Homework 9

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Problem 19.7: The voltage output (in volts) of an electrical circuit at time t seconds is given by the function

$$V(t) = 2^{3\sin(5\pi t - 3\pi)+1}$$

(a) What is the initial voltage output of the circuit?

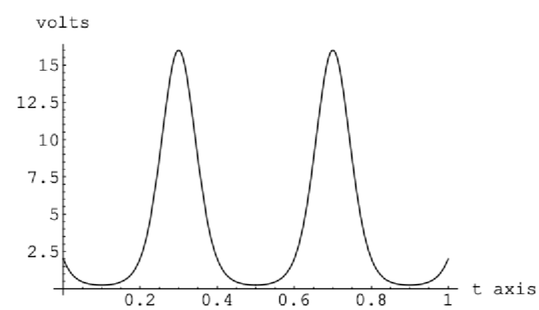
(b) Is the voltage output of the circuit ever equal to zero? Explain.

(c) The function $V(t) = 2^{p(t)}$, where $p(t) = 3\sin(5\pi t - 3\pi) + 1$. Put the sinusoidal function $p(t)$ in standard form and sketch the graph for $0 \le t \le 1$. Label the coordinates of the extrema on the graph.

(d) Calculate the maximum and minimum voltage output of the circuit.

(e) During the first second, determine when the voltage output of the circuit is 10 volts.

(f) A picture of the graph of $y = V(t)$ on the domain $0 \le t \le 1$ is given; label the coordinates of the extrema of the graph.



(g) Restrict the function $V(t)$ to the domain $0.1 \le t \le 0.3$; explain why this function has an inverse and find the formula for the inverse rule. Restrict the function $V(t)$ to the domain $0.3 \le t \le 0.5$; explain why this function has an inverse and find the formula for the inverse rule.

Part (a) Solution: The initial voltage output of the circuit is the voltage output at time $t = 0$. Substituting 0 for t into $V(t)$, we find that the intitial voltage output of the circuit is

$$\boxed{V(0) = 2^{3\sin(0-3\pi)+1} \text{ volts.}}$$

Part (b) Solution: We know that the voltage output of the circuit can never be 0 becuase the voltage output is found by taking 2 to the power of some number. We know that taking any number to the power of another number can never result in 0, so $\boxed{\text{the voltage output can never be 0.}}$

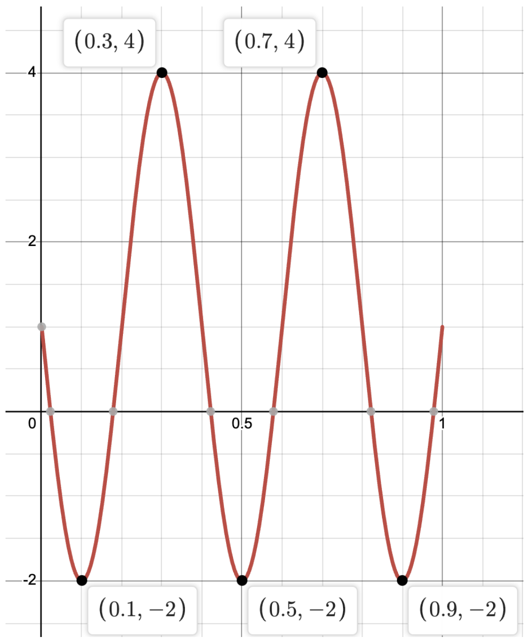
Part (c) Solution: The standard form of a sinusoidal is

$$p(t) = A(B(x - C)) + D$$

where A, B, C, and D are constants. $p(t)$ is almost in standard form; we just have to factor out 5π to get

$$p(t) = 3\sin\left(5\pi\left(t - \frac{3}{5}\right)\right) + 1$$

The graph of this function on the domain $0 \leq t \leq 1$ is included below. Coordinates of the extrema have been labeled.



Part (d) Solution: The maximum voltage of $p(t)$ can be found by adding the mean (D) and the amplitude (A) in the sinusoidal. The maximum would then be $3 + 1 = 4$. The minimum of $p(t)$ can be found by subtracting the amplitude from the mean, which would make the minimum $1 - 3 = -2$. To find the maximum and minimum of $V(t)$, we can raise 2 to the power of the maximum and minimum of $p(t)$. The maximum of $V(t)$ would be $\boxed{2^4 = 16}$ and the minimum of $V(t)$ would be $\boxed{2^{-2} = \frac{1}{4}.}$

Part (e) Solution: To solve this problem, we can set $V(t)$ equal to 10:

$$10 = 2^{3\sin(5\pi t - 3\pi) + 1}$$

Solving for t:

$$3\sin(5\pi t - 3\pi) + 1 = \log_2(10) - 1$$

$$\sin(5\pi t - 3\pi) + 1 = \frac{\log_2(10) - 1}{3}$$

Applying arcsin to both sides of the equation, we have

$$5\pi t - 3\pi = \arcsin\left(\frac{\log_2(10) - 1}{3}\right) + 2\pi n \text{ and } 5\pi t - 3\pi = \pi - \arcsin\left(\frac{\log_2(10) - 1}{3}\right) + 2\pi n$$

Thus, solving for t, we get

$$t = \frac{\arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi n + 3\pi}{5\pi} \text{ and } t = \frac{\pi - \arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi n + 3\pi}{5\pi}$$

Note that n is any integer. We know that we are looking for values of t that are in between 0 and 1. Values of n that produce t values between 0 and 1 are -1 and 0. Substituting 0 and -1 for n in both of the solutions for t, our solutions are

$$\boxed{t = \frac{\arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi(-1) + 3\pi}{5\pi} \approx 0.26}$$

$$\boxed{t = \frac{\arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi(0) + 3\pi}{5\pi} \approx 0.66}$$

$$\boxed{t = \frac{\pi - \arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi(-1) + 3\pi}{5\pi} \approx 0.34}$$

$$\boxed{t = \frac{\pi - \arcsin\left(\frac{\log_2(10)-1}{3}\right) + 2\pi(0) + 3\pi}{5\pi} \approx 0.74}$$

Thus, in the first second, the voltage output is 10 volts after approximately 0.26, 0.66, 0.34, and 0.74 seconds.

Part (f) Solution: On the graph, the extrema occur at $(0.1, 0.25)$, $(0.3, 16)$, $(0.5, 0.25)$, $(0.7, 16)$, and $(0.9, 0.25)$.

Part (g) Solution: When restricted to $0.1 \le t \le 0.3$ or $0.3 \le t \le 0.5$, $V(t)$ is a one to one function (every input has exactly one output), and one to one functions have inverses. To find the inverse of this function, we can start by switching $V(t)$ and t in the equation for $V(t)$. For simplicity, we can let $y = V(t)$ and rewrite the equation using y:

$$t = 2^{3\sin(5\pi y - 3\pi)+1}$$

Now, we can solve for y to find the inverse:

$$\log_2 t = 3\sin(5\pi y - 3\pi) + 1$$

$$\frac{\log_2 t - 1}{3} = \sin(5\pi y - 3\pi)$$

Applying arcsin to both sides, we have two different equations:

$$\arcsin\left(\frac{\log_2 t - 1}{3}\right) + 2\pi n = 5\pi y - 3\pi \text{ and } \pi - \arcsin\left(\frac{\log_2 t - 1}{3}\right) + 2\pi n = 5\pi y - 3\pi$$

Solving both for y:

$$y = \frac{\arcsin\left(\frac{\log_2 t-1}{3}\right) + 2\pi n + 3\pi}{5\pi} \text{ and } y = \frac{\pi - \arcsin\left(\frac{\log_2 t-1}{3}\right) + 2\pi n + 3\pi}{5\pi}$$

To find the value of n for which one of these solutions is an inverse for our domain restrictions, we can try different n values. In order to try different n values, we have to pick some t to try them with. We can take an t value of 5, for example, as it is an t value in between the maximum and minimum y values of the sinusoidal. The sinusoidal on the domain is $0.1 \le x \le 0.3$ is increasing, so we will use our principal solution to find the inverse for this domain. When $n = -1$, $\frac{\arcsin\left(\frac{\log_2 5-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \approx 0.23$. 0.23 is in between 0.1 and 0.3, so we know that $\boxed{y = \frac{\arcsin\left(\frac{\log_2 t-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \text{ is the inverse for } 0.1 \le t \le 0.3.}$

The sinusoidal on the domain $0.3 \le t \le 5$ is decreasing, so we will use the symmetry solution to find the inverse for this domain. When substituting 5 for t and -1 for n in the symmetry solution, we get $\frac{\pi-\arcsin\left(\frac{\log_2 5-1}{3}\right)+2\pi(-1)+3\pi}{5\pi} \approx 0.37$. This falls in the domain $0.3 \le t \le 0.5$, so we know that $\boxed{y = \frac{\pi-\arcsin\left(\frac{\log_2 t-1}{3}\right)+2\pi n+3\pi}{5\pi} \text{ is the inverse for } 0.3 \le t \le 0.5.}$

Problem 20.10: Answer the following questions:

(a) If $y = \sin(x)$ on the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, what is the domain D and range R of $y = 2\sin(3x-1)+3$? How many solutions does the equation $4 = 2\sin(3x-1)+3$ have on the domain D and what are they?

(b) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, what is the domain D and range R of $y = 8\sin\left(\frac{2\pi}{1.2}(t-0.3)\right)+18$. How many solutions does the equation $22 = 8\sin\left(\frac{2\pi}{1.2}(t-0.3)\right)+18$ have on the domain D and what are they?

(c) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, what is the domain D and range R of $y = 27\sin\left(\frac{2\pi}{366}(t-0.85)\right)+45$? How many solutions does the equation $40 = 27\sin\left(\frac{2\pi}{366}(t-0.85)\right)+45$ have on the domain D and what are they?

(d) If $y = \cos(x)$ on the domain $0 \le x \le \pi$, what is the domain D and range R of $y = 4\cos(2x+1)-3$? How many solutions does the equation $-1 = 4\cos(2x+1)-3$ have on the domain D and what are they?

(e) If $y = \tan(x)$ on the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$, what is the domain D and range R of $y = 2\tan(-x+5)+13$? How many solutions does the equation $100 = 2\tan(-x+5)+13$ have on the domain D and what are they?

Part (a) Solution: To find the domain of this function, we have to consider the horizontal transformations it has undergone. From equation, it appears that it has first undergone a shift 1 unit to the right (hence the -1) and then a vertical compression by a factor of 3 (hence the coefficient of 3). We can apply these transformations to the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Moving it one unit to the right, the domain becomes $-\frac{\pi}{2}+1 \le x \le \frac{\pi}{2}+1$. Compressing this domain by a factor of 3, we can say $\boxed{\text{D is } \frac{1}{3}\left(-\frac{\pi}{2}+1\right) \le x \le \frac{1}{3}\left(\frac{\pi}{2}+1\right).}$

To find the range R of this function, we have to consider the vertical transformations it has undergone. The function appears to have undergone a vertical stretch by a factor of 2 and a shift upwards by 3 units. The range of an untransformed sine function is $-1 \le y \le 1$. Stretching this by a factor of 2 and then moving it up 2 units, we can say $\boxed{\text{R is } 1 \le y \le 1.}$

To find how many solutions the equation $4 = 2\sin(3x-1)+3$ has on the domain D, we can solve for x. Doing this, we have

$$0 = 2\sin(3x-1)-1$$
$$\sin(3x-1) = \frac{1}{2}$$

Applying arcsin to both sides, we have the following two equations:

$$3x-1 = \arcsin\left(\frac{1}{2}\right)+2\pi n \text{ and } 3x-1 = \pi - \arcsin\left(\frac{1}{2}\right)+2\pi n$$

Solving for x:

$$x = \frac{\arcsin\left(\frac{1}{2}\right)+2\pi n+1}{3} \text{ and } x = \frac{\pi-\arcsin\left(\frac{1}{2}\right)+2\pi n+1}{3}$$

We need to look for values of n for which these solutions fall in our domain. In our first solution, the only value of n that satisfies this condition is 0. In our second solution, there are no values of n which satisfy this condition. Thus, there is $\boxed{1}$ solution that this equation has on the domain D. This solution is

$$\boxed{x = \frac{\arcsin\left(\frac{1}{2}\right) + 2\pi(0) + 1}{3}}$$

Part (b) Solution: The horizontal transformations that this function has undergone is a compression by a factor of $\frac{2\pi}{1.2}$ followed by a shift 0.3 units to the right. Applying these transformations to the domain $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, we can say $\boxed{\text{D is } \frac{1.2}{2\pi}\left(-\frac{\pi}{2}\right) + 0.3 \le t \le \frac{1.2}{2\pi}\left(\frac{\pi}{2}\right) + 0.3.}$ The vertical transformations that the equation has undergone are a dilation by a factor of 8 and a shift upwards by 18 units. Applying these transformations to the range of an untransformed sine function, we can say $\boxed{\text{R is } 10 \le t \le 26.}$

To find how many solutions the equation $22 = 8\sin\left(\frac{2\pi}{1.2}(t - 0.3)\right) + 18$ has on the domain D, we can solve the equation:

$$0 = 8\sin\left(\frac{2\pi}{1.2}(t - 0.3)\right) - 4$$

$$\sin\left(\frac{2\pi}{1.2}(t - 0.3)\right) = \frac{1}{2}$$

Applying arcsin to both sides, we have the following two equations:

$$\frac{2\pi}{1.2}(t - 0.3) = \arcsin\left(\frac{1}{2}\right) + 2\pi n \text{ and } \frac{2\pi}{1.2}(t - 0.3) = \pi - \arcsin\left(\frac{1}{2}\right) + 2\pi n$$

$$t = \frac{1.2}{2\pi}\left(\arcsin\left(\frac{1}{2}\right) + 2\pi n\right) + 0.3 \text{ and } t = \frac{1.2}{2\pi}\left(\pi - \arcsin\left(\frac{1}{2}\right) + 2\pi n\right) + 0.3$$

We need to look for values of n for which these solutions fall in our domain. In our first solution, the only value of n that satisfies this condition is 0. In our second solution, there are no values of n which satisfy this condition. Thus, there is $\boxed{1}$ solution that this equation has on the domain D. This solution is

$$t = \frac{1.2}{2\pi}\left(\arcsin\left(\frac{1}{2}\right) + 2\pi(0)\right) + 0.3$$

Part (c) Solution: We can repeat the process used in part (b). The horizontal transformations that the function has undergone are a horizontal compression by a factor of $\frac{2\pi}{366}$ and then a shift to the right by 0.85 units. Using these transformations on the domain, we can say $\boxed{\text{D is } \frac{366}{2\pi}\left(-\frac{\pi}{2}\right) + 0.85 \le t \le \frac{366}{2\pi}\left(\frac{\pi}{2}\right) + 0.85.}$ The vertical transformations the function has undergone are a stretch by a factor of 27 and a shift upwards by 45 units. Applying these transformations to the range $-1 \le y \le 1$, we find that $\boxed{\text{R is } 18 \le y \le 72.}$

To find the solutions that $40 = 27\sin\left(\frac{2\pi}{366}(t - 0.85)\right) + 45$ has on the domain, we can solve for t:

$$0 = 27\sin\left(\frac{2\pi}{366}(t - 0.85)\right) + 5$$

$$-\frac{5}{27} = \sin\left(\frac{2\pi}{366}(t - 0.85)\right)$$

Applying arcsin to both sides of the equation, we have the following two equations:

$$\frac{2\pi}{366}(t - 0.85) = \arcsin(-\frac{5}{27}) + 2\pi n \text{ and } \frac{2\pi}{366}(t - 0.85) = \pi - \arcsin(-\frac{5}{27}) + 2\pi n$$

Solving for t:

$$t = \frac{366}{2\pi}\left(\arcsin(-\frac{5}{27}) + 2\pi n\right) + 0.85 \text{ and } t = \frac{366}{2\pi}\left(\pi - \arcsin(-\frac{5}{27}) + 2\pi n\right) + 0.85$$

We need to look for values of n for which these solutions fall in our domain. In our first solution, the only value of n that satisfies this condition is 0. In our second solution, there are no values of n which satisfy this condition. Thus, there is $\boxed{1}$ solution that this equation has on the domain D. This solution is

$$t = \frac{366}{2\pi}\left(\arcsin(-\frac{5}{27}) + 2\pi(0)\right) + 0.85$$

Part (d) Solution: The horizontal transformations the function has undergone is first a shift to the left by one unit and then a compression by a factor of 2. Applying these transformations to the domain, we can say $\boxed{\text{D is } -\frac{1}{2} \leq x \leq \frac{\pi-1}{2}.}$ The vertical transformations the function has undergone is a stretch by a factor of 4 and a shift upwards by 45 units. Applying these transformations to the range of a cosine function $(-1 \leq y \leq 1)$, we can say $\boxed{\text{R is } 41 \leq y \leq 49.}$

To find the solutions that fall in domain D, we can solve the equation:

$$0 = 4\cos(2x+1) - 3$$
$$\cos(2x+1) = \frac{3}{4}$$

Applying arccos to both sides, we have the following two solutions:

$$2x+1 = \arccos\left(\frac{3}{4}\right) + 2\pi n \text{ and } 2x+1 = -\arccos\left(\frac{3}{4}\right) + 2\pi n$$

$$x = \frac{\arccos\left(\frac{3}{4}\right) + 2\pi n - 1}{2} \text{ and } x = \frac{-\arccos\left(\frac{3}{4}\right) + 2\pi n - 1}{2}$$

We need to look for values of n for which these solutions fall in our domain. In our first solution, the only value of n that satisfies this condition is 0. In our second solution, there are no values of n which satisfy this condition. Thus, there is $\boxed{1}$ solution that this equation has on the domain D. This solution is

$$x = \frac{\arccos\left(\frac{3}{4}\right) + 2\pi(0) - 1}{2}$$

Part (e) Solution: The horizontal transformations the function has undergone are first a shift 5 units to the left and then a flip over the y-axis (multiplication by -1). Applying these transformations to the domain, we can say $\boxed{\text{D is } -\left(\frac{\pi}{2} - 5\right) \leq x \leq -\left(-\frac{\pi}{2} - 5\right).}$ The range of a tangent function is all real numbers, so $\boxed{\text{R is all real numbers.}}$

To find its solutions on the domain D, we can solve the equation:

$$0 = 2\tan(-x+5) - 87$$
$$\frac{87}{2} = \tan(-x+5)$$
$$-x+5 = \arctan\left(\frac{87}{5}\right) + 2\pi n$$
$$x = -\arctan\left(\frac{87}{5}\right) - 2\pi n + 5$$

We need to look for values of n for which these solutions fall in our domain. In our first solution, the only value of n that satisfies this condition is 0. In our second solution, there are no values of n which satisfy this condition. Thus, there is $\boxed{1}$ solution that this equation has on the domain D. This solution is

$$x = -\arctan\left(\frac{87}{5}\right) - 2\pi(0) + 5$$

Problem 2.11: Tiffany is a model rocket enthusiast. She has been working on a pressurized rocket filled with laughing gas. According to her design, if the atmospheric pressure exerted on the rocket is less than 10 pounds/sq.in., the laughing gas chamber inside the rocket will explode. Tiff worked from a formula $p = (14.7)e^{-h/10}$ pounds/sq.in. for the atmospheric pressure h miles above sea level. Assume that the rocket is launched at an angle of α above level ground at sea level with an initial speed of 1400 feet/sec. Also, assume the height (in feet) of the rocket at time t seconds is given by the equation $y(t) = -16t^2 + 1400\sin(\alpha)t$.

(a) At what altitude will the rocket explode?

(b) If the angle of launch is $\alpha = 12°$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?

(c) If the angle of launch is $\alpha = 82°$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?

(d) Find the largest launch angle α so that the rocket will not explode.

Part (a) Solution: The rocket will explode when the atmospheric pressure is less than 10. To find when the atmospheric pressure reaches 10, we can set p equal to 10 in the equation for p:

$$10 = (14.7)e^{-h/10}$$

Solving for h:

$$\frac{10}{14.7} = e^{-h/10}$$
$$-\frac{h}{10} = \log_e\left(\frac{10}{14.7}\right)$$

Because h is the number of miles above sea level, we can say that the rocket will explode if it reaches an altitude higher than

$$h = -10\log_e\left(\frac{100}{147}\right) \approx 20343.84 \text{ feet.}$$

Part (b) Solution: Substituting 12 degrees into the equation for the height of the rocket at time t, we get $y(t) = -16t^2 + 1400\sin(12)t$. This is a quadratic equation, meaning that the y-coordinate of the vertex (maximum) of this quadratic will be the highest altitude that the rocket reaches. The x-coordinate of the vertex is $\frac{1400\sin(12)}{32}$. Substituting this back into the quadratic, the y-coordinate of the vertex and the rocket's highest altitude is at $-16\left(\frac{1400\sin(12)}{32}\right)^2 + 1400\sin(12)\left(\frac{1400\sin(12)}{32}\right)$, or approximately 1323.84 feet.

We know that the equation for the atmospheric pressure is an exponential decay function, meaning that a higher altitude corresponds to lower pressure. Thus the rocket will experience its lowest pressure when it is at its vertex. The atmospheric pressure at this time would be

$$(14.7)e^{\frac{-\left(-16\left(\frac{1400\sin(12)}{32}\right)^2+1400\sin(12)\left(\frac{1400\sin(12)}{32}\right)\right)}{10}} \approx 14.37 \text{ pounds/sq. in}$$

The rocket's highest altitude is not higher than 20343.84 feet, so the rocket will not combust in midair.

Part (c) Solution: We can repeat the process we used in part (b).Substituting 82 degrees into the equation for the height of the rocket at time t, we get $y(t) = -16t^2 + 1400\sin(82)t$. This is a quadratic equation, meaning that the y-coordinate of the vertex (maximum) of this quadratic will be the highest altitude that the rocket reaches. The x-coordinate of the vertex is $\frac{1400\sin(82)}{32}$. Substituting this back into the quadratic, the y-coordinate of the vertex and the rocket's highest altitude is at $-16\left(\frac{1400\sin(82)}{32}\right)^2 + 1400\sin(82)\left(\frac{1400\sin(12)}{32}\right)$, or approximately 30031.82 feet.

We know that the equation for the atmospheric pressure is an exponential decay function, meaning that a higher altitude corresponds to lower pressure. Thus the rocket will experience its lowest pressure when it is at its vertex. The atmospheric pressure at this time would be

$$(14.7)e^{\frac{-\left(-16\left(\frac{1400\sin(82)}{32}\right)^2+1400\sin(82)\left(\frac{1400\sin(82)}{32}\right)\right)}{10}} \approx 8.32 \text{ pounds/sq. in}$$

The rocket's highest altitude would be higher than 20343.84 feet, so the rocket will combust in midair.

Part (d) Solution: If we want the rocket to not explode, its height has to remain below $-10\log_e\left(\frac{100}{147}\right)$ miles throughout its flight. The highest point of the rocket is at its maximum, so we need to find launch angles for which the maximum height is less than or equal to this value. The following equation sets the y-coordinate of the vertex (using some arbitrary angle α) equal to $-10\log_e\left(\frac{100}{147}\right)\cdot 5280$. Note that we multiply by 5280 because our maximum is in feet while $-10\log_e\left(\frac{100}{147}\right)$ is in miles.

$$-16\left(\frac{1400\sin(\alpha)}{32}\right)^2 + 1400\sin(\alpha)\left(\frac{1400\sin(\alpha)}{32}\right) = -10\log_e\left(\frac{100}{147}\right)$$

Solving for α, we get

$$\alpha = \arcsin\left(\frac{\sqrt{-\frac{8448\ln\left(\frac{100}{147}\right)}{1225}}}{2}\right) \approx 54.586 \text{ degrees}$$

Thus the largest launch angle so the rocket doesn't explode is approximately 54.586 degrees.

Problem 20.12: Let's make sure we can handle the symbolic and mechanical aspects of working with the inverse trigonometric functions:

(a) Find four solutions of $\tan\left(2x^2+x-2\right)=5$.

(b) Solve for x: $\arctan\left(2x^2+x-1\right)=0.5$.

Part (a) Solution: Applying arctan to both sides of the equation, we have

$$2x^2+x-2=\arctan(5)+2\pi n$$

If we let $n=0$, we have

$$2x^2+x-2=\arctan(5)$$

Solving for x, two of our solutions are

$$x=\frac{1}{4}\left(-1-\sqrt{17+8\arctan(5)}\right) \text{ and } x=\frac{1}{4}\left(-1+\sqrt{17+8\arctan(5)}\right)$$

If we let $n=1$, we have

$$2x^2+x-2=\arctan(5)+2\pi$$

Solving for x, another two solutions are

$$x=\frac{1}{4}\left(-1-\sqrt{17+16\pi+8\arctan(5)}\right) \text{ and } x=\frac{1}{4}\left(-1+\sqrt{17+16\pi+8\arctan(5)}\right)$$

Part (b) Solution: First, we can apply tan to both sides of the equation:

$$2x^2+x-1=\tan(0.5)$$

Solving for x, we get

$$x=\frac{-1+\sqrt{1-4\cdot 2\cdot(-1-\tan(0.5))}}{4} \text{ and } x=\frac{-1-\sqrt{1-4\cdot 2\cdot(-1-\tan(0.5))}}{4}$$