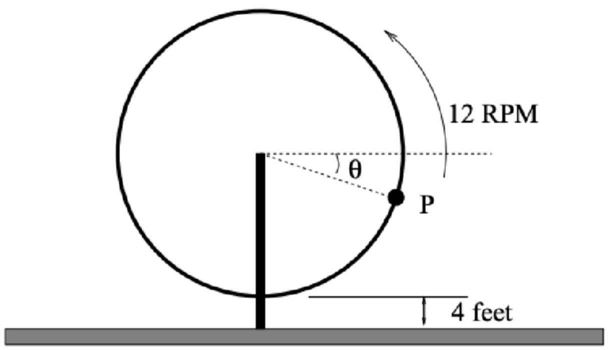
Homework 6

Sahana Sarangi

6 May 2024

Problem 17.1: John has been hired to design an exciting carnival ride. Tiff, the carnival owner, has decided to create the world's greatest ferris wheel. Tiff isn't into math; she simply has a vision and has told John these constraints on her dream: (i) the wheel should rotate counterclockwise with an angular speed of 12 RPM; (ii) the linear speed of a rider should be 200 mph; (iii) the lowest point on the ride should be 4 feet above the level ground. Recall, we worked on this in Exercise 16.5.



(a) Impose a coordinate system and find the coordinates $T(t) = (x(t), y(t))$ of Tiff at time t seconds after she starts the ride.

(b) Tiff becomes a human missile after 6 seconds on the ride. Find Tiff's coordinates the instant she becomes a human missile.

(c) Find the equation of the tangential line along which Tiff travels the instant she becomes a human missile. Sketch a picture indicating this line and her initial direction of motion along it when the seat detaches.

Part (a) Solution: We can let the center of the ferris wheel be the origin of our coordinate system. The horizontal diameter of the ferris wheel would be the x-axis and the vertical diameter of the ferris wheel would be the y-axis. 1 unit on the x or y axes is equivalent to 1 mile. We are given that the ferris wheel rotates at 12 RPM, which is equivalent to 720 RPH (revolutions per hour). We also know that a rider on the ferris wheel travels a linear distance of 200 miles in an hour. If 720 revolutions is equivalent to 200 miles, then we know that a rider making one revolution on the ferris wheel would have traveled $\frac{200}{720} = \frac{5}{18}$ miles. In other words, the circumference of the wheel is $\frac{5}{18}$ miles. Circumference is 2π times the radius, so the radius of the ferris wheel is $\frac{5}{18 \cdot 2\pi} = \frac{5}{36\pi}$ miles.

The unit circle is a regular circle with radius 1 and centered at $(0, 0)$ in a coordinate system. The ferris wheel is also a circle centered at $(0, 0)$, but has a radius of $\frac{5}{36\pi}$. Therefore we can think of the ferris wheel as a regular unit circle, but scaled up by a factor of $\frac{5}{36\pi}$. Any point on the unit circle can be written in the form $(\cos\theta, \sin\theta)$ where θ is an angle. If the ferris wheel is just scaled up from the unit circle, then any coordinate on the circumference of the ferris wheel can be written as $\left(\frac{5}{36\pi}\cos\theta, \frac{5}{36\pi}\sin\theta\right)$. In our answer to problem 16.5, we found that Tiff starts at a point $-\frac{\pi}{50}$ radians counterclockwise from the rightmost point of the wheel. In other words, the value of θ before Tiff starts moving is $-\frac{\pi}{50}$ radians.

Tiff travels at a speed of 12 RPM, which is equivalent to 24π radians per minute, or $\frac{2\pi}{5}$ radians per second. This means that for each second that passes, Tiff moves $\frac{2\pi}{5}$ radians away from her starting position. So her angular speed is $\frac{2\pi}{5}t$. Hence the angle representing the distance between the rightmost point of the wheel and Tiff's location (or θ) is $\left(-\frac{\pi}{50}+\frac{2\pi}{5}t\right)$ radians. If this is θ, we can plug this value in for θ in our formula for any point on the circumference of the ferris wheel. Doing this, we can say Tiff's coordinates at time t are $\boxed{\left(\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{2\pi}{5}t\right), \frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{2\pi}{5}t\right)\right).}$

Part (b) Solution: To find Tiff's coordinates when she becomes a human missile, we can substitute 6 for t in our answer for part (a). She becomes a human missile when she is at the coordinates $\boxed{\left(\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right), \frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\right).}$

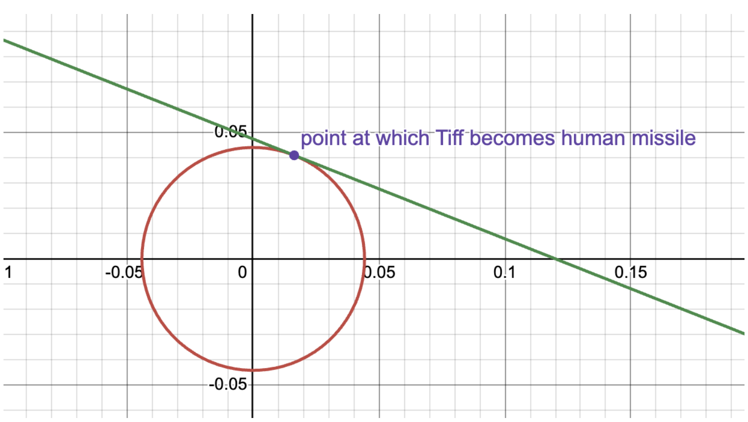
Part (c) Solution: To solve this problem, we need to find the line that is tangent to the ferris wheel at the point $\left(\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right), \frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\right)$. The tangential line will be perpendicular to the radius, so the slope of the radius is the negative reciprocal of the tangential line. When Tiff becomes a human missile, she is $\left(\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\right)$ miles to the right of and $\left(\frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\right)$ miles above the center of the wheel. So the slope of the radial line that intersects the point that Tiff becomes a human missile at is

$$\frac{\frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)}{\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)} = \tan\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)$$

The negative reciprocal of this, or the slope of the tangent line, is $-\cot\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)$. We also know that the tangent line must go through the point $\left(\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right), \frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\right)$. Using this point and the slope, we can say that the equation for the tangential line in point slope form is

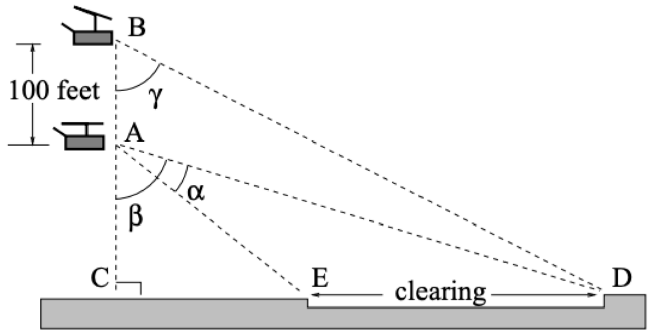
$$\boxed{y-\frac{5}{36\pi}\sin\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right) = -\cot\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\cdot\left(x-\frac{5}{36\pi}\cos\left(-\frac{\pi}{50}+\frac{12\pi}{5}\right)\right)}$$

A diagram of this tanential line is shown below:



In the diagram, the red circle represents the ferris wheel, the green line is the line along which Tiff travels, and the purple poin is the point at which Tiff's seat detaches. After the seat detaches, Tiff travels to the left, or upwards along this green line.

Problem 17.3: The crew of a helicopter needs to land temporarily in a forest and spot a flat horizontal piece of ground (a clearing in the forest) as a potential landing site, but are uncertain whether it is wide enough. They make two measurements from A (see picture) finding $\alpha = 25°$ and $\beta = 54°$. They rise vertically 100 feet to B and measure $\gamma = 47°$. Determine the width of the clearing to the nearest foot.



Solution: We can let h be the height of the helicopter at point A. We can let c be the distance from point E to point D. We can let x be the distance between point C and point E. We know that tangent of an angle is equivalent to dividing the length of the side of of the right triangle opposite to that angle by the length of the side adjacent to that angle. We can then construct the following equations that we know to be true:

$$\tan(\beta - \alpha) = \frac{x}{h}$$
$$\tan(\beta) = \frac{x + c}{h}$$
$$\tan(\gamma) = \frac{x + c}{h + 100}$$

Plugging in the values that we are given for α, β, and γ, our new equations are

$$\tan(29) = \frac{x}{h}$$
$$\tan(54) = \frac{x + c}{h}$$
$$\tan(47) = \frac{x + c}{h + 100}$$

Now that we have a system of three equations, with three unknowns, we can solve for x, c, and h. Our solutions are $h \approx 353$, $x \approx 196$, and $c \approx 290$. c represents the distance between E and D, or the length of the clearing, so the length of the clearing is $\boxed{290 \text{ feet.}}$

Problem 18.3: Solve the following:

(a) If $\cos(\theta) = \frac{24}{25}$, what are the two possible values of $\sin(\theta)$?

(b) If $\sin(\theta) = -0.8$ and θ is in the third quadrant of the xy plane, what is $\cos(\theta)$?

(c) if $\sin(\theta) = \frac{3}{7}$, what is $\sin\left(\frac{\pi}{2} - \theta\right)$?

Part (a) Solution: Given the trigonometric identity $\cos^2(\theta)+\sin^2(\theta) = 1$, we know that $\left(\frac{24}{25}\right)^2+\sin^2(\theta) = 1$. Simplifying this, we have that $\sin^2(\theta) = \frac{49}{625}$. We then have that $\boxed{\sin(\theta) = \frac{7}{25} \text{ and that } \sin(\theta) = -\frac{7}{25}.}$

Part (b) Solution: Given the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$, we know that $(-0.8)^2 + \cos^2(\theta) = 1$. Simplifying , we have that $\cos^2(\theta) = 0.36$ and $\cos(\theta) = \pm 0.6$. If θ is in the third quadrant, we know that $\cos(\theta)$ and $\sin(\theta)$ must be negative, so our only solution is $\boxed{\cos(\theta) = 0.6.}$

Part (c) Solution: One of the confunction identities states that $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$. So we only need to find $\cos(\theta)$. Using the trigonometric identity $\cos^2(\theta)+\sin^2(\theta)=1$, we have that $\frac{9}{49}+\cos^2(\theta)=1$, or $\cos^2(\theta)=\frac{40}{49}$. This simplifies to $\boxed{\cos(\theta)=\sin\left(\frac{\pi}{2}-\theta\right)=\pm\sqrt{\frac{40}{49}}.}$

Problem 18.5: Start with the equation $\sin(\theta)=\cos(\theta)$. Use the unit circle interpretation of the circular functions to find the solutions of this equation; make sure to describe your reasoning.

Solution: In the unit circle, all coordinates can be expressed as $(\cos(\theta),\sin(\theta))$ where $\cos(\theta)$ is the x-coordinate of the point and $\sin(\theta)$ is the y-coordinate of the point. When we are looking for solutions to the equation $\sin(\theta)=\cos(\theta)$, we are looking for values of θ for which the x and y-coordinates of a point on the unit circle are the same. This only occurs when $\theta=\frac{\pi}{4}$ and $\theta=\frac{5\pi}{4}$. When $\theta=\frac{\pi}{4}$, $\sin(\theta)=\cos(\theta)=\frac{\sqrt{2}}{2}$. When $\theta=\frac{5\pi}{4}$, $\sin(\theta)=\cos(\theta)=-\frac{\sqrt{2}}{2}$. So the only solutions to the equation are $\boxed{\frac{\pi}{4}\text{ and }\frac{5\pi}{4}.}$