	5	20/0	(2) ←
20	40/20	30		
(2)	(1)	(1)		

[20] 1	2	1	30/10	(0)
3	3	2	40	(1)
20/0	20	30		
(2) ↑	(1)	(1)		

2	[10] 1	10/0	(1) ←
3	2	40	(1)
20	30/20		
(1)	(1)		

[20] 3	2	40/20	(1)
20/0	20		
(3) ↑	(2)		

[20] 2	20

[20] 1	2	[10] 1	4	30
3	[20] 3	[20] 2	[10] 1	50
4	[20] 2	5	9	20
20	40	30	10	

$20\times1 + 10\times1 + 20\times3 + 20\times2 + 10\times1 + 20\times2$

Ans = 180

Optimality test : Apply MODI Method

6 sol's no loop

Find u_i & v_j for basic cells

1		1		$u_1 = -1$
	3	2	1	$u_2 = 0$
	2			$u_3 = -1$

$v_1 = 2$ $v_2 = 3$ $v_3 = 2$ $v_4 = 1$

$u_1 + v_1 = 1$

$u_1 + v_3 = 1$

$u_2 + v_2 = 3$ $\boxed{v_2 = 3}$ $\boxed{u_1 = -1}$

$u_2 + v_3 = 2$ $\boxed{v_3 = 2}$ $\boxed{u_3 = -1}$

$u_2 + v_4 = 1$ $\boxed{v_4 = 1}$ $\boxed{v_1 = 2}$

$u_3 + v_2 = 2$

Find new c_{ij} for nonbasic cells

x	2	x	0	-1
2	x	x	x	0
1	x	1	0	-1
2	3	2	1	

Find the difference between ~~new c_{ij}~~ old c_{ij} & new c_{ij}

old c_{ij}

x	2	x	4
3	x	x	x
4	x	5	9

−

x	2	x	0
2	x	x	x
1	x	1	0

=

x	0	x	4
1	x	x	x
3	x	4	9

This shows that all empty cell evaluation are non-negative.

Hence the sol^n under test is Optimal

$c_{12} \to 0$ ∴ it indicates

Example 5: Hindustan construction company needs 3, 3, 4 & 5 million cubic feet of fill at four earthen dam-sites in punjab. It can transfer the fill from three mounds A, B & C where 2, 6 & 7 million cubic feet of fill is available respectively. Costs of transporting one million cubic feet of fill from mounds to the four sites in lakhs are given in the table.

(a) Solve the pblm using ~~trans~~ transportation algorithm for minimum cost

(b) formulate the problem as LPP.

TO

15	10	17	18	2
16	13	12	13	6
12	17	20	11	7
3	3	4	5	

Using Vogel's Approximation Method

					Penalty I	II	III	IV
15	[2] 10	17	18	2/0	(5) ←	—	—	—
16	[1] 13	[4] 12	[1] 13	6/2/1/0	(1)	(1)	(3)	(0) 0
[3] 12	17	20	[4] 11	7/4/0	(1)	(1)	(1)	(6) ←
3/0	3/1/0	4/0	5/1/0					

I	(3)	(3)	(5)	(2)
II	(4)	(4)	(8) ↑	2
III	(4) ↑	(4)	—	(2)
IV	—	(4)	—	(2)
	13 ↑	13		

Sol^n $2\times10 + 1\times13 + 4\times12 + 1\times13 + 3\times12 + 4\times11$

$= 20 + 13 + 48 + 13 + 36 + 44$

$= \cancel{189}\ 174$

Find u_i & v_j for basic cells

	10	×		$u_1 = -3$
	13	12	13	$u_2 = 0$
12			11	$u_3 = -2$

$v_1 = 14$ $v_2 = 13$ $v_3 = 12$ $v_4 = 13$

Find c_{ij} for non basic cells

11	×	9	10	-3
14	×	×	×	0
×	11	10	×	-2
14	13	12	13	

Difference bet^n old c_{ij} & new c_{ij}

4		8	8
2			
	6	10	

Since non negative values, the previously obtained is optimal sol^n

$x_{12} = 2,\ x_{22} = 1,\ x_{23} = 4,\ x_{24} = 1,\ x_{31} = 3,\ x_{34} = 4$

L.P formulation: Let x_{ij} be the amount of fill transfered from mounds to dam-sites. Then formulation of problem becomes

$Min\ Z = (15x_{11} + 10x_2 + 17x_3 + 18x_4) + (16x_{21} + 13x_{22} + 12x_{23} + 13x_{24}) + (12x_{31} + 17x_{32} + 20x_{33} + 11x_{34})$

S.T.

PTO

Subject to constraints

$x_{11}+x_{12}+x_{13}+x_{14}=2$ — (1)

$x_{21}+x_{22}+x_{23}+x_{24}=6$ — (2)

$x_{31}+x_{32}+x_{33}+x_{34}=7$ — (3)

$x_{11}+x_{21}+x_{31}=3$

$x_{12}+x_{22}+x_{32}=3$

$x_{13}+x_{23}+x_{33}=4$

$x_{14}+x_{24}+x_{34}=5$

& all $x_{ij} \geq 0$ $(i=1,2,3;\ j=1,2,3,4)$

* optimality Sol^n

Moving towards optimality

After obtaining an initial basic feasible sol^n to a a give T.P., the next question is how to arrive at the optimum solution. The basic steps are:

Step 1: Examination of ~~the~~ initial basic feasible solution for non-degeneracy. If it is degenerate, some modification is required to make it non-degenerate. — no loops & allocation allowed m+n−1

Step 2: (i) Determination of net-evaluations (Cost-difference) for empty cells. — (u, v method)

Step 3: Selection of the entering variable, provided Step 2 (ii) indicates that current solution can be improved.

Step 4: Selection of the leaving variable.

Step 5: Finally, repeating the steps 1 ~~to~~ through 4 until an optimum sol^n is obtained.

To examine the initial basic feasible sol^n for non-degeneracy

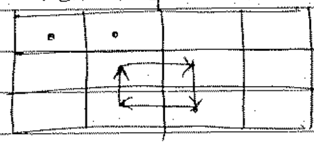
A basic feasible solution of $m \times n$ transportation problem is said to be non-degenerate, if it has following two properties:

(i) Initial BFS must contain $m+n-1$ number of individual allocations.

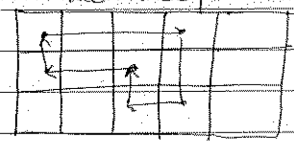
(ii) These allocations must be independent positions. Independent positions of a set of allocations mean that it is always impossible to form any closed loop through these allocations.

Eg.

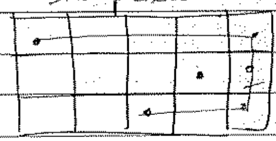
Non-independent



Non-independent



Independent



No loop formation

Determination of Net-Evaluations (u, v method)

Theorem 1. If we have a feasible solution consisting of $m+n-1$ independent allocations, and of numbers u_i & v_j satisfying $c_{rs}=u_r+v_s$ for each occupied cell (r,s), then the evaluation d_{ij} corresponding to each empty cell (i,j) is given by

$$d_{ij} = c_{ij} - (u_i + v_j)$$

The optimality test

If the cost difference $d_{ij} \geq 0$ (which implies increase in cost for each empty cell), then the BFS under test must be optimal. Otherwise, if $d_{ij} < 0$ ($\because$ negative difference implies decrease in cost) for one or more empty cells, then it would be better to reduce the cost more by allocating as much as possible to the cell with largest negative (smallest) value of d_{ij}. This way, it is possible to improve the BFS successively for reduced cost till the optimal solⁿ is obtained, for which $d_{ij} > 0$ for each empty cell.

Selection of Entering variable

Here our aim is to minimize the cost of transportation. So the current basic feasible solⁿ will not be optimum so long as any of the net evaluation d_{ij} is negative.

Thus if all d_{ij} are non-negative, the current solution is an optimum one.

$$\boxed{d_{rs} = \min_{i,j} \{ d_{ij} < 0 \}}$$

Selection of leaving Variable

Our next step will be to determine the leaving basic variable and then to determine the new improved basic solution.

The Simplex like leaving criterion in the notations of transportation problem states that if the variable x_{rs} is selected to enter the basis, then the basic variable x_{Bi} corresponding to the minimum ratio: $\min \left\{ \frac{x_{Bi}}{y_{irs}}, y_{irs} > 0 \right\}$ will leave the basis.

Working rule to obtain leaving variable and improved Basic feasible Solution

Step 1: After identifying the entering variable x_{rs}, describe a loop which starts and ends at the non-basic cell (r,s) connecting only the basic cells. Such a closed path exists and is unique for any non-degenerate basic solⁿ.

Step 2: The amount (say θ) to be allocated to the entering variable is interchangeably subtracted from and added to the successive end points of the closed loop so that the supply and demand constraints always remain satisfied.

Step 3: Then the minimum value of θ, which will render non-negative values for all the basic variables in the new solution is obtained. This consequently, determines the leaving variable.

VTU 2003 The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.

		I	II	III	IV	Supply
Warehouse	A	5	2	4	3	22
	B	4	8	1	6	15
	C	4	6	7	5	8
Requirement		7	12	17	9	45

The shipping clerk has worked out the following schedule from experience: 12 units from A to II, 1 unit from A to III, 9 units from A to IV, 15 units from B to III, 7 units from C to I, and 1 unit from C to III.

ⓐ Check & see if the clerk has optimal schedule.

ⓑ Find the optimal schedule and minimum total shipping cost.

ⓒ If the clerk is approached by a carrier of route C to II who offers to reduce his rate in the hope of getting some business, by how much the rate is reduced before the clerk should consider giving him an order.

Sol^n: The basic feasible sol^n is

	I	II	III	IV	
A	5	[12] 2	[1] 4	[9] 3	22
B	4	8	[15] 1	6	15
C	[7] 4	6	[1] 7	5	8
	7	12	17	9	45

IBFS $= 2\times12 + 4\times1 + 3\times9 + 1\times15 + 4\times7 + 7\times1$

$= 24 + 4 + 27 + 15 + 28 + 7$

$= 105$

	2	4	3	$u_1 = 0$
		1		$u_2 = -3$
4		7		$u_3 = 3$
$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 3$	

$1 = u_2 + v_3$

$1 = u_2 + 4$

$u_2 + 4 = 1$

$u_2 = 1 - 4$

$\boxed{= -3}$

$7 = u_3 + v_3$ $\quad 4 = v_1 + u_3$

$7 = u_3 + 4$ $\quad 4 = v_1 + 3$

$u_3 + 4 = 7$ $\quad \boxed{v_1 = 1}$

$\boxed{u_3 = 3}$

New cost of non basic cells

1	x	x	x	0
-2	-1	x	0	-3
x	5	x	6	3
1	2	4	3	

Cost difference is

5	x	x	x
4	8	x	6
x	6	x	5

−

1	x	x	x
-2	-1	x	0
x	5	x	6

=

4	x	x	x
6	9	x	6
x	1	x	-1

	x	+θ	-θ
x		-θ	θ

min(1, 9) = 1

5	[12] 2	[2] 4	[8] 3
4	8	[15] 1	
[7] 4		[0] ~~7~~	5 [1]

∴ Total minimum shipping cost is

$Z = 2\times12 + 4\times2 + 3\times8 + 1\times15 + 4\times7 + 5\times1$

$\boxed{Z = 104}$

ⓒ If the clerk decides to transport at the most 8 units from C to II (instead of 7 to I & 1 to IV), then II may reduce his cost from $C_{32} = 6$ atleast 4 in order to have the improved cost. So according to the given proposal, the total

Matrix form of LPP

Sol^n ~~Matrix C~~=

Max $Z = Cx$ (objective function)

Subject to $Ax \le b$ (constraints)

& $x \ge 0$ (Non-negativity Restriction)

where $C \geq (C_1, C_2, \ldots C_n)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Eg: using North-west corner Rule find the IBFS and find out the optimum cost for the following Matrix

(7) 6	(12) 3	(3) 5	4	22/15/3/0
5	9	(14) 2	(1) 7	15/1/0
5	7	8	(8) 6	8/0
7/0	12/0	17/14/0	9/8/0	45

IBFS $= 7\times6 + 12\times3 + 3\times5 + 14\times2 + 1\times7 + 8\times6$

$= 42 + 36 + 15 + 28 + 7 + 48$

$= 176$

Here $m+n-1 = 6$,

∴ Sol^n is non degenerated

Find out the ~~cost~~ u & v values for basic cells

6	3	5	*	$u_1 = 0$
*	*	2	7	$u_2 = -3$
*	*	*	6	$u_3 = -4$
$v_1 = 6$	$v_2 = 3$	$v_3 = 5$	$v_4 = 10$	

we have $C_{11} = u_1 + v_1$

$6 = 0 + v_1$

$\boxed{v_1 = 6}$ III^{ly} $v_2 = 3$, $v_3 = 5$

$C_{23} = u_2 + v_3$	$C_{24} = u_2 + v_4$	$C_{34} = u_3 + v_4$
$2 = u_2 + 5$	$7 = -3 + v_4$	$6 = u_3 + 10$
∴ $u_2 = -3$	$v_4 = 10$	$u_3 = -4$

now C_{ij} for non basic cells

*	*	*	10
3	0	*	*
2	1	1	*

difference is

*	*	*	-6
2	9	*	*
3	6	6	*

Entering variable

*	*	$-\theta$	θ
		$+\theta$	$-\theta$

=

(7) 6	(12) 3	(2) 5	(1) 4
		(15) 2	(0) ~~7~~
			(8) 6

Min(1) min(3, 1, 14)

$= \theta = 1$

∴ Sol^n $= 7\times6 + 12\times3 + 2\times5 + 1\times4 + 15\times2 + 8\times6$

$= 42 + 36 + 10 + 4 + 30 + 48$

$= 170$

Continued

$Z = x_1 + x_2 + x_3$

s.t

$x_1 + x_2 \le 8$

$-x_2 + x_3 \le 0$

5 -2

-3

$x_1 + x_2 + x_4 = 8$

$-x_2 + x_3 + x_5 = 0$

BV	NBV	value of BV	z	Feas.
x_1, x_2	$x_3 x_4 x_5 = 0$	$x_1 = 8$, $x_2 = 0$		$(8,0,0,0,0)$
x_1, x_3	$x_2 x_4 x_5 = 0$	$x_1 = 8$, $x_3 = 0$		$(8,0,0,0,0)$
x_2, x_3	$x_1 x_4 x_5 = 0$	$x_2 = 8$, $x_3 = 0$		$(0,8,0,0,0)$
x_2, x_4	$x_1 x_3 x_5 = 0$	$x_4 = 8$, $x_1 = 0$		$(0,0,0,8,0)$
x_1, x_5	$x_1, x_3, x_4 = 0$			

$x_1 + x_2 + x_4 = 8$

$-x_2 + x_3 + x_5 = 0$

$\cancel{x_1} = 8$

$x_5 = 0$

$x_1 + \cancel{x_2} = 8$

$\cancel{x_2} = 0$

$x_1 = 8$

$x_1 +$

$x_2 + x_4 = 8$

$-x_2 = 0$

$x_1 = 8$

$x_2 = 0$

Again apply MODI method

6	3	5	4	$u_1 = 0$
x	x	2	x	$u_2 = -3$
x	x	x	6	$u_3 = 2$
$v_1 = 6$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$	

Find out the new cost C_{ij} for ~~to~~ non basic cells

x	x	x	x	0
3	0	x	1	-3
8	5	7	x	2
6	3	5	4	

difference 6

x	x	x	x
2	9	x	6
-3	2	1	x

=

x -θ	x	x	+θ
		x	
+θ			-θ

θ = 8

Min 7

$\boxed{0}$ 6	$\boxed{12}$ 3	$\boxed{2}$ 5	$\boxed{7}$ 4
		$\boxed{15}$ 2	
$\boxed{7}$ 5			$\boxed{1}$ 6

$3 \times 12 + 2 \times 5 + 8 \times 4 + 2 \times 15 + 7 \times 5 + 1 \times 6$

$= 149$

Vogel's Approximation method (Unit Cost penalty method)

Step1: In lowest cost entry method, usually it is not possible to make an allocation to the cell (1,1) which has the second lowest cost in the matrix. It is trivial that allocation should be made in atleast one cell of each row and each column.

Step2: Next enter the difference between the lowest & second lowest cost entries in each column beneath the corresponding column, & put the difference between the lowest & second lowest entries of each row to the right of that row. Such individual differences can be thought of a penalty for making allocations in second lowest cost entries instead of lowest cost entries in each row or column.

* In case, the lowest & second lowest costs in row/column are equal, the penalty will be taken zero.

Eg:-

2	6	6	11	50
10	8	7	5	70
13	3	9	12	30
4	6	8	3	50
25	35	105	35	

(i) $\Sigma avail = 50 + 70 + 30 + 50 = 200$

$\Sigma req. = 25 + 35 + 105 + 35 = 200$

∴ The problem is balanced.

Row penalty

					I	II	III	IV
[25] 2	6	[25] 6	11	50/25/0	(4)	(4)	(0)	(0) (6)
10	8	[70] 7	5	70/0	(3)	(3)	(10)	(1) (7)
13	[30] 3	9	12	30/0	(6)	—	—	—
4	[5] 6	[10] 8	[35] 3	50/15/10/0	(1)	(1)	(2)	(2) (8)
25/0	35/5/0	105/95/25	35/0	200				

Col penalty				
I	(2)	(3)	(1)	(2)
II	(1)	(2)	(1)	(2)
III	—	(0)	(1)	(2)
IV	—	(0)	(1)	—
		(0)	(1)	

$2\times25 + 6\times25 + 7\times70 + 3\times30 + 6\times5 + 10\times8 + 35\times3$

Rs. ~~896~~ Rs 995

					Penalty I	II	III	IV
4	[20] 6	8	8	40/20/0	(2)	(2)	—	—
6	8	[50] 6	[10] 7	60/10/0	(0)	(1)	(1)	(1) —
5	[10] 7	6	[40] 8	50/40/0	(1)	(1)	(1)	(1) (1)
20/0	30/10/0	50/0	50/40/0	150				
(1)	(1)	(0)	(1)					
	(1)	(0)	(1)					
—	(1)	(0)	(1)					
—	(1)	—	(1)					

$20\times4 + 20\times6 + 50\times6 + 10\times7$

$+ 10\times7 + 40\times8$

$= 960$

$\boxed{\text{optimal sol}^n \text{ is } 920}$

Degeneracy in Transportation problems

The solution procedure for non-degenerate basic feasible solution with exactly m+n-1 strictly positive allocations in independent positions has been discussed so far. However, sometimes it is not possible to get such initial feasible solution to start with. Thus degeneracy occurs in transportation problem whenever a number of occupied cells is less than m+n-1.

Degeneracy in transportation problems can occur in two ways:

(1) Basic feasible solutions may be degenerate from the initial stage onwards.

(2) They may become degenerate at any intermediate stage.

Resolution of Degeneracy during the initial stage

To resolve degeneracy, allocate an extremely small amount of goods (close to zero) to one or more of the empty cells so that a number of occupied cells becomes m+n-1. The cell containing this extremely small allocations is, of course, considered to be an occupied cell.

Rule: The extremely small quantity usually denoted by the Greek letter 'Δ' [(or 'ε' epsilon)] is introduced in the least cost independent cell subject to the following assumptions. If necessary, two or more Δ's can be introduced in the least & second least cost independent cells.

(1) $\Delta < x_{ij}$ for $x_{ij} > 0$

(2) $x_{ij} + \Delta = x_{ij} - \Delta$, $x_{ij} > 0$

(3) $\Delta + 0 = \Delta$.

* (4) If there are more than Δ's in the solution $\Delta < \Delta'$, when Δ is to the left of Δ'.

Ex: A company has three plants A, B & C and three warehouses X, Y & Z. Number of units available at the plants is 60, 70 & 80, respectively. Demands at X, Y & Z are 50, 80, 80, respectively, unit costs of transportation are as follows.

	X	Y	Z	
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
	50	80	80	

Solⁿ: $\Sigma a_i = \Sigma b_j = 210$ Hence it is balanced pblm.

Solve by Vogel's approximation method

Penalty

8	7	[60] 3	60/0	(4) (4) (4) (3)
[50] 3	8	[20] 9	70/20/0	(5) (1) (9)
11	[80] 3	5	80/0	(2) (2)
50/0	80/0	80/60/0		
(5)	(4)	(2)		
	(4)	(2)		
		(6)		

Solⁿ: $50\times3 + 60\times3 + 20\times9 + 80\times3$

$= 750$

allocation should be

$m+n-1$

$3+3-1 = 5$

$\therefore 4 < 5$

But there are only 4 solⁿ & hence this is degeneracy in transportation problem.

Resolving degeneracy:

		[60] 3	60
[50] 3		[20] 9	70
	[80] 3		80
50	80	80	

Choose the least cost cell in the independent position & allocate value 'Δ' to the cell (it should not form any loops).

		[60] 3	60
[50] 3		[20] 9	70
	[80] 3	Δ (5)	80
50	80	80	

$80 + \Delta = 80$

Find out u_i & v_j values

		[60] 3	$u_1 = -2$
[50] 3		[20] 9	$u_2 = 4$
	[80] 3	5	$u_3 = 0$
$v_1 = -1$	$v_2 = 3$	$v_3 = 5$	

Step 4: Values of u_i & v_j will be obtained as shown previously once a unique set of u_i & v_j values has been determined, Various steps of the transportation algorithm can be applied in a routine manner to obtain an optimal solⁿ

Matrix C_{ij} for empty cells u_i+v_j

$$\begin{array}{ccc|c} -3 & 1 & x & -2 \\ x & 7 & x & 4 \\ -1 & x & x & 0 \\ \hline -1 & 3 & 5 & \end{array}$$

old C_{ij} for empty cells   new C_{ij} for empty cells

$$\begin{bmatrix} 8 & 7 & x \\ x & 8 & x \\ 11 & x & x \end{bmatrix} - \begin{bmatrix} -3 & 1 & x \\ x & 7 & x \\ -1 & x & x \end{bmatrix}$$

Matrix $[C_{ij} - (u_i+v_j)]$

$$\begin{bmatrix} 11 & 6 & x \\ x & 1 & x \\ 12 & x & x \end{bmatrix}$$

Since all cell evaluation are +ve from the table the Solⁿ under test is optimal.

The real total cost of subsequent Solⁿ didn't happen to change in the Eg after 'Δ' was introduced.

$$= 60\times3 + 50\times3 + 20\times9 + 80\times3 + 5(\Delta)$$

$$= 750 + 5(0) \quad \text{Where } \Delta = 0$$

$$= 750$$

The solution under test is optimal. The real total cost of subsequent solutions did not happen to change in the example after Δ was introduced.

In general, this will not be the case. In as much as the infinitesimal quantity Δ plays only an auxiliary role and has no significance, it is removed when the optimal solⁿ is obtained. Hence the final answer is 750

Resolution of degeneracy during the solution stages

The transportation problem may also become degenerate during the solution stages. This happens when most favourable quantity is allocated to the empty cells having the largest negative cell-evaluation resulting in simultaneous vacation of two or more of currently occupied cells. To resolve degeneracy, allocate Δ to one or more of recently vacated cells so that the number of occupied cells is m+n-1 in the new solution. This type of degeneracy can be explained below by

Eg:- The cost requirement table for the TP is given below.

4	3	1	2	6	40
5	2	3	4	5	30
3	5	6	3	2	20
2	4	4	5	3	10
30	30	15	20	5	100

$\Sigma a_i = \Sigma b_i = 100$

∴ Hence it is balanced.

Solⁿ: By North-West Corner Method

4

PTO

8

[30] 4	[10] 3	1	2	6	40/10/0
[20] 2	[10] 3	4	5		30/10/0
8	15	[5] 6	[15] 3	2	20/15/0
2	4	4	[5] 5	[5] 3	10/5/0
30/0	30/20/0	15/5/0	20/5/0	5/0	

$m+n-1 = 8$

allocation: 8 ∴ It is non degenerate

Step 3: Find u_i & v_j for basic cells

4	3	.	.	.	$u_1 = 0$
.	2	3	.	.	$u_2 = -1$
.	.	6	3	.	$u_3 = 2$
.	.	.	5	5	$u_4 = 4$

$v_1 = 4$ $v_2 = 3$ $v_3 = 4$ $v_4 = 1$ $v_5 = -1$

Step 4: Matrix old C_{ij} for empty cells   new $C_{ij} = u_i + v_j$ for empty cells

×	×	1	2	6
5	×	×	4	5
3	5	×	×	2
2	4	4	×	×

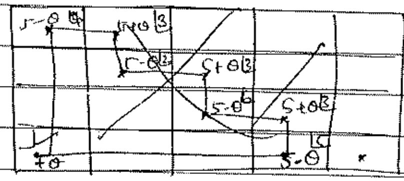
−

×	×	4	1	−1
3	×	×	0	−2
6	5	×	×	1
8	7	8	×	×

old C_{ij} − new C_{ij}

×	×	−3	1	7
2	×	×	4	7
−3	0	×	×	1
✓−6	−3	−4	×	×

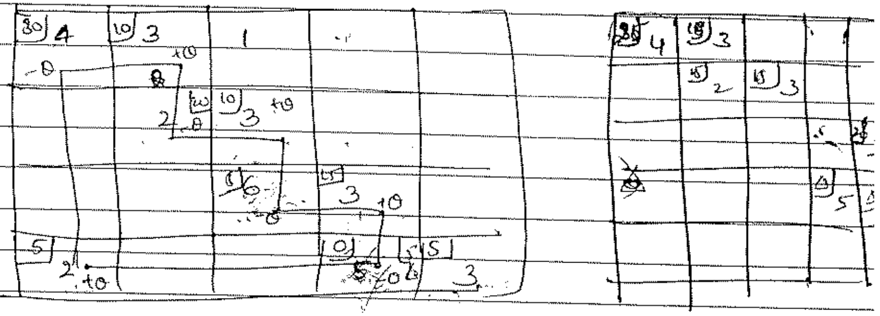
Since, the largest negative cell evaluation is $d_{41} = -6$, allocate as much as possible to this cell $(4,1)$.



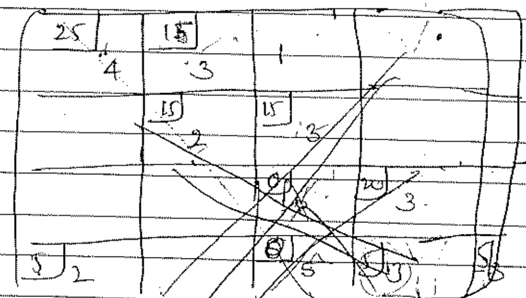
Min $(30-\theta, 20-\theta, 5-\theta, 5-\theta) = 0$

∴ $\theta = 5$

The revised solⁿ become



revised solⁿ is



Solution

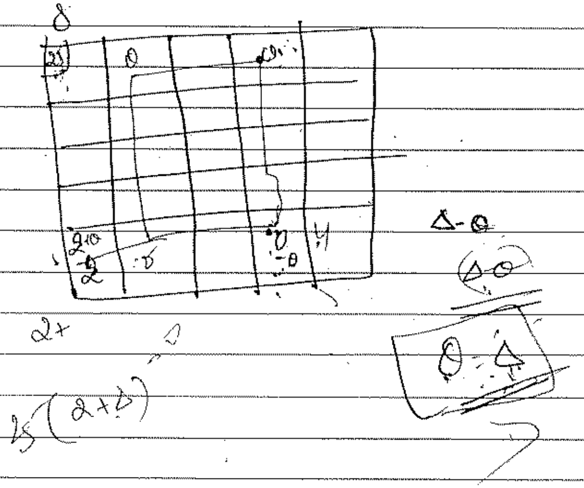
$25\times4 + 13\times3 + 15\times2 + 15\times3 + 20\times3 + 5\times3 + 5\times2$

~~T sol~~ⁿ $8 \neq 7$

• Now this degeneracy may resolve by adding Δ to one of the recently vacated cells (3,3) or (4,4).

But in minimization problem, add Δ to recently vacated cell (4,4) only ∵ it has the lowest shipping cost of Rs 5 per unit.

The rest of the procedure will be exactly same as explained earlier. This way the optimal sol^n can be ~~explained~~ obtained.



Eg: 2 VAM method

							Pen I II III IV V
9	12	[5] 9	6	9	10	5	(3)
7	[4] 3	7	7	5	[2] 5	6/4	(2)
[1] 6	5	[1] 9	12	3	11	2	(2)
[3] 6	8	11	[2] 2	[4] 2	10	9	(0)
4	4	6	2	4	2/0		
(0)	(2)	(2)	(4)	(1)	(5)		

I
II
III
IV
V

8 < 9

Using VAM method the initial basic feasible sol^n having transportation cost Rs 112 is given below

		5(9)				5
	4(3)	Δ(7)			2(5)	6
1(6)		1(9)				2
3(6)			2(2)	4(2)		9
4	4	6	2	4	2	

Since the no of allocations (= 8) is less than m+n-1 = 9 a very small quantity Δ may be introduced in the independent cell (2,3), although least cost independent cell is (2,5).

		•5			
	•4	•Δ			•2
•1		•1			
•3			•2	•4	

Since all the net evaluations are non-negative, the current solⁿ is an optimum one. Hence the optimum solⁿ is

$$5(9) + 4(3) + \Delta(7) + 2(5) + 1(6) + 1(9) + 3(6) + 2(2) + 4(2)$$
$$= Rs.112$$

NOTE: In above optimum table, Δ may also be introduced in least cost independent cell (2,1)

③ Solve the transportation problem for minimisation

2	2	3	10
4	1	2	15
1	3	1	40
20	15	30	65

Solⁿ

$\Sigma a_i = \Sigma b_i$, the pblm is balanced.

Apply Northwest Corner rule

(10) 2	2	3	10/0
(10) 4	(5) 1	2	15/5/0
1	(10) 3	(30) 1	40/30
20/10/0	15/10/0	30	

3+3−1
6−1
= 5

$10\times2 + 10\times4 + 5\times1 + 10\times3 + 30\times1$

$\boxed{= Rs.125}$

Since the no. of occupied cell = 5 = m+n−1 and all the allocations are independent, we get

To find the optimal solution (MODI Method)

We determine a set of numbers u_i & v_j for each row and each column with $u_i + v_j = c_{ij}$ for each occupied cell. ~~To start with we give 0 to the third column as it has the maximum number of allocation~~.

2	x	x	$u_1 = 2$
4	1	x	$u_2 = 4$
x	3	1	$u_3 = 6$
$v_1 = 0$	$v_2 = -3$	$v_3 = -5$	

new c_{ij} for non basic =

x	−1	3
x	x	−1
6	x	x

x	2	3
x	x	2
1	x	x

=

x	−3	0
x		−3
5	x	x

old c_{ij} − new c_{ij}

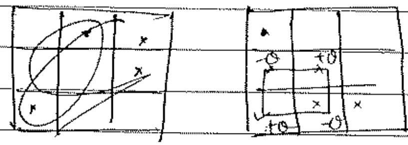
x	2	3
x	x	2
1	x	x

−

x	−1	3
x	x	−1
6	x	x

=

x	3	0
x	x	3
−5	x	x



5

(10) 2		
(5) 4	(10) 1	
(5) 1	(5) 3	(30) 1

min (5, 1)

Determine the optimal sol^n to each of the following degenerate transportation problem.

4	7	3	8	2	4
1	4	7	3	8	7
7	2	4	7	7	9
4	8	2	4	7	2
8	3	7	2	2	

Ans $x_{11}=1$, $x_{13}=1$, $x_{21}=7$, $x_{32}=3$, $x_{15}=2$, $x_{23}=6$, $x_{44}=2$

Min cost = 56

Unbalanced Transportation problem:

So far we have discussed the balanced type of transportation problems where the total destination requirement equals the total origin capacity ($\Sigma a_i = \Sigma b_j$). But, sometimes in practical situations, the demand may be more than availability, & viceversa ($\Sigma a_i \neq \Sigma b_j$)

$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then such pblm is unbalanced transportation pblm.

To modify unbalanced T.P to balanced type

An unbalanced T.P may occur in two different ways.

(i) Excess of availability
(ii) Shortage in availability

Case(i) (Excess of availability i.e. $\Sigma a_i \geq \Sigma b_j$)

Whenever $\Sigma a_i \geq \Sigma b_j$, we introduce a dummy destination - column in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to the zero. The requirement at this dummy destination is assumed to be equal to the difference $\Sigma a_i - \Sigma b_j$.

Case (ii) (Shortage in Availability i.e. $\Sigma a_i \leq b_j$)

whenever $\Sigma a_i \leq \Sigma b_j$, introduce a dummy source in the transportation table. The cost of transportation from this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference ($\Sigma b_i - \Sigma a_i$)

Eg:

						Avail
	4	2	3	2	6	8
	5	4	5	2	1	12
	6	5	4	7	3	14
req	4	4	6	8	8	

Avail = 8+12+14 = 34
req = 4+4+6+8+8 = 30
∴ difference is 4

Sol^n Since the total requirement are 30 & total capacities is 34, the problem is of unbalanced type ∴ the problem can be modified as.

Step 1. (Modifying the given problem to balanced type)
Since the capacities are four more than the total req

4	2	3	2	6	0	8
5	4	5	2	1	0	12
6	5	4	7	3	0	14
4	4	6	8	8	4	34

dummy destn

Step 2: To find the initial solution, Apply the Vogel's method as usual.

							Penalty I	II	III	IV
4	[4] 2	3	[4] 2	6	0	8/4/0	(2) (0)	(0)	(0)	(1)
5	4	5	[4] 2	[8] 1	0	12/4/0	(1)	(1)	(2)	— —
[4] 6	5	[6] 4	7	3	[4] 0	14/10/6/0	(3)	(1)	(1)	(1) (1)
4/0	4/0	6/0	8/4/0	8/0	4/0					
(1)	(2)	(1)	(0)	(2)	(0)					
(1)	(2)	(1)	(0)	(2)	—					
(1)	(2)	(1)	(0)	—						
(1)	(3)	(1)	(5)	—	—					

This gives the transportation cost =
$4(2)+4(2)+4(2)+8(1)+4(6)+6(4)+4(0)$
$= 80$

Step 3: (To test the initial sol^n for optimality)

Since the total no of allocations is 7 instead of $6+3-1 = 8$. This is a degenerate basic sol^n.
∴ allocate an infinitesimal quantity Δ to empty cell (1,1). Then proceeding in the usual manner, following tables for testing the optimality of the sol^n are obtained.

Δ 4	[4] 2	3	[4] 2	6	0	8
5	4	5	[4] 2	[8] 1	0	12
[4] 6	5	[6] 4	7	3	[4] 0	4
4	4	6	8	8	4	

4	2	3	2	6	0	$u_1 = 0$
5	4	5	[4] 2	[8] 1	0	$u_2 = 0$
[4] 6	5	[6] 4	7	3	[4] 0	$u_3 = 2$

$v_1 = 4$ $v_2 = 2$ $v_3 = 2$ $v_4 = 2$ $v_5 = 1$ $v_6 = -2$

New c_{ij}

•	•	2	•	1	-2	0
4	2	2	•	•	-2	0
•	4	•	4	3	•	2
4	2	2	2	1	-2	

old c_{ij} − new c_{ij}

.	.	1	.	5	8
1	2	3	.	.	2
.	1	.	3	0	.

Since all d_{ij} values for empty cells are non-negative, the sol^n under test is optimal.

---

(2)

5	8	6	6	3	800
4	7	7	6	5	500
8	4	6	6	4	900
400	400	500	400	800	

2200

2500

= 300 unit

5	8	6	6	3	800
4	7	7	6	5	500
8	4	6	6	4	900
0	0	0	0	0	300
400	400	500	400	800	

		800(6)		300(3)	800
400(4)			100(6)	Δ5	500 + Δ = 500
	400(4)			500(4)	900
			300(0)		300
400	400	500	400	800 + Δ = 800	

(3)

5	1	7	10
6	4	6	80
3	2	5	15
75	20	50	

Supply - 105

demand = 145

(40)

Consider the following unbalanced T.P there is not enough supply some of the demands at this destination may not be satisfied. Suppose there are penalty costs for every unsatisfied demand units which are given by 5, 3 & 2 for 1, 2 & 3 respectively find the opt. sol^n

1	2	3		I	II	III	IV
5	[10] 1	7	10/0	(4)	–	–	–
[60] 6	[10] 4	[10] 6	80	(2)	(2)		
[15] 3	2	5	15	(1)	(1)		
5	3	[40] 2	40/0	(1)	(1)		
75	20/10	50/10	145				
(2)	(1)	(3)					

$m+n-1$

$= 4+3-1$

$= 6 = 6$

10 + 360 + 40 + 60 + 45 + 80

= 595

5 3 (2)	(1) 10	(7) 3	−3
(6) 60	(4) 10	(6) 10	0
(3) 15	(2) 1	(1) 3	−3
(5) 2	(3)	(2) 40	−4
6	4	6	

Since all the net evaluations are non-negative, the O.S is given by

$x_{12}=10,\ x_{21}=60,\ x_{22}=10,\ x_{23}=10,\ x_{31}=15$

(allocations in dummy row is not considered)

$\therefore$ the optimum transportation cost is given by

$(10\times1)+(60\times6)+(10\times4)+(10\times6)+(15\times3)$

$=515$

The Assignment problem

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit). Suppose that we have 'n' jobs to be performed on 'm' machines (one job to one machine) & our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be in the form of $m\times n$ matrix (c_{ij}) called a cost matrix (or) effectiveness matrix where c_{ij} is the cost of assigning i^{th} Machine to the j^{th} job.

(jobs j)

(Machine i)	1	2	3	....	n
1	c_{11}	c_{12}	c_{13}	....	c_{1n}
2	c_{21}	c_{22}	c_{23}	....	c_{2n}
3	c_{31}	c_{32}	c_{33}	....	c_{3n}
⋮					
m	c_{m1}	c_{m2}	c_{m3}	...	c_{mn}

Formulation of an assignment problem

Consider an assignment problem of assigning n jobs to n machines (one job to one machine)

Let c_{ij} be the unit cost of assigning i^{th} Machine to the j^{th} job &

Let $x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is } i^{th} \text{ M/c} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ M/c} \end{cases}$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2 \cdots n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2 \cdots n \ \& \ x_{ij} = 0 \text{ (or) } 1$$

Assignment Model (comparison with transportation Model)

The assignment ~~model~~ problem may be considered as a special case of the transportation problem.

Consider a transportation problem with 'n' destination & 'n' sources.

Destination

	1	2	3	....	n	Supply (a_i)
1	C_{11}	C_{12}	C_{13}	....	C_{1n}	a_1
2	C_{21}	C_{22}	C_{23}	....	C_{2n}	a_2
3	C_{31}	C_{32}	C_{33}	....	C_{3n}	a_3
Source :	.	.	.	—	.	.
:	.	.	.	—	.	.
:	.	.	.	—	.	.
n	C_{n1}	C_{n2}	C_{n3}	....	C_{nn}	a_n
Demand (b_j)	b_1	b_2	b_3	....	b_n	

the total transportation cost

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

is minimized

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2 \cdots n$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, 2 \cdots n$$

$$\sum a_i = \sum b_j, \quad i, j, 1, 2 \cdots n$$

$$\& \ x_{ij} \geq 0, \quad i, j = 1, 2, 3 \cdots n$$

Hence the 'sources' represent 'facilities' & 'destinations' represents 'jobs'.

Suppose that the supply available at each source is 1 i.e., $a_i = 1$ & demand required at each destination is 1 i.e., $b_j = 1$

Let C_{ij} be the unit transportation cost from the i^{th} source to the j^{th} destination. Here it means the cost of assigning the i^{th} M/c to the j^{th} job.

Let x_{ij} is the amount to be shipped from i^{th} source to the j^{th} destination. Here it means the assignment of the i^{th} M/c to the j^{th} job.

P.T.O

We can restrict the value of x_{ij} to be either 0 (or) 1.
$x_{ij}=0$ means that the i^{th} Machine does not get the j^{th} job and $x_{ij}=1$ means that the i^{th} Machine gets the j^{th} job.

Since each machine should be assigned to only one job & each job requires only machine, the total assignment value of i^{th} M/c is 1, (i.e)

$\sum_{j=1}^{n} x_{ij}=1$ & the total assignment value of the j^{th} job is (i.e) $\sum x_{ij}=1$.

Hence the assignment problem can be expressed as

$$\text{Minimize } Z=\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

where c_{ij} is the cost of assigning i^{th} M/c to the j^{th} job subject to the constraints.

$$x_{ij}=\begin{cases}1, & \text{if } i^{th} \text{ m/c is assigned the } j^{th} \text{ job}\\ 0, & \text{if } i^{th} \text{ M/c is not " " "}\end{cases}$$

i.e. $x_{ij}=0$ (or) 1

$$\Rightarrow x_{ij}(x_{ij}-1)=0$$

$$\Rightarrow x_{ij}^2=x_{ij}$$

$$\sum_{j=1}^{n} x_{ij}=1,\ i=1,2\ldots n \ \&\ \sum_{i=1}^{n} x_{ij}=1,\ j=1,2\ldots n$$

From this we see that assignment problem represents a transportation problem with all demands and supplies equal to 1.

The units available at each source & units demanded at each destination are equal to 1. It means exactly one occupied cell in each row & each column of the transportation table i.e. only 'n' occupied cells in place of the required $n+n-1=2n-1$ occupied cells. Hence, an assignment problem is a completely degenerate form of a transportation problem.

But the transportation technique (or) simplex method cannot to be used to solve the assignment problem ∵ of degeneracy. In fact a very convenient iterative procedure is available for solving an assignment pblm.

The technique used for solving assignment problem makes use of the following two theorems.

Theorem 1: The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of the row or column of the assignment of cost matrix.

Theorem 2: If for an assignment problem all $c_{ij}\geq 0$ then an assignment schedule (x_{ij}) which satisfies $\sum c_{ij}x_{ij}=0$ must be optimal.

Differences between the transportation & Assignment pblm

T.P	A.P
(1) ~~Supply~~ The no of rows & columns need not be equal	(1) The number of rows & columns should necessarily be equal
(2) Supply at any source may be any positive quantity a_i.	(2) Supply at any source (m/c) will be 1, i.e. $a_i=1$
(3) Demand at any destination may be any positive quantity b_j	(3) Demand at any destination (job) will be 1, i.e. $b_j=1$
(4) One source to any number of	(4) One source (M/c) to only one

A Special Algorithm for the Assignment problem (Hungarian Method)

Various steps of the computational procedure for obtaining an optimal assignment is summarized as follows:

Step1: Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows.

Step2: Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective column. Thus obtain the first modified matrix.

Step3: Then, draw the minimum number of horizontal and vertical lines to cover all the zeroes in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities.

(i) If $N = n$, the number of rows (columns) of given matrix, then an optimal assignment can be made. So make the zero assignment to get the required sol^n.

(ii) If $N < n$, then proceed to step 4.

Step4: Determine the smallest element in the matrix, not covered by the N lines. Subtract this minimum element from all uncovered elements & add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step5: Again repeat steps 3 & 4 until minimum number of lines become equal to the number of rows (colms) of the given matrix i.e. $N = n$.

Step6: (To make zero-assignment). Examine the rows successively until a row-wise exactly single zero is found, mark this zero by '□' to make the assignment. Then, mark a cross (x) over all zeroes if lying in the column of the marked '□' zero, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

Step7: Repeat the step6 successively until one of the following situations arise:

(i) if no marked zero is left, then the process ends; or

(ii) if there lie more than one of the unmarked zeroes in any column or row, then mark '□' one of the ~~two~~ unmarked zeroes arbitrarily & mark a cross in the cells of remaining zeroes in its row & column. Repeat the process until no unmarked zero is left in the matrix.

Step8: Thus exactly one marked '□' zero in each row & each column of the matrix is obtained. The assignment corresponding to these marked '□' zeroes will give the optimal assignment.

A Rule to Draw Minimum Number of lines

A very convenient rule of drawing minimum number of lines to cover all the 0's of the reduced matrix is given in the following steps.

(i) Step1: Tick (✓) rows do not have any marked (□) zero.

Step2: Tick (✓) columns having marked (□) zeroes or otherwise in ticked rows.

Step 3: Tick (✓) rows having marked 0's in ticked columns.

Step 4: Repeat steps 2 & 3 until the chain in ticking is complete.

Step 5: Draw lines through all unticked rows & ticked columns.

Ep: (1): Solve the Assignment problem.

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Sol^n: No of rows = No of column.
∴ The problem is balanced.

Step 1: Select the smallest cost element in each row & subtract this from all the elements of the corresponding row, we get the reduced matrix.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step 2: Select the smallest cost in each column ~~from~~ & subtract this from all the elements of the corresponding column, we get the following reduced matrix.

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	$\cancel{0}$
9	12	4	$\boxed{0}$

Step 3: Now we shall examine the rows successively.
1st row contains a single unmarked zero, encircle this zero & cross all the other zeroes in its column.

i.e., (1) Starting with the row - (1) of the matrix examine the row one by one until a row containing a single '0' element is found.
An experimental assignment indicated by '□' is mark to that cell.
(2) Now cross all other 0's in column in which the assignment has made this eliminate the possibility of marking this assignment.

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	$\cancel{0}$
D	9	12	4	$\boxed{0}$

∴ A - I
B - III   8 + 4 + 19 + 10
C = II   = 41
D : IV

Ep 2: Four jobs are to be done on 4 different machines the cost of producing i^th job on j^th machine is given below.

Assign the job as to different M/c so as to minimize the assignment cost.

	M/c			
job	15	11	13	15
	17	12	12	13
	14	15	10	14
	16	13	11	17

Sol^n: No of rows = No of columns

Step 1: Select the smallest cost in each row & subtract this from all elements.

15	11	13	15
17	12	12	13
14	15	10	14
16	13	11	17

⇓

Row element − Row Minimizing

4	0	2	4
5	0	0	1
4	5	0	4
5	4	0	6

Step 2: Select the smallest cost in each column & subtract this from all the elements of the corresponding column & the matrix is reduced as

I column element − column minimum

	I	II	III	IV
A	$\cancel{0}$	$\boxed{0}$	2	3
B	1	0	0	$\boxed{0}$
C	$\boxed{0}$	5	0	3
D	1	4	$\boxed{0}$	5

A − 2
B − 4
C − 1
D − 3

∴ 11 + 13 + 14 + 11

= 49

Problem 3:

15	17	14	16
11	12	15	13
13	12	10	11
15	13	14	17

⇓

Row element − Row Minima

1	3	0	2
0	1	3	2
3	2	0	1
2	0	1	4

⇓

Column element − Column Minimum

1	3	$\boxed{0}$	1
$\boxed{0}$	1	3	1
3	2	$\cancel{0}$	$\boxed{0}$
2	$\boxed{0}$	1	3

A − 3
B − 1
C − 4
D − 2

14 + 11 + 11 + 13

= 49

(4) Solve the following problem

	1	2	3	4	5
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Sol^n Step1: Row Minimization

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

Step2: Column Minization

30	$\boxed{0}$	~~35~~	~~30~~	~~15~~
15	0	$\boxed{0}$	10	0
30	0	35	30	20
$\boxed{0}$	0	20	0	5
20	0	25	15	15

Choose the row element & column element 1's & 2's

In this type of problem we go to "Hungarian method"

Step1: Mark (✓) for rows having no assignments
Step2: Mark (✓) for columns having zeros in the marked row
Step3: Mark (✓) rows having assigned zeros in marked column.
Step4: Repeat 2 & 3, continue till no further marking
Step5: Draw lines through a marked column & unmarked rows (should cover all assignments).

Step6: Find the minimum in the uncovered element [no lines] say x.

$\boxed{x = 15}$

Step7: Subtract x from all uncovered elements & add it to the intersection lines cell.

Till Step 6.

30	0	35	30	15
15	0	$\boxed{0}$	10	~~0~~
30	0	35	30	20
$\boxed{0}$	0	20	~~0~~	5
20	0	25	15	15

$x = 15$

Step7:

	I	II	III	IV	V
A	15	0	20	15	$\boxed{0}$
B	15	15	$\boxed{0}$	10	0
C	15	$\boxed{0}$	20	15	5
D	$\boxed{0}$	15	20	0	5
E	5	~~0~~	10	$\boxed{0}$	~~0~~

A → V 5 $\quad$ 200 + 130 + 110 + 50 + 80
B → III 3
C → II 2 $\quad$ = 570
D → I 1
E → IV 4

② Write the procedure of Hungarian Method

	I	II	III	IV	V
A	11	7	10	17	10
B	13	21	7	11	13
C	13	13	15	13	14
D	18	10	13	16	14
E	12	8	16	19	10

Row Minimization

Step 1:

4	0	3	10	3
6	14	0	4	6
0	0	2	0	1
8	0	3	6	4
4	0	8	11	2

Column Minimization

Step 2:

4	$\boxed{0}$	3	10	2
6	14	$\boxed{0}$	4	5
$\boxed{0}$	$\cancel{0}$	2	0	0
8	0	3	6	3
4	0	8	11	1

Step 3:

4	$\boxed{0}$	3	10	2
6	14	$\boxed{0}$	4	5
0	0	2	0	0
8	0	3	6	3
4	0	8	11	1

$\boxed{\min = 1}$
x

3	$\boxed{0}$	2	9	1
6	15	$\boxed{0}$	4	5
$\boxed{0}$	1	2	0	0
7	0	2	5	2
3	0	7	10	$\boxed{0}$

$x = 1$

2	$\cancel{0}$	1	8	$\boxed{0}$
6	16	$\boxed{0}$	4	5
$\boxed{0}$	2	2	0	0
6	$\boxed{0}$	1	4	1
3	1	7	10	$\cancel{0}$

2	$\cancel{0}$	1	8	$\boxed{0}$
6	16	$\boxed{0}$	4	5
$\boxed{0}$	2	2	0	0
6	$\boxed{0}$	1	4	1
3	1	7	10	$\cancel{0}$

$x = 1$

1	$\cancel{0}$	$\cancel{0}$	7	0
6	16	$\boxed{0}$	4	6
$\boxed{0}$	2	2	0	1
6	$\boxed{0}$	1	4	2
2	$\cancel{0}$	6	9	$\boxed{0}$

$x = 1$

	I	II	III	IV	V
A	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	6	$\cancel{0}$
B	5	15	$\boxed{0}$	3	4
C	$\cancel{0}$	3	3	$\boxed{0}$	2
D	5	$\boxed{0}$	1	3	2
E	3	$\cancel{0}$	6	8	$\boxed{0}$

Ans:

A – I
B – III
C – IV
D – II
E – V

$11 + 7 + 13 + 10 + 10$
$= 51$

(3) using the following cost matrix, determine (a) optimal job assignment (b) cost of assignments.

$$
\begin{array}{c c}
 & \text{Job} \\
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\text{Mechanic} & \begin{bmatrix} 10 & 3 & 3 & 2 & 8 \\ 9 & 7 & 8 & 2 & 7 \\ 7 & 5 & 6 & 2 & 4 \\ 3 & 5 & 8 & 2 & 4 \\ 9 & 10 & 9 & 6 & 10 \end{bmatrix}
\end{array}
$$

Sol^n: Select the smallest cost in each row & subtract this

(4) A company has 5 jobs to be done on 5 machines. Any job can be done on any machine. The cost of doing the jobs in different m/c's are given below. Assign the jobs for different m/s so as to minimize the total cost.

	A	B	C	D	E
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

Step I. Row minimization

$$
\begin{bmatrix} 5 & 0 & 8 & 10 & 19 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 4 & 0 & 8 \end{bmatrix}
$$

Step L: Column minimization:

Since each column has minimum element 0, we have the first modified matrix.

$$\begin{bmatrix} 5 & \boxed{0} & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & \cancel{0} & 0 & \boxed{0} \\ \boxed{0} & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & \boxed{0} & 8 \end{bmatrix}$$

∴ Covered zero is 4 which is < the order of matrix

∴ Apply Hungarian method:

$$\begin{bmatrix} 5 & \boxed{0} & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & \boxed{0} & 0 & 0 \\ \boxed{0} & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & \boxed{0} & 8 \end{bmatrix}$$

$x = 2$

Find minimum in uncovered element

$$\begin{bmatrix} 7 & \boxed{0} & 8 & 13 & 11 \\ 0 & 3 & 12 & 0 & \boxed{0} \\ M & 5 & \boxed{0} & 3 & 0 \\ \boxed{0} & 3 & 1 & 2 & 4 \\ 3 & 2 & 3 & \boxed{0} & 5 \end{bmatrix}$$

Job	M/c
1	B
2	E
3	C
4	A
5	D

All the five jobs have been assigned to 5 M/cs

∴ Minimum (Total cost) = 8+12+4+6+12 = 42

(4) Solve the following Assignment problem

$$\begin{bmatrix} M & 6 & 12 & 6 & 4 \\ 3 & M & 7 & 2 & 1 \\ 8 & 5 & M & 11 & 3 \\ 3 & 2 & 9 & M & 3 \\ 6 & 3 & 7 & 8 & M \end{bmatrix}$$

Step 1: Row element − Row minimum

$$\begin{bmatrix} M & 2 & 8 & 2 & 0 \\ 2 & M & 6 & 1 & 0 \\ 5 & 2 & M & 8 & 0 \\ 1 & 0 & 7 & M & 1 \\ 3 & 0 & 4 & 5 & M \end{bmatrix}$$

Step 2: Column element − Column minimum

$$\begin{bmatrix} M & 2 & 4 & 1 & 0 \\ 1 & M & 2 & 0 & 0 \\ 4 & 2 & M & 7 & 0 \\ 0 & 0 & 3 & M & 1 \\ 2 & 0 & 0 & 4 & M \end{bmatrix}$$

Step 3: Find a row with one zero, make an assignment

$$\begin{bmatrix} M & 2 & 4 & 1 & \boxed{0} \\ 1 & M & 2 & \boxed{0} & 0 \\ 4 & 2 & M & 7 & 0 \\ \boxed{0} & 0 & 3 & M & 1 \\ 2 & \boxed{0} & 0 & 4 & M \end{bmatrix}$$

(Assignment) 4 < 5 (order)

Step 4: Hungarian method

$$\begin{bmatrix} M & 2 & 4 & 1 & \boxed{0} \\ 1 & M & 2 & \boxed{0} & 0 \\ 4 & 2 & M & 7 & 0 \\ \boxed{0} & 0 & 3 & M & 1 \\ 2 & \boxed{0} & 0 & 4 & M \end{bmatrix}$$

$x = 1$

$$\begin{bmatrix} M & 1 & 3 & 0 & 0 \\ 1 & M & 2 & \boxed{0} & 1 \\ 3 & 1 & M & 6 & 0 \\ \boxed{0} & 0 & 3 & M & 2 \\ 2 & \boxed{0} & 0 & 4 & M \end{bmatrix}$$

Still 4 < 5

Apply Hungarian method

$$\begin{bmatrix} M & 1 & 3 & 0 & 0 \\ 1 & M & 2 & \boxed{0} & 1 \\ 3 & 1 & M & 6 & \boxed{0} \\ \boxed{0} & 0 & 3 & M & 2 \\ 2 & \boxed{0} & 0 & 4 & M \end{bmatrix} \quad x = 1$$

$$\begin{bmatrix} M & 0 & 2 & \boxed{0} & 0 \\ \boxed{0} & M & 1 & 0 & 1 \\ 2 & 0 & M & 6 & \boxed{0} \\ 0 & \boxed{0} & 3 & M & 3 \\ 2 & 0 & \boxed{0} & 5 & M \end{bmatrix}$$

$1 \to 4$
$2 \to 1$
$3 \to 5$
$4 \to 2$
$5 \to 3$

OK

Cost = $6+3+3+2+7 = 21$

Alternate sol^n

$$\begin{bmatrix} M & \boxed{0} & 2 & 0 & 0 \\ 0 & M & 1 & \boxed{0} & 1 \\ 2 & 0 & M & 6 & \boxed{0} \\ \boxed{0} & 0 & 3 & M & 3 \\ 2 & 0 & \boxed{0} & 5 & M \end{bmatrix}$$

$6+2+3+3+7 = 21$

$1 \to 2$
$2 \to 4$
$3 \to 5$
$4 \to 1$
$5 \to 3$

(5) $$\begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix}$$

Ans:

Job	M/c
J_1	M_4
J_2	M_2
J_3	M_1
J_4	M_3

Minimum cost - $6+5+4+8 =$ Rs. 23

Alternate sol^n

$J_1 \to M_1$; $J_2 \to M_2$, $J_3 \to M_3$, $J_4 \to M_4$

Minimum cost = $5+5+10+3 =$ Rs. 23

Unbalanced Assignment problem:

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix i.e., the number of rows & the number of columns are not equal. To make it balanced we add a dummy row or dummy column with all the entries as zero.

Ex: (1) $$\begin{bmatrix} 9 & 14 & 19 & 15 \\ 7 & 17 & 20 & 19 \\ 9 & 18 & 21 & 18 \\ 10 & 12 & 18 & 19 \\ 10 & 15 & 21 & 16 \end{bmatrix} \Longrightarrow \begin{bmatrix} 9 & 14 & 19 & 15 & 0 \\ 7 & 17 & 20 & 19 & 0 \\ 9 & 18 & 21 & 18 & 0 \\ 10 & 12 & 18 & 19 & 0 \\ 10 & 15 & 21 & 16 & 0 \end{bmatrix}$$

Row minimization

$$\begin{bmatrix} 9 & 14 & 19 & 15 & 0 \\ 7 & 17 & 20 & 19 & 0 \\ 9 & 18 & 21 & 18 & 0 \\ 10 & 12 & 18 & 19 & 0 \\ 10 & 15 & 21 & 16 & 0 \end{bmatrix}$$

Column minimization

$$\begin{bmatrix} 2 & 2 & 1 & \boxed{0} & 0 \\ \boxed{0} & 5 & 2 & 4 & 0 \\ 2 & 4 & 3 & 3 & \boxed{0} \\ 3 & 0 & 0 & 4 & 0 \\ 3 & 3 & 3 & 1 & 0 \end{bmatrix} \quad x = 2$$

$$\begin{bmatrix} 2 & 2 & 1 & \boxed{0} & 0 \\ \boxed{0} & 5 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 & \boxed{0} \\ 3 & \boxed{0} & 0 & 4 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$x = 1$

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 1 & 1 & 0 & 0 & 1 \\ B & 0 & 5 & 2 & 5 & 2 \\ C & 0 & 1 & 0 & 1 & 0 \\ D & 3 & 0 & 0 & 5 & 2 \\ E & 1 & 1 & 1 & 0 & 0 \end{array}$$

$A \to 4$
$B \to 1$
$C \to 3$.
$D \to 2$
$E \to 5$

$15 + 7 + 21 + 12 + 0$
$= 55$

54

(2)

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline 1 & 4 & 3 & 6 & 2 & 7 \\ 2 & 10 & 12 & 11 & 14 & 16 \\ 3 & 4 & 3 & 2 & 1 & 5 \\ 4 & 8 & 7 & 6 & 9 & 6 \end{array}$$

Solⁿ: Since the cost matrix is not a square matrix, the problem is unbalanced. We add a dummy job 5 with corresponding entries zero.

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline 1 & 4 & 3 & 6 & 2 & 7 \\ 2 & 10 & 12 & 11 & 14 & 16 \\ 3 & 4 & 3 & 2 & 1 & 5 \\ 4 & 8 & 7 & 6 & 9 & 6 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

~~Row min.~~ ~~Column~~ Row Minimization

$$\begin{bmatrix} 2 & 1 & 4 & \boxed{0} & 5 \\ \boxed{0} & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4 assignments

5 jobs

4 < 5

Hungarian method

$$\begin{bmatrix} 2 & 1 & 4 & \boxed{0} & 5 \\ \boxed{0} & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & \cancel{0} & 4 \\ 2 & 1 & \boxed{0} & 3 & \cancel{0} \\ 0 & 0 & \cancel{0} & \cancel{0} & \boxed{0} \end{bmatrix}$$

x = 1

$$\begin{bmatrix} 1 & \boxed{0} & 3 & 0 & 4 \\ \boxed{0} & 2 & 1 & 5 & 6 \\ 2 & 1 & \cancel{0} & \boxed{0} & 3 \\ 2 & 1 & \boxed{0} & 4 & 0 \\ 0 & \cancel{0} & 0 & \cancel{0} & \boxed{0} \end{bmatrix}$$

1 → 2
2 → 1
3 → 4
4 → 3

3 + 10 + 1 + 6

20

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline A & 12 & 10 & 15 & 22 & 18 & 8 \\ B & 10 & 18 & 25 & 15 & 16 & 12 \\ C & 11 & 10 & 3 & 8 & 5 & 9 \\ D & 6 & 14 & 10 & 13 & 13 & 12 \\ E & 8 & 12 & 11 & 7 & 13 & 10 \end{array}$$

(3)

Solⁿ add dummy row

Row Min

$$\begin{bmatrix} 12 & 10 & 15 & 22 & 18 & 8 \\ 10 & 18 & 25 & 15 & 16 & 12 \\ 11 & 10 & 3 & 8 & 5 & 9 \\ 6 & 14 & 10 & 13 & 13 & 12 \\ 8 & 12 & 11 & 7 & 13 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 7 & 14 & 10 & \boxed{0} \\ \boxed{0} & 8 & 15 & 5 & 6 & 2 \\ 8 & 7 & \boxed{0} & 5 & 2 & 6 \\ \cancel{0} & 8 & 4 & 7 & 7 & 6 \\ 1 & 5 & 4 & \boxed{0} & 6 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \boxed{0} & \cancel{0} \end{bmatrix}$$

No Column Minim

$$\begin{bmatrix} 4 & 2 & 7 & 14 & 10 & \boxed{0} \\ \boxed{0} & 8 & 15 & 5 & 6 & 2 \\ 8 & 7 & \boxed{0} & 5 & 2 & 6 \\ \cancel{0} & 8 & 4 & 7 & 7 & 6 \\ 1 & 5 & 4 & \boxed{0} & 6 & 3 \\ \cancel{0} & 0 & \cancel{0} & \cancel{0} & \boxed{0} & 0 \end{bmatrix}$$

x = 2

$$\begin{bmatrix} 6 & 2 & 7 & 14 & 10 & \boxed{0} \\ \boxed{0} & 6 & 13 & 3 & 4 & \cancel{0} \\ 10 & 7 & \boxed{0} & 5 & 2 & 6 \\ \cancel{0} & 6 & 2 & 5 & 5 & 4 \\ 3 & 5 & 4 & \boxed{0} & 6 & 3 \\ 2 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & 0 \end{bmatrix}$$

x = 2

$$\begin{bmatrix} 9 & 2 & 8 & 14 & 10 & \boxed{0} \\ \cancel{0} & 3 & 11 & \cancel{0} & 1 & \cancel{0} \\ 12 & 6 & \boxed{0} & 4 & 1 & 5 \\ \boxed{0} & 3 & \cancel{0} & 2 & 2 & 1 \\ 6 & 5 & 5 & \boxed{0} & 6 & 3 \\ 5 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

x = 1

$$\begin{bmatrix} 8 & 2 & 7 & 14 & 10 & \boxed{0} \\ \boxed{0} & 4 & 11 & 1 & 2 & \cancel{0} \\ 12 & 7 & \boxed{0} & 5 & 2 & 6 \\ \cancel{0} & 4 & \cancel{0} & 3 & 3 & 2 \\ 5 & 5 & 4 & \boxed{0} & 6 & 3 \\ 4 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

x = 1

$$\left[\begin{array}{cccccc} 9 & 2 & 8 & 14 & 10 & \boxed{0} \\ \cancel{0} & 3 & 11 & \cancel{0} & 1 & \cancel{0} \\ 12 & 6 & \boxed{0} & 4 & 1 & 5 \\ \boxed{0} & 3 & \cancel{0} & 2 & 2 & 1 \\ 6 & 5 & 5 & \boxed{0} & 6 & 3 \\ 5 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \quad x = 1$$

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline A & 8 & 2 & 7 & 14 & 9 & \boxed{0} \\ B & \cancel{0} & 2 & 10 & \cancel{0} & \boxed{0} & \cancel{0} \\ C & 13 & 6 & \boxed{0} & 5 & 1 & 6 \\ D & \boxed{0} & 2 & \cancel{0} & 9 & 1 & 1 \\ E & 6 & 4 & 4 & \boxed{0} & 5 & 3 \\ F & 5 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}$$

$A \to 6$
$B \to 5$
$C \to 3$
$D \to 1$
$E \to 4$
$F \to 2$

$8 + 16 + 3 + 6 + 13 + 0$

$= 46$

(4) A company has 4 machines to do 3 jobs. Each job can be assigned to one & only one m/c. The cost of each job on each machine is given below. Determine the job alignments which will minimise total cost.

$$\begin{array}{c|cccc} & \multicolumn{4}{c}{M/c} \\ & W & X & Y & Z \\ \hline A & 18 & 24 & 28 & 32 \\ B & 8 & 13 & 17 & 18 \\ C & 10 & 15 & 19 & 22 \end{array}$$

Sol: Since cost matrix is not a square matrix we add a dummy row D.

$$\begin{array}{c|cccc} A & 18 & 24 & 28 & 32 \\ B & 8 & 13 & 17 & 18 \\ C & 10 & 15 & 19 & 22 \\ D & 0 & 0 & 0 & 0 \end{array}$$

Row minimization

$$\left[\begin{array}{cccc} \boxed{0} & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 10 \\ 0 & 5 & 9 & 12 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \end{array}\right] \quad x = 5$$

$$\left[\begin{array}{cccc} \boxed{0} & 1 & 5 & 9 \\ \cancel{0} & \boxed{0} & 4 & 5 \\ \cancel{0} & \cancel{0} & 4 & 7 \\ 5 & \cancel{0} & \boxed{0} & \cancel{0} \end{array}\right] \quad x = 4$$

$$\begin{array}{c|cccc} A & \boxed{0} & 1 & 1 & 5 \\ B & \cancel{0} & \cancel{0} & \boxed{0} & 1 \\ C & \cancel{0} & \boxed{0} & \cancel{0} & 3 \\ D & 9 & 4 & 0 & \boxed{0} \end{array} \quad \text{or} \quad \left[\begin{array}{cccc} \boxed{0} & 1 & 1 & 5 \\ \cancel{0} & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 \\ 9 & 4 & \cancel{0} & \boxed{0} \end{array}\right]$$

$(18 + 17 + 15 + 0)$ $\qquad$ $(18 + 13 + 19)$

Cost : Rs 50 $\qquad$ Rs 50

Maximization in Assignment problem

In this, the objective is to maximise the profit. To solve this we first convert the given profit matrix into the loss matrix by subtracting all the elements from the highest element of the given profit matrix. For this converted loss matrix we apply the steps in Hungarian method to get the optimum assignment.

Eg:- A company has 5 jobs to be done, if following matrix shows the return in Rs on assigning i^{th} M/C to j^{th} job. Assign the 5 jobs to the 5 M/C so as to maximize the total profit.

$$\begin{array}{c|c} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 5 & 11 & 10 & 12 & 4 \\ 2 & 4 & 6 & 3 & 5 \\ 3 & 12 & 5 & 14 & 6 \\ 6 & 14 & 4 & 11 & 7 \\ 7 & 9 & 8 & 12 & 5 \end{bmatrix} \end{array}$$

Step 1. Convert maximization to minimization problem. Subtract every element by table max element.

Here Max = 14

$$\begin{bmatrix} 9 & 3 & 4 & 2 & 10 \\ 12 & 10 & 8 & 11 & 9 \\ 11 & 2 & 9 & 0 & 8 \\ 8 & 0 & 10 & 3 & 7 \\ 7 & 5 & 6 & 2 & 9 \end{bmatrix}$$

Step 2: Row element – Row Minimization

$$\begin{bmatrix} 7 & 1 & 2 & 0 & 8 \\ 4 & 2 & 0 & 3 & 1 \\ 11 & 2 & 9 & 0 & 8 \\ 8 & 0 & 10 & 3 & 7 \\ 5 & 3 & 4 & 0 & 7 \end{bmatrix}$$

Step 3: Column element – Column minimization

$$\begin{bmatrix} 3 & 1 & 2 & \boxed{0} & 7 \\ \boxed{0} & 2 & 0 & 3 & 0 \\ 7 & 2 & 9 & 0 & 7 \\ 4 & \boxed{0} & 10 & 3 & 6 \\ 1 & 3 & 4 & 0 & 6 \end{bmatrix}$$

x = 1

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 6 \\ 0 & 2 & 0 & \boxed{4} & 0 \\ 6 & 1 & 8 & 0 & 6 \\ 4 & 0 & 10 & 4 & 6 \\ \boxed{0} & 2 & 3 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 6 \\ 0 & 2 & 0 & 4 & \boxed{0} \\ 6 & 1 & 8 & \boxed{0} & 6 \\ 4 & \boxed{0} & 10 & 4 & 6 \\ \boxed{0} & 2 & 3 & 0 & 0 \end{bmatrix}$$

x = 1

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 1 & 0 & \boxed{0} & 0 & 5 \\ 0 & 3 & 0 & 4 & \boxed{0} \\ 5 & 2 & 7 & \boxed{0} & 5 \\ 3 & \boxed{0} & 9 & 4 & 6 \\ \boxed{0} & 3 & 2 & 0 & 5 \end{bmatrix} \end{array}$$

A → 3
B → 5
C → 4
D → 2
E → 1

Minimum Cost = 10 + 5 + 14 + 14 + 7
= 50

Eg 2: The owner of a small machine shop has four mechanics available to assign jobs for the day. Five jobs are offered with expected profit for each mechanic on each job which are as follows:-

mechanic	A	B	C	D	E
1	62	78	50	~~100~~ 101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Solⁿ The given profit matrix is not a square matrix as the number of jobs is not equal to the number of mechanics. Hence we introduce a dummy ~~mtc~~ mechanic 5 with all the elements 0.

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80
5	0	0	0	0	0

Now we convert this profit matrix to into loss matrix by subtracting all the elements from the highest element 111.

Loss Matrix

	A	B	C	D	E
1	49	33	61	10	29
2	40	27	50	38	52
3	24	19	0	40	30
4	63	47	24	34	31
5	111	111	111	111	111

We subtract the smallest element from all the elements in the respective row.

	A	B	C	D	E	
1	39	23	51	$\boxed{0}$	19	
2	13	$\boxed{0}$	23	12	25	
3	24	19	$\boxed{0}$	40	30	✓
4	39	23	$\cancel{0}$	10	7	✓
5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	

~~x=10~~

x=7

~~No column~~ Minima:

39	23	61	0	19
13	0	33	12	25
14	9	0	30	20
29	13	0	0	

	1	2	3	4	5
A	39	23	58	$\boxed{0}$	19
B	13	$\boxed{0}$	30	12	25
C	14	12	$\boxed{0}$	23	23
D	32	16	$\cancel{0}$	3	$\boxed{0}$
E	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	0

Job Mtc

A → 4

B → 2

C → 3

D → 5

E → 1

101 + 84 + 111 + 80 + 0

Rs. 376

Example 2: (Maximization problem). A company has four territories open and four salesmen available for assignment. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	I	II	III	IV
Annual sales (Rs)	60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability; it is estimated that, working under the same conditions, their yearly sales would be proportionately as follows:

Salesman	A	B	C	D
proportion	7	5	5	4

If the criterion is maximum expected total sales, then intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest and so on. Verify this answer by the assignment technique.

Soln. Step 1: To construct the effectiveness matrix.

In order to avoid the fractional values of annual sales of each salesman in each territory, it will be rather convenient to consider the sales for 21 years (the sum of proportions: $7+5+5+4=21$), taking Rs. 10,000 as one unit. Divide the individual sales in each territory by 21, if the annual sales by salesman are required.

Thus, the sales matrix for maximization is obtained as follows:

Table 1

Sales in 10 Thousand of rupees

	Sales proportion	6	5	4	3
		1	2	3	4
7	A	42	35	28	21
5	B	30	25	20	15
5	C	30	25	20	15
4	D	24	20	16	12

Step 2: To convert 'the maximum sales matrix' to 'minimum sales matrix')

The problem of 'maximization' can be converted to minimization one by simply multiplying each element of above matrix by -1. Thus, result matrix becomes

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -42 & -35 & -28 & -21 \\ -30 & -25 & -20 & -15 \\ -30 & -25 & -20 & -15 \\ -24 & -20 & -16 & -12 \end{bmatrix} \end{array}$$

Step 3: Row minimization

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 7 & 14 & 21 \\ 0 & 5 & 10 & 15 \\ 0 & 5 & 10 & 15 \\ 0 & 4 & 8 & 12 \end{bmatrix} \end{array}$$

Step 4: Column Minimization

$$\begin{pmatrix} \boxed{0} & 3 & 6 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & \boxed{0} & 0 & 0 \end{pmatrix} \qquad n=1 \begin{pmatrix} \boxed{0} & 2 & 5 & 8 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{bmatrix} \boxed{0} & 2 & 5 & 8 \\ \cancel{0} & \boxed{0} & 1 & 2 \\ \cancel{0} & \cancel{0} & 1 & 2 \\ 1 & \cancel{0} & \boxed{0} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{0} & 2 & 4 & 7 \\ 0 & \boxed{0} & 0 & 1 \\ \cancel{0} & 0 & \boxed{0} & 1 \\ 2 & 1 & \cancel{0} & \boxed{0} \end{bmatrix} \quad 42+25+20+12 = 99$$

or

$$\begin{bmatrix} \boxed{0} & 2 & 4 & 7 \\ \cancel{0} & \cancel{0} & \boxed{0} & 1 \\ \cancel{0} & \boxed{0} & \cancel{0} & 1 \\ 2 & 1 & \cancel{0} & \boxed{0} \end{bmatrix} \quad 42+20+25+12 = 99$$

Thus, two possible solutions are

(i) A → I (ii) A → I
B → II B → III
C → III C → II
D → IV D → IV

Both the solutions shows that the best salesman A is assigned to the richest territory I, the worst salesman D to the poorest territory IV.
Salesman B & C being equally good, so they may be assigned to either II or III. This verifies the answer.

Prohibited Assignment

(2) Sales ~~proportion~~:

The criteria of max expected total sales is to be met by assigning the best engg to the richest zone, the next best to the second richest zone

Sales proportion	Sales Engineer 10	8	7	11
14	140	112	98	154
9	90	72	63	99
11	110	88	77	121
8	80	64	56	88

14×10
14×8

Maximum e

$$\begin{bmatrix} 14 & 42 & 56 & 0 \\ 64 & 82 & 91 & 55 \\ 44 & 66 & 77 & 33 \\ 74 & 90 & 98 & 66 \end{bmatrix} \Rightarrow$$

Row minimization

Colm

$$\begin{matrix} 6 & 18 & 24 & 0 \\ 1 & 3 & 4 & 0 \\ 3 & 9 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 5 & 17 & 23 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & 8 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

L_1, L_2, L_3, L_4

$$\begin{matrix} 5 & 15 & 21 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 3 & 13 & 19 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{matrix} \quad \begin{matrix} 3 & 13 & 19 & \boxed{0} \\ \cancel{0} & \boxed{0} & 1 & 2 \\ \boxed{0} & 4 & 7 & 0 \\ 2 & \cancel{0} & \boxed{0} & 5 \end{matrix}$$

P → D is richest zone;
S → C poorest zone
Max sales = 154 + 72 + 110 + 56
= 392

M

Prohibited Assignment

Sometimes technical, legal or other restrictions donot permit the assignment of a particular facility to a particular job. Such difficulty can be overcome by assigning a very high cost (say, infinite cost) to the corresponding cell, so that the activity will be automatically excluded from the optimal solution.

Ep..

1) A job shop has purchased 5 new machines of different type. There are 5 available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centres which would have a heavy work flow to & from these machines. ∴ the objective is to assign the new machines to the available locations in order to minimize the total cost of material handling. The estimated cost per unit time of materials handling involving each of the M/c's is given below for the respective locations. Locations 1,2,3,4 & 5 are not considered suitable for M/c's A, B, C, D, & E respectively. Find the optimal sol^n

Location (cost in K)

		1	2	3	4	5
	A	X	10	25	25	10
	B	1	X	10	15	2
M/c	C	8	9	X	20	10
	D	14	10	24	X	15
	E	10	8	25	27	X

How would the optimal sol^n get modified if location 5 is also unsuitable for M/c A?

Sol^n: Since locations 1,2,3,4 & 5 are not suitable for M/c A, B, C, D & E respectively, an extremely large cost (say ∞) should be attached to these locations. The cost matrix of resulting assignment problem becomes as shown below

$$\begin{pmatrix} \infty & 10 & 25 & 25 & 10 \\ 1 & \infty & 10 & 15 & 2 \\ 8 & 9 & \infty & 20 & 10 \\ 14 & 10 & 24 & \infty & 15 \\ 10 & 8 & 25 & 27 & \infty \end{pmatrix}$$

Row Minimum of ats

Colum

Find Matrix

$$\begin{pmatrix} \infty & 2 & 6 & 3 & \boxed{0} \\ \cancel{0} & \infty & \boxed{0} & 2 & 1 \\ \cancel{0} & 3 & \infty & \boxed{0} & 2 \\ 2 & \boxed{0} & 3 & \infty & 3 \\ \boxed{0} & \cancel{0} & 6 & 5 & \infty \end{pmatrix}$$

Following the usual procedure of solving an assignt pblm, the optimum assignment is obtained as shown

$A \to 5,\ B \to 3,\ C \to 4,\ D \to 2,\ E \to 1$ Min cost Rs = 60

Now if location 5 is also not suitable for the M/c A, we attach an extremely large cost ($= \infty$) to cell (1,5). Again apply the assignment procedure to this modified pblm, the following assignment solution can be easily obtained

$A \to 4,\ B \to 3,\ C \to 5,\ D \to 2,\ E \to 1$
or $A \to 2,\ B \to 3,\ C \to 4,\ D \to 5,\ E \to 1$ } $\boxed{\text{cost Rs. 65}}$

Sensitivity in Assignment problems

The structure of assignment pblm is of such type that there is very little scope for sensitivity analysis. Modest alterations in the conditions (such as one being able to do two jobs) can be considered by repeating the man's row & adding a dummy column to square up the matrix.

Addition of a constant throughout any row or column also makes no difference to the position of optimal assignment. However sometimes equiproportionate change throughout a row or column can make a difference. So in reference to assignment problems there is no scope for altering the level of an assignment.

Travelling salesman problem (shortest Acyclic route Model)

~~Assigning~~ Assuming a salesman has to visit n cities. He wishes to start from particular city, visit each city once & then return to his starting point. His objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized.

To visit 2 cities (A & B) there is no choice, to visit 3 cities we have 2! possible routes. For 4 cities we have 3! possible routes. In general to visit n cities there are (n-1)! possible routes.

→ Same as assignment problem with 2 additions.

1. No assignment should be made along the diagonal line.
2. Solving the problem as a routine assignment pblm.
3. Scrutinizing the solⁿ obtained under (2) to see if the 'route' condition is satisfied.
4. If not, making adjustments in assignments to satisfy the condition "next best solution" may be required to be considered.

(1) Solve the following TSP

		To			
		A	B	C	D
	A	-	46	16	40
	B	41	-	50	40
from	C	82	32	-	60
	D	40	40	36	-

Solⁿ The cost matrix of the given TSP is

$$\begin{bmatrix} \infty & 46 & 16 & 40 \\ 41 & \infty & 50 & 40 \end{bmatrix}$$

*A travelling salesman problem is very similar to the assignment problem except that in former case, there is an additional restriction that x_{ij} is so chosen that no city is visited twice before the tour of all the cities is completed.

Row reduction

$$\begin{bmatrix} \infty & 30 & 0 & 24 \\ 1 & \infty & 10 & 0 \\ 50 & 0 & \infty & 28 \\ 4 & 4 & 0 & \infty \end{bmatrix}$$

Column reduction

$$\begin{bmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ 3 & 4 & 0 & \infty \end{bmatrix}$$

Make assignment

$$\begin{bmatrix} \infty & 30 & \boxed{0} & 24 \\ \boxed{0} & \infty & 10 & \cancel{0} \\ 49 & \boxed{0} & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{bmatrix}$$

$x = 3$

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & \infty & 27 & \boxed{0} & 21 \\ B & \cancel{0} & \infty & 13 & \boxed{0} \\ C & 49 & \boxed{0} & \infty & 28 \\ D & \boxed{0} & 1 & \cancel{0} & \infty \end{array}$$

$A \to C \to B \to D \to A$

$A \to C$

$B \to D$

$C \to B$

$D \to A$

$A \to C \to B \to D \to A$

$16 + 32 + 40 + 40 = 128/-$

~~$A \to C \to B \to A$~~

(2) Solve the following T.S.P so as to minimize the cost per cycle.

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & - & 3 & 6 & 2 & 3 \\ B & 3 & - & 5 & 2 & 3 \\ C & 6 & 5 & - & 6 & 4 \\ D & 2 & 2 & 6 & - & 6 \\ E & 3 & 3 & 4 & 6 & - \end{array}$$

Sol^n row reduction

$$\begin{bmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{bmatrix}$$

Column reduction

$$\begin{bmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{bmatrix}$$

Assignment

$$\begin{bmatrix} \infty & 1 & 3 & \boxed{0} & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & \boxed{0} \\ \cancel{0} & \boxed{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 & \infty \end{bmatrix}$$

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & \cancel{0} & 2 & \boxed{0} & \cancel{0} \\ B & \boxed{0} & \infty & 1 & \cancel{0} & \cancel{0} \\ C & 2 & 1 & \infty & 3 & \boxed{0} \\ D & \cancel{0} & \boxed{0} & 3 & \infty & 4 \\ E & \cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty \end{array}$$

A → D
B → A
C → E
D → B
E → C

A → D → B → A

Ex.

A → B, B → D, C → E, D → A, E → C

Path:- A → B → D → A C → E → C

Cost = 3 + 2 + 4 + 2 + 4 = 15

But this assignment does not provide the sol^n of TSP ∵ it does not satisfy the routing condition. So we try to find the next best sol^n which satisfies the routing condition also. The next min (non zero) cost element in the cost matrix is 1 so we try to bring 1 into the sol^n. but 1 occurs at 2 places, we shall consider all the cases separately until the sol^n is reached.

We start with making an assignment at (2,3) instead of few assignment (2,1)

$$\begin{bmatrix} \infty & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & \infty & 3 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty \end{bmatrix}$$

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & \cancel{0} & 2 & \boxed{0} & \cancel{0} \\ B & \cancel{0} & \infty & \boxed{1} & \cancel{0} & \cancel{0} \\ C & 2 & 1 & \infty & 3 & \boxed{0} \\ D & \cancel{0} & \boxed{0} & 3 & \infty & 4 \\ E & \boxed{0} & \cancel{0} & \cancel{0} & 4 & \cancel{\infty} \end{array}$$

A → D
B → C
C → E
D → B
E → A

A → D → B → C → E → A

2 + 5 + 4 + 2 + 3

= 16

or

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & \cancel{0} & 2 & \cancel{0} & \boxed{0} \\ B & \cancel{0} & \infty & 1 & \boxed{0} & \cancel{0} \\ C & 2 & \boxed{1} & \infty & 3 & \cancel{0} \\ D & \boxed{0} & \cancel{0} & 3 & \infty & 4 \\ E & \cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty \end{array}$$

A → E
B → D
C → B
D → A
E → C

A → E → C → B → D → A

Cost = 3 + 4 + 5 + 2 + 2

= 16

	A	B	C	D	E
③ A	–	2	5	7	1
B	6	–	3	8	2
C	8	7	–	4	7
D	12	4	6	–	5
E	1	3	2	8	–

$$\Rightarrow \begin{bmatrix} \infty & 2 & 5 & 7 & 1 \\ 6 & \infty & 3 & 8 & 2 \\ 8 & 7 & \infty & 4 & 7 \\ 12 & 4 & 6 & \infty & 5 \\ 1 & 3 & 2 & 8 & \infty \end{bmatrix}$$

Row reduction

$$\begin{bmatrix} \infty & 1 & 4 & 6 & 0 \\ 4 & \infty & 1 & 6 & 0 \\ 4 & 3 & \infty & 0 & 3 \\ 8 & 0 & 2 & \infty & 1 \\ 0 & 2 & 1 & 7 & \infty \end{bmatrix}$$

Colu^n reduction

	A	B	C	D	E	
A	∞	1	3	6	$\boxed{0}$	A → E
B	4	∞	$\boxed{0}$	6	~~0~~	B → C
C	4	3	∞	$\boxed{0}$	3	C → D
D	8	$\boxed{0}$	1	∞	1	D → B
E	$\boxed{0}$	2	~~0~~	7	∞	E → A

$1+3+4+4+1 = 13$ Rs.

Total cost – Rs. 13

A → E → A

Next best sol^n is 6

Next minimum is 1

	A	B	C	D	E
A	∞	$\boxed{1}$	3	6	~~0~~
B	4	∞	$\boxed{0}$	6	~~0~~
C	0	3	∞	$\boxed{0}$	3
D	6	~~0~~	1	∞	$\boxed{1}$
E	$\boxed{0}$	2	~~0~~	7	∞

A → B
B → C
C → D
D → E
E → A

A → B → C → D → E → A

$2+3+4+5+1$

Rs. 15

③ A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to his starting point. Cost of going from one city to another city is shown below. You are required to find the least cost route.

	A	B	C	D	E
A	∞	4	10	14	2
B	12	∞	6	10	4
C	16	14	∞	8	14
D	24	8	12	∞	10
E	2	6	4	16	∞

Sol^n:- Row Reduction

$$\begin{bmatrix} \infty & 2 & 8 & 12 & 0 \\ 8 & \infty & 2 & 6 & 0 \\ 8 & 6 & \infty & 0 & 6 \\ 16 & 0 & 4 & \infty & 2 \\ 0 & 4 & 2 & 14 & \infty \end{bmatrix}$$

Column reduction

	A	B	C	D	E
A	∞	2	6	12	$\boxed{0}$
B	8	∞	$\boxed{0}$	6	~~0~~
C	8	6	∞	$\boxed{0}$	6
D	16	$\boxed{0}$	2	∞	2
E	$\boxed{0}$	4	~~0~~	14	∞

A → E
B → C
C → D
D → B
E → A

A → E → A

As the salesman should go from A to E and then come back to A without covering B, C, D which is contradicting the fact that no city is visited twice before all the cities are visited.

~~Head~~

P 70

Hence we obtain the next best sol^n by bringing the next minimum non-zero element namely 4.

$$\begin{bmatrix} \infty & \boxed{0} & 6 & 12 & 0 \\ 8 & \infty & \boxed{0} & 6 & 0 \\ 8 & \cancel{4} & \infty & \boxed{0} & 6 \\ 18 & \cancel{8} & \cancel{4} & \infty & \boxed{4} \\ \boxed{0} & 4 & 0 & 14 & \infty \end{bmatrix}$$

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$

$4 + 6 + 8 + 10 + 2 = \boxed{\text{Rs. } 30}$

UNIT - 3

Simplex Method — 2: Adapting to other model forms; post optimality analysis; computer implementation, Foundation of the simplex method.

Adapting to other Model forms (Two-phase Method, Big-M Method)

Artificial variable technique:

In simplex method, starting the simplex iterations at a basic feasible sol^n guarantees that all the succeeding iterations will be feasible.

For the LPs in which all the constraints are of the (≤) type (with non-negative right hand sides), the slacks offer a convenient starting basic feasible sol^n. A natural question then arises.

How can we find a starting basic solution for models that involve (=) & (>=) constraints?

The most common procedure for starting LPs that do not have convenient slacks is to use artificial variables.

These are variables that assume the role of slacks at the first iteration, only to be disposed of at a later iteration. Two closely related methods are proposed for effecting this result.

(1). The Big-M method (penalty Method) &
(2) Two phase method.

The Big-M method (penalty Method)

The Big-M method starts with the LP in the standard form. For any equation i that does not have a slack, we augment an artificial variable a_i. Such a variable then becomes part of the starting basic solution.

However, because artificial are extraneous of the LP model, we assign them a penalty in the objective function to force them to zero level at a later iteration of the simplex algorithm.

Given M is a sufficiently large positive value, the variable a_i is penalized in the objective function using
$-Ma_i$ in case of maximization
$+Ma_i$ " " minimization.

Because of this penalty, the nature of the optimization process will logically attempt to drive a_i to zero level during the course of the simplex iterations.

Computational steps of Big-M Method

① Express the LPP in the standard form.

② Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type (≥) & '='. When artificial variables are added it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists).

~~are~~ on the other hand, if [illegible] the pblm does not have a solution, atleast one of the artificial variables will appear in the final sol^n with positive value. This is achieved by assigning a very large price (per unit penalty) to these variables in the objective function. Such large price will be designated by $-M$ for maximization problems ($+M$ for minimization problems), where $M > 0$

(3) In the last, use the artificial variables for the starting sol^n & proceed with usual simplex routine until the optimal sol^n is obtained arrived.

Steps) BIG M

→ Express the given LPP in standard form. Then
→ The constraints having $\geq$ & $=$ sign & non-ve variables $a_1, a_2, \ldots a_k$ called artificial variables.
→ The purpose of introducing these artificial variables to obtain an initial basic feasible sol^n

But their additional causes violation of the corresponding constraints as such we would like to get rid of these variables and would not be allowed to appear in final sol^n

For this purpose we assign a very large penalty '$-M$' for these variables in objective function. Hence the objective f^n is now given by

$$Z = C_1x_1 + C_2x_2 + \ldots C_mx_m - Ma_1 - Ma_2 - \ldots Ma_k$$

$-M$ for Max problems
M for Min "

NOTE:- The artificial variables are only a computational device for getting starting sol^n. Once an artificial variable is the basic sol^n, it has served its purpose & we forget about it by deleting the column headed by this outgoing

→ Whenever $\geq$ sign appears in the constraint, we use both surplus & artificial variable

→ Whenever $=$ sign appears in the constraint, we make use of only artificial variable

Eg-1 Max $Z = 3x_1 + 5x_2$

S.T

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 = 18$ where $x_1, x_2 \geq 0$

The augmented form of the problem becomes (slack variables)

$Z - 3x_1 - 5x_2 = 0$ — (0)
$x_1 + S_1 = 4$ — (1)
$2x_2 + S_2 = 12$ — (2)
$3x_1 + 2x_2 = 18$ — (3)

We can't use slack variable to use as the initial basic variable for eqn (3). It is necessary to find an initial BF sol^n to start the simplex method.

This difficulty can be circumvented in the following way.

Obtaining an Initial BF-sol^n: The procedure is to construct an artificial problem that has the same optimal sol^n as the real problem by making two modification of the real problem.

(1) Apply the artificial-variable technique by by introducing a nonnegative artificial variable (call it as a_1 or $\bar{x}_5$) into Eq (3), just as if it were a slack variable.

Eqn (3) $3x_1 + 2x_2 + a_1 = 18$

② Assign an overwhelming penalty to having $a_1 > 0$ by changing the objective fn

$Z = 3x_1 + 5x_2$ ~~to~~

$\downarrow$

$$Z = 3x_1 + 5x_2 - Ma_1$$

where M symbolically represents a huge positive number. (This method of forcing a_1 to be $a_1 = 0$ in the optimal solution is called the Big M method).

Now find the optimal solⁿ for the real problem by applying the simplex method to the artificial problem, starting with the following initial BF solⁿ.

I BFS:

Non basic variables $x_1 = 0,\ x_2 = 0$

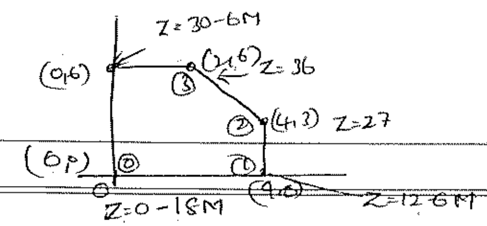
Basic " $s_1 = 4$

$s_2 = 12$

$a_1 = 18$

∵ a_1 plays the role of slack variable for the 3rd constrn in the artificial problem, this constraint is equivalent to $3x_1 + 2x_2 \le 18$

Real pblm	Artificial pblm
Max $Z = 3x_1 + 5x_2$	Define $a_1 = 18 - 3x_1 - 2x_2$
s.t	Max $Z = 3x_1 + 5x_2 - Ma_1$
$x_1 \le 4$	s.t
$2x_2 \le 12$	$x_1 \le 4$
$3x_1 + 2x_2 = 18$	$2x_2 \le 12$
& $x_1, x_2 \ge 0$	$3x_1 + 2x_2 \le 18$
	or $(3x_1 + 2x_2 + a_1 = 18)$



The only portion of this feasible region that coincides with the feasible region for the real pblm is $a_1 = 0$ & so $3x_1 + 2x_2 = 18$

fig shows CPF solⁿ

Converting equation (0) to proper form:

The system of equations after the artificial problem is augmented is

(0) $Z - 3x_1 - 5x_2 + Ma_1 = 0$

(1) $x_1 + s_1 = 4$

(2) $2x_2 + s_2 = 12$

(3) $3x_1 + 2x_2 + a_1 = 18$

where the initial basic variables (s_1, s_2, a_1) are shown

However this system is not yet in proper form of gaussian elimination ∵ a_1 has nonzero coefficient in eqⁿ (0).

→ Recall that all basic variables must be algebraically eliminated from eqⁿ (0), before the simplex method can either apply the optimality test

The algebraically eliminate a_1 from eqⁿ (0), we need to subtract from eqⁿ (0) the product, M times eqⁿ (3)

$$Z - 3x_1 - 5x_2 + Ma_1 = 0$$
$$-\ M(3x_1 + 2x_2 + a_1 = 18)$$

i.e

$$Z - 3x_1 - 5x_2 + Ma_1 = 0$$
$$-\ M(3x_1 + 2x_2 + a_1 = 18)$$

New (0) $\quad Z - (3M+3)x_1 - (2M+5)x_2 = -18M$ —— (0)

Application of the Simplex Method. This new eqⁿ (0) gives Z in terms of just the nonbasic variables (x_1, x_2)

$$Z = -18M + (3M+3)x_1 + (2M+5)x_2$$

	BV	eqn	Z	x_1	x_2	S_1	S_2	a_1	RHS	Min ratio
	Z	0	1	$-(3M+3)$	$-(2M+5)$	0	0	0	$-18M$	
LV	S_1	1	0	1	0	1	0	0	4	$4/1=4$
	S_2	2	0	0	2	0	1	0	12	$12/0=X$
	a_1	3	0	3	2	0	0	1	18	$18/3=6$

	BV	eqn	Z	x_1	x_2	S_1	S_2	a_1	RHS	
$R_0=R_0+(3M+3)R_1$	Z	0	1	0	$-(2M+5)$	$(3M+3)$	0	0	$-6M+12$	
$R_1\to R_1$	x_1	1	0	1	0	1	0	0	4	
$R_2\to R_2$	S_2	2	0	0	2	0	1	0	12	$12/2=6$
$R_3\to R_3-3R_1$	a_1	3	0	0	2	-3	0	1	6	$6/2=3$

	BV	eqn	Z	x_1	x_2	S_1	S_2	a_1	RHS	Min ratio
	Z	0	1	0	$-(2M+5)$	$(3M+3)$	0	0	$-6M+12$	
	x_1	1	0	1	0	1	0	0	4	$4/1=4$
	S_2	2	0	0	2	0	1	0	12	$6/3=2$
$R_3\to R_3/2$	x_2	3	0	0	1	$-3/2$	0	$1/2$	3	3

$R_0=R_0+(2M+5)R_3$	Z	0	1	0	0	$-9/2$	0	$M+5/2$	27	
$R_1\to R_1$	x_1	1	0	1	0	1	0	0	4	$4/1=4$
$R_2\to R_2-2R_3$	S_2	2	0	0	0	3	1	-1	6	$6/3=2$
$R_3\to$	x_2	3	0	0	1	$-3/2$	0	$1/2$	3	$3/-3/2=X$

	Z	0	1	0	0	$-9/2$	0	$M+5/2$	27
	x_1	1	0	1	0	1	0	0	4
$R_2\to R_2/3$	S_1	2	0	0	0	1	$1/3$	$-1/3$	2
	x_2	3	0	0	1	$-3/2$	0	$1/2$	3

$R_0\to R_0+\frac{9}{2}R_2$	Z	0	1	0	0	0	$3/2$	$M+1$	36
$R_1\to R_1-R_2$	x_1	1	0	1	0	0	$-1/3$	$1/3$	2
$R_2\to R_2$	S_1	2	0	0	0	1	$1/3$	$-1/3$	2
$R_3\to R_3+\frac{3}{2}R_2$	x_2	3	0	$3/2$	1	0	0	$1/2$	6

$\boxed{Z_{max} = 36 \text{ at } x_1 = 2,\ x_2 = 6}$

(2) Maximize $Z = 3x_1 + 2x_2$

S.T $2x_1 + x_2 \le 2$

$3x_1 + 4x_2 \ge 12$

where $x_1, x_2 \ge 0$

Sol^n Big M Method:

$2x_1 + x_2 + S_1 = 2$ —— (2)

$3x_1 + 4x_2 - S_2 = 12$

Now introduce artificial variable

$3x_1 + 4x_2 - S_2 + a_1 = 12$ —— (3)

$\therefore Z = 3x_1 + 2x_2 + 0\cdot S_1 + 0\cdot S_2 - Ma_1$ $\quad x_1, x_2, \underbrace{S_1, S_2, a_1}_{B.V} \ge 0$

$Z - 3x_1 - 2x_2 - 0\cdot S_1 + 0\cdot S_2 + Ma_1 = 0$ —— (1)

$\therefore$ eqn(1) $-$ M eqn(3)

$Z - 3x_1 - 2x_2 - 0\cdot S_1 + 0\cdot S_2 + Ma_1 = 0$

$-\boxed{3Mx_1 + 4Mx_2 - MS_2 + Ma_1 = 12M}$

New eqn

$\therefore Z - (3M+3)x_1 - (4M+2)x_2 + MS_2 + 0 = -12M$ —— (1)

Coefficient of $S_2 = -1$ ∴ so we can't consider it as BV

(Identity Matrix consideration)

	BV	eqn	Z	x_1	x_2	S_1	S_2	a_1	RHS	Min Ratio
	Z	1	1	$-(3M+3)$	$-(4M+2)$	0	M	0	$-12M$	
LV	S_1	2	0	2	1	1	0	0	2	$2/1=2$
	a_1	3	0	3	4	0	-1	1	12	$12/4=3$

$R_0\to R_0+(4M+2)R_1$	Z	1	1	$5M+1$	0	$4M+2$	M	0	$-4M+M$
$R_1\to R_1$	x_2	2	0	2	1	1	0	0	2
$R_2\to R_2-4R_1$	a_1	3	0	-5	0	-4	-1	1	4

(3) Max $Z = 2x_1 + 3x_2$

S.T $x_1 + 2x_2 \le 4$

$x_1 + x_2 = 3$

$x_1, x_2 \geq 0$

Augmented form

$x_1 + 2x_2 + S_1 = 4$ —— (2)

$x_1 + x_2 + a_1 = 3$ —— (3)

$Z = 2x_1 + 3x_2 - Ma_1$

$Z - 2x_1 - 3x_2 + Ma_1 = 0$ —— (1)

New eqn (1) = Eqn (1) − M [eqn (3)]

$Z - 2x_1 - 3x_2 + 0 \cdot S_1 + Ma_1 = 0$

$- M(x_1 + x_2 + a_1 = 3)$

$Z - (M+2)x_1 - (M+3)x_2 + 0 \cdot S_1 = -3M$ —— New (1)

	BV	eqn	Z	x_1	x_2	S_1	a_1	RHS	Min ratio
	Z	1	1	$-(M+2)$	$-(M+3)$	0	0	$-3M$	
	S_1	2	0	1	2	1	0	4	$4/2 = 2$
	a_1	3	0	1	1	0	1	3	$3/1 = 3$
$R_1 \to R_1 + (M+3)R_2$	Z	1	1	$-(M+1)/2$	0	$(M+3)/2$	0	$-M+6$	-
$R_2 \to R_2/2$	x_2	2	0	$1/2$	1	$1/2$	0	2	$2/\frac{1}{2} = 4$
$R_3 \to R_3 - R_2$	a_1	3	0	$1/2$	0	$-1/2$	1	1	$1/\frac{1}{2} = 2$
$R_1 \to R_1 + \frac{(M+1)}{2}R_3$	Z	1	1	0	0	1	$M+1$	7	
$R_2 \to R_2 - R_3/2$	x_2	2	0	0	1	1	-1	1	
$R_3 \to R_3 \times 2$	x_1	3	0	1	0	-1	2	2	

$\boxed{Z_{max} = 7 \text{ at } x_1 = 2,\ x_2 = 1}$

or $\boxed{\text{another Method of writing simplex table}}$

Max $Z = 3x_1 + 5x_2$

S.T

$x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 = 18$

Max $Z = 3x_1 + 5x_2 + 0 \cdot S_1 + 0 \cdot S_2 - Ma_1$

$x_1 + S_1 = 4$

$2x_2 + S_2 = 12$

$3x_1 + 2x_2 + a_1 = 18$

		3	5	0	0	$-M$		
C_B		x_1	x_2	S_1	S_2	a_1	b	θ
0	S_1	1	0	1	0	0	4	$4/1$
0	S_2	0	2	0	1	0	12	$12/0$
$-M$	a_1	3	2	0	0	1	18	$18/3$
Z_j		$-3M$	$-2M$	0	0	$-M$	$-18M$	

$Z_j = \Sigma C_B \times a_{ij}$

		x_1	x_2	S_1	S_2	a_1	b	θ
3	x_1	1	0	1	0	0	4	$4/0$
0	S_2	0	2	0	1	0	12	$12/2$
$-M$	a_1	0	2	-3	0	1	6	$6/2$
Z_j	Z	3	$-2M$	$3M+3$	0	$-M$	$-6M+12$	

		x_1	x_2	S_1	S_2	a_1	b	θ
3	x_1	1	0	1	0	0	4	$4/1$
0	S_2	0	0	6	1	-1	6	$6/6$
5	x_2	0	1	$-3/2$	0	$1/2$	3	-
Z_j		3	5	$-9/2$	0	$5/2$	27	

		x_1	x_2	s_1	s_2	a_1	b	θ
3	x_1	1	0	0	-1	1	2	
0	s_1	0	0	1	1/6	-1/6	1	
5	x_2	0	0	0	3/2	-1	6	
Z_j		3	5	0	9/2		36	

$x_1 = 2, \; x_2 = 6$

(4) Solving the radiation therapy example

Minimize $Z = 0.4x_1 + 0.5x_2$

$0.3x_1 + 0.1x_2 \le 2.7$

$0.5x_1 + 0.5x_2 = 6$

$0.6x_1 + 0.4x_2 \ge 6 \qquad x_1, x_2 \ge 0$

Maximize $Z' = -Z = -0.4x_1 - 0.5x_2$

s.t. $0.3x_1 + 0.1x_2 + s_1 = 2.7$ — (2)

$0.5x_1 + 0.5x_2 + a_1 = 6$ — (3)

$0.6x_1 + 0.4x_2 - S_2 + a_2 = 6$ — (4)

$Z' = -0.4x_1 - 0.5x_2 + 0.S_1 + 0.S_2 - Ma_1 - Ma_2$

$\therefore Z' + 0.4x_1 + 0.5x_2 - 0.S_2 + 0.S_2 + Ma_1 + Ma_2 = 0$ — (1)

Eqn (1) $- M$[eqn 3] $- M$[eqn 4]

$Z' + 0.4x_1 + 0.5x_2 + \cancel{Ma_1} + \cancel{Ma_2} - 0.5Mx_1 - 0.5Mx_2 - \cancel{Ma_1} = -6M$

$- 0.6Mx_1 - 0.4Mx_2 + MS_2 - \cancel{Ma_2} = -6M$

$Z' + (0.4 - 0.5M)x_1 + (0.5 - 0.9M)x_2 + 0.S_1 + 0.a_1 + MS_2 + 0.a_2 = -12M$

Initial Simplex table:

BV	Z	x_1	x_2	S_1	a_1	S_2	a_2	RHS	Ratio
Z	-1	0.4-1.1M	0.5-0.9M	0	0	M	0	-12M	-
S_1	0	0.3	0.1	1	0	0	0	2.7	2.7/0.3 = 9 ←
a_1	0	0.5	0.5	0	1	0	0	6	6/0.5 = 12
a_2	0	0.6	0.4	0	0	-1	1	6	6/0.6 = 10

				x_2 ↓						
$R_0 \to R_0 - (0.4-1.1M)R_1$	Z	-1	0	$\frac{11}{30} - \frac{16}{30}M$	$\frac{11}{3}M - \frac{4}{3}$	0	M	0	$\frac{-21}{10}M - \frac{36}{10}$	
$R_1/0.3$	x_1	0	1	1/3	10/3	0	0	0	9	9 × 3 = 27
$R_2 - 0.5R_2$	a_1	0	0	1/3	-5/3	1	0	0	3/2	9/2 = 4.5
$R_3 - 0.6R_3$	a_2	0	0	1/5	-2	0	-1	1	3/5	3 ←

Z	-1	0	$\frac{-5}{3}M + \frac{7}{3}$	0	$-\frac{5}{3}M + \frac{11}{}$	$\frac{-11}{6} + \frac{8}{3}M$
x_1	0	1	20/3	0	5/3	
a_1	0	0	5/3	1	5/3	
x_2	0	0	-10	0	-5	

							S_2 ↓			
$R_0 \to R_0 - \left(\frac{M}{30} - \frac{16M}{30}\right)$	Z	-1	0	0	$\frac{-5}{3}M + \frac{7}{3}$	0	$\frac{-5M}{3} + \frac{11}{6}$	$\frac{-11}{6} + \frac{8}{3}M$	$\frac{-1}{2}M - \frac{47}{10}$	
$R_1 - 1/3R_3$	x_1	0	1	0	20/3	0	5/3	$-\frac{5}{3}$	8	24/5 = 4.8
$R_2 - 1/3R_3$	a_1	0	0	0	5/3	1	5/3	-5/3	1/2	3/10 = 0.3 ←
$R_3 \times 5$	x_2	0	0	1	-10	0	-5	5	3	×

$R_0 \to R_0 + \frac{5M}{3} + \frac{11}{6}$	Z	-1	0	0	$\frac{10}{3}M - \frac{19}{6}$	$M - \frac{11}{10}$	0	M	-5.25
$R_1 - 5/3R_2$	x_1	0	1	0	5/3	-1	0	0	15/2
$R_2 \times 3/5$	S_2	0	0	0	3	3/5	1	-1	3/10
$R_3 + 5R_2$	x_2	0	0	1	-5	3	0	0	9/2

$(-Z)$ Max $= -5.25$

$\therefore Z_{min} = 5.25$ at $x_1 = 15/2$ & $x_2 = 9/2$

Two-phase Method

Working procedure of 2-phase simplex Method :-

The solⁿ is obtained in two phase as follows:-

phase I :- In this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solⁿ to the original problem.

Step 1: Assign a cost -1 (mul with -1) to each artificial variable and a cost 0 (mul with 0) to all other variables (in place of their original cost) in the objective function

$$Z^* = -A_1 - A_2 - A_3 \cdots \quad \text{where } A_i \text{ are artificial variables}$$

Step 2: Construct the auxiliary linear programming problem in which the new objective function Z is to be maximized subject to the given set of constraints.

Step 3: Solve the auxiliary problem by simplex method using either of the following 3 possibilities arise:

(1) Max $Z^* < 0$ and atleast one artificial vector appear in the optimum basis at a positive level. In this case given problem does not posses any feasible solⁿ.

(2) Max $Z^* = 0$ and atleast one artificial vector appear in the optimum basis at zero level. In this case proceed to phase-II.

(3) Max $Z^* = 0$ and no artificial vector appear in the optimum basis. In this case also proceed to phase-II.

Phase II : Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solⁿ has been attained. The artificial variables which are non-basic at the end of phase-I are removed.

Eg (1): use two phase method to solve the problem.

$$\min Z = 0.4x_1 + 0.5x_2$$

S.T

$$0.3x_1 + 0.1x_2 \le 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \ge 6 \qquad x_1, x_2 \ge 0$$

$$0.3x_1 + 0.1x_2 + S_1 = 2.7$$
$$0.5x_1 + 0.5x_2 + a_1 = 6$$
$$0.6x_1 + 0.4x_2 - S_2 + a_2 = 6$$

Phase I : objective $\min Z = a_1 + a_2$ —— (0) [until $a_1 = 0, a_2 = 0$]

S.T $0.3x_1 + 0.1x_2 + S_1 = 2.7$ —— (1)

$0.5x_1 + 0.5x_2 + a_1 = 6$ —— (2)

$0.6x_1 + 0.4x_2 - S_2 + a_2 = 6$ —— (3)

$x_1, x_2, S_1, a_1, a_2, S_2 \ge 0$

0

Eqⁿ (0) $\times -1 \Rightarrow$ Max

$\therefore -Z = -a_1 - a_2$

$\therefore -Z + a_1 + a_2 = 0$

New

$-Z + 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot S_1 + 0 \cdot S_2 + a_1 + a_2 = 0$ —— (0)

$\therefore S_1, S_2, a_1, a_2$ are basic variables

Co-efficient of basic Variables in (0) eqⁿ must be zero

Eqⁿ(0) $- \left[\text{eq}^n(2) + \text{eq}^n(3) \right]$

$-Z + a_1 + a_2 - \{0.5x_1 + 0.5x_2 + a_1 + 0.6x_1 + 0.4x_2 - S_2 + a_2\}$
$= 0 - \{6+6\}$

$-Z + \cancel{a_1} + \cancel{a_2} - 1.1x_1 - 0.9x_2 - \cancel{a_1} + S_2 - \cancel{a_2} = -12$

$-Z - 1.1x_1 - 0.9x_2 + S_2 = -12$

$-Z - 1.1x_1 - 0.9x_2 - 0.S_1 - 0.a_1 + S_2 - 0.a_2 = -12$

Phase I:

	BV	eqⁿ	Z	x_1	x_2	S_1	a_1	S_2	a_2	RHS	Min ratio
	Z	0	-1	-1.1	-0.9	0	0	1	0	-12	
⟷	S_1	1	0	0.3	0.1	1	0	0	0	2.7	9
	a_1	2	0	0.5	0.5	0	1	0	0	6	12
	a_2	3	0	0.6	0.4	0	0	-1	-1	6	10
$R_0 \to R_0 + 1.1R_1$	Z	0	-1	0	-16/30	11/3	0	1	0	-2.1	
$R_1 \to R_1/0.3$	x_1	1	0	1	1/3	10/3	0	0	0	9	9x3=27
$R_2 \to R_2 - 0.5R_1$	a_1	2	0	0	1/3	-5/3	1	0	0	1.5	9/2=4.5
$R_3 \to R_3 - 0.6R_1$	a_2	3	0	0	0.2	-2	0	-1	1	6	3
	Z	0	-1	0	0	-5/3	0	-5/3	8/3	-0.5	6.3
	x_1	1	0	1	0	20/3	0	5/3	-5/3	8	6/5
	a_1	2	0	0	0	5/3	1	5/3	-5/3	0.5	0.3 ←
	x_2	3	0	0	1	-10	0	-5	5	3	-0.3

	BV	eqⁿ	Z	x_1	x_2	S_1	a_1	S_2	a_2	RHS
$R_0 \to R_0 + 5/3 R_2$	Z	0	-1	0	0	0	1	0	1	0
	x_1	1	0	1	0	0	-4	-5	5	6
	S_1	2	0	0	0	1	3/5	1	-1	0.3
	x_2	3	0	0	1	0	6	5	-5	6

In the last Table of phase I The RHS value of Z^{th} row must be '0'.

This is the first table of phase II with original objective fun.

Phase 2:

$-1 \times$ Eqⁿ(0) max $-Z = -0.4x_1 - 0.5x_2$

Max $-Z + 0.4x_1 + 0.5x_2 = 0 \rightarrow$ is objective fⁿ

Substitute the original objective fⁿ.

Final phase table

BV	eqⁿ	Z	x_1	x_2	S_1	a_1	S_2	a_2	RHS
Z	0	-1	0	0	0	1	0	1	0
x_1	1	0	1	0	0	-4	-5	5	6
S_1	2	0	0	0	1	3/5	1	-1	0.3
x_2	3	0	0	1	0	6	5	-5	6

Drop a_1 & a_2

BV	eqⁿ	Z	x_1	x_2	S_1	a_1	S_2	a_2	RHS
Z	0	-1	0	0	0		0		0
x_1	1	0	1	0	0		-5		6
S_1	2	0	0	0	1		1		0.3
x_2	3	0	0	1	0		5		6

Sub phase 2 objectiveⁿ

BV	eqⁿ	Z
Z	0	-1
x_1	1	0
S_1	2	0
x_2	3	0

Min $z = 0.4x_1 + 0.5x_2$

s.c. $0.3x_1 + 0.1x_2 \le 2.7$

$0.5x_1 + 0.5x_2 = 6$

$0.6x_1 + 0.4x_2 \ge 6$

$x_1, x_2 \ge 0$

Multiply by 10

Max $z' = -4x_1 - 5x_2$

s.c. $3x_1 + x_2 \le 27$

$5x_1 + 5x_2 = 60$

$6x_1 + 4x_2 \ge 60$

$a_1 = a_1 - 5(x_1)$

$= 5 - 5(1/3) = 10/3$

$= 60 - 5(9) = 15$

$a_2 = a_2 - 6(x_1)$

$= 4 - 6(1/3)$

$60 - 6(9)$

$\frac{5}{3}+2$ $\quad \frac{5+6}{3}$

$-\frac{10}{3} + 2$ $\quad \frac{-10-6}{3}$ $\left(\frac{-16}{3}\right)$

Phase 1 : $z^* = 0x_1 - 0x_2 + 0S_1 - 0S_2 - 1A_1 - 1A_2$

	z_j	0	0	0	0	-1	-1		
C_B	BV	x_1	x_2	s_1	s_2	a_1	a_2	b	θ
0	s_1	3	1	1	0	0	0	27	27/3
-1	a_1	5	5	0	0	1	0	60	60/5
-1	a_2	6	4	0	1	0	1	60	60/6
$\Sigma c_B a_{ij}$		-11	-9	0	-1	~~0~~ 1	-1	-120	
		↑							

0	x_1	1	1/3	1/3	0	0	0	9	27
-1	a_1	0	10/3	-5/3	0	1	0	15	45/10
-1	a_2	0	2	-2	1	0	1	6	6/2
$\Sigma c_B a_{ij}$		0	-16/3	11/3	-1	-1	-1	-21	
			↑						

0	x_1	1	0	2/3	-1/4	0	-1/4	8	24/2
-1	a_1	0	0	5/3	10/6	1	10/6	5	15/3 ←
0	x_2	0	1	-1	1/2	0	1/2	3	
$\Sigma c_B a_{ij}$		0	0	-5/3	-10/6	-1	-10/6	-5	
				↑					

$a_1 = a_1 - \frac{10}{6}(x_1)$ $\quad$ $x_1 = x_1 - \frac{1}{4}(a_1)$ $\quad$ 1 $\quad$ -5 $\quad$ 10(-1)

$\frac{8}{3} \cdot 5$ $\quad \frac{10}{6} \times \frac{3}{5}$ $\quad -5 + \frac{7}{3}$ $\quad \frac{5}{5/3}$ $\quad \frac{5 \times 3}{5}$ $\quad x_2 = x_2 + \frac{1}{2}(3)$

$8 - \frac{15}{3}$ $\quad -\frac{2}{3}$ $\quad = 6 + \frac{5}{4}(3)$ $\quad = 6 + \frac{15}{4} = \frac{39}{4}$ $\quad -\frac{15}{2} - \frac{8}{3}$ $\quad 5 - \frac{15}{2}$ $\quad \frac{5}{2}$ $\quad 8 - \frac{2}{3}(3)$ $\quad x_1 = x_1 - \frac{2}{3}(S_1)$ $\quad -\frac{1}{4} - \frac{2}{3}\left(\frac{3}{2}\right)$

$= 6 - \frac{3}{2}(3) = -\frac{1-4}{4}$

$= \frac{12-9}{2}$

		0	0	0	0	-1	-1		
C_B	BV	x_1	x_2	s_1	s_2	a_1	a_2	b	θ
0	x_1	1	-2/3	0	-5/4	0	0	6	
0	s_1	0	0	1	1	3/5	1	3	
0	x_2	0	1	0	3/2	0	3/2	6	
$\Sigma c_B a_{ij}$		0	0	0	0	0	0	0	

Phase 2

		-4	-5	0	0		θ
-4	x_1	1	-0	0	-5/4	6	—
0	s_1	0	0	1	1	3	3/1 ←
-5	x_2	0	1	0	3/2	6	12/3
$\Sigma c_B a_{ij}$		-4	-7/3	0	5/2	-54	
$z_j - c_j$		0	8/3	0	-5/2		
					↑		

$-39 - \frac{15}{2}$

$\frac{-78-15}{2}$

$-\frac{93}{2}$

-4	x_1	1	-0	5/4	0	39/4
0	s_2	0	0	1	1	3
-5	x_2	0	1	-3/2	0	3/2
$\Sigma c_B a_{ij}$		-4	-5	9/2	0	

$8 - \frac{2}{3}(3)$

Degeneracy problem (Vogel's Approxi.)

6	4	1	5	14
8	9	2	7	16
4	3	6	7	5
6	10	15	4	

Operations Research (10CS661)

Scheme & Solution

2nd Test

Date: 12/4/16

For A, B & C section

a): Part A

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} - & 2 & 5 & 7 & 1 \\ 6 & - & 3 & 8 & 2 \\ 8 & 7 & - & 4 & 7 \\ 12 & 4 & 6 & - & 5 \\ 1 & 3 & 2 & 8 & - \end{bmatrix} \end{array}$$

Row reduction

$$\begin{bmatrix} \infty & 1 & 4 & 6 & 0 \\ 4 & \infty & 1 & 6 & 0 \\ 4 & 3 & \infty & 0 & 3 \\ 8 & 0 & 2 & \infty & 1 \\ 0 & 2 & 1 & 7 & \infty \end{bmatrix}$$ — $1\frac{1}{2}$

Column reduction

$$\begin{bmatrix} \infty & 1 & 3 & 6 & 0 \\ 4 & \infty & 0 & 6 & 0 \\ 4 & 3 & \infty & 0 & 3 \\ 8 & 0 & 1 & \infty & 1 \\ 0 & 2 & 0 & 7 & \infty \end{bmatrix}$$ — $1\frac{1}{2}$

Assignment

	A	B	C	D	E
A	∞	1	3	6	$\boxed{0}$
B	4	∞	$\boxed{0}$	6	~~0~~
C	4	3	∞	$\boxed{0}$	3
D	8	$\boxed{0}$	1	∞	1
E	$\boxed{0}$	2	~~0~~	7	∞

(2)

A → E
B → C
C → D
D → B
E → A

Path
A → E → A   pathes is cyclic

Cost 13

Select next minimum cost

	A	B	C	D	E
A	∞	$\boxed{1}$	3	6	~~0~~
B	4	∞	$\boxed{0}$	6	~~0~~
C	4	3	∞	$\boxed{0}$	3
D	8	~~0~~	~~1~~	∞	$\boxed{1}$
E	$\boxed{0}$	2	0	7	∞

(2)

A → B → C → D → E → A

Cost = 15

3): Determine the minimum cost assignment for the following problem. (2)

	J_1	J_2	J_3	J_4	J_5
M_1	7	5	9	8	11
M_2	9	12	7	11	10
M_3	8	5	4	6	9
M_4	7	3	6	9	5
M_5	4	6	7	5	11

Soln Here the problem is balanced as number of rows equal to no of columns

Row reduction matrix

2	0	4	3	6
2	5	0	4	3
4	1	0	2	5
4	0	3	6	2
0	2	3	1	7

— (2)

Column reduction

2	$\boxed{0}$	4	3	6	
2	5	$\boxed{0}$	3	1	✓
4	1	~~0~~	1	3	✓
4	~~0~~	3	5	$\boxed{0}$	
$\boxed{0}$	2	3	~~0~~	5	
		✓			

— (2)

x=1 Minimum element

$$\begin{bmatrix} 1 & \boxed{0} & 4 & 1 & 3 \\ 1 & 5 & \boxed{0} & 2 & 0 \\ 3 & 1 & \cancel{0} & \boxed{0} & 2 \\ 4 & 1 & 4 & 5 & \boxed{0} \\ \boxed{0} & 3 & 4 & \cancel{0} & 5 \end{bmatrix}$$

2

∴ Min cost is $5+7+6+5+4 = 27$

---

29. Find the initial basic feasible solⁿ using North-West corner and Vogel's approximation methods for the following transportation problem. 3

N-W-corner.

$\boxed{5}$ 19	$\boxed{2}$ 30	50	10	7/2/0
70	$\boxed{6}$ 30	$\boxed{3}$ 40	60	9/3/0
40	8	$\boxed{4}$ 70	$\boxed{14}$ 20	18/14/0
5/0	8/6/0	7/4/0	14/0	

4

$5 \times 19 + 2 \times 30 + 6 \times 30 + 3 \times 40 + 4 \times 70$
$+ 14 \times 20 = 1015$ Rs.

VAM :-

Step I :-

				Avail	Pen.
19	30	50	10	7	(9)
70	30	40	60	9	(10)
40	$\boxed{8}$ 8	70	20	18/10	(12)
Req: 5	8	7	14		
Penalties: (21)	(22)	(10)	(10)		

5

Step2!-

			Avail	Pent
19	50	10	7/2	(9)
70	40	60	9	(20)
40	70	20	10	(20)
Req 5/0	7	14		
Pen (21)	(10)	(10)		

Step3!-

		Avail	Pent
50	10	2	40
40	60	9	20
70	$\boxed{10}$ 20	10/0	50
Req 7	14/4		

Step4!-

		Avail	Pen
50	$\boxed{2}$ 10	2/0	40
$\boxed{7}$ 40	$\boxed{2}$ 60	9/7/0	20
Req 7/0	4/2/0		
Pen 10	50		

40	$\boxed{7}$ 60	9/7	20
7	2		

$\boxed{7}$ 40	7
7	

$8\times8 + 5\times19 + 2\times10 + 7\times40 + 2\times60 + 0\times20$

$= 779$

D) Write the procedure for Hungarian method!-

Step1!- Row reduction.

Step2!- column Reduction.

Step3!- Assignment → if not satisfied go for Hungarian method.

1) Make (✓) for rows having no assignments.

2) Make (✓) for columns having zeroes in the marked row.

3) Make (✓) rows having assigned zeros in the marked column.

4) Repeat 2 & 3, continue till no further marking.

5) Draw lines through a marked column & unmarked rows. (should cover all assignments)

6) Find minium in the uncovered element [no lines] say x.
Sub 'x' from all uncovered elements & add it to the intersection lines cell.

4

... of VAM.

				Av	Pen P1	P2	P3	P4
[5] 19	30	50	[2] 10	7/2/0	9	9	40	40
70	30	40	60	9	10	20	20	20
40	[8] 8	70	[10] 20	18/10/0	12	20	50	–
Req. 5/0	8/0	7	14/4/2					

← Factors

Pen				
P1	21	22 ↑	10	10
P2	21 ↑	—	10	10
P3	—	—	10	10
P4	—	—	10	10

3

PART - B

Project site

				Supply
2	3	11	7	6
1	0	6	1	1
5	8	15	9	10
7	5	3	2	17

5

Sum of req = Sum of supply.

∴ It is a balanced transportation problem

					Penalties I	II	III	IV
[1] 2	[5] 3	11	7	6	(1)	(1)	(5) ←	–
1	0	6	[1] 1	1	(1)	–	–	–
[6] 5	8	[3] 15	[1] 9	10	(3)	(3)	(4)	(4)
7	5	3	2					

Penal.				
I	(1)	(3)	(5)	(6) ↑
II	(3)	(1) ↑	(4)	(2)
III	(3)	–	(4)	(2)
IV	(1)	–	(15) ↑	(9)

2

$(1\times 2) + (5\times 3) + (1\times 1) + (6\times 5) + (3\times 15) + (1\times 9) = \underline{102}$

UV Method

2	3			$u_1 = -3$
			1	$u_2 = -8$
5		15	9	$u_3 = 0$
$v_1 = 5$	$v_2 = 6$	$v_3 = 15$	$v_4 = 9$	

$m+n-1 =$
$3+4-1$
$= 6$ sol^n

P.T.O

New cost for non basic cells

		12	6
-3	-2	-7	
	6		

difference (old c_{ij} − new c_{ij} = d_{ij})

		-1	1
4	2	-1	
	2		

1	5		
		+θ	1-θ
		3-θ	+θ

min{1-θ, 3-θ}
∴ θ = 1

1 2	1 3		
		1 6	0 X
6 5		2 15	2 9

(1×2) + (5×3) + (1×6) + (6×1) + (2×15) + (2×9) +

= 101

4

All solutions are in independent positions
∴ no closed loops.
∴ optimal soln = 101

VAM − (1×θ)
102 − (1×1) = 101

Again Apply MODI Method.

6

2	3			$u_1 = -3$
		6		$u_2 = -9$
5		15	9	$u_3 = 0$
$v_1 = 5$	$v_2 = 6$	$v_3 = 15$	$v_4 = 9$	

New cost for non basic cells

x	x	12	6
-4	-3	x	0
x	6	x	x

4

Cost difference

x	x	-1	+1
5	+3	x	
x	2	x	x

1			+θ
-θ			-θ 2

new (θ-1) (θ-2)
∴ θ = 1

0 ~~2~~	5 3	11 4	
		1 6	
1 5		2 15 1	2 9

5×3 + 11×1 + 1×6 + 5×7 + 15×1
+ 2×9

~~86~~

100 4

	0	11	
		U_6	
U_5		U_{15}	U_9

1) 6 soln.
2) No loop formation

∴ Again apply Modi method

find out u & v values

	3	11		$u_1 = -4$
		6		$u_2 = -9$
5		15	9	$u_3 = 0$
$v_1 = 5$	v_2	$v_3 = 15$	$v_4 = 9$	

New cost matrix (C_{ij}) for non basic cells

1	x	x	5
-4	-6	x	0
x	3	x	+

2

Cost difference: ~~new~~ old c_{ij} - new c_{ij} = d_{ij}

1	x	x	2
5	6	x	1
x	5	x	x

∴ previously obtained is optimal
∵ no negative values

4) LPP using Big-M method

Maximize $Z = -2x_1 - x_2$

S.T

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4 \quad x_1, x_2 \geq 0$

Soln : $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

Max $Z = -2x_1 - x_2 - Ma_1 - Ma_2$

$Z + 2x_1 + x_2 + Ma_1 + Ma_2 = 0$ — 3

$-[3Mx_1 + Mx_2 + Ma_1 = 3M]$

$-[4Mx_1 + 3Mx_2 - Ms_1 + Ma_2 = 6M]$

$Z + (2-7M)x_1 + (1-4M)x_2 + Ms_1 = -9M$

BV	x_1	x_2	s_1	s_2	a_1	a_2	RHS	Min ratio
Z	2-7M	1-4M	M	0	0	0	-9M	
a_1	3	1	0	0	1	0	3	3/3
a_2	4	3	-1	0	0	1	6	6/4
s_2	1	2	0	1	0	0	4	4/1
Z	0	1-5M/3	M	0	-(2+7M)/3	0	-2-2M	
x_1	1	1/3	0	0	1/3	0	1	1/ 1/3 = 3
a_2	0	5/3	-1	0	-4/3	1	2	2/ 5/3 = 1.2
s_2	0	5/3	0	1	-1/3	0	3	3/ 5/3 = 1.8

3

3

1st case

2) phase Method

1) use Two-phase method to solve

$$\text{Max } Z = 5x_1 + 3x_2$$

S.T

$$2x_1 + x_2 \le 1$$

$$x_1 + 4x_2 \ge 6$$

$$x_1, x_2 \ge 0$$

Soln: we convert the given problem into a standard form by adding slack, surplus and artificial variables. we form the auxiliary LPP by assigning the cost -1 to the artificial variable & 0 to all the other variables.

$$2x_1 + x_2 + S_1 = 1$$

$$x_1 + 4x_2 - S_2 + a_1 = 6$$

Phase I:

$$\text{Max } Z^* = \cancel{5x_1 + 3x_2}\ 0x_1 + 0x_2 + 0 \cdot S_1 + 0 \cdot S_2 - 1 \cdot a_1 \quad -(1)$$

S.T

$$2x_1 + x_2 + S_1 = 1 \quad -(2)$$

$$x_1 + 4x_2 - S_2 + a_1 = 6 \quad -(3)$$

Initial basic feasible is given by $S_1 = 1,\ a_1 = 6$

coefficient of basic variable should be zero

$$Z - 0x_1 - 0 \cdot x_2 - 0 \cdot S_1 - 0 \cdot S_2 + a_1 - x_1 - 4x_2 + S_2 - \cancel{a_1} = 0 - 6$$

So $\quad = Z - x_1 - 4x_2 + S_2 = -6$

BV	eqn	Z	x_1	x_2	S_1	S_2	a_1	RHS	Minratio
Z	1	1	-1	-4	1	0	0	-6	x
S_1	2	0	2	1	1	0	0	1	$1/1 = 1$
a_1	3	0	1	4	-1	0	1	6	$6/4 = 3/2 = 1.5$
$R_1 = R_1 + 4R_2$ Z	1	1	7	0	4	0	0	-2	$-2/1 = 2$
x_2	2	0	2	1	1	0	0	1	$1/2 = 0.5$

BV	x_1	x_2	S_1	S_2	a_1	a_2	RHS	Minratio
Z	0	0	1/5	0	M-2/5	M-1/5	-12/5	
x_1	1	0	1/5	0	3/5	-1/5	3/5	
x_2	0	1	-3/5	0	-4/5	3/5	6/5	
S_2	0	0	1	1	1	-1	1	

-(3)

$$\boxed{\text{Max } Z = -12/5 \quad x_1 = 3/5 \quad x_2 = 6/5}$$

~~Max Z is negative~~

~~It is Infeasible soln~~

	BV	eqn	Z	x_1	a_2	s_1	s_2	a_1	RHS	Ratio
	Z	1	1	-1	0	4	0	0	-2	$-2/1 = 2$
$R_2 = R_2/2$	x_2	2	0	2	1	1	0	0	1	$1/2 = 0.5$
	a_1	3	0	1	0	-8	0	1	2	$2/1 = 2$
$R_1 = R_1 + R_2$	Z	1	1	-1	0	4	0	0	-2	
	x_1	2	0	1	1/2	1/2	0	0	1/2	
$R_3 = R_3 - R_2$	a_1	3	0	1	0	-8	0	1	2	
									Z^*	
	Z	1	1	0	1/2	9/2	0	0	-3/2	
	x_1	2	0	1	1/2	1/2	0	0	1/2	
	a_1	3	0	0	-1/2	-17/2	0	1	3/2	

$Z^* < 0$ and an artificial variable A_1 is in the basis at a positive level, the original LPP does not possess any feasible soln.

Example ② Min $Z = x_1 + x_2$

S-T

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7 \quad x_1, x_2 \geq 0$

Phase I:

$2x_1 + x_2 - s_1 + a_1 = 4$ — ①

$x_1 + 7x_2 - s_2 + a_2 = 7$ — ② $\quad x_i \geq 0$

Obj ~~Min Z~~ ~~$a_1 + a_2$~~

Min $Z = -a_1 - a_2$

Max $Z =$ ~~$a_1 + a_2$~~

$Z - a_1 a_2$

Soln Convert the objective function to maximization using

Min $Z = -$ Max $(-Z)$

$=$ Max $Z = -x_1 - x_2$

$-\frac{1}{2}$

$\frac{-4+1}{2} = -\frac{3}{2}$

$\frac{-8-\frac{1}{2}}{2-\frac{1}{2}} = \frac{-17/2}{3/2}$

cost -1 &

Introducing the artificial variable a_1 & a_2 i.e. cost 0 to other variables, the objective function of auxiliary LPP is

Max $-Z^* = 0 \cdot s_1 + 0 \cdot s_2 - 1 \cdot a_1 - 1 \cdot a_2$

$= Z + a_1 + a_2 - 2x_1 - x_2 + s_1 - a_1 - x_1 - 7x_2 + s_2 - a_2 = 0 - 4 - 7$

$-Z - 3x_1 - 8x_2 + s_1 + s_2 = -11$

$-Z - 3x_1 - 8x_2 + s_1 + s_2 - 0 \cdot a_1 - 0 \cdot a_2 = -11$ — ③

The basic variables are a_1, a_2.

	BV	eqn	Z	x_1	x_2	s_1	a_1	s_2	a_2	RHS	Ratio
	Z	0	-1	-3	-8	1	0	1	0	-11	
	a_1	1	0	2	1	-1	1	0	0	4	$4/1 = 4$
	← a_2	2	0	1	7	0	0	-1	1	7	$7/7 = 1$
$R_1 + 8$	Z	0	-1	-13/7	0	1	0	-1/7	8/7	-3	—
$-R_2$	a_1	1	0	13/7	0	-1	1	1/7	-1/7	3	$21/13 = 1.$
/7	x_2	2	0	1/7	1	0	0	-1/7	1/7	1	7
R_1	Z	0	-1	0	0	0	1	0	1	0	
3^*	x_1	1	0	1	0	-7/13	7/13	1/13	-1/13	21/13	
	x_2	2	0	0	1	1/13	-1/13	-2/13	2/13	10/13	

* RHS of Z^{th} row is zero
* Drop a_1 & a_2

coefficients of BV's x_1 & x_2 should be '0'.
Restore to proper form using Gaussian elimination

Phase 2:

original objective function

Min Z:

$\text{Max} -Z = -x_1 - x_2$

$-Z + x_1 + x_2 = 0$

drop artificial variables

BV	eqn	Z	x_1	x_2	S_1	a_1	S_2	a_2	RHS	Min
Z	0	-1	1	1	0	X	0	X	0	
x_1	1	0	1	0	$-7/13$	X	$1/13$	X	$21/13$	
x_2	2	0	0	1	$1/13$	X	$-2/13$	X	$10/13$	

$R_0 - [R_1 + R_2]$

Z	0	-1	0	0	$6/13$	X	$1/13$	X	$-31/13$
x_1	1	0	1	0	$-7/13$	X	$1/13$	X	$21/13$
x_2	2	0	0	1	$1/13$	X	$-2/13$	X	$10/13$

$\therefore Z_{max} = -31/13$

$\therefore Z_{min} = 31/13$

at

$x_1 = 21/13$ & $x_2 = 10/13$

$R_0 \to [R_1 + R_2]$

$1 - [1 + 1]$

$R_0 \to 1R_1 \quad x - 1$

$0 + -7/13 + 1/13$

$= \frac{6}{13}$

$R_0 + R_1 + 0$

$\frac{1}{13} + \frac{-2}{13} = \frac{1-2}{13}$

Starting from the final table, we drop the artificial variables a_1 & a_2, substitute the phase 2 objective function (original form) into row 0 & then restore the proper form Gaussian elimination (by algebraically eliminating the basic variables x_1 & x_2 from row 0). Thus row 0 in the last table is obtained by performing the following elementary

from row 0 subtract both the product, ...

(1) ~~Max~~ Min $Z = x_1 - 2x_2 - 3x_3$

S.T

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Example case 3

(2) ~~Question paper~~ use 2-phase simplex method

Minimize $Z = 15/2\, x_1 - 3x_2$

S.T

$3x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$ &

$x_1, x_2, x_3 \geq 0$

Soln: Convert the objective function to the max form

$\text{Max } Z' = -15/2\, x_1 + 3x_2$

Introducing the ~~slack~~ surplus variables $S_1 \geq 0$ & $S_2 \geq 0$ and artificial variables $a_1 \geq 0$ & $a_2 \geq 0$, the constraints of the given problem become.

$3x_1 - x_2 - x_3 - S_1 + a_1 = 3$ — (2)

$x_1 - x_2 + x_3 - S_2 + a_2 = 2$ — (3)

$x_1, x_2, x_3, S_1, S_2, a_1, a_2 \geq 0$

Phase 1: Assigning a cost -1 to artificial variables a_1 and a_2 and cost 0 to all other variables. $\therefore$ the new objective function for auxiliary problem becomes.

$\text{Max } Z' = -a_1 - a_2$ — (1)

$$-Z + \cancel{a_1} + \cancel{a_2} - 3x_1 + x_2 + x_3 + S_1 - \cancel{a_1} - x_1 + x_2 + S_2 - \cancel{a_2} = 0 - 3 - 2$$

$$-Z - 4x_1 + 2x_2 + S_1 + S_2 = -5$$

BV	eq^n	Z	x_1	x_2	x_3	S_1	S_2	a_1	a_2	RHS	Min
Z	1	-1	-4	2	0	1	1	0	0	-5	
a_1	2	0	3	-1	-1	-1	0	1	0	3	3/3
a_2	3	0	1	-1	1	0	-1	0	1	2	2/1
$R_1: R_1+4(R_2)$ Z	1	-1	0	2/3	-4/3	-1/3	1	4/3	0	-1	×
$R_2: R_2/3$ x_1	2	0	1	-1/3	-1/3	-1/3	0	1/3	0	1	×
$R_3: R_3-R_1$ a_2	3	0	0	-2/3	4/3	1/3	-1	-1/3	1	1	
Z	1	~~-1~~	0	0	0	0	0	1	1	0	
x_1	2	0	1	-1/2	0	-1/4	-1/4	1/4	1/4	5/4	
x_2	3	0	0	-1/2	1	1/4	-3/4	-1/4	1/4	3/4	

No artificial variable appears in the basis an optimum sol^n to the auxiliary problem has been attained

Phase 2: In this phase, now consider the actual costs associated with the original variables, the objective function thus becomes

$$\text{Min } Z = 15/2\, x_1 - 3x_2$$

$$\text{Max } z' = -15/2\, x_1 + 3x_2 + 0.S_1 + 0.S_2$$

$$\text{Max } z' + 15/2\, x_1 - 3x_2 - 0.S_1 - 0.S_2$$

BV	eq^n	Z	x_1	x_2	x_3	S_1	S_2	RHS
Z	1	-1	15/2	-3	0	0	0	
x_1	2	0	1	-1/2	0	-1/4	-1/4	
x_2	3	0	0	-1/2	1	1/4	-3/4	

$R_0 \longrightarrow R_0 - 15/2\, R_1$ (no R_2 bec x_3)

Z	1	-1	0	3/4	0	15/8	15/8	-75/8
x_1	2	0	1	-1/2	0	-1/4	-1/4	
x_2	3	0	0	-1/2	1	1/4	-3/4	

$\therefore Z' = -75/8$

$$\boxed{\begin{array}{l} \text{Min } Z = 75/8 \\ x_1 = 5/4 \\ x_2 = 0 \\ x_3 = 3/4 \end{array}}$$

UNIT - 5

Duality in Linear programming

Introduction:

Every LPP (called the primal) is associated with another LPP (called its dual). Either of the problem can be considered as primal with the other one as dual.

The importance of the duality concept is due to two main reasons

(i) If the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it in to the dual problem & the solving it.

(ii) The interpretation of the dual variables from the cost or economic point of view, proves extremely useful in making future decisions in the activities being programmed.

Formation of Dual problems

First we bring the problem in the canonical form. The following changes are used in formulating the dual problem.

(1) change the objective function of maximization in the primal into minimization one in the dual & vice versa.

(2) The number of variable in the primal will be number of contraints in the dual & vice versa.

(3) The cost coefficients $c_1, c_2, \ldots c_n$ in the objective function of the primal will be the RHS constant of the constraints in the dual & vice versa.

(4) In forming the constraints for the dual, we consider the transpose of the body matrix of the primal problem.

(5) The variables in both problems are non-negative

(6) If the variable in the primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation & vice versa.

<u>Definition of the dual problem</u>

Let the primal ~~LPP~~ problem be

$$\text{Max } Z = c_1x_1 + c_2x_2 + c_3x_3 \cdots + c_nx_n$$

S.T

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n \leq b_m$$

$$x_1, x_2 \cdots x_n \geq 0$$

<u>Dual</u> The dual problem is defined as

$$\text{Min } Z' = b_1w_1 + b_2w_2 + \cdots + b_mw_m$$

S.T

$$a_{11}w_1 + a_{21}w_2 + a_{31}w_3 + \cdots + a_{m1}w_m \geq c_1$$
$$a_{12}w_1 + a_{22}w_2 + \cdots \quad a_{m2}w_m \geq c_2$$
$$\vdots$$
$$a_{1n}w_1 + a_{2n}w_2 + \cdots \quad a_{mn}w_m \geq c_n$$

$$w_1, w_2, \cdots w_m \geq 0$$

Where w_1, w_2, w_3 are dual Variables

Eg:- Write the dual of the following primal LP pblm.

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

S.T

$$2x_1 + x_2 - x_3 \leq 2$$
$$-2x_1 + x_2 - 5x_3 \geq -6 \Longrightarrow 2x_1 - x_2 + 5x_3$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

<u>Soln</u> Since the problem is not in the canonical form, we interchange the inequality of second constraint

Dual ~~let w_1, w_2, w_3 be the dual variables~~

$$\cancel{\text{Min } Z' = 2w_1 + 6w_2 + 6w_3}$$

~~S.T~~

$$\cancel{2w_1 + 2w_2 + 4w_3 \geq 1}$$

Max $\rightarrow \leq$

Min $\rightarrow \geq$

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

S.T

$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

∴ Dual let w_1, w_2, w_3 be the dual variables

$$\text{Min } Z' = 2w_1 + 6w_2 + 6w_3$$

S.T

$$2w_1 + 2w_2 + 4w_3 \geq 1$$
$$w_1 - w_2 + w_3 \geq 2$$
$$-w_1 + 5w_2 + w_3 \geq 1$$
$$w_1, w_2, w_3 \geq 0$$

(2) Find the dual of the following LPP

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

S.T

$$4x_1 - x_2 \le 8$$
$$8x_1 + x_2 + 3x_3 \ge 12$$
$$5x_1 - 6x_3 \le 13$$
$$x_1, x_2, x_3 \ge 0$$

Sol^n: Since the problem is not in the canonical form, we interchange the inequality of the second contraint.

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

S.T

$$4x_1 - x_2 + 0, x_3 \le 8$$
$$-8x_1 - x_2 - 3x_3 \le -12$$
$$5x_1 + 0x_2 - 6x_3 \le 13$$
$$x_1, x_2, x_3 \ge 0$$

Max $Z = Cx$

S.T

$$Ax \le B$$
$$x \ge 0$$

$$C = \begin{pmatrix} 3 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$$

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & -6 \end{bmatrix}$$

Dual: let w_1, w_2, w_3 be the dual variables. The dual problem is

$$\min Z' = b^T w$$

ST

$$A^T w \ge C^T \;\&\; w \ge 0$$

i.e $\text{Min } Z' = \begin{pmatrix} 8 & -12 & 13 \end{pmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

S.T $\begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \text{Min } Z' = 8w_1 - 12w_2 + 13w_3$

S.T

$$4w_1 - 8w_2 + 5w_3 \ge 3$$
$$-w_1 - w_2 + 0w_3 \ge -1$$
$$0w_1 - 3w_2 - 6w_3 \ge 1$$
$$w_1, w_2, w_3 \ge 0$$

(3) Write the dual of the following LPP

$$\text{Min } Z = 2x_2 + 5x_3$$

S.T

$$x_1 + x_2 \ge 2$$
$$2x_1 + x_2 + 6x_3 \le 6$$
$$x_1 - x_2 + 3x_3 = 4$$
$$x_1, x_2, x_3 \ge 0$$

Sol^n: Since the given primal problem is not in the canonical form, we interchange the inequality of the constraint. Also the third constraint is an equation. This equation can be converted into two equations.

$$\text{Min } Z = 0x_1 + 2x_2 + 5x_3$$

S.T

$$x_1 + x_2 + 0x_3 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \leq 4$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Again on rearranging the constraint, we have

$$\text{Min } Z = 0x_1 + 2x_2 + 5x_3$$

S.T

$$x_1 + x_2 + 0x_3 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1\, x_2\, x_3 \geq 0$$

Dual: Since there are four constraints in the primal, we have four dual variable namely
w_1, w_2, w_3', w_3''

$\therefore$ $\text{Max } Z' = 2w_1 - 6w_2 + 4w_3' - 4w_3''$

S.T

$$w_1 - 2w_2 + w_3' - w_3'' \leq 0$$
$$w_1 - w_2 - w_3' + w_3'' \leq 2$$
$$0w_1 - 6w_2 + 3w_3' - 3w_3'' \leq 5$$

$$\text{Max } Z' = 2w_1 - 6w_2 + 4\underbrace{(w_3' - w_3'')}_{w_3}$$

S.T

$$w_1 - 2w_2 + (w_3' - w_3'') \leq 0$$
$$w_1 - w_2 - (w_3' - w_3'') \leq 2$$
$$0w_1 - 6w_2 + 3(w_3' - w_3'') \leq 5$$

$$\boxed{\begin{aligned} &\text{Max } Z' = 2w_1 - 6w_2 + 4w_3 \\ &\text{S.T} \\ &w_1 - 2w_2 + w_3 \leq 0 \\ &w_1 - w_2 - w_3 \leq 2 \\ &0w_1 - 6w_2 + 3w_3 \leq 5 \\ &w_1, w_2 \geq 0,\ w_3 \text{ is unrestricted} \end{aligned}}$$

$\boxed{\text{or}}$

Sol^n: $\Rightarrow$ First convert the problem into standard form as follows:-

(1) change the objective function min to max

i.e. $\text{Min } Z = \text{Max } Z'$

$\therefore \text{Max } Z' = -2x_2 - 5x_3$

(2) The inequality can be written as
$x_1 + x_2 \geq 2$ $(x -1)$
will become

$$-x_1 - x_2 \leq -2$$

(3) The eq^n $x_1 - x_2 + 3x_3 = 4$ can be expressed as a pair of inequalities $[x = w \Rightarrow x \leq w,\ x \geq w]$

$$\therefore \begin{cases} x_1 - x_2 + 3x_3 \leq 4 \\ x_1 - x_2 + 3x_3 \geq 4 \end{cases}$$

Thus original problem now becomes of the standard primal form

$$\text{Max } Z' = 0\cdot x_1 - 2x_2 - 5x_3$$

S.T

$$-x_1 - x_2 \le -2$$
$$2x_1 + x_2 + 6x_3 \le 6$$
$$x_1 - x_2 + 3x_3 \le 4$$
$$-x_1 + x_2 - 3x_3 \le -4$$
$$x_1, x_2, x_3 \ge 0$$

Thus by using the rules required dual is given as

$$\text{Min } Z = -2w_1 + 6w_2 + 4w_3 - 4w_4$$

S.T

$$-w_1 + 2w_2 + w_3 - w_4 \ge 0$$
$$-w_1 + w_2 - w_3 + w_4 \ge -2$$
$$0\cdot w_1 + 6w_2 + 3w_3 - 3w_4 \ge -5$$
$$\therefore\ w_1, w_2, w_3, w_4 \ge 0$$

(4) Construct the dual problem

$$\text{Max } Z = 16x_1 + 14x_2 + 36x_3 + 6x_4$$

S.T

$$14x_1 + 4x_2 + 14x_3 + 8x_4 = 21$$
$$13x_1 + 17x_2 + 80x_3 + 2x_4 \le 48$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Sol^n:- First convert the problem into standard form as follows:

(1) change the objective function max to min

$$Z = 21w_1 + 48w_2$$

(2) The inequality can be written as

$$14x_1 + 4x_2 + 14x_3 + 8x_4 \le 21$$
$$14x_1 + 4x_2 + 14x_3 + 8x_4 \ge 21$$
i.e., $-14x_1 - 4x_2 - 14x_3 - 8x_4 \le -21$

(3) The original problem becomes

$$\text{Max } Z = 16x_1 + 14x_2 + 36x_3 + 6x_4$$

~~Max Z = 21w...~~

S.T

$$14x_1 + 4x_2 + 14x_3 + 8x_4 \le 21$$
$$-14x_1 - 4x_2 - 14x_3 - 8x_4 \le -21$$
$$13x_1 + 17x_2 + 80x_3 + 2x_4 \le 48$$
$$x_1, x_2, x_3, x_4 \ge 0$$

dual is

$$\text{Min } Z = \underbrace{21w_1 - 21w_2 + 48w_3}_{21(w_1 - w_2) + 48w_3}$$ w_1, w_2 is unrestricted

S.T

$14(w_1 - w_2) + 48w_3$

$$14w_1 - 14w_2 + 13w_3 \ge 16$$
$$4w_1 - 4w_2 + 17w_3 \ge 14$$
$$14w_1 - 14w_2 + 80w_3 \ge 36$$
$$8w_1 - 8w_2 + 2w_3 \ge 6$$

w_1, w_2 unrestricted

(5) $\text{Max } Z = \begin{bmatrix} 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

S-T $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 10 \\ 20 \end{bmatrix}$

$x_i \geq 0$

Soln.

Step1: $\text{Max } Z = 5x_1 + 8x_2$

S-T

$x_1 + 2x_2 \leq 5$

$x_1 + 3x_2 \leq 10$

$3x_1 + 5x_2 \leq 20$ $\quad x_i \geq 0$

Step2: $\text{Min } Z' = 5w_1 + 10w_2 + 20w_3$

S-T

$w_1 + w_2 + 3w_3 \geq 5$

$2w_1 + 3w_2 + 5w_3 \geq 8$

$w_i \geq 0$

$\text{Min } Z' = \begin{bmatrix} 5 & 10 & 20 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

S-T $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \geq \begin{matrix} 5 \\ 8 \end{matrix}$

(6) $\text{Max } Z = 2x_1 + 3x_2 + x_3$

S-T

$4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$ & $x_1, x_2, x_3 \geq 0$

Soln. $\text{Max } Z = 2x_1 + 3x_2 + x_3$

S-T

$4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$-x_1 - 2x_2 - 5x_3 \leq 4$ $\quad x_1, x_2, x_3 \geq 0$

Dual form is

(7) Give the dual of the following problem

$$\text{Max } Z = x + 2y$$

S.T

$$2x + 3y \geq 4$$
$$3x + 4y = 5$$
$$x \geq 0 \text{ \& } y \text{ unrestricted}$$

Sol^n. Since the variable y is unrestricted, it can be expressed as

$$y = y' - y''$$
where $y', y'' \geq 0$

on reformulating the given problem, we have

$$\text{Max } Z = x + 2(y' - y'')$$

S.T

$$-2x - 3(y' - y'') \leq -4$$
$$3x + 4(y' - y'') \leq 5$$
$$+3x + 4(y' - y'') \geq 5$$
$$x, y', y'' \geq 0$$

Since the problem is not in the canonical form we rearrange the constraints

$$\text{Max } Z = x_1 + 2y' - 2y''$$

S.T

$$-2x - 3y' + 3y'' \leq -4$$
$$3x + 4y' - 4y'' \leq 5$$
$$-3x - 4y' - 4y'' \leq -5$$

~~$x, y', y'' \geq 0$~~

Dual: Since there are three variables & three constraints, in dual we have three variables namely $\omega_1, \omega_2', \omega_2''$

$$\text{Min } Z' = -4\omega_1 + 5\omega_2' - 5\omega_2''$$

S.T

$$-2\omega_1 + 3\omega_2' - 3\omega_2'' \geq 1$$
$$-3\omega_1 + 4\omega_2' - 4\omega_2'' \geq 2$$
$$+3\omega_1 - 4\omega_2' + 4\omega_2'' \geq -2$$
$$\omega_1, \omega_2', \omega_2'' \geq 0$$

Let $\omega_2 = \omega_2' - \omega_2''$, so that the dual variable ω_2 is unrestricted in sign, Finally the dual is

$$\text{Min } Z' = -4\omega_1 + 5(\omega_2' - \omega_2'')$$

S.T

$$-2\omega_1 + 3(\omega_2' - \omega_2'') \geq 1$$
$$-3\omega_1 + 4(\omega_2' - \omega_2'') \geq 2$$
$$3\omega_1 - 4(\omega_2' - \omega_2'') \geq -2$$

$$\text{Min } Z' = -4\omega_1 + 5\omega_2$$

S.T

$$-2\omega_1 + 3\omega_2 \geq 1$$
$$-3\omega_1 + 4\omega_2 \geq 2$$
$$3\omega_1 - 4\omega_2 \geq -2$$
$$\omega_1 \geq 0 \text{ \& } \omega_2 \text{ is unrestricted}$$

i.e. $\text{Min } Z' = -4\omega_1 + 5\omega_2$

S.T

$$-2\omega_1 + 3\omega_2 \geq 1$$
$$-3\omega_1 + 4\omega_2 = 2$$
$$\omega_1 \geq 0 \text{ \& } \omega_2 \text{ is unrestricted}$$

(8) Write the dual of the following primal LPP

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_3$$

S.T

$$x_1 + x_2 + x_3 = 22$$
$$3x_1 + 5x_2 - 2x_3 \leq 65$$
$$x_1 + 7x_2 + 4x_3 \geq 120$$
$x_1 + x_2 \geq 0$ & x_3 is unrestricted.

Soln:- Since the Variable x_3 is unrestricted $x_3 = x_3' - x_3''$. Also bring the problem into canonical form by rearranging the constraints

$$\text{Min } Z = 4x_1 + 5x_2 - 3(x_3' - x_3'')$$

S.T

$$x_1 + x_2 + (x_3' - x_3'') \leq 22$$
$$x_1 + x_2 + (x_3' - x_3'') \geq 22$$
$$-3x_1 - 5x_2 + 2(x_3' - x_3'') \geq -65$$
$$x_1 + 7x_2 + 4(x_3' - x_3'') \geq 120$$
$$x_1, x_2, x_3' - x_3'' \geq 0$$

$$\text{Min } Z = 4x_1 + 5x_2 - 3x_1' + 3x_3''$$

S.T

$$x_1 + x_2 + x_3' - x_3'' \geq 22$$
$$-x_1 - x_2 - x_3' + x_3'' \geq -22$$
$$-3x_1 - 5x_2 + 2x_3' - 2x_3'' \geq -65$$
$$x_1 + 7x_2 + 4x_3' - 4x_3'' \geq 120$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

Dual: Since there are four constraints in the primal problem, in dual there are four Variables namely w_1', w_1'', w_2, w_3 so that the dual is given by

$$\text{Max } Z' = 22(w_1' - w_1'') - 65w_2 + 120w_3$$

S.T
$$w_1' - w_1'' - 3w_2 + w_3 \leq 4$$
$$w_1' - w_1'' - 5w_2 + 7w_3 \leq 5$$
$$w_1' - w_1'' + 2w_2 + 4w_3 \leq -3$$
$$-w_1' + w_1'' - 2w_2 - 4w_3 \leq 3$$
$$w_1', w_1'', w_2, w_3 \geq 0$$

let $w_1 = w_1' - w_1''$ i.e the variable w_1 is unrestricted

i.e $\text{Max } Z' = 22(w_1' - w_1'') - 65w_2 + 120w_3$

S.T
$$(w_1' - w_1'') - 3w_2 + w_3 \leq 4$$
$$(w_1' - w_1'') - 5w_2 + 7w_3 \leq 5$$
$$\cancel{(w_1' - w_1'') + 2w_2 + 4w_3 \leq -3}$$
$$-(w_1' - w_1'')$$
$$-w_1' + w_1'' - 2w_2 - 4w_3 \geq 3$$
$$\Downarrow$$
$$-(w_1' - w_1'') - 2w_2 - 4w_3 \geq 3$$
$$-(w_1' - w_1'') - 2w_2 - 4w_3 \leq 3$$

Finally we have

$$\text{Min } Z' = 22w_1 - 65w_2 + 120w_3$$

S.T
$$w_1 - 3w_2 + w_3 \leq 4$$
$$w_1 - 5w_2 + 7w_3 \leq 5$$
$$-w_1 - 2w_2 - 4w_3 = 3$$
$w_2, w_3 \geq 0$, w_1 is unrestricted

Important results in duality

1. The dual of the dual is primal.
2. If one is a maximization problem then the other is a minimization one.
3. The necessary and sufficient condition for any LPP and its dual to have an optimal solution is that both must have feasible solution.
4. Fundamental duality theorem states if either the primal or dual problem has a finite optimal solution and also the optimal values of the objective function in both the problems are the same i.e., $Max\ Z = Min\ Z'$. The solution of the other problem can be read from the $Z_j - C_j$ (CB(S)-Z) row below the columns of the slack, surplus variable.
5. Existence Theorem states that, if either problem has an unbounded solution then the other problem has no feasible solution.
6. Complementary slackness theorem: According to which
   (i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum & vice versa.
   (ii) If a primal constraint is a strict inequality then the corresponding dual variable is zero at the optimum & vice versa.

Eg(1): Find the dual of the problems & solve by Simplex or Big M method:

$$Max\ Z = 3x_1 + 5x_2$$
S.T
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

Soln: The dual form is

$$Min\ Z' = 4y_1 + 12y_2 + 18y_3$$
S.T
$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$
$$y_i \ge 0$$

Step 2: Solve using BIG M Method

$$Max\ -Z' = -4y_1 - 12y_2 - 18y_3$$
S.T
$$y_1 + 3y_3 - S_1 + a_1 = 3 \quad (1)$$
$$2y_2 + 2y_3 - S_2 + a_2 = 5 \quad (2) \quad y_i \ge 0$$

$$\therefore -Z' = -4y_1 - 12y_2 - 18y_3 - Ma_1 - Ma_2$$
$$-Z' + 4y_1 + 12y_2 + 18y_3 + Ma_1 + Ma_2 = 0 \quad (3)$$

$$Eq^n\ (3) \quad -M[eq^n(1) + eq^n(2)]$$

$$-Z' + 4y_1 + 12y_2 + 18y_3 + \cancel{Ma_1} + \cancel{Ma_2} - My_1 - 3My_3 + MS_1 - \cancel{Ma_1} - 2My_2 - 2My_3 + MS_2 - \cancel{Ma_2} = -3M - 5M - 0$$

$$-Z' + (4-M)y_1 + (12-2M)y_2 + (18-5M)y_3 + MS_1 + MS_2$$

	BV	eq^n	z	y_1	y_2	y_3	S_1	a_1	S_2	a_2	RHS	Min ratio
	z'	0	-1	$(4-M)$	$(2-2M)$	$(18-5M)$	M	0	M	0	-8M	
	a_1	1	0	1	0	3	-1	1	0	0	3	1
	a_2	2	0	0	2	2	0	0	-1	1	5	$5/2 = 2.5$
$R_0 \to R_0 - (18-5M)R_1$	z'	0	-1	$-2+\frac{2}{3}M$	$2-2M$	0	$\frac{6-2M}{3}$	$-6+\frac{5M}{3}$	M	0	$-18-3M$	
$R_1 \to R_1/3$	y_3	1	0	1/3	0	1	-1/3	1/3	0	0	1	x
$R_2 \to R_2 - 2R_1$	a_2	2	0	-2/3	2	0	2/3	-2/3	-1	1	3	3/2
	z'	0	-1	2	0	0	2	$-2+M$	6	$M-6$	-36	
	y_3	1	0	1/3	0	1	-1/3	1/3	0	0	1	
	y_2	2	0	-1/3	1	0	1/3	-1/3	-1/2	1/2	3/2	

$Z'_{Max} = -36$

$\therefore \boxed{Z_{Min} = 36}$ $\quad y_2 = 3/2,\ y_3 = 1$

dual form is

Max $W = w_1 + w_2 + w_3$

S.T

$2w_1 + w_2 + 2w_3 \leq 2$

$4w_1 + 2w_2 + w_3 \leq 2$

$w_1, w_2, w_3 \geq 0$

Apply simplex method to solve

Ans

$$\boxed{\begin{array}{l} \text{Max } Z = 4/3 \\ x_1 = 1/3,\ x_2 = 1/3 \end{array}}$$

(2) Use duality to solve the following LPP

Minimize $Z = 2x_1 + 2x_2$

S.T

$2x_1 + 4x_2 \geq 1$

$-x_1 - 2x_2 \leq -1$

$2x_1 + x_2 \geq 1$ & $x_1, x_2 \geq 0$

Sol^n: ~~The dual problem is~~

~~Max~~ =

Standard form is

Min $Z = 2x_1 + 2x_2$

S.T

$2x_1 + 4x_2 \geq 1$

$x_1 + 2x_2 \geq 1$

Ans:

$\boxed{\begin{array}{l} Max\ Z = 21.23 \\ x_1 = 7 \\ x_2 = 0 \end{array}}$

(3) Apply the principle of duality to solve the LPP

$$Max\ Z = 3x_1 + 2x_2$$

S.T

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7 \qquad \text{(MU, BE, Oct 96)}$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3, \quad x_1, x_2 \geq 0$$

Sol^n: First we convert the given (primal) problem into its dual. As there are 4 constraints in the primal problem we have four variables w_1, w_2, w_3, w_4 in its dual. We convert the given problem into its canonical form by rearranging some of the constraints

$$Max\ Z = 3x_2 + 2x_2$$

S.T

$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$0 \cdot x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

∴ Dual

$$Min\ Z' = -w_1 + 7w_2 + 10w_3 + 3w_4$$

S.T

$$-w_1 + w_2 + w_3 + 0 \cdot w_4 \geq 3$$
$$-w_1 + w_2 + 2w_3 + w_4 \geq 2$$
$$w_1, w_2, w_3, w_4 \geq 0$$

We apply BIG M method to get the sol^n of the dual problem as it involves artificial variables.

Convert Min to Max:

$$Max\ z' = w_1 - 7w_2 - 10w_3 - 3w_4 + 0 \cdot s_1 + 0 \cdot s_2 - Ma_1 - Ma_2 \quad \text{— } 1$$

S.T

$$-w_1 + w_2 + w_3 + 0 \cdot w_4 - s_1 + a_1 = 3 \quad \text{— } 2$$
$$-w_1 + w_2 + 2w_3 + w_4 - s_2 + a_2 = 2 \quad \text{— } 3$$
$$w_1, w_2, w_3, w_4, s_1, s_2, a_1, a_2 \geq 0$$

BVO eqn

$$Max\ z' - w_1 + 7w_2 + 10w_3 + 3w_4 - 0 \cdot s_1 - 0 \cdot s_2 + Ma_1 + Ma_2 = 0 \quad 4$$

$$\therefore eqn\ 1 - M\left[eqn\ 2 + eqn\ 3\right]$$

$$Max\ Z' - w_1 + 7w_2 + 10w_3 + 3w_4 - 0 \cdot s_1 - 0 \cdot s_2 + Ma_1 + Ma_2 + Mw_1 - Mw_2 - Mw_3 + Ms_1 - Ma_1 + Mw_1 - Mw_2 - 2Mw_3 - Mw_4 + M \cdot s_2 - Ma_2 = -3M - 2M$$

$$\cancel{Z' = (1 - 2M)w_1 + (7 + 2M)w_2}$$

(3) prove using duality theory that the following LPP is feasible but has no optimal sol^n

$$\text{Min } Z = x_1 - x_2 + x_3$$

S.T

$$x_1 - x_3 \geq 4$$

$$x_1 - x_2 + 2x_3 \geq 3$$

$$\& \; x_1, x_2, x_3 \geq 0$$

Dual. Since there are two constraints, there are two variables w_1, w_2 in the dual.

$$\text{Max } Z' = 4w_1 + 3w_2$$

S.T

$$-w_1 + w_2 \leq 1$$

$$0w_1 - w_2 \leq -1 \qquad -0 \cdot w_1 + w_2 \geq 1$$

$$-w_1 + 2w_2 \leq 1$$

$$w_1, w_2 \geq 0$$

To solve the dual problem

$$\text{Max } Z' = 4w_1 + 3w_2$$

S.T

$$w_1 + w_2 + S_1 = 1$$

$$0 \cdot w_1 - w_2 - S_2 + A_1 = 1$$

$$-w_1 + 2w_2 + S_3 = 1$$

where S_1, S_3 are slack variables, S_2 the surplus variable and A_1 the artificial variable.

$$\text{Max } w = 4w_1 + 3w_2 + 0 \cdot S_1 + 0S_2 + 0S_3 - Ma_1$$

S.T

$$w_1 + w_2 + S_1 = 1 \qquad \text{PTO}$$

$$0w_1 - w_2 - S_2 + A_1 = 1$$

$$-w_1 + 2w_2 + S_3 = 1 \quad \& \quad w_1, w_2, w_3, S_1, S_2, S_3, a_1 \geq 0$$

The initial basic feasible solution is given by
$(s_1 = 1,\ a_1 = 1,\ s_3 = 1$ (basic)) ($w_1, w_2 = s_2 = 0$ nonbasic)

Initial Iteration:

C_B	W_B	X_B	w_1 (4)	w_2 (3)	S_2 (0)	S_1 (0)	a_1 (−M)	S_3 (0)	?
0	s_1	1	1	1	0	1	0	0	1
−M	a_1	1	0	1	−1	0	1	0	1
0	S_3	1	−1	2	0	0	0	1	1/2 ←
$w_j - b_j$		−M	−4	−M−3	M	0	0	0	

First iteration: Introduce w_2 & drop S_3

C_B	W_B	X_B	w_1 (4)	w_2 (3)	S_2 (0)	S_1 (0)	a_1 (−M)	S_3 (0)	θ
0	S_1	1/2	3/2	0	0	1	0	−1/2	1/3 ←
−M	a_1	1/2	1/2	0	−1	0	1	−1/2	
3	w_2	1/2	−1/2	1	0	0	0	1/2	
$(w_j - b_j)$		$\frac{-M+3}{2}$	$\frac{-M-1}{2}$	0	M	0	0	$\frac{M+3}{2}$	

Second Iteration:

C_B	W_B	X_B	w_1 (4)	w_2 (3)	S_2 (0)	S_1 (0)	a_1 (−M)	S_3 (0)
4	w_1	1/3	1	0	0	2/3	0	−1/3
−M	a_1	1/3	0	0	−1	−1/3	1	−1/3
3	w_2	2/3	0	1	0	1/3	0	1/3
$w_j - b_j$		$\frac{10-M}{3}$	0	0	M	$\frac{M+11}{3}$	0	$\frac{M+11}{3}$

Since all $(w_j - b_j) \geq 0$ and an artificial variable a_1 appears in the basis at non-zero level, the dual problem has no optimal ~~sol~~ basic feasible solⁿ.

Dual Simplex Method

The dual simplex Method is very similar to the regular simplex method, the only difference lies in the criterion used for selecting a variable to enter the basis and to leave the basis. In dual simplex Method, we first select the variable to leave the basis and then the variable to enter the basis. This method yields an optimal solⁿ to the given LPP in a finite number of steps provided, no basis is repeated.

The dual simplex method is used to solve problems which start dual feasible (i.e, whose primal is optimal but infeasible). In this method the solution starts optimum, but infeasible and remains infeasible until the true optimum is reached, at which the solution becomes feasible. The advantage of this method is avoiding the artificial variables introduced in the constraints along with the surplus variables as all '≥' constraints are converted into '≤' type.

Simplex Method	Dual simplex
→ Max z (only) $b_i \geq 0$	→ Min z (also) $\xrightarrow{\text{converted to by } \times -1}$ Max $b_i \leq 0$
→ Step 1: Entering variable	→ Step 1: leaving variable
→ Step 2: leaving "	→ Step 2: Entering "
→ Step 3: Gaussian Method	→ Step 3: Gaussian method, convert all eqⁿˢ to ≤ & add slack variables to it

~~Step~~ 1:

Dual Simplex:

(most negative value)

leaving variable ⟹ row with ~~largest~~ negative b_i value

Entering variable → min value: min (0^{th} row / leaving variable)

Eg: Max $Z = -4y_1 - 12y_2 - 18y_3$

S.T

$y_1 + 3y_3 \geq 3$ — (1)

$2y_2 + 2y_3 \geq 5$ — (2) $\quad y_1, y_2 \geq 0$

Eqn (1) → $-y_1 - 3y_3 \leq -3$

(2) → $-2y_2 - 2y_3 \leq -5$ $\quad y_1, y_2 \geq 0$

$Z + 4y_1 + 12y_2 + 18y_3 = 0$ — (0)

$-y_1 - 0.y_2 - 3y_3 + S_1 = -3$ — (1)

$-0.y_1 - 2y_2 - 2y_3 + S_2 = -5$ — (2)

Initial Simplex Table

→ Entering variable

BV	Z	y_1	y_2	y_3	S_1	S_2	RHS	
Z	1	4	12	18	0	0	0	
S_1	0	-1	0	-3	1	0	-3	
S_2	0	0	-2	-2	0	1	-5 ← leaving variable	
0/LV		4/0	12/2=6	18/2=9	-	0		
Z	1	4	0	6	0	6	-30	$R_0 \to R_0 - 12R_2$
S_1	0	-1	0	-3	1	0	-3 ←	$R_1 \to R_1$
y_2	0	0	1	1	0	-1/2	5/2	$R_2 \to R_2/-2$
0/LV		4/1=4	-	6/3=2	0	-	-	

Z	1	2	0	0	+2	6	-36	$R_0 \to R_0 - 6R_1$
y_3	0	1/3	0	1	-1/3	0	1	$R_1 \to R_1/3$
y_2	0	-1/3	1	0	1/3	-1/2	3/2	$R_2 \to R_2 - R_1$

∴ Non negative (optimum)

$Z_{max} = -36$ at $y_1 = 0,\ y_2 = 3/2,\ y_3 = 1$

(2) Max $Z = 3x_1 + 2x_2$

S.T

$3x_1 + x_2 \geq 3$ — (1)

$4x_1 + 3x_2 \geq 6$ — (2)

$x_1 + x_2 \leq 3$ — (3)

where $x_1, x_2 \geq 0$

Step 1: Convert '≥' type constraints if any to '≤' type by multiplying both sides by -1

$-3x_1 - x_2 \leq -3$ — (1)

$-4x_1 - 3x_2 \leq -6$ — (2)

$x_1 + x_2 \leq 3$ — (3)

Step 2: add slack variables

$-3x_1 - x_2 + S_1 = -3$

$-4x_1 - 3x_2 + S_2 = -6$

$x_1 + x_2 + S_3 = 3$

Max $Z - 3x_1 - 2x_2 = 0$

BV	Z	x_1	x_2	S_1	S_2	S_3	RHS	
Z	1	-3	-2	0	0	0	0	
S_1	0	-3	-1	1	0	0	-3	
S_2	0	-4	-3	0	1	0	-6	← L.V
S_3	0	1	1	0	0	1	3	
Ratio	–	3/4	2/3	–	–	–		
Z	1	-1/3	0	0	-2/3	0	4	$R_0 \to R_0 + 2R_2$
S_1	0	-5/3	0	1	-1/3	0	-1	L.V $R_1 \to R_1 + R_2$
x_2	0	4/3	1	0	-1/3	0	2	$R_2 \to R_2 / -3$
S_3	0	-1/3	0	0	1/3	1	1	$R_3 \to R_3 - R_1$
Ratio	–	1/5	–	–	2	–		
Z	1	0	0	-1/5	-3/5	0	21/5	$R_0 \to R_0 + 1/3 R_1$
x_1	0	1	0	-3/5	1/5	0	3/5	$R_1 \to R_1 \times -3/5$
x_2	0	0	1	4/5	-3/5	0	6/5	$R_2 \to R_2 - 4/3 R_1$
S_3	0	0	0	-1/5	2/5	1	6/5	$R_3 \to R_3 + 1/3 R_1$

E.V

non-negative ($\because$ optimal)

$\therefore Z_{max} = 21/5$ at $x_1 = 3/5$
$x_2 = 6/5$

(3) Working procedure for dual simplex

Step 1: Convert the problem to maximization form if it is initially in the minimization form.

Step 2: Convert ($\geq$) type constraints, if any to ($\leq$) type by multiplying both sides by -1

Step 3: Express the problem in standard form by introducing slack variables. Obtain the initial basic solution & express this information in simplex table.

Step 4: Optimal condition

case (i): if all non basic variables is ≥ 0 and then RHS ≥ 0 then the current solⁿ is an optimum

case (ii): if all non basic variables ≥ 0 & atleast one RHS < 0, then the current solⁿ is not an optimum basic feasible solⁿ $\therefore$ go to next step.

Step 5: feasibility condition)

(i) Leaving variable: The leaving variable is the basic variable corresponding to the most negative value of RHS.

(ii) Entering variable: Compute the ~~ratio~~ Min ratio & then entering variable will be the least value

$\therefore$ Min ratio = $\dfrac{Z^{th}\ \text{row}}{\text{abs(leaving variable row)}}$

Step 6: Carryout the row operations as in the regular simplex method & repeat the procedure until either an optimum feasible solution or there is an indication of non-existence of a

The Role of Duality in Sensitivity Analysis

A very important question in using the optimal solution to a ~~pro~~ linear programming problem is the question as to how the solution would be affected if the parameters b_i, C_j or a_{ij} of the problem change. This question is answered by the sensitivity analysis.

Sensitivity Analysis: The investigation that deals with changes in the optimal solution due to changes in the parameters a_{ij}, b_i or C_j is called sensitivity analysis or post optimality analysis.

The objective of sensitivity analysis is to considerably reduce the additional computational effort which arise in solving the problem by treating it as new LPP.

Changes in LPP which are studied.

(a) changes in the co-efficient (C_j) of the objective function
- co-efficients of basic variables
- co-efficients of non-basic variables.

(b) changes in the right - hand side constraints (b_i)

(c) changes in (a_{ij})
- Coefficients of basic variables
- " " non " "

(d) addition of new variable to the problem

(e) addition of new or secondary constraint.

P.T.O.

In general, the result could be one of the three
(i) The basic solution changes completely
(ii) The basic variables remain the same but their values are changed.
(iii) The solution is same (basic variable as well as their values).

Results in duality

Result 1: The dual of the dual is the primal.

Result 2: If one is maximization problem then other is minimization problem.

Result 3: The necessary and sufficient condition for any Lpp and its dual to have an optimal solution is that both have feasible solution.

Result 4: Fundamental theory of duality: If either primal or the dual problem has a finite ~~optimum~~ optimal solution, then the other problem also has finite optimal solution and the values of the objective functions are equal i.e., Max Z = Min W.

The solution of the other problem can be read from the $(Z_j - C_j)$ row below the column of slack, surplus variables. The values of dual variables are called shadow prices.

Result 5: Existence Theorem: If either the primal or dual problem has an unbounded solution, then the other problem has no feasible solution.

Result 6: If i^{th} dual constraint is multiplied by -1, then i^{th} primal variable computed from $(Z_j - C_j)$ row as the dual problem has no feasible solution.

Result 7: If dual has no feasible solⁿ, then primal also admits no feasible solution.

Result 8: The value of the objective function Z for any feasible solution of the primal is less than or equal to the value of the objective function W for any feasible solution of the dual.

Result 9: complementary slackness theorem

(i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum & vice versa.

(ii) If primal constraint is strict inequality, then the corresponding dual variable is zero at the optimum & vice versa.

The Essence of Sensitivity Analysis

The work of the operations research team usually is not even nearly done when the simplex method has been successfully applied to identify an optimal solution for the model. One assumption of linear programming is that all the parameters of the model (a_{ij}, b_j, & C_j) are known constants.

Actually, the parameter values used in the model normally are just estimates based on prediction for future conditions.

The data obtained to develop these estimates often are rather crude or nonexistent, so that the parameters in the original formulation may represent little more than quick rules of thumb provided by harassed line personnel. The data may even represent deliberate overestimates or underestimates to protect the interests of the estimators.

UNIT - 6

Game Theory, Decision Analysis

Syllabus

Game Theory: The formulation of two person, zero sum games, Solving simple games - examples, Games with mixed strategies, Graphical solution procedure, Solving by linear programming, extension.

Decision Analysis: A prototype example, Decision making without experimentation, Decision making with " , Decision Trees.

Introduction to Game Theory

Game theory is a decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcome results.

[Father of game theory was J. Vo Neumann]

Many practical problems requires decision making in a competitive situation where there are two or more opposing parties with conflicting interests & where the action of one depends upon the action taken by the opponent.

Eg:- candidates for an election of countries involved in battle have different conflicting interests. In a competitive situation the courses of action (alternatives) for each competition may be finite & infinite. A competitive situation will be called a 'game' if it has the following properties.

Properties of Game

1. There are finite no of competitors called 'players'
2. Each player has a finite no of possible courses of action called 'strategies'.

(3) All the strategies and their effects are known to the players but player does not know which strategy is to be choosen.

(4) A game is played when each player chooses one of his strategies. The strategies are assumed to be made simultaneously with an outcome such that no player knows his opponents strategy until he decides his own strategy.

(5) The game is a combination of the strategies and in certain units (usually in terms of money) which determines the gain (shown in positive figures) or loss (shown in negative figures).

(6) The figures (either gain or loss) shown as the outcomes of strategies in a matrix form is called "pay off matrix".

(7) The player playing the game always tries to choose the best course of action which results in optimal pay off, called 'optimal strategy".

(8) The expected pay off when all the players of the game follow their optimal strategies is known as "value of the game". The main objective of a problem of game is to find the value of the game.

(9) The game is said to be 'fair game' if the value of game is zero, otherwise it is known as 'unfair'.

Characteristics & Terms used in Game Theory

(1) Game: A game involves game of strategy as well as a game of chance.

(i) Game of strategy: The activities which are evaluated by skill is known as a game of strategy. Generally game of strategy is dealt with the subject point of view.

(ii) Game of chance: Activities which are determined by chance is known as game of chance.

(2) Number of Activities: May be finite or infinite.

(3) Number of persons: n-person game if the number of persons playing in a game is 'n'.

(4) Number of Alternatives Available to Each person: In a particular course of action, may be finite or infinite, a finite game has finite no of activities involving a finite no of alternatives, otherwise the game is said to infinite.

(1) Pay off: A quantitative measure of satisfaction a person gets at the end of each play is known as a "pay off". It is real-valued function of variable in the game.

Let V_i be the pay off to the player P_i, $1 \leq i \leq n$ when

n = An n-person game

If $\sum_{i=1}^{n} V_i = 0$, then the game is said to be "Zero-sum game".

Two person Zero-sum game is also known as rectangular game.

* The representation of gains & losses resulting from different actions of the competitions & is represented in the form of matrix is pay off matrix.

Strategy:

A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides he should adopt. In other words, a strategy for a given ~~gam~~ player is a set of rules (programs) that specifies which of the available course of action he should make at each play. This strategy may be of two kinds.

(i) pure Strategy: If a player knows exactly what the other player is going to do, a deterministic situation is obtained & objective function is to maximize the gain.

∴, the pure strategy is a decision rule always to select a particular course of action.

∴ A pure strategy is usually represented by a number with which the course of action is associated.

(ii) Mixed Strategy: If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained & objective function is to maximize the expected gain.

Thus, the mixed strategy is a selection among pure strategies with fixed probabilities. Mathematically a mixed-strategy for a player with ~~fin~~ $m (\geq 2)$ possible courses of action, is denoted by the set S of m non-negative real nos whose sum is unity, representing probabilities with which each course of action is chosen. If $x_i (i = 1, 2, 3 \ldots m)$ is the probability of choosing the course i, then

$$S = (x_1, x_2, x_3 \ldots x_m)$$

Subject to the conditions

$$x_1 + x_2 + x_3 + \cdots x_m = 1$$

$$\& \quad x_1 \geq 0, x_2 \geq 0, \cdots x_m \geq 0$$

~~Saddle point~~ :

Optimum strategy: A course of action which puts the player in the most preferred position, irrespective of the strategy of his competitions, is referred as an optimum strategy

Two-person zero sum game: when only two players are involved in the game & if the gain made by the one player is equal to the loss of the other then it is called two persons zero-sum game.

Value of game:- It is expected pay off of the player when all the players of the game follow their optimum strategies

Fair game: If the value of game is zero, it is referred as a fair game.

Classification of Games

- Game
  - Two-person games
    - Zero-sum game
      - Deterministic
        - Fair
        - unfair
      - probabilistic
        - fair
        - unfair
    - Non-Zero sum games
  - Multiperson games
    - zero-sum games
    - Non-Zero sum games

Saddle point:

A Saddle point of a pay of matrix is the position of such an element in the pay-off matrix which is minimum ~~not~~ in its row & maximum in its column.

Formulation of Two persons & Zero sum games [Rectangular game]

A game with only two players (say, player A & player B) is called two person, zero sum game if the losses of one player are equivalent to the gains of the other, so that sum of their net gains is zero. [also called rectangular games ∵ payoff matrix is rectangular form]

i.e. [Zero sum Game — is one in which the sum of the payment to all the competitors is zero for every possible outcome of the game is in a game if the sum of the points won equals the sum of the points lost]

Characteristics of such a game are

(i) only two player participate in the game.

(ii) Each player has a finite number of strategies to use.

(iii) Each specific strategy results in payoff

(iv) Total payoff to the two players at the end of each play is zero.

10

p.r.e

Pay off matrix:

Suppose the player A has m activities & the player B has n activities. Then a pay off matrix can be formed by adopting the following rules.

(i) Row ~~destinations~~ designations for each matrix are activities available to player A.

(ii) Column ~~destin~~ designations for each matrix are activities available to player B.

(iii) cell entry 'V_{ij}' is the payment to player A in A's pay off matrix when A chooses the activity i & B chooses the activity j.

(iv) With a 'zero-sum, two person game', the cell entry in the player B's pay off matrix will be negative of the corresponding cell entry 'V_{ij}' in the player A's pay off matrix so that sum of pay off matrices for player A & player B is ultimately zero.

Player A's pay off Matrix

Player B

Player A	1	2	...	j	...	n
1	V_{11}	V_{12}	...	V_{1j}	...	V_{1n}
2	V_{21}	V_{22}		V_{2j}		V_{2n}
:	:					
i	V_{i1}	V_{i2}	...	V_{ij}	--	V_{in}
:	:					
m	V_{m1}	V_{m2}	...	V_{mj}	--	V_{mn}

8 7 9 7

Player B's pay of Matrix is

Player B

A	1	2	---	j	---	n
1	$-V_{11}$	$-V_{12}$	---	$-V_{1j}$	---	$-V_{1n}$
2	$-V_{21}$	$-V_{22}$	---	$-V_{2j}$	---	$-V_{2n}$
:	:					
i	$-V_{i1}$	$-V_{i2}$	---	$-V_{ij}$	---	$-V_{in}$
:						
m	$-V_{m1}$	$-V_{m2}$	...	$-V_{mj}$		$-V_{mn}$

Assumption of game

(1) Each player has finite number of possible courses of actions (strategies) with him.

2) The list of strategies of each player need not be same.

3) player in rows attempts to maximize his gain while player in columns tries to minimize his losses.

4) The decision of both players are made individually prior to the play without any communication between them.

5) The decision is supposed to be made simultaneously & also announced simultaneously so that neither player has any advantage over the other due to the direct knowledge of the other player's decision.

6) Both players know their own pay offs as well as the pay off of each other.

7) The positive sign of the pay off indicates the gain to row player or loss to column player & negative sign indicates loss to row player or gain to column

Solving simple (Two person - Zero sum) Games Ep

Two person zerosum games may be deterministic or probabilistic. The deterministic games will have saddle points & pure strategies exist in such games.

In contrast, the probabilistic games will have no saddle points & Mixed strategies are taken with help of probabilities.

~~Games with saddle point~~ - ~~Deterministic~~ Games - (Pure Strategies)

Solⁿ of the game

These games can be solved by using one of the following methods

(1) MiniMax & MaxiMin principle
(2) Dominance principle
(3) Graphical Method

Solⁿ of the game having saddle point (pure strategy)

1) ~~Maxi~~ MiniMax - MaxiMin criterion

Ep: consider the following game

Player B

$$\text{player A}\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}$$

Solⁿ First find Row Minima & Column Maxima

	Player B		Row Minima
~~Player A~~	1	1	1
	4	-3	-3
Column Maxima	4	1	

→ MinMax = 1

MaxMin = 1

∴ Saddle point (1,1)

Value of Game i.e. V=1 //

we denote minimum of Maximum by (*) and maximum of minimum by (*). This selection of strategies by A and B was based upon maximin, MiniMax principle which guarantees the best of worst result.

when Maximin value of the game = minimax value of the game, the corresponding pure strategies are called optimum strategies.

MiniMax (MaxiMin) Criterion and optimal strategy

The 'Minimax criteria of optimality' states that if a player lists the worst possible outcomes of all his potential strategies, he will chose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

(2) Solve the following two persons zerosum game with the following 3x2 payoff matrix for player A.

$$\text{player A}\quad\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & 9 & 2 \\ A_2 & 8 & 6 \\ A_3 & 6 & 4 \end{array}$$

Solⁿ.

Player B

		B_1	B_2	Row Minima
A	A_1	9	2	2
	A_2	8	6	$\boxed{6}$ → MaxMin
	A_3	6	4	4
Column Maxima		9	$\boxed{6}$	

MinMax

∴ Min Max = MaxMin

Optimal Strategy for player A is A_2

" " " B is B_2

Value of game = 6

* This principle is used for the selection of optimal strategies by two players.
Consider A & B,
A is player who wishes to Maximize his gain while player B Minimize his losses.
Since A would like to maximize his minimum gain we obtain for player A, the value called MaxiMin value & the corresponding strategy is called Maximin strategy.
On the other hand, Since player B wishes to minimize his losses, a value called the MiniMax value which is the minimum of the maximum losses is found. The corresponding strategy is called the minimax strategy.
When these two are equal (maximum value = MiniMax value), the corresponding strategies are called optimal strategy & the game is said to have a saddle point. The value of the value of the game is given by the saddle point.

NOTE:- Maximin value = $\underline{v}$ → value of game
MiniMax value = $\bar{v}$

(i) A game is said to be fair if
$X = Y = 0$

(ii) A game is said to be strictly determinable if
$X = Y \neq 0$ $X = Z = Y$

(3) For what value of λ, the game with the following matrix is strictly determinable?

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & \lambda & 6 & 2 \\ A_2 & -1 & \lambda & -7 \\ A_3 & -2 & 4 & \lambda \end{array}$$

Sol: ignoring the value of λ, the payoff matrix is

$$\begin{array}{c|c} \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} & \begin{matrix} \text{Row Minima} \\ 2 \\ -7 \\ -2 \end{matrix} \end{array}$$

MaxMin = 2

Column maxima -1 6 2

Min Max = -1

$v \neq \bar{v}$

The game is strictly determinable, if
$X = Y = Z$ Hence $X = 2$
$Y = -1$

(4) Determine which of the following two person zero sum games are strictly determinable & fair. Given the optimum strategy for each player in the case of strictly determinable games.

(a) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{cc} B_1 & B_2 \\ \hline -5 & 2 \\ -7 & -4 \end{array}$$

Sol:

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row Minima} \\ \hline A_1 & -5 & 2 & -5 \\ A_2 & -7 & -4 & -7 \\ \hline \text{Column Maxim} & -5 & 2 & \end{array}$$

Maximinma = -5

Minimax = -5

Since $X = Y = -5 \neq 0$, the game is strictly determinable ∴ Saddle point = -5
The optimal strategy is the position of Saddle point given by (A_1, B_1)

(b)

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & 1 \\ A_2 & 4 & -3 \end{array}$$

$$\begin{array}{c|cc|c} & & & \text{Row Minima} \\ \hline & 1 & 1 & 1 \\ & 4 & -3 & -3 \\ \hline \text{Column Maxima} & 4 & 1 & \end{array}$$

Maxi(Minimum) = 1
Mini(Max) = 1

⟹ Same (A_1, B_2)

Games without saddle points. (Mixed strategies)

Consider a 2×2 two-person zero sum game without any saddlepoint having the payoff matrix for player A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

(theory in next page)

Method 1: (Algebraic Method)

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where $\boxed{p_1 = \dfrac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}}$, $p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$

$\boxed{q_1 = \dfrac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}}$, $q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$

∴ The value of the game is

$$\boxed{= \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}}$$

Eg: Solve the following payoff matrix, determine the optimal strategies & the value of game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

$$\begin{array}{cc|c} 5 & 1 & 1 \\ 3 & 4 & \boxed{3} \\ \hline 5 & \boxed{4} & \end{array}$$

No saddle point

Soln: $\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$ let this be

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

∴ The optimum mixed strategy

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where $p_1 = \dfrac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \dfrac{4-3}{(5+4)-(1+3)} = \dfrac{1}{5}$

∴ $p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \dfrac{1}{5} = \dfrac{4}{5}$

∴ $q_1 = \dfrac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \dfrac{4-1}{(5+4)-(1+3)} = \dfrac{3}{5}$

∴ $q_2 = 1 - q_1 \Rightarrow q_2 = 1 - \dfrac{3}{5} = \dfrac{2}{5}$

∴ Value of game $\nu = \dfrac{(5\times4) - (1\times3)}{(5+4) - (1+3)} = \dfrac{17}{5}$

∴ Optimum mixed strategies

$S_A = \left(\dfrac{1}{5}, \dfrac{4}{5}\right)$ $S_B = \left(\dfrac{3}{5}, \dfrac{2}{5}\right)$

Value: $17/5$

Eg2) Solve the following game & determine the value of game.

$$\begin{array}{cc|c} & B & \text{Row minima} \\ A & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} & \begin{matrix} -4 \\ -4 \end{matrix} \\ \text{Column Max} & 4 \quad 4 & \end{array}$$

MiniMax = -4

MaxiMin = 4

It is clear that the pay off matrix does not posses any saddle point. The players will use mixed strategy.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}, \quad P_1 + P_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}, \quad q_1 + q_2 = 1$$

$$\therefore P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{4-(-4)}{4+4-(-4-4)} = \frac{8}{16} = \frac{1}{2}$$

$$\therefore P_2 = 1 - P_1 = P_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{4(-4)}{4+4-(-4-4)} = \frac{8}{16} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore$ optimum strategy is $S_A = \left[\frac{1}{2}, \frac{1}{2}\right]$ $S_B = \left[\frac{1}{2}, \frac{1}{2}\right]$

The value of game is
$$Z = \frac{a_{22}a_{11} - a_{12}a_{21}}{(a_{22}+a_{11}) - (a_{12}+a_{21})}$$
$$= \frac{(4 \times 4) - [-4 \times (-4)]}{(4+4) - [-4+(-4)]} = 0$$

Method - 2

(1) Two players 'A' & 'B' throw 2 coins on a table A wins 8 when both coins show heads & 1 when both are tail. 'B' wins 3 when coin does not match. prepare the pay off matrix & determine strategies for each player

$$\begin{array}{ccc|c} & & B & \\ & & H \quad T & \text{Row minima} \\ A & \begin{matrix} H \\ T \end{matrix} & \begin{bmatrix} 8 & 3 \\ 3 & 1 \end{bmatrix} & \begin{matrix} 3 \\ 1 \end{matrix} \\ \text{Column Max} & & 8 \quad 3 & \end{array}$$

MaxMin = 3

MiniMax = 3

~~Sol^n 5a) 2x2 Games without Saddle point Method~~

Example for fair game

Eg The pay of matrix in respect of a two-person-zerosum game is

A's Strategy	B's Strategy B_1	B_2	B_3	B_4	B_5
A_1	8	10	-3	-8	-12
A_2	3	6	0	6	12
A_3	7	5	-2	-8	17
A_4	-11	12	-10	10	20
A_5	-7	0	0	6	2

(a) write the maximin & minimax strategies.
(b) is it a strictly determinable game?
(c) what is the value of game?

$$\begin{array}{c|c}
\begin{bmatrix} 8 & 10 & -3 & -8 & -12 \\ 3 & 6 & 0 & 6 & 12 \\ 7 & 5 & -2 & -8 & 17 \\ -11 & 12 & -10 & 10 & 20 \\ -7 & 0 & 0 & 6 & 2 \end{bmatrix} & \begin{array}{c} \text{Row Minima} \\ -12 \\ \boxed{0} \\ -8 \\ -11 \\ -7 \end{array}
\end{array}$$

MaxiMin = 0

Column Maxima: 8 10 $\boxed{0}$ 10 20

MiniMax = 0

Using the maximin - Minimax principle, player A will select that strategy which is Maximum of the minimum gains & player B will select that strategy which is minimum of maximum gains

From that table, it is apparent that the value of Maximin coincides with the value of MiniMax,
∴ saddle point is determined.

Sol^n of game is
- player A → optimum strategy is A_2
- " B " " B_3

∴ (A_2, B_3)

∴ value of game is 0 for both the players i.e., game ends in a draw. Since the value of game is zero, the game is fair.

Games with Mixed Strategies

In certain cases, no pure strategy solution exists for the game, In other words, saddle points does not exist. In all such game, both players ~~must~~ may adopt an optimal blend of the strategies called Mixed strategies to find a saddle (equilibrium) point.

The optimal mix for each player may be determined by assigning each strategy a probability of it being chosen. Thus, these mixed strategies are probabilistic combination of available better (or dominant) strategies & these games hence called probabilistic games.

III^ly, when saddle points does not exist in the game, we can use the dominance principle to reduce to 2x2 game or atleast 2xn or mx2 ~~method~~ matrices. If this is also not possible we can formulate it as linear programming problem.

Thus the probabilistic mixed strategy games without saddle points are commonly solved by any of the following method.

		Applicable to
1)	Algebraic method	2 x 2 games
2)	Graphical "	2x2, mx2 & 2xn games
3)	Linear programming method	2x2, mx2, 2xn & mxn games

Algebraic method: suppose a general game without saddle points as given below

$$\begin{array}{c|cc} & \multicolumn{2}{c}{\text{Player } B} \\ & B_1 & B_2 \\ A_1 & a & b \\ A_2 & c & d \end{array}$$

Since, it is assumed that there no saddle points, A wishes to use a mixed strategy of A_1 & A_2 with the probabilities of p_1 & p_2. & B to use a mixed strategy of B_1 & B_2 with the probabilities of q_1 & q_2 respectively.

According to the assumptions that A always tries to maximise his minimum returns & B always tries to minimise his maximum losses, they employ their strategies most optimally such that the B's minimax losses will be equal to A's maximin gains.

Hence, outcome of probability min (of p_1 & p_2) by A in his A_1 & A_2 strategies when B uses his B_1 strategy i.e.,

$ap_1 + cp_2 \rightarrow$ is the value of game.

Otherwise, when B uses his B_2 strategy, A's returns will be

$bp_1 + dp_2 \rightarrow$ is also value of game.

These both must be equal

$$\therefore ap_1 + cp_2 = bp_1 + dp_2$$

$$\Rightarrow (a-b)p_1 = (d-c)p_2$$

$$\Rightarrow \boxed{\frac{p_1}{p_2} = \frac{(d-c)}{(a-b)}}$$

$|||^{ly}$, from B's point of view

$$aq_1 + bq_2 = cq_1 + dq_2$$

$$(a-c)q_1 = (d-b)q_2$$

$$\boxed{\frac{q_1}{q_2} = \frac{d-b}{a-c}}$$

From the rule of probabilities we know that

$$p_1 + p_2 = 1 \quad \& \quad q_1 + q_2 = 1$$

Thus, we have and

$p_1 = 1 - p_2$ (or)

$p_2 = 1 - p_1$ & $q_1 = 1 - q_2$ (or)

$q_2 = 1 - q_1$

Thus, we have (on solving the equations)

$$P_1 = \frac{(d-c)}{(d-c)+(a-b)} = \frac{c-d}{(c-d)-(a-b)} = \frac{c-d}{(b+c)-(a+d)}$$

$$\text{Ie } q_1 = \frac{(d-b)}{(d-b)+(a-c)} = \frac{b-d}{(b-d)-(a-c)} \cdot \frac{b-d}{(b+c)-(a+d)}$$

where $P_2 = 1-P_1$

$q_2 = 1-q_1$

$\therefore$ Ie $V = ap_1+cp_2$ or dp_1+dp_2 or aq_1+bq_2 or cq_1+dq_2)

i.e. $\boxed{V = \dfrac{ad-bc}{(a+d)-(b+c)}}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\longrightarrow \boxed{\dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}}$

Eg. Let the payoff matrix be as follows: $A \to$ player

$$A \quad \begin{array}{c|cc} & H & T \\ \hline H & 2 & -1 \\ T & -1 & 0 \end{array} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Determine strategy for A, B, value of game.

Soln

$S_A = \begin{pmatrix} A_1 & A_2 \\ P_1 & P_2 \end{pmatrix}$ $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

where $P_1 = \dfrac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$ where $P_1+P_2 = 1$

$P_2 = 1-P_1$

$$P_1 = \frac{0-(-1)}{2-(-2)} = \boxed{\frac{1}{4}}$$

$$\therefore P_2 = 1-\frac{1}{4} = \boxed{\frac{3}{4}}$$

$$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-(-1)}{2-(-2)} = \boxed{\frac{1}{4}}$$

$$\therefore \boxed{q_2 = \frac{3}{4}}$$

$\therefore$ A's strategy $= \left(\frac{1}{4}, \frac{3}{4}\right)$

B's strategy $= \left(\frac{1}{4}, \frac{3}{4}\right)$

$\therefore$ Value of game is

$$\frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$= \frac{(2\times 0) - (-1\times -1)}{(2+0) - (-2)} = \frac{-1}{2+2} = \frac{-1}{4}$$

Value of game is $\dfrac{-1}{4}$

Method 2:

$$\begin{array}{c|cc|c} & H & T & \text{Row Minima} \\ \hline H & 2 & -1 & -1 \\ T & -1 & 0 & -1 \end{array} \qquad \text{MaxiMin} = -1$$

Column Maxima 2 0 MiniMax = 0

Here maximin value $\underline{V} = -1$

MinMax value $\overline{V} = 0$

$\therefore$ The matrix is without saddle point.

Let A play H with probability x and T with probability $1-x$ $\quad$ $1/4$, $1-1/4$

If player B plays H all the times then A's expected gain is

$$E(A,H) = x.2 + (1-x)(-1) = 3x-1$$

111^{ly} if the player B plays T all the times, then A's expected gain is

$$E(A,T) = x.(-1) + (1-x).0 = -x$$

We can show that

$E(A,H) = E(A,T) = E(A)$ will determine best strategy for A.

$$\Rightarrow 3x-1 = -x \qquad \text{i.e. } x = \frac{1}{4}, \quad 1-\frac{1}{4} = \frac{3}{4}$$

$\therefore$ The best strategy for player A is to play H & T with probability $\frac{1}{4}$ & $\frac{3}{4}$ respectively.

We apply the same procedure for B. Let the probability of the choice of H be denoted by y & ~~that~~ of T be $(1-y)$.

$$E(B,H) = E(B,T) = E(B) \text{ say}$$

or

$$y.2 + (1-y)(-1) = y(-1) + (1-y)0$$

or

$$4y = 1 \quad \boxed{y = 1/4}$$

Now using x, y in $E(A), E(B)$

$$\Rightarrow E(A) = -\frac{1}{4}$$

$$E(B) = -\frac{1}{4} \qquad \text{A's strategy} = \left(\frac{1}{4}, \frac{3}{4}\right)$$

value of the Game $= -\frac{1}{4}$

(5) In a game of matching coins with two players suppose A wins one unit of value when there are two heads, wins nothing when there are two tails & losses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player & the value of game to A.

Sol^n The payoff matrix for the player A is given by

$$\begin{array}{c|cc|c} & \multicolumn{2}{c|}{B} & \\ & H & T & \\ \hline H & 1 & -1/2 & -1/2 \\ T & -1/2 & 0 & -1/2 \\ \hline & 1 & 0 & \end{array}$$

Let this be

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{cc} B_1 & B_2 \\ \multicolumn{2}{c}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \end{array}$$

The optimum mixed strategies $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$, $P_1 + P_2 = 1$

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$, $q_1 + q_2 = 1$

$$\therefore P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{0 - (-1/2)}{1+0-(-1/2-1/2)} = \frac{1/2}{2} = \frac{1}{4}$$

$$P_2 = 1 - P_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{0 - (-1/2)}{(1+0) - (-1/2 - 1/2)} = 1/4$$

$$\therefore q_2 = 1 - q_1 = 1 - 1/4 = 3/4$$

$$\therefore \text{Value of game} = \frac{1 \times 0 - (-1/2)(-1/2)}{1+0-(-1/2-1/2)} = \frac{-1/4}{2}$$

$$= -1/8$$

$\therefore$ The optimal mixed strategy is given by

$S_A = \left(\frac{1}{4}, \frac{3}{4}\right)$ $S_B = \left(\frac{1}{4}, \frac{3}{4}\right)$ Value $= -1/8$

Dominance property

Sometimes it is observed that one of the pure strategies of either player is always inferior to atleast one of the remaining ones. The superior strategies are said to dominate the inferior ones. In such cases of dominance, we reduce the size of payoff matrix by deleting those strategies which are dominated by others.

The general rule for dominance are:

(1) If all the elements of a row, say k^{th} row, are less than are equal to the corresponding elements of any other row say r^{th} row, then k^{th} row is dominated by the r^{th} row.

(2) If all the elements of a column, say k^{th} column, are greater than or equal to the corresponding elements of any other column, say r^{th} column, then the k^{th} column is dominated by the r^{th} column.

(3) Dominated rows & columns may be deleted to reduce the size of the pay-off matrix as the optimal strategies will remain unaffected.

(4) If some linear combinations of some rows dominates i^{th} row, then the i^{th} row will be deleted.
IIIlar arguments follow for column also

Eg: (1) Row dominance: Every element in a particular row is less than or equal to the corresponding element of another row,
∴ row is said to be inferior or dominated by the later row, Hence, this can be deleted.

(1) Row

$$\begin{array}{c|ccc|} R_1 & 4 & 6 & -5 \\ R_2 & 5 & 6 & 3 \end{array}$$

$\therefore R_2 \geq R_1$ Hence R_1 can be deleted.

$$\begin{bmatrix} 4 & 6 & -3 \\ 5 & 4 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

4 6 2

4 -3
5 2
3 2

(2) Column dominance:
Every element in a particular column is greater than or equal to corresponding elements of another column.

$$A \quad \begin{array}{c|cc|} & \multicolumn{2}{c}{B} \\ & C_1 & C_2 \\ R_1 & 4 & 2 \\ R_2 & 0 & -1 \\ R_3 & 3 & -3 \end{array}$$

$C_1 \geq C_2$

$\boxed{\text{delete } C_1}$

(3) Mixed Row dominance & (4) Mixed Column dominance

Eg: (1)

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} B_1 \quad B_2 \quad B_3 \\ \begin{bmatrix} 3 & 5 & 4 \\ 5 & 6 & 3 \\ 8 & 7 & 9 \\ 4 & 2 & 8 \end{bmatrix} \end{array}$$

→ while checking rowwise retain the max row
→ while checking columnwise retain the min column

row A_3

	A_1	A_2	A_4
	8>3	8>5	8>4
	7>5	7>6	7>2
	9>4	9>3	9>8

eliminate A_1, A_2, A_4

$[8 \quad 7 \quad 9]$ 9>8, delete B_3

Since 7<8 7<9 $[8 \quad 7]$ 8 Value of game = 7

Ex(1): Find the best strategy & the value of the following game.

$$\begin{array}{cc} & \begin{array}{ccc} & B & \\ I & II & III \end{array} \\ A \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix} \end{array}$$

Soln: First iteration

$$\begin{array}{cc} \begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix} & \begin{array}{c} \text{Row Minima} \\ -2 \\ -1 \\ \boxed{0} \end{array} \end{array}$$

MaxMin = 0

Column Max 7 5 12

MinMax: 7

∴ MinMax ≠ MaxMin

And it is not 2x2 Matrix. (reduce to 2x2 Matrix) by applying dominance rule

∴ Hence, we check the principle of dominance.

It is found that row A(III) is dominating A(I): Since every element in A(III) is > the corresponding element of A(I)

∴ A will never use the strategy A(I), hence deleted.

Now, the new iterated payoff is

2nd iteration

$$\begin{bmatrix} 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix}$$

Column B(I) dominates B(II) since every element in B(II) is less than corresponding element of B(I). Hence B(I) is deleted (B will never use B-I as it yields more losses against any strategy played by A).

3rd Iteration

$$\begin{array}{cc} & \begin{array}{cc} II & III \end{array} \\ A \begin{array}{c} II \\ III \end{array} & \begin{bmatrix} 5 & -1 \\ 0 & 12 \end{bmatrix} \end{array} \quad \begin{array}{c} -1 \\ 0 \end{array}$$

5 12

Now, they play a mixed strategies with ~~probabil~~ probabilities. Since there are neither saddle points nor any dominance.

Let A use the strategies II & III with probabilities P_1 and P_2 respectively and B use the strategies II & III with probabilities q_1 & q_2 respectively.

~~Max $P_1 + P_2$~~

apply the algebraic Method

∴ Mixed strategies of A are (II, III) & probabilities (2/3, 1/3)

" " B are (II, III) " " (13/18, 5/18)

Value of game is $\frac{10}{3}$ (A wins the game)

(2) Solve the following game by dominance principle rule.

$$\begin{array}{ccc} & \begin{array}{cccc} & \text{Player B} & & \\ B_1 & B_2 & B_3 & B_4 \end{array} & \text{Row Minima} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix} & \begin{array}{c} 0 \\ \boxed{2} \rightarrow \text{MaxMin} \\ 0 \\ 0 \end{array} \end{array}$$

Column Maxima $\boxed{4}$ 4 4 8

MinMax = 4

Since, no saddle point, use dominance concept.

Step 1: Row 3 > Row 1

$\therefore$ delete R_1

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 8 \end{array}$$

$C_1 > C_3$

$\therefore$ delete C_1

$$\begin{array}{c|ccc} & B_2 & B_3 & B_4 \\ \hline A_2 & 4 & 2 & 4 \\ A_3 & 2 & 4 & 0 \\ A_4 & 4 & 0 & 8 \end{array}$$

There is no pure dominance of rows & columns

Mixed dominance can be considered i.e.

average of 2 rows or 2 columns are consid.

$\frac{4+2}{2} = 3$, $\frac{2+4}{2} = 3$, $\frac{4+0}{2} = 2$

$\frac{2+4}{2} = 3$, $\frac{4+0}{2} = 2$, $\frac{0+8}{2} = 4$

$$\begin{bmatrix} 4 & 3 \\ 2 & 3 \\ 4 & 4 \end{bmatrix}$$

$C_1 > \left(\frac{C_2+C_3}{2}\right)$

4 3
2 2
4 4

$\therefore$ delete C_1

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_2 & 2 & 4 \\ A_3 & 4 & 0 \\ A_4 & 0 & 8 \end{array} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$$

$\frac{2+4}{2} = \frac{6}{2} = 3$

$\frac{4+0}{2}, \frac{0+8}{2}$ $\frac{4+0}{2} = 2$

$= 2, 4$

$$\begin{bmatrix} 3 & 2 \\ 0 & 8 \end{bmatrix}$$

apply formula

	I	II	III	IV		I	II	III	IV
Probability of A =	0	0	2/3	1/3	B:	0	0	2/3	1/3

Value of game = 8/3

Graphical solutions: are helpful for the two-person zero-sum games of order $2 \times n$ or $m \times 2$ i.e. when one the players have only two dominant strategies to mix while other has many strategies to play. However, graphical solutions used with an assumption that optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies.

$\therefore$ If one player has only two strategies, the other also should have to use same (two) number of strategies. This method is helpful in finding out which two strategies can be used.

Here we have two cases viz $2 \times n$ or $m \times 2$ cases.

$2 \times n$ Games: In this case, row player has two strategies to play while column player has n strategies. The graphical method is applied to find which two strategies of column player are to be used.

Suppose a payoff given in table

	B_1	B_2	B_3	....	B_n
A_1	a_{11}	a_{12}	a_{13}	- - -	a_{1n}
A_2	a_{21}	a_{22}	a_{23}	- - -	a_{2n}

If player A mixes his strategies A_1 & A_2 with probabilities P_1 & $P_2 \geq 0$ where $P_1 + P_2 = 1$, then for each strategy of B, A will have the expected payoff as follows.

B's pure strategy	A's payoff
B_1	$a_{11}P_1 + a_{21}P_2$
B_2	$a_{12}P_1 + a_{22}P_2$
B_3	$a_{13}P_1 + a_{23}P_2$
$\vdots$	$\vdots$
B_n	

As A is assumed to always try to maximize his minimum gains. The highest point of lower envelop (lower boundary) formed by drawing these as straight lines represents the optimal probability mix. The lines corresponding to this point will yield a 2x2 matrix from which the value of the game, optimal strategies & their probability mix can be found.

Thus the two series are to represented on two parallel axes & these are joined to represent respective pay off. Then the lower envelop is identified & upon the highest point is located. The coefficients of these lines make a 2x2 Matrix for further iteration / calculation.

(1) Solve the following game graphically

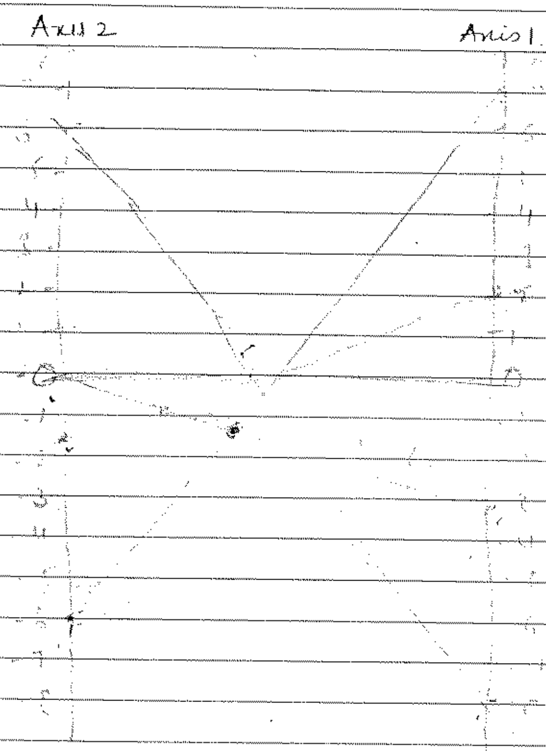
$$\begin{bmatrix} -6 & 0 & 6 & -3/2 \\ 7 & -3 & -8 & 2 \end{bmatrix}$$

Sol^n: Let Row player's (say A) strategies are A_1 & A_2 are used with the probabilities P_1 & P_2, then his expected pay off when his opponent (column player) uses his pure strategies are shown below

column player pure strategies	Row player, Expected pay off
I	$-6p_1 + 7p_2$
II	$0p_1 + 7p_2$
III	$6p_1 - 8p_2$
IV	$-3/2\, p_1 + 2p_2$

NOTE:- If 2 rows n columns $2\times n \rightarrow$ lower region
if n rows 2 columns $n\times 2 \rightarrow$ upper region

These are graphically represented as follows as two parallel axis of unit distance apart.



The highest point on lower envelope appears at the intersection of the lines represented by column player's I & II strategies.

$-6p_1 + 7p_2$ & $0p_1 - 3p_2$

∴ The required 2x2 Matrix is

$$\begin{array}{c|cc} & q_1 & q_2 \\ \hline p_1 & -6 & 0 \\ p_2 & 7 & -3 \end{array}$$

PTO

we have $-6p_1 + 7p_2 = 0p_1 - 5p_2 \Rightarrow$

$6p_1 = 10p_2$ $\quad p_1 = \frac{10p_2}{6}$

$\therefore \frac{p_1}{p_2} = \frac{10}{6}$ & $p_1 + p_2 = 1$ $\quad p_1 = 1 - p_2$

$1 - p_2 = \frac{10p_2}{6}$

$\therefore$

$6 - 6p_2 = 10p_2$

$p_1 = \frac{10}{16} = \frac{5}{8}$

$6 = 16p_2$

$p_2 = \frac{6}{16} = \frac{3}{8}$ $\quad \boxed{p_2 = \frac{6}{16}}$

$\therefore 6p_1 = 10p_2$

$6p_1 = \frac{10 \times 6}{16}$

$\boxed{p_1 = \frac{10}{16}}$

and $-6q_1 + 0q_2 = 7q_1 - 3q_2$

$\therefore \Rightarrow 13q_1 = 3q_2$

$\frac{q_1}{q_2} = \frac{3}{13}$ & $q_1 + q_2 = 1$

$\therefore q_1 = \frac{3}{16}$ & $q_2 = \frac{13}{16}$

Row player will use his strategies with the probabilities as $(5/8, 3/8)$ respectively where the column player uses the first two strategies mix at the probabilities of $\left(\frac{3}{16}, \frac{13}{16}\right)$

$\therefore$ The value of game $-6 \times \frac{5}{8} + 7 \times \frac{3}{8} = \frac{-9}{8}$

or

$0 \times \frac{5}{8} - 3 \times \frac{3}{8} = \frac{-9}{8}$

$\therefore$ The column player wins the game with a gain of 9/8 units or row player loses the game with a loss of 9/8 units.

mx2 Games: In this case a column player has two strategies while row player can choose optimal mix of best two among m strategies. The graphical method is used to find which two strategies of row player would be the best.

Suppose a pay off two player A & B is given below

$$
\begin{array}{c|cc}
 & \multicolumn{2}{c}{\text{Player B}} \\
 & B_1 & B_2 \\
\hline
A_1 & a_{11} & a_{12} \\
A_2 & a_{21} & a_{22} \\
\vdots & & \\
A_m & a_{m1} & a_{m2}
\end{array}
$$

When player B mixes his strategies B_1 & B_2 with non-zero probabilities q_1 & q_2 where $q_1 + q_2 = 1$, then for each strategy of A, B's expected payoff is given by

A's pure strategy	B's payoff
A_1	$a_{11}q_1 + a_{12}q_2$
A_2	$a_{21}q_1 + a_{22}q_2$
$\vdots$	$\vdots$
A_m	$a_{m1}q_1 + a_{m2}q_2$

Now, as B is assumed to be the loser always, he will try to minimize his maximum losses. Thus when these expected payoffs of B, when drawn as straight lines

PTO

between two parallel axes the lowest point (minimum) of the upper envelop (maximum losses) represents his optimal strategy. The line consisting of two point will represent the best two strategies at its edge on the two parallel axis of unit distance apart. The perpendicular drawn from the minimum point of envelop on to x-axis divides segment on x-axis into two parts equal to the probabilities used by B.

Eg:- Solve the game by Graphical method

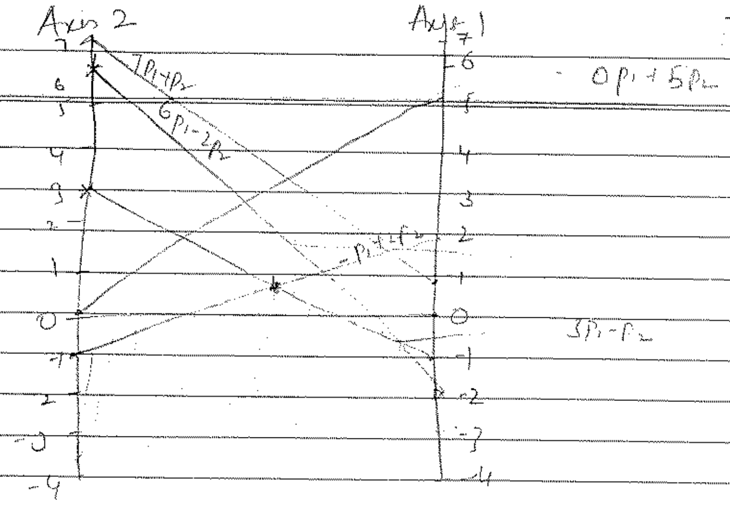
B

		1	2	3	4	5
A	1	3	0	6	-1	7
	2	-1	5	-2	2	1

Let row player's (say A) strategies are A_1 & A_2 are used with the probabilities p_1 & p_2, then his expected pay off when his opponent (column player) uses his pure strategies

Column player pure strategies	Row player (A's) Expected payoff
1	$3p_1 - p_2$
2	$0p_1 + 5p_2$
3	$6p_1 - 2p_2$
4	$-1p_1 + 2p_2$
5	$7p_1 + 1p_2$

These are graphically represented as follows as two parallel axis of unit distance apart



The highest point