Homework 8

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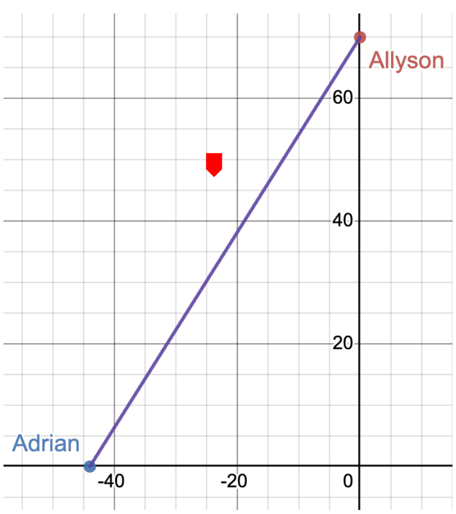
October 30th, 2023

Problem 4.7: Allyson and Adrian have decided to connect their ankles with a bungee cord; one end is tied to each person's ankle. The cord is 30 feet long, but can stretch up to 90 feet. They both start from the same location. Allyson moves 10 ft/sec and Adrian moves 8 ft/sec in the directions indicated. Adrian stops moving at time $t = 5.5$ sec, but Allyson keeps on moving 10 ft/sec in the indicated direction.

Part (a): Sketch an accurate picture of the situation at time $t = 7$ seconds. Make sure to label the locations of Allyson and Adrian; also, compute the length of the bungee cord at $t = 7$ seconds.

Part (b): Where is Allyson when the bungee reaches its maximum length?

Part (a) Solution: To draw the picture of this situation, we can impose a coordinate system where Allyson travels on the y-axis, Adrian travels on the y-axis, and intervals are in terms of feet. Allyson and Adrian both start at the origin of the system. We know that Allyson moves at a rate of 10 ft/sec and has been traveling for 7 seconds. Therefore, using the formula $d = rt$, we can say that she is located 70 feet north of the origin at $(0, 70)$. We know that Adrian moves at a rate of 8 ft/sec, but also stops after 5.5 seconds. This means that even after 7 seconds, she will be the same distance from the origin as she was after 5.5 seconds. Using the formula $d = rt$ again, Adrian will be located 44 feet left of the origin at $(-44, 0)$. The purple line segment in the picture represents the bungee cord connecting Allyson and Adrian.



To find the length of the bungee cord, we can use the coordinates of Allyson and Adrian's locations and plug them into the distance formula.

$$d = \sqrt{(0 + 44)^2 + (70 - 0)^2}$$

Simplifying this:

$$d = \sqrt{6836}$$

Therefore, the length of the bungee cord after 7 seconds will be $\boxed{\sqrt{6836} \text{ feet.}}$

Part (b) Solution: We know that Adrian does not move from the point $(-44, 0)$ after 5.5 seconds. We also know that the bungee cord has a maximum length of 90 feet. Therefore, we have to find Allyson's coordinates when the distance between her location and $(-44, 0)$ is 90 feet. However, we also have to take into account the fact that the bungee cord will be interrupted by the corner of the building at the point $(-20, 30)$. Therefore, we have to find Allyson's coordinates when the distance between the corner of the building and Adrian plus the distance between the corner of the building and Allyson is 90 feet.

Allyson's x-coordinate will always be 0, as she travels along the y-axis. We can use the variable y to represent Allyson's y-coordinate. To find this y-coordinate, we can use the distance formula.

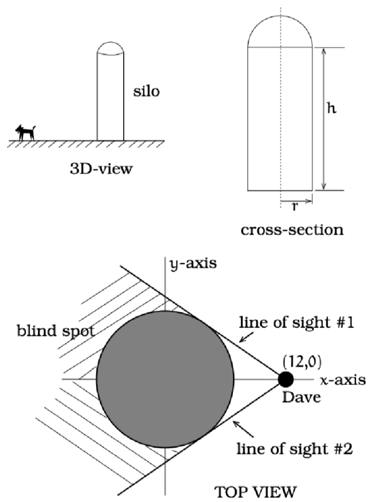
$$90 = \sqrt{(-20+44)^2 + (30-0)^2} + \sqrt{(-20-0)^2 + (30-y)^2}$$

Simplifying:

$$\begin{gathered}
90 = \sqrt{1476} + \sqrt{400 + (30-y)^2} \\
90 - 38.4187 = \sqrt{400 + (30-y)^2} \\
51.5813^2 = 400 + (30-y)^2 \\
2260.63051 = (30-y)^2 \\
\sqrt{2260.63051} = |30 - y| \\
y = 77.5468,\ y = -17.5468
\end{gathered}$$

We know that Allyson's y-coordinate cannot be negative because she is moving north from the origin. This means that Allyson's y-coordinate when the bungee cord in 90 feet long is 77.5468. Therefore, Allyson is at the point $\boxed{(0, 77.5468)}$ when the bungee cord reaches its maximum length.

Problem 4.8: Dave is going to leave academia and go into business building grain silos. A grain silo is a cylinder with a hemispherical top, used to store grain for farm animals. Here is a 3D view, a cross-section, and the top view:



If Dave is standing next to a silo of cross-sectional radius $r = 8$ feet at the indicated position, his vision will be partially obstructed. Find the portion of the y-axis that Dave cannot see. (Hint: Let a be the x-coordinate of the point where line of sight #1 is tangent to the silo; compute the slope of the line using two points (the tangent point and $(12, 0)$). On the other hand, compute the slope of line of sight #1 by noting it is perpendicular to a radial line through the tangency point. Set these two calculations of the slope equal and solve for a.)

4.8 Solution: Using the hint in the problem, we can say the point at which the line of sight #1 is tangent to the silo is (a, b). Using the formula to calculate the slope of a line, we can say the slope of the line that both $(12, 0)$ and (a, b) belong to is $\frac{b}{a-12}$. We also know that line of sight #1 is perpendicular to the radial line that intersects (a, b) and the origin (center of the circle). Therefore, the slope of line of sight #1 is the negative reciprocal of the slope of the radial line. Using the formula for the slope of a line, we can say the radial line's slope is $\frac{b}{a}$. Therefore, the slope of line of sight #1 is $-\frac{a}{b}$. Because we know that the slope of line of sight #1 is $-\frac{a}{b}$ and $\frac{b}{a-12}$, we can set these values equal to each other and solve for a.

$$-\frac{a}{b} = \frac{b}{a-12}$$

$$b^2 = 12a - a^2$$

We can now use the the standard form of the equation of a circle to write the equation for the silo. We know that it is centered at $(0, 0)$ and has a radius of 8, so the equation would be $x^2 + y^2 = 64$. Now, we can substitute a and b into this equation to find the point of tangency. The equation would then be

$$a^2 + 12a - a^2 = 64$$

Solving for a:

$$a = \frac{16}{3}$$

We can substitute this value for a back into the equation $b^2 = 12a - a^2$ to find the value of b:

$$b^2 = \frac{16}{3} \cdot 12 - \left(\frac{16}{3}\right)^2$$

$$b^2 = \frac{320}{9}$$

$$b = \frac{\sqrt{320}}{3},\ b = -\frac{\sqrt{320}}{3}$$

Therefore, the point of tangency is at $\left(\frac{16}{3}, \pm\frac{\sqrt{320}}{3}\right)$. We can find the slopes of lines of lines of sights #1 and #2 using this point and $(12, 0)$. The slopes would be $\frac{\pm\frac{\sqrt{320}}{3}}{\frac{16}{3}}$, which simplifies to $\pm\frac{2\sqrt{5}}{5}$. Therefore, the equations for lines of sight #1 and #2 in point slope form would be $y = \pm\frac{2\sqrt{5}}{5}(x - 12)$. To find the portions of the y-axis that are not visible, we need to find the y-coordinates of this equation when the x-coordinates are zero. To do this, we can substitute 0 for x into the equations for the lines.

$$y = \pm\frac{2\sqrt{5}}{5}(0 - 12)$$

Solving for y:

$$y = \pm\frac{24\sqrt{5}}{5}$$

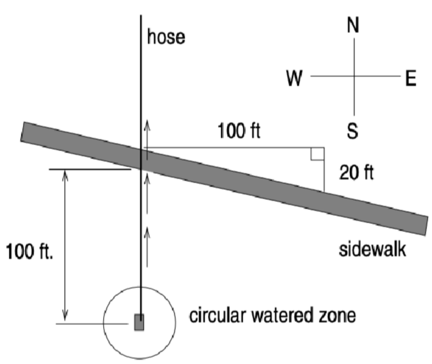
These are the two y-coordinates between which Dave's view is obstructed. Therefore, Dave cannot see the y-axis when $-\frac{24\sqrt{5}}{5} \leq y \leq \frac{24\sqrt{5}}{5}$.

Problem 4.12: The infamous crawling tractor sprinkler is located as pictured below, 100 feet South of a 10 ft. wide sidewalk; notice the hose and sidewalk are not perpendicular. Once the water is turned on, the sprinkler waters a circular disc of radius 20 feet and moves North along the hose at the rate of $\frac{1}{2}$ inch/second.

Part (a): Impose a coordinate system. Describe the initial coordinates of the sprinkler and find the equation of the line forming the southern boundary of the sidewalk.

Part (b): After 33 minutes, sketch a picture of the wet portion of the sidewalk; find the length of the wet portion of the Southern edge of the sidewalk.

Part (c): Find the equation of the line forming the northern boundary of the sidewalk. (Hint: You can use the properties of right triangles.)



Part (a) Solution: To impose a coordinate system, we can let the origin be the intersection between the tractor and the southern boundary of the sidewalk. The y-axis would be the horizontal line that intersects the origin, and the x-axis would be the hose. Therefore, the initial coordinates of the sprinkler would be at $\boxed{(0, -100).}$ To find the equation of the southern boundary of the sidewalk, we can find its slope, which would be $-\frac{20}{100}$, or $-\frac{1}{5}$. We know the y-intercept of the sidewalk is at $(0, 0)$. This means that the equation of the southern boundary in slope intercept form would be $\boxed{y = -\frac{1}{5}x.}$

Part (b) Solution: We know that the tractor moves at a speed of $\frac{1}{2}$ inch/second, which is equivalent to $\frac{5}{2}$ feet/minute. Therefore, after 33 minutes, the tractor will have moved $\frac{5}{2} \cdot 33 = 82.5$ feet north of $(0, -100)$ to the point $(0, -17.5)$. We can use the standard form for the equation of a circle to describe the circular disc the sprinkler waters. We know that the center (or the location of the tractor) is at $(0, -17.5)$ and the radius of the disc is 20 feet. Therefore, the equation would be:

$$x^2 + (y + 17.5)^2 = 400$$

To find the portion of the southern edge of the sidewalk that is watered, we have to find the two points where the edge of the circular disc intersects the southern edge of the sidewalk. To do this, we can substitute $-\frac{1}{5}$ for y in the equation for the circle.

$$x^2 + (-\frac{1}{5}x + 17.5)^2 = 400$$

Solving for x:

$$x^2 + \frac{1}{25}x^2 - \frac{35}{5}x + \frac{1225}{4} = 400$$
$$\frac{26}{25}x^2 - \frac{35}{5}x - \frac{375}{4} = 0$$
$$x = \frac{175 \pm 25\sqrt{439}}{52}$$

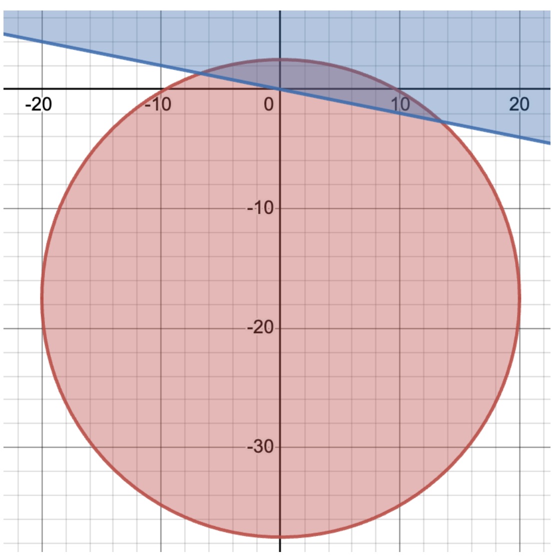
To find the y-coordinates of the intersection points, we can plug these x-values back into the equation for the southern boundary:

$$y = -\frac{1}{5}\left(\frac{175 + 25\sqrt{439}}{52}\right), \; y = -\frac{1}{5}\left(\frac{175 - 25\sqrt{439}}{52}\right)$$
$$y = -\frac{175 + 25\sqrt{439}}{260}, \; y = -\frac{175 - 25\sqrt{439}}{260}$$

Therefore, the southern edge of the sidewalk is wet between the points $\left(\frac{175+25\sqrt{439}}{52}, -\frac{175+25\sqrt{439}}{260}\right)$ and $\left(\frac{175-25\sqrt{439}}{52}, -\frac{175-25\sqrt{439}}{260}\right)$. Converting these values into decimals, we can use the distance formula to find the distance between these two points.

$$\sqrt{(13.4386 + 6.7078)^2 + (-2.6877 - 1.3416)^2} \approx 20.5454$$

Therefore, the length of the southern boundary of the sidewalk that is watered is $\boxed{\text{20.5454 feet.}}$



Part (c) Solution: The northern and southern boundaries of the sidewalk are parallel to each other, so the northern boundary also has a slope of $-\frac{1}{5}$. We can create a right triangle to solve this problem. Vertex A is located at the origin, vertex B is located on the northern boundary, and vertex C is located directly south of B on the x-axis. $\overline{AB}$ is perpendicular to the northern and southern boundaries of the sidewalk. $\overline{BC}$ is a vertical line segment. Because $\overline{AB}$ is perpendicular to the boundaries of the sidewalk and has a y-intercept at the origin, its equation is $y = 5x$. Because the slope is 5, we know that the ratio of the legs of the right triangle is $\frac{1}{5}$. Therefore, we can represent $\overline{BC}$ as x and $\overline{AC}$ as $5x$. If $\overline{AC}$ is $5x$, $\overline{BC}$ is x, and the length of $\overline{AB}$ is 10, then we can use the Pythagorean Theorem to solve for x.

$$25x^2 + x^2 = 100$$
$$x = \pm\sqrt{\frac{50}{13}}$$

This means that the x-coordinate of A is $\sqrt{\frac{50}{13}}$ because the x-coordinate must be positive. To find the y-coordinate, we can substitute this back into the equation $y = 5x$:

$$y = 5 \cdot \sqrt{\frac{50}{13}}$$

Therefore, the coordinates of A are $\left(\sqrt{\frac{50}{13}}, 5\sqrt{\frac{50}{13}}\right)$. Because point A lies on the northern boundary of the sidewalk and the northern boundary has a slope of $-\frac{1}{5}$, the equation for the boundary in point slope form would be $\boxed{y - 5\sqrt{\frac{50}{13}} = -\frac{1}{5}\left(x - \sqrt{\frac{50}{13}}\right)}$.