Homework 9

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Problem 14.2: Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4.

Part (a): Give a linear-to-linear rational model relating the meter reading y to how many feet x Oscar has gone into the room.

Part (b): How far must he go for the meter to reach 10? 100?

Part (c): Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

Part (a) Solution: The standard form of a linear to linear rational function is given as $\frac{ax+b}{x+c}$ where a, b, and c are constants. We are given that when Oscar is some far length down the hallway the meter reads 0.2, so we can assume that the horizontal asymptote will be $y = 0.2$. In the rational function, a represents the value of the horizontal asymptote, so we can say $a = 0.2$. Our function is then $y = \frac{0.2+b}{x+c}$.

We are also given that 6 feet into the room, the meter reads 2.3 and 8 feet into the room the meter reads 4.4. In other words, if y is the meter reading and x is the number of feet into the room, we have the coordinates $(6, 2.3)$ and $(8, 4.4)$. By plugging these coordinates into our rational function, we can set up the following equations:

$$\frac{0.2(6)+b}{6+c} = 2.3$$

$$\frac{0.2(8)+b}{8+c} = 4.4$$

Solving both of these equations for b, we have $b = 12.6 + 2.3c$ and $b = 33.6 + 4.4c$. Setting these two equations equal to each other, we can solve for c:

$$12.6 + 2.3c = 33.6 + 4.4c$$

$$c = -10$$

Plugging c back into the first equation, we can solve for b:

$$12.6 + 2.3(-10) = b$$

$$b = -10.4$$

Using these values for a, b, and c, our linear to linear rational function is $\boxed{y = \frac{0.2x-10.4}{x-10}.}$

Part (b) Solution: To find how far he must go for the meter to reach 10, we can set our equation from part (a) equal to 10:

$$\frac{0.2x - 10.4}{x - 10} = 10$$

Solving for x:

$$-9.8x = -89.6$$
$$x = \frac{89.6}{9.8} = \frac{896}{98}$$

Therefore he must go $\boxed{\frac{896}{98} \text{ feet}}$ for the meter to reach 10.

To find how far he must go for the meter to reach 100, we can set our equation equal to 100:

$$\frac{0.2x - 10.4}{x - 10} = 100$$

Solving for x:

$$-99.8x = -989.6$$
$$x = \frac{989.6}{99.8} = \frac{9896}{998}$$

Therefore he must go $\boxed{\frac{9896}{998} \text{ feet}}$ for the meter to reach 100.

Part (c) Solution: The magnet is 10 feet into the room. When we consider our function $y = \frac{0.2x-10.4}{x-10}$, we know that x cannot be 10 because then the denominator of the function will be 0. In other words, the meter will not produce a reading when Oscar is 10 feet into the room. This suggests that the magnet is 10 feet into the room, as when Oscar reaches the magnet, the meter won't be able to produce a reading.

Problem 14.4: Isobel is producing and selling casette tapes of her rock band. When she had sold 10 tapes, her net profit was \$6. When she had sold 20 tapes, however, her net profit had shrunk to \$4 due to increased production expenses. But when she had sold 30 tapes, her net profit had rebounded to \$8.

Part (a): Give a quadratic model relating Isobel's net profit y to the number of tapes sold x.

Part (b): Divide the profit function in part (a) by the number of tapes sold x to get a model relating average profit w per tape to the number of tapes sold.

Part (c): How many tapes must she sell in order to make \$1.20 per tape in net profit?

Part (a) Solution: From the information given in the problem, we have three points: $(10, 6)$, $(20, 4)$ and $(30, 8)$. The standard form of a quadratic functions is $y = ax^2 + bx + c$. We can substitute these three points into this equation to get a system of 3 equations:

$$\begin{cases} 100a + 10b + c = 6 \\ 400a + 20b + c = 4 \\ 900a + 30b + c = 8 \end{cases}$$

First, we can subtract the first equation from the second one to get $-300a - 10b = 2$. Next, we can subtract the third equation from the second to get $-500a - 10b = -4$. Subtracting this equation from the last one, we get $200a = 6$. Solving this, we have $a = \frac{3}{100}$.

Next, we can substitute this value of a into the first equation in the system to get $100\left(\frac{3}{100}\right) + 10b + c = 6$, or $10b + c = 3$. Now, we can substitute this value of a into the second equation in the system to get

$400\left(\frac{3}{100}\right) + 20b + c = 4$, or $20b + c = -8$. Subtracting this equation from the last one, we have $-10b = 11$. Solving for b, we have $b = -1.1$.

Substituting $\frac{3}{100}$ for a and -1.1 for b into the first equation in the system, we have $3 - 11 + c = 6$. Solving for c, we have $c = 14$. Using these values for a, b, and c, our quadratic model is $\boxed{y = 0.03x^2 - 1.1x + 14.}$

Part (b) Solution: If we divide the profit function by x, the model relating the average profit w to the number of tapes sold is

$$\boxed{w = \frac{0.03x^2 - 1.1x + 14}{x}}$$

Part (c) Solution: In order to find the amount of tapes that need to be sold to make an average of \$1.20 per tape, we have to set w equal to 1.2:

$$\frac{0.03x^2 - 1.1x + 14}{x} = 1.2$$

Solving for x:

$$0.03x^2 - 1.1x + 14 = 1.2x$$
$$0.03x^2 - 2.3x + 14 = 0$$
$$x = \frac{20}{3},\ x = 70$$

Therefore she must sell either $\boxed{\frac{20}{3} \text{ or } 70}$ tapes to make \$1.20 per tape in net profit. However, it would be difficult to sell a fractional amount of tapes, so it makes more sense for her to sell 70 tapes.

Problem 14.7: The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score is a linear-to-linear function of the number of hours that you study. If you want a score of 80%, how long do you need to study?

Solution: The standard form for a linear-to-linear function is $y = \frac{ax+b}{x+c}$ where a, b, and c are constants. Here, x is the number of hours we have studied for and y is our score. From the information given in the problem, we know that as we study more hours, the closer our score will get to perfect, or 100%. Therefore, the horizontal asymptote of the function is 100, so $a = 100$. This makes our function

$$y = \frac{100x + b}{x + c}$$

We are also given two coordinates from the information in the problem: $(10, 65)$ and $(20, 95)$. We can plug both of these coordintaes into the function to get two different equations:

$$\frac{100(10) + b}{10 + c} = 65$$
$$\frac{100(20) + b}{20 + c} = 95$$

Solving the first equation for b, we have:

$$1000 + b = 650 + 65c$$
$$b = -350 + 65c$$

Solving the second equation for b, we have:

$$2000 + b = 1900 + 95c$$
$$b = -100 + 95c$$

Setting these two equations for b equal to each other, we have

$$-350 + 65c = -100 + 95c$$

Solving for c:

$$-30c = 250$$
$$c = -\frac{25}{3}$$

Substituting this value of c back into the equation $b = -350 + 65c$, we have $b = -350 - \frac{1625}{3} = -\frac{2675}{3}$. Plugging these values for b and c back into our linear-to-linear model, our answer is

$$\boxed{y = \frac{100x - \frac{2675}{3}}{x - \frac{25}{3}}}$$

If we want to find how many hours we have to study to score 80%, we can set y equal to 80:

$$\frac{100x - \frac{2675}{3}}{x - \frac{25}{3}} = 80$$

Solving for x:

$$100x - \frac{2675}{3} = 80x - \frac{2000}{3}$$
$$20x = \frac{675}{3}$$
$$x = \frac{675}{60}$$

Therefore, you have to study $\boxed{\frac{675}{60} \text{ hours}}$ to get an 80%.

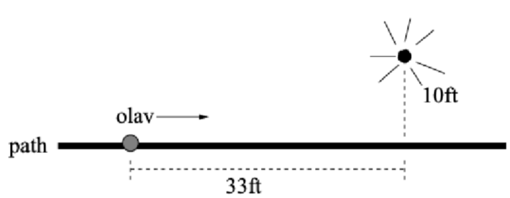
Problem 14.8: A street light is 10 feet above a straight bike path. Olav is bicycling down the path at a rate of 15 MPH. At midnight, Olav is 33 feet from the point on the bike path directly below the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source.

Part (a): From 20 feet away, the street light has an intensity of 1 candle. What is k?

Part (b): Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.

Part (c): When will the light on Olav have maximum intensity?

Part (d): When will the intensity of the light be 2 candles?



Part (a) Solution: To find k, we can substitute 20 for d and 1 for C in the equation $C = \frac{k}{d^2}$:

$$1 = \frac{k}{400}$$

$$\boxed{k = 400}$$

Part (b) Solution: Currently, we have an equation which gives the intensity of light as a function of distance. Distance is just rate times time. We are given that Olav travels at a rate of 15 miles per hour. We can let the time (in seconds) he has been traveling for be represented by the variable t.

If he travels at 15 miles per hour, this is the same thing as $\frac{1}{240}$ miles per second, or 22 feet per second. Now, we need to calculate Olav's distance from the streetlight. We know that the horizontal distance between Olav and the streetlight as he is moving is $33 - 22t$ feet. The vertical distance between him and the streetlight is 10 feet. Using the Pythagorean theorem (as Olav's location, the point below the streetlight, and the streetlight are 3 points of a triangle), the distance between him and the streetlight is $d = \sqrt{(33 - 22t)^2 + 10^2}$.

Substituting this value of d into the equation $C = \frac{k}{d^2}$, we have

$$C = \frac{k}{(33 - 22t)^2 + 100}$$

However, we already found that $k = 400$, so we can substitute 400 for k to give us the function

$$\boxed{C = \frac{400}{(33 - 22t)^2 + 100}}$$

Part (c) Solution: The light on Olav will have maximum intensity when Olav is at the spot directly below the street light. Or, the distance from him to the spot directly below the street light is 0. When this distance is 0, Olav will have traveled 33 feet from his starting point, or $\frac{33}{5280}$ miles. He travels at a rate of 15 miles per hour, so it will take him $\boxed{\frac{\frac{33}{5280}}{15} \text{ hours after he starts}}$ to be directly under the streetlight.

Part (d) Solution: To find when the intensity will be 2 candles, we can substitute 2 for C in the function $C = \frac{400}{(33-22t)^2+100}$:

$$2 = \frac{400}{(33 - 22t)^2 + 100}$$

Solving for t:

$$(33 - 22t)^2 + 100 = 200$$

$$(33 - 22t)^2 = 100$$

$$|33 - 22t| = 10$$

$$33 - 22t = 10,\ 33 - 22t = -10$$

$$t = \frac{23}{22},\ t = \frac{43}{22}$$

The intensity of the light will be 2 candles $\boxed{\frac{23}{22} \text{ and } \frac{43}{22} \text{ seconds after he starts.}}$