Homework 9

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Problem 5.1: For each of the following functions, find the expression for

$$\frac{f(x+h)-f(x)}{h}$$

Simplify each of your expressions far enough so that plugging in $h = 0$ would be allowed.

Part (a): $f(x) = x^2 - 2x$

Part (b): $f(x) = 2x + 3$

Part (c): $f(x) = x^2 - 3$

Part (d): $4 - x^2$

Part (e): $-\pi x^2 - \pi^2$

Part (f): $f(x) = \sqrt{x-1}$ (Hint: rationalize the numerator)

Part (a) Solution: To find $f(x+h)$, we can substitute $(x+h)$ for x in the function $f(x) = x^2 - 2$. This would look like:

$$f(x+h) = (x+h)^2 - 2(x+h)$$

Simplifying:

$$f(x+h) = x^2 + 2xh + h^2 - 2x - 2h$$

We can now substitute this value for $f(x+h)$ as well as $x^2 - 2x$ for $f(x)$ into the initial equation:

$$\frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

Simplifying:

$$\frac{h^2 + 2xh - 2h}{h}$$

Now, we have to write this equation to ensure that $h = 0$ can be plugged in. This means that h cannot be in the denominator, as we cannot divide anything by zero. To do this, we can divide both the numerator and denominator of the fraction by h. The expression would then be:

$$\boxed{h + 2x - 2}$$

Part (b) Solution: To find $f(x+h)$, we can substitute $(x+h)$ for x in the function $f(x) = 2x+3$. This would look like:

$$f(x+h) = 2(x+h)+3$$

Simplifying:

$$f(x+h) = 2x+2h+3$$

We can now substitute this value for $f(x+h)$ as well as $2x+3$ for $f(x)$ into the initial equation:

$$\frac{2x+2h+3-2x-3}{h}$$

Simplifying:

$$\frac{2h}{h}$$

Now, we have to write this equation to ensure that $h=0$ can be plugged in. This means that h cannot be in the denominator, as we cannot divide anything by zero. To do this, we can divide both the numerator and denominator of the fraction by h. The expression would then be $\boxed{2, \text{ if } h \neq 0.}$

Part (c) Solution: To find $f(x+h)$, we can substitute $(x+h)$ for x in the function $f(x) = x^2-3$. This would look like:

$$f(x+h) = (x+h)^2-3$$

Simplifying:

$$f(x+h) = x^2+2xh+h^2-3$$

We can now substitute this value for $f(x+h)$ as well as x^2-3 for $f(x)$ into the initial equation:

$$\frac{x^2+2xh+h^2-3-x^2+3}{h}$$

Simplifying:

$$\frac{2xh+h^2}{h}$$

Now, we have to write this equation to ensure that $h=0$ can be plugged in. This means that h cannot be in the denominator, as we cannot divide anything by zero. To do this, we can divide both the numerator and denominator of the fraction by h. The expression would then be:

$$\boxed{2x+h}$$

Part (d) Solution: To find $f(x+h)$, we can substitute $(x+h)$ for x in the function $f(x) = 4-x^2$. This would look like:

$$f(x+h) = 4 - (x+h)^2$$

Simplifying:

$$f(x+h) = 4 - x^2 - 2xh - h^2$$

We can now substitute this value for $f(x+h)$ as well as $4 - x^2$ for $f(x)$ into the initial equation:

$$\frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h}$$

Simplifying:

$$\frac{-2xh - h^2}{h}$$

Now, we have to write this equation to ensure that $h = 0$ can be plugged in. This means that h cannot be in the denominator, as we cannot divide anything by zero. To do this, we can divide both the numerator and denominator of the fraction by h. The expression would then be:

$$\boxed{-2x - h}$$

Part (e) Solution: To find $f(x+h)$, we can substitute $(x+h)$ for x in the function $f(x) = -\pi x^2 - \pi^2$. This would look like:

$$f(x+h) = -\pi(x+h)^2 - \pi^2$$

Simplifying:

$$f(x+h) = -\pi x^2 - \pi 2xh - \pi h^2 - \pi^2$$

We can now substitute this value for $f(x+h)$ as well as $-\pi x^2 - \pi^2$ for $f(x)$ into the initial equation:

$$\frac{-\pi x^2 - \pi 2xh - \pi h^2 - \pi^2 + \pi x^2 + \pi^2}{h}$$

Simplifying:

$$\frac{-\pi 2xh - \pi h^2}{h}$$

Now, we have to write this equation to ensure that $h = 0$ can be plugged in. This means that h cannot be in the denominator, as we cannot divide anything by zero. To do this, we can divide both the numerator and denominator of the fraction by h. The expression would then be:

$$\boxed{-\pi 2x - \pi h}$$

Part (f) Solution: To find $f(x+h)$, we can substitute $(x+h)$ for x in the function $f(x) = \sqrt{x-1}$. This would look like:

$$f(x+h) = \sqrt{x+h-1}$$

We can now substitute this value for $f(x+h)$ as well as $\sqrt{x-1}$ for $f(x)$ into the initial equation:

$$\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}$$

Now, we have to write this equation to ensure that $h = 0$ can be plugged in. This means that h cannot be in the denominator, as we cannot divide anything by zero. To do this, we can rationalize the numerator of the expression by multiplying both the numerator and denominator by its conjugate, $\sqrt{x+h-1} + \sqrt{x-1}$ The expression would then be:

$$\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}$$

Using the formula for the difference of two two squares, we ca write this as:

$$\frac{x+h-1-x+1}{h(\sqrt{x+h-1} + \sqrt{x-1})}$$

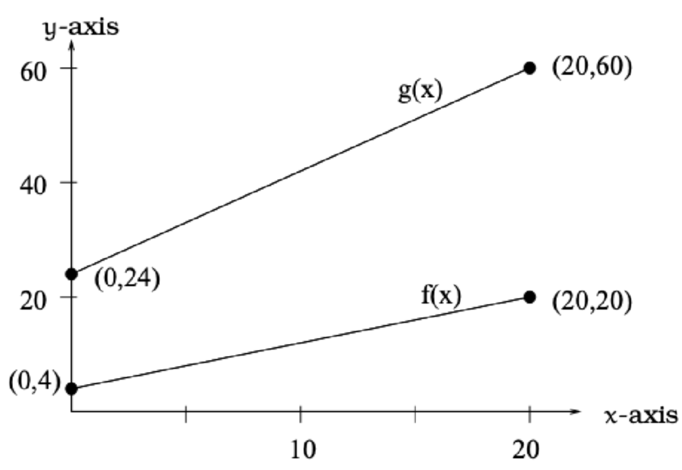
Simplifying:

$$\frac{h}{h(\sqrt{x+h-1} + \sqrt{x-1})}$$

We can now divide both sides by h to remove h from the denominator. The expression would then be:

$$\boxed{\frac{1}{\sqrt{x+h-1} + \sqrt{x-1}}}$$

Problem 5.2: Here are the graphs of two linear functions on the domain $0 \leq x \leq 20$. Find the formula for each of the rules $y = f(x)$ and $y = g(x)$. Find the formula for a NEW function $v(x)$ that calculates the vertical distance between the two lines at x. Explain in terms of the picture what $v(x)$ is calculating. What is $v(5)$? What is $v(20)$? What are the smallest and largest values of $v(x)$ on the domain $0 \leq x \leq 20$?



Solution: To find the rules of $f(x)$, we can first find its slope. We are given two points on the line, $(0, 4)$ and $(20, 20)$. The slope of the line would be $\frac{4}{5}$. We can use this slope in the standard form for the equation of the line, so it would be: $4x - 5y = C$. Now, we can plug in the point $(0, 4)$ to find C.

$$4(0) - 5(4) = C$$
$$C = -20$$

Therefore the equation for $f(x)$ is $\boxed{f(x) = 4x - 5y = -20.}$ We can repeat this process to find the rule for $g(x)$. We can use the given points, $(20, 60)$ and $(0, 24)$ to find the slope of the line, which would be $\frac{9}{5}$. The standard form for the equation of this line would then be: $9x - 5y = C$. We can then plug in the point $(20, 20)$ to find the value of C:

$$9(20) - 5(20) = C$$
$$C = 80$$

Therefore, the equation of $g(x)$ would be $\boxed{g(x) = 9x - 5y = 80.}$

Now, we can write the formula for $v(x)$. $v(x)$ is the function that calculates the distance between the y-coordinates of a point belonging to $g(x)$ and a point belonging to $f(x)$ when the points have the same x-coordinate. To find $v(x)$, we can use a point belonging to $g(x)$ and a point belonging to $f(x)$ that have the same x-coordinate. One of these pairs is $(0, 24)$ and $(0, 4)$. In this case, the y-coordinates of the points is 0 and their vertical difference is 20. Therefore, we can write the coordinate $(x, v(x))$ where x is the x-coordinate and $v(x)$ is the vertical distance between these two points as $(0, 20)$. To write a formula for $v(x)$, we will need another point written as $(x, v(x))$. Two other points belonging to $f(x)$ and $g(x)$ that have the same x-coordinate are $(20, 60)$ and $(20, 20)$. We could write the coordinate for their x-values and vertical distances as $(20, 40)$. Now that we have the points $(0, 20)$ and $(20, 40)$, we can write the formula for the line that runs through these points. First we can find the slope of the lines, which is 1. The equation would then be:

$$v(x) = x + b$$

Substituting $(0, 20)$ into the equation:

$$20 = 0 + b$$
$$b = 20$$

We can say

$$\boxed{v(x) = x + 20}$$

Now, we can find $v(5)$ by substituting 5 for x in the equation:

$$v(5) = 5 + 20$$
$$\boxed{v(5) = 25}$$

Similarly, to find $v(20)$, we can substitute 20 for x in the equation:

$$v(20) = 20 + 20$$

$$\boxed{v(20) = 40}$$

Now, we need to find the smallest and largest values of $v(x)$ on the domain $0 \leq x \leq 20$. The smallest value of $v(x)$ on this domain would be when $x = 0$, as the the slope is positive and x increases as $v(x)$ increases. The largest value of $v(x)$ would be when $x = 20$. To find $v(0)$, we can substitute 10 for x in the equation:

$$v(10) = 10 + 20$$

Meaning that the smallest value of $v(x)$ on this domain is

$$\boxed{v(0) = 30}$$

The largest value is when $x = 20$. As we found earlier, $v(20) = 40$, meaning that the largest value of $v(x)$ on this domain is $\boxed{v(20) = 40.}$

Problem 5.4: At 5 AM one day, a monk began a trek from his monastery by the sea to the monastery at the top of a mountain. He reached the mountain-top monastery at 11 AM, spent the rest of the day in meditation, and then slept the night there. In the morning, at 5 AM, he began walking back to the seaside monastery. Though walking downhill should have been faster, he dawdled in the beautiful sunshine, and ending up getting to the seaside monastery at exactly 11 AM.

Part (a): Was there necessarily a time during each trip when the monk was in exactly the same place on both days? Why or why not?

Part (b): Suppose the monk walked faster on the second day, and got back at 9 AM. What is your answer to part (a) in this case?

Part (c): Suppose the monk started later, at 10 AM, and reached the seaside monastery at 3 PM. What is your answer to part (a) in this case?

Part (a) Solution: There has to be a time during each trip when the monk was in the exactly the same place on both days. We can imagine the monk walking to and from the mountain as two different people walking in opposite directions (person 1 and person 2) along the same line segment starting and ending at the same time. Person 1 starts on the end of the segment represented by the seaside monastery, and person 2 starts on the end of the segment representing the mountain. Regardless of their speeds, they will have to intersect once if they start and end walking at the same times.

Part (b) Solution: We can imagine the monk's trip on the first day as person 1 and the monk's trip on the second day as person 2. This means that person 2 finished walking at 9 AM instead of 11 AM. We know that person 1 is en route to the mountain by 9 AM, so this means that our answer to Part (a) will be the same and the monk will be in the same place at the same time at some point.

Part (c) Solution: We know that person 1 reaches the mountain at 11 AM and person 2 leaves the mountain at 10 AM. This means that there is a one-hour period in which the two people will have to intersect at some point. Our answer to Part (a) will not change.