Homework 13

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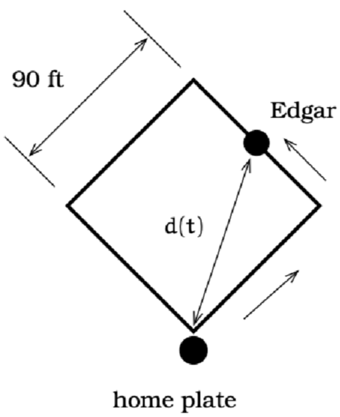
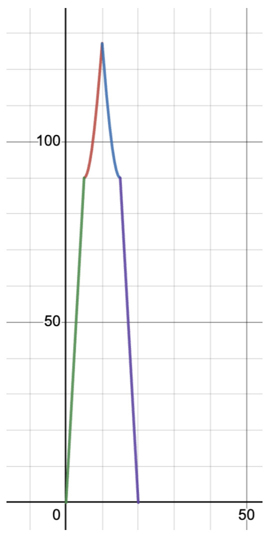
Problem: 6.9: A baseball diamond is a square with sides of length 90 ft. Assume Edgar hits a home run and races around the bases (counterclockwise) at a speed of 18 ft/sec. Express the distance between Edgar and home plate as a function of time t. (Hint: This will be a multipart function.) Try to sketch a graph of this function.



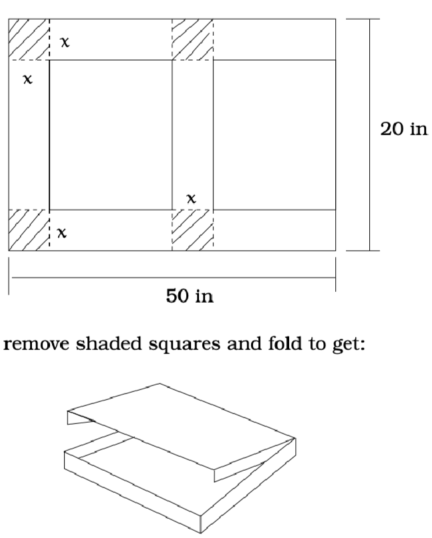
Solution: To solve this problem, we can rotate the baseball field such that in our coordinate system, home plate is the origin, the southeast corner is at $(90, 0)$, the northeast corner is at $(90, 90)$, and the northwest corner is at $(0, 90)$. Therefore, the distance between Edgar and the home plate will be a multipart function with four parts: when he travels east, north, west, and south. We can start by finding the part of the function when Edgar is traveling east. He travels east for 90 feet at a rate of 18ft/sec, meaning that he travels for a total of 5 seconds east. Because he starts at time 0 and ends after 5 seconds, the domain for this part will be $0 \leq t \leq 5$. His distance from home plate can be expressed as $18t$ (his rate times the time he has been traveling). Next, Edgar travels north for 90 feet at 18 ft/sec. He starts running north after 5 seconds, and runs north for 5 seconds. Therefore, the domain for this part of the function will be $5 < t \leq 10$. To find his distance from home plate, we can use the distance formula. Home plate is located at $(0, 0)$ and Edgar's position after t seconds will be $(90, 18(t-5))$. Therefore, Edgar's distance from home plate will be $\sqrt{90^2 + (18(t-5))^2}$. Next, Edgar travels west. He starts traveling west after 10 seconds and travels west for 5, so the domain for this part of the function will be $10 < t \leq 15$. Edgar's position while traveling west is $(90, 90 - 18(t-10))$. Using the distance formula to find the distance between Edgar and home plate, we can say the distance is $\sqrt{90^2 + (90 - 18(t-10))^2}$. Edgar travels south last. He starts traveling south after 15 seconds and travels south for 5 seconds, meaning the domain for this part of the function is $15 < t \leq 20$. Edgar's position while traveling south is $(0, 90 - 18(t-15))$. Because he is traveling vertically towards the origin, his distance from home plate will be $90 - 18(t-15)$. Using these four parts, we can write the multipart function $f(t)$ representing Edgar's distance from home plate after t seconds as

$$y = \begin{cases} 18t & \text{if } 0 \le t \le 5 \\ \sqrt{90^2 + (18(t-5))^2} & \text{if } 5 < t \le 10 \\ \sqrt{90^2 + (90 - 18(t-10))^2} & \text{if } 10 < t \le 15 \\ 90 - 18(t-15) & \text{if } 15 < t \le 20 \end{cases}$$

The graph of this function would be:



Problem 6.10: Pagliacci Pizza has designed a cardboard delivery box from a single piece of cardboard, as pictured.

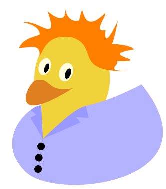


Part (a): Find a polynomial function $v(x)$ that computes the volume of the box in terms of x. What is the degree of v?

Part (b): Find a polynomial function $a(x)$ that computes the exposed surface area of the closed box in terms of x. What is the degree of a? What are the explicit dimensions if the exposed surface area of the closed box is 600 sq. inches?

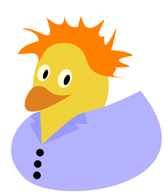
Part (a) Solution: We know that the volume of a rectangular prism is its length times its width times its height. We are given that the width of the unfolded pizza box is 20 inches. In the unfolded image, two flaps with length x are added to the width to make the total width 20 inches. Therefore, the actual width of the pizza box is $20 - 2x$ inches. Because the length of the unfolded pizza box includes parts of the box with length x that will not be part of the actual length of the folded pizza box, the length of the folded pizza box should be $50 - 2x$. However, we have to keep in mind that the pizza box will be folded over, meaning that the true length will be half of $50 - 2x$, or $25 - x$. We know the length, width, and height of the box, meaning we can now write the function $v(x)$ that computes the volume of the box in terms of x.

$v(x) = x(20 - 2x)(25 - x)$. There are three x terms being multiplied, so the degree of v is 3.



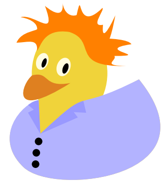
Part (b) Solution: In part (a), we found that the height of the box was x, the width was $20 - 2x$, and the height was $25 - x$. There are a total of 6 exposed sides on the box. The surface area of the top and bottom of the box are the length and width multiplied—meaning their surface area would be $(20 - 2x)(25 - x)$. The two sides of the box whose edges are determined by the length and height of the box have a surface area of $x(25 - x)$. The last two sides of the box have a surface area that is the width and height multiplied, meaning their surface area is $x(20 - 2x)$. Adding all 6 of these values together is the formula for the polynomial function $a(x)$ which computes the exposed surface area of the closed box in terms of x.

$a(x) = 2(20 - x)(25 - x) + 2x(25 - x) + 2x(20 - 2x)$



There are a total of six x terms in the formula for the function, but they are not all multiplied together. Instead, there will only be one x^2 term.

The degree of $a(x)$ will be 2.

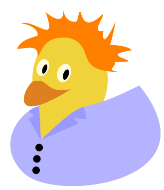


To find the explicit dimensions of the surface area of the closed box when it is 600 square inches, we can use the formula we found for $a(x)$ and set it equal to 600:

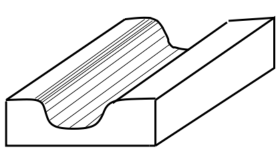
$$2(20-2x)(25-x)+2x(25-x)+2x(20-2x)=600$$
$$-2x^2-50x+400=0$$
$$x=-\frac{25\pm 5\sqrt{27}}{2}$$

The solution $-\frac{25-5\sqrt{27}}{2}$ is negative and therefore does not fit within this context. This means that $x=-\frac{25+5\sqrt{27}}{2}$. Therefore, the dimensions of the closed pizza box when the surface area is 600 square inches are

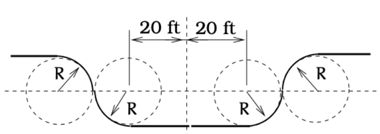
$$-\frac{25+5\sqrt{27}}{2}\times\left(25+\frac{25+5\sqrt{27}}{2}\right)\times\left(45+5\sqrt{27}\right)$$



Problem 6.11: The vertical cross-section of a drainage ditch is pictured below:



3D−view of ditch



vertical cross-section

Here, R indicates a circle of radius 10 feet and all of the indicated circle centers lie along the common horizontal line 10 feet above and parallel to the ditch bottom. Assume that water is flowing into the ditch so that the level above the bottom is rising 2 inches per minute.

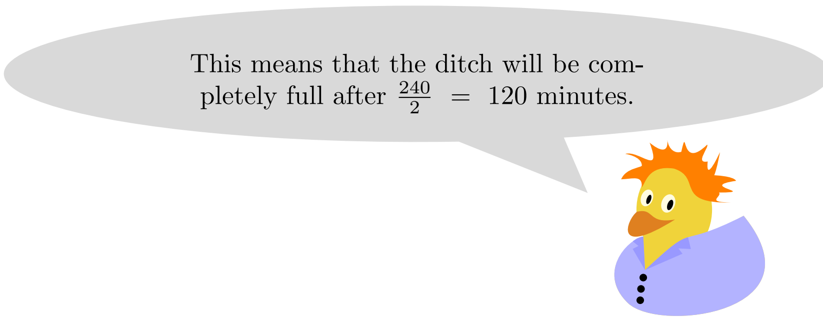
Part (a): When will the ditch be completely full?

Part (b): Find a multipart function that models the vertical cross-section of the ditch.

Part (c): What is the width of the filled portion of the ditch after 1 hour and 18 minutes?

Part (d): When will the filled portion of the ditch be 42 feet wide? 50 feet wide? 73 feet wide?

Part (a) Solution: The radius of each circle is 10 feet and the ditch is one circle deep. Therefore, the ditch is 20 feet, or 240 inches, deep. The water level is rising at 2 inches per minute.



Part (b) Solution: There are 6 different parts to this multipart function. To solve this, we can let the horizontal dashed line represent the x-axis and the vertical dashed line represent the y-axis. To identify the circles, we can have the leftmost circle be Circle 1, the second to left circle be Circle 2, the second to right circle be Circle 3, and the right most circle be Circle 4.

The first part of the multipart is when $x \leq -40$. For these x-values, the cross section of the ditch is a straight line 10 feet above the x-axis, as it is tangent to Circle 1. This line is $y = 10$. We can also see the line $y = 10$ reappear on the other side of the ditch, when $x > 40$, as $y = 10$ is also tangent to Circle 4.

The second part of the multipart is when $-40 < x \leq -30$. This part of the ditch is the upper right quarter of Circle 1. Circle 1 has a center at $(-40, 0)$ and a radius of 10, meaning the standard form equation of Circle 1 is $(x+40)^2 + y^2 = 100$. This equation can be rewritten as $y = \pm\sqrt{-(x+40)^2 + 100}$. Because we are only looking for the upper half of Circle 1, we will only consider the equation $y = +\sqrt{-(x+40)^2 + 100}$, as it represents values above the horizontal line that splits the circle in half.

The third part of the multipart is when $-30 < x \leq -20$. This part of the ditch is the lower left quarter of Circle 2. Circle 2 has a center at $(-20, 0)$ and a radius of 10, meaning the standard form equation of Circle 2 is $(x+20)^2 + y^2 = 100$. This equation can be rewritten as $y = \pm\sqrt{-(x+20)^2 + 100}$. Because we are only looking for the lower half of Circle 2, we will only consider the equation $y = -\sqrt{-(x+20)^2 + 100}$, as it represents values below the horizontal line that splits the circle in half.

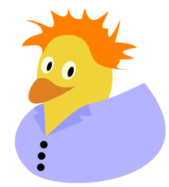
The fourth part of the multipart is when $-20 < x \leq 20$. This part of the ditch is the bottom horizontal line. This line is tangent to circles 2 and 3 at their lowest points, meaning that the line is 10 feet below the x-axis. Therefore, this line is $y = -10$.

The fifth part of the multipart is when $20 < x \leq 30$. This part of the ditch is the lower right quarter of Circle 3. Circle 3 has a center at $(20, 0)$ and a radius of 10, meaning the standard form equation of Circle 2 is $(x-20)^2 + y^2 = 100$. This equation can be rewritten as $y = \pm\sqrt{-(x-20)^2 + 100}$. Because we are only looking for the lower half of Circle 2, we will only consider the equation $y = -\sqrt{-(x-20)^2 + 100}$, as it represents values below the horizontal line that splits the circle in half.

The sixth part of the multipart is when $30 < x \leq 40$. This part of the ditch is the upper left quarter of Circle 4. Circle 4 has a center at $(40, 0)$ and a radius of 10, meaning the standard form equation of Circle 4 is $(x-40)^2 + y^2 = 100$. This equation can be rewritten as $y = \pm\sqrt{-(x-40)^2 + 100}$. Because we are only looking for the upper half of Circle 4, we will only consider the equation $y = +\sqrt{-(x-40)^2 + 100}$, as it represents values above the horizontal line that splits the circle in half.

We can now take all six parts of this multipart and write the function $f(x)$ that represents the vertical cross-section of the ditch.

$$y = \begin{cases} 10 & \text{if } x \leq -40 \text{ and } x > 40 \\ \sqrt{-(x+40)^2+100} & \text{if } -40 < x \leq -30 \\ -\sqrt{-(x+20)^2+100} & \text{if } -30 < x \leq -20 \\ -10 & \text{if } -20 < x \leq 20 \\ -\sqrt{-(x-20)^2+100} & \text{if } 20 < x \leq 30 \\ \sqrt{-(x-40)^2+100} & \text{if } 30 < x \leq 40 \end{cases}$$



Part (c) Solution: 1 hour and 18 minutes is equivalent to 78 minutes. Because the ditch fills at a rate of 2 inches per minute, we can say that the ditch will have filled $78 \cdot 2 = 156$ inches, or 13 feet, after 78 minutes. Because the bottom of the ditch is 10 feet below the x-axis, we know that the water will have filled to $13 - 10 = 3$ feet above the x-axis. This means the water level is currently at the line $y = 3$. To find the width of the ditch, we need to find the distance from where $y = 3$ intersects the y-axis (at the point $(0, 3)$) to where $y = 3$ intersects Circle 4 as well as where $y = 3$ intersects Circle 1. Because Circle 1 and Circle 4 are equidistant from the y-axis, we can choose to only find the distance from $(0, 3)$ and Circle 4 and multiply this distance by 2. First, we need to find the first time $y = 3$ intersects Circle 4 (or the left most intersection). In part (b), we found that the equation for Circle 4 was $(x - 40)^2 + y^2 = 100$. Substituting 3 for y in the equation:

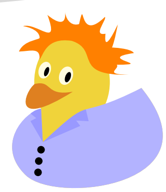
$$(x-40)^2 + 3^2 = 100$$

Solving for x:

$$x^2 - 80x + 1509 = 0$$
$$x = 40 \pm \sqrt{91}$$

The first time that $y = 3$ intersects Circle 4 is the left most intersection point (lesser x-value), which is when $x = 40 - \sqrt{91}$. Therefore, the distance from $(0, 3)$ to Circle 4 is $40 - \sqrt{91}$ feet. The total width is double this value.

The width of the filled portion after 78 minutes is $90 - 2\sqrt{91}$ feet.



Part (d) Solution: We know that when the water level reaches the x-axis, the width of the ditch is 60 feet.

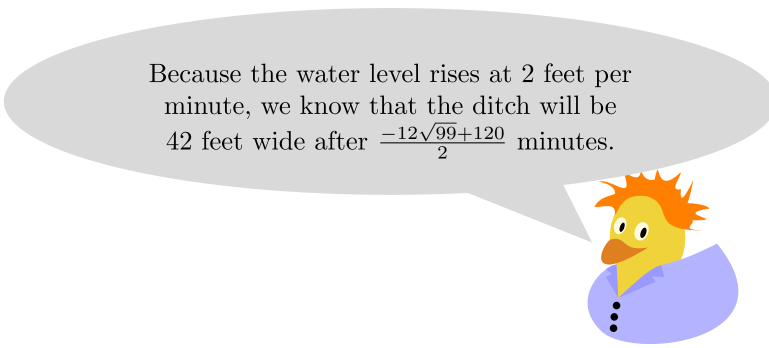
When the width of the ditch is 42 feet, we know that the water level is still below the x-axis, as the width of the ditch at the x-axis is 60 feet. This means we need to look at circles 2 and 3. If the width of the ditch is 42 feet, we know that the distance from the y-axis to either to lower left quarter of circle 2 or lower right quarter of circle 3 is 21 feet. If we use circle 3, we can say that the x-coordinate of the intersection of the water level with the lower right quarter of this circle must be 21. Therefore, we can use the formula we found for the lower right quarter of Circle 3, $y = -\sqrt{-(x-20)^2+100}$, and substitute 21 for x:

$$y = -\sqrt{-(21-20)^2+100}$$

Solving for y:

$$y = -\sqrt{99}$$

Therefore, the ditch has a width of 42 feet when the water level reaches the line $y = -\sqrt{99}$. The bottom of the ditch is at $y = -10$, meaning that this line is $-\sqrt{99}+10$ feet, or $-12\sqrt{99}+120$ inches above the bottom of the ditch.

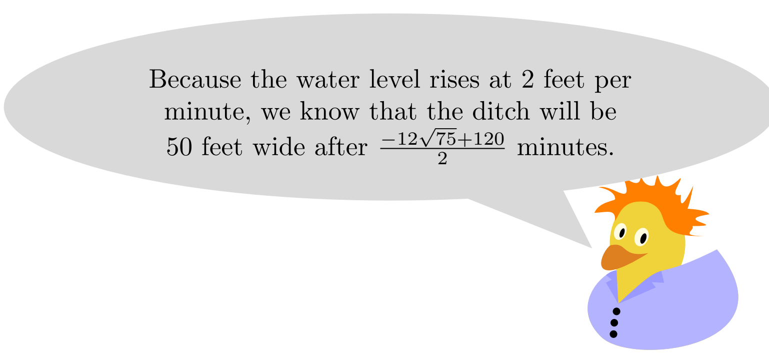


We can repeat the same process to find when the width of the ditch is 50 feet. Using Circle 3 again, we know that the distance from the y-axis to the lower right quarter of Circle 3 must be 25 feet. Therefore, we can substitute 25 for x into the formula for the lower right quarter of Circle 3:

$$y = -\sqrt{-(25-20)^2+100}$$

$$y = -\sqrt{75}$$

The line $y = -\sqrt{75}$ is $-\sqrt{75}+10$ feet, or $-12\sqrt{75}+120$ inches above the bottom of the ditch.



When the width of the ditch is 73 feet, we know that the water level will have risen above the x-axis, as the width of the ditch at the x-axis is 60 feet. Therefore, we will need to find the distance from the y-axis to either the upper left quarter of Circle 4 or the upper right quarter of Circle 1. The formula for the upper right quarter of Circle 4 is $y = \sqrt{-(x-40)^2+100}$. We need to find when the distance from the y-axis to the intersection of the water level with this formula is $\frac{73}{2} = 36.5$ feet. The x-coordinate of this intersection point must be 36.5. This means that we can substitute 36.5 for x into the formula:

$$y = \sqrt{-(36.5-40)^2+100}$$

$$y = \sqrt{87.25}$$

The line $y = \sqrt{87.25}$ is $\sqrt{87.25}+10$ feet, or $12\sqrt{87.25}+120$ inches, above the bottom of the ditch.