Homework 16

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Problem 7.11: A Norman window is a rectangle with a semicircle on top. Suppose that the perimeter of a particular Norman window is to be 24 feet. What should its dimensions be in order to maximize the area of the window and, therefore, allow in as much light as possible?

Solution: To solve this problem, we can let x be the side of the rectangle parallel to the diameter of the semicircle. We can now find the length of the arc of the semicircle. If x is the diameter, the perimeter of the whole circle would be πx. The length of the arc of the semicircle would be half of this perimeter, so the length is $\frac{1}{2}x\pi$. If we know the length of the semicircular part, the length of the side of the rectangle parallel to the diameter of it, and that the total perimeter is 24, we know that the combined length of the two other sides of the rectangle would be $24 - x - \frac{1}{2}x\pi$. Because the length of each of the sides is equivalent, the length of each is

$$\frac{24 - x - \frac{1}{2}x\pi}{2}, \text{ or } \frac{48 - 2x - x\pi}{4}.$$

Now, we need to find the equation that represents the total area of the window. To do this, we need an expression that represents the area of the semicircle and another expression that represents the area of the rectangle. The rectangle has sides $\frac{48-2x-x\pi}{4}$ and x, so its total area would be

$$\frac{48x - 2x^2 - x^2\pi}{4}$$

The semicircle has radius $\frac{1}{2}x$, so we can use the formula for the area of a semicircle, $\frac{\pi r^2}{2}$ where r is the radius, to find the area. The area of the semicircle would be $\frac{1}{8}x^2\pi$. Combining both the area of the semicircle and rectangle, we can say the equation that represents the area of the Norman window (where y is the area) is

$$y = \frac{48x - 2x^2 - x^2\pi}{4} + \frac{1}{8}x^2\pi$$

Simplifying this equation would result in

$$y = 12x - \left(\frac{4+\pi}{8}\right)x^2$$

To find x when the window is at its maximum area, we need to find the vertex of this parabola. We can find the x-coordinate by using the formula $-\frac{b}{2a}$:

$$-\frac{b}{2a} = \frac{12}{\frac{4+\pi}{4}} = \frac{48}{4+\pi}$$

Therefore, $x = \frac{48}{4+\pi}$ when the Norman window has the maximum area. To find the length of the other side of the rectangle, we need to substitute this value for x into the equation for the other side:

$$\frac{48 - 2\left(\frac{48}{4+\pi}\right) - \left(\frac{48}{4+\pi}\right)\pi}{4} = \frac{24}{4+\pi}$$

This means that the dimensions of the Norman window are

$$\boxed{\frac{48}{4+\pi} \times \frac{24}{4+\pi}}$$

Problem 7.13: You have \$6000 with which to build a rectangular enclosure with fencing. The fencing material costs \$20 per meter. You also want to have two paritItions across the width of the enclosure, so that there will be three separated spaces in the enclosure. The material for the partitions costs \$15 per meter. What is the maximum area you can achieve for the enclosure?

Solution: We can let w represent the width of the rectangular enclosure in meters. We can let l represent the length of the rectangular enclosure in meters. The price of building the lengths and widths are \$20 per meter, meaning we can express their total cost as $40w + 40l$. There are two partitions (each the length of the width) that cost \$15 per meter. We can express their cost as $30w$. We know that the total cost of all parts of the enclosure is \$6000, so we can write the following equation:

$$70w + 40l = 6000$$

Now, we can solve this equation for l:

$$l = 150 - \frac{7}{4}w$$

We know that the area of this enclosure is its length times its width. If its length is $150 - \frac{7}{4}w$ and its width is w, we can multiply these to find the area (y):

$$y = w(150 - \frac{7}{4}w)$$

Simplifying:

$$y = 150w - \frac{7}{4}w^2$$

The maximum area of this enclosure will be at the vertex of this parabola. We can start by using the formula $-\frac{b}{2a}$ to find the x-coordinate of the vertex:

$$-\frac{b}{2a} = \frac{150}{\frac{14}{4}} = \frac{300}{7}$$

Now, to find the maximum area of the enclosure, we need to substitute $\frac{300}{7}$ for w in this equation and solve for the y-coordinate:

$$y = 150\left(\frac{300}{7}\right) - \frac{7}{4}\left(\frac{300}{7}\right)^2$$

$$y \approx 3214.2857$$

Therefore, the maximum area of this enclosure is $\boxed{\text{3214.2857 square meters.}}$

Problem 7.15: Two particles are moving in the xy-plane. They move along straight lines at constant speed. At time t, particle A's position is given by

$$x = t + 2,\ y = \frac{1}{2}t - 3$$

and particle B's position is given by

$$x = 12 - 2t,\ y = 6 - \frac{1}{3}t$$

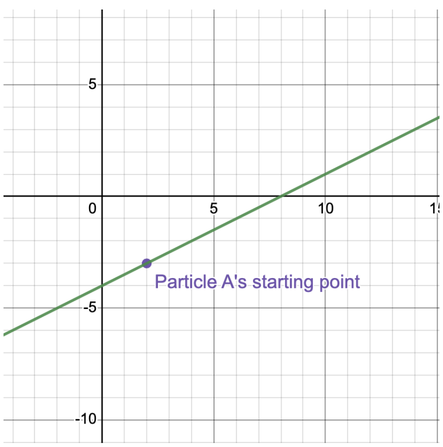
Part (a): Find the equation of the line along which particle A moves. Sketch this line, and label A's starting point and direction of motion.

Part (b): Find the equation of the line along which particle B moves. Sketch this line on the same axes, and label B's starting point and direction of motion.

Part (c): Find the time (i.e., the value of t) at which the distance between A and B is minimal. Find the locations of particles A and B at this time, and label them on your graph.

Part (a) Solution: Given the two equations that represent particle A's x and y coordinates at time t, we know that the coordinates of particle A at time t are $\left(t+2, \frac{1}{2}t-3\right)$. To find this line, we need at least 2 different coordinates. To find different coordinates, we can simply substitute 2 different values for t and take the resulting coordinates. Any random 2 values will work, as there are no restrictions on what time t is. For simplicity, we can take $t=0$ and $t=2$. Plugging in $t=0$, the coordinates of particle A would be $(2,-3)$. Plugging in $t=2$, the coordinates of particle A would be $(4,-2)$. The slope of the line formed by these two points would be $\frac{1}{2}$. Plugging in the point $(2,-3)$, the equation of this line in slope intercept form would then be $\boxed{y=0.5x-4.}$

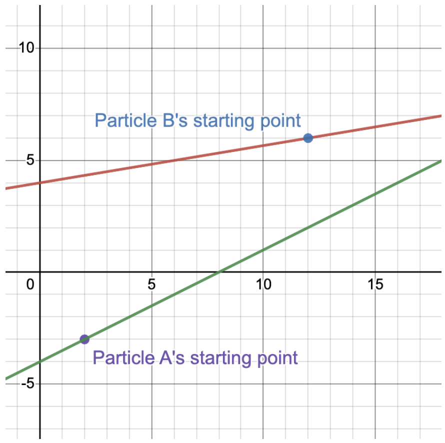
Graph of this line:



We found that Particle A's coordinates at time 0 were $(2,-3)$, meaning this is its starting point. Particle A's location at time 0 has lesser x and y coordinates than its location at time 2, meaning that Particle A moves upward along the line. Its x and y coordinates are increasing.

Part (b) Solution: To find the line that particle B travles on, we can repeat the same process as we used in part (a). The coordinates of particle B at time t is $(12-2t, 6-\frac{1}{3}t)$. Two find two points along the line, we can substitute any two values for t. For simplicity, we can use $t=0$ and $t=3$. When $t=0$, the coordinates for particle B are $(12,6)$. When $t=3$, the coordinates for particle B are $(6,5)$. Using these two points, we can find that the slope of the line that particle B travels on is $\frac{1}{6}$. If we use the point $(6,5)$, the equation of the line that particle B travels on in point slope form is $\boxed{y-5=\frac{1}{6}(x-6)}$.

Graph of this line:



We found that Particle B's coordinates at time 0 were $(12, 6)$, meaning this is its starting point. Particle B's location at time 0 has a greater x and y coordinates than its location at time 3, meaning that Particle B moves downward along the line. Its x coordinate and y coordinate are decreasing.

Part (c): We want to minimize the distance between particles A and B. We can start by writing the distance between the two using the distance formula. Recall that particle A's coordinates are $(t+2, 0.5t-3)$ and particle B's coordinates are $(12-2t, 6-\frac{1}{3}t)$. Plugging in these two points, the distance between them would be:

$$d = \sqrt{(t+2-12+2t)^2 + \left(0.5t-3-6+\frac{1}{3}t\right)^2}$$

Simplifying:

$$d = \sqrt{(3t-10)^2 + \left(\frac{5}{6}t-9\right)^2}$$

$$d = \sqrt{9t^2 - 60t + 100 + \frac{25}{36}t^2 - 15t + 81}$$

$$d = \sqrt{\frac{349}{36}t^2 - 75t + 181}$$

We cannot find the maximum of a function that is a quadratic under a square root, meaning that we cannot use this function as is. Instead, we can take the equation for the distance squared, or

$$d^2 = \frac{349}{36}t^2 - 75t + 181$$

This formula is simply squaring the output of the function, so the x-coordinate at the vertex of the function will be the same as that of the equation for just the distance. We can use the formula $-\frac{b}{2a}$ to find the x-coordinate of the vertex:

$$-\frac{b}{2a} = \frac{75}{\frac{349}{18}} = \frac{1350}{349}$$

The x-coordinate of the vertex of this parabola represents the time at which there is a minimal distance between particles A and B, so A and B will be the closest at time $\boxed{\frac{1350}{349}.}$

To find the location of particle A at this time, we can just substitute $\frac{1350}{349}$ for t in particle A's coordinates. Particle A's coordinates at this time would be $\left(\frac{1350}{349}+2, \frac{1350}{698}-3\right)$. To find the location of particle B at this point, we can substitute $\frac{1350}{349}$ for t in its coordinates as well. Particle B's coordinates at this time would be $\left(12-\frac{2700}{349}, 6-\frac{1350}{1047}\right)$.

Particles A and B at this time on our graph: