Homework 6

Sahana Sarangi

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Problem 12.2: As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function.

Part (a): If the intensity of light at the water's surface is I_o, find a formula for $I(d)$, the intensity of light at a depth of d meters. Your formula should depend on I_o and d.

Part (b): At what depth will the light intensity be decreased to 1% of its surface intensity?

Part (a) Solution: We can write an exponential function $I(d)$ that will represent the intensity of light at a depth of d meters. The intensity of the light at the water's surface is I_o, which will be our intial value (a) in the exponential equation. We know that the intensity decreases by 15%, so our b value must be $(1-0.15)$, or 0.85. However, we know that the intensity decreases by 15% every three years, not every year, so our exponent cannot be just d. The exponent must be $\frac{d}{3}$ to represent the intensity decreasing by 15% every three years. Using these values, we can write our exponential function as

$$\boxed{I(d) = I_0(0.85)^{\frac{d}{3}}}$$

Part (b) Solution: The surface intensity of light is I_o, so 1% of this intensity is $0.01I_o$. To find the depth of the water at which this is the light intensity, we can substitute this value for $I(d)$ in the equation:

$$0.01I_o = I_0(0.85)^{\frac{d}{3}}$$

Solving for d:

$$0.01 = (0.85)^{\frac{d}{3}}$$

Writing this as a logarithm:

$$\frac{d}{3} = \log_{0.85} 0.01$$

$$d = 3\log_{0.85} 0.01$$

Therefore, the intensity of light will be 1% of the surface intensity at the depth $\boxed{3\log_{0.85} 0.01}$ meters.

Problem 12.7: Your Grandfather purchased a house for \$55,000 in 1952 and it has increased in value according to a function $y = v(x)$, where x is the number of years owned. These questions probe the future value of the house under various mathematical models.

Part (a): Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a linear function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

Part (b): Suppose the value of the house is \$75,000 in 1962 and \$120,000 in 1967. Assume $v(x)$ is a quadratic function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

Part (c): Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a function of exponential type. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

Part (a) Solution: We know that the house was valued at \$55000 in 1952 (when the house as been owned for 0 years) and \$75000 in 1962 (when the house has been owned for 10 years), meaning we have two coordinates: $(0, 55000)$ and $(10, 75000)$. If this is a linear function, we can find the slope using these two coordinates. The slope would be $\frac{75000-55000}{10} = 2000$. We already know the y-intercept is at $(0, 55000)$ because this is when the x-value is 0. Therefore, the linear function $v(x)$ in slope-intercept form is

$$\boxed{v(x) = 2000x + 55000}$$

To find the value of the house in 1995—which is 43 years after it was bought—we can substitute 43 for x in this function. This would give us $v(43) = 2000(43) + 55000 = 141000$. Therefore, the value of the house in 1995 would be $\boxed{\$141000.}$

To find when the house is valued at \$200000, we can substitute 200000 for $v(x)$ in the function. Our equation would then be $200000 = 2000x + 55000$. Solving this, we can say $x = 72.5$. This means the house will be valued at \$200000 $\boxed{\text{72.5 years after 1952.}}$

Part (b) Solution: Our quadratic function $v(x)$ will be in the form $ax^2 + bx + c$. We know that the house is worth \$50000 zero years after it was bought, which is the y-intercept of this function. Therefore, $c = 55000$. However, we still need to solve for a and b in this function. We are also given that the value of the house is \$75000 ten years after 1952 and \$120000 fifteen years after 1952. This means the two other coordinates we have are $(10, 75000)$ and $(15, 120000)$. We can use these coordinates (plug them into the equation) to set up a system of equations to solve for a and b:

$$\begin{cases} 75000 = 10^2a + 10b \\ 120000 = 15^2a + 15b \end{cases}$$

The first equation in this system can be simplified to $75000 = 100a + 10b$ which is the same equation as $7500 = 10a + b$. Solving this equation for b, we have $b = 7500 - 10a$. Substituting this value for b in the second equation, we can rewrite the second equation as $120000 = 225a + 15(7500 - 10a)$. If we solve this equation for a, we can find that $a = \frac{15}{7}$. If we plug this value of a into the equation $b = 7500 - 10a$, we can say that $b = 7500 - 10(\frac{15}{7}) = -\frac{8}{3}$. Writing the quadratic function using these values for a, b, and c, we can say that $\boxed{v(x) = \frac{15}{7}x^2 - \frac{8}{3}x + 55000.}$

To find the value of the house in 1995—or 43 years after 1952—we can substitute 43 for x in the equation. Our equation would then be $v(43) = \frac{15}{7}(43)^2 - \frac{8}{3}(43) + 55000 = 803200$. Therefore, the value of the house in 1995 will be $\boxed{\$803200.}$ To find when the house will be valued at \$200000, we can subsitute 200000 for $v(x)$ in the function. Our equation would then be $200000 = \frac{15}{7}x^2 - \frac{8}{3}x + 55000$. Solving for x, we can say $x = -15$ or $x = \frac{145}{7}$. We are not considering the years before the house was bought (meaning we can disregard -15 as a solution), so the house will be worth \$200000 $\boxed{\frac{145}{7} \text{ years after 1952.}}$

Part (c) Solution: Our exponential model will be in the form $v(x) = a(b)^x$. We can let our initial year be 1952, so the value of the house in 1952 (which is \$55000) is our a value. We are also given that the value of the house is \$75000 in 1962, or 10 years after 1952. This means that the value of the house increases by a factor of $\frac{75000}{55000}$ every 10 years. Therefore, we can say our b value is $\frac{75000}{55000}$, or $\frac{15}{11}$. Because the value is increasing by this factor every 10 years, not every one year, our exponent must be $\frac{x}{10}$ instead of just x in order to reflect this. Using the values, we can say that $v(x) = 55000\left(\frac{15}{11}\right)^{\frac{x}{10}}$.

To find the value of the house in 1995, which is 43 years after 1952, we can substitute 43 for x in the function. This would mean $v(43) = 55000\left(\frac{15}{11}\right)^{\frac{43}{10}}$. Therefore, the value of the house in 1995 would be $\boxed{55000\left(\frac{15}{11}\right)^{\frac{43}{10}} \text{ dollars.}}$

To find when the house will be worth \$200000, we can substitute 200000 for $v(x)$. This would give us the equation $200000 = 55000\left(\frac{15}{11}\right)^{\frac{x}{10}}$. To solve for x, we can start by rewriting this equation:

$$\frac{200000}{55000} = \left(\frac{15}{11}\right)^{\frac{x}{10}}$$

Now, rewriting this as a logarithm:

$$\frac{x}{10} = \log_{\left(\frac{15}{11}\right)} \frac{40}{11}$$

$$x = 10\log_{\left(\frac{15}{11}\right)} \frac{40}{11}$$

Therefore, the value of the house will be \$200000 $\boxed{10\log_{\left(\frac{15}{11}\right)} \frac{40}{11} \text{ years after 1952.}}$

Problem 12.9: A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time.

Part (a): How long did it take the population to double?

Part (b): How long did it take the population to triple?

Part (c): When were there be 10,000 ants on board?

Part (d): There also was an exponentially-growing population of anteaters on board. At the start of the voyage there were 17 anteaters, and the population of anteaters doubled every 2.8 weeks. How long into the voyage were there 200 ants per anteater?

Part (a) Solution: We can start by writing an exponential function $f(x)$ to represent the situation. We can let x be the amount of weeks since the ship embarked on the voyage. There were 500 ants onboard when the ship started, so our initial value will be 500. We know that in one week the ants increased by a factor of $\frac{800}{500}$, or $\frac{8}{5}$. Therefore, our b value is $\frac{8}{5}$. We know that the amount of ants increases by this factor each week, so our exponent is simply x, the amount of weeks since the ship started. Using all these values, our exponential function is $f(x) = 500\left(\frac{8}{5}\right)^x$. We want to find how long it took for the population to double, or for it to each 1000 ants. To find this, we can substitute 1000 for x in the equation:

$$1000 = 500\left(\frac{8}{5}\right)^x$$

Solving for x will give us the amount of weeks it took for the population to reach 1000:

$$2 = \left(\frac{8}{5}\right)^x$$

Writing this as a logarithm:

$$x = \log_{\left(\frac{8}{5}\right)} 2$$

Therefore, it took $\boxed{\log_{\left(\frac{8}{5}\right)} 2 \text{ weeks for the population to double.}}$

Part (b) Solution: We can use the same process as part (a) to find how long it takes for the population to triple. Three times 500 is 1500, so we want to find when the population is 1500; we can do this by substituting 1500 for $f(x)$:

$$1500 = 500\left(\frac{8}{5}\right)^x$$

Again, solving for x:

$$3 = \left(\frac{8}{5}\right)^x$$

Writing this as a logarithm:

$$x = \log_{\left(\frac{8}{5}\right)} 3$$

Therefore, it took $\boxed{\log_{\left(\frac{8}{5}\right)} 3 \text{ weeks for the population to triple.}}$

Part (c) Solution: We can find when there will be 10000 ants on board by substituting 10000 for $f(x)$ in the function:

$$10000 = 500\left(\frac{8}{5}\right)^x$$

Again, solving for x:

$$20 = \left(\frac{8}{5}\right)^x$$

Writing this as a logarithm:

$$x = \log_{\left(\frac{8}{5}\right)} 20$$

Therefore, it took $\boxed{\log_{\left(\frac{8}{5}\right)} 20 \text{ weeks for the population to reach 10000.}}$

Part (d) Solution: We can write an exponential function $g(x)$ to represent the amount of anteaters on board. There were 17 at the start of the voyage, so our initial value will be 17. We know that the population doubles every 2.8 weeks, so our b value will be 2. The population doubles every 2.8 weeks, not every week, so our exponent must be $\frac{x}{2.8}$ instead of just x in order to reflect this. Using these values, we can write our exponential function as $g(x) = 17(2)^{\frac{x}{2.8}}$.

To find when there were 200 anteaters, we can substitute 200 for $g(x)$ in the function:

$$200 = 17(2)^{\frac{x}{2.8}}$$

Solving for x:

$$\frac{200}{17} = (2)^{\frac{x}{2.8}}$$

Writing this as a logarithm:

$$\frac{x}{2.8} = \log_2 \frac{200}{17}$$

$$x = 2.8 \log_2 \frac{200}{17}$$

Therefore, it took $\boxed{2.8 \log_2 \frac{200}{17} \text{ weeks for the population to reach 200.}}$

Problem 12.10: The populations of termites and spiders in a certain house are growing exponentially. The house contains 100 termites the day you move in. After 4 days, the house contains 200 termites. Three days after moving in, there are two times as many termites as spiders. Eight days after moving in, there were four times as many termites as spiders. How long (in days) does it take the population of spiders to triple?

Solution: To find the spider population, we can first write an exponential function $f(x)$ to represent the population of termites. We can let x be the days since moving in. The termite population starts at 100, so our initial value is 100. We are given that it takes 4 days for the termite population to go from 100 to 200—for for the termite population to double. If it is doubling, our b value must be 2. However, it doubles every 4 days, not every day, so our exponent must be $\frac{x}{4}$, not x, to represent this. Using these values, we can say the equation that represents the termite population is $f(x) = 100(2)^{\frac{x}{4}}$.

The termite population after 3 days is the value of $f(x)$ when $x = 3$. Substituting 3 for x, we can say the termite poulation on the 3rd day is $f(3) = 100(2)^{\frac{3}{4}}$. If the spider population is half the termite population on the third day, the spider population on the third day must be $50(2)^{\frac{3}{4}}$. The termite population on the 8th day can be found by substituting 8 for x in $f(x)$. Doing this, we can say the termite population on the 8th day is $f(8) = 100(2)^{\frac{8}{4}} = 400$. If the spider population is a fourth of this on the 8th day, the spider population on the 8th day must be 100.

Using the two populations of the spiders that we found, we can construct an exponential function $g(x)$ that represents the spider population after x days. We know that the spider population is $50(2)^{\frac{3}{4}}$ on the 3rd day, so we can take this population as our initial value. We also found that the spider population is 100 on the 8th day. Therefore, we know that the population grows from $50(2)^{\frac{3}{4}}$ to 100 in five days. This means that the spider population is growing by a factor of $\frac{100}{50(2)^{\frac{3}{4}}}$.

However, it grows by this factor every five days, not every day, so our exponent must have a denominator of 5. We need to keep in mind that our initial value was the spider population on the 3rd day, not on the first day. Therefore, we need to subtract 3 from x in the exponent so that the numerator of the exponent represents the days since the 3rd day, not the 1st day. Our exponent then is $\frac{x-3}{5}$. Using these values, we can say that our exponential function is

$$g(x) = 50(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{\frac{x-3}{5}}$$

Now, we need to find the initial spider population (the spider population on the move in day). The spider population on this day is when $x = 0$, so we can substitute 0 for x in the equation we just found:

$$g(x) = 50(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}}$$

The initial spider population is $50(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}}$, so triple this population is $150(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}}$. We can find when the population is this value by subsituting this value for $g(x)$ in the function:

$$150(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}} = 50(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{\frac{x-3}{5}}$$

Solving for x:

$$\frac{150(2)^{\frac{3}{4}} \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}}}{50(2)^{\frac{3}{4}}} = \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{\frac{x-3}{5}}$$

Writing this as a logarithm:

$$\frac{x-3}{5} = \log_{\frac{100}{50(2)^{\frac{3}{4}}}} \left(3 \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}} \right)$$

$$x = 5 \log_{\frac{100}{50(2)^{\frac{3}{4}}}} \left(3 \left(\frac{100}{50(2)^{\frac{3}{4}}} \right)^{-\frac{3}{5}} \right) + 3$$

Therefore, it will take $\boxed{\left(5\log_{\frac{100}{50(2)^{\frac{3}{4}}}}\left(3\left(\frac{100}{50(2)^{\frac{3}{4}}}\right)^{-\frac{3}{5}}\right)+3\right) \text{ days for the population to triple.}}$