Homework 3

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Problem 10.2: Put each equation in standard exponential form:

Part (a): $y = 3(2^{-x})$

Part (b): $y = 4^{-\frac{x}{2}}$

Part (c): $y = \pi^{\pi x}$

Part (d): $y = 1\left(\frac{1}{3}\right)^{3+\frac{x}{2}}$

Part (e): $y = \frac{5}{0.345^{2x-7}}$

Part (f): $y = 4(0.0003467)^{-0.4x+2}$

Part (a) Solution: Standard exponential form is $y = a(b)^x$. Because $a^{mn} = (a^m)^n$, we can rewrite $y = 3(2^{-x})$ as $y = 3(2^{-1})^x$. $y = 3(2^{-1})^x = \boxed{3\left(\frac{1}{2}\right)^x.}$

Part (b) Solution: Because $a^{mn} = (a^m)^n$, we can rewrite $y = 4^{-\frac{x}{2}}$ as $y = \left(4^{-1}\right)^{\frac{x}{2}}$. $y = \left(4^{-1}\right)^{\frac{x}{2}} = \left(\frac{1}{4}\right)^{\frac{x}{2}}$. Because $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$, we can rewrite this as $y = \left(\sqrt{\frac{1}{4}}\right)^x$. $y = \left(\sqrt{\frac{1}{4}}\right)^x = \boxed{\left(\frac{1}{2}\right)^x.}$

Part (c) Solution: This equation is already in standard form, as π^π is a constant. Because $a^{mn} = (a^m)^n$, we can say $y = \pi^{\pi x}$ in standard exponential form is $\boxed{y = (\pi^\pi)^x.}$

Part (d) Solution: We can get rid of the coefficient 1 because multiplying anything by 1 does not have any value. Because $a^{m+n} = a^m \cdot a^n$, we can rewrite the equation as $y = \left(\frac{1}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^{\frac{x}{2}}$. Simplifying this results in $y = \frac{1}{27}\left(\frac{1}{3}\right)^{\frac{x}{2}}$. Similarly to how we used the exponent rule $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ in part (b), we can say this equation is equivalent to $\boxed{y = \frac{1}{27}\left(\sqrt{\frac{1}{3}}\right)^x.}$

Part (e) Solution: Making use of the exponent rule $a^{-k} = \frac{1}{a^k}$, this equation can be rewritten as as $y = 5 \cdot 0.345^{7-2x}$. We can use the exponent rule $a^{m+n} = a^m \cdot a^n$ to rewrite this as $y = 5 \cdot 0.345^7 \cdot 0.345^{-2x}$. The rule $(a^m)^n = a^m n$ allows us to rewrite this as $y = 5 \cdot 0.345^7 \cdot \left(0.345^{-2}\right)^x$, which is equivalent to $\boxed{y = \left(5 \cdot 0.345^7\right)\left(\frac{1}{0.345^2}\right)^x.}$

Part (f) Solution: The exponent rule $a^{m+n} = a^m \cdot a^n$ allows us to rewrite $y = 4(0.0003467)^{-0.4x+2}$ as $y = 4 \cdot 0.0003467^{-0.4x} \cdot 0.0003467^2$. Because $a^{mn} = (a^m)^n$, we can say $y = 4 \cdot 0.0003467^2 \cdot \left(0.0003467^{-0.4}\right)^x$, which is equivalent to $\boxed{y = 4 \cdot 0.0003467^2 \cdot \left(\frac{1}{0.0003467^{0.4}}\right)^x.}$

Problem 11.1: In 1968, the U.S. minimum wage was \$1.60 per hour. In 1976, the minimum wage was \$2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where t represents the time in years after 1960.

Part (a): Find a formula for $w(t)$.

Part (b): What does the model predict for the minimum wage in 1960?

Part (c): If the minimum wage was \$5.15 in 1996, is this above, below or equal to what the model predicts.

Part (a) Solution: The standard form of an exponential function is $y = a(b)^x$ where a is the initial value (or minimum wage in 1960), b is the amount the minimum wage increases by per year, and t is the amount of years since 1960. We are given the minimum wage in the years 1968 and 1976, which are 8 and 16 years since 1960, respectively. To solve this problem, we can let a (the initial value) be the minimum wage in 1968, or \$1.60.

To find b, we can divide the minimum wage in 1976 by the minimum wage in 1968, which will result in $b = \frac{2.3}{1.6} = 1.4375$. However, this does not mean that the minimum wage increases by a factor of 1.4375 every year; it means that it increases by a factor of 1.4375 every 8 years (as 1968 and 1976 are 8 years apart). Therefore, b's exponent cannot be x, but must be $\frac{t}{8}$ if the initial year is 1960. However, we need to keep in mind that our initial value was the minimum wage in 1968, not 1960. So, we need to subtract 8 from t (t represents the years since 1960) so that the numerator in b's exponent represents the time in years since 1968, not 1960. This means b's exponent is $\frac{t-8}{8}$. Using the values for a, b, and b's exponent, we can say $\boxed{w(t) = 1.6\,(1.4375)^{\frac{x-8}{8}}.}$

Part (b) Solution: In 1960, the amount of years since 1960 will be 0. Plugging in 0 for x in the model:

$$w(t) = 1.6\,(1.4375)^{\frac{0-8}{8}} = 1.6\,(1.4375)^{-1} = \frac{1.6}{1.4375} \approx 1.11$$

Therefore, the model predicts that the minimum wage in 1960 is $\boxed{\$1.11.}$

Part (c) Solution: 1996 is 36 years after 1960. Substituting 36 for x in the model:

$$w(t) = 1.6\,(1.4375)^{\frac{36-8}{8}} = 1.6\,(1.4375)^{\frac{7}{2}} \approx 5.70$$

The model predicts that the minimum wage in 1996 will be \$5.70, meaning that \$5.15 or the minimum wage in 1996 is $\boxed{\text{below}}$ what the model predicts.

Problem 11.2: The town of Pinedale, Wyoming, is experiencing a population boom. In 1990, the population was 860 and five years later it was 1210.

Part (a): Find a linear model $l(x)$ and an exponential model $p(x)$ for the population of Pinedale in the year $1990 + x$.

Part (b): What do these models estimate the population of Pinedale to be in the year 2000?

Part (a) Solution: In these models, x represents the number of years since 1990.First, we can find the linear model. We know that in 1990 (where x would equal 0), the population is 860 and 5 years after 1990, the population was 1210. This means we have the coordinates $(0, 860)$ and $(5, 1210)$. The slope of the linear model would then be $\frac{1210-860}{5} = 70$. The y-intercept would be at 860, as one of the coordinates is $(0, 860)$. Therefore, the linear model that represents the population in the year $1990 + x$ in slope-intercept form is $\boxed{l(x) = 70x + 860.}$

Now, we can find the exponential model. Because the population in 1990 was 860, the initial value (or the value of a) in the exponential model will be 860. To find b, we can divide 1210 by 860 to find that the population increases by a factor of $\frac{121}{86}$ every 5 years. This mean that the exponent of b is $\frac{x}{5}$, as $\frac{121}{86}$ is the the factor the population increases by every 5 years, not every 1 year. Using the values for a, b, and b's exponent, we can say that $\boxed{p(x) = 860\left(\frac{121}{86}\right)^{\frac{x}{5}}.}$

Part (b) Solution: First, we can find the linear model's estimate of the population in the year 2000. 2000 is 10 years after 1990, so we can substitute 10 for x in the linear model:

$$l(x) = 70(10) + 860 = 1560$$

The linear model predicts the population in 2000 will be $\boxed{1560.}$

To find the exponential model's prediction of the population, we can also substitute 10 for x in the model:

$$p(x) = 860\left(\frac{121}{86}\right)^{\frac{10}{5}} = 860\left(\frac{121}{86}\right)^{2} = \frac{73205}{43} \approx 1702.44$$

Therefore, the exponential model predicts the population in 2000 will be $\boxed{\frac{73205}{43} \approx 1702.44.}$

Problem 11.3: In 1989, research scientists published a model for predicting the cumulative number of AIDS cases reported in the United States:

$$a(t) = 155\left(\frac{t-1980}{10}\right)^3, \text{ (thousands)}$$

where t is the year. This paper was considered a "relief", since there was a fear the correct model would be of exponential type. Pick two data points predicted by the research model $a(t)$ to construct a new exponential model $b(t)$ for the number of cumulative AIDS cases. Discuss how the two models differ and explain the use of the word "relief".

Solution: To make a new exponential model $b(t)$ we can choose the data points from the model $a(t)$ from years 1990 and 2000. When $t = 1990$, $a(t)$ will be $155\left(\frac{1990-1980}{10}\right)^3 = 155$. When $t = 2000$, $a(t)$ will be $155\left(\frac{2000-1980}{10}\right)^3 = 1240$.

We can let the amount of cases in the year 1990 be our initial value, or a. To find b, we need to divide the amount of cases in 2000 by the amount of cases in 1990, which will result in b being $\frac{1240}{155} = 8$. However, b's exponent will not be t, as 8 is the factor that the amount of cases increases by every 10 years (as 1990 and 2000 are 10 years apart). The exponent seems to be $\frac{t}{10}$ for it to describe the exponent by which the cases increase by every 1 year. However, we need to keep in mind that we used the population in 1990, not year 0, as our initial value. This means that the numerator in the exponent currently represents the years since year 0 instead of the years since 1990. Therefore, the exponent must be $\frac{t-1990}{10}$ to account for this. Using these new values, we can say that

$$\boxed{b(t) = 155(8)^{\frac{t-1990}{10}}}$$

These two models differ because $a(t)$ is a cubic function while $b(t)$ is an exponential one. The "relief" expressed when it was found that $a(t)$ was not an exponential model was due to the fact that an exponential model would result in significantly higher numbers of cases as the years pass. Although $b(t)$ will produce a smaller number of cases than $a(t)$ will for the year 2000, it will produce much higher numbers of cases than $a(t)$ later on because $a(t)$ is simply taking some number to the power of three while $b(t)$ has the year itself (t) in the exponent (which could be incredibly high). As the years pass, t will get larger and larger, making the outputs of $b(t)$ also significantly larger than that of a cubic function where the exponent remains 3. For example, in year 2050, $a(t)$ will output 53165 cases while $b(t)$ will output 40632320 cases.