Homework 8

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Problem 20.1: Let's make sure we can handle the symbolic and mechanical aspects of working with the inverse trigonometric functions:

(c) Find four values of x that satisfy the equation $5\sin(2x^2 + x - 1) = 2$.

(d) Find four values of x that satisfy the equation $5\tan(2x^2 + x - 1) = 2$.

Part (c) Solution: Dividing both sides of the equation by 5, we have

$$\sin(2x^2 + x - 1) = \frac{2}{5}$$

Applying arcsin to both sides of the equation, we now have the two following equations in terms of x. Note that the first equation is our principal solution and the second equation is our symmetry solution (π minus the arcsin).

$$2x^2 + x - 1 = \arcsin\left(\frac{2}{5}\right) + 2\pi n$$

$$2x^2 + x - 1 = \pi - \arcsin\left(\frac{2}{5}\right) + 2\pi n$$

In these two equations, n is any integer. For simplicity, when solving for x in both of these equations, we can let $n = 0$. Solving for x in the first equation, we have

$$2x^2 + x - 1 = \arcsin\left(\frac{2}{5}\right)$$

Solving this quadratic equation, we get

$$\boxed{x = \frac{1}{4}\left(-1 - \sqrt{9 + 8\arcsin\left(\frac{2}{5}\right)}\right),\ x = \frac{1}{4}\left(-1 + \sqrt{9 + 8\arcsin\left(\frac{2}{5}\right)}\right)}$$

These are two possible solutions for x. To find the other two, we can solve the second equation we found and while letting $n = 0$:

$$2x^2 + x - 1 = \pi - \arcsin\left(\frac{2}{5}\right)$$

Solving this quadratic equation, we get

$$\boxed{x = \frac{1}{4}\left(-1 - \sqrt{9 + 8\pi - 8\arcsin\left(\frac{2}{5}\right)}\right),\ x = \frac{1}{4}\left(-1 + \sqrt{9 + 8\pi - 8\arcsin\left(\frac{2}{5}\right)}\right)}$$

Part (d) Solution: To solve this equation, we can use a similar process as the one in part (a). First, we can divide both sides of the equation by 5:

$$\tan(2x^2 + x - 1) = \frac{2}{5}$$

Next, we can apply arctan to both sides of the equation. When we do so, we get

$$2x^2 + x - 1 = \arctan\left(\frac{2}{5}\right) + \pi n$$

where n is any real integer. Solving for x, we get

$$\boxed{x = \frac{1}{4}\left(-\sqrt{8\pi n + 9 + 8\arctan\left(\frac{2}{5}\right)} - 1\right),\ x = \frac{1}{4}\left(\sqrt{8\pi n + 9 + 8\arctan\left(\frac{2}{5}\right)} - 1\right)}$$

Problem 20.4: Hugo bakes world famous scones. The key to his success is a special oven whose temperature varies according to a sinusoidal function; assume the temperature (in degrees Fahrenheit) of the oven t minutes after inserting the scones is given by

$$y = s(t) = 15\sin\left(\frac{\pi}{5}t - \frac{3\pi}{2}\right) + 415$$

(a) Find the amplitude, phase shift, period and mean for $s(t)$, then sketch the graph on the domain $0 \le t \le 20$ minutes.

(b) What is the maximum temperature of the oven? Give all times when the oven achieves this maximum temperature during the first 20 minutes.

(c) What is the minimum temperature of the oven? Give all times when the oven achieves this minimum temperature during the first 20 minutes.

(d) During the first 20 minutes of baking, calculate the total amount of time the oven temperature is at least 410°F.

(e) During the first 20 minutes of baking, calculate the total amount of time the oven temperature is at most 425°F.

(f) During the first 20 minutes of baking, calculate the total amount of time the oven temperature is between 410°F and 425°F.

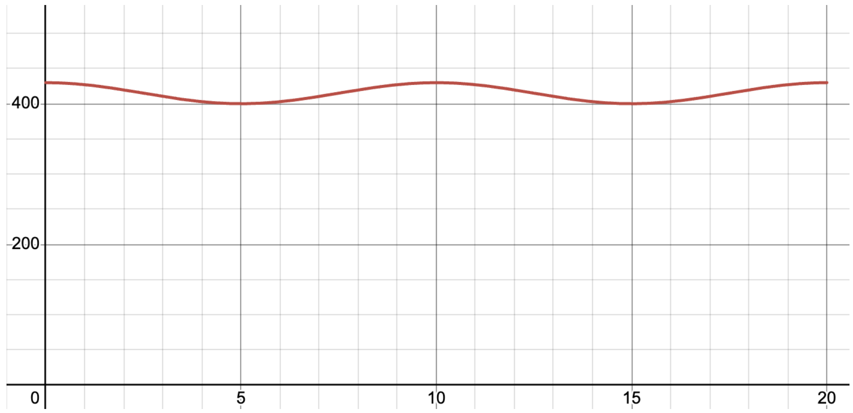
Part (a) Solution: First, we need to write $s(t)$ in the standard form of a sinusoidal, which is

$$y = A\sin(B(x - C)) + D$$

where A is the amplitude, B is 2π divided by the period, C is the phase shift, and D is the mean. Doing this, we have

$$s(t) = 15\sin\left(\frac{\pi}{5}(t - 7.5)\right) + 415$$

From this equation, we can now say that 15 is the amplitude, 7.5 is the phase shift, and 415 is the mean. The period would be 2π divided by $\frac{\pi}{5}$, which is 10. The graph of $s(t)$ on the domain $0 \le t \le 20$ would be



Part (b) Solution: In our current equation for $s(t)$, $s(t)$ has been shifted 7.5 units to the right. If $s(t)$ hadn't shifted, then $s(t)$ would have intersected the y-axis halfway between the minimum and maximum (like an unshifted sinusoidal), or at its mean (415 degrees).

The period of $s(t)$ is 10, meaning that the horizontal distance between a point with a y-coordinate at the mean and the maximum of $s(t)$ is one fourth of the period, or 2.5 units. This means that if $s(t)$ hadn't been shifted, the first maximum it would've had is at 2.5 minutes. However, because it is shifted, the first maximum it would have (after 0 minutes) $2.5 + 7.5 = \boxed{\text{10 minutes.}}$ Because the period of $s(t)$ is also 10 minutes, we know that there had to have been a maximum 10 minutes before as well, or at $\boxed{\text{0 minutes.}}$ There would also have to be a maximum 10 minutes after, or at $\boxed{\text{20 minutes.}}$

Part (c) Solution: The minimum can be found by subtracting the amplitude from the mean, which in this case is 400 degrees. A minimum occurs halfway between two maximums, meaning that a minimum occurs 5 minutes (half the period) after each maximum. If a maximum occurs at 0, 10, and 20 minutes, then in the first 20 minutes a minimum only occurs at $\boxed{\text{5 and 15 minutes.}}$

Part (d) Solution: To solve this question, we can find the amount of time the temperature is at least 410 degrees in the first 5 minutes, then multiply this time by 4 to find when the temperature is at least 410 degrees in the first 20 minutes. This works because our period is 10; a maximum or minimum occurs every 5 minutes, meaning in every 5 minute interval, there is a consistent amount of time for which the temperature is at least 410 degrees. To do this, we can set our sinusoidal equal to 410:

$$410 = 15\sin\left(\frac{\pi}{5}t - \frac{3\pi}{2}\right) + 415$$

Solving for t:

$$-\frac{1}{3} = \sin\left(\frac{\pi}{5}t - \frac{3\pi}{2}\right)$$

Applying arcsin to both sides of the equation, we have the following two equations (one is the principal and one is the symmetry solution):

$$\frac{\pi}{5}t - \frac{3\pi}{2} = \arcsin\left(-\frac{1}{3}\right) + 2\pi n$$

$$\frac{\pi}{5}t - \frac{3\pi}{2} = \pi - \arcsin\left(-\frac{1}{3}\right) + 2\pi n$$

Solving for t in both equations:

$$t = \frac{5\left(\arcsin\left(-\frac{1}{3}\right) + \frac{3\pi}{2} + 2\pi n\right)}{\pi}$$

$$t = \frac{5\left(\pi - \arcsin\left(-\frac{1}{3}\right) + \frac{3\pi}{2} + 2\pi n\right)}{\pi}$$

In both of these equations, n is any real integer. Because we are looking for the first time that the temperature reaches 410 degrees, we need to find the least positive value of t that either of these equations can produce. This least positive t value occurs when $n = -1$ in the second equation. This results in

$$t = \frac{5\left(\pi - \arcsin\left(-\frac{1}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi} \approx 3.04087$$

This means the first time the temperature is 410 degrees is after about 3 minutes. From our answer in part (a), we know that at time 0, the temperature is at a maximum. This means that the temperature was at least 410 degrees from $t = 0$ to $t \approx 3$. Therefore, in every 5 minutes, the temperature is eat least 410 degrees for approximately 3 minutes. This means that the temperature is at least 410 degrees for four times this number of minutes in the first 20 minutes. Using the exact value for the amount of time the temperature is at least 410 degrees in the first 5 minutes, we know that the temperature is at least 410 degrees for

$$\boxed{4\left(\frac{5\left(\pi - \arcsin\left(-\frac{1}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi}\right) \approx 12.16 \text{ minutes}}$$

in the first 20 minutes.

Part (e) Solution: To solve this problem, we can use a similar process to part (d). First, we can find the amount of time for which the temperature is at least 425 degrees in the first 20 minutes. Then, we can subtract this time from 20 to find the amount of time the temperature was at most 425 degrees. Using the same reasoning from part (d), we can find the amount of time the temperature is at least 425 degrees in the first 5 minutes, then multiply this by 4. To do this, we can set our sinusoidal equal to 425:

$$425 = 15\sin\left(\frac{\pi}{5}t - \frac{3\pi}{2}\right) + 415$$

Solving for t:

$$\frac{2}{3} = \sin\left(\frac{\pi}{5}t - \frac{3\pi}{2}\right)$$

Applying arcsin to both sides of the equation, we have the following two equations (one is the principal and one is the symmetry solution):

$$\frac{\pi}{5}t - \frac{3\pi}{2} = \arcsin\left(\frac{2}{3}\right) + 2\pi n$$

$$\frac{\pi}{5}t - \frac{3\pi}{2} = \pi - \arcsin\left(\frac{2}{3}\right) + 2\pi n$$

Solving for t in both equations:

$$t = \frac{5\left(\arcsin\left(\frac{2}{3}\right) + \frac{3\pi}{2} + 2\pi n\right)}{\pi}$$

$$t = \frac{5\left(\pi - \arcsin\left(\frac{2}{3}\right) + \frac{3\pi}{2} + 2\pi n\right)}{\pi}$$

In both of these equations, n is any real integer. Because we are looking for the first time that the temperature reaches 425 degrees, we need to find the least positive value of t that either of these equations can produce. This least positive t value occurs when $n = -1$ in the second equation. This results in

$$t = \frac{5\left(\pi - \arcsin\left(\frac{2}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi} \approx 1.339$$

This means the first time the temperature is 425 degrees is after about 1 minute. We know that at time 0, the temperature is at a maximum. This means that the temperature was at least 420 degrees from $t = 0$ to $t \approx 1$. Therefore, in every 5 minutes, the temperature is eat least 425 degrees for approximately 1 minute. This means that the temperature is at least 425 degrees for four times this number of minutes in the first

20 minutes. Using the exact value for the amount of time the temperature is at least 425 degrees in the first 5 minutes, we know that the temperature is at least 425 degrees for

$$4\left(\frac{5\left(\pi - \arcsin\left(\frac{2}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi}\right) \approx 5.354 \text{ minutes}$$

To find the number of minutes that the temperature is at most 425 degrees, we have to subtract the time that the temperature is at least 425 degrees from 20 minutes. This means that the temperature is at most 425 degrees for

$$\boxed{20 - 4\left(\frac{5\left(\pi - \arcsin\left(\frac{2}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi}\right) \approx 14.646 \text{ minutes}}$$

in the first 20 minutes.

Part (f) Solution: To solve this problem, we can subtract the total amount of time the temperature was at least 425 degrees (found in part (d)) from the total amount of time the temperature was at least 410 degrees (found in part (e)) in the first 20 minutes. Doing this, we can say the amount of time the temperature was between 410 and 425 degrees was

$$\boxed{4\left(\frac{5\left(\pi - \arcsin\left(-\frac{1}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi}\right) - 4\left(\frac{5\left(\pi - \arcsin\left(\frac{2}{3}\right) + \frac{3\pi}{2} - 2\pi\right)}{\pi}\right) \approx 6.8 \text{ minutes}}$$

in the first 20 minutes.

Problem 20.5: The temperature in Gavin's oven is a sinusoidal function of time. Gavin sets his oven so that it has a maximum temperature of 300°F and a minimum temperature of 240°. Once the temperature hits 300°, it takes 20 minutes before it is 300° again. Gavin's cake needs to be in the oven for 30 minutes at temperatures at or above 280°. He puts the cake into the oven when it is at 270° and rising. How long will Gavin need to leave the cake in the oven?

Solution: First, we can write a sinusoidal for Gavin's oven. We know that the maximum and minimum are 300 and 240 respectively, so the amplitude is $\frac{300-240}{2} = 30$. It takes 20 minutes between two maximums, so the period of the function is 20. The mean of the function is $\frac{300+240}{2} = 270$. There is no phase shift specified, so we can assume that there is no phase shift in the sinusoidal. Putting all of these pieces together, the sinusoidal the models the temperature of the oven is

$$y = 30\sin\left(\frac{2\pi}{20}x\right) + 270$$

To find how long he needs to leave the cake in the oven, we can set the sinusoidal equal to 280:

$$280 = 30\sin\left(\frac{2\pi}{20}x\right) + 270$$

Solving for x:

$$\frac{1}{3} = \sin\left(\frac{2\pi}{20}x\right)$$

Applying arcsin to both sides, we get our principal and symmetry solutions (written in that order):

$$\frac{2\pi}{20}x = \arcsin\frac{1}{3}$$

$$\frac{2\pi}{20}x = \pi - \arcsin\frac{1}{3}$$

Solving for x in both equations:

$$x = \arcsin \frac{1}{3} \cdot \frac{10}{\pi}$$

$$x = \frac{10}{\pi}\left(\pi - \arcsin \frac{1}{3}\right)$$

These solutions represent the two times in a period that the oven's temperature is exactly 280 degrees. To find the time that passes between these times, we can subtract the symmetry solution from the principal solution. Doing this, we can say that in every period, the oven has

$$\arcsin \frac{1}{3} \cdot \frac{10}{\pi} - \frac{10}{\pi}\left(\pi - \arcsin \frac{1}{3}\right) \text{ minutes}$$

during which its temperature is at most 280 degrees. Because each period is 20 minutes, the oven's temperature is at least 280 degrees for the rest of the time, or for

$$20 - \arcsin \frac{1}{3} \cdot \frac{10}{\pi} + \frac{10}{\pi}\left(\pi - \arcsin \frac{1}{3}\right) \text{ minutes each period.}$$

We know that the the oven needs to be at least 280 degrees for 30 minutes. If the above expression represents how long the oven is at least 280 degrees in each period, then it will take

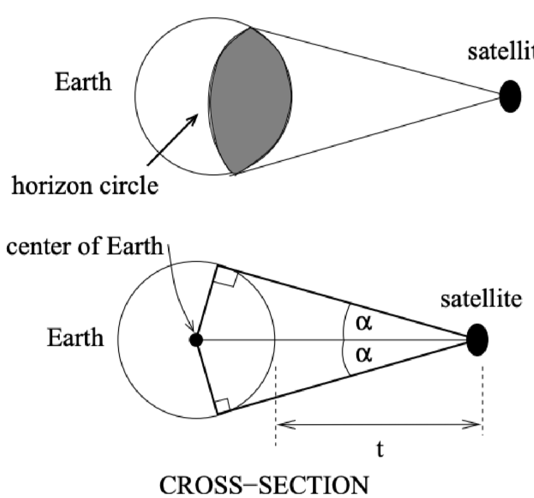
$$\frac{30}{20 - \arcsin \frac{1}{3} \cdot \frac{10}{\pi} + \frac{10}{\pi}\left(\pi - \arcsin \frac{1}{3}\right)} \text{ periods}$$

for the oven to have been at least 280 degrees for 30 minutes. Because each period is 20 minutes, we can multiply this amount of periods by 20 to find that the cake will need to be in the oven for

$$\boxed{\frac{30 \cdot 20}{20 - \arcsin \frac{1}{3} \cdot \frac{10}{\pi} + \frac{10}{\pi}\left(\pi - \arcsin \frac{1}{3}\right)} \text{ minutes}}$$

to cook.

Problem 20.9: A communications satellite orbits the earth t miles above the surface. Assume the radius of the earth is 3,960 miles. The satellite can only "see" a portion of the earth's surface, bounded by what is called a horizon circle. This leads to a two-dimensional cross-sectional picture we can use to study the size of the horizon slice:



(a) Find a formula for α in terms of t.

(b) If $t = 30000$ miles, what is alpha? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

(c) If $t = 1000$ miles, what is alpha? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

(d) Suppose you wish to place a satellite into orbit so that 20% of the circumference is covered by the satellite. What is the required distance t?

Part (a) Solution: We can first consider the upper triangle in the cross section. We know that the radius of the circle is 3960, so the length of the horizontal line connecting the center of the earth to the satellite (hypotenuse of the right triangle) is $(t + 3960)$. The other leg of the right triangle is the radius, which has length 3960. Sine is the length of the opposite leg divided by the length of the hypotenuse, so we can say that

$$\sin(\alpha) = \frac{3960}{t + 3960}$$

Applying arcsin to both sides of the equation, we can say that one formula for α is

$$\boxed{\alpha = \arcsin\left(\frac{3960}{3960 + t}\right)}$$

Part (b) Solution: To solve this, we can substitute 3000 for t in our equation for α:

$$\boxed{\alpha = \arcsin\left(\frac{3960}{3960 + 30000}\right) = \arcsin\left(\frac{33}{283}\right) \approx 0.117 \text{ radians}}$$

When considering one of the right triangles in the cross section, the triangle is composed of one right angle, α, and a central angle. Because the angles in a triangle sum to π radians, the sum of $\frac{\pi}{2}$ radians, $\arcsin\left(\frac{33}{283}\right)$ radians, and the central angle should be π. Hence we know that the central angle is π - $\frac{\pi}{2}$ - $\arcsin\left(\frac{33}{283}\right)$ radians. This central angle is also half the length of the arc that the satellite encompasses. The length of the full arc would be

$$2\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{33}{283}\right)\right)$$

A full circle is 2π radians, so this arc is

$$\frac{2\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{33}{283}\right)\right)}{2\pi}$$

of the earth, and also that fraction of the circumference of the earth. Writing this as a percentage, the satellite would cover

$$\boxed{\frac{2\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{33}{283}\right)\right)}{2\pi} \cdot 100 \approx 46.28\%}$$

of the earth's circumference. If this satellite covers 46.28% of the circumference, then two satellites will cover roughly 92.56% of the circumference, and $\boxed{\text{3 satellites}}$ will be necessary to cover the entire circumference.

Part (c) Solution: To solve this problem, we can repeat the process of part (b). First, we can substitute 1000 for t:

$$\boxed{\alpha = \arcsin\left(\frac{3960}{3960 + 1000}\right) = \arcsin\left(\frac{99}{124}\right) \approx 0.925 \text{ radians}}$$

We know that our central angle is π - $\frac{\pi}{2}$ - α, so our central angle (and half the length of the arc) is $\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{99}{124}\right)\right)$ radians. The length of the full arc would be

$$2\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{99}{124}\right)\right)$$

A full circle is 2π radians, so this arc is

$$\frac{2\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{99}{124}\right)\right)}{2\pi}$$

of the circumference of the earth. As a percentage, this arc is

$$\boxed{\frac{2\left(\pi - \frac{\pi}{2} - \arcsin\left(\frac{99}{124}\right)\right)}{2\pi} \cdot 100 \approx 20.57\%}$$

of the earth's circumference. If this satellite covers 20.57% of the earth's circumference, it will take $\boxed{5}$ satellites to cover the entirety of the earth's circumference.

Part (d) Solution: If the satellite covered 20% of the earth, it would mean that the arc that the satellite encompasses would be $0.2 \cdot 2\pi = 0.4\pi$ radians. Half this arc would be 0.2π radians. Because a triangle sums to π radians, we can say $\alpha = \pi - \frac{\pi}{2} - 0.2\pi$ radians. Because sine is the opposite leg divided by the hypotenuse, we can say that

$$\sin(\alpha) = \sin\left(\pi - \frac{\pi}{2} - 0.2\pi\right) = \frac{3960}{t + 3960}$$

Solving for t, we have

$$(t + 3960) \cdot \sin\left(\pi - \frac{\pi}{2} - 0.2\pi\right) = 3960$$

$$t + 3960 = \frac{3960}{\sin\left(\pi - \frac{\pi}{2} - 0.2\pi\right)}$$

$$\boxed{t = \frac{3960}{\sin\left(\pi - \frac{\pi}{2} - 0.2\pi\right)} - 3960}$$