Homework 4

Sahana Sarangi

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Problem 11.4: Define two new functions:

$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

and

$$y = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

These are called the basic hyperbolic trigonometric functions.

Part (a): Sketch rough graphs of these two functions.

Part (b): The graph of the equation $x^2 - y^2 = 1$ is shown below; this is called the unit hyperbola. For any value a, show that the point $(x, y) = (\cosh(a), \sinh(a))$ is on the unit hyperbola. (Hint: Verify that $[\cosh(x)]^2 - [\sinh(x)]^2 = 1$, for all x.

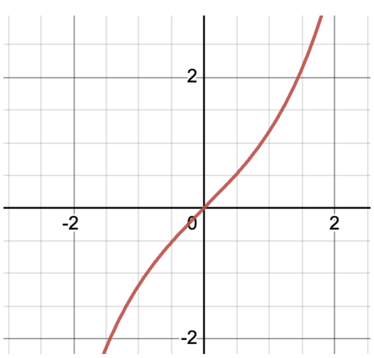
Part (c): A hanging cable is modeled by a portion of the graph of the function:

$$y = a \cosh\left(\frac{x-h}{a}\right) + C,$$

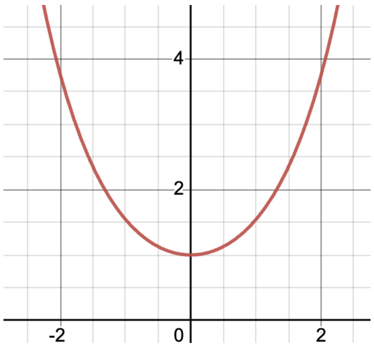
for appropriate constants a, h and C. The constant h depends on how the coordinate system is imposed. A cable for a suspension bridge hangs from two 100 ft. high towers located 400 ft. apart. Impose a coordinate system so that the picture is symmetric about the y-axis and the roadway coincides with the x-axis. The hanging cable constant is $a = 500$ and $h = 0$. Find the minimum distance from the cable to the road.

Part (a) Solution:

Graph of $y = \sinh(x)$:



Graph of $y = \cosh(x)$:



Part (b) Solution: If the point $(\cosh(a), \sinh(a))$ is a point on the unit hyperbola, then substituting $\cosh(a)$ for x and $\sinh(a)$ for y in the unit hyperbola equation, $x^2 - y^2 = 1$, will let the equation hold true. The problem defines $\sinh(a)$ as $\left(\frac{e^x - e^{-x}}{2}\right)^2$ and $\cosh(a)$ as $\left(\frac{e^x + e^{-x}}{2}\right)^2$. Substituting these values for x and y in the equation:

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$$

Simplifying:

$$\frac{e^{2x} + 2e^0 + e^{-2x}}{4} - \frac{e^{2x} - 2e^0 + e^{-2x}}{4} = 1$$

$$1 = 1$$

The equation $1 = 1$ is always true, meaning that we can confirm that the point $(\cosh(a), \sinh(a))$ is in fact on the unit hyperbola.

Part (c) Solution: If we know that $a = 500$ and $h = 0$, we can plug these values into the function to rewrite it as:

$$y = 500 \cosh\left(\frac{x}{500}\right) + C$$

The point at which the suspension bridge is closes to the roadway (x-axis) is at its lowest point, or minimum. This graph is symmetric about the y-axis, meaning that the minimum must be on the y-axis. If the minimum were not to be on the y-axis, there would have to be two minimums, as the bridge is symmetric about the y-axis (if the minimum were on one side of the y-axis, another minimum would have to appear on the other side). However, this is a suspension bridge, so it is not reasonable to infer that there are two minimums in the graph (a suspension bridge doesn't dip and come back up as gravity doesn't work that way). Therefore, the minimum of the graph must be on the y-axis, or on the line $x = 0$ for there to be only one minimum.

If the minimum occurs when $x = 0$, then we can plug $x = 0$ into the function to find the minimum:

$$y = 500 \cosh\left(\frac{0}{500}\right) + C = 500 \cosh(0) + C = 500\left(\frac{e^0 + e^0}{2}\right) + C = 500 + C$$

Therefore, the minimum of this function is at $(0, 500 + C)$. This also means that the minimum distance from the bridge/cable is $500 + C$ feet. To find the value of C, we can remember that we are given that the bridge hangs from two towers 100 feet high and 400 feet apart. Because the picture is symmetric about the y-axis, the bridge hangs from the points $(-200, 100)$ and $(200, 100)$. If we plug one of these points into the original function, we can find the value of C. Plugging in $(200, 100)$:

$$100 = 500 \cosh\left(\frac{200}{500}\right) + C$$

Solving for C:

$$100 = 500\left(\frac{e^{\frac{2}{5}} + e^{-\frac{2}{5}}}{2}\right) + C$$

$$C = \frac{100}{500\left(\frac{e^{\left(\frac{2}{5}\right)} + e^{\left(-\frac{2}{5}\right)}}{2}\right)} = \frac{1}{5\left(\frac{e^{\left(-\frac{2}{5}\right)} + e^{\left(\frac{2}{5}\right)}}{2}\right)} = \frac{2}{5e^{\left(\frac{2}{5}\right)} + 5e^{\left(-\frac{2}{5}\right)}}$$

As we found earlier, the minimum distance between the cable and the roadway is $500 + C$ feet. Therefore, plugging in this value for C, we can say the minimum distance is $\boxed{500 - \frac{2}{5e^{\left(\frac{2}{5}\right)} + 5e^{\left(-\frac{2}{5}\right)}}}$ feet.

Additonal Problem Set Question 1: Solve the equation

$$2^{3x-6} = 8^{4-x^2}$$

Explain how one could solve this equation without using logarithms.

Solution: We know that $8 = 2^3$, so this equation can be rewritten as

$$2^{3x-6} = \left(2^3\right)^{4-x^2}$$

which is the same as

$$2^{3x-6} = 2^{12-3x^2}$$

For this equation to be true, the exponents of 2 must be equal to each other. This is because two exponential expressions with the same base must have the same exponent to be equivalent. Therefore, we can say that $3x - 6 = 12 - 3x^2$. This equation is equivalent to $3x^2 + 3x - 18 = 0$. Solving for x, we can find that $\boxed{x = 2 \text{ and } x = -3.}$

Additional Problem Set Question 2: Write the function

$$\frac{9^x\left(2^{6x-3} + 3^{x-3}\sqrt{25^{3x-4}}\right)}{3 \cdot \sqrt{49^x} \cdot 7^{-x}}$$

using as few exponents involving variables as possible. Your solution may omit elementary algebra, but show any steps that involve the use of exponent laws.

Solution: Simply taking the square root of the numbers under the square roots, this can be rewritten as

$$\frac{9^x\left(2^{6x-3} + 3^{x-3} \cdot 5^{3x-4}\right)}{3 \cdot 7^x \cdot 7^{-x}}$$

Because $a^m \cdot a^n = a^{m+n}$, we can rewerite this as

$$\frac{9^x\left(2^{6x-3} + 3^{x-3} \cdot 5^{3x-4}\right)}{3}$$

We know that $a^{mn} = (a^m)^n$, so this equivalent to

$$\frac{3^{2x}\left(2^{6x-3} + 3^{x-3} \cdot 5^{3x-4}\right)}{3}$$

We know that $\frac{1}{a^k} = a^{-k}$, so this is the same as

$$3^{2x-1}\left(2^{6x-3} + 3^{x-3} \cdot 5^{3x-4}\right)$$

or

$$3^{2x-1} \cdot 2^{6x-3} + 3^{2x-1} \cdot 3^{x-3} \cdot 5^{3x-4}$$

$a^m \cdot a^n = a^{mn}$, so this is the same as

$$3^{2x-1} \cdot 2^{6x-3} + 3^{3x-4} \cdot 5^{3x-4}$$

$a^{mn} = (a^m)^n$, so we can rewrite this as

$$3^{2x-1} \cdot 8^{2x-1} + 3^{3x-4} \cdot 5^{3x-4}$$

$m^a \cdot n^a = (mn)^a$, so we can rewrite this as

$$\boxed{24^{2x-1} + 15^{3x-4}}$$

Additional Problem Set Question 3: An ecologist is studying the populations of two species of deer in a particular region. She notices that the population of one species is doubling every eight years, and the population of the other species is tripling every twelve years. The current populations are 500 and 900 respectively.

(a) Which population is growing more quickly?

(b) Write a formula for the total number of deer in the region, counting both species, after t years.

(c) Is the function you wrote in (b) exponential?

Part (a) Solution: First, we can find equations that represent each species of deer. The first species has an initial population of 500 and doubles every two years. This means in the standard form for an exponential equation, $a = 500$ and $b = 2$. However, the exponent cannot be t years, as the population doesn't double every year, it doubles every 8 years. Therefore, the exponent must be $\frac{t}{8}$ to represent this. This means the equation modeling the first deer species is $f(t) = 500(2)^{\frac{t}{8}}$. The second deer species has an inital population of 900 ($a = 900$) and triples ($b = 3$). However, it triples every 12 years, not every 1 year, so the exponent must be $\frac{t}{12}$. Therefore, the equation modeling this species' population is $g(t) = 900(3)^{\frac{t}{12}}$.

The question asks which population is growing more quickly (or has a higher rate of growth) so we can disregard the initial populations (500 and 900) in the equations. However, we cannot just compare 2 and 3—what seem like the rate of growth—because these equations aren't in standard exponential form (their exponent is not a single variable). We can convert the first equation (without the initial population) into standard exponential form:

$$(2)^{\frac{t}{8}} = \left(\sqrt[8]{2}\right)^t$$

Converting the second equation:

$$(3)^{\frac{t}{12}} = \left(\sqrt[12]{3}\right)^t$$

Their rate of growths respectively are $\sqrt[8]{2} \approx 1.090508$ and $\sqrt[12]{3} \approx 1.095873$. From these approximations, it is clear that $\sqrt[12]{3}$ is larger, meaning that the $\boxed{\text{deer population with initial population of 900}}$ (or the second one) is growing more quickly.

Part (b) Solution: The equations we found in part (a) represent the deer populations of each species after t years, so we can simply add them together so that our formula for the total deer population after t years is $\boxed{500(2)^{\frac{t}{8}} + 900(3)^{\frac{t}{12}}.}$

Part (c) Solution: $\boxed{\text{No.}}$ This equation cannot be reweritten in the standard form for an exponential equation : $a(b)^x$. Although the two addends can be, the whole function cannot. The only situation where the sum of two exponentials could also be an exponential is if when written in exponential form, the bases of both addends are the same, which is not true in this case.