Homework 15

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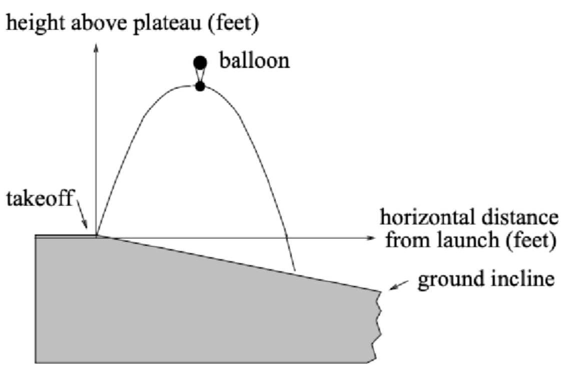
Problem 7.7: A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. Answer the following questions:



Part (a): What is the maximum height of the balloon above plateau level?

Part (b): What is the maximum height of the balloon above ground level?

Part (c): Where does the balloon land on the ground?

Part (d): Where is the balloon 50 feet above the ground?

Part (a) Solution: First, we can impose a coordinate system where the balloon's takeoff point is the origin, the line in the diagram labeled as "height above plateau" is the y-axis, and the line in the diagram labeled as "horizontal distance from launch" is the x-axis. Intervals on both axes are in terms of feet. To find the maximum height of the balloon above plateau level, we only need to find the vertex of the parabola that the balloon's path forms, as "plateau level" is also the x-axis. The y-coordinate of the vertex will be the maximum height the balloon reaches. We can start by finding the x-coordinate of the vertex, which we can find by using the formula $-\frac{b}{2a}$:

$$-\frac{b}{2a} = \frac{\frac{4}{5}}{\frac{8}{2500}} = 250$$

To find the y-coordinate of the vertex, we can substitute this x-coordinate into the equation for the parabola:

$$f(x) = -\frac{4}{2500}(250)^2 + \frac{4}{5}(250)$$
$$f(x) = -100 + 200 = 100$$

Therefore, the maximum height of the balloon above plateau level is $\boxed{\text{100 feet.}}$

Part (b) Solution: We know that the equation $f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$ represents the distance between the balloon and the plateau. Now, we need to find the equation that models the distance from the plateau to the ground. The ground incline is 1 vertical foot for each 5 horizontal feet, meaning the slope of the ground is $-\frac{1}{5}$. This line also intersects the origin, so the line is $y = -\frac{1}{5}x$. If this equation represents the distance from the plateau to the ground, then we know that the distance from the ground to the balloon will have to be the sum of the distances represented by these equations. Therefore, the equation that represents the distance from the balloon to the ground will be $y = -\frac{4}{2500}x^2 + \frac{4}{5}x + \frac{1}{5}x$, or $y = -\frac{4}{2500}x^2 + x$. To find the maximum height of the balloon above ground level, we need to find the vertex of this new parabola. We can start by finding the x-coordinate:

$$-\frac{b}{2a} = \frac{1}{\frac{8}{2500}} = \frac{625}{2}$$

Now, to find the maximum height, we need to substitute $\frac{625}{2}$ for x into the equation:

$$y = -\frac{4}{2500}\left(\frac{625}{2}\right)^2 + \frac{625}{2} = -\frac{4}{2500}\left(\frac{390625}{4}\right) + \frac{625}{2} = -156.25 + 312.5 = 156.25$$

Therefore, the maximum height of the balloon above ground level is $\boxed{\text{156.25 feet.}}$

Part (c) Solution: There are two points at which the balloon is on the ground: one when the balloon takes off, and one where the balloon lands. The ground is represented by the line $y = -\frac{1}{5}x$, meaning that that those two points when the balloon is on the ground are when the balloon's equation, $y = -\frac{4}{2500}x^2 + \frac{4}{5}x$ intersects $y = \frac{1}{5}x$. Because we are finding the point when the balloon lands, we are looking for the intersection that is farthest right. We can find these intersections by setting the two equations equal to each other:

$$-\frac{4}{2500}x^2 + \frac{4}{5}x = -\frac{1}{5}x$$
$$-\frac{4}{2500}x^2 + x = 0$$
$$x = 0,\ x = 625$$

The right most intersection will be when $x = 625$, so we have to substitute 625 for x into either the equation for the ground or the balloon's path to find the y-coordinate of where the balloon lands. For simplicity, we can take the equation that represnts the ground:

$$y = -\frac{1}{5}(625)$$
$$y = -125$$

This means that the balloon lands on the ground at the point $\boxed{(625, -125).}$

Part (d) Solution: We found that the equation that represents the balloon's height above the ground is $y = -\frac{4}{2500}x^2 + x$. To find when the balloon is 50 feet above the ground, we can substitute 50 for y in this equation:

$$50 = -\frac{4}{2500}x^2 + x$$
$$0 = -\frac{4}{2500}x^2 + x - 50$$
$$x = \frac{625 + 125\sqrt{17}}{2},\ x = \frac{625 - 125\sqrt{17}}{2}$$

Now, we can substitute both of these values for x back into the equation for the balloon's path to find the locations at which the balloon is 50 feet above the ground. We can start by substituting $x = \frac{625-125\sqrt{17}}{2}$:

$$y = -\frac{4}{2500}\left(\frac{625-125\sqrt{17}}{2}\right)^2 + \frac{4}{5}\left(\frac{625-125\sqrt{17}}{2}\right) \approx 39.04$$

Next, we can substitute $x = \frac{625+125\sqrt{17}}{2}$:

$$y = -\frac{4}{2500}\left(\frac{625+125\sqrt{17}}{2}\right)^2 + \frac{4}{5}\left(\frac{625+125\sqrt{17}}{2}\right) \approx -64.04$$

Therefore, the two points at which the balloon will be 50 feet above the ground are $\boxed{\left(\frac{625-125\sqrt{17}}{2}, 39.04\right)}$ and $\boxed{\left(\frac{625+125\sqrt{17}}{2}, -64.04\right).}$

Problem 7.9: Sylvia has an apple orchard. One season, her 100 trees yielded 140 apples per tree. She wants to increase her production by adding more trees to the orchard. However, she knows that for every 10 additional trees she plants, she will lose 4 apples per tree (i.e., the yield per tree will decrease by 4 apples). How many trees should she have in the orchard to maximize her production of apples?

Solution: We can let x be the amount of trees that Sylvia increases her orchard by. Let $f(x)$ be the amount of apples she produces. We know that Sylvia starts at 100 trees that each produce 140 apples. If she loses 4 apples per tree for every 10 trees she plants, the equation that represents her apple yield will be $f(x) = (100+x)(140-\frac{4}{10}x)$. To find Sylvia's maximum profit, we will need to find the vertex of this parabola. First, we can expand the equation to $f(x) = 14000 + 100x - \frac{2}{5}x^2$. We can find the x-coordinate of the vertex:

$$-\frac{b}{2a} = \frac{100}{\frac{4}{5}} = 125$$

125 trees is the amount of trees Sylvia has to plant after already planting 100. This means that Sylvia actually has to plant $125 + 100 = \boxed{225 \text{ trees}}$ to maximize her profit.

Problem 7.10: Rosalie is organizing a circus performance to raise money for a charity. She is trying to decide how much to charge for tickets. From past experience, she knows that the number of people who will attend is a linear function of the price per ticket. If she charges 5 dollars, 1200 people will attend. If she charges 7 dollars, 970 people will attend. How much should she charge per ticket to make the most money?

Solution: From the problem, we know that when Rosalie increases her price by 2 dollars, she loses 230 people. Or, if we simplified this, when Rosalie increases her price by a dollar, she loses 115 people. We can let x be the amount that Rosalie increases her ticket price by. Let $f(x)$ be the amount of people that attend. We can model the situation as a quadratic function: $f(x) = (5+x)(1200-115x)$. Expanding this equation yields $f(x) = 6000 + 625x - 115x^2$. To find when Rosalie makes the most money, we need to find the x-coordinate of the vertex of this parabola:

$$-\frac{b}{2a} = \frac{625}{230}$$

However, $\frac{625}{230}$ dollars represents the amount *on top of 5 dollars* that Rosalie has to charge. This means that for Rosalie to make the most money, she has to charge $5 + \frac{625}{230} \approx \boxed{7.72 \text{ dollars.}}$