Connections 1

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October 9th, 2023

Problem C: Alice and Dmitri are going on a hike through a park when they come to a fork in the road. One route goes in a straight line to the place they plan to stop for lunch. The other route is longer and more twisty, but also goes to that stopping point. Alice, who walks 1 mph faster than Dmitri, decides that she wants to take the long route; Dmitri decides to take the short route. Unsurprisingly, when Dmitri arrives at the lunch spot, Alice isn't there yet. To kill time, he heads back to the fork (along the short route) and then returns to the lunch spot, over and over until Alice gets there. After a few laps, Alice arrives at the same moment Dmitri does. Dmitri walks a mile in 25 minutes, and the long route is 10 miles longer than the short route. The sign for the short route was missing, but Dmitri knows it had to be between 3 and 4 miles long. How long was it?

Question 1: To approach this problem, I first collected all the information that was obvious in the problem. I knew that Dmitri's pace was 25 minutes/mile, Alice's speed is 1mph faster than Dmitri's, and that the longer route is 10 miles longer than the shorter one. The problem also states that the shorter route is within 3 to 4 miles long, but this piece of information can make the problem anxiety-inducing from the beginning, so I disregarded it for the moment.

Because this problem deals with distance, rate, and time, I know that I'll have to keep the formula $d = rt$ in mind. Because we don't know the length of the shorter route, I used the variable x for it. And because Alice's distance is 10 miles longer than the shorter route, it seems reasonable to use $x + 10$ to represent the distance she traveled. To set up any equations, I know I'll need either both of their rates or both of their times. The problem already gives us information regarding both of their rates, so I decided to use those. Unfortunately, I got Dmitri's pace instead of his rate, so I have to convert 25 minutes/mile into miles per hour. I know that 25 minutes is $\frac{5}{12}$ of an hour, so I flipped the fraction to say that his speed is $\frac{12}{5}$ miles/hour. Because Alice's speed is 1pmh faster than Dmitri's, I added 1 to $\frac{12}{5}$ and got that her speed was $\frac{17}{5}$ miles/hour.

Knowing both Alice's and Dmitri's distances and rates and that their times were equivalent, I felt confident setting up this proportion:

$$\frac{x}{\frac{12}{5}} = \frac{x+10}{\frac{17}{5}}$$

However, I quickly realized that I was forgetting a crucial part of the problem: that Dmitri reached the lunch spot *before* Alice, and walked laps back and forth before finally reaching the lunch spot at the same time as her. I realized that x represented the length of the shorter route, not the actual distance he traveled. I created the variable y to represent how many laps he made, and rewrote my proportion as:

$$\frac{xy}{\frac{12}{5}} = \frac{x+10}{\frac{17}{5}}$$

I wasn't sure how I was going to approach this because I had one equation and only two variables, but I decided to solve for x and go from there. Simplifying, I found that

$$x = \frac{120}{17y - 12}$$

I was unsure about what to do from here, but I reread the problem and remembered how Dmitri knew that the shorter route was between **3 to 4 miles long.** The only reasonable thing to do was to set up a double inequality as such:

$$3 \leq \frac{120}{17y - 12} \leq 4$$

Simplifying this, I found that

$$\frac{42}{17} \leq y \leq \frac{52}{17}$$

In this part of the process, I had to remember what was practical for this problem. y represented the amount of laps that Dmitri took, which meant that it had to be an *odd, whole number.* The only number that satisfied these conditions as well as the inequality was **3**, meaning that Dmitri reached the lunch spot at the same time as Alice at the end of his third lap. If I knew the value of y, I knew I could just plug this value into the equation where I solved for x. My equation then looked like:

$$x = \frac{120}{17(3) - 12}$$

And I simplified to find that

$$x = \frac{120}{39}$$

This meant that the length of the shorter route was $\frac{120}{39}$ miles long, which satisfies the condition that the length of the route is between 3 and 4 miles long.

Question 2: A number key to my thought process that would yield unpractical results if changed was the number 10. The longer route is 10 miles longer than the shorter route. This directly affects the inequality later in the problem–we know that the amount of laps Dmitri takes has to be a whole, odd number, but if the longer route is not 10 miles longer than the shorter route, the value of y may not fit the criteria. For example, if the longer route was instead 15 miles longer than the shorter one, the inequality would yield that the value of y is in between $\frac{52}{17}$ and $\frac{72}{17}$. There is no whole, odd number between these values. The whole number between them is 4, but Dmitri could not have made an even number of laps because it would mean he would reach the *starting point* at the exact moment Alice reaches the *lunch spot.* Numbers such as 14, 15, and 16 would not be reasonable substitutes for the number 10, but numbers such as 11 and 20 could be used instead and still yield sensible results because the value of y could still be a whole, odd number.

Question 3: A qualitative feature that was key to my thought process was which route Alice and Dmitri took. If instead Dmitri had taken the longer route and Alice had taken the shorter route, my thought process would have changed.

New version of problem: Alice and Dmitri are going on a hike through a park when they come to a fork in the road. One route goes in a straight line to the place they plan to stop for lunch. The other route is longer and more twisty, but also goes to that stopping point. Alice, who walks 1 mph faster than Dmitri, decides that she wants to take the short route; Dmitri decides to take the long route. Unsurprisingly, when Alice arrives at the lunch spot, Dmitri isn't there yet. To kill time, she heads back to the fork (along the short route) and then returns to the lunch spot, over and over until Dmitri gets there. After a few laps, Dmitri arrives at the same moment Alice does. Dmitri walks a mile in 25 minutes, and the long route is 10 miles longer than the short route. The sign for the short route was missing, but Alice knows it had to be between 3 and 4 miles long. How long was it?

Question 4 (Solution): As stated earlier, we can express the length of the shorter route as x, and the length of the longer route as $x + 10$. We also found earlier, Dmitri's speed is $\frac{12}{5}$ mph and Alice's speed is $\frac{17}{5}$ mph. The variable y represents the amount of laps Alice took back and forth before finally meeting Dmitri at the lunch spot. As explained in question 2, the value of y has to be a whole, odd number. Alice cannot take a non-whole number of laps because she would not still be walking on the road when Dmitri arrives at the lunch spot. If Alice walks an even number of laps, she will reach the starting point at the same moment Dmitri reaches the lunch spot.

We know that the time Dmitri and Alice took is the same, and using the formula $d = rt$ where d is distance, r is rate or speed, and t is time, we can write these equations:

$$t = \frac{xy}{\frac{17}{5}}$$

$$t = \frac{x + 10}{\frac{12}{5}}$$

Using substitution, we can write the equation

$$\frac{xy}{\frac{17}{5}} = \frac{x + 10}{\frac{12}{5}}$$

Now, we can solve for x to find the length of the shorter route in terms of y.

$$xy\left(\frac{12}{5}\right) = \frac{17}{5}(x + 10)$$

$$12xy = 17x + 170$$

$$x = \frac{170}{12y - 17}$$

Now that we solved for x, we have to remember that Alice knows that the length of the short route, or x, is within 3 to 4 miles. Using this information, we can write this inequality:

$$3 \leq \frac{170}{12y - 17} \leq 4$$

Solving this inequality for y:

$$\frac{170}{4} \leq 12y - 17 \leq \frac{170}{3}$$

$$\frac{238}{48} \leq y \leq \frac{221}{36}$$

We know that y has to be a whole, odd number and the only one that satisfies this inequality is **5**. To find the length of the shorter route, we have to now plug this value for y back into the equation we wrote that solves for x. The equation would then look like:

$$x = \frac{170}{12(5) - 17}$$

Solving for x:

$$x = \frac{170}{43}$$

This means that the length of the shorter route is $\boxed{\frac{170}{43} \text{ miles}}$, which satisfies Alice's criteria that the length of the shorter route is between 3 and 4 miles.