Homework 1

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January 8th, 2024

Problem 8.3:

Part (a): Let $f(x)$ be a linear function, $f(x) = ax + b$ for constants a and b. Show that $f(f(x))$ is a linear function.

Part (b): Find a function $g(x)$ such that $g(g(x)) = 6x - 8$.

Part (a) Solution: To find $f(f(x))$, we must substitute $f(x)$ for x in the function $f(x)$. This results in

$$f(f(x)) = a(ax + b) + b$$

Expanding this, we get

$$f(f(x)) = a^2x + ab + b$$

The function $f(f(x))$ appears to have a degree of 2, but a and b are constants. Therefore, a^2 is also a constant. This means that the function has a degree of 1, as the x-term has a degree 1.

Part (b) Solution: In part (a), we found a formula for $f(f(x))$ where $f(x)$ is a linear function. We are given that $g(g(x))$ is a linear equation, so $g(x)$ must also be a linear function. We also found that the formula for $f(f(x))$ when $f(x) = ax + b$, a linear function, is $f(f(x)) = a^2x + ab + b$. If $g(x)$ is linear and $g(g(x)) = 6x - 8$, we can say that $a^2x + ab + b$ must be equal to $6x - 8$. We know that for these expressions to be equal, both of their variable terms must be equal and their constant terms must be equal. The variable terms are a^2x and $6x$. The constant terms are $ab + b$ and -8. First, we can deal with the variable terms. If $a^2x = 6x$, a^2 must be equal to 6. Therefore, $a = \pm\sqrt{6}$. This leaves two possibiilities for a, meaning that we could result in two functions $g(x)$. The problem only asks for one function, however, so we can choose to either work with $a = \sqrt{6}$ or $a = -\sqrt{6}$. For simplicity, we can choose $a = \sqrt{6}$. Now, we can deal with the constant terms. We know that $ab + b = -8$. Substituting our new value for a, our equation will be $b\sqrt{6} + b = -8$. Solving for b, we can say that $b = -\frac{8}{\sqrt{6}+1}$. We know that $g(x)$ must be written in the form $ax + b$, and we have found a and b. Therefore, we can say $\boxed{g(x) = x\sqrt{6} - \frac{8}{\sqrt{6}+1}.}$

Problem 8.4: Let $f(x) = \frac{1}{2}x + 3$.

Part (a): Sketch the graphs of $f(x)$, $f(f(x))$, $f(f(f(x)))$ on the interval $2 \le x \le 10$.

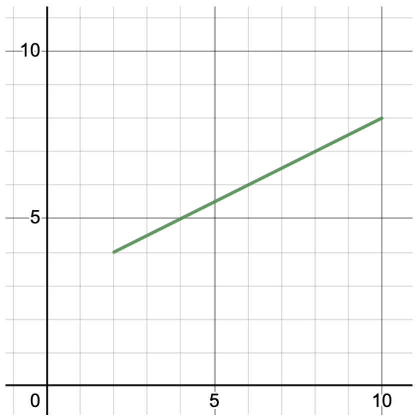
Part (b): Your graphs should all intersect at the point (6,6). The value $x = 6$ is called a fixed point of the function $f(x)$ since $f(6) = 6$; that is, 6 is fixed - it doesn't move when f is applied to it. Give an explanation for why 6 is a fixed point for any function $f(f(f(...f(x)...)))$.

Part (c): Linear functions (with the exception of $f(x) = x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g(x) = x^2 - 2$.

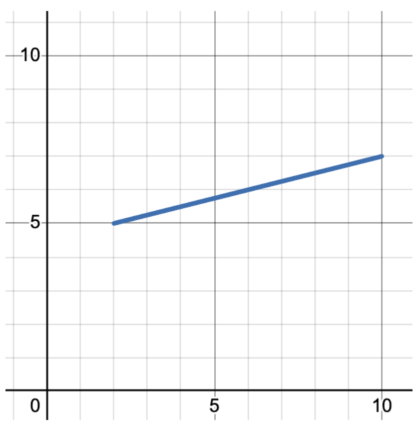
Part (d): Give a quadratic function whose fixed points are $x = -2$ and $x = 3$.

Part (a) Solution:

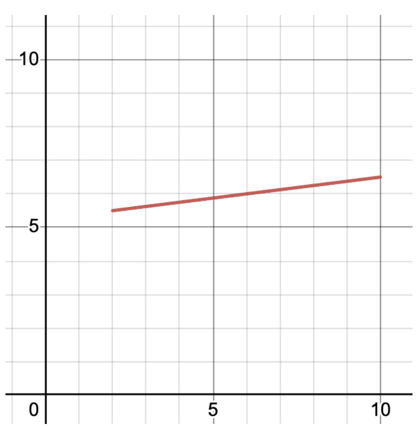
Graph of $f(x)$:



To find $f(f(x))$, we need to substitute $\frac{1}{2}x + 3$ for x into $f(x)$. Doing this, we can say $f(f(x)) = \frac{1}{2}\left(\frac{1}{2}x + 3\right) + 3$. Graphing $f(f(x))$:



To find $f(f(f(x)))$, we need to substitute $\frac{1}{2}x + 3$ for x in $f(f(x))$. Doing this, we can say $f(f(f(x))) = \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}x + 3\right) + 3\right) + 3$. Graphing $f(f(f(x)))$:



Part (b) Solution: In the function $f(x) = \frac{1}{2}x + 3$, $f(6)$ will always have an output of 6. If we are given the function $f(f(6))$, $f(6)$ has an output of 6, meaning that the input for the next iteration of f would be

6. This would mean that $f(f(6))$ also equals 6. Therefore, for no matter how many times the function is iterated, if $x = 6$, the output of the composition will always be 6, making 6 a fixed point.

Part (c) Solution: We know that a fixed point of any function occurs when the input of the function is equal to its output. If x is the input of the function and $g(x)$ is the output, this can be written as $x = g(x)$. To find the fixed points of $g(x) = x^2 - 2$, we can find points on this parabola that satisfy the equation $g(x) = x$. To do this, we can set these equations equal to each other:

$$x^2 - 2 = x$$

Solving this equation for x:

$$x = 2,\ x = -1$$

Therefore, the fixed points of the function $g(x) = x^2 - 2$ are $\boxed{-1 \text{ and } 2.}$

Part (d) Solution: If -2 and 3 are fixed points on this parabola, we know that the parabola contains the points $(-2, -2)$ and $(3, 3)$. Given that the standard form for the equation of a parabola is $y = ax^2+bx+c$, we can say that (by plugging in $(-2, -2)$ and $(3, 3)$ into this equation) $4a-2b+c = -2$ and $9a+3b+c = 3$. Now, we have a system equations. Using elimination, we can subtract these two equations from each other. Doing this results in $5a + 5b = 5$, which simplifies to the equation $a + b = 1$. We are not given enough information to determine the exact parabola that these two points pass through (as there are many parabolas which contain both of these points), so we can pick any a and b that we like. For simplicity, we can let $a = 1$ and $b = 0$. We can now substitute these values of a and b back into either of our initial equations. Substituting them into $4a - 2b + c = -2$, we will result in the equation $4 - 0 + c = -2$. Solving this equation, we can say $c = -6$. Now that we have a, b, and c, our final quadratic equation which contains these fixed points is $\boxed{y = x^2 - 6.}$

Problem 8.7: Let $y = f(z) = \sqrt{4 - z^2}$ and $z = g(x) = 2x + 3$. Compute the composition $y = f(g(x))$. Find the largest possible domain of x-values so that the composition $y = f(g(x))$ is defined.

Solution: To find $f(g(x))$, we need to substitute $2x + 3$ for z in the function f. Doing this results in

$$f(g(x)) = \sqrt{4 - (2x + 3)^2} = \sqrt{-4x^2 - 12x - 5}$$

In a square root function, the expression under the square root must be positive, or else the function is undefined. Therefore, we need to solve the inequality $-4x^2 - 12x - 5 \geq 0$ to find the domain of this function. Factoring the expression, the inequality can be rewritten as $-(x + 2.5)(x + 0.5) \geq 0$. For this quadratic to be greater than or equal to 0, exactly one of the factors must be less than or equal to 0. $(x + 2.5)$ cannot be negative while $(x + 0.5)$ is positive, so we need to find conditions for x so that $(x + 0.5)$ is negative while $(x + 2.5)$ is positive. Conditions for x that would make this true is $-2.5 \leq x \leq -0.5$. Therefore, the domain of this function is $\boxed{-2.5 \leq x \leq -0.5.}$

Problem 8.8: Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \leq x \leq 6$ and the range of $f(x)$ is $-3 \leq y \leq 5$.

Part (a) What is the domain of $f(2(x - 3))$?

Part (b) What is the range of $f(2(x - 3))$?

Part (c) What is the domain of $2f(x) - 3$?

Part (d) What is the range of $2f(x) - 3$?

Part (e) Can you find constants B and C so that the domain of $f(B(x - C))$ is $8 \leq x \leq 9$?

Part (f) Can you find constants A and D so that the range of $Af(x) + D$ is $0 \leq y \leq 1$?

Part (a) Solution: We are given that the domain of this function is $1 \leq x \leq 6$. We can find the domain of the function when x is transformed to $2(x-3)$ by substituting $2(x-3)$ for x in this inequality. This means we now have to solve the inequality $1 \leq 2(x-3) \leq 6$. Simplifying, we can find that the domain of this function is $\boxed{\frac{7}{2} \leq x \leq 6.}$

Part (b) Solution: We are given that the range of this function is $-3 \leq y \leq 5$. The expression $2(x-3)$ has a domain of all real numbers, so it does not affect the prexisting domain for the function. The range of $f(2(x-3))$ is $\boxed{-3 \leq y \leq 5.}$

Part (c) Solution: Multiplying $f(x)$ by 2 and then subtracting 3 does not have any extra domain restrictions, as it only affects the output of $f(x)$. The domain of this will remain the original domain, or $\boxed{1 \leq x \leq 6.}$

Part (d) Solution: The range of this function is $-3 \leq y \leq 5$. When multiplying this function by 2 and subtracting 3, the range of the function also gets multiplied by 2 and then subtracted by 3. Applying this, our new range is $2(-3) - 3 \leq 2y - 3 \leq 2(5) - 3$, or $\boxed{-9 \leq y \leq 7.}$

Part (e) Solution: To find the domain of $f(B(x-C))$, we can use the same process as part (a). We can substitute $B(x-C)$ for x into the original domain and solve for x.

$$1 \leq B(x-C) \leq 6$$

Solving for x:

$$\frac{1+BC}{B} \leq x \leq \frac{6+BC}{B}$$

We want to find B and C so that the domain of the function is $8 \leq x \leq 9$. This means that $\frac{1+BC}{B} = 8$ and $\frac{6+BC}{B} = 9$. Solving both of these equations for B, we have $B = -\frac{1}{C-8}$ and $B = -\frac{6}{C-9}$. To solve these equations, we can set them equal to each other:

$$-\frac{1}{C-8} = -\frac{6}{C-9}$$

Simplifying:

$$C - 9 = 6C - 48$$

$$C = \frac{39}{5}$$

Now, we can find B by substituting this value of C back into any of our previous equations. Using the equation $B = -\frac{6}{C-9}$, we can find that $B = 5$. Therefore, for the domain of the function to be $8 \leq x \leq 9$, $\boxed{C = \frac{39}{5} \text{ and } B = 5.}$

Part (f) Solution: Similar to what we described in part (d), for the range of $Af(x)+D$ to be $0 \leq y \leq 1$, the original range must be multiplied by A and then added to D. The original range was $-3 \leq y \leq 5$. Therefore, we can say that $A(-3) + D \leq y \leq 5A + D$. For this inequality to be equivalent to $0 \leq y \leq 1$, $A(-3) + D = 0$ and $5A + D = 1$. Using elimination to solve, we can subtract the two equations to find that $-8A = -1$. Simplifying, we can say that $A = \frac{1}{8}$. Now, we can substitute this back into either of the previous equations to solve for D. Substituting $\frac{1}{8}$ for A into the equation $A + D = 1$, the equation would then be $\frac{1}{8} + D = 1$. Therefore, $D = \frac{7}{8}$. Therefore, for the range of this function to be $0 \leq y \leq 1$, $\boxed{A = \frac{1}{8} \text{ and } D = \frac{7}{8}.}$