Homework 17

Sahana Sarangi

December 4th, 2023

Problem 7.16: Sven starts walking due south at 5 feet per second from a point 120 feet north of an intersection. At the same time Rudyard starts walking due east at 4 feet per second from a point 150 feet west of the intersection.

Part (a): Write an expression for the distance between Sven and Rudyard t seconds after they start walking.

Part (b): When are Sven and Rudyard closest? What is the minimum distance between them?

Part (a) Solution: We can impose a coordinate system where the intersection is the origin, Sven is walking south along the y-axis, and Rudyard is walking east along the x-axis. If intervals are in terms of feet, Sven starts at $(0, 120)$ and Rudyard starts at $(-150, 0)$. Sven walks south at a rate of 5 feet per second from $(0, 120)$, so his location after t seconds will be $(0, 120 - 5t)$. Rudyard walks east at a rate of 4 feet per section from $(-150, 0)$, so his location after t seconds is $(-150 + 4t, 0)$. To find the distance between Rudyard and Sven, we can plug these coordinates into the distance formula, where d is the distance. This formula is the expression for the idstance between Sven and Rudyard after they start walking.

$$\boxed{d = \sqrt{(-150 + 4t)^2 + (120 - 5t)^2}}$$

Part (b) Solution: To find when Sven and Rudyard are closest, we need to optimize the equation we obtained by finding the value of t at the functions maximum/minimum. However, the formula we have involves a square root, which we cannot optimize. Instead we can square this formula, as taking the square of the output of this function does not change the time at which the optimum occurs. This means we can rewrite the equation as

$$d^2 = (-150 + 4t)^2 + (120 - 5t)^2$$

Expanding this equation:

$$d^2 = 22500 - 1200t + 16t^2 + 14400 - 1200t + 25t^2$$

$$d^2 = 41t^2 - 2400t + 36900$$

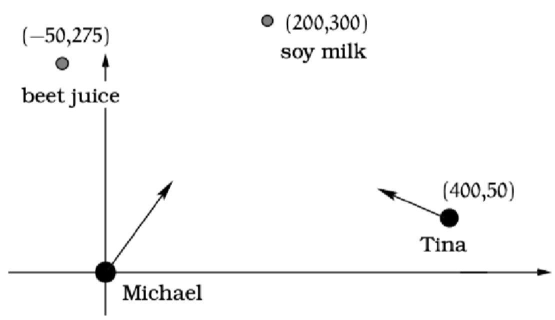
Now, we need to find the x coordinate of the vertex of this new quadratic equation. To do this, we can use the formula $-\frac{b}{2a}$:

$$-\frac{b}{2a} = \frac{2400}{82} = \frac{1200}{41}$$

Because this is the x-coordinate of the vertex at this function's maximum, we know that Sven and Rudyard are closest after $\boxed{\frac{1200}{41} \text{ seconds.}}$ The minimum distance between them will occur at this time, so we can plug this value for t back into our original equation, $d = \sqrt{(-150 + 4t)^2 + (120 - 5t)^2}$. When we plug this in, we will find that the minimum distance between the two is

$$\boxed{\sqrt{\left(-150+4\left(\frac{1200}{41}\right)\right)^2+\left(120-5\left(\frac{1200}{41}\right)\right)^2} \text{ feet.}}$$

Problem 7.17: After a vigorous soccer match, Tina and Michael decide to have a glass of their favorite refreshment. They each run in a straight line along the indicated paths at a speed of 10 ft/sec. Parametrize the motion of Tina and Michael individually. Find when and where Tina and Michael are closest to one another; also compute this minimum distance.



We can start by parametrizing Michael's path of motion. Michael starts at the point $(0,0)$ and runs toward the soy milk located at $(200,300)$. The slope of the line that runs through these points is $\frac{3}{2}$. We can let the variable y represent the time Michael spends to run 10 feet. Using the Pythagorean theorem as well as the slope of Michael's line of motion, we can express the time Michael takes to run 10 feet in the equation $(3y)^2+(2y)^2=10^2$. $3y$ represents the distance Michael travels vertically and $2y$ represents the distance Michael travels horizontally. Solving for y:

$$9y^2+4y^2=100$$
$$y=\sqrt{\frac{100}{13}}$$

If Michael is traveling for $\sqrt{\frac{100}{13}}$ seconds, then the distance he travels vertically is $3\sqrt{\frac{100}{13}}$ feet and the distance he travels horizontally is $2\sqrt{\frac{100}{13}}$ feet. Therefore, if t represents the time Michael has been traveling for, Michael's coordinates after t seconds are $\boxed{\left(2t\sqrt{\frac{100}{13}},3t\sqrt{\frac{100}{13}}\right).}$

We can use a similar process to find Tina's parameters. We know that she starts at the point $(400,50)$ and travels towards the beet juice located at $(-50,275)$. Therefore, the slope of her line of motion is $-\frac{1}{2}$. We can let y represent the time it takes Tina to run 10 feet. Again, using the Pythagoraean theorem, we can express the time Tina takes to run 10 feet in the equation $(-y)^2+(2y)^2=10^2$. $(-y)$ represents the distance Tina travels vertically and $2y$ represnts thte distance Tina travels horizontally. Solving for y:

$$y^2+4y^2=100$$
$$y=\sqrt{20}$$

If Tina travels for $\sqrt{20}$ seconds, then she travels $\sqrt{20}$ feet vertically and $2\sqrt{20}$ feet horizontally. However, when writing coordinates for her location, we must remember that she starts from $(400,50)$. This means that if t represents the time Tina has been traveling for, her coordinates after t seconds are

$$\boxed{\left(400-2t\sqrt{20},50+t\sqrt{20}\right)}$$

Using the coordinates we found, we can plug them into the distance formula to find an equation that represents the distance between Tina and Michael:

$$d = \sqrt{\left(2t\sqrt{\frac{100}{13}} - 400 + 2t\sqrt{20}\right)^2 + \left(3t\sqrt{\frac{100}{13}} - 50 - t\sqrt{20}\right)^2}$$

Similar to part (b) of problem 7.16, we now need to optimize this equation. However, we cannot optimize a radical equation, so we can square this equation and find the vertex of the resulting parabola. Squaring the equation results in

$$d^2 = \left(2t\sqrt{\frac{100}{13}} - 400 + 2t\sqrt{20}\right)^2 + \left(3t\sqrt{\frac{100}{13}} - 50 - t\sqrt{20}\right)^2$$

Expanding this equation:

$$d^2 = 4t^2\left(\frac{100}{13}\right) - 800t\sqrt{\frac{100}{13}} + 4t^2\sqrt{\frac{2000}{13}} - 800t\sqrt{\frac{100}{13}} + 160000 - 800t\sqrt{20} + 4t^2\sqrt{\frac{2000}{13}} - 800t\sqrt{20} + 80t^2$$

$$+9t^2\left(\frac{100}{13}\right) - 150t\sqrt{\frac{100}{13}} - 3t^2\sqrt{\frac{2000}{13}} - 150t\sqrt{\frac{100}{13}} + 2500 + 50t\sqrt{20} - 3t^2\sqrt{\frac{2000}{13}} + 50t\sqrt{20} + 20t^2$$

Simplifying:

$$d^2 = t^2\left(\frac{200\sqrt{13} + 40\sqrt{5}}{\sqrt{13}}\right) - t\left(\frac{19000 + 3000\sqrt{65}}{\sqrt{13}}\right) + 162500$$

To optimize this function, we need to find the vertex of this parabola. We can find the x-coordinate of the vertex using the formula $-\frac{b}{2a}$:

$$-\frac{b}{2a} = \frac{\frac{19000+3000\sqrt{65}}{\sqrt{13}}}{2\left(\frac{200\sqrt{13}+40\sqrt{5}}{\sqrt{13}}\right)} = \frac{19000 + 3000\sqrt{65}}{400\sqrt{13} + 80\sqrt{5}} = \frac{475 + 75\sqrt{65}}{10\sqrt{13} + 2\sqrt{5}} = \frac{25\sqrt{13} + 55\sqrt{5}}{8}$$

Therefore, we know that Tina and Michael are closest together after $\boxed{\frac{25\sqrt{13}+55\sqrt{5}}{8} \text{ seconds.}}$ If we know the time at which they are the minimum distance apart, we can plug this time back into our original equation, $d^2 = \left(2t\sqrt{\frac{100}{13}} - 400 + 2t\sqrt{20}\right)^2 + \left(3t\sqrt{\frac{100}{13}} - 50 - t\sqrt{20}\right)^2$ and solve for d to find the minimum distance between them:

$$d^2 = \left(2\left(\frac{25\sqrt{13} + 55\sqrt{5}}{8}\right)\sqrt{\frac{100}{13}} - 400 + 2\left(\frac{25\sqrt{13} + 55\sqrt{5}}{8}\right)\sqrt{20}\right)^2 +$$

$$\left(3\left(\frac{25\sqrt{13} + 55\sqrt{5}}{8}\right)\sqrt{\frac{100}{13}} - 50 - \left(\frac{25\sqrt{13} + 55\sqrt{5}}{8}\right)\sqrt{20}\right)^2$$

Simplifying:

$$d^2 = \left(\frac{2400\sqrt{5} - 1600\sqrt{13}}{8\sqrt{13}}\right)^2 + \left(\frac{450\sqrt{13} + 500\sqrt{5}}{4\sqrt{13}}\right)^2$$

$$d = \pm\sqrt{\left(\frac{2400\sqrt{5} - 1600\sqrt{13}}{8\sqrt{13}}\right)^2 + \left(\frac{450\sqrt{13} + 500\sqrt{5}}{4\sqrt{13}}\right)^2}$$

We know that the distance between Tina and Michael must be positive (as distance is always positive) so the minimum distance between the two is

$$\boxed{\sqrt{\left(\frac{2400\sqrt{5}-1600\sqrt{13}}{8\sqrt{13}}\right)^2+\left(\frac{450\sqrt{13}+500\sqrt{5}}{4\sqrt{13}}\right)^2}\text{ feet.}}$$

We can find where Tina and Michael are closest to each other by plugging the time at which they are closest together into their respective coordinates. We can start by finding Michael's coordinates. Substituting $\frac{25\sqrt{13}+55\sqrt{5}}{8}$ for t into $\left(2t\sqrt{\frac{100}{13}}, 3t\sqrt{\frac{100}{13}}\right)$, we know that Michael's coordinates when they are closest together are:

$$\boxed{\left(2\left(\frac{25\sqrt{13}+55\sqrt{5}}{8}\right)\sqrt{\frac{100}{13}}, 3\left(\frac{25\sqrt{13}+55\sqrt{5}}{8}\right)\sqrt{\frac{100}{13}}\right)}$$

Now, we can find Tina's coordinates by substituting this time $\left(\frac{25\sqrt{13}+55\sqrt{5}}{8}\right)$ for t into her coordinates $\left(400-2t\sqrt{20}, 50+t\sqrt{20}\right)$. We know that Tina's coordinates when they are closest together are:

$$\boxed{\left(400-2\left(\frac{25\sqrt{13}+55\sqrt{5}}{8}\right)\sqrt{20}, 50+\left(\frac{25\sqrt{13}+55\sqrt{5}}{8}\right)\sqrt{20}\right)}$$

Problem 7.18: Consider the equation: $ax^2+2a^2x+1=0$. Find the values of x that make this equation true (your answer will involve a). Find values of a that make this equation true (your answer will involve x).

Solution: To find values of x that make this equation true, we can solve for x in terms of a.

$$ax^2+2a^2x+1=0$$

We can solve for x by using the quadratic formula. Values of x that make this equation true are

$$\boxed{\frac{-2a^2\pm\sqrt{4a^4-4a}}{2a}}$$

Now, we can find values of a that makes this equation true. We can repeat the same process as earlier by using the quadratic formula to solve for a.

$$ax^2+2a^2x+1=0$$

Values of a that make this equation true are

$$\boxed{\frac{-x^2\pm\sqrt{x^4-8x}}{4x}}$$