Homework 7

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Problem 1: Find all solutions to the equation $\log_2(x+4) + \log_2|2x-5| = \log_2|2x+3|$.

Solution: We can apply the logarithmic rule $\log_a x + \log_a y = \log_a(xy)$:

$$\log_2((x+4)|2x-5|) = \log_2|2x+3|$$

Because logarithmic functions are one to one, we can rewrite this as

$$(x+4)|2x-5| = |2x+3|$$

We now have two options. The value of $(2x+3)$ could either be negative or positive. If we first consider the case where it is positive, our equation will be

$$(x+4)|2x-5| = 2x+3$$

We now have an equation where a binomial is multiplied by an absolute value expression that produces some positive value. $(2x-5)$ is in absolute value, so the value of $(2x-5)$ can be either negative or positive. We can first take the case where $(2x-5)$ is also positive. If $2x+3 \geq 0$ and $2x-5 \geq 0$, then we can solve these to find restrictions on our domain. Solving these, we have $x \geq -\frac{3}{2}$ and $x \geq \frac{5}{2}$. $\frac{5}{2}$ is greater than $-\frac{3}{2}$, so our domain restriction is just $x \geq \frac{5}{2}$.

Now that we have a domain restriction on x, we can solve this equation by dropping the absolute value signs:

$$(x+4)(2x-5) = 2x+3$$

Solving for x:

$$x = \frac{-1 \pm \sqrt{185}}{4}$$

These values of x satisfy the domain $x \geq \frac{5}{2}$, so we can conclude that $\frac{-1+\sqrt{185}}{4}$ and $\frac{-1-\sqrt{185}}{4}$ are solutions to this equation.

We now have to consider the case where the value of $(2x+3)$ is positive and $(2x-5)$ is negative. If $(2x+3) \geq 0$, we have $x \geq -\frac{3}{2}$. If $(2x-5) \leq 0$, we have $x \leq \frac{5}{2}$. This means the domain of this function is between these two values, or $-\frac{3}{2} \leq x \leq \frac{5}{2}$. Now that we have these domain restrictions, we can drop the absolute value signs. We also have to keep in mind that $(2x-5)$ is negative, so we have to multiply it by -1 to reflect that:

$$-(x+4)(2x-5) = (2x+3)$$

Solving for x:

$$x = \frac{-5-\sqrt{161}}{4} \approx -4.42,\ x = \frac{-5+\sqrt{161}}{4} \approx 1.92$$

Our domain restriction is that $-\frac{3}{2} \leq x \leq \frac{5}{2}$ and we have found that $\frac{-5-\sqrt{161}}{4}$ is a value that is not between these two values. Therefore this is an extraneous solution and can be disregarded. We can conclude that $\frac{-5+\sqrt{161}}{4}$ is a solution to this equation.

Now, we have to consider the case where $(2x+3)$ is negative. We have to remember that $(2x-5)$ is also in absolute value, so $(2x-5)$ can be either negative or positive. We can first take the case where $(2x+3)$ is negative and $(2x-5)$ is positive. If $(2x+3)$ is negative, when we drop the absolute value signs to solve the equation we will need to multiply $(2x+3)$ by -1. Doing this will make our equation

$$(x+4)(2x-5) = -(2x+3)$$

which is equivalent to the equation we found when $(2x+3)$ was positive and $(2x-5)$ was negative, which was

$$-(x+4)(2x-5) = (2x+3)$$

This means the solutions to $(x+4)(2x-5) = -(2x+3)$ will be the same as that of the equation when $(2x+3)$ is positive and $(2x-5)$ is negative. This means we have already accounted for the solutions of this equation and we don't have to solve it.

Lastly, we have to consider the case wehre $(2x+3)$ is negative and $(2x-5)$ is negative. If they are both negative, then when we drop the absolute value signs our equation will be

$$(x+4)(2x-5) = (2x+3)$$

This is the exact same equation we found when $(2x+3)$ and $(2x-5)$ were both positive, which means we have already accounted for the solutions to this equation. Therefore, we can conclude that the solutions to this equation are $\boxed{\frac{-1+\sqrt{185}}{4}, \frac{-1-\sqrt{185}}{4}, \text{ and } \frac{-5+\sqrt{161}}{4}.}$

Problem 2: Find all solutions to the equation $\log_2 \sqrt{v^2+3} + \log_4 (v^2-7) = 1$.

Solution: We can start by converting the logarithms in this function so that they have the same base. We can convert them so that they both have a base of 2. To convert $\log_4 (v^2-7)$ into a logarithm with base 2, we can use the change of base formula

$$\log_a b = \frac{\log_c b}{\log_c a}$$

to rewrite $\log_4 (v^2-7)$ as

$$\frac{\log_2 (v^2-7)}{\log_2 4} = \frac{1}{2} \log_2 (v^2-7)$$

Using the rule $a \log_c b = \log_c (b)^a$, this expression can be rewritten as

$$\log_2 \left((v^2-7)^{\frac{1}{2}} \right)$$

Rewriting the original equation using this expression, we have

$$\log_2 \sqrt{v^2+3} + \log_2 \left((v^2-7)^{\frac{1}{2}} \right) = 1$$

The number 1 can be expressed as a logarithm with base 2 as well. We know that $\log_2 2 = 1$, so we can rewrite the original equation substitute $\log_2 2$ for 1:

$$\log_2 \sqrt{v^2+3} + \log_2 \left((v^2-7)^{\frac{1}{2}} \right) = \log_2 2$$

Using the logarithmic rule $\log_a x + \log_a y = \log_a (xy)$, this can be rewritten as

$$\log_2 \left(\left(\sqrt{v^2+3} \right) (v^2-7)^{\frac{1}{2}} \right) = \log_2 2$$

Because logarithms are one to one, this can be rewritten as

$$\left(\sqrt{v^2+3}\right)\left(v^2-7\right)^{\frac{1}{2}}=2$$

Taking both sides of the equation to the power of 2:

$$\left(v^2+3\right)\left(v^2-7\right)=4$$

To solve this equation, we can let $x=v^2$. Rewriting this equation using x:

$$(x+3)(x-7)=4$$
$$x=2+\sqrt{29},\ x=2-\sqrt{29}$$

Therefore, we can say that $v^2=2+\sqrt{29}$ and $v^2=2-\sqrt{29}$. Or, that $v=\pm\sqrt{2+\sqrt{29}}$ and $v=\pm\sqrt{2-\sqrt{29}}$. However, we must keep in mind that we cannot take the square root of a negative number, so the value of the expression under the square root must be greater than or equal to 0. We know that $2-\sqrt{29}$ is some negative value, so $2-\sqrt{29}$ cannot be an expression under a square root. As a result, the soltuion $v=\pm\sqrt{2-\sqrt{29}}$ is extraneous. Therefore the solution to this equation is $\boxed{v=\pm\sqrt{2+\sqrt{29}}.}$

Problem 3: True or false: When an exponential function is translated horizontally by any amount, the result is still an exponential function.

Solution: If this statement is true, then when any function written in standard exponential form is translated some number of units horizontally, the resulting function can also be written in standard exponential form. To prove this, we can consider the standard exponential function which is $f(x)=a(b)^x$. We can then translate this function c units horizontally (where c is an arbitrary constant) by adding c to the input of the function. Our translated function would then be $f(x+c)=a(b)^{x+c}$. Now, we can try to write this translated function in standard exponential form using exponent rules:

$$f(x+c)=a(b)^{x+c}=a\left(b^x\cdot b^c\right)=\left(a\cdot b^c\right)(b)^x$$

We know that a, b, and c are all arbitrary constants. Therefore, the value of $\left(a\cdot b^c\right)$ will also be some arbitrary constant. Therefore, in our translated function, we have some arbitrary constant multiplied by b^x. In the standard form of an exponential function, we also have some arbitrary constant a multiplied by b^x. This proves that translating an exponential function by some number of units horizontally still allows us to write the resulting function in standard exponential form (some constant multiplied by b^x.) Therefore, the answer is $\boxed{\text{true.}}$

Problem 4: True or false: When a function of the form $f(x)=\log_a(x)$ for constant a is dilated vertically by any (positive) factor, the result is another function of that form.

Solution: This function only accepts real numbers as inputs. In other words, x is a variable, but represents real numbers that can be plugged into this function. This function, then, is a logarithm with some real number base and some real number argument.

To show a vertical dilation of this function, we can let this function be dilated by a factor of b and write the translated function as $b(f(x))=b\log_a(x)$. Using a log rule, this is the equivalent of $b(f(x))=\log_a\left(x^b\right)$. b is some positive real number, so we know that when we plug in some real number for x, the value of x^b will also be some real number. Therefore, our translated function is also a logarithm with some real number base and real number argument. This means that a vertical dilation to $f(x)=\log_a(x)$ can always as a function of the same form. The answer is $\boxed{\text{true.}}$