Homework 12

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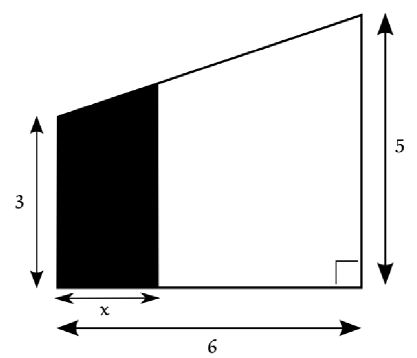
Problem 6.5: Express the area of the shaded region below as a function of x. The dimensions in the figure are centimeters.



Solution: We can solve this problem by first rotating this trapezoid 90 degrees clockwise. Then, we could impose a coordinate system where the bottom left corner of the trapezoid would be at $(0, 0)$, the bottom right corner would be at $(5, 0)$, the top right corner would be at $(3, 6)$, and the top left would be at $(0, 6)$. We know that the area of a trapezoid is given by the formula $A = \frac{h(b_1+b_2)}{2}$, where h is the height of the trapezoid, b_1 is the shorter base of the trapezoid, and b_2 is the longer base of the trapezoid. The diagram gives us that b_1 of the shaded trapezoid is 3 and the height is x. To find the area of the shaded region, we need to find b_2 of the shaded trapezoid. To do this, we can imagine the line $x = 3$ being drawn on the graph of this figure. This would mean the length of b_2 of the whole trapezoid that is left of the line $x = 3$ would be 3, while the length to the right of the line would be 2. Also, the length of b_2 of the shaded trapezoid that is to the left of $x = 3$ would also be 3. However, we do not know the length of b_2 of the shaded trapezoid that is right of $x = 3$, so we can use the variable y to represent it. When this line is drawn, we can say a shaded triangle with vertices at $(3, 6)$, $(3, 6 - x)$ and $(3 + y, 6 - x)$ is similar to the triangle with vertices at $(3, 6)$, $(3, 0)$, and $(5, 0)$ because each of their corresponding angles are congruent. To find the length of b_2 of the shaded trapezoid, we have to find y. To do this, we can set up a proportion where corresponding sides of similar triangles are proportional:

$$\frac{x}{y} = \frac{6}{2}$$

Solving for y:

$$y = \frac{1}{3}x$$

We said earlier that the length of b_2 of the shaded trapezoid that is to the left of $x = 3$ is 3. This means that the total length of b_2 of the shaded trapezoid is $3 + y$, or $3 + \frac{1}{3}x$. We can now plug in the values for b_1, b_2, and h for the shaded trapezoid into the formula for the area of a trapezoid:

$$f(x) = \frac{x(3 + 3 + \frac{1}{3}x)}{2}$$

Simplifying:

$$f(x) = \frac{1}{6}x^2 + 3x$$

We also need to taken into account the domain of this function. We rotated the graph of the function 90 degrees clockwise, which could change its domain. We can revert it back and see that points belonging to the shaded trapezoid are only when $0 \leq x \leq 6$, which means that is the domain. Therefore, the area of the shaded region below as a function of x is

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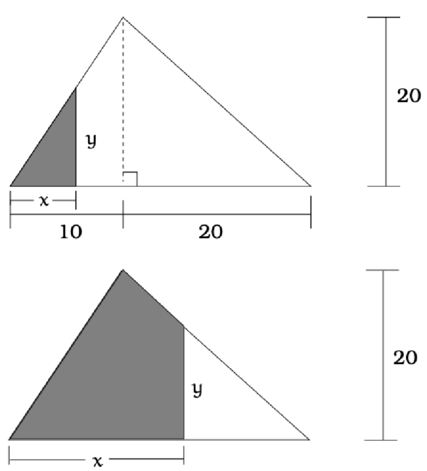
Problem 6.6: Pizzeria Buonapetito makes a triangular-shaped pizza with base width of 30 inches and height 20 inches as shown. Alice wants only a portion of the pizza and does so by making a vertical cut through the pizza and taking the shaded portion. Letting x be the bottom length of Alice's portion and y be the length of the cut as shown, answer the following questions:



Part (a): Find a formula for y as a multipart function of x, for $0 \leq x \leq 30$. Sketch the graph of this function and calculate the range.

Part (b): Find a formula for the area of Alice's portion as a multipart function of x, for $0 \leq x \leq 30$.

Part (c): If Alice wants her portion to have half the area of the pizza, where should she make the cut?

Part (a) Solution: We can start by imposing a coordinate system in which the left most vertex of the triangle is at the origin, the top vertex of the triangle is at $(10, 20)$, and the right most vertex of the triangle is at $(30, 0)$. In the first diagram, we can see that there is an altitude drawn from the top vertex of the triangle to the base. The piecewise function will have two parts: one where $x \leq 10$ and one where $x > 10$. The triangle formed when $x \leq 10$ is left of Alice's cut and can be named as triangle A. The triangle formed when $x > 10$ is right of Alice's cut and can be named as triangle B. To find the formula for y, we can start by finding the equation for the hypotenuse of A. The slope of the line would be $\frac{y}{x}$, or 2. The hypotenuse also has a y-intercept at the origin, meaning the equation in slope intercept form for the hypotenuse is $y = 2x$ when $0 \leq x \leq 10$. Now, we can find the equation for the hypotenuse of B. The slope of the hypotenuse is $\frac{y}{x} = -1$. The hypotenuse also intersects the point $(30, 0)$, meaning the equation in point slope form for the hypotenuse of B is $y = -(x - 30)$ if $10 < x \leq 30$. The formula for y as a multipart function would then be:

$$y = \begin{cases} 2x & \text{if } 0 \leq x \leq 10 \\ 30 - x & \text{if } 10 < x \leq 30 \end{cases}$$

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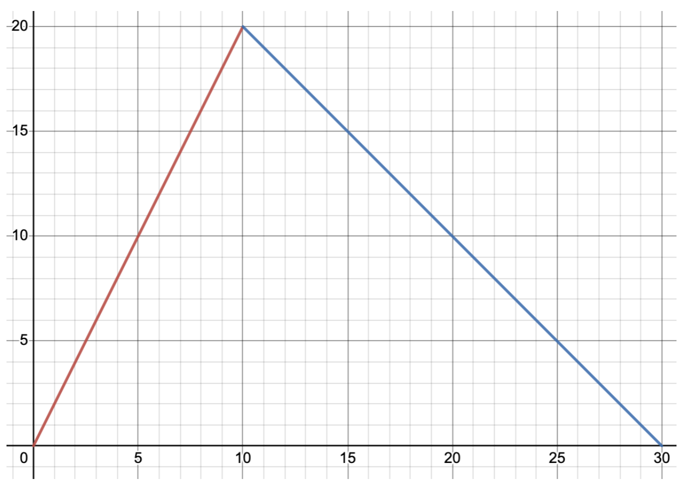
The graph of this function would be:



This function only has y-values between 0 and 20, meaning the range would be

$$0 \leq y \leq 20$$

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Part (b) Solution: In part (a), we found y in terms of x for triangles A and B. The formula for the area of a triangle is $\frac{bh}{2}$, where b is the length of the base and h is the height of the triangle. Therefore, in triangle A, the formula for the area would be $\frac{xy}{2}$. Because $y = 2x$, the formula for the area of Alice's portion when $0 \leq x \leq 10$ is $\frac{2x^2}{2}$, or x^2. The area of triangle B is composed of the total area of the triangle left of the altitude plus the area of the trapezoidal shaded area from the altitude to Alice's cut. The area of the triangle is $\frac{20 \cdot 10}{2}$, or 100. The formula for the area of a trapezoid is $\frac{h(b_1+b_2)}{2}$, where b_1 is the length of the shorter base of the trapezoid, b_2 is the length of the longer base of the trapezoid, and h is the height of the trapezoid. In this case, b_1 is $30 - x$ (defined in the multipart function), b_2 is 20 (the length of the altitude) and the height would be $x - 10$. Plugging these values into the formula for the area of a trapezoid:

$$\frac{(x-10)(30-x+20)}{2}$$

Simplifying:

$$\frac{60x - 500 - x^2}{2}$$

Adding the area of the left triangle to the area of the trapezoid:

$$30x - 250 - \frac{1}{2}x^2 + 100 =$$

$$30x - 150 - \frac{1}{2}x^2$$

We can use $f(x)$ to represent the area of Alice's portion in terms of x. Therefore, the multipart function would be:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 10 \\ 30x - 150 - \frac{1}{2}x^2 & \text{if } 10 < x \leq 30 \end{cases}$$

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Part (c) Solution: The area of the full triangle is $\frac{30 \cdot 20}{2} = 300$ square units. Half of this is 150 square units, meaning we need to find the x value when $f(x) = 150$. Because $f(x)$ is a multipart, we need to set 150 equal to both parts of the function and solve for x. Starting with the first part of the function:

$$x^2 = 150$$
$$x = \pm\sqrt{150}$$

Because the domain for this part of the function is $0 \leq x \leq 10$, we know that x cannot be either $-\sqrt{150}$ or $\sqrt{150}$, as they do not fit the domain. Now, we can set 150 equal to the second part of the function and solve for x:

$$30x - 150 - \frac{1}{2}x^2 = 150$$

$$30x - 300 - \frac{1}{2}x^2 = 0$$

$$x = 30 + 10\sqrt{3},\ x = 30 - 10\sqrt{3}$$

$30 + 10\sqrt{3}$ is approximately 47, and $30 - 10\sqrt{3}$ is approximately 13. The domain of this part of the function is $10 < x \leq 30$. The only value of x that fits this domain is $30 - 10\sqrt{3}$. This means that if Alice wants to have exactly half the area of the pizza, she should make her cut along the line

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$$x = 30 - 10\sqrt{3}$$

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Problem 6.8: Arthur is going for a run. From his starting point, he runs due east at 10 feet per second for 250 feet. He then turns, and runs north at 12 feet per second for 400 feet. He then turns, and runs west at 9 feet per second for 90 feet. Express the (straight-line) distance from Arthur to his starting point as a function of t, the number of seconds since he started.

Solution: We can start by imposing a coordinate system in which Arthur starts at the origin of the system and intervals on the x and y-axes are in feet. There are three parts of the multipart: the first when Arthur travels east for 250 feet, when Arthur travels north for 400 feet, and when Arthur travels west for 90 feet. When Arthur travels east, he travels along the x-axis at 10 ft/sec, meaning his position can be expressed as $(10t, 0)$. The distance from this point to the origin is $10t$ feet. We know that Arthur travels east for 250 feet, meaning that the time he spends traveling east is $\frac{250}{10} = 25$ seconds. Therefore, the domain for this part of the function would be $0 \le t \le 25$. When Arthur travels north, he travels along the line $x = 250$ at a rate of 12 ft/sec, meaning his position would be at $(250, 12(t-25))$. The distance from this point to the origin (using the distance formula) would be $\sqrt{250^2 + 144(t-25)^2}$. Arthur is only traveling west for 400 feet, meaning that the total time he spends traveling west would be $\frac{400}{12} = \frac{100}{3}$ seconds. However, we have to add this time to the time it took him to travel east to find the interval of seconds at which he was traveling north. $25 + \frac{100}{3} = \frac{175}{3}$. Therefore, the domain for this part of the function would be $25 < t \le \frac{175}{3}$. When Arthur travels west, he travels along the line $y = 400$ at a rate of 9 ft/sec, meaning his position would be at $\left(250 - 9\left(t - \frac{175}{3}\right), 400\right)$. Using the distance formula again, we can say the distance from this point to the origin would be $\sqrt{\left(250 - 9\left(t - \frac{175}{3}\right)\right)^2 + 400^2}$. Arthur travels west for $\frac{90}{9} = 10$ seconds. However, we need to add this time to the time it took him to travel east and north. Therefore, the interval of seconds at which Arthur traveled west is $\frac{175}{3} < t \le \frac{205}{3}$. Using $g(t)$ to represent the multipart function, we can express the distance between Arthur and his starting point as

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$$g(t) = \begin{cases} 10t & \text{if } 0 \le t \le 2 \\ \sqrt{250^2 + 144(t-25)^2} & \text{if } 25 < t \le \frac{175}{3} \\ \sqrt{\left(250 - 9\left(t - \frac{175}{3}\right)\right)^2 + 400^2} & \text{if } \frac{175}{3} < t \le \frac{205}{3} \end{cases}$$

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