Project 5

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1 Introduction

The topic for this paper revolves around the three trigonometric functions we have been discussing the most throughout our trigonometry unit—sine, cosine, and tangent. Specifically, when looking at graphs of equations like $\sin(x^b) + \cos(y^b) = \tan(\pi)$ where b is some integer, some interesting things start to appear. For example, we can graph this equation when $b = 2$:

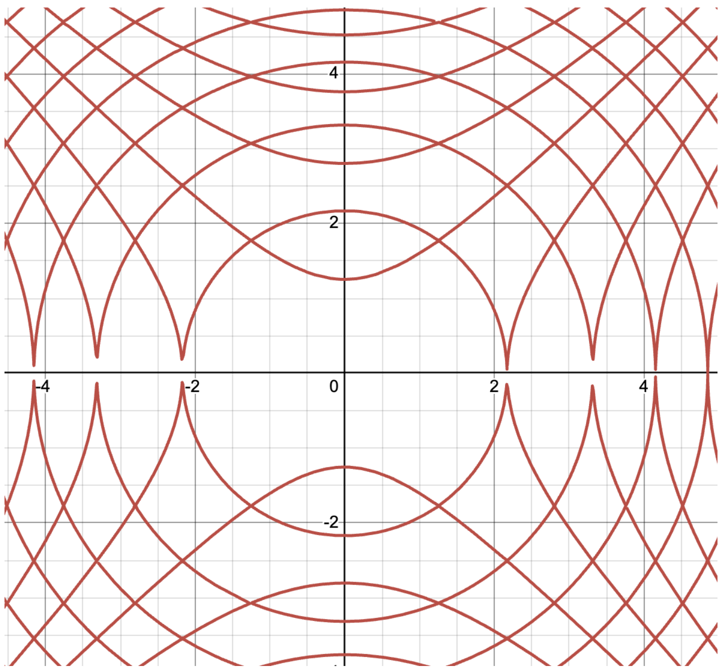

Figure 1: Graph of $\sin(x^2) + \cos(y^2) = \tan(\pi)$

This graph appears to consist of a variety of different shapes and curves. When we let $b = 3$, the graph of this equation changes drastically:

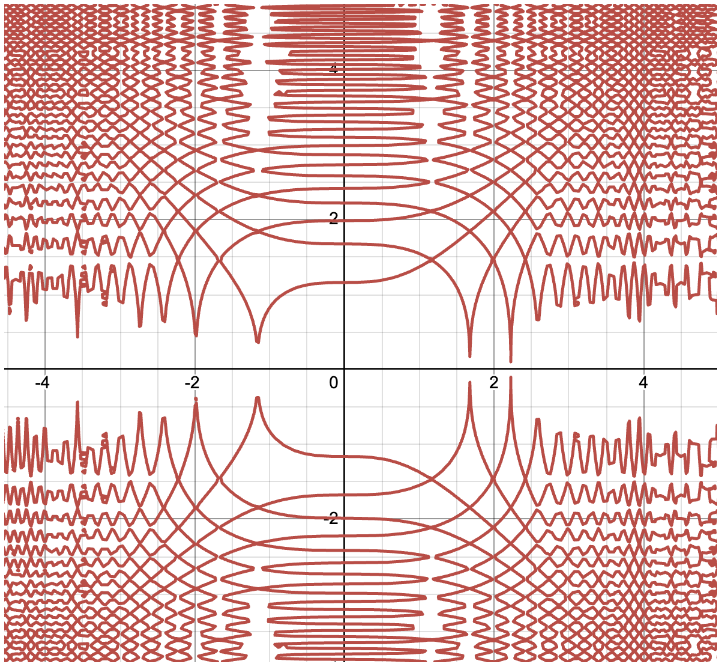

Figure 2: Graph of $\sin(x^3) + \cos(y^3) = \tan(\pi)$

And this changes again when we let $b = 4$:

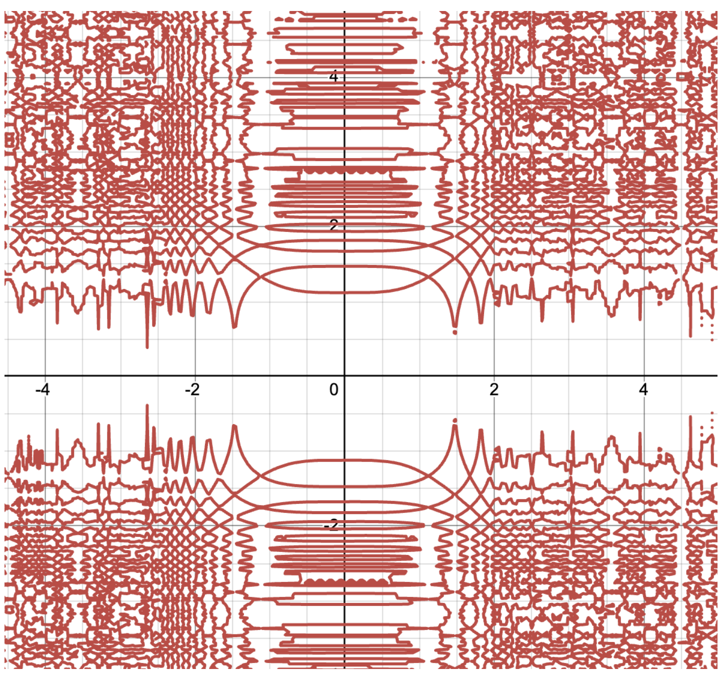

Figure 3: Graph of $\sin(x^4) + \cos(y^4) = \tan(\pi)$

It appears that as b gets larger, the graph of $\sin(x^b) + \cos(y^b) = \tan(\pi)$ seems to become more and more complicated. "Complicated" because the graph becomes more and more unconventional, which could be due to rendering errors in graphing such a complex equation. If we set b to a smaller number, like 1 for example, we get the following graph:

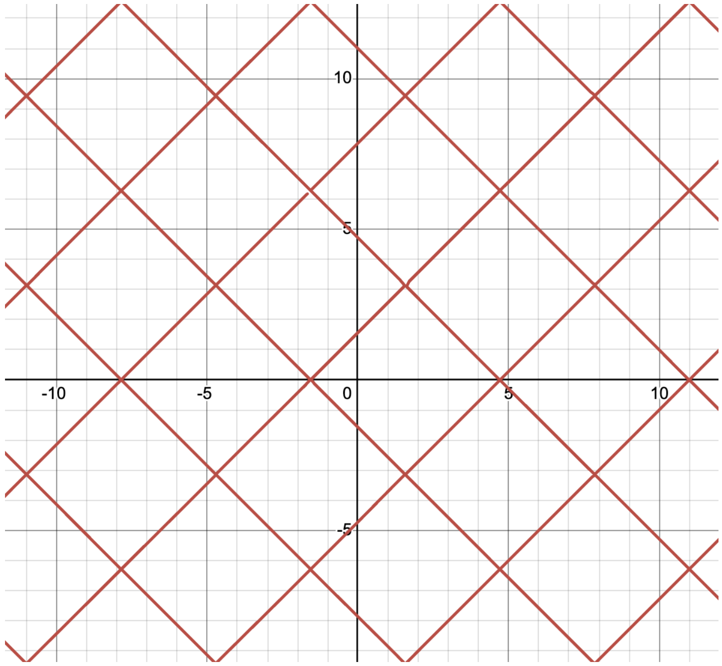

Figure 4: Graph of $\sin(x) + \cos(y) = \tan(\pi)$

This graph appears to be many different lines (potentially infinite) that intersecct at 90 degree angles. The diamonds formed by these intersections seem to be almost perfect squares. Judging by the curved nature of the graph of this equation for other values of b, it would be interesting if the graph of $\sin(x)+\cos(y) = \tan(\pi)$ actually did produce square-like shapes, or did consist of straight lines. Thus, the first goal of this paper is to find out whether the graph of $\sin(x) + \cos(y) = \tan(\pi)$ creates at least one square.

2 Solving

2.1 Approach

From the grpah, we can observe that a diamond that looks like a square is created by four different intersection points. Each of these four intersections serves as a vertex for this hypothesized square shape.

A square is a quadrilateral (specifically a rectangle) where all four sides have the same side length. All four internal angles of a square are 90 degrees. Thus, if we can find four lines that each intersect another two lines perpendicularly, and the distance between adjacent intersection points is equivalent, then we can confirm that a square is produced in the graph of this equation.

In order to do this, we can first estimate (based on the graph of $\sin(x)+\cos(y) = \tan(\pi)$) the coordinates of each of the four intersection points. Using these intersection points, we can construct the equations for four potential intersecting lines that belong to $\sin(x)+\cos(y) = \tan(\pi)$. We can then substitute the equation for each of these lines into $\sin(x) + \cos(y) = \tan(\pi)$ to find whether they satisfy the equation. The result we expect when we substitue each of these equations into $\sin(x) + \cos(y) = \tan(\pi)$ is explained in further detail in **2.4 Checking Lines**.

Once we have confirmed that each of these equations do in fact belong to the graph of $\sin(x) + \cos(y) = \tan(\pi)$, we can compare the equations of the lines to determine if lines are perpendicular or parallel to each other. In order for a square to be formed, each of the four lines must be perpendicular to two other lines and parallel to exactly one line. When we have proved that the four lines form 90 degree angles where they intersect, we can then check whether the sides of our hypothesized square all have the same length. To do this, we can calculate the distance between adjacent intersection points. If the distance is equivalent for all adjacent intersection points, then we know that the graph of $\sin(x) + \cos(y) = \tan(\pi)$ contains a square.

2.2 Intersection Points

First, we can estimate the coordinates of four intersection points that could possibly form a square. From the graph of $\sin(x) + \cos(y) = \tan(\pi)$ in Figure 4, it appears that possible intersection points that form a square occur at $\left(\frac{\pi}{2}, \pi\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(-\frac{\pi}{2}, 0\right)$, and $\left(\frac{\pi}{2}, -\pi\right)$. To visualize these points, we can graph them alongside $\sin(x) + \cos(y) = \tan(\pi)$:

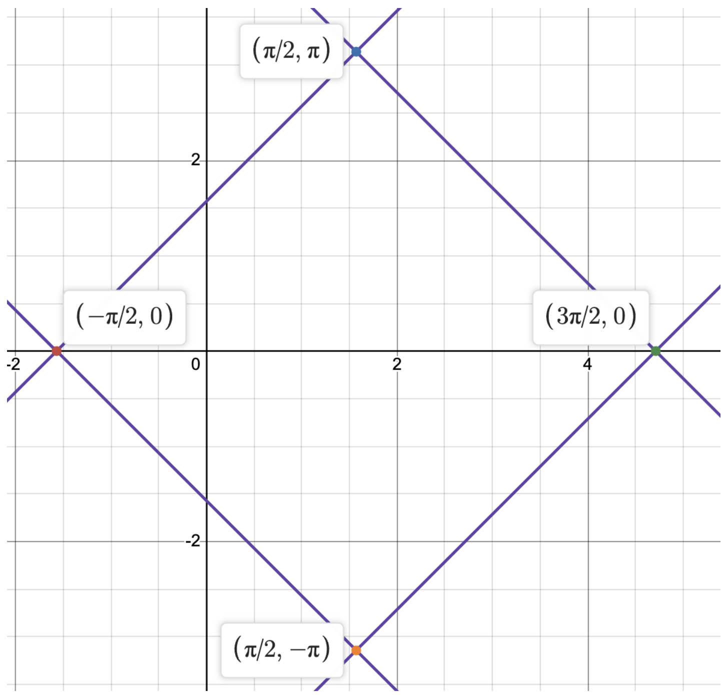

Figure 5: Graph of $\sin(x) + \cos(y) = \tan(\pi)$ with $\left(\frac{\pi}{2}, \pi\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(-\frac{\pi}{2}, 0\right)$, and $\left(\frac{\pi}{2}, -\pi\right)$

The four coordinates are highlighted in blue, orange, green, and red in the graph, as well as labeled. These are points that we can hypothesize belong to $\sin(x) + \cos(y) = \tan(\pi)$. We now need to use these four points to construct four intersecting lines that we can hypothesize belong to the graph of $\sin(x) + \cos(y) = \tan(\pi)$.

2.3 Constructing Lines

One line will be constructed using the points $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, another will be constructed using the points $\left(\frac{3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, -\pi\right)$, another will be constructed using the points $\left(-\frac{\pi}{2}, -\pi\right)$ and $\left(\frac{\pi}{2}, 0\right)$, and the last will be constructed using the points $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \pi\right)$.

The line that goes through $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ would have a slope of -1 and a y-intercept at $\left(0, \frac{3\pi}{2}\right)$, meaning the equation of the line in slope intercept form would be $y = -x + \frac{3\pi}{2}$. The line that goes through $\left(\frac{3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, -\pi\right)$ has a slope of 1 and a y-intercept of $\left(0, -\frac{3\pi}{2}\right)$, so its equation is $y = x - \frac{3\pi}{2}$. The line that goes through $\left(\frac{\pi}{2}, -\pi\right)$ and $\left(-\frac{\pi}{2}, 0\right)$ has a slope of -1 and a y-intercept at $\left(0, -\frac{\pi}{2}\right)$, so its equation is $y = -x - \frac{\pi}{2}$. The line that goes through $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ has a slope of 1 and a y-intercept at $\left(0, \frac{\pi}{2}\right)$, so its equation is $y = x + \frac{\pi}{2}$.

2.4 Checking Lines

Now that we have an equation for each of the four lines we predict belong to the graph of $\sin(x) + \cos(y) = \tan(\pi)$, we need to check whether they actually do. To do this, we can substitute each of the equations for y in the equation $\sin(x) + \cos(y) = \tan(\pi)$. If we find that after substituting the equation of a line, our equation comes out to be true, then we can confirm that the line does belong to $\sin(x) + \cos(y) = \tan(\pi)$. If we find that the equation is true, then we know that all points that belong to the line we substituted also belong to

$\sin(x)+\cos(y)=\tan(\pi)$. Thus, we can confirm that the substituted line belongs to $\sin(x)+\cos(y)=\tan(\pi)$.

First we can substitute the first equation, $y=-x+\frac{3\pi}{2}$, for y in $\sin(x)+\cos(y)=\tan(\pi)$:

$$\sin(x)+\cos\left(-x+\frac{3\pi}{2}\right)=\tan(\pi)$$

To solve for x here, we can make use of the following trigonometric identity: $\cos\left(\theta-\frac{3\pi}{2}\right)=-\sin(\theta)$. If we apply this to our equation, we can rewrite it as

$$\sin(x)-\sin(x)=\tan(\pi)$$

We know that $\tan(\pi)=0$, so this equation simplifies to

$$0=0$$

This statement is true, meaning that our equation is true. Therefore, all points that belong to $y=-x+\frac{3\pi}{2}$ also belong to $\sin(x)+\cos(y)=\tan(\pi)$, so $y=-x+\frac{3\pi}{2}$ belongs to $\sin(x)+\cos(y)=\tan(\pi)$.

Next, we can substitute the second line, $y=x-\frac{3\pi}{2}$ into $\sin(x)+\cos(y)=\tan(\pi)$:

$$\sin(x)+\cos\left(x-\frac{3\pi}{2}\right)=\tan(\pi)$$

To solve for x here, we can make use of the same trigonometric identity that $\cos\left(\theta-\frac{3\pi}{2}\right)=-\sin(\theta)$. If we apply this to our equation, we can rewrite it as

$$\sin(x)-\sin(x)=\tan(\pi)$$

$\tan(\pi)=0$, so this equation simplifies to

$$0=0$$

which is a true statement. Hence, we can conclude that all points on the line $y=x-\frac{3\pi}{2}$ also belong to $\sin(x)+\cos(y)=\tan(\pi)$.

We can repeat this process with our third line, $y=-x-\frac{\pi}{2}$. Substituting this for y in $\sin(x)+\cos(y)=\tan(\pi)$, we get

$$\sin(x)+\cos\left(-x-\frac{\pi}{2}\right)=\tan(\pi)$$

The expression $\cos\left(-x-\frac{\pi}{2}\right)$ can be rewritten as $\cos\left(-\left(x+\frac{\pi}{2}\right)\right)$. Using the trigonometric identity $\cos(\theta)=\cos(-\theta)$, we can rewrite $\cos\left(-\left(x+\frac{\pi}{2}\right)\right)$ as $\cos\left(x+\frac{\pi}{2}\right)$. Using another trig identity, $\cos\left(\theta+\frac{\pi}{2}\right)=-\sin(\theta)$, we can rewrite $\cos\left(x+\frac{\pi}{2}\right)$ as $-\sin(x)$. Hence we can rewrite our whole equation as

$$\sin(x)-\sin(x)=\tan(\pi)$$

This simplifies to

$$0=0$$

which is a true statement. Therefore all points that belong to $y=-x-\frac{\pi}{2}$ also belong to $\sin(x)+\cos(y)=\tan(\pi)$.

Lastly, we can check to see if our fourth equation, $y=x+\frac{\pi}{2}$ belongs to $\sin(x)+\cos(y)=\tan(\pi)$. Substituting the line for y in the equation, we have

$$\sin(x)+\cos\left(x+\frac{\pi}{2}\right)=\tan(\pi)$$

Using the same trigonometric identity as before, that $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta)$, we can rewrite the equation as

$$\sin(x) - \sin(x) = \tan(\pi)$$

Again, this simplifies to

$$0 = 0$$

Therefore we can confirm that our fourth line, $y = x + \frac{\pi}{2}$, also belongs to $\sin(x) + \cos(y) = \tan(\pi)$.

Thus, we know that all four of our hypothesized lines (and subsequently their intersection points) all belong to the graph of $\sin(x)+\cos(y) = \tan(\pi)$. Now that we have this information, we have to confirm that the four lines intersect perpendicularly, as the interior angles of a square are 90 degrees.

2.5 Perpendicular Lines

In order for two lines to be perpendicular, their slopes have to be negative reciprocal of each other. From the way that our four lines are oriented to form a square, each line must be perpendicular to two of the other three lines, and parallel to the last line.

The line $y = x + \frac{\pi}{2}$ has to be perpendicular to $y = -x - \frac{\pi}{2}$ and $y = -x - \frac{\pi}{2}$. The slopes of these two lines is -1, which is the negative reciprocal of the slope of $y = x + \frac{\pi}{2}$, which is 1. The line $y = -x + \frac{3\pi}{2}$ has to be perpendicular to $y = x + \frac{\pi}{2}$ and $y = x - \frac{3\pi}{2}$. The slopes of these two lines is 1, which is the negative reciprocal of the slope of $y = -x + \frac{3\pi}{2}$, which is -1. The line $y = x - \frac{3\pi}{2}$ has to be perpendicular to $y = -x + \frac{3\pi}{2}$ and $y = -x - \frac{\pi}{2}$. The slopes of these two lines is -1, which is the negative reciprocal of the slope of $y = x - \frac{3\pi}{2}$, which is 1.The line $y = -x - \frac{\pi}{2}$ has to be perpendicular to $y = x - \frac{3\pi}{2}$ and $y = x + \frac{\pi}{2}$. The slope of these two lines is 1, which is the negative reciprocal of the slope of $y = -x - \frac{\pi}{2}$, which is -1.

Thus, we have confirmed that each of the four lines is perpendicular to the two lines it is supposed to be perpendicular to. As a result, lines that form the opposite sides of a square are parallel to each other (have the same slope).

2.6 Equal Sides

Now that we have confirmed that each of the four lines is perpendicular to the correct lines, we need to check whether the sides of our hypothesized square are actually equal. This would mean that the distance between adjacent intersection points would have to be equivalent. To check this, we can recall the intersection points that we predicted in **2.2 Intersection Points**. These points were $\left(\frac{\pi}{2}, \pi\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(-\frac{\pi}{2}, 0\right)$, and $\left(\frac{\pi}{2}, -\pi\right)$. Since we have proved that these points all belong to $\sin(x)+\cos(y) = \tan(\pi)$, we just need to use the distance formula to find the distance between adjacent intersection points.

Based on the orientation of the square visible in Figure 5, we know that the distances between $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, -\pi\right)$, $\left(\frac{\pi}{2}, -\pi\right)$ and $\left(-\frac{\pi}{2}, 0\right)$, and $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ have to be equivalent.

The distance formula is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

where d is distance, x_1 and y_1 and the x and y coordinates of the first point, and x_2 and y_2 are the x and y coordinates of the second point. Plugging the coordinates $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ into the formula, we have $d = \pi\sqrt{2}$. Plugging $\left(\frac{3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, -\pi\right)$ in, we also get $d = \pi\sqrt{2}$. Plugging in $\left(\frac{\pi}{2}, -\pi\right)$ and $\left(-\frac{\pi}{2}, 0\right)$, we also get $d = \pi\sqrt{2}$. When plugging in $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, \pi\right)$, d is again $\pi\sqrt{2}$. Because we have confirmed the distance between adjacent intersection points is consistently $\pi\sqrt{2}$, we know that all the sides of our hypothesized square are equal.

3 Analysis

So far, we were able to hypothesize four possible intersection points of four lines in the graph of $\sin(x) + \cos(y) = \tan(\pi)$. From these intersection points, we were able to derive four possible lines that intersect one another, and then confirm that they actually do belong to $\sin(x) + \cos(y) = \tan(\pi)$. Because we found that each of the four lines intersects two other lines perpendicularly and that the distance between adjacent intersection points was consistent ($\pi\sqrt{2}$), we can confirm that those four intersection points are vertices of a square. This is because the interior angles of a square are all 90 degrees and the sides of a square are all equivalent in length. Thus, the answer to our original question is yes, the graph of $\sin(x) + \cos(y) = \tan(\pi)$ creates at least one square.

This result is interesting, given that trigonometric functions are usually represented as curves/waves. As we observed in **1 Introduction**, for other values of b, the equation $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ produced curved, intricate lines. However, when $b = 1$, the graph actually produces straight lines and even a square. This demonstrates the versatitility of trigonometric function—they do not just have to be thought of as wavy or curved functions. When used in certain ways, they can actually produce shapes that are not commonly associated with trigonometric functions.

4 Extension

4.1 Introduction

Now that we have considered the properties of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ when $b = 1$, we generalize our thinking by considering the equation without assigning a value to b. As shown in figure 1, 2, 3, the graph of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ becomes increasingly complex as the value of b increases. Noticeably, in all three graphs, $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ never seems to intersect the x-axis. Interestingly, when $b = 1$, we know that $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ does in fact intersect the x-axis, as we have shown that certain lines belong to the equation. The fact that no x-intercepts appear in figures 1, 2, and 3 could simply be a rendering error, but this has to be investigated further. Thus, the second part of this paper focuses on finding the x-intercepts of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ for some arbitrary integer b.

4.2 Solving

We know that x-intercepts occur when the y-coordinate of a point is 0. So, to find the x-intercepts of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$, we can substitute 0 for y:

$$\sin\left(x^b\right) + \cos\left(0^b\right) = \tan\left(\pi\right)$$

Zero to the power of any constant is just 0, so we have

$$\sin\left(x^b\right) + \cos\left(0\right) = \tan\left(\pi\right)$$

$\cos(0)$ is 1, so we can rewrite the equation as

$$\sin\left(x^b\right) + 1 = \tan\left(\pi\right)$$

Simplifying this, we have

$$\sin\left(x^b\right) = \tan\left(\pi\right) - 1$$

We can now apply arcsin to both sides of this equation. When applying arcsin, we have to keep in mind that we get a principal and symmetric solution. As a result, when applying arcsin, we have two equations:

$$x^b = \arcsin\left(\tan(\pi) - 1\right) + 2\pi n$$

$$x^b = \pi - \arcsin\left(\tan(\pi) - 1\right) + 2\pi n$$

We know that $\tan(\pi)$ is 0. Thus, we can rewrite these equations as

$$x^b = \arcsin(-1) + 2\pi n$$

$$x^b = \pi - \arcsin(-1) + 2\pi n$$

$\arcsin(-1)$ is $\frac{\pi}{2}$, so we can again rewrite these equations as

$$x^b = -\frac{\pi}{2} + 2\pi n$$

$$x^b = \pi + \frac{\pi}{2} + 2\pi n$$

We can now solve these equations for x. To do this, we need to take the bth root of $\frac{3\pi}{2} + 2\pi n$ and $-\frac{\pi}{2} + 2\pi n$. However, we need to keep in mind that when b is even, we need to add a $\pm$ to the beginning of our solutions, as both positive and negative solutions are correct. Hence, we can say that when b is odd, x-intercepts occur at

$$x = \sqrt[b]{-\frac{\pi}{2} + 2\pi n}$$

and

$$x = \sqrt[b]{\frac{3\pi}{2} + 2\pi n}$$

When b is even, x-intercepts occur at

$$x = \pm\sqrt[b]{-\frac{\pi}{2} + 2\pi n}$$

and

$$x = \pm\sqrt[b]{\frac{3\pi}{2} + 2\pi n}$$

5 Analysis

From the four solutions that we found, we know that there are x-intercepts of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ when b is an integer that is not 1. When b is an odd number, there should be an x-intercept that occurs for any value of n.This is because we can take the bth root of some number, even if the number is negative, only if b is odd. When b is even, x-intercepts do not occur for all values of n. If a value of n causes the expression under the square root to be less than 0, then we cannot take the bth root of that expression when b is even. However, when b is even, every value of n that results in the expression under the square root being at least 0 will produce two solutions—a positive and negative one—due to the $\pm$ sign in front of the solution.

When looking at figure 1, the graph of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ when $b = 2$, it appears that the graph has no x-intercepts. Though it cannot be displayed in this paper, when graphed in a graphing calculator, the graph never appears to reach the x-axis for any value of x. We can conclude that this is a rendering error, as we have found that there are some x-intercepts for when b is even. The same holds true for the graphs in figures 2 and 3. Figure 4, when $b = 1$, is the only graph where the x-intercepts appeared to render correctly.

6 Further Inquiry

An interesting change to this extension question would be to allow b to be any real number and not just an integer. We could compare the the nature of the x-intercepts when b is any real number to when b is just an integer. It would also be meaningful to find the y-intercepts of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan\left(\pi\right)$ and see how similar our solutions are to the x-intercepts we found.

Based on the first question asked on this paper, there are some different directions for further inquiry that we could take. For example, in figure 2, there appears to be a circle-like shape that appears in the center of the graph. It would be meaningful to find whether or not this circle is actually a real circle. Similarly, the graphs of $\sin\left(x^b\right) + \cos\left(y^b\right) = \tan(\pi)$ for different values of b produce increasingly interesting shapes. We could find shapes that appear the most intriguing and check whether they are actually the shapes they appear to be. Doing this gives us greater insight to the diversity of trigonometric functions beyond the notion of a simple sine wave. They would add to our understanding of trigonometric graphs and the types of graphs they are able to produce when manipulated in certain ways.