Homework 7

Sahana Sarangi

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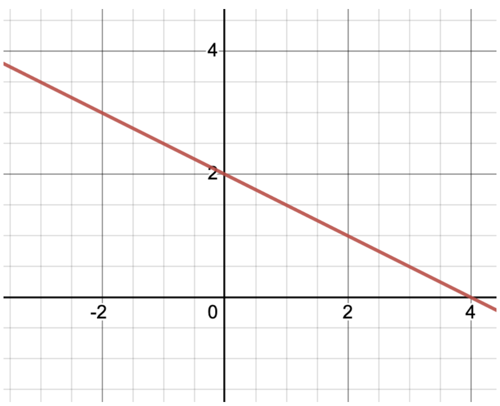
Problem 4.2: Sketch an accurate picture of the line having equation $y = 2 - \frac{1}{2}x$. Let α be an unknown constant.

Part (a): Find the point of intersection between the line you have graphed and the line $y = 1 + \alpha x$; your answer will be a point in the xy plane whose coordinates involve the unknown α.

Part (b): Find α so that the intersection point in (a) has x-coordinate 10.

Part (c): Find α so that the intersection point in (a) lies on the x-axis.



Part (a) Solution: We can use substitution to find the x and y-coordinates of the intersection of the two lines. Because we have the equations $y = 1 + \alpha x$ and $y = 2 - \frac{1}{2}x$, we can start by substituting $1 + \alpha x$ for y in the second equation. The equation would look like:

$$1 + \alpha x = 2 - \frac{1}{2}x$$

We can now solve for x in terms of α in this equation to find the x-coordinate of the lines' intersection:

$$\alpha x + \frac{1}{2}x = 1$$

$$x\left(\alpha + \frac{1}{2}\right) = 1$$

$$x = \frac{1}{\alpha + \frac{1}{2}}$$

Simplifying:

$$x = \frac{2}{2\alpha + 1}$$

Now that we know the x-coordinate of the intersection, we can find the y-coordinate by substituting this value back into either of the initial equations. We can use the equation $y = 2 - \frac{1}{2}x$.

$$y = 2 - \frac{1}{2}\left(\frac{2}{2\alpha+1}\right)$$

Simplifying:

$$\begin{aligned} y &= 2 - \frac{2}{4\alpha+2} \\ y &= 2 - \frac{1}{2\alpha+1} \\ y &= \frac{4\alpha+1}{2\alpha+1} \end{aligned}$$

Therefore, the coordinates of the intersection between these two lines is $\boxed{\left(\frac{2}{2\alpha+1}, \frac{4\alpha+1}{2\alpha+1}\right).}$

Part (b) Solution: We found in Part (a) that the x-coordinate of the point of intersection is $\frac{2}{2\alpha+1}$. To find α when the x-coordinate is 10, we have to solve this equation:

$$10 = \frac{2}{2\alpha+1}$$

Solving for α:

$$\begin{aligned} 20\alpha + 10 &= 2 \\ \alpha &= -\frac{2}{5} \end{aligned}$$

Therefore, the value of α when the x-coordinate is 10 is $\boxed{-\frac{2}{5}.}$

Part (c) Solution: We know that any point on the x-axis will have a y-value of 0. Therefore, if the intersection were to lie on the x-axis, its y-coordinate would also be 0. As we found in Part (a), the y-coordinate of the intersection in terms of α was $\frac{4\alpha+1}{2\alpha+1}$. We can solve this equation to find the value of α when the y-coordinate is 0:

$$\frac{4\alpha+1}{2\alpha+1} = 0$$

Simplifying:

$$\begin{aligned} 4\alpha + 1 &= 0 \\ \alpha &= -\frac{1}{4} \end{aligned}$$

Therefore, the value of α when the intersection lies on the x-axis is $\boxed{-\frac{1}{4}.}$

Problem 4.3:

Part (a): What is the area of the triangle determined by the lines $y = -\frac{1}{2}x + 5$, $y = 6x$ and the y-axis?

Part (b): If $b > 0$ and $m < 0$, then the line $y = mx + b$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of m and b.

Part (c): The lines $y = mx + 5$, $y = x$ and the y-axis form a triangle in the first quadrant. Suppose this triangle has an area of 10 square units. Find m.

Part (a) Solution: To find the area of the triangle determined by the given lines, we will first need to find the coordinates of intersection of all of these lines, then use the formula for the area of a triangle. We can start by finding all the coordinates of intersection. The y-axis is also defined as the line $x = 0$, so we can start by substituting 0 for x in the equation $y = -\frac{1}{2}x + 5$. The equation would then look like:

$$y = -\frac{1}{2}(0) + 5$$

Simplifying:

$$y = 5$$

Therefore, the first coordinate of intersection is the point $(0, 5)$. We can repeat this process to find the other two points of intersection. We can substitute 0 for x again into the second equation, $y = 6x$. The equation would look like:

$$y = 6(0)$$
$$y = 0$$

We can say the second point of intersection is at $(0, 5)$. Both $(0, 0)$ and $(0, 5)$ are intersections on the line $x = 0$. Therefore, the last intersection point must be between $y = -\frac{1}{2}x + 5$ and $y = 6x$. To find this point, we can substitute $6x$ for y into the first equation.

$$6x = -\frac{1}{2}x + 5$$

Solving for x:

$$x = \frac{10}{13}$$

To find the y-coordinate of the intersection, we can plug this value of x back into either of the initial equations. We can use $y = 6x$ for simplicity:

$$y = 6 \cdot \frac{10}{13}$$
$$y = \frac{60}{13}$$

Therefore the coordinates of the last point of intersection are $\left(\frac{10}{13}, \frac{60}{13}\right)$. To solve for the area of the triangle, we can use the formula $A = \frac{1}{2}bh$, where A is the area of the triangle, b is one of the bases of the triangle, and h is the height perpendicular to that base.

Because we can take any base of the triangle, we can choose the base that is on the line $x = 0$. To find the length of the base, we have to find the distance between the two intersection points that belong to this line. As we found earlier, they are $(0, 0)$ and $(0, 5)$. Because the points are on the same vertical line, the length of this base is 5 units.

Next, we can find the length of the height of the triangle that is perpendicular to the line $x = 0$. We can say the length of the height is the distance between $x = 0$ and the intersection point $\left(\frac{10}{13}, \frac{60}{13}\right)$. The height will run along the line that is perpendicular to $x = 0$ and that includes the point $\left(\frac{10}{13}, \frac{60}{13}\right)$. To do this, we have to find the point along $x = 0$ that is directly left of this intersection point. The y-coordinate of the intersection is $\frac{60}{13}$, so the coordinates of this point are $\left(0, \frac{60}{13}\right)$. Because both $\left(0, \frac{60}{13}\right)$ and $\left(\frac{10}{13}, \frac{60}{13}\right)$ are along the same horizontal line, the distance between them is the difference of their x-coordinates, which in this case is $\frac{10}{13}$.

Now that we have the lengths of the base and height, we can plug them into the formula for the area of the triangle.

$$A = \frac{1}{2} \cdot 5 \cdot \frac{10}{13}$$
$$A = \frac{25}{13}$$

Therefore, the area of the triangle is $\boxed{\frac{25}{13} \text{ square units.}}$

Part (b) Solution: We know that the triangle that the line cuts off is a right triangle because its legs are formed by the x and y axes which are perpendicular to each other (forming a $90°$ angle). Therefore, if we use the formula for the area of the triangle, we need to find the lengths of the base and height, which in this case are the two legs of the right triangle. Because the legs are formed by the x and y axes, we can find their lengths by calculating the distance from the origin $(0, 0)$ and the x and y-intercepts. In slope intercept form, b represents the y-intercept of the line, meaning that the y-intercept is at the point $(0, b)$. Because all x-intercepts have a y-coordinate of 0, we can find the x-intercept of this line by substituting 0 for y in the equation. This would look like:

$$0 = mx + b$$

Solving for x:

$$x = -\frac{b}{m}$$

Therefore, the x-intercept of this line is at $\left(-\frac{b}{m}, 0\right)$. To find the lengths of the legs, we have to find the distance between the origin and each of the intercepts. Because the origin and y-intercept are on the same vertical line, we can say they are b units apart. The origin and the x-intercept are on the same horizontal line, so we can say they are $-\frac{b}{m}$ units apart. We can now plug these values into the equation for the area of a circle:

$$A = \frac{1}{2} \cdot b \cdot -\frac{b}{m}$$

Simplifying:

$$A = -\frac{b^2}{2m}$$

The area of this triangle is $\boxed{-\frac{b^2}{2m} \text{ square units.}}$

Part (c) Solution: To solve this problem, we can use a similar process to Part (a). We can start by finding the intersection points between these three lines. We can first find the intersection between the lines $x = 0$ and $y = x$ by substituting 0 for x in the second equation.

$$y = 0$$

Therefore, the first intersection is at the point $(0, 0)$. The second intersection point will be between $x = 0$ and $y = mx + 5$. We can again substitute 0 for x into this equation:

$$y = m(0) + 5$$
$$y = 5$$

Therefore, the second intersection is at the point $(0, 5)$. The third intersection is between the lines $y = mx + 5$ and $y = x$. We can substitute y by setting the equations equal to one another:

$$x = mx + 5$$

Solving for x:

$$x(1 - m) = 5$$
$$x = \frac{5}{1 - m}$$

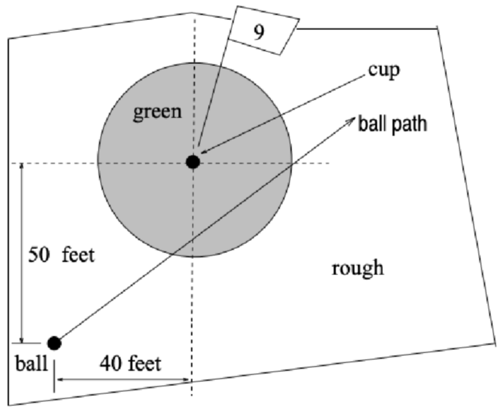
To find the y-coordinate of the intersection, we can plug this back value into either of the initial equations. For simplicity, we can use the equation $y = x$. Using this equation, we can say the coordinates of the intersection are $\left(\frac{5}{1-m}, \frac{5}{1-m}\right)$. Now, we need to find the lengths of the base and height of the triangle to use the formula $A = \frac{1}{2}bh$ to find the area. We can again use $x = 0$ as the base of the triangle. The two intersections along this line are $(0, 0)$ and $(0, 5)$. Therefore, the length of the base is 5 units. To find the length of the height, we have to find the distance between the intersection that does not belong to the base, $\left(\frac{5}{1-m}, \frac{5}{1-m}\right)$, and the line $x = 0$. The point directly to the left of $\left(\frac{5}{1-m}, \frac{5}{1-m}\right)$ that also belongs to $x = 0$ is the point $\left(0, \frac{5}{1-m}\right)$. Because these points are on the same horizontal line, the distance between them is the difference of their x-coordinates, which in this case is $\frac{5}{1-m}$. We have found the lengths of the base and height of the triangle, but we are also given that the area of the triangle is 10 square units. If we plugged in all of these values into the formula, it would look like:

$$10 = \frac{1}{2} \cdot \frac{5}{1 - m} \cdot 5$$

Now we can solve for m.

$$10 = \frac{25}{2 - 2m}$$
$$20 - 20m = 25$$
$$\boxed{m = -\frac{1}{4}}$$

Problem 4.6: The cup on the 9th hole of a golf course is located dead center in the middle of a circular green that is 70 feet in diameter. Your ball is located as in the picture below:



The ball follows a straight line path and exits the green at the right-most edge. Assume the ball travels a constant rate of 10 ft/sec.

Part (a): Where does the ball enter the green?

Part (b): When does the ball enter the green?

Part (c): How long does the ball spend inside the green?

Part (d): Where is the ball located when it is closest to the cup and when does this occur.

Part (a) Solution: We can impose a coordinate system where the 9th hole is the origin, the vertical line through the center of the green is the y-axis, and the horizontal line through the center of the green is the x-axis. Intervals along the axes are in terms of feet, where one unit is one foot. Using this system, we can write an equation for the boundary of the circle. We defined the center of the circle as $(0, 0)$ and the problem states the diameter of the circle is 70 feet. If the diameter is 70 feet, the radius must be 35 feet. Plugging these into the standard form of a circle, we can write the equation

$$x^2 + y^2 = 35^2$$

In this system, the ball would be located at point $(-40, -50)$ as per the diagram. We also know that the ball exits the green at the point $(35, 0)$, the eastern most edge of the circle. These are two points along the line which the ball follows. Using these points, we can write the equation for this line in standard form, $y = mx + b$. First, we can find the slope.

$$\frac{-50 - 0}{-40 - 35}$$

Meaning the slope is $\frac{2}{3}$. As of right now, the equation of the line is

$$y = \frac{2}{3}x + b$$

Now, we can plug in either point into this equation to find the value of b. We can use the point $(35, 0)$.

$$0 = \frac{2}{3} \cdot (35) + b$$

Solving for b:

$$b = -\frac{70}{3}$$

Therefore, the equation of this line is

$$y = \frac{2}{3}x - \frac{70}{3}$$

Now, we can substitute the value of y in this equation into the equation for the circle to find the point at which the line intersects the circle. The equation would look like:

$$x^2 + \left(\frac{2}{3}x - \frac{70}{3}\right)^2 = 35^2$$

Solving for x:

$$\begin{aligned} x^2 + \frac{4}{9}x^2 - \frac{280}{9}x + \frac{4900}{9} &= 1225 \\ \frac{13}{9}x^2 - \frac{280}{9}x - \frac{6125}{9} &= 0 \\ x = -\frac{175}{13}&, x = 35 \end{aligned}$$

Therefore, the ball intersects the boundary of the green at x-coordinates of $-\frac{175}{13}$ and 35. However, the ball enters the green from the west, so the x-coordinate of the intersection must be $-\frac{175}{13}$. To find the y-coordinate, we can now plug this value back into the equation for the line that the ball follows. The equation would look like:

$$y = \frac{2}{3} \cdot -\frac{175}{13} - \frac{70}{3}$$

Simplifying:

$$y = -\frac{1260}{39}$$

Therefore, the ball enters the green at the point $\boxed{\left(-\frac{175}{13}, -\frac{1260}{39}\right).}$

Part (b) Solution: To find when the ball enters the green, we have to find the time the ball takes to reach the point $\left(-\frac{175}{13}, -\frac{1260}{39}\right)$. To do this, we can first find the distance from this point and the ball's initial starting point, $(-40, -50)$. To do this, we can plug the values into the distance formula, where d is the distance the ball travels:

$$d = \sqrt{\left(-\frac{175}{13} + 40\right)^2 + \left(-\frac{1260}{39} + 50\right)^2}$$

Simplifying:

$$d = \sqrt{\left(\frac{345}{13}\right)^2 + \left(\frac{690}{39}\right)^2}$$

$$d = \sqrt{\frac{119025}{169} + \frac{476100}{1521}}$$

$$d = \sqrt{\frac{13225}{13}}$$

To find when the ball enters the green, we have to calculate the time it takes for it to travel $\sqrt{\frac{13225}{13}}$ feet. The problem states that the ball moves at a rate of 10 feet/second. We know that distance is equal to rate multiplied by time, so we can say that the ball will enter the green after

$$\boxed{\frac{\sqrt{\frac{13225}{13}}}{10} \text{ seconds.}}$$

Part (c) Solution: We know that the ball enters the green at the point $\left(-\frac{175}{13}, -\frac{1260}{39}\right)$ and exits at the point $(35, 0)$. We can start by calculating the distance between the two points by using the distance formula:

$$d = \sqrt{\left(-\frac{1260}{39}\right)^2 + \left(-\frac{175}{13} - 35\right)^2}$$

Simplifying:

$$d = \sqrt{\frac{1587600}{1521} + \frac{396900}{169}}$$
$$d = \sqrt{\frac{44100}{13}}$$

We know that the distance that the ball rolls for inside the green is $\sqrt{\frac{44100}{13}}$ feet and that the ball moves at 10 feet/second. Because distance is speed multiplied by time, we can say the ball spends

$$\boxed{\frac{\sqrt{\frac{44100}{13}}}{10} \text{ seconds inside the green.}}$$

Part (d) Solution: The ball is closest to the cup at the point where the line that runs perpendicular to the ball's path and includes the the point at which the cup is located at intersects the ball's path. To start, we can find the equation for the line that runs perpendicular to the ball's path. We know that the ball's path is defined by the line $y = \frac{2}{3}x - \frac{70}{3}$. Therefore, the slope of the line perpendicular to this will be $-\frac{3}{2}$, the negative reciprocal. The equation so far looks like:

$$y = -\frac{3}{2} + b$$

To find b, we have to find a point that belongs to the perpendicular line. In this case, one of those points is the location of the cup, $(0, 0)$. Plugging in these coordinates into the equation:

$$0 = -\frac{3}{2}(0) + b$$

Solving for b:

$$b = 0$$

Therefore, the equation for the perpendicular line is

$$y = -\frac{3}{2}x$$

The point at which the ball is closest to the cup is the point where this perpendicular line intersects the ball's path. To find this point, we can substitute $-\frac{3}{2}x$ for y in the equation for the ball's path. The equation would then look like:

$$-\frac{3}{2}x = \frac{2}{3}x - \frac{70}{3}$$

Solving for x:

$$-\frac{13}{6}x = -\frac{70}{3}$$
$$x = \frac{140}{13}$$

Therefore, the x-coordinate of this intersection is $\frac{140}{13}$. To find the y-coordinate, we can plug this x-value back into the equation for the ball's path. The equation would be:

$$y = \frac{2}{3} \cdot \frac{140}{13} - \frac{70}{3}$$

Simplifying:

$$y = -\frac{630}{39}$$

This means that the $\boxed{\text{ball will be closest to the cup at the point } \left(\frac{140}{13}, -\frac{630}{39}\right).}$ To find when this occurs, we have to again find the distance between this point and the ball's starting point and then divide by the ball's speed, 10 ft/sec. We can use the distance formula to find the distance between the two points:

$$d = \sqrt{\left(\frac{140}{13} + 40\right)^2 + \left(-\frac{630}{39} + 50\right)^2}$$

Simplifying:

$$d = \sqrt{\left(-\frac{660}{13}\right)^2 + \left(\frac{1320}{39}\right)^2}$$
$$d = \sqrt{\frac{435600}{169} + \frac{1742400}{1521}}$$
$$d = \sqrt{\frac{48400}{13}}$$

Because the distance the ball has to travel to this point is $\sqrt{\frac{48400}{13}}$ and its speed is 10 ft/sec, we can say that the ball reaches this point after

$$\boxed{\frac{\sqrt{\frac{48400}{13}}}{10} \text{ seconds.}}$$