Homework 6

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Problem 3.5: A crawling tractor sprinkler is located as pictured below, 100 feet South of a sidewalk. Once the water is turned on, the sprinkler waters a circular disc of radius 20 feet and moves North along the hose at the rate of $\frac{1}{2}$ inch/second. The hose is perpendicular to the 10 ft. wide sidewalk. Assume there is grass on both sides of the sidewalk.

Part (a): Impose a coordinate system. Describe the initial coordinates of the sprinkler and find equations of the lines forming the North and South boundaries of the sidewalk.

Part (b): When will the water first strike the sidewalk?

Part (c): When will the water from the sprinkler fall completely North of the sidewalk?

Part (d): Find the total amount of time water from the sprinkler falls on the sidewalk.

Part (e): Sketch a picture of the situation after 33 minutes. Draw an accurate picture of the watered portion of the sidewalk.

Part (f): Find the area of GRASS watered after one hour.

Part (a) Solution: To impose a coordinate system, we can define the x-axis as the southern border of the sidewalk and the hose as the y-axis. The origin of the system would be the point at which the hose and the southern border of the sidewalk intersect. Intervals on the axes would be in terms of feet, where one unit is equal to one foot. Therefore, the $\boxed{\text{initial point of the sprinkler would be at } (0, -100)}$ and the North and South boundaries of the sidewalk would be defined by the lines $\boxed{y = 10 \text{ and } y = 0}$, respectively.

Part (b) Solution: We know that because the radius of the circle is 20 feet, the farthest north the water spray will reach at its starting point is to the point $(0, -80)$, because the tractor starts at the point $(0, -100)$. We know that the water will first strike the sidewalk when the spray reaches the southern-most boundary of the sidewalk. Because we already defined the line $y = 0$, or the x-axis as the southern boundary, we know that the tractor will be at the point $(0, -20)$ when the water strikes the sidewalk. To find when the water hits the sidewalk, we have to calculate the time the tractor takes to travel from its original point, $(0, -100)$ to the point $(0, -20)$. We know that the tractor moves at a rate of $\frac{1}{2}$ inches per second. Converting into feet and minutes:

$$\frac{1\,\text{inch}}{2\,\text{second}} \cdot \frac{60\,\text{second}}{1\,\text{minute}} \cdot \frac{1\,\text{foot}}{12\,\text{inches}} = \frac{5}{2}\,\text{feet per minute}$$

Both $(0, -100)$ and $(0, -20)$ are on the same vertical line because both of their x-coordinates are zero. Therefore, the distance between the two points is 80 feet. We can use the distance formula to solve for the time the tractor takes because we already know the distance the tractor travels and its speed. If the distance is 80 feet and the tractor moves at $\frac{5}{2}$ feet per minute, we can say the water will strike the sidewalk after

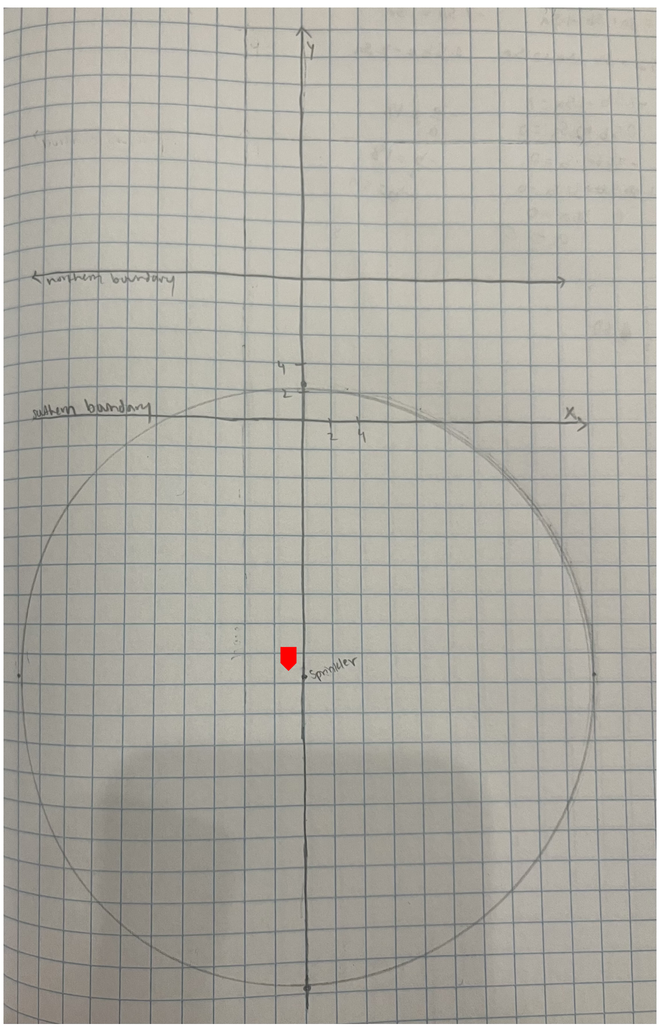
$$\boxed{\frac{80}{2.5} = 32\,\text{minutes.}}$$

Part (c) Solution: The water will be completely North of the sidewalk when the southern-most point of the water spray is past the northern border, which we defined as $y = 10$. Because the tractor is driving along the x-axis, the water spray will fall completely north of the sidewalk when the southern-most point of the water is at the point $(0, 10)$. At the tractor's starting point, we know that the southern-most point of the water spray will be at $(0, -120)$ because the tractor starts at $(0, -100)$ and the radius is 20 feet. To find when all the water will be north of the sidewalk, we have to find the time that the tractor takes to travel from $(0, -120)$ and $(0, 10)$. Because these two points are on the x-axis, the distance between them is 130 feet. We know that the tractor travels at a speed of 2.5 feet per second, so we can use the distance formula to say the water spray will be completely north of the sidewalk just after

$$\boxed{\frac{130}{2.5} = 52\,\text{minutes.}}$$

Part (d) Solution: To solve this problem, we can use the answers we found in Parts (a) and (b). We know that the time that the water falls on the sidewalk is the amount of time between the first time water touches the southern border and the last time the water touches the northern border. As we found in Part (a), the water will first strike the sidewalk in 1920 seconds. In Part (b), we found that the water reaches the edge of the northern boundary of the sidewalk after 3120 seconds. Therefore, the total amount of time the water from the sprinkler falls from the sidewalk is $3120 - 1920 = \boxed{1200\,\text{seconds.}}$

Part (e) Solution:



Part (f) Solution: To solve this problem, we can say the area of the total grass watered is the area of the sidewalk watered subtracted from the total land watered. The total area the water has covered includes the bottom semicircle of the tractor's initial starting point, the top semicircle of the tractor's ending point, and the rectangle in between these two semicircles. The rectangle's width is the diameter of the semicircles, or 40 feet. The rectangle's height is the distance between the centers of the diameters of each of the semicircles.

We can start by finding the area of this rectangle. We already know its width, 40 feet. To find the height of the rectangle, we have to find the centers of the diameters of each of the semicircles. The center of the diameter of the bottom semicircle is the initial location of the tractor, at the point $(0, -100)$. To find the center of the other diameter, we have to remember the problem is asking about the area of grass watered after one hour. Because we are given the amount of time and we already calculated the tractor has a speed of 2.5 feet per second, we can say the distance the tractor will travel is:

$$2.5 \cdot (60\,\text{minutes}) = 150\,\text{feet}$$

Therefore, because the tractor travels 150 feet, the height of the rectangle is also 150 feet. The area of the rectangle would then be

$$150 \cdot 40 = 6000\,\text{square feet.}$$

Now, we can find the area of the top and bottom semicircles. We know that two semicircles will have the same area as one circle, so we can just find how much area the water spray covers when the tractor is in one place. The formula to solve for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Solving for area:

$$A = \pi 20^2$$
$$A = 400\pi$$

Adding the areas of the rectangle and the circle, we can say the total area covered by the water spray is $(6000 + 400\pi)$ square feet. However, we still have to subtract the area of the sidewalk that water was sprayed on. The width of this area would be the same as the circle's diameter, 40 feet. The problem states that the height of the sidewalk is 10 feet. Therefore, the area of the sidewalk that was sprayed is 400 square feet. Therefore, the area of the grass covered after one hour will be

$$\boxed{(5600 + 400\pi)\,\text{square feet.}}$$

Problem 3.6: Erik's disabled sailboat is floating stationary 3 miles East and 2 miles North of Kingston. A ferry leaves Kingston heading toward Edmonds at 12 mph. Edmonds is 6 miles due east of Kingston. After 20 minutes the ferry turns heading due South. Ballard is 8 miles South and 1 mile West of Edmonds. Impose coordinates with Ballard as the origin.

Part (a): Find the equations for the lines along which the ferry is moving and draw in these lines.

Part (b): The sailboat has a radar scope that will detect any object within 3 miles of the sailboat. Looking down from above, as in the picture, the radar region looks like a circular disk. The *boundary* is the "edge" or circle around this disc, the *interior* is the inside of the disk, and the *exterior* is everything outside of the disk (i.e. outside of the circle). Give a mathematical (equation) description of the boundary, interior and exterior of the radar zone. Sketch an accurate picture of the radar zone by determining where the line connecting Kingston and Edmonds would cross the radar zone.

Part (c): When does the ferry enter the radar zone?

Part (d): Where and when does the ferry exit the radar zone?

Part (e): How long does the ferry spend inside the radar zone?

Part (a) Solution: Ballard is located at the origin, $(0,0)$. Therefore, Edmonds is located at $(1,8)$, Kingston is located at $(-5,8)$, and Erik's disabled sailboat is located at $(-2,10)$. We know that the ferry is traveling from Kingston towards Edmonds, which both have y-coordinates of 8. Therefore, one of the lines the ferry is traveling on is $\boxed{y = 8.}$ To find the line the ferry is traveling on when it turns south, we have to find the point at which it turns south at. The problem states the ferry turns south after 20 minutes and is traveling at a speed of 12 mph. Using the distance formula, we can say the distance the ferry travels on the line $y = 8$ is

$$\frac{1}{3} \cdot 12 = 4\,\text{miles}.$$

We know that Kingston is located at $(-5,8)$, so four miles east of this point will be located at $(-1,8)$. Because the x-coordinate of the point at which the boat turns south at is -1, the other line that the boat travels on will be $\boxed{x = -1.}$

Part (b) Solution: As we defined in Part (a), the sailboat is located at $(-2,10)$. The radar zone detects any object within 3 miles of the sailboat, meaning that $(-2,10)$ is the center of the circular disk and the radius of the circular disk is 3 miles. Plugging these values into the standard form of a circle, the mathematical equation that describes the boundary of this disk is

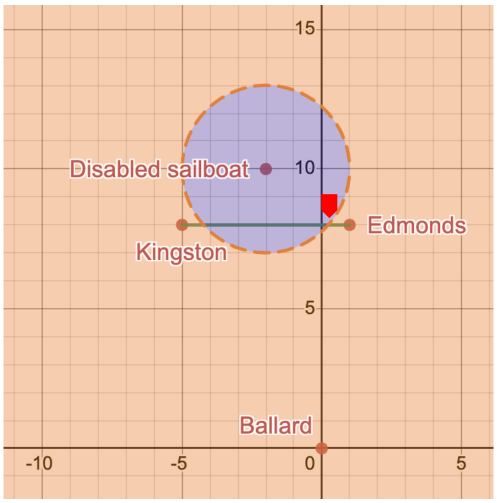
$$\boxed{(x+2)^2 + (y-10)^2 = 9}$$

The interior of this circle is the area inside this boundary. Therefore, we would define the interior using an inequality with the "less than" symbol. The interior of the circle can be described as:

$$\boxed{(x+2)^2 + (y-10)^2 < 9}$$

Similarly, the exterior of the circle would also be an inequality, but instead using a "greater than" symbol. The exterior can be described as:

$$\boxed{(x+2)^2 + (y-10)^2 > 9}$$



Part (c) Solution: We know that the ferry is traveling along the line $y = 8$ from Kingston to Edmonds. This means that the ferry will first enter the radar zone at the first intersection between $y = 8$ and the boundary of the circle, $(x+2)^2 + (y-10)^2 = 9$. To find the intersection, we can substitute 8 for y in the equation for the circle:

$$(x+2)^2 + (8-10)^2 = 9$$

Solving for x:

$$x^2 + 4x - 1 = 0$$
$$x = -2 + \sqrt{5},\ x = -2 - \sqrt{5}$$

Between these two values of x, we need the leftmost intersection because we are looking for the first time the ferry's path intersects with the radar. Therefore, we can say the first intersection will be at $(-2-\sqrt{5}, 8)$. We know that the ferry starts at the point $(-5, 8)$ in Kingston. The distance between these two coordinates will be $(|-5+2+\sqrt{5}|)$ units, because they are located on the same horizontal line. The problem states that the ferry is traveling at a speed of 12 mph. We can use the distance formula to find when the ferry will intersect the boundary. The ferry will enter the radar zone after

$$\boxed{\frac{|-3+\sqrt{5}|}{12} \approx \frac{19}{300} \text{ hours.}}$$

Part (d) Solution: In Part (a), we found that the ferry turns south at $(-1, 8)$ and travels along the line $x = -1$. To find the location at which the ferry exits the radar zone, we have to find the point at which the line $x = -1$ intersects the boundary of the circle. We can do this by substituting -1 for x into the equation for the boundary of the circle:

$$(-1+2)^2 + (y-10)^2 = 9$$

Solving for y:

$$y^2 - 20y + 92 = 0$$

$$y = 10 + 2\sqrt{2},\ y = 10 - 2\sqrt{2}$$

Therefore, the line $x = -1$ intersects the boundary of the circle at $(-1, 10 + 2\sqrt{2})$ and $(-1, 10 - 2\sqrt{2})$. The ferry exits the circle at the coordinate with the smaller y-value (the intersection farther down the y-axis), meaning that the $\boxed{\text{ferry exits the radar at } (-1, 10 - 2\sqrt{2}).}$

To find the when the ferry exits this radar zone, we have to find the distance the ferry travels due east as well as due south. We know that the ferry turns south at the point $(-1, 8)$ and begins from Kingston at the point $(-5, 8)$. This means the ferry travels due east for 4 units. As we found earlier, the ferry exits the radar zone at the point $(-1, 10 - 2\sqrt{2})$. The distance between this point and $(-1, 8)$ is $-2 + 2\sqrt{2}$, which is also the distance the ferry travels south for. To find the time the ferry takes to cover these distances, we can use the distance formula. Plugging in the distances and the ferry's speed (12 mph), the ferry will exit the radar zone after

$$\boxed{\frac{4 - 2 + 2\sqrt{2}}{12} = \frac{1 + \sqrt{2}}{6} \text{ hours.}}$$

Part (e) Solution: We found in Part (d) that the ferry will exit the radar zone $\left(\frac{1+\sqrt{2}}{6}\right)$ hours after it starts from Kingston. In Part (c), we found that the ferry enters the radar zone after approximately $\left(\frac{19}{300}\right)$ hours. Therefore, the time the ferry spends inside the radar zone is the time it takes to enter the radar zone subtracted from the time it took for the ferry to exit the radar zone. The ferry spends

$$\boxed{\left(\frac{1+\sqrt{2}}{6} - \frac{19}{300}\right) \text{ hours inside the radar zone.}}$$

Problem 3.7: Nora spends part of her summer driving a *combine* during the wheat harvest. Assume she starts at the indicated position heading east at 10 ft/sec toward a circular wheat field of radius 200 ft. The combine cuts a swath 20 feet wide and begins when the corner of the machine labeled "a" is 60 feet north and 60 feet west of the western-most edge of the field.

Part (a): When does Nora's rig first start cutting the wheat?

Part (b): When does Nora's rig first start cutting a swath 20 feet wide?

Part (c): Find the total amount of time wheat is being cut during this pass across the field.

Part (d): Estimate the area of the swath cut during this pass across the field.

Part (a) Solution: For this problem, we can impose a coordinate system where the westernmost edge of the wheat field is the origin, the x-axis includes the diameter of the wheat field, and the y-axis is the line that is tangent to the wheat field at the point $(0, 0)$. Intervals on the axes are in terms of feet, where one unit is equal to one foot. We are given that the corner of the machine, "a", is located 60 feet north and 60 feet west of the origin of this system. Therefore, “a” is located at the point $(-60, 60)$. However, the combine is also moving east at a rate of 10 ft/sec, which means that the combine's x-coordinate is not stationary. Using the variable t to represent the time since the combine started moving, we can express the coordinates for “a” as $(-60 + 10t, 60)$.

To find when the rig will first start cutting the wheat, we need to find when “a” will first touch the wheat field. We can use the standard form of the equation of a circle to write the equation for the wheat field. We are given that the wheat field has a radius of 200 feet, which also means that the center of the field will be located at $(200, 0)$. Plugging in these values into the equation, it would look like:

$$(x - 200)^2 + y^2 = 40000$$

To find when “a” first intersects the field, we have to plug in the coordinates for “a” into the equation for the wheat field. The equation would then look like:

$$(-60 + 10t - 200)^2 + 60^2 = 40000$$

Solving for t:

$$100t^2 - 5200t + 31200 = 0$$

$$t = 26 + 2\sqrt{91},\ t = 26 - 2\sqrt{91}$$

We need to use the smaller value of t because we are solving for the first time Nora's combine will reach the wheat field. Therefore, the rig will first start cutting wheat $\boxed{\text{after } (26 - 2\sqrt{91}) \text{ seconds.}}$

Part (b) Solution: Nora's rig will first start cutting a swath 20 feet wide when the upper right hand corner of the rig in the diagram reaches the edge of the wheat field. Because the combine cuts a swath 20 feet wide, the rig itself is 20 feet wide. This means that the upper right hand corner of the rig will be 20 feet above “a”, and its coordinates will be at $(-60, 80)$. The rig is still moving at a rate of 10 feet/second, so the actual coordinates of this corner are $(-60 + 10t, 80)$. To find when the rig will first start cutting the 20-foot swath, we need to find when these coordinates will intersect the equation of the wheat field. We can solve for this by substituting these coordinates into the equation, similar to Part (a).

$$(-60 + 10t - 200)^2 + 80^2 = 40000$$

Solving for t:

$$100t^2 - 5200t + 34000 = 0$$

$$t = 26 + 4\sqrt{21},\ t = 26 - 4\sqrt{21}$$

Similar to Part (a), we need the value of t that is smaller because we are looking for the first time that the rig touches the edge of the wheat field. Therefore, the rig will start cutting a swath 20 feet wide $\boxed{\text{after } (26 - 4\sqrt{21}) \text{ seconds.}}$

Part (c) Solution: To find the total amount of time wheat is being cut for, we need to know the time at which “a” enters the wheat field and the time at which “a” exits the wheat field. In Part (a), we determined two solutions that defined the times at which “a” intersected the boundary of the wheat field. We found that “a” entered the field after $(26 - 2\sqrt{91})$ seconds, and “a” exited the field after $(26 + 2\sqrt{91})$ seconds. To find the time that wheat was being cut for, we have to subtract $(26 - 2\sqrt{91})$ from $(26 + 2\sqrt{91})$. Doing this, we can say the wheat was being cut for

$$\boxed{26 + 2\sqrt{91} - 26 + 2\sqrt{91} = 4\sqrt{91}\,\text{seconds.}}$$

Part (d) Solution: The area of the swath cut during this pass is between the area of a rectangle whose length is determined by the distance between the intersections of "a" and the boundary of the circle, and the rectangle whose length is determined by the distance between the intersections of the upper-right corner of the combine and the boundary of the circle.

To find the intersections of "a" and the boundary of the circle, we have to find the intersections of the line $y = 60$ and the boundary of the circle. This is because "a" consistently travels on this line throughout its path. To find the intersections, we can substitute 60 for y in the equation for the circle. The equation would then look like:

$$(x - 200)^2 + 60^2 = 40000$$

Solving for x:

$$x^2 - 400x + 3600 = 0$$
$$x = 200 + 20\sqrt{91},\ x = 200 - 20\sqrt{91}$$

Therefore, the coordinates of where "a" intersects the circle are $(200 + 20\sqrt{91}, 60)$ and $(200 - 20\sqrt{91}, 60)$. The distance between these two points is

$$200 + 20\sqrt{91} - 200 + 20\sqrt{91} = 40\sqrt{91}\,\text{feet.}$$

We are given that the width of the swath is 20 feet. We calculated the length of the rectangle to be $40\sqrt{91}$ feet. Therefore, the area of the rectangle whose length is the distance between the intersections of "a" and the boundary of the circle is

$$20 \cdot 40\sqrt{91} = 800\sqrt{91}\,\text{square feet.}$$

The second rectangle we have to consider is the rectangle whose length is the distance between the intersections of the upper right corner of the combine and the boundary of the circle. As stated in Part (b) of this solution, the upper right corner of the combine is moving along the line $y = 80$ because it is located 20 feet above "a". Using the same process as before, we can substitute 80 for y in the equation for the boundary of the circle:

$$(x - 200)^2 + 80^2 = 40000$$

Solving for x:

$$x^2 - 400x + 6400 = 0$$
$$x = 200 + 40\sqrt{21},\ x = 200 - 40\sqrt{21}$$

The coordinates of the intersections of the upper right corner and the boundary of the circle are $(200 + 40\sqrt{21}, 0)$ and $(200 + 40\sqrt{21}, 0)$. The distance between these two points is

$$200 + 40\sqrt{21} - 200 + 40\sqrt{21} = 80\sqrt{21}\,\text{feet.}$$

Because the width of the swath is 20 feet, we can say the area of this rectangle is

$$80\sqrt{21} \cdot 20 = 1600\sqrt{21}\,\text{square feet.}$$

We know that the area of the swath cut during this pass of the field must be between the areas of these two rectangles because the area of one of the rectangles would be an overestimate, while the area of the other rectangle would be an underestimate. Therefore, the area of the swath is $\boxed{\text{between } 1600\sqrt{21} \text{ and } 800\sqrt{91} \text{ feet.}}$