Homework 5

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Problem 3.2: Find the center and radius of each of the following circles.

Part (a): $x^2 - 6x + y^2 + 2y - 2 = 0$

Part (b): $x^2 + 4x + y^2 + 6y + 9 = 0$

Part (c): $x^2 + \frac{1}{3}x + y^2 - \frac{10}{3}y = \frac{127}{9}$

Part (a) Solution: To solve this problem, we have to rewrite this equation in the standard form for a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$

To do this, we have to use completing the square. Rewriting to create two expanded perfect square binomials would look like:

$$x^2 - 6x + 9 - 9 + y^2 + 2y + 1 - 1 - 2 = 0$$

Now, we can simplify this equation by unexpanding the binomials and rewriting the equation to match the standard form of a circle.

$$(x-3)^2 + (y+1)^2 - 12 = 0$$

We know that the center of a circle is defined as (h, k). Therefore, using this equation, we can say $\boxed{\text{the center of the circle is at } (3, -1).}$ According to the standard form of a circle, r^2 is equal to 12. Therefore, $\boxed{\text{the radius of the circle is } \sqrt{12}\text{, or } 2\sqrt{3}.}$

Part (b) Solution: To find the center and radius of this circle, we can repeat the process of Part (a). We can start by completing the square:

$$x^2 + 4x + 4 - 4 + y^2 + 6y + 9 = 0$$

Simplifying:

$$(x+2)^2 + (y+3)^2 = 4$$

If (h, k) is the center of the circle in standard form, we can say $\boxed{\text{the center of the circle is at } (-2, -3).}$ The radius of the circle is the square root of the constant on the right side of the equal sign, so the radius is $\boxed{\sqrt{4}\text{, or } 2.}$

Part (c) Solution: We can solve this problem by following the same process as the last two parts. Starting by using completing the square to rewrite the equation:

$$x^2 + \frac{1}{3}x + \frac{1}{36} - \frac{1}{36} + y^2 - \frac{10}{3}y + \frac{25}{9} - \frac{25}{9} = \frac{127}{9}$$

Simplifying:

$$\left(x + \frac{1}{6}\right)^2 + \left(y - \frac{5}{3}\right)^2 = \frac{609}{36}$$

Because the center of the circle is at (h, k), $\boxed{\text{the center of this circle is at } \left(-\frac{1}{6}, \frac{5}{3}\right).}$ The radius of this circle is $\boxed{\sqrt{\frac{609}{36}}.}$

Problem 3.3: Water is flowing from a major broken water main at the intersection of two streets. The resulting puddle of water is circular and the radius r of the puddle is given by the equation $r = 5t$ feet, where t represents time in seconds elapsed since the the main broke.

Part (a): When the main broke, a runner was located 6 miles from the intersection. The runner continues toward the intersection at the constant speed of 17 feet per second. When will the runner's feet get wet?

Part (b): Suppose, instead, that when the main broke, the runner was 6 miles east, and 5000 feet north of the intersection. The runner runs due west at 17 feet per second. When will the runner's feet get wet?

Part (a) Solution: We know that the runner's feet will get wet when they reach the edge of the puddle. To start, we can write an expression that represents the total distance the runner runs until they reach the puddle. We know that the runner is 6 miles from the intersection and the radius of the puddle is $5t$. However, because $5t$ is in feet, we have to convert 6 miles into feet to keep the units consistent. 6 miles is 31680 feet. Therefore, we can express the runner's total distance as:

$$31680 - 5t$$

In order to solve for t, we need to write another expression that represents the runner's distance. Because we are given that t represent the runner's time, the speed of the runner is 17 feet/second, and we know that distance equals rate multiplied by time, we an also express the runner's distance as:

$$31680 - 5t = 17t$$

Now, we can set these values equal to each other and solve for t.

$$31680 - 5t = 17t$$
$$t = 1440$$

Therefore, $\boxed{\text{the runner's feet will get wet 1440 seconds after they start running.}}$

Part (b) Solution: To solve this problem, we can write an equation for the puddle, using the point (0,0) as the center of the circle (center of circle/puddle is the intersection). As given in the problem, we know that the radius of the circle is $5t$. Using these values, we can write the equation for the circle:

$$x^2 + y^2 = (5t)^2$$

The problem states that the runner is 6 miles (31680 feet) east and 5000 feet north of the intersection. 31680 feet east of the intersection is the horizontal distance from the center, and 5000 feet north of the intersection is the vertical distance from the center. Because the runner's distance from the puddle is constantly shrinking by $17t$ feet/second, we can substitute his x-coordinate position, $(31680 - 17t)$ for the x-value in the equation. We can substitute 5000, his y-coordinate position, for the y-value in the equation. The equation would then look like:

$$(31680 - 17t)^2 + 5000^2 = (5t)^2$$

Solving for t:

$$1028622400 - 1077120t + 289t^2 = 25t^2$$

$$t = \frac{67320 + 50\sqrt{115566}}{33},\ t = \frac{67320 - 50\sqrt{115566}}{33}$$

These two values for t represent the two times at which the runner will reach the edges of the puddle. However, we know that the runner's feet will get wet when he reaches the edge of the puddle the first time, meaning that whichever time is shorter is correct. $\boxed{\text{The runner's feet will get wet after } \frac{67320-50\sqrt{115566}}{33} \text{ seconds.}}$

Problem 3.4: An amusement park Ferris Wheel has a radius of 60 feet. The center of the wheel is mounted on a tower 62 feet above the ground (see picture). For these questions, the wheel is not turning.

Part (a): Impose a coordinate system.

Part (b): Suppose a rider is located at the point in the picture, 100 feet above the ground. If the rider drops an ice cream cone straight down, where will it land on the ground?

Part (c): The ride operator is standing 24 feet to one side of the support tower on the level ground at the location in the picture. Determine the location(s) of a rider on the Ferris Wheel so that a dropped ice cream cone lands on the operator. (Note: There are two answers.)

Part (a) Solution: To impose a coordinate system, we can define the tower as the y-axis and the ground as the x-axis. The point at which the building and the ground intersect is the origin of the coordinate system. The intervals of both axes will be in terms of feet, where one unit is equal to one foot.

Part (b) Solution: We know that the location where the ice cream cone will land is on the ground. Therefore, we know that the y-coordinate of the cone's location will be 0. This means we only have to solve for the x-coordinate. We know that the rider is 100 feet above the ground, or 100 units above the x-axis. Therefore, we can imagine that the rider is located on the line:

$$y = 100$$

We also know that the rider is located at the exact point at which the line $y = 100$ intersects the Ferris wheel. To find the location of this point, we have to create an equation that represent the Ferris wheel. We will need to use the standard form of a circle, because the Ferris wheel is in the shape of one:

$$(x - h)^2 + (y - k)^2 = r^2$$

We defined the intersection of the tower and ground as the origin of the coordinate system. Because the center of the circle is located at the top of the tower and we know the tower is 62 feet tall, we can say the center of the circle is at point (0, 62). In the image, we are also given that the radius of the circle is 60 feet. We can now plug these values into the standard form of the equation:

$$x^2 + (y - 62)^2 = 60^2$$

Now that we know the equation of the circle and the equation of the horizontal line at which the runner is located on, we can substitute the value for y in the horizontal line equation into the circle equation. The equation for the circle would then look like:

$$x^2 + (100 - 62)^2 = 60^2$$

Solving for x:

$$x^2 = 2156$$
$$x = \sqrt{2156},\ x = -\sqrt{2156}$$

However, we know that the value of x must be positive because the cone lands on the positive side (right side) of the origin. Therefore, $\boxed{\text{the ice cream cone will land at the point } (\sqrt{2156}, 0).}$

Part (c) Solution: Because the operator is located on the ground and 24 feet left of the tower, the operator is located at (-24, 0). We can say the point (-24, 0) is on the line $x = -24$. The possible locations of the rider will be at the intersections of this line and the circle. To find these intersections, we can substitute -24 for x in the equation for the circle. The equation would then look like:

$$24^2 + (y - 62)^2 = 60^2$$

Solving for y:

$$y^2 - 124y + 820 = 0$$
$$y = 62 + 12\sqrt{21},\ y = 62 - 12\sqrt{21}$$

Because we found the y-coordinates at which the rider could drop an ice cream cone and it could land on the operator, we can say $\boxed{\text{those two points are located at } (-24,\ 62 + 12\sqrt{21}) \text{ and } (-24,\ 62 - 12\sqrt{21}).}$