Homework 8

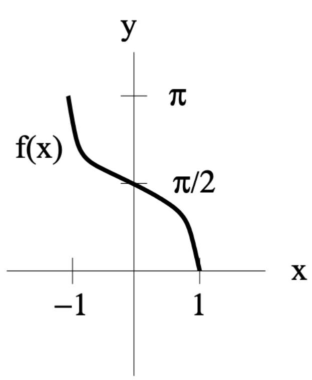
Sahana Sarangi

26 February 2024

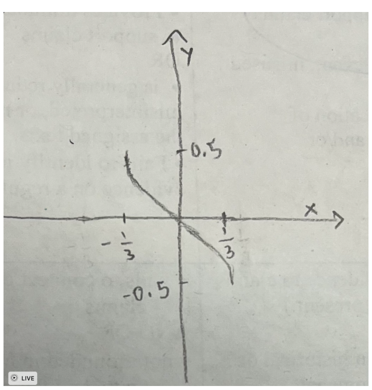
Problem 13.3: The graph of a function $y = f(x)$ is pictured with domain $-1 \leq x \leq 1$. Sketch the graph of the new function

$$y = g(x) = \frac{1}{\pi} f(3x) - 0.5$$

Find the largest possible domain of the function $y = \sqrt{g(x)}$.



Solution: The graph of $g(x)$ will be



To find the largest possible domain of $\sqrt{g(x)}$, we have to consider the fact that $g(x)$ is under a square root. We know that we cannot have any negative values under a square root, so we can say $g(x) \geq 0$. The original domain of $f(x)$ was $-1 \leq x \leq 1$. After undergoing a horizontal compression by a factor of 3, the domain of $g(x)$ becomes $-\frac{1}{3} \leq x \leq \frac{1}{3}$. Also after undergoing transformations, the graph of $g(x)$ intersects the y-axis at the origin and produces only positive values for x-values less than 0. Therefore, for $g(x) \geq 0$, x must be less than 0. Therefore the largest possible domain of $y = \sqrt{g(x)}$ is $\boxed{x < 0.}$

Problem 13.5: Consider the function $y = f(x)$ with multipart definition

$$f(x) = \begin{cases} 0 \text{ if } x \leq -1 \\ 2x+2 \text{ if } -1 \leq x \leq 0 \\ -x+2 \text{ if } 0 \leq x \leq 2 \\ 0 \text{ if } x \geq 2 \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

(b) Is $y = f(x)$ an even function? Is $y = f(x)$ an odd function? (A function $y = f(x)$ is called even if $f(x) = f(-x)$ for all x in the domain. A function $y = f(x)$ is called odd if $f(-x) = -f(x)$ for all x in the domain.)

(c) Sketch the reflection of the graph across the x-axis and y-axis. Obtain the resulting multipart equations for these reflected curves.

(d) Sketch the vertical dilations $y = 2f(x)$ and $y = \frac{1}{2}f(x)$.

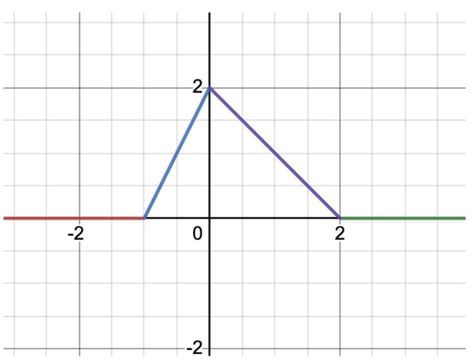
(e) Sketch the horizontal dilations $y = f(2x)$ and $y = f(\frac{1}{2}x)$.

(f) Find a number $c > 0$ so that the highest point on the graph of the vertical dilation $y = cf(x)$ has y-coordinate 11.

(g) Using horizontal dilation, find a number $c > 0$ so that the function values $f(\frac{x}{c})$ are non-zero for all $-\frac{5}{2} < x < 5$.

(h) Using horizontal dilation, find positive numbers c, $d > 0$ so that the function values $f(\frac{1}{c}(x - d))$ are non-zero precisely when $0 < x < 1$.

Part (a) Solution: The graph of $y = f(x)$ is



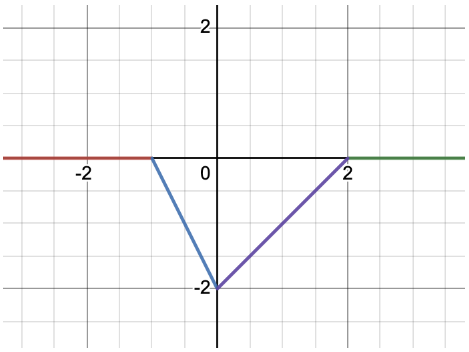
Part (b) Solution: We can first check if $f(x)$ is even. $f(x)$ is even if $f(x) = f(-x)$. In other words, it is even if $f(x)$ looks like the same function after reflecting it over the y-axis. Or, $f(x)$ is symmetric across the y-axis. $f(x)$ is not symmetric about the y-axis, so $\boxed{f(x) \text{ is not even.}}$

We can now check whether $f(x)$ is odd. If $f(x)$ is odd, then $f(x) = -f(x)$. In other words, it looks the same after reflecting it over the x-axis. Or, $f(x)$ is symmetric about the x-axis. $f(x)$ is not symmetric about the x-axis, so $\boxed{f(x) \text{ is not odd.}}$

Part (c) Solution: We can first find the multipart function for when $f(x)$ is reflected over the x-axis. None of the domain restrictions will need to be altered (as we are not changing the x-coordinates of our graph), but need to multiply each of the parts in the multipart by -1 (as the y-coordinates are being multiplied by -1). The resulting multipart would be:

$$-f(x) = \begin{cases} 0 \text{ if } x \leq -1 \\ -2x - 2 \text{ if } -1 \leq x \leq 0 \\ x - 2 \text{ if } 0 \leq x \leq 2 \\ 0 \text{ if } x \geq 2 \end{cases}$$

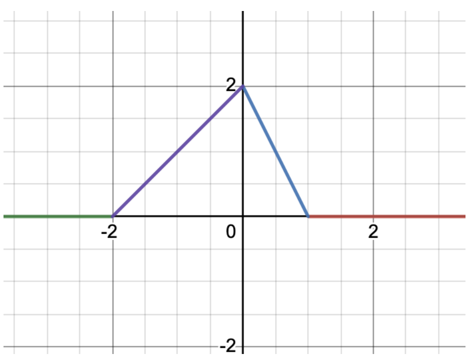
The graph of $-f(x)$ would then be:



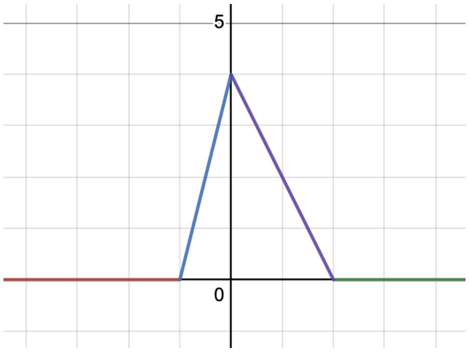
Now, we can find the multipart function for when $f(x)$ is reflected over the y-axis. When the function is reflected over the y-axis, the function becomes $f(-x$, so all of the x's in the function are replaced with $-x$. Reflecting over the y-axis also changes the domain restrictions on each part of the multipart. The domain restrictions on each multipart are multiplied by -1. Our resulting multipart, then, would be:

$$f(-x) = \begin{cases} 0 \text{ if } x \geq 1 \\ -2x + 2 \text{ if } 0 \leq x \leq 1 \\ x + 2 \text{ if } -2 \leq x \leq 0 \\ 0 \text{ if } x \leq -2 \end{cases}$$

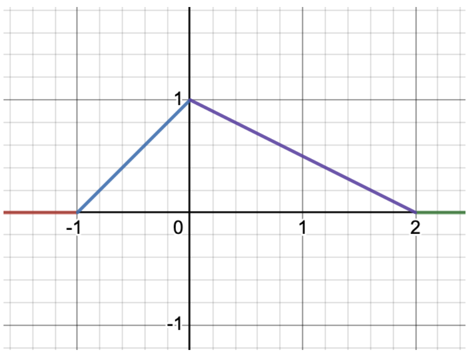
We can graph this to find the graph of $f(-x)$ which is:



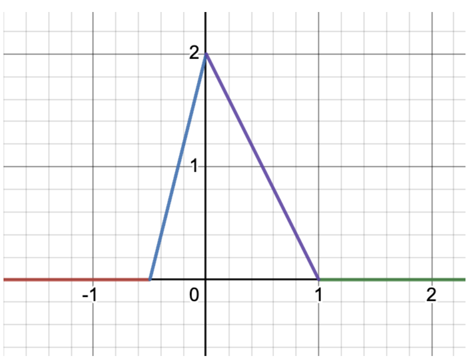
Part (d) Solution: The graph of $2f(x)$ would be the same as the graph of $f(x)$, but stretched vertically by a factor of 2. All of the outputs of $2f(x)$ would be double that of $f(x)$. The graph would be



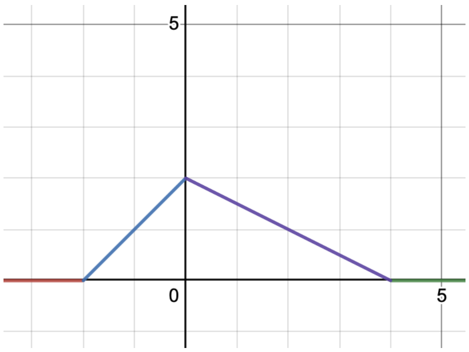
The graph of $\frac{1}{2}f(x)$ would be the same as the graph of $f(x)$, but compressed vertically by a factor of 2. All of the outputs of $\frac{1}{2}f(x)$ would be half that of $f(x)$. The graph would be



Part (e) Solution: The graph of $f(2x)$ would be the same as the graph of $f(x)$, but compressed horizontally by a factor of 2. All of the x-coordinates in the graph of $f(2x)$ would be half that of $f(x)$. The graph would be



The graph of $f(\frac{1}{2}x)$ would be the same as the graph of $f(x)$, but stretched horizontally by a factor of 2. All of the x-coordinates in the graph of $f(\frac{1}{2}x)$ would be double that of $f(x)$. The graph would be



Part (f) Solution: In the graph of $f(x)$, the highest point on the graph has a y-coordinate of 2. We know that when a function is dilated vertically by some factor c, all of the y-coordinates in the function get multiplied by a factor of c. Therefore, if dilating $f(x)$ by a factor of c results in the highest y-coordinate being 11 instead of 2, then c must be $\frac{11}{2}$.

Part (g) Solution: The nonzero domain of $f(x)$ is $(-1, 2)$. The question is asking for a value of c such that the nonzero domain will stretch to $\left(\frac{5}{2}, 5\right)$. We have to remember that we are multiplying the inputs of the function by $\frac{1}{c}$. In that case, one possible value of $\frac{1}{c}$ that will stretch the domain to $\left(\frac{5}{2}, 5\right)$ is $\frac{5}{2}$. If $\frac{1}{c} = \frac{5}{2}$, the nonzero domain of $f(\frac{x}{c})$ would be $\left(-\frac{5}{2}, 5\right)$. The question does not state that the nonzero domain cannot include more values than $\left(\frac{5}{2}, 5\right)$, so $\frac{1}{c}$ could equal any number greater than or equal to $\frac{5}{2}$. If $\frac{1}{c} \geq \frac{5}{2}$, then c could be any number that satifies the inequaltiy $0 < c \leq \frac{5}{2}$.

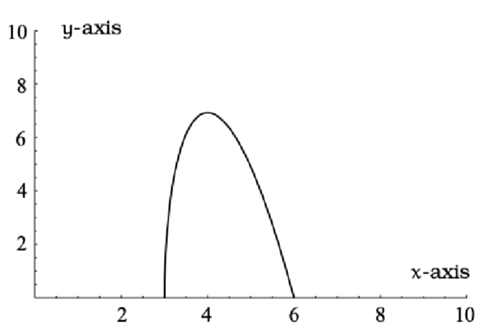
Part (h) Solution: The current nonzero domain of the function is $(-1, 2)$. If we were to move this domain 1 unit to the right, we would get the domain $(0, 3)$. If we were then to compress this domain by a factor of 3, we would have the domain $(0, 1)$, which is the domain that we want according to the question. Therefore, a horizontal translation 1 unit to the right followed by a horizontal compression by a factor of 3 will result in our domain being $(0, 1)$. This can be written as $f(3x - 1)$. We want this to be written in the format $f(\frac{1}{c}(x - d))$, so we can rewrite this as $f(3(x - \frac{1}{3}))$. This would mean $\boxed{c = d = \frac{1}{3}.}$

Problem 13.6: An isosceles triangle has sides of length x, x and y. In addition, assume the triangle has perimeter 12.

(a) Find the rule for a function that computes the area of the triangle as a function of x. Describe the largest possible domain of this function.

(b) Assume that the maximum value of the function $a(x)$ in (a) occurs when $x = 4$. Find the maximum value of $z = a(x)$ and $z = 2a(3x + 3) + 1$.

(c) The graph of $z = a(x)$ from part (a) is given below. Sketch the graph and find the rule for the function $z = 2a(3x + 3) + 1$; make sure to specify the domain and range of this new function.



Part (a) Solution: This triangle has two sides of length x and a base of length y. To find the area of this isosceles triangle, we can think of it as two right triangles created by drawing an altitutde from the top of the triangle to the base. The base of each right triangle would then have length $\frac{y}{2}$ and hypotenuse x. We can find the height of each right triangle by using Pythagorean theorem; the heights would be $\sqrt{x^2 - \frac{y^2}{4}}$.

We know that the perimeter of the whole triangle is 12 units, or $x+x+y=12$. Solving for y in terms of x, we can find that $y = 12-2x$. We can replace y with $12-2x$ to find that the height of each right triangle is $\sqrt{x^2 - \frac{(12-2x)^2}{4}}$.

Multiplying height by base, the area of each right triangle is $\frac{y}{2}\sqrt{x^2 - \frac{(12-2x)^2}{4}}$. Again substituting $12-2x$ for y, we have $\frac{12-2x}{2}\sqrt{x^2 - \frac{(12-2x)^2}{4}}$. There are two of these right triangles, so we have to multiply this expression by 2 to find that the rule for the total area of the isosceles triangle is $\boxed{a(x) = (12-2x)\sqrt{x^2 - \frac{(12-2x)^2}{4}}.}$

The only domain restriction we have to consider is the possible values that $a(x)$ can output. $a(x)$ calculates the area of the isoscles triangle, so it cannot output a negative number. The value of a square root will always be positive, but if $(12-2x)$ is negative, it will cause $a(x)$ to output a negative number. Therefore, $12-2x \geq 0$. Solving this, we can say $x \leq 6$.

We also know that any expression under a square root must be greater than or equal to 0. In this case, $x^2 - \frac{(12-2x)^2}{4} \geq 0$. Simplifying this:

$$4x^2 - (12-2x)^2 \geq 0$$

$$-144 + 48x \geq 0$$

$$x \geq 3$$

Therefore, the domain restriction on $a(x)$ is $\boxed{3 \leq x \leq 6.}$

Part (b) Solution: The maximum of $a(x)$ occurs when $x = 4$, so to find the maximum of $a(x)$ we can just substitute 4 for x in $a(x)$:

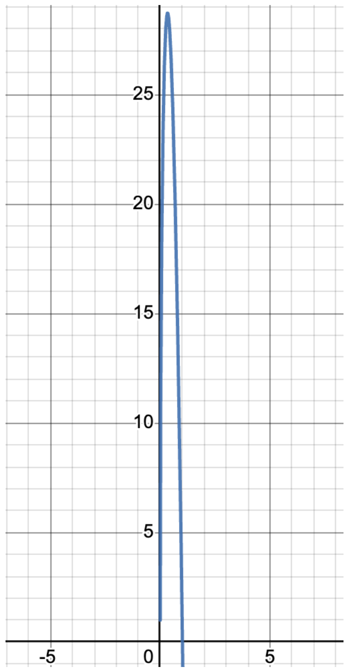
$$a(4) = (12-2(4))\sqrt{4^2 - \frac{(12-2(4))^2}{4}}$$

$$a(4) = 4\sqrt{3}$$

Therefore the maximum of $a(x)$ is at $\boxed{4\sqrt{3}.}$

To find the maximum of $2a(3x+3)+1$, we can use the maximum that we found for $a(x)$. Because we are considering the y-coordinates of the maximums, we can look at the vertical transformations being done to $a(x)$. The only vertical transformations are a vertical stretch by a factor of 2, then a shift upwards by 1 unit. To find the maximum of transformed f then, we can apply the transformations that have been done to f to the y-coordinate of the maximum of $a(x)$. Stretching $4\sqrt{3}$ by a factor of 2, we have $8\sqrt{3}$. Shifting this one unit up, we have $8\sqrt{3}+1$. Therefore, the maximum of $2a(3x+3)+1$ is $\boxed{8\sqrt{3}+1.}$

Part (c) Solution: The graph of $2a(3x+3)+1$ would be the same as that of $a(x)$, but with a horizontal shift 3 units to the left, horizontal compression by a factor of 3, vertical stretch by a factor of 2, and a vertical shift 1 unit up. The resulting graph would be:



To find the domain of this new function, we can consider the horizontal transformations it has undergone. The initial domain of $a(x)$ was $[3, 6]$. If we translate this 3 units to left, we have $[0, 3]$. Compressing this by a factor of 3, we have $\boxed{[0, 1].}$

To find the range of this function by considering the vertical transformations it has undergone. The original range of this function was from $[0, 4\sqrt{3}]$. The function $2a(3x + 3) + 1$ has undergone a vertical stretch by a factor of 2 then a translation 1 unit upward. First applying the vertical stretch, we have the range $[0, 8\sqrt{3}]$. Then, applying the translation, we have the range $\boxed{[1, 8\sqrt{3} + 1].}$

Additional Problemset Question 1: Functions can be functions of numbers, but they can also be functions of all sorts of things. In this problem, we'll consider functions of functions, commonly called "functionals".

(a) Let $\mathcal{F}$ be the function which takes in a function and compresses it horizontally by a factor of 2. For example, if $f(x) = x^2$, then $\mathcal{F}(f) = g$, where $g(x) = (2x)^2$. Is $\mathcal{F}$ a one-to-one function? If so, describe its inverse function.

(b) Let $\mathcal{G}$ be the function which takes in a function and stretches it vertically by a factor of three, and then shifts it to the right by four units. For example, if $f(x) = x^2$, then $\mathcal{G}(f) = g$, where $g(x) = 3(x - 4)^2$. Is $\mathcal{G}$ a one-to-one function? If so, describe its inverse function.

(c) Is there a function $\mathcal{H}$ so that $\mathcal{H}$ takes a transformed function and produces the parent function? For example, since $g(x) = 3(x - 4)^2$ is a transformation of $f(x) = x^2$, $\mathcal{H}(g)$ would be f.

Part (a) Solution: We can check if $\mathcal{F}$ is a one to one function by trying to find the inverse of $\mathcal{F}$. If $\mathcal{F}$ has a valid inverse, it is one to one. We know that $\mathcal{F}$ horizontally compresses functions, so its inverse must horizontally stretch functions. If $\mathcal{F}(f) = f(2x)$, then $\mathcal{F}^{-1}(f) = f(\frac{1}{2}x)$. To check whether or not this is the inverse, we can apply both $\mathcal{F}$ and $\mathcal{F}^{-1}$ to the function $f(x)$:

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))) = f\left(\frac{1}{2} \cdot 2 \cdot x\right) = f(x)$$

Because applying both $\mathcal{F}$ and $\mathcal{F}^{-1}$ to $f(x)$ gave us $f(x)$, we can conclude that $\mathcal{F}^{-1}$ is a valid inverse of $\mathcal{F}$ and therefore $\boxed{\mathcal{F} \text{ is one to one.}}$

Part (b) Solution: $\mathcal{G}$ is one to one if it has a valid inverse. We know that $\mathcal{G}$ takes a function and stretches it vertically by a factor of 3 then shifts it to the right by four units. Therefore, its inverse should compress it vertically by a factor of 3 then shifts it to the left by four units. If $\mathcal{G}(f(x)) = 3(x-4)$, then $\mathcal{G}^{-1}(f(x)) = \frac{1}{3}(x+4)$. To check whether or not this is the inverse, we can apply both $\mathcal{G}$ and $\mathcal{G}^{-1}$ to the function $f(x)$:

$$\mathcal{G}(\mathcal{G}^{-1}(f(x))) = f\left(\frac{1}{3} \cdot 3(x+4-4)\right) = f(x)$$

Because applying both $\mathcal{G}$ and $\mathcal{G}^{-1}$ to $f(x)$ gave us $f(x)$, we can conclude that $\mathcal{G}^{-1}$ is a valid inverse of $\mathcal{G}$ and $\boxed{\mathcal{G} \text{ is one to one.}}$

Part (c) Solution: $\boxed{\text{No,}}$ there is no funtion $\mathcal{H}$ that can take a transformed function and produce the parent because there are not set mathematical operations $\mathcal{H}$ could do to find the parent. In the example of the function $g(x) = 3(x-4)^2$, $\mathcal{H}$ would have to do a translation 4 units to the left and a vertical compression by a factor of 3 in order to the find the parent function. However, doing a translation 4 units to the left and a vertical compression by a factor of 3 is not going to be able to find the parent function of *any* transformed function. Therefore, $\mathcal{H}$ doesn't have the capability to take a transformed function and find the parent function.