Homework 2

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Problem 9.5: Show that, for every value of a, the function $f(x) = a + \frac{1}{x-a}$ is its own inverse.

Solution: First, we can find the inverse of the function $f(x) = a + \frac{1}{x-a}$. The inverse would be $x = a + \frac{1}{f(x)-a}$. For simplicity, we can let $y = f(x)$ and rewrite this equation using y: $x = a + \frac{1}{y-a}$. Now, we can solve this equation for y:

$$x(y-a) = a(y-a) + 1$$
$$xy - ay = xa - a^2 + 1$$
$$y(x-a) = xa - a^2 + 1$$
$$y = \frac{xa - a^2 + 1}{x-a}$$

To check if this is the same as the original function, we can set this equation equal to the original one:

$$\frac{xa - a^2 + 1}{x-a} = a + \frac{1}{x-a}$$

Simplifying:

$$xa - a^2 + 1 = ax - a^2 + 1$$
$$1 = 1$$

The statement $1 = 1$ is always true, meaning that $f(x)$ is its own inverse and a does not affect whether $f(x)$ is its own inverse. Therefore, for any value of a, $f(x)$ is its own inverse.

Problem 9.6: Clovis is standing at the edge of a cliff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. With the origin of the coordinate system located where he is standing, and the x-axis extending horizontally, the path of the rocket is described by the formula $y = -2x^2 + 120x$.

Part (a): Give a function $h = f(x)$ relating the height h of the rocket above the sloping ground to its x-coordinate.

Part (b): Find the maximum height of the rocket above the sloping ground. What is its x-coordinate when it is at its maximum height?

Part (c): Clovis measures its height h of the rocket above the sloping ground while it is going up. Give a function $x = g(h)$ relating the x-coordinate of the rocket to h.

Part (d): Does this function still work when the rocket is going down? Explain.

Part (a) Solution: The line that represents the cliff that Clovis is standing on is $y = -4x$ because the cliff has a slope of 4 and y-intercept at the origin. At any given x-coordinate, the the y-coordinate of the rocket's path would be $-2x^2 + 120x$ and the y-coordinate on the line representing the cliff is $-4x$.

We need to find the distance between these two y-coordinates. We can say that the distance from the rocket to the ground at any x-coordinate is the difference between the y-coordinates of the two functions, or $-2x^2 + 120x + 4x$. Therefore, the function h would be $\boxed{h = -2x^2 + 124x.}$

Part (b) Solution: The maximum height of the rocket above the ground is the maximum of the function for the height of the rocket that we just found, $h = -2x^2 + 124x$. We can find the x-coordinate of the vertex of this function:

$$-\frac{b}{2a} = \frac{124}{4} = 31$$

Therefore, the x-coordinate of the rocket when it is at maximum height is $\boxed{31.}$ To find this maximum height, we can plug 29 back into the function:

$$h = -2(31)^2 + 124(31) = 1922$$

Therefore, the maximum height of the rocket above the ground is $\boxed{\text{1922 feet.}}$

Part (c) Solution: The function that will relate the x-coordinate of the rocket to its height is the inverse of the function that will relate the height of the rocket to its x-coordinate. Therefore, we need to find the inverse of the function we found in part (a) by first swapping x and h:

$$x = -2h^2 + 124h$$

Now, we need to solve this equation for h:

$$h = \frac{-124 \pm \sqrt{124^2 - 4 \cdot -2 \cdot x}}{-4}$$

We flipped x and h to find the inverse function, so now x describes the height of the rocket while h describes the x-coordinate of the rocket. To fix this, we can swap h and x again so that h represents the height of the rocket and x represents its x-coordinate. Swapping, or equation is

$$g(h) = \frac{-124 \pm \sqrt{15376 - 8h}}{-4}$$

Because Clovis is measuring when the rocket is going up, or increasing until it hits the vertex, he is looking at the smaller set of x-coordinates (the x-coordinates of the rocket from 0 to 31 are less than x-coordinates of the rocket greater than 31). Therefore, we will choose the negative sign from $\pm$ as it will give us the smaller x-coordinates. This means that $\boxed{g(h) = \frac{-124-\sqrt{15376-8h}}{-4}.}$

Part (d) Solution: The function $g(h) = \frac{-124-\sqrt{15376-8h}}{-4}$ will not work when the rocket is going down because this function is only for the height of the rocket when the rocket is going up. In part (c), we picked the negative sign when given an equation with the $\pm$ sign because we wanted to look at the smaller half of the rocket's x-coordinates. When the rocket is going down, we will need to look at the larger half of the x-coordinates, meaning that we would have needed to pick the + sign. Hence, the function we got in part (c) will not work unless the negative sign is changed to a positive one.

Problem 9.8: A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour.

Part (a): Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.

Part (b): After how many hours will the surface of the water have width of 6 feet?

Part (c): Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.

Part (a) Solution: We can start by imposing a coordinate system in which the southern most point of the semicircle is located at the origin and intervals are in terms of feet. In this coordinate system, the center of the circle that the semicircle is part of is located at $(0, 5)$. We also know that the radius of the circle is 5 feet. Therefore, we know that the equation that represents this circle is $x^2 + (y - 5)^2 = 25$. We also know that the water level in the trough is a straight line. The trough fills at a rate of 2 inches per hour, which is equivalent to $\frac{1}{6}$ feet per hour. This means that after t hours, the water level is described by the line $y = \frac{t}{6}$. To find the width of the surface of the water, we can find where the line $y = \frac{t}{6}$ intersects the circle equation. To do this, we can substitute $\frac{t}{6}$ for y in the circle equation:

$$x^2 + \left(\frac{t}{6} - 5\right)^2 = 25$$

Solving for x:

$$x^2 = 25 - \left(\frac{t}{6} - 5\right)^2$$

$$x = \pm\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}$$

This means that the line that represents the surface of the water intersects the semicircle at x-coordinates $-\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}$ and $\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}$ after t hours. The horizontal distance between these two x-coordinates is $2\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}$ feet. Now, we need to calculate the domain of this function. We know that the range (where $f(t)$ equals y) is $0 \leq y \leq 10$ because the minimum width of the surface of the water is 0 and the maximum width is 10. We can find the domain by solving for t when the function is equal to the upper and lower bounds of the range. We already know that $t = 0$ when $f(t) = 0$ because after 0 hours there will be no water and therefore the surface of the water will be 0 feet in width. Hence, we only need to solve for when $f(t) = 10$, the upper bound of the range:

$$10 = 2\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}$$

$$25 = 25 - \left(\frac{t}{6} - 5\right)^2$$

$$0 = \frac{t}{6} - 5$$

$$t = 30$$

This means the upper bound of the domain is 30 and the lower bound of the domain is 0. Therefore, the function relating the width of the surface of the water to the time in hours is $\boxed{f(t) = 2\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}}$ on the domain $\boxed{0 \leq t \leq 30}$ with range $\boxed{0 \leq f(t) \leq 10.}$

Part (b) Solution: To find when the surface of the water is 6 feet in width, we can use the function we found in part (a) that represents the width of the surface of the water and set it equal to 6:

$$6 = 2\sqrt{25 - \left(\frac{t}{6} - 5\right)^2}$$

$$9 = 25 - \left(\frac{t}{6} - 5\right)^2$$

$$16 = \left(\frac{t}{6} - 5\right)^2$$

$$4 = \left|\frac{t}{6} - 5\right|$$

$$t = 6,\ t = 54$$

Our solutions mean that the water will be 6 feet in width after 6 hours and after 54 hours. However, 54 hours does not fit within the function's domain, $0 \le t \le 30$, so the water will only be 6 feet in width after $\boxed{\text{6 hours.}}$

Part (c) Solution: To find the function $t = f^{-1}(w)$, we need to find the inverse of the function we found in part (a). The domain of the original function is the same as the range of the inverse, so the range of $t = f^{-1}(w)$ is $\boxed{0 \le t \le 30.}$ The range of the original function is the same as the domain of the inverse, so the domain of $t = f^{-1}(w)$ is also $\boxed{0 \le w \le 10.}$ To find the inverse, we can start by switching t and $f(t)$ in the function:

$$t = 2\sqrt{25 - \left(\frac{f(t)}{6} - 5\right)^2}$$

For simplicity, we can let $f(t) = y$ and rewrite the function:

$$t = 2\sqrt{25 - \left(\frac{y}{6} - 5\right)^2}$$

Solving for y:

$$\frac{t^2}{4} = 25 - \left(\frac{y}{6} - 5\right)^2$$

$$25 - \frac{t^2}{4} = \left(\frac{y}{6} - 5\right)^2$$

$$\sqrt{25 - \frac{t^2}{4}} = \left|\frac{y}{6} - 5\right|$$

$$y = 6\sqrt{25 - \frac{t^2}{4}} + 30,\ y = -6\sqrt{25 - \frac{t^2}{4}} + 30$$

Now that we have two possible equations for the inverse, we need to keep the range of the function in mind. The range is $0 \le t \le 30$. Only the second equation, or the equation with a coefficient of -6 can give us outputs that are less than 30. The first equation can only output values greater than or equal to 30. Therefore, we need to use the second equation, as it will give us values that are within the range. However, we need to keep in mind that earlier we swapped y (or $f(t)$) with t, so now y represents the time since the trough has been filling while t represents the width of the surface of the water. Switching thse back so that y represents the width of the surface of the water and t represents the time:

$$t = -6\sqrt{25 - \frac{y^2}{4}} + 30$$

We used y to represent $f(t)$ (output of original function), which is also equivalent to w (input of inverse). Because we are finding the function $f^{-1}(w)$, we need to replace y with w. The function that relates the time to the width of the surface of the water would then be:

$$\boxed{t = -6\sqrt{25 - \frac{w^2}{4}} + 30}$$

Problem 9.9: A biochemical experiment involves combining together two protein extracts. Suppose a function $\Phi(t)$ monitors the amount (nanograms) of extract A remaining at time t (nanoseconds). Assume you know these facts:

1. The function Φ is invertible; i.e., it has an inverse function.

2. $\Phi(0) = 6$, $\Phi(1) = 5$, $\Phi(2) = 3$, $\Phi(3) = 1$, $\Phi(4) = 0.5$, $\Phi(10) = 0$.

Part (a): At what time do you know there will be 3 nanograms of extract A remaining?

Part (b): What is $\Phi^{-1}(0.5)$ and what does it tell you?

Part (c): (True or False) There is exactly one time when the amount of extract A remaining is 4 nanograms.

Part (d): Calculate $\Phi(\Phi^{-1}(1))$.

Part (e): Calculate $\Phi^{-1}(\Phi(6))$.

Part (f): What is the domain and range of Φ?

Part (a) Solution: When $t = 2$, or after $\boxed{\text{2 nanoseconds.}}$ The function is invertible and $\Phi(2) = 3$, meaning that 2 is the only value of t when $\Phi(t) = 3$.

Part (b) Solution: $\boxed{\Phi^{-1}(0.5) = 4}$ because when 4 was an input in Φ, 0.5 was the output. This tells us that after 0.5 nanoseconds there will be 4 nanograms of extract A.

Part (c) Solution: $\boxed{\text{True,}}$ because this function is invertible. This means the function is 1 to 1 and there is exactly one time at which there is 4 nanograms remaining.

Part (d) Solution: $\boxed{\Phi(\Phi^{-1}(1)) = 1}$ because plugging a number into the inverse of a function and then plugging the result into the actual function results in the output just being the input (in this case, 1).

Part (e) Solution: $\boxed{\Phi^{-1}(\Phi(6)) = 6.}$ This uses the exact same logic as part (d). Regardless of whether a number is inputted into the function or its inverse function first, the output will still be that number, or 6.

Part (f) Solution: The domain of this function is $\boxed{0 \leq t \leq 10.}$ It is unreasonable for times before 0 to be included in the domain and the amount of extract reaches 0 after 10 seconds. Having a function that describes negative amounts of extract also seems unreasonable, which means the upper restriction of the domain must be 10. However, it is possible that for values of t greater than 10, $\Phi(t)$ is not negative but instead remains at 0. In this case, the domain of the function would be $0 \leq t$.

The range of this function is $\boxed{0 \leq \Phi(t) \leq 6.}$ At time 0, $\Phi(t) = 6$. Times before time 0 are not counted in the domain, so 6 is the upper bound of the range. The smallest amount of extract possible is 0 (as it is unreasonable to have a negative amount of extract), so 0 is the lower bound of the range.