Connections 4

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Question C: Why does the equation

$$x^6 + 3x^4y^2 - 12x^4y + 3x^2y^4 - 24x^2y^3 + 44x^2y^2 = -y^6 + 12y^5 - 44y^4 + 48y^3$$

produce a graph that consists of three circles?

Rewritten Question C: How many circles does the equation

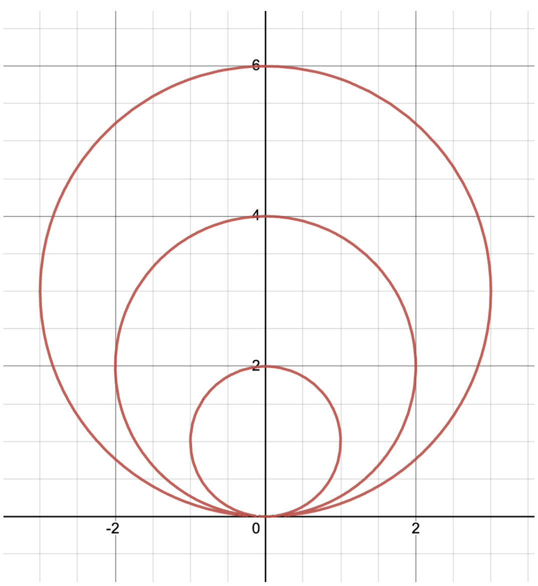
$$x^6 + 3x^4y^2 - 12x^4y + 3x^2y^4 - 24x^2y^3 + 44x^2y^2 = -y^6 + 12y^5 - 44y^4 + 48y^3$$

produce?

Solution: We can start by writing this equation so that all the terms are on the left side for simplicity. The equation would then be:

$$x^6 + 3x^4y^2 - 12x^4y + 3x^2y^4 - 24x^2y^3 + 44x^2y^2 + y^6 - 12y^5 + 44y^4 - 48y^3 = 0$$

To find how many circles this equation produces, we can use a graphing calculator to graph the equation:



The graph that the calculator produces is of three circles. However, we have to verify this equation graphs all three of these. To do this, we can start by writing the equations for each circle individually. As per the graph, the largest circle has a center of $(0, 3)$ and radius of 3. The second largest circle has a center of $(0, 2)$ and a radius of 4. The smallest circle has a center of $(0, 1)$ and a radius of 1. Using the standard form for the equation of a circle, we can say the equations for each of these of these three circles are:

$$x^2 + (y-3)^2 = 9$$

and

$$x^2 + (y-2)^2 = 4$$

and

$$x^2 + (y-1)^2 = 1$$

Now, we can simplify each of the equations so that each is set equal to 0. The equations would then be:

$$x^2 + (y-3)^2 - 9 = 0$$

and

$$x^2 + (y-2)^2 - 4 = 0$$

and

$$x^2 + (y-1)^2 - 1 = 0$$

If $x^2 + (y-3)^2 - 9$, $x^2 + (y-2)^2 - 4$, and $x^2 + (y-1)^2 - 1$ all equal to zero, we know that the product of all three will also equal zero. We also know that multiplying the equations of these three circles will result in an equation that will graph all three. Therefore, if we multiply these three equations and the product is equivalent to $(x^6 + 3x^4y^2 - 12x^4y + 3x^2y^4 - 24x^2y^3 + 44x^2y^2 + y^6 - 12y^5 + 44y^4 - 48y^3 = 0)$, then we can say that this equation could represent the graph of three circles. If we multiplied these equations, it would look like:

$$(x^2 + (y-3)^2 - 9)(x^2 + (y-2)^2 - 4)(x^2 + (y-1)^2 - 1) = 0$$

Expanding the first two parentheses:

$$(x^4 + x^2(y-2)^2 - 4x^2 + x^2(y-3)^2 + (y-3)^2(y-2)^2 - 4(y-3)^2 - 9x^2 - 9(y-2)^2 + 36)(x^2 + (y-1)^2 - 1) = 0$$

Simplifying:

$$(x^4 + x^2y^2 - 4x^2y + 4x^2 - 4x^2 + x^2y^2 - 6x^2y + 9x^2 + (y^2 - 6y + 9)(y^2 - 4y + 4) - 4y^2 + 24y - 36 - 9x^2 - 9y^2 + 36y - 36 + 36) \cdot$$
$$(x^2 + (y-1)^2 - 1) = 0$$

Simplifying even more:

$$(x^4 + 2x^2y^2 - 10x^2y + y^4 - 10y^3 + 24y^2) \cdot (x^2 + (y-1)^2 - 1) = 0$$

Multiplying again:

$$x^6 + x^4(y-1)^2 - x^4 + 2x^4y^2 + 2x^2y^2(y-1)^2 - 2x^2y^2 - 10x^4y - 10x^2y(y-1)^2 + 10x^2y + x^2y^4 + y^4(y-1)^2 - y^4 - 10x^2y^3$$
$$-10y^3(y-1)^2 + 10y^3 + 24x^2y^2 + 24y^2(y-1)^2 - 24y^2 = 0$$

Simplifying:

$$x^6+x^4y^2-2x^4y+x^4-x^4+2x^4y^2+2x^2y^4-4x^2y^3+2x^2y^2-2x^2y-10x^4y-10x^2y^3+20x^2y^2-10x^2y+10x^2y+x^2y^4$$
$$+y^6-2y^5+y^4-y^4-10x^2y^3-10y^5+20y^4-10y^3+10y^3+24x^2y^2+24y^4-48y^3+24y^2-24y^2=0$$

Simplifying even more:

$$x^6+3x^4y^2-12x^4y+3x^2y^4-24x^2y^3+44x^2y^2+y^6-12y^5+44y^4-48y^3=0$$

The equation we got when multiplying the equations of the three circles is the same as the initial equation, meaning that the graph could represent three circles. However, we also need to ensure that the equation does not graph more than three circles. To do this, we can represent a fourth circle as the standard form of the equation of a circle, $(x-h)^2+(y-k)^2=r^2$. This can be rewritten as $(x-h)^2+(y-k)^2-r^2=0$. Therefore, if the initial equation were to graph four circles, then the product of all four circles would also be equal to zero:

$$(x^2+(y-3)^2-9)(x^2+(y-2)^2-4)(x^2+(y-1)^2-1)((x-h)^2+(y-k)^2-r^2)=0$$

Each of the factors in the equation, $(x^2+(y-3)^2-9)$, $(x^2+(y-2)^2-4)$, $(x^2+(y-1)^2-1)$, and $((x-h)^2+(y-k)^2-r^2)$ have a degree of 2. We know that the initial equation given has a degree of 6. When expanding this new equation, it will give us an expression of the 8th degree, a result of four 2nd degree expressions being multiplied. Because the new equation will result in a higher degree than the initial one, we know that the initial equation cannot represent the graph of 4 circles. Using the same reasoning, any more than the product of three circles will result in a degree higher than 6. This means we can be sure that this equation $\boxed{\text{produces three circles.}}$