Intercepts of Infinite Iterations

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1 Introduction

1.1 Overview

The topic we have chose to study revolves around iterations of a certain quadratic function. We noticed that for the function $f(x) = x^2 - a$ for some arbitrary value a, as the number of iterations of f increased, the greatest zero of the function would increase as well, but at a seemingly slower rate.

For example, if we have $a = 1$ (an arbitrary value), our graph for $f(x)$ when compared with the line for fixed points ($y = x$) would be:

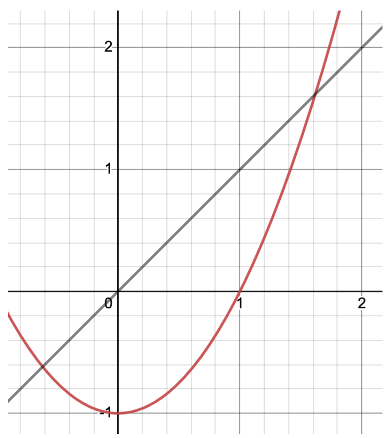

Fig. 1: Graph of $f(x) = x^2 - 1$ compared to $y = x$

In this graph, the greatest zero, or intersection with the x-axis, is about at $(1, 0)$. However, in our graph for $f^2(x)$, we can see a change:

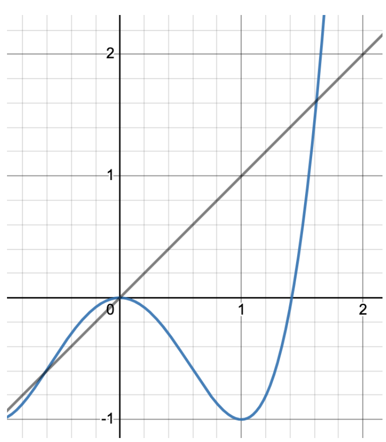

Fig. 2: Graph of $f^2(x)$ where $f(x) = x^2 - 1$ compared to $y = x$

As we observe from the graph, the greatest zero of $f^2(x)$ is greater than the greatest zero of $f(x)$, and is at approximately $(1.4, 0)$ as compared to the greatest zero of $f(x)$, which is about $(1, 0)$. We see a general trend in where as the iterations of f increase, the greatest zero also increases.

For the iteration $f^4(x)$, our graph is:

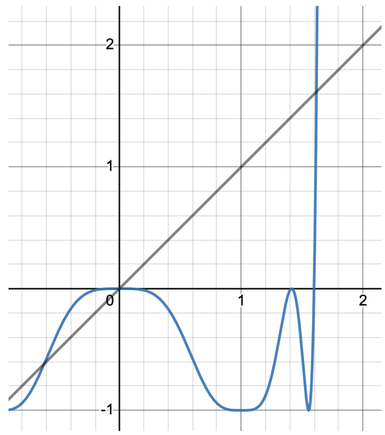

Fig. 3: Graph of $f^4(x)$ where $f(x) = x^2 - 1$ compared to $y = x$

Here, we can see a clear shift in the "shape" of the graph. Not only is the greatest zero greater than previous iterations (as we have graphed $f(x)$ and $f^2(x)$), but the part of the graph going from this zero to the fixed point appears to be getting steeper and steeper, and in this iteration, appears to almost be vertical. We can also notice that as the iterations of $f(x)$ increase, the distance between the x-value of the fixed point and the x-value of the greatest zero decreases by less and less (in other words, the zero "moves" less after each iteration). This leads us to ask: what is the limit that the greatest zero of increasing iterations of $f(x)$ is "moving" towards?

1.2 Question

If the function $f(x)$ is defined as $f(x) = x^2 - a$ for some arbitrary positive constant a, then in terms of a, what is the lowest limit of the greatest zero (intersection with the x-axis) of the iteration $f^n(x)$ as n approaches infinity?

1.3 Motivation

In the iterations of some quadratic functions, we noticed that there seemed to be a boundary or limit of the greatest (rightmost) intersection of the iteration with the x -axis that each iteration got closer to but never seemed to ever reach. The question is a specification of this idea, in which we have a more specific quadratic function. Our answer may give us more insight on the nature of limits and asymptotes on quadratic functions, as well as perhaps a relationship between the greatest zeros of iterations.

2 Solving

2.1 Approach

To start, we can notice an interesting phenomena that occurs; the x-intercepts seem to be approaching the x-value for the fixed point present throughout *all* iterations, but never seems to actually hit this x-value on the x-axis (as in have its greatest zero have the exact x-value as the fixed point). The fact that it must never have the same x-value intercept is proven by the fact that this is a function, and thus, cannot have the same x-value correspond to two different y-values (and would thus fail the vertical line test). Thus, our hypothesis is that the x-value for the fixed point of $f(x)$ seems to be this lowest limit of the function's

x-intercepts - as in, more and more iterations of $f(x)$ would get closer and closer to this x-value as the greatest zero but would never actually hit it - hence, the limit.

The below graph demonstrates the tendency towards the x-coordinate of this fixed point.

Fig. 4: First 8 iterations of the function $f(x) = x^2 - a$ for $a = 1$

The upper point through which all lines pass through is the fixed point, and the point where the blue line passes through the x-axis is the greatest x-intercept of the 8^{th} iteration. The black iteration and red iteration can be seen passing through the x-axis at slightly "earlier" points, and both are "earlier" iterations.

To find whether our prediction on the limit is true, we can first determine the x-value of that fixed point in terms of a for our function $f(x) = x^2 - a$. Then, we can find the pattern of distance "moved" of the greatest x-intercepts throughout the iterations of the function (as in, the pattern of change in x-value across iterations). Finally, we can determine whether this pattern proves that the x-value of that fixed point is the lowest limit to the greatest x-intercept in infinite iterations.

2.2 Finding the Fixed Point

To start, we can determine the x-value of the fixed point. Fixed points only occur when $f(x)$ is equal to x, meaning that we will have:

$$f(x) = x$$

Thus, we have:

$$x = x^2 - a$$

We can then isolate x to find the x-value of the fixed point in terms of a. We can do this by arranging into a quadratic equation and then solving for x using the quadratic formula:

$$x^2 - x - a = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4(-a)}}{2}$$

$$x = \frac{1 \pm \sqrt{1 + 4a}}{2}$$

We are only considering the positive values of a, meaning that the number under the square root will always result in a number bigger than 1. This means if we choose the minus sign in the $\pm$, the x-value of the fixed point is negative. However, we are considering the greatest x-intercept, which, when a is positive, is always greater than 0 due to difference of squares in $x^2 - a$. This means that we must choose the plus sign from the $\pm$ as we are focusing on the positive fixed point of our function $f(x)$.

$$x = \frac{1 + \sqrt{1 + 4a}}{2}$$

2.3 The Pattern of Intercepts

Next, we can start to try and identify the pattern of change in greatest x-intercept across iterations. To do this, we must first find the x-intercepts of the first iterations and note if we can notice a pattern. Then, we can try generalizing the process of finding this x-intercept to $f^n(x)$. To start, we can find the greatest x-intercept of $f(x)$. To do this, we can set $f(x)$ to 0 and solve for x. Doing this is shown below.

$$0 = x^2 - a$$

$$a = x^2$$

$$\pm\sqrt{a} = x$$

We are only considering the greatest x-intercept, meaning that we should choose the positive value:

$$x = \sqrt{a}$$

Now, we have found the greatest x-intercept of the first iteration of $f(x)$ (as in just $f(x)$). We can now continue this process for greater iterations to see if we can notice a pattern.

We can define $f^2(x)$ as:

$$f^2(x) = (x^2 - a)^2 - a$$
$$f^2(x) = x^4 - 2x^2a + a^2 - a$$

To find the x-intercept of this function, we need to set $f^2(x)$ equal to 0:

$$0 = x^4 - 2x^2a + a^2 - a$$

Solving for x, we get:

$$x = \pm\sqrt{a \pm \sqrt{a}}$$

Again, we are only solving for the greatest possible x-intercept, so we must choose the $+$ sign whenever there is a $\pm$ sign. Therefore, the greatest x-intercept of $f^2(x)$ is

$$\sqrt{a + \sqrt{a}}$$

Now, we can use the same process to find the x-intercepts of $f^3(x)$. We can define $f^3(x)$ as:

$$f^3(x) = ((x^2 - a) - a)^2 - a$$
$$f^3(x) = (x^2 - a)^2 - 2a(x^2 - a) + a^2 - a$$
$$f^3(x) = x^4 - 4ax^2 + 4a^2 - a$$

Solving for x, we get:

$$x = \pm\sqrt{a \pm \sqrt{a \pm \sqrt{a}}}$$

We are only looking for the greatest possible x-intercept of the function, so we must pick the $+$ signs again. The greatest x-intercept of $f^3(x)$ is

$$\sqrt{a + \sqrt{a + \sqrt{a}}}$$

By now, we can start to see a pattern. Given that the x-intercept of the nth iteration of f is x_n, it seems that the following equation can be used to find the x-intercept of the nth iteration:

$$x_n = \sqrt{a + x_{n-1}}$$

Note that x_{n-1} would represent the value of the previous iteration of the function.

2.4 Proving Across Infinite Iterations

Now, we need to prove whether this equation holds for the x-intercept of $f^n(x)$ for any value of n. To do this, we can first assume that what we just found is true: that the x-coordinate of the x-intercept of $f^n(x)$ is $x_n = \sqrt{a + x_{n-1}}$. Now, we can let k be the value of the x-coordinate of the x-intercept of the $(n+1)$th iteration of f. In other words,

$$f^{n+1}(k) = 0$$

If we are assuming that the x-intercept of the nth iteration is $\sqrt{a + x_{n-1}}$, then we need to prove that this formula is applicable to the x-intercept of the $(n+1)$th iteration as well. If we can prove that the x-intercept is this value for the subsequent iteration of any nth iteration, it means that the x-intercepts of all iterations of f is also this value. Therefore, we need to solve for k to check whether $k = \sqrt{a + x_{n-1}}$.

However, we need to keep in mind that the expression $\sqrt{a + x_{n-1}}$ is equal to the x-intercept of the nth iteration of f, and k is the $(n+1)$th iteration. The expression is made of the square root of a plus the x-intercept of the previous iteration. The x-intercept of the iteration before the $(n+1)$th iteration is simply x_n, so we actually need to check whether $k = \sqrt{a + x_n}$. To start solving for k, we can apply the inverse function to both sides of the equation $f^{n+1}(k) = 0$:

$$f^n(k) = f^{-1}(0)$$

The value of $f^{-1}(0)$ is simply the value of x in the function $f(x) = 0$, or the x-intercept of the first iteration of f. As we found earlier, the x-intercept of the first iteration is $\sqrt{a}$. Substituting $\sqrt{a}$ for $f^{-1}(0)$ in the equation:

$$f^n(k) = \sqrt{a}$$

To isolate k, we now need to apply the inverse of the function n times:

$$k = f^{-n}(\sqrt{a})$$

To apply the inverse function n times, we need to find the inverse function itself. We can find the inverse of $f(x)$ by solving for x in terms of $f(x)$. For simplicity, we can let $y = f(x)$:

$$y = x^2 - a$$

Solving for x:

$$a + y = x^2$$

$$x = \pm\sqrt{a + y}$$

So, our x input in terms of $f(x)$ output is $x = \pm\sqrt{a + y}$. Thus, our inverse is:

$$f^{-1}(x) = \pm\sqrt{a + x}$$

As we are looking for the greatest x-intercept, our inverse would be, in more exact terms:

$$f^{-1}(x) = \sqrt{a + x}$$

If we iterate the inverse function again, we will get

$$f^{-2}(x) = \sqrt{a + \sqrt{a + x}}$$

If we iterate the inverse yet again, we will get

$$f^{-3}(x) = \sqrt{a + \sqrt{a + \sqrt{a + x}}}$$

This pattern will continue no matter how many times we iterate the inverse function as we are simply substituting $\sqrt{a+x}$ for x every time we iterate. Therefore, we can make a generalization for this pattern: that after n iterations of the inverse function, the function can be written as

$$f^{-n}(x) = x_n = \sqrt{a + x_{n-1}}$$

where x_{n-1} is the value of the previous iteration of the inverse function.

If this is the formula for $f^{-1}(x)$, we can find $f^{-1}(\sqrt{a})$ by substituting $\sqrt{a}$ for x in the function. If were to substitute, the resulting function could be modeled as

$$f^{-n}(\sqrt{a}) = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}} \ldots$$

Therefore,

$$k = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}} \ldots$$

Recall that we were trying to check whether k equals the x-intercept of the nth iteration of f. The equation that modeled the nth iteration of f was

$$x_n = \sqrt{a + x_{n-1}}$$

This equation is also equal to

$$\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}} \ldots$$

so we have confirmed that the formula $x_n = \sqrt{a + x_{n-1}}$ represents the x-intercept of $f^n(x)$ for any value of n. Now, we need to find the limit on the maximum value this x-intercept can be.

2.5 Proving Our Prediction

First, we will define d_n to be the n^{th} term of the sequence (recall that the sequence is finding the x-intercepts of successive iterations of $f(x)$), where $d_1 = \sqrt{a}$ (d_1 is the initial term).

We can define the sequence recursively as follows (using the recursive definition for the inverse found previously):

$$d_n = \sqrt{a + d_{n-1}}$$

To prove that that the x-intercepts do not go past the x-coordinate of the fixed point for all iterations of $f(x)$, we can use the concept of limits. We will define a variable L to represent the term when $n = \infty$ as follows, and we can use a limit to represent this:

$$L = \lim_{n \to \infty} d_n$$

Now, we can substitute the recursive definition of d_n into the equation:

$$L = \lim_{n \to \infty} \sqrt{a + d_{n-1}}$$

When n is ∞, we know that d_n will be equal to L. Similarly, the same will hold true for $\infty - 1$, because the difference between the two terms will be practically nothing. So, by this logic, $d_{n-1} = L$ as n tends to ∞. Now, we can substitute this relation into the equation:

$$L = \lim_{n \to \infty} \sqrt{a + L}$$

However, the limit becomes irrelevant as L itself is defined using the same limit, and it is the only term that is actually relevant. So, we can remove it, giving us the equation:

$$L = \sqrt{a + L}$$

Solving for L, we get:

$$L^2 = a + L$$

$$L^2 - L - a = 0$$

Now that we have a quadratic equation, we can use the quadratic formula to solve for L, which represents the x-intercept when $n = \infty$:

$$L = \frac{1 \pm \sqrt{1^2 + 4a}}{2}$$

$$L = \frac{1 \pm \sqrt{4a + 1}}{2}$$

We will take the positive value, since we are looking for the greater value, giving us $L = \frac{1+\sqrt{4a+1}}{2}$.

So, we have discovered the value that the greatest x-intercept approaches as n tends to infinity (note that this is the value of L). Since this lines up with the x-coordinate of the fixed point that we solved for in **2.2 Finding a Fixed Point**, this means the greatest x-intercepts approach but never reach the x-value of the fixed point. So, the lowest limit of the greatest zero of $f^n(x)$ in terms of a as n approaches ∞ is thus $L = \frac{1+\sqrt{4a+1}}{2}$, or the x-coordinate of the positive fixed point of the function $f(x) = x^2 - a$.

3 Analysis

After solving, we have derived our answer on the limit of the rightmost zero in terms of a (see **2 Solving**), and proved that it is indeed the limit. The fact that the lowest limit of the greatest x-intercept is the x-value of the fixed point (positive fixed point) of our function is intriguing, as it further demonstrates how as iterations increase, for a certain value of x near the fixed point, the y value also gets "closer and closer" to the fixed point.

Our answer and proof also display our function as, well, a function – it makes sense logically that the x-value of the fixed point is our limit, as the greatest zero will increasingly approach this value but never reach it due to the laws of a function (and it would not pass the vertical line test). Additionally, our answer shows us a concrete relationship between the x-intercepts throughout iterations, establishing the grounds for further investigation about iterations of certain quadratics and perhaps the relationship in the pattern that they display.

4 Extension

4.1 Motivation

In our original problem, we explored the limit on the x-intercepts of the iterations of our function $f(x) = x^2 - a$. In this extension, we can instead focus on the y-intercepts of our function. However, unlike for our original problem, there isn't a limit on the y-intercept for increasing iterations of the function.

As we can see, for $a = 1$, our graph looks like:

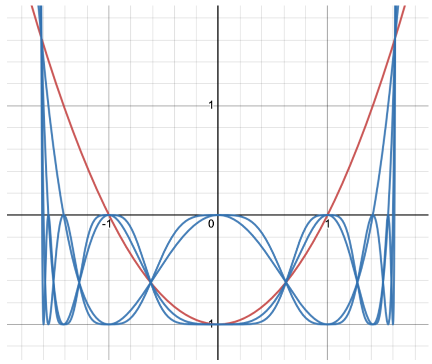

Fig. 5: Graph of $f(x) = x^2 - 1$

We can see that there seem to be two y-intercepts - one at $y = 0$ and one at $y = -1$.

Similarly, in the graph of $a = 2$, we also have 2 y-intercepts:

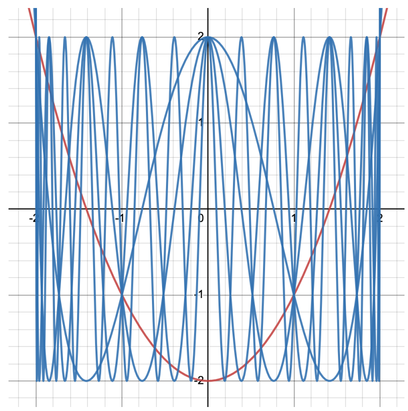

Fig. 6: Graph of $f(x) = x^2 - 2$

We can see that for certain values of a, we have two y-intercepts. For some other values of a, such as $a = 1.5$, we have multiple:

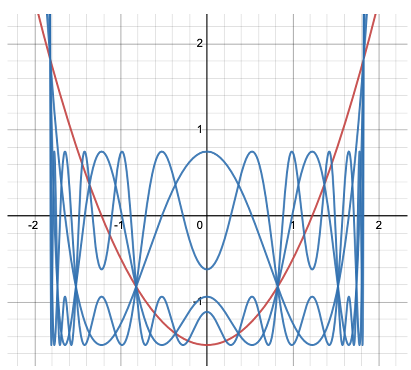

Fig. 7: Graph of $f(x) = x^2 - 1.5$

So, instead of looking for a limit as we did for our original problem, for our extension, we can look for certain values of a for when there are only two y-intercepts across all iterations.

4.2 Question

This gives us the main question for our extension: given the function $f(x) = x^2 - a$ for some arbitrary positive constant a, for what values of a do **all** (infinite) iterations of f have exactly **two** y-intercepts?

4.3 Intercepts of Two Iterations

We are solving for values of a where there are only two y-intercepts throughout infinite (all) iterations of $f(x)$.

Let us first just consider the first and second iterations of $f(x)$. Either the y-intercepts for the iterations must be equal (the same), or they must be different. If they are different, they must represent the two possibilities of y-intercepts for *all* future iterations - as we are considering when there is a total of two y-intercepts throughout all iterations. For both of these possibilities, we must calculate the y-intercepts for the first two iterations in terms of a. Finding the y-intercepts of the first iteration, we get:

$$(0)^2 - a = f(x)$$
$$-a = f(x)$$

Finding the y-intercepts of the second iteration, we have:

$$f(f(x)) = \left(x^2 - a\right)^2 - a$$
$$f(f(x)) = \left(0^2 - a\right)^2 - a$$
$$f(f(x)) = a^2 - a$$

We can set these equal to each other to determine which values of a, if any, would have it so the y-intercepts for the first and second iterations are same:

$$a^2 - a = -a$$
$$a^2 = 0$$
$$a = 0$$

This is not in the possible values that we are considering in the problem, as we are only considering when a is positive. This means that the first two y-intercepts must be different, as otherwise our solution does not exist in the scope of our restrains of a. This also means that these two intercepts must be the only two appearing throughout all iterations of $f(x) = x^2 - a$. That means that the third iteration's y-intercept must be equal to either the first or second iteration's y-intercept for there to be *only* 2 intercepts.

4.4 The Third Iteration

First, we can consider what the y-intercept of the third iteration is. Using the second iteration, we can find the third iteration. To recall, the second iteration is:

$$f(f(x)) = \left(x^2 - a\right)^2 - a$$

Iterating this again gives us:

$$f(f(f(x))) = \left(\left(x^2 - a\right)^2 - a\right)^2 - a$$

Finding the y-intercept of this iteration, we get:

$$f(f(f(x))) = \left(\left(0^2 - a\right)^2 - a\right)^2 - a$$
$$f(f(f(x))) = \left(a^2 - a\right)^2 - a$$
$$f(f(f(x))) = a^4 - 2a^3 + a^2 - a$$

This must be equal to either the first or second iteration's y-intercept for there to be *only* 2 intercepts, as if it is some other value, we would have more than 2 y-intercepts. Setting it equal to the first iteration's y-intercept and solving:

$$a^4 - 2a^3 + a^2 - a = -a$$
$$a^4 - 2a^3 + a^2 = 0$$
$$a^2\left(a^2 - 2a + 1\right) = 0$$
$$a^2(a-1)^2 = 0$$
$$a = 0,\ 1$$

Since we are only considering positive values of a, this means that one potential solution is $a = 1$. We can now find values of a when the third iteration's y-intercept is equal to the second iteration's y-intercept:

$$a^4 - 2a^3 + a^2 - a = a^2 - a$$
$$a^4 - 2a^3 = 0$$
$$a^3(a-2) = 0$$
$$a = 0,\ 2$$

Again, we are only considering positive values of a, so we will disregard 0 as a solution. The y-intercept of the n^{th} iteration of f is equivalent to iterating $f(-a)$ an $n-1$ amount of times. This is because substituting 0 for x into the n^{th} iteration of f (to find the y-intercept) is equivalent to substituting $-a$ for x in the $(n-1)^{\text{th}}$ iteration. Because the y-intercept of the current iteration is dependent on the previous iterations, we can conclude that the only two possible solutions to our problem are $a = 1$ and $a = 2$, and that there are no other values of a that will produce two y-intercepts of this function.

4.5 Proving Values of a

Now that we have found cases for which the third iteration is equal to the first or second iteration, we must prove that solutions to the two previously mentioned cases will be the values that we are looking for. In the scenario where the third iteration is equal to the first, we can expect a graph of the y-intercepts to look similar to this, where the x-axis is the number of times f has been iterated (n) and the y-axis is the y-intercept of $f^n(x)$:

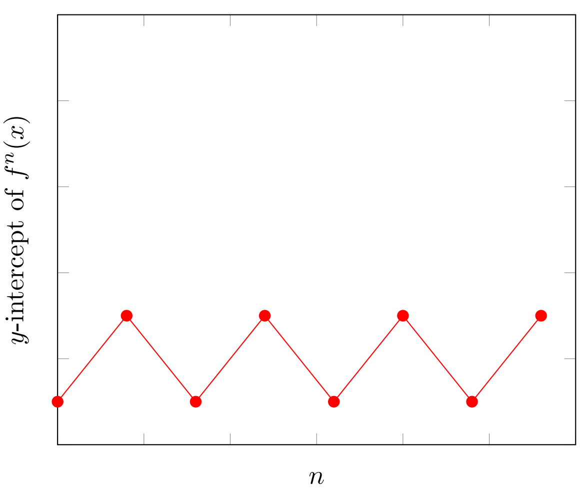

Fig. 7: the y-intercept of $f^n(x)$ vs. n

We know that evaluating $f(-a)$ yields the y-intercept of the second iteration of f based on **4.3 Solution**, and that the y-intercept of the nth iteration of f can be found by iterating $f(-a)$ an $(n-1)$ amount of times. In this section, we found the y-intercepts of the first and second iterations; we also found that the y-intercept of the third iteration of f was equivalent to that of the first iteration of f. The reason we know that the y-intercepts of the first two iterations are the only two possible y-intercepts that iterations of f can have is because we have effectively found a cycle of values. The y-intercepts of iterations of f oscillate between the y-intercepts of the first and second iterations of f. Since we are applying the same function again and again in multiple iterations, we can expect to see the same zigzag pattern continuing.

As is shown in the graph, the graph continues to alternate between the two values, meaning that in this case, there will only be two y-intercepts across iterations of f.

By similar logic, if the third iteration is equal to the second iteration, we will instead get a graph (of y-intercepts versus the number of iterations) like shown below. Note that the graph depicts the y-intercept of the second iteration and subsequent iterations as less than that of the first, but it is possible the first iteration has a lesser y-intercept than that of the second iteration and onwards. We have simply chosen only one of those scenarios in order to highlight that the y-intercept of the first iteration would be different from that of the rest.

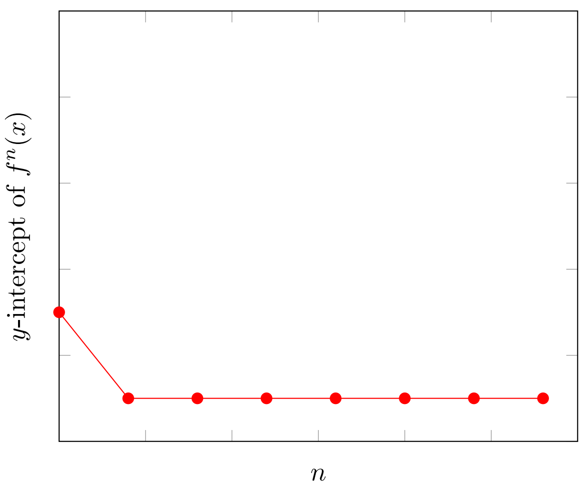

Fig. 8: the y-intercept of $f^n(x)$ vs. n

Again, because we are iterating the same value in the same function, we can expect this straight line to continue.

There is another situation where, in the first three iterations of $f(x)$, the y-intercepts of the first and second iterations of $f(x)$ would be equal. However, this means that the graph of the y-intercepts in relation to n for $f^n(x)$ would then just be a straight line, meaning that there would only be one y-intercept over all iterations of $f(x)$. This does not apply because we are looking for situations with two y-intercepts.

We also know that if no two of the first three iterations are equal, they would have 3 distinct values, so we can exclude the cased where there is no equality between the first three iterations. This means that the two situations that we have discussed are the only applicable situations.

4.6 Analysis

The fact that we only found 1 and 2 as our solutions makes sense as they are values of a that when plugged into the y-intercepts of iterations of f (that are in terms of a) will give us exactly two possible outputs. When

$a = 1$, the value of the y-intercept of consecutive iterations of this function oscillates between 0 and -1. For example, when we input 0, we get an output -1, and when we input -1, we get an output 0. Inputting -1 gives us 0 again, and this is where we get a “cycle”:

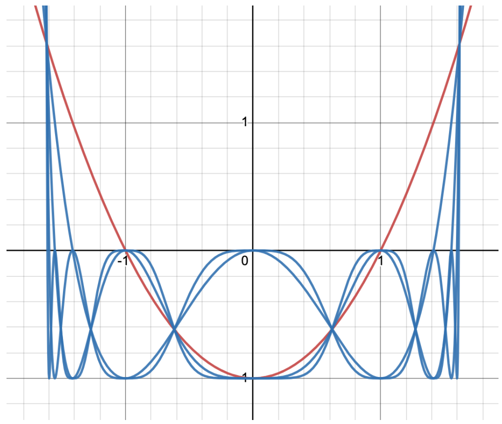

Fig. 5: Graph of $f(x) = x^2 - 1$

When $a = 2$, the first iteration of f has a y-intercept of -2 and all subsequent iterations have a y-intercept of 2. When we first input 0, we get -2 as out output, and when we input -2, we get 2 as our output. Inputting 2 gets us 2 as our output - thus initiating a stable point:

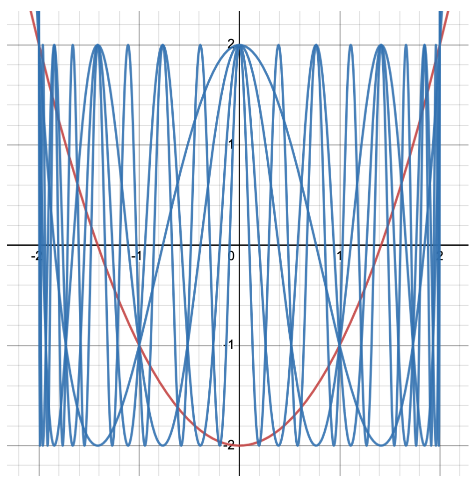

Fig. 6: Graph of $f(x) = x^2 - 2$

This displays the interesting importance of the values 1 and 2 within our quadratic and their nature throughout multiple iterations. This also gives us a concrete restriction on a for a specific case in our function—having 2 y-intercepts—which is significant because it lays the groundwork for further inquiry on a more general relationship between a and the function’s y-intercept.

5 Further Inquiry

An interesting direction for further investigation would be regarding whether there was a concrete relation between a and the y-intercepts of this function. Finding a relation (or no relation) can give us insight into how the number of intercepts that remain constant through infinite iterations of a function are affected by values in the function itself. Additionally, another path of pursuit could be on patterns of y- or x- intercepts throughout multiple iterations, and how changing a or the amount of iterations affects the amount of change within the intercept. Our work is a stepping stone for inquiries in these directions, as we have already identified a pattern in greatest x-intercepts for our function, and our work provides solid groundwork for further study around restrictions, patterns, limits, and values of y- and x-intercepts.