Homework 7

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Problem 19.2: A weight is attached to a spring suspended from a beam. At time $t = 0$, it is pulled down to a point 10 cm above the ground and released. After that, it bounces up and down between its minimum height of 10 cm and a maximum height of 26 cm, and its height $h(t)$ is a sinusoidal function of time t. It first reaches a maximum height 0.6 seconds after starting.

(a) Follow the procedure outlined in this section to sketch a rough graph of $h(t)$. Draw at least two complete cycles of the oscillation, indicating where the maxima and minima occur.

(b) What are the mean, amplitude, phase shift and period for this function?

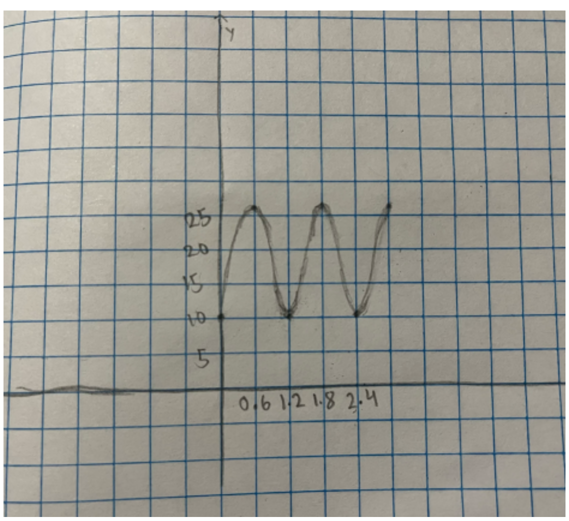
(c) Give four different possible values for the phase shift.

(d) Write down a formula for the function $h(t)$ in standard sinusoidal form, as in 19.1.1 on Page 254.

(e) What is the height of the weight after 0.18 seconds?

(f) During the first 10 seconds, how many times will the weight be exactly 22 cm above the floor? (Note: This problem does not require inverse trigonometry.)

Part (a) Solution: A rough graph of $h(t)$ is:



In this graph, the maxima occur at the point $(0.6, 26), (1.8, 26)$, and $(3, 26)$. The minima occur at $(0, 10), (1.2, 10)$, and $(2.4, 10)$.

Part (b) Solution: The mean is the average of the maximum and minimum, which in this case is 18. The amplitude is half the difference between the minimum and the maximum, which in this case is 8. The period of a function is the time that passes between two consecutive maxima or minima. We know that the time between one minimum and the following maximum is 0.6 seconds. This time is half the period (as in another 0.6 seconds the next minimum will be reached). So, the period is 1.2. The phase shift is either one fourth of the period subtracted from the x-coordinate of a maximum or one fourth of the period added to the x-coordinate of a minimum. We know that a maximum occurs at an x-coordinate of 0.6. Therefore the phase shift is $0.6 - \frac{1}{4} \cdot 1.2 = 0.3$.

Part (c) Solution: One possible value of the phase shift is 0.3, as we found above. If a maximum occurs at $t = 0.6$, a minimum must occur 0.6 seconds later, or at $t = 1.2$. Subtracting one fourth of the period from this, another possible phase shift is $1.2+0.3 = 1.5$. After this minimum, the next minimum must occur another 1.2 seconds later, or at $t = 2.4$. Adding one fourth of the period to this, another possible phase shift is $2.4+0.3 = 2.7$. After that minimum, the next minimum would occur at $t = 3.6$. Adding one fourth of the period to this, another possible phase shift is $3.6 + 0.3 = 3.9$.

Part (d) Solution: The standard form for a sinusoidal is

$$y = A\sin\left(B(x - C)\right) + D$$

where A is the amplitude, B is 2π divided by the period, C is the phase shift, and D is the mean. Substituting all the values we found in part (a) for these variables, our sinusoidal is

$$\boxed{h(t) = 8\sin\left(\frac{5\pi}{3}(t - 0.3)\right) + 18}$$

Part (e) Solution: To find the height of the weight after 0.18 seconds, we can substitute 0.18 for t in our sinusoidal. Doing this, we can say the height of the weight would be

$$\boxed{\left(8\sin\left(\frac{5\pi}{3}(0.18 - 0.3)\right) + 18\right)\ \text{cm.}}$$

Part (f) Solution: We know that it takes 1.2 seconds for the weight to complete one cycle of oscillation. In 10 seconds, the weight will have completed $8 + \frac{1}{3}$ cycles of oscillation. We know that the weight is 22 cm above the floor two times during a cycle of oscillation—once when it is going up, and once when it is going down. In the 8 full cycles that it completes, the weight will be at 22 cm 16 times. When $t = 10$, $h(t)$ is exactly 22, meaning that the weight is 22 cm above the ground once in the last $\frac{1}{3}$ of a cycle it completes. So it is 22 cm above the ground a total of $\boxed{\text{17 times.}}$

Problem 19.4: Suppose the high tide in Seattle occurs at 1:00 a.m. and 1:00 p.m. at which time the water is 10 feet above the height of low tide. Low tides occur 6 hours after high tides. Suppose there are two high tides and two low tides every day and the height of the tide varies sinusoidally.

(a) Find a formula for the function $y = h(t)$ that computes the height of the tide above low tide at time t. (In other words, $y = 0$ corresponds to low tide.)

(b) What is the tide height at 11:00 a.m.?

Part (a) Solution: We know that the standard form of a sinusoidal is

$$y = A\sin(B(x - C)) + D$$

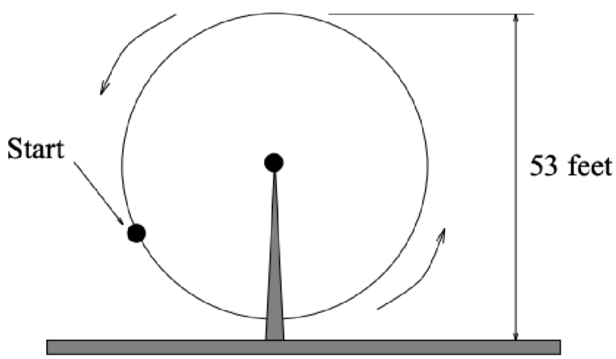
Given that $y = 0$ corresponds to low tide, we know that $y = 10$ corresponds to high tide. This means the minimum and maximum of this sinusoidal function are 0 and 10, respectively. The amplitude of a sinusoidal, or A, is half the difference of its minimum and maximum, which in this case is 5. We also know that the horizontal distance between two peaks (maxima or minima), also called the period, is 12 hours. B is 2π divided by the period, which in this case is $\frac{\pi}{6}$. C is either the x-coordinate of a maximum minus one fourth of the period, or the x-coordinate of a minimum plus one fourth of the period. If we let $t = 0$ occur when the time is 12 AM, a maximum occurs at $t = 1$. One fourth of the period is $\frac{1}{4} \cdot 12 = 3$, so $C = 1 - 3 = -2$. The value of D is the average of the maximum and minimum, which is also 5. Putting all of these together, our sinusoidal is

$$\boxed{h(t) = 5 \sin\left(\frac{\pi}{6}(t+2)\right) + 5}$$

Part (b) Solution: 11:00 AM is 11 hours after 12 AM, so it occurs at $t = 11$. Substituting 11 for t in the sinusoidal, the tide height at 11:00 AM is

$$\boxed{5 \sin\left(\frac{\pi}{6}(13)\right) + 5 \text{ feet.}}$$

Problem 19.5: Your seat on a Ferris Wheel is at the indicated position at time $t = 0$.



Let t be the number of seconds elapsed after the wheel begins rotating counterclockwise. You find it takes 3 seconds to reach the top, which is 53 feet above the ground. The wheel is rotating 12 RPM and the diameter of the wheel is 50 feet. Let $d(t)$ be your height above the ground at time t.

(a) Argue that $d(t)$ is a sinusoidal function, describing the amplitude, phase shift, period and mean.

(b) When are the first and second times you are exactly 28 feet above the ground?

(c) After 29 seconds, how many times will you have been exactly 28 feet above the ground?

Part (a) Solution: $d(t)$ is a sinusoidal function because a seat's height changes in a cyclic manner, reaching a maximum and minimum in an alternating manner. This can be further corroborated by the fact that if we assume that $d(t)$ is sinusoidal, we can find the amplitude, phase shift, period, and mean of the function. The amplitude of a sinusoidal is half the difference between the sinusoidal's maximum and minimum. We know that the maximum height of a seat on the wheel is 53 feet above the ground. The diameter of the wheel is 50 feet, meaning that the minimum height of a seat is 3 feet above the ground. Half the difference between 50 and 3 is 23.5, meaning the amplitude is 23.5.

The period of a sinusoidal is the horizontal distance between two consecutive maxima or two consecutive minima. In this case, where our independent variable is time in seconds, it is number of seconds it takes for the ferris wheel to make one full rotation. We know tthat the ferris wheel's speed is 12 RPM, meaning that it makes 1 rotation in $\frac{1}{12}$ of a minute, or 5 seconds. Therefore our period is 5.

We know that the phase shift can be found by adding a fourth of the period to the x-coordinate of a minimum, or subtracting a fourth of the period from the x-coordinate of a maximum. We know that at

$t = 0$, it would take the seat another 3 seconds to reach the top of the wheel. Given that the wheel takes 5 seconds to make one full rotation, it must take 2.5 seconds to make half a rotation (or travel from the bottom most point of the wheel to the top most point). Hence, we know that at $t = 0$, it would take the seat $3 - 2.5 = 0.5$ seconds to reach the bottom of the wheel. Therefore the first minimum that the seat reaches occurs when $t = 0.5$. Adding one fourth of the period to this, our phase shift is $0.5 + \frac{5}{4} = 1.75$.

The mean of a sinusoidal is calculated by finding the mean of the maximum and minimum, which in this case would be $\frac{53+3}{2} = 28$. Because we can find the amplitude, phase shift, period, and mean of $d(t)$, it makes sense that $d(t)$ would be a sinusoidal.

Part (b) Solution: The standard form for a sinusoidal function is

$$y = A\sin(B(x - C)) + D$$

Substituting the amplitude for A, 2π divided by the period for B, the phase shift for C, and the mean for D, we have the following sinusoidal function:

$$d(t) = 23.5\sin\left(\frac{2\pi}{5}(t - 1.75)\right) + 28$$

To find when the seat will be 28 feet above the ground, we can substitute 28 for $d(t)$ and solve for t:

$$28 = 23.5\sin\left(\frac{2\pi}{5}(t - 1.75)\right) + 28$$

Simplifying:

$$0 = \sin\left(\frac{2\pi}{5}(t - 1.75)\right)$$

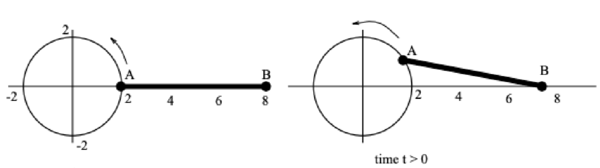
We know that in the unit circle, the first two times that the sine of an angle is 0 is when that angle is 0 or π radians. Hence we have the following two equations:

$$\frac{2\pi}{5}(t - 1.75) = 0$$
$$\frac{2\pi}{5}(t - 1.75) = \pi$$

Solving for t in both equations, our solutions are $t = 1.75$ and $t = 4.25$. Therefore the first and second times that the seat is exactly 28 feet above the ground is $\boxed{\text{after 1.75 seconds and after 4.25 seconds.}}$

Part (c) Solution: Following the first time that the seat reaches 28 feet above the ground, the seat is 28 feet high every 2.5 seconds (as $4.25 - 1.75 = 2.5$). In the first 1.75 seconds, then, the seat has already reached a height of 28 feet once. In the remaining 27.25 seconds, the seat will reach a height of 28 feet $\frac{27.25}{2.5} = 10.9$ times. Obviously the seat cannot reach 28 feet 10.9 times; it actually reaches 28 feet only 10 times. Including the time the seat reaches 28 feet at 1.75 seconds, the seat reaches 28 feet a total of $\boxed{\text{11 times.}}$

Problem 19.8: A six foot long rod is attached at one end A to a point on a wheel of radius 2 feet, centered at the origin. The other end B is free to move back and forth along the x-axis. The point A is at $(2, 0)$ at time $t = 0$, and the wheel rotates counterclockwise at 3 rev/sec.



(a) As the point A makes one complete revolution, indicate in the picture the direction and range of motion of the point B.

(b) Find the coordinates of the point A as a function of time t.

(c) Find the coordinates of the point B as a function of time t.

(d) What is the x-coordinate of the point B when $t = 1$? You should be able to find this two ways: with your function from part (c), and using some common sense (where is point A after one second?).

(e) Is the function you found in (c) a sinusoidal function? Explain.

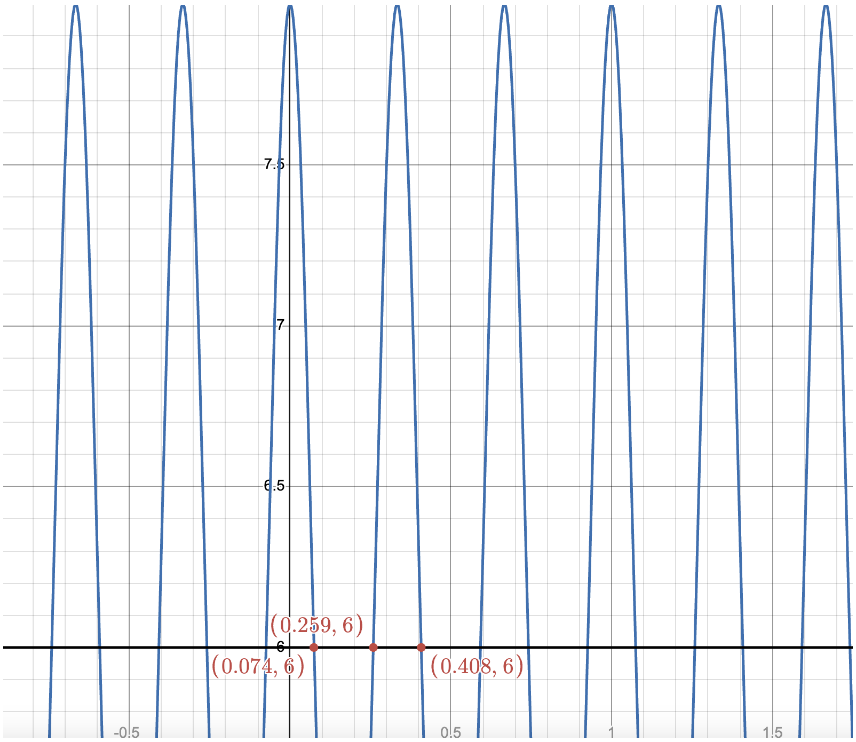
Part (a) Solution: As point A makes one complete rotation, point B will move left and right along the x-axis. While the y-coordinate of A is positive, B will move leftward until it reaches the x-coordinate of 4. While the y-coordinate of A is negative, B will move to the right until it reaches the y-coordinate of 6.

Part (b) Solution: We can think about the circle that A revolves around as a scaled up version of the unit circle (scaled up by a factor of 2, as the radius of the circle is twice that of the unit circle). In the unit circle, all points can be expressed using the coordinates $(\cos(\theta), \sin(\theta))$. Thus A's coordinates can be expressed in the form $(2\cos(\theta), 2\sin(\theta))$. Point A's starting point has an angle of 0 radians, as it starts at the rightmost point on the circle. Point A moves at 3 revolutions per second, or $3 \cdot 2\pi = 6\pi$ radians per second. This means that at time t, Point A is $6\pi t$ radians away from its starting position. Substituting this for θ, Point A's coordinates at time t are $\boxed{(2\cos(6\pi t), 2\sin(6\pi t)).}$

Part (c) Solution: B's y-coordinate will always remain 0, as B travels along the x-axis. However, B's x-coordinate can be expressed as the sum of A's x-coordinate and the horizontal distance between A and B at time t. In part (b), we found that A's x-coordinate at time t is $2\cos(6\pi t)$. We can find the horizontal distance between A and B by considering the triangle that is created by the line segment AB, the x-axis, and a vertical line between the x-axis and point A. The length of this vertical line is simply A's y-coordinate, or $2\sin(6\pi t)$. We know that the length of AB is 6 feet. Given that AB is the hypotenuse of this triangle and the vertical line is a leg, the horizontal distance between A and B (using Pythagorean theorem) can be expressed as $\sqrt{36 - 4\sin^2(6\pi t)}$. Thus B's coordinates at time t are $\left(2\cos(6\pi t) + \sqrt{36 - 4\sin^2(6\pi t)}, 0\right)$.

Part (d) Solution: Because Point A moves at a rate of 3 revolutions per second, after one second Point A will be back at its starting position. At Point A's starting position, Point B's coordinates are $(6, 0)$. Thus B's x-coordinate when $t = 1$ is $\boxed{6.}$

Part (e) Solution: No, the function is not a sinusoidal because the width of peaks and valleys is not consistent. To visualize this, we can graph a line at $y = 6$, or at the mean of this sinusoidal:



Looking at the points highlighted in red, we can see that the horizontal distance between consecutive points along the mean is not consistent. This distance from 0.074 to 0.259 is greater than the distance from 0.259 to 0.408. A characteristic of sinusoidal functions is that these distances are all the same. Because that is not reflected in this graph, we can conclude that $\boxed{\text{it is not sinusoidal.}}$