Intersections of Exponentials and Logarithms

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1 Introduction

From studying systems of linear equations to finding fixed points through a functions' intersections with $y = x$, we have observed and studied a lot of intersections throughout this year. We recently finished our units regarding exponentials and logarithms; this leaves us to consider: what intersections are there to study in these two fields? We can consider a logarithmic function and an exponential function together, like $f(x) = \log_a x$ and $g(x) = a^x$ for some arbitrary positive constant, a. Looking at these two functions can give us further insight into the nature of exponentials and logarithms. To give us a starting point, we can ask: where do these two functions intersect? To answer this, let's look at some values of a.

For $a = 0$, we get the following

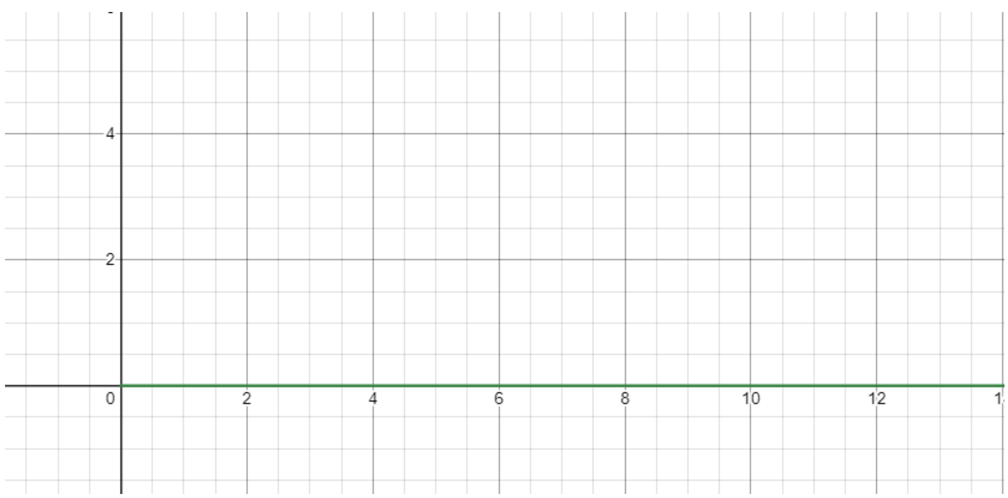


Figure 1: The intersection(s) of $f(x) = \log_0 x$ and $g(x) = 0^x$.

Notably, there are no intersections of $f(x)$ and $g(x)$ for $a = 0$.

Next, let's look at $a = 0.2$.

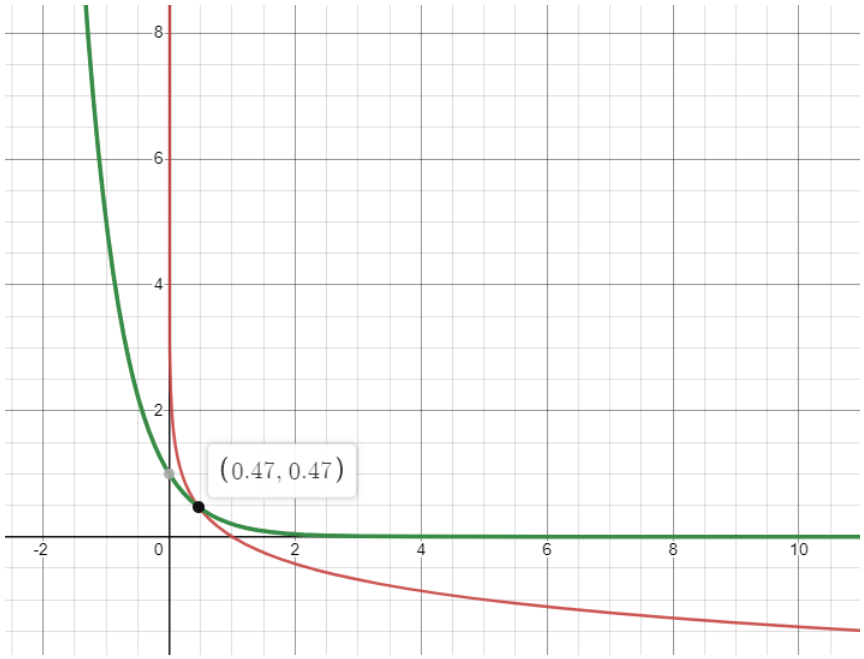


Figure 2: The intersection(s) of $f(x) = \log_{0.2} x$ and $g(x) = 0.2^x$.

Here, $f(x)$ and $g(x)$ intersect once, at $(0.47, 0.47)$.
What about at $a = 1$?

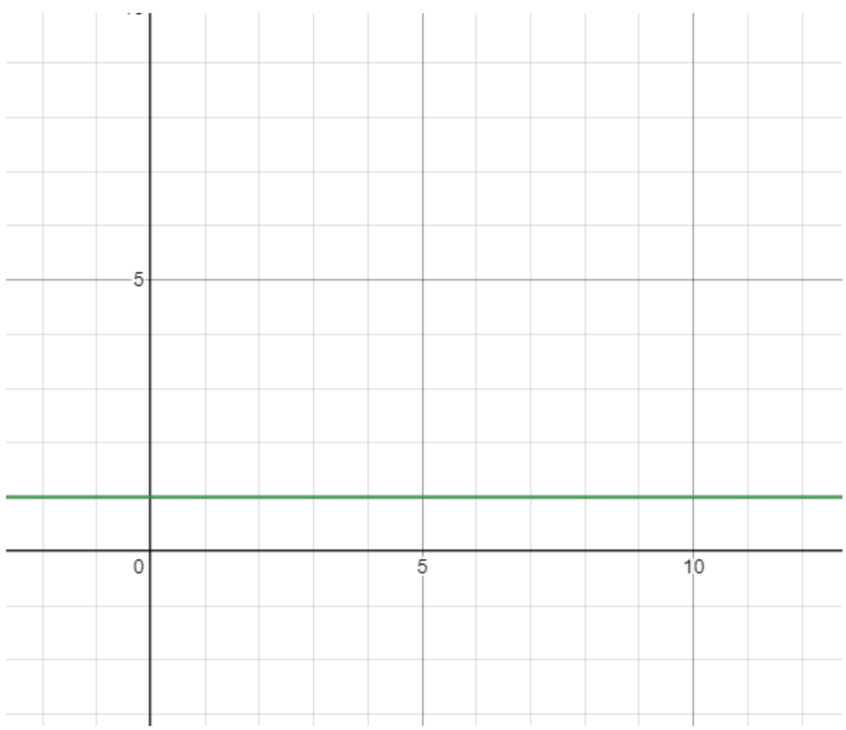


Figure 3: The intersection(s) of $f(x) = \log_1 x$ and $g(x) = 1^x$.

There are no intersections for $a = 1$. Namely, this is a transformation of $g(x) = a^x$ for $a = 0$ by one unit up.

That wasn't terribly interesting; what about 1.2?

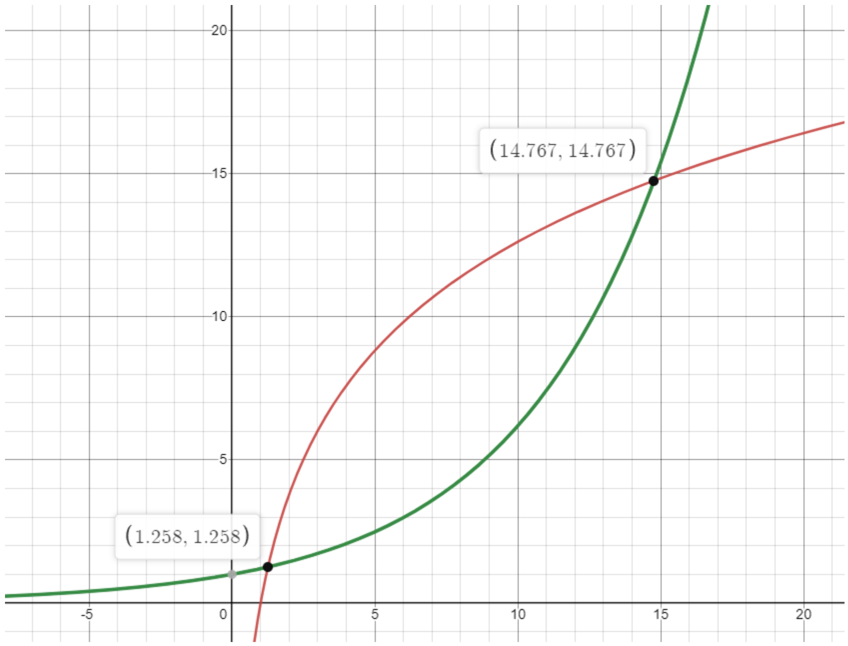


Figure 4: The intersection(s) of $f(x) = \log_{1.2} x$ and $g(x) = 1.2^x$.

For $a = 1.2$, $f(x)$ and $g(x)$ intersect twice, once at $(1.258, 1.258)$, and once at $(14.767, 14.767)$.

Repeat this process – of testing different values of a in each function and finding the resulting intersections of the functions – and we find that this pattern of intersections eventually subsides. The pattern appears to stop around 1.45-ish. An interesting exercise is finding the exact value for which this pattern appears to start and stop; we see that the pattern appears at $a > 1$ and, as stated, stops around $a < 1.45$. That leads to the central question of this paper: **for what values of a are two intersections between $g(x) = a^x$ and $f(x) = \log_a x$?**

2 Solving

2.1 Lower Bound

We know that the graph of $\log_a x$ has vertical asymptote at 0 when a is some positive constant. Thus, our graph would be increasing to the negative y values for positive ones. The graph of an exponential function would increase from small positive y values into larger y values as x increases. Hence, when these graphs have two intersections, we know that for positive x before the intersection with the smallest x-coordinate (or leftmost intersection), the outputs of our exponential would have to be greater than the outputs of our logarithmic function. At some x-coordinate between the two intersections, the outputs of the logarithmic function would be greater than the outputs of the exponential. At some x-coordinate greater than the x-coordinate of the intersections with the greatest x-coordinate, the outputs of the exponential would be greater than the outputs of the logarithmic function. In this scenario, the value of the exponential starts off greater than the logarithm, but as the x-values increase, the value of the exponential becomes less than that of the logarithm. If we increase the x-values even more, the value of the exponential again becomes greater than the logarithm. In other words, the pattern between these two functions is that the exponential is greater, then less than, and finally greater than the logarithm *only when $g(x)$ and $f(x)$ have two intersections.*

For the sake of simplicity, we can let "the first intersection" refer to the intersection between $f(x)$ and $g(x)$ that has the smallest x-coordinate, and let "the second intersection" refer to the intersection with the greater x-coordinate. Hence, the exponential function has greater outputs than the logarithmic function before the first intersection, has smaller outputs than the logarithmic function between the first and second intersections, and again has greater outputs than the logarithmic function after the second intersection (given that $f(x)$ and $g(x)$ intersect twice).

Using this logic, we can find our lower bound on values of a that produce two intersections by constructing two lines. Our "first" line will intersect $f(x)$ and $g(x)$ *before* the first intersection, and our "second" line will intersection $f(x)$ and $g(x)$ *between* the first and second intersections. We know that if the exponential function intersects the first line at a greater y-coordinate than the logarithmic function does, but intersects the second line at a smaller y-coordinate than the logarithmic function does, we can confirm that the exponential and logarithmic functions would have had to intersect at some x-coordinate between the first and second lines. This is because the outputs of the exponential would have switched from being greater than to less than the outputs of the logarithmic function, and they would need to intersect in order for this to happen.

Using these two lines, we want to find the values of a that allow this first intersection to happen. We can construct our first line to be $x = 1$ and our second line to be $x = e$. As stated earlier, if $f(x)$ and $g(x)$ have two intersections, our first criterion is that the y-coordinate of the intersection of $g(x)$ with $x = 1$ will be greater than the y-coordinate of the intersection of $f(x)$ with $x = 1$. The y-coordinate of the intersection with $g(x)$ is simply $g(1) = a^1 = a$. The y-coordinate of the intersection with $f(x)$ is $f(1) = \log_a 1$. If $a = 1$, the solution to this logarithm is 1. However, for all a greater than 1, the solution is 0. We are only considering positive values for a, so we know that a will always be greater than 0. However, if $a = 1$, the value of $g(1)$ and $f(1)$ would be equivalent (as a and $\log_a 1$ would be equivalent), and hence the y-coordinate of the intersection with $g(x)$ would not be greater than the y-coordinate of the intersection with $f(x)$. Thus the values of a that satisfy this first criterion are all values of a as long as $a \neq 1$.

From our reasoning earlier, our second criterion was that the y-coordinate of the intersection of $g(x)$ with our second line, $x = e$ would be less than the y-coordinate of the intersection of $f(x)$ with $x = e$. To solve for values of a that satisfy this criterion, we can set up an inequality: $\log_a e > a^e$. To solve this inequality, we can first consider that a cannot be equal to 1, as $\log_1(e)$ is undefined.

For values of a that are less than one, a number between 0 and 1 would be base of our logarithm. If a number between 0 and 1 is our base while e is our argument, the value of the logarithm will be negative. The value of our exponential, on the other hand, would always be greater than 0, as a fractional value would be raised to the power of e, resulting in a smaller fractional value. Thus, for $a < 1$ (given that a is strictly positive), our exponent would have positive y-coordinate for its intersection with $x = e$, but our logarithm would have a negative y-coordinate. Hence, values of a that are less than 1 do not satisfy our inequality.

Lastly, we can consider when $a > 1$. Certain values of a greater than 1 satisfy our inequality, but at some point, the values of a stop satisfying the inequality. For example, when $a = 1.1$, our inequality is $\log_{1.1} e > (1.1)^e$. This inequality is true, as it simplifies to (approximately) $10.492 > 1.296$. Similarly, when $a = 1.2$, our inequality is $\log_{1.2} e > (1.2)^e$, which is true because it simplifies to approximately $5.485 > 1.641$. Again, when $a = 1.3$, our inequality is $\log_{1.3} e > (1.3)^e$, which is true because it simplifies to approximately $3.811 > 2.197$. However, at some point, values of a no longer satisfy the inequality. For example, when $a = 2$, our inequality is $\log_2 e > 2^e$, which is false because it simplifies to approximately $1.443 > 6.581$. The reason why there is a point at which a values do not satisfy this inequality can be found by considering what logarithms and exponentials represent. In the inequality $\log_a e > a^e$, we are saying that the power to which we have to take a in order to get e must be greater than taking a to the power of e. It makes sense that for smaller values of a, or values of a that are closer to 1, this inequality would hold true. This is because when a is smaller, the value of $\log_a e$ would be a larger number, while a^e would be a smaller number. However, as a increases, the value of a^e will increase and eventually be greater than $\log_a e$.

It is also unreasonable that there are one-off values of a between 1.1 and 1.2, or 1.2 and 1.3, that don't satisfy this inequality. This is because we know that as a increases, the value of a^e is also increasing and the value of $\log_a e$ is also decreasing, so there cannot be random values of a between 1 and 1.3 that suddenly don't work. We can confirm that all values of a included in $(1, 1.3]$ satisfy this inequality. However, we do not know the value of a for which this inequality no longer holds true. There could be (and likely are) values of a greater than 1.3 that satisfy the inequality. Hence, all we can confirm is that the values of a for which the y-coordinate of the intersection between $g(x)$ and $x = e$ is less than the y-coordinate of the intersection between $f(x)$ and $x = e$ are *some* values of a that satisfy $a > 1$. In other words, 1 is our lower bound for values of a that produce *one intersection* between $f(x)$ and $g(x)$.

Through logical reasoning, we can also show that when $f(x)$ and $g(x)$ intersect once, they must also intersect a second time. Note that when considering when $f(x)$ and $g(x)$ have one intersection, this is disregarding the case when $f(x)$ and $g(x)$ are tangent (as if they are tangent, they will not intersect another time). Hence, for **2.2 Lower Bound**, when $f(x)$ and $g(x)$ are referred to as having one intersection, we are not considering the case where they are tangent.

To show that they must intersect a second time if they intersect once, we can consider the nature of the graphs of $f(x)$ and $g(x)$. $g(x)$ is an exponential function that, as the name suggests, increases exponentially as x increases. On the other hand, $f(x)$ is a logarithmic function that does not increase exponentially as x increases—it increases at a rate that is slower than an exponential function because it is logarithmic. Because $g(x)$ has a greater rate of change than $f(x)$, we know that in the long run the outputs of $g(x)$ will eventually be greater than the outputs of $f(x)$. Hence, even if for some values of $a > 1$ the outputs of $g(x)$ are smaller than the outputs of $f(x)$ for a certain set of x-values after the first time $f(x)$ and $g(x)$ intersect, we know that eventually this will change so that outputs of $g(x)$ are greater than the outputs of $f(x)$. Because we know that the function producing greater outputs will eventually have to switch, there must be an intersection between them at some point (as we reasoned at the beginning of this section). Therefore, we can confirm that if $f(x)$ and $g(x)$ intersect once (not including tangency) for $a > 1$, they must always intersect another time.

Therefore, we have shown that for the same values of a that let $f(x)$ and $g(x)$ intersect once also make them intersect another time. Thus, **1 must also mark the lower bound of when our functions begin to have two intersections.** We know that our lower bound is $a > 1$, but we still must find the upper bound on a for values of a that produce two intersections.

2.2 Upper Bound

Now that we have a lower bound on the values of a that produce 2 intersections, we must look for an upper bound. To do this, we can look for the value of a when $f(x)$ and $g(x)$ are tangent to each other. The reason why this is useful to us is because of the nature of the graphs of $f(x)$ and $g(x)$. For certain values of a that are greater than 1, the functions appear to intersect twice, as shown in Figure 4. As we increase the value of a, there appears to be some value of a that causes $f(x)$ and $g(x)$ to be tangent, as such:

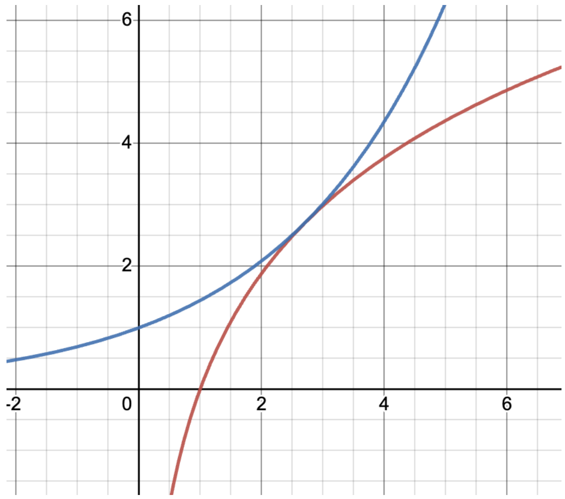


Figure 5: Graph of $f(x)$ and $g(x)$ when they are tangent

As we increase a even more, the graphs of $f(x)$ and $g(x)$ appear to have no intersections at all. In this case, the graphs of $f(x)$ and $g(x)$ resemble this:

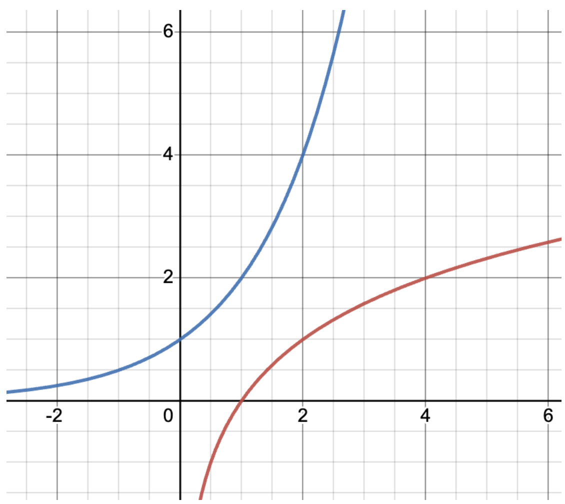


Figure 6: Graph of $f(x)$ and $g(x)$ when they have no intersections

Therefore, the value of a when $f(x)$ and $g(x)$ are tangent to each other is the upper bound of a. This is because the values of a from our lower bound ($a = 1$) to the value of a when the graphs are tangent produce two intersections between $f(x)$ and $g(x)$. We know that this is true because of what we know about the nature of exponential and logarithmic functions. As a increases past 1, $g(x) = a^x$ becomes steeper and steeper, causing the bottom of the graph to flatten out and a sharper bend in the curve. The graph of $f(x) = \log_a x$ will also get steeper, causing the top of the graph to flatten out and a sharper bend in the curve to be formed as well. Hence, for certain values of $a > 1$, two intersections between $f(x)$ and $g(x)$ will be formed, as the vertical distance in the "gap" between these two functions will increase.

However, we can see that this rule does not hold true for all $a > 1$. When a is greater than the value of a that causes $f(x)$ and $g(x)$ to be tangent, $f(x)$ and $g(x)$ have no intersections with one another. Hence, the value of a when the two functions are tangent can be interpreted as the "first" value of a where $a > 1$ such that there are *not* two intersections between the functions. Our goal, then, is to find the point at which $f(x)$ and $g(x)$ are tangent and to use this to find the value of a for which they are tangent.

To find this point of tangency, we can consider the properties of an exponential and logarithmic function. Specifically, we can note that $f(x) = \log_a x$ and $g(x) = a^x$ are inverses of each other. Because they are inverses, they are reflections of each other across the line $y = x$. Thus, if $f(x)$ and $g(x)$ were to intersect each other at exactly one point, it would be on the line $y = x$. Hence, we are trying to find the point at which the line $y = x$ intersects with $f(x)$ and $g(x)$.

To find the point at which they are tangent, we can find the derivative of $f(x)$ and $g(x)$. The derivative tells us the slope of the tangent line of a function in terms of x. We know that the line that is tangent to both $f(x)$ and $g(x)$ is $y = x$ (which has a slope of 1), so we want the derivatives of $f(x)$ and $g(x)$ to equal 1. We can start by finding the derivative of $g(x) = a^x$. To do this, we can rewrite a^x.

Using exponent rules, we can say that

$$y = a^x = e^{\ln(a^x)} = e^{x(\ln(a))}$$

Now, to find the derivative of $e^{x(\ln(a))}$, we can use the chain rule. We can let the function $u = x(\ln(a))$ and $y = e^u$. When we are finding the derivative, we are looking at the change in y in respect to the change in x, so we need to differentiate y with respect to x. In other words, we need to find $\frac{dy}{dx}$. In this scenario, the chain rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The value of $\frac{dy}{du}$ is simply the change in the function y with respect to u, hence the value of $\frac{dy}{du}$ is just e^u. The value of $\frac{du}{dx}$ is the change in u with respect to x, hence the value of $\frac{du}{dx}$ is just $\ln(a)$. Plugging these values back into the equation, we have:

$$\frac{dy}{dx} = e^u \cdot \ln(a) = e^{x(\ln(a))} \cdot \ln(a) = a^x \ln(a)$$

This means that the slope of the tangent line to $g(x) = a^x$ is $a^x \ln(a)$. We want this slope to be 1 (as the slope of $y = x$ is 1), so we can say

$$a^x \ln(a) = 1$$

Now, we need to find the derivative of $f(x) = \log_a x$. First, we can rewrite $\log_a x$ using the change of base formula:

$$\log_a x = \frac{\ln(x)}{\ln(a)}$$

Now, we need to find $\frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right)$. To find this value, we can use the derivative quotient rule, which is

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Using this rule, we have

$$\frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{\ln(x)'\ln(a) - \ln(x)\ln(a)'}{\ln^2(a)}$$

Using more derivative rules, we know that $\ln(x)' = \frac{1}{x}$ and $\ln(a)' = 0$. Using these values in our equation, we have

$$\frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{\frac{\ln(a)}{x} - 0}{\ln^2(a)} = \frac{\ln(a)}{x(\ln^2(a))} = \frac{1}{x(\ln(a))}$$

Therefore the slope of the tangent line to $f(x) = \log_a x$ is $\frac{1}{x(\ln(a))}$. We want this tangent line to be $y = x$, which has a slope of 1, so we can say

$$\frac{1}{x(\ln(a))} = 1$$

Now that we have an equation representing the slope of $f(x)$ and $g(x)$ in terms of x, we want to solve for a in terms of x for each of these equations. Then, we can set the resulting equations equal to each other to find the value of x when $f(x)$ and $g(x)$ are tangent. The issue here is that it would be difficult to solve for a in terms of x given the current set of tools we have. However, there is a way to find a without solving for a in terms of x. We can consider substituting some constant value for x in each equation. Then, we could solve for a in each equation. If the values of a that we get from each equation are equivalent, then we know that this value of a is what allows $f(x)$ and $g(x)$ to be tangent to a line with slope 1 at the same x-coordinate that we substituted for x at the start. However, if the values of a are different, it means that $f(x)$ and $g(x)$ are tangent to a line with slope 1 at this x-coordinate when the value of a in both of these functions is different. We cannot have them be tangent at this x-coordinate for different values of a (as a must be the same in both functions), so we are looking for when the values of a are equivalent.

For us to solve for the point of tangency and a in this way, we need to find some value that we could substitute for x in each equation. If we look back at Figure 5, where $f(x)$ and $g(x)$ appear to be tangent, it appears that their point of tangency has an x-coordinate of about 2.71, or about e. Hence, we can try substituting e for x in each equation. We can start by substituting it into our first equation, $a^x \ln(a) = 1$:

$$a^e \ln(a) = 1$$

Now, we can solve this equation for a. We can start by rewriting this as

$$a^e \cdot \log_e a = 1$$

We know that a^e and $\log_e a$ are inverses and, when multiplied, equal 1. Therefore a^e and $\log_e a$ must be reciprocals of each other. The only case in which these two are reciprocals is when $a = e^{\frac{1}{e}}$, as then a^e would equal e and $\log_e a$ would equal $\frac{1}{e}$. This is shown below:

$$(e)^{\frac{1}{e} \cdot e} \cdot \log_e \left(e^{\frac{1}{e}}\right) = 1$$

$$e \cdot \frac{1}{e} = 1$$

$$1 = 1$$

Therefore, we can conclude that $a = (e)^{\frac{1}{e}}$ is the solution to this equation. We can also conclude that this is the *only* solution to this equation because there no other values of a for which a^x and $\log_e a$ are reciprocals.

Now, we can substitute e for x in our second equation, $\frac{1}{x(\ln(a))} = 1$, then solve for a:

$$\frac{1}{e(\ln(a))} = 1$$

$$e(\ln(a)) = 1$$

$$\log_e a = \frac{1}{e}$$

Rewriting this logarithm as an exponential equation, we have

$$(e)^{\frac{1}{e}} = a$$

Therefore, we have found that the value of a that satisfies our second equation is the same as the value of a that satisfies our first equation ($a = (e)^{\frac{1}{e}}$). We have found that when both $f(x)$ and $g(x)$ have the same a value (which they always do), they are tangent to a line with slope 1 at a point with x-coordinate e. However, a restriction here is that we know that each function is tangent to a line with slope 1 at the x-coordinate e, but we do not know that both functions are tangent to the *same* line with slope 1. If they are tangent to the same line with slope 1, the functions will be tangent to each other. This line must be $y = x$ because $f(x)$ and $g(x)$ are inverses (and therefore reflections of each other across the line $y = x$), so their point of tangency cannot be anywhere but on the line $y = x$. Otherwise, they would no longer be reflections of each other across $y = x$. To check whether they are tangent to the same line (and therefore to each other), we can substitute $(e)^{\frac{1}{e}}$ for a and e for x into both $f(x)$ and $g(x)$ and then solve. We can start by substituting into $g(x)$:

$$g(e) = (e)^{\frac{1}{e}\cdot e} = e$$

Therefore, the y-coordinate is e when x-coordinate is e in the function $g(x)$. In other words, $g(x)$ is tangent to the line with slope 1 at the point (e, e).

Now, we can substitute $(e)^{\frac{1}{e}}$ for a and e for x into both $f(x)$ and then solve:

$$f(e) = \log_{e^{\frac{1}{e}}}(e) = e$$

Therefore, the y-coordinate is e when x-coordinate is e in the function $f(x)$. Or, $f(x)$ is also tangent to a line with slope 1 at the point (e, e).

If $f(x)$ and $g(x)$ are both tangent to a line with slope 1 at the same point, then they are therefore tangent to each other. We know that $f(x)$ and $g(x)$ can only be tangent at this one point, as from what we have reasoned earlier, the nature of the graphs of these functions as a changes means they can either have two intersections, be tangent, or have no intersections (for $a > 1$). Now that we have found that $a = (e)^{\frac{1}{e}}$ when $f(x)$ and $g(x)$ are tangent, we know that $a = (e)^{\frac{1}{e}}$ is the "first" a value (for $a > 1$) for which $f(x)$ and $g(x)$ do not have two intersections.

Hence, we can conclude that $(e)^{\frac{1}{e}}$, or $\sqrt[e]{e}$, is the upper bound of values of a that produce two intersections.

2.3 Analysis

Thus, between $a = 1$ and $a = \sqrt[e]{e}$, the functions $f(x) = \log_a(x)$ and $g(x) = a^x$ intersect twice. This is because of the nature of exponential and logarithmic graphs: exponential functions increase from small positive y-values to very large positive $y-$values for positive x (we are only considering $g(x) = a^x$ and $f(x) = \log_a(x)$ for $x > 0$ and $a > 0$). Contrarily, logarithmic functions increase to negative y-values for positive $x-$values. Because of this fact, it can be deduced that, for some positive x before the intersection of the two functions, the value of the exponential function exceeds the value of the logarithmic function, then the logarithmic function exceeds the value of the exponential function, then the exponential function finally exceeds the value of the logarithmic function. We observed identifiable bounds for these excessive cases, which appeared to be at $a = 1$ and $a = \sqrt[e]{e}$. Then, taking into account these values of a, we constructed vertical lines at $x = 1$ and $x = e$ to display this pattern of up-down, down-up, then up-down again, observing these lines' intersections with the functions $f(x)$ and $g(x)$. For values after $a = 1$, the exponential and logarithmic functions switch places, in terms of values of the function, three times, thus satisfying the observed pattern. This solidified the observed lower bound, which appeared to be at $a = 1$, and was proven, in **2.1 Lower Bound** to be $a = 1$. The upper bound was found to be $a = \sqrt[e]{e}$, because the point of tangency of the two functions $f(x)$ and $g(x)$ with $y = x$ is located at $x = e$. From this information, we found for what value of a did the logarithm exceed the exponential, which turned out to be $a = \sqrt[e]{e}$. Hence, the values of a that allow $f(x)$ and $g(x)$ to have two intersections are a between $(1, \sqrt[e]{e})$.

3 Extension

After exploring the intersections of $g(x) = a^x$ and $f(x) = \log_a x$, we noticed some peculiarities when these functions are transformed. One especially interesting one occurs when the logarithm function is transformed up by e units, as in, $f(x) = \log_a x + e$. If we consider our original functions, the value $x = 1.3$ would render a graph as follows:

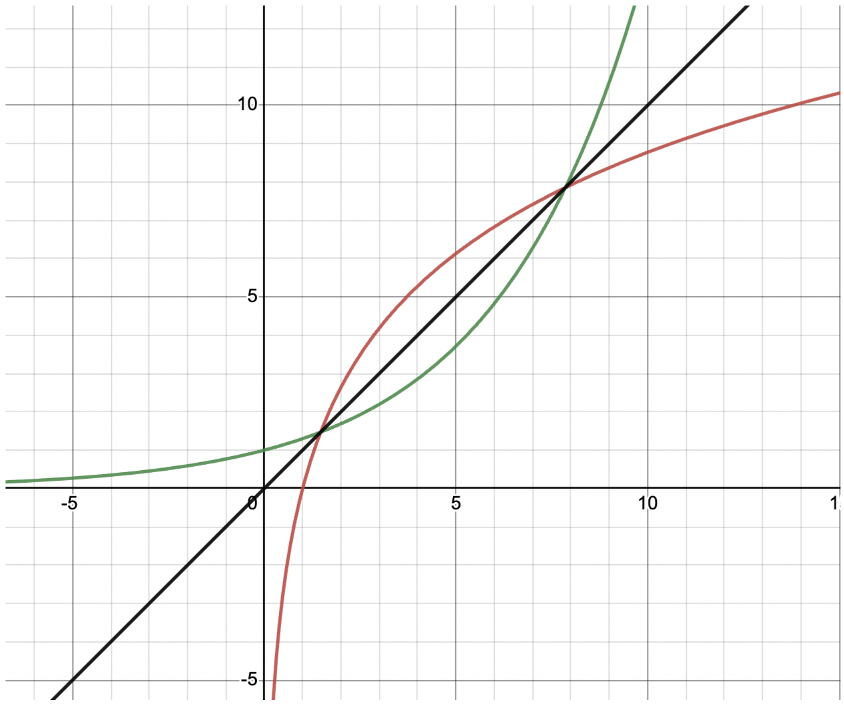


Figure 7: $f(x) = \log_a 1.3$ and $g(x) = a^{1.3}$

However, the graphs of $f(x) = \log_a x + e$ and $g(x) = a^x$ render a unique graph:

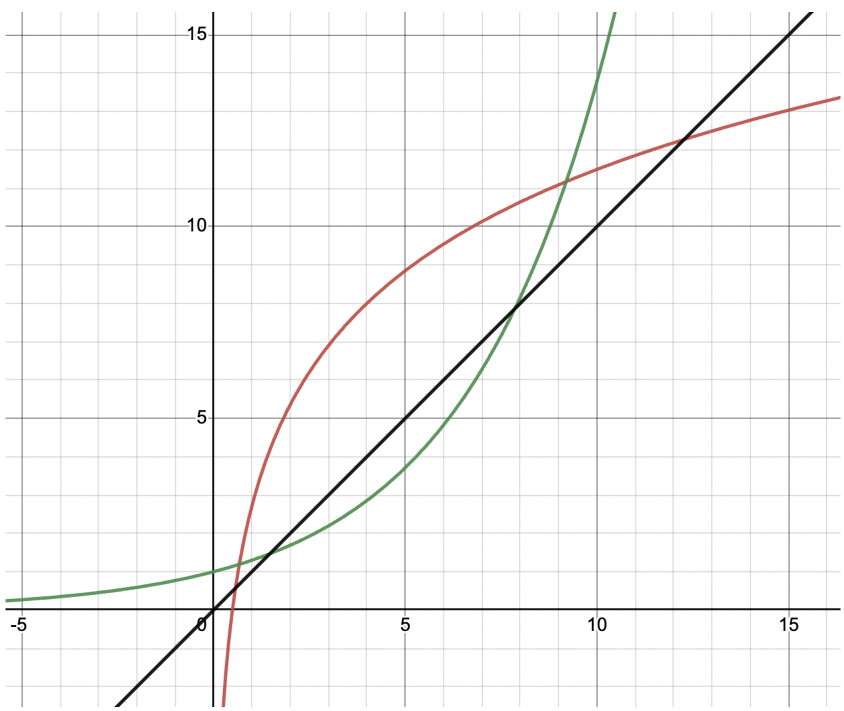
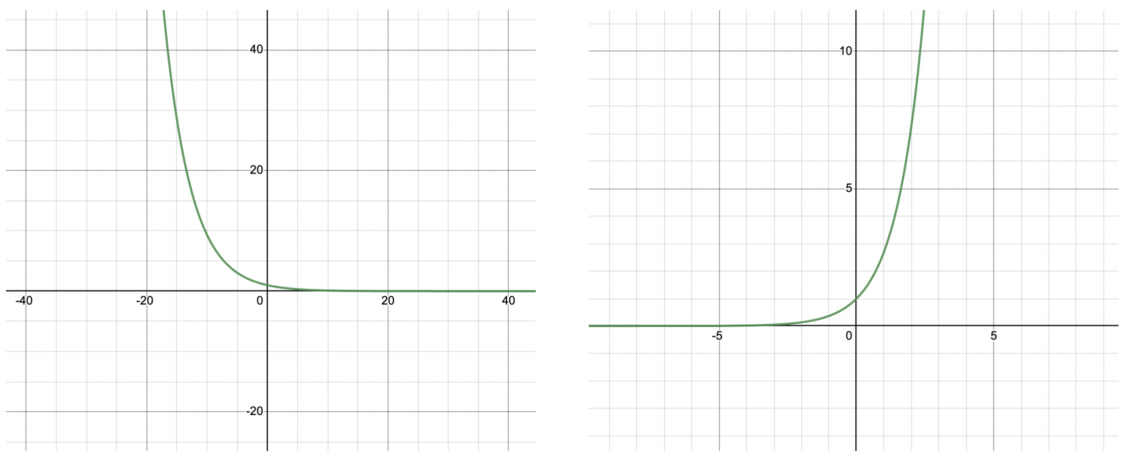


Figure 8: $f(x) = \log_a 1.3 + e$ and $g(x) = a^{1.3}$

If we consider the logarithmic functions, we can see that the translation up by e units marked a change between our graphs, making it so they are longer inverses to each other. If we compare this function with our original function, we can consider that our approach would no longer work. While we would be able to use our approach to find two intersections of this set of function by constructing three lines, we can't find the upper limit using derivatives anymore since our functions are no longer inverses and don't have the line $y = x$ going through their intersections. Thus, we must consider how to alter our original approach to understand **for what values do the functions $f(x) = log_a x + e$ and $g(x) = a^x$ have two intersections**.

3.1 Lower Bound

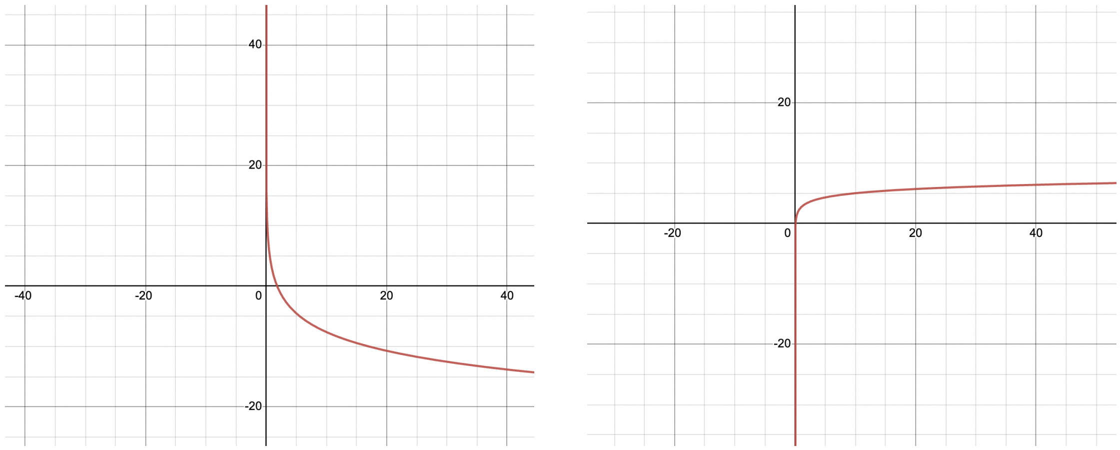
To consider the value at which our function starts intersecting, we can consider the lower bound we found in our previous problem, $a = 1$. This is the point at which we reasoned that the logarithmic functions starts being greater than the exponential function for all values of a. This would constitute the lower bound of our intersection area since for two intersections to occur, we'd need to the log to first be less then the exponential, and then greater than it and than less than it again. If we consider when $a < 1$, our exponential function goes from producing high outputs to increasingly smaller outputs. If $a > 1$, then our exponential will have very small outputs but then increasing to have very great outputs. Thus, the shape of the exponential change to go from an high values to low and then to low values to high:



(a) Graph of $a < 1$ (b) Graph of $a > 1$

Figure 9: The shape of $g(x) = a^x$

If we consider the shape of our logarithm, due to the small base, when $a < 1$ the values of the log would go from extremely high to very small. When $a > 1$, the base would get bigger and thus the graph would flip and have extremely small output increase into higher outputs as x increases.



(a) Graph of $a < 1$ (b) Graph of $a > 1$

Figure 10: The shape of $f(x) = \log_a(x)$

Considering these ideas, when $a < 1$ both of functions have relatively similar shapes. Therefore, the maximum times they could intersect would be one since they would reach be leveling out at each of there ends and produce an L sort of shape and thus be simultaneously moving away from each other. When $a > 1$ then the graphs of our logarithm and our exponential don't have to same shape: our exponential is going from going to extremely small, positive outputs to extremely large positive outputs while our logarithm goes from extremely small negative outputs to greater positive outputs. Thus, the shape of the graphs are basically reflections of each other along the line $y = x$, making it so at maximum, our graphs can have two intersections which only begins to be possible when $a > 1$. Thus, the lower bound of when our functions can have two intersections must be 1, or when $a > 1$.

3.2 Upper Bound

Now that we have found a lower bound for a when $g(x) = a^x$ and $f(x) = \log_a x$ have two intersections, we can find an upper bound on a. When looking at the graph of $f(x)$ and $g(x)$, it appears that as we continue to increase a past 1, these functions never appear to *not* have two intersections. For example, here is the graph when $a = 4$:

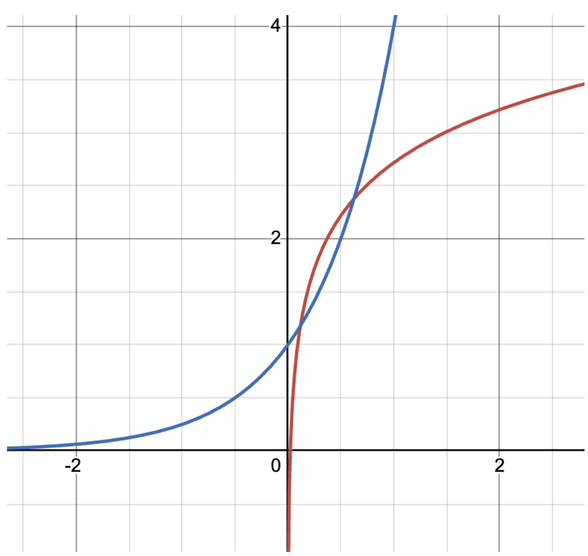


Figure 9: Graph of $g(x) = a^4$ and $f(x) = \log_a 4$

When $a = 4$, the functions appear to have two intersections. We can try increasing a to a much larger value, such as $a = 1000000$:

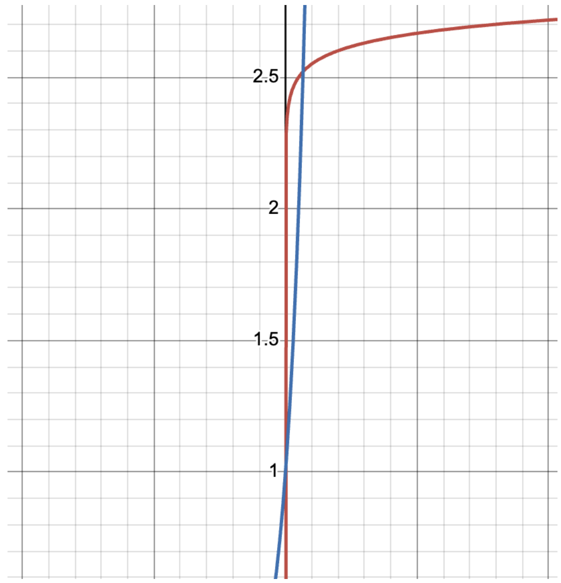


Figure 10: Graph of $g(x) = a^{1000000}$ and $f(x) = \log_a(1000000)$

Even when a is a large number, $f(x)$ $g(x)$ still have two intersections. Unlike our initial problem, there does not seem to be a clear point at which $g(x)$ and $g(x)$ *stop* having two intersections. Hence, we seek to find whether or not $f(x)$ and $g(x)$ will have two intersections for any value of a such that $a > 1$.

To do this, we can consider the derivatives of $f(x)$ and $g(x)$ that we found earlier. The derivatives of $f(x)$ and $g(x)$ tell us the slope of the line that is tangent to $f(x)$ and $g(x)$ at a certain x-coordinate. If the value of the derivative were to decrease as the value of the x-coordinate increases, it would mean the slope of the tangent line is getting smaller and smaller, or the graph of the function is leveling out horizontally. On the other hand, if the value of the derivative were to increase as the value of the x-coordinate increases, it would meant this tangent line is getting steeper and steeper, or the graph is leveling out vertically.

First, we can consider the derivative of $g(x) = a^x$. In **2.2 Upper Bound**, we found that the derivative of this was $a^x \ln(a)$. We can break this derivative down into two pieces: a^x and $\ln(a)$. Because a is some positive constant value greater than 1, we know that $\ln(a)$ is also some positive constant value. The value of a^x will change exponentially as x gets larger because a^x is an exponential function. Like any exponential function, if the value of the base of the exponential (in this case a) is between 0 and 1, we know that it will be an exponential decay function. For bases that are greater than 1, it will be an exponential growth function. In our scenario, we are only concerned with values of a that are greater than 1 (as 1 is our lower bound), and hence we can confirm that a^x is increasing exponentially. $a^x \ln(a)$, then, is an expression where an exponentially increasing value is being multiplied by a constant. Regardless of what this constant is (as it is negligible compared to an exponentially increasing value), the value of $a^x \ln(a)$ will always increase as x increases. In other words, like stated earlier, the line tangent to $g(x) = a^x$ will get steeper and steeper as x increases, meaning the graph of $g(x)$ is leveling out vertically.

Next, we can consider the derivative of $f(x) = \log_a x + e$. Using the derivative law

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

we can say

$$\frac{d}{dx}(\log_a x + e) = \frac{d}{dx}(\log_a x) + \frac{d}{dx}(e)$$

From our work in **2.2 Upper Bound**, we know that the derivative of $\log_a x$ is $\frac{1}{x(\ln(a))}$. We also know that the derivative of a constant is 0, so $\frac{d}{dx}(e) = 0$. Hence, the derivative of $\log_a x + e$ is just $\frac{1}{x(\ln(a))}$.

We can consider the denominator of this derivative in two pieces: x and $\ln(a)$. Because a is some constant value greater than 1, the value of $\ln(a)$ will always be some constant value. Hence, as we increase the value of x, the value of $x(\ln(a))$ becomes greater and greater. When we are dividing 1 by $x(\ln(a))$ as x increases, we are dividing 1 by a number that is increasing. Dividing 1 by a number that is increasing will result in quotients that are decreasing. Hence, as the value of x increases, the slope of the tangent line to $f(x)$ is decreasing, or $f(x)$ is getting less steep leveling out horizontally.

Because $g(x)$ is getting steeper and steeper and $f(x)$ is getting less and less steep as x increases, it means that eventually the outputs of $g(x)$ will be larger than the outputs of $f(x)$.

As we have shown in **2.1 Lower Bound**, after the first intersection between a logarithm and an exponential function, the outputs of the exponential function start to become less than the outputs of the logarithmic function. However, here, we have proved that eventually the outputs of this exponential will become larger than the outputs of the logarithmic function. Because the function with greater outputs will eventually switch again after the first time a logarithm and exponential intersect, it means that the functions must intersect again after the first time for this to happen. Hence, because $f(x)$ and $g(x)$ will intersect a second time (with the value of a having no bearing on this as long as $a > 1$), there is no upper bound on a. For all values of a greater than 1, there will be two intersections between $f(x)$ and $g(x)$. In other words, the values of a that will produce two intersections are between $(1, \infty)$.

3.3 Analysis

We have thus derived the upper and lower bounds of the values of a for which $f(x) = \log_a x + e$ and $g(x) = a^x$ intersect. Namely, the lower bound, as reasoned in **section 3.1 Lower Bound**, occurs at $a = 1$ because, for this value, the exponential function is going from small, positive outputs to really large positive outputs very quickly, while the logarithm goes from small negative outputs to larger positive outputs. This process only starts occurring at $a > 1$, and continues until $a = \infty$, as we proved with our construction of lines in **2.1 Lower Bound**. This is because, for all values of a such that $a > 1$, the graph of the logarithm is above the graph of the exponential – however, as discussed and proven in **3.2 Upper Bound** eventually, the graph of the exponential becomes above the graph of the logarithm. This suggests that there are two intersections that occur for values of $a > 1$. Thus it was derived that the lower bound for which two intersections between $f(x) = \log_a x + e$ and $g(x) = a^x$ occurs was $a > 1$, and the upper bound was none; or, in other words, $a = \infty$.

3.4 Further Inquiries

To further explore this field of study – namely, intersections between logarithmic and exponential equations, – one may consider applying other transformations to each function. Transformations may include, but are not limited to, the following:

$$f(x) = e \log_a x,\ g(x) = ea^x$$

$$f(x) = \frac{\log_a x}{e},\ g(x) = \frac{a^x}{e}$$

$$f(x) = \log_a(x + e),\ g(x) = a^{ex}$$

Or, one may consider more generalized transformations, such as the following:

$$f(x) = \log_a x + a,\ g(x) = a^x + a$$

$$f(x) = \frac{\log_a x}{a},\ g(x) = \frac{a^x}{a}$$

for the same arbitrary constant a.

Exploring the intersections of transformed functions could pose more interesting discoveries, just as transforming $f(x)$ with a vertical transformation up by e units did.

Another potential avenue of further exploration may include studying different aspects of previously-explored functions; for example, procuring an equation for the intersections of $f(x) = \log_a x + e$ and $g(x) = a^x$ when $a > 1$ could pose an intriguing and challenging exercise.