Project 1: On Peanuts, Circles, and Infinites

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1 Introduction

An interesting quality of the circle $x^2 + y^2 = 1$ is that the distance from all points on this curve to its center, $(0, 0)$, is 1. Drawing inspiration from this, we could consider the point $(0, 0)$ and take any other random point (a, b) in the coordinate system. A possible question would be: what is the set of points so that the products of the distances from each point to these two points is 1? But this question is too simple; we can complicate this even further. We do not know the nature of the graph of this situation, so it could be virtually any curve we can think of. What would be interesting to consider is possible values of (a, b) so that this curve is closed, or just one closed shape. And there we have our question: consider the origin and pick any other point (a, b). Consider the set of points such that the products of the distances from each point to the origin and (a, b) are 1. For what values of a and b will the resulting figure be one closed shape? This is what we will address in this paper.

2 Finding an Equation

Solving this problem will be much simpler if we are able to make a graph that accurately represents the situation. We know that the product of two distances will be equal to 1—meaning that the formula for this graph will consist of two "distance formulas" multiplied and set equal to one, as such:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \cdot \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} = 1$$

However, this formula has to be slightly altered to fit the situation. Instead of using x_1 and y_1, we can simply use the variables x and y where (x, y) is any point belonging the set of points such that the products of the distances from each point to $(0, 0)$ and (a, b) are 1. Our formula would then be:

$$\sqrt{(x - x_2)^2 + (y - y_2)^2} \cdot \sqrt{(x - x_3)^2 + (y - y_3)^2} = 1$$

In the question, the two points we are finding the distances to are $(0, 0)$ and (a, b). The point $(0, 0)$ can be substituted for (x_2, y_2) in the equation and (a, b) can be substituted for (x_3, y_3). Doing this would result in the equation being

$$\sqrt{x^2 + y^2} \cdot \sqrt{(x - a)^2 + (y - b)^2} = 1$$

3 Graphing

Ideally, we would want to be able to graph this equation that we found. However, because a and b are arbitrary, we need to plug in some value for (a, b) for the equation to result in a graph. To do this, we can start by finding the graph of this figure for different values of a when $b = 0$. We can graph this function when (a, b) is $(3, 0)$, $(2, 0)$, and $(1.9, 0)$.

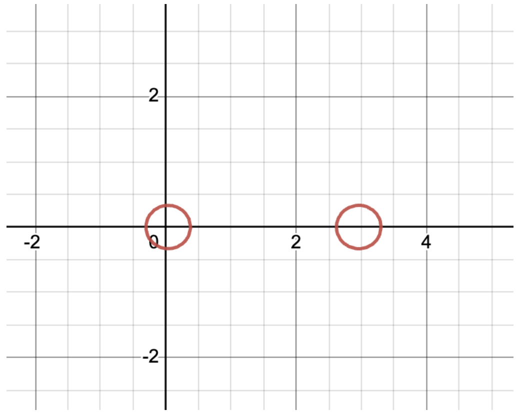

Figure 1: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-3)^2+y^2}=1$

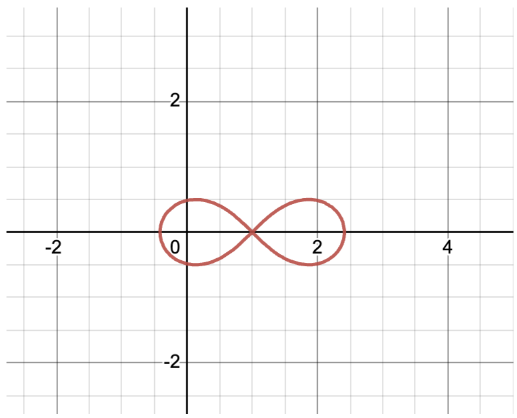

Figure 2: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-2)^2+y^2}=1$

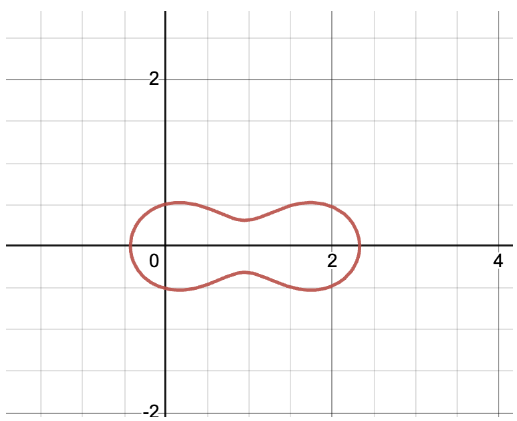

Figure 3: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-1.9)^2+y^2}=1$

In the graphs of these three equations, we can see how when $a=3$ and $b=0$, the resulting figure is of two separate enclosed shapes (which look like circles) instead of only one. This is because the domain of $\sqrt{x^2+y^2}\cdot\sqrt{(x-3)^2+y^2}=1$ is on one interval. In figure 2 (graph that looks like an infinity symbol) there appears to be no gaps in the graph's domain or range, meaning that the resulting figure is a closed shape. Figure 3 (which looks like a peanut), for the same reasons, is also a closed shape. Therefore, we have identified the defining factor for a closed shape in this problem: that the figure's domain and range is on only on one interval.

4 Using a Line

Now, we need to find a way to test whether this graph's domain and range are on only one interval. We can recall that we are looking for the set of points such that the products of their distances from the origin and (a, b) is 1. Therefore, we can try graphing the line that runs through $(0, 0)$ and (a, b) alongside the graph of the equation. The slope of this line would be $\frac{b}{a}$ and it has a y-intercept of 0. This means the equation in slope intercept form for the line that runs through these two points is $y = \frac{b}{a}x$. Note that this line is not applicable for when $a = 0$, as the line would be undefined. Instead, if (a, b) is on the same vertical line as the origin, we can use the line $x = 0$.

We can try graphing $y = \frac{b}{a}x$ in the same plane as $\sqrt{x^2 + y^2} \cdot \sqrt{(x-a)^2 + (y-b)^2} = 1$. We can look at 3 different graphs: one where (a, b) is $(1.5, 1.2)$, another where (a, b) is $(2, 4)$, and the last where (a, b) is $(2, 0)$:

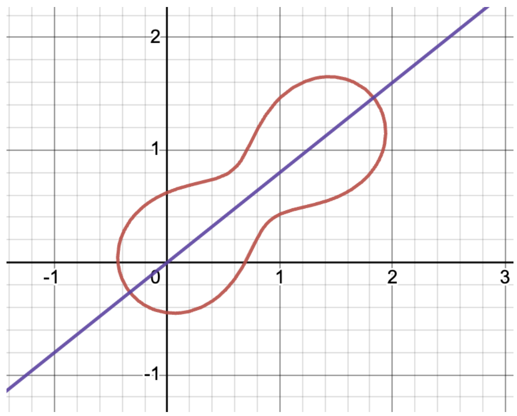

Figure 5: Graph of $\sqrt{x^2 + y^2} \cdot \sqrt{(x-1.5)^2 + (y-1.2)^2} = 1$ and $y = \frac{4}{5}x$

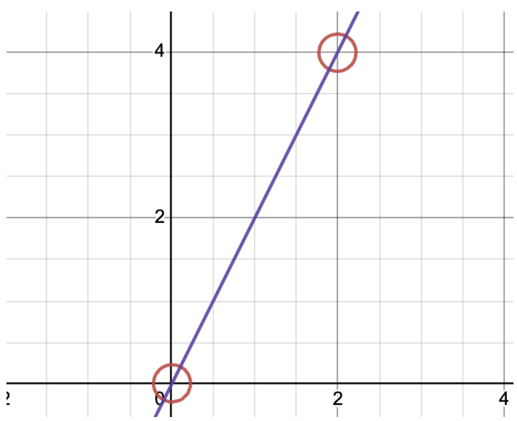

Figure 6: Graph of $\sqrt{x^2 + y^2} \cdot \sqrt{(x-2)^2 + (y-4)^2} = 1$ and $y = 2x$

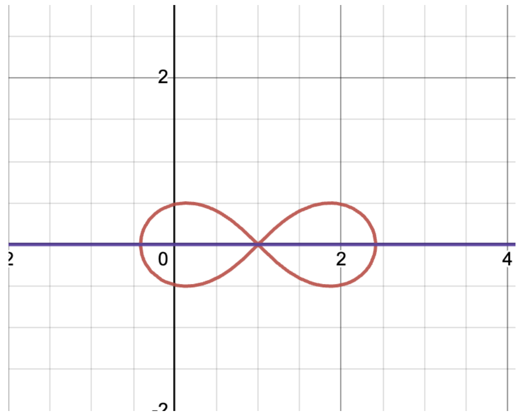

Figure 7: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-2)^2+y^2}=1$ and $y=0$

In these examples, we can see that Figure 5 is a closed shape, Figure 6 is not a closed shape, and Figure 7 is again a closed shape. Something interesting we can find just by looking at the graphs of these equations is the amount of intersections with the line $y=\frac{b}{a}x$ they have. In Figure 5, the graph of the equation appears to have two intersections with the line. In Figure 6, the graph of the equation appears to have four intersections with the line. This is true for all non-closed figures, as their domains must be on more than one interval. For there to be multiple intervals, the graph must have at least four intersections with the line, as any less would result in only one interval. Figure 7 shows the graph of the equation appearing to have 3 intersections with the line. This is interesting, as Figure 7 also appears to be a closed shape.

Now, we have another pressing question: can the graph of a closed shape have 3 intersections with the line $y=\frac{b}{a}x$?

5 Finding Intersections

It is possible that the graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-2)^2+y^2}=1$ in fact intersects $y=0$ at four points, but middle two of those points are so close together we cannot see the gap between them. It appears that the middle intersection occurs at $(1,0)$. To check whether this is a true intersection point, we can plug in $y=0$ into the equation $\sqrt{x^2+y^2}\cdot\sqrt{(x-2)^2+y^2}=1$. This is because the graph appears to have 3 intersections with the line $y=0$, meaning that if we plug this point in, it should result in 3 solutions for x.

$$\sqrt{x^2+0^2}\cdot\sqrt{(x-2)^2+0^2}=1$$

Simplifying this:

$$|x|\cdot|x-2|=1$$

$$x=1+\sqrt{2},\ x=1-\sqrt{2},\ x=1$$

We were able to get 3 different solutions for x, meaning that there are in fact 3 intersections between this equation and the line. In our solutions, we can see that one is $x=1$, meaning that middle intersection is in fact $(1,0)$.

We can analyze the graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-2)^2+y^2}=1$ a bit further. In Figure 7, we can see that a third intersection point occurs at $(1,0)$. But what if we changed a from 2 to 2.5? Or from 2 to 1.5? We can figure this out by graphing again:

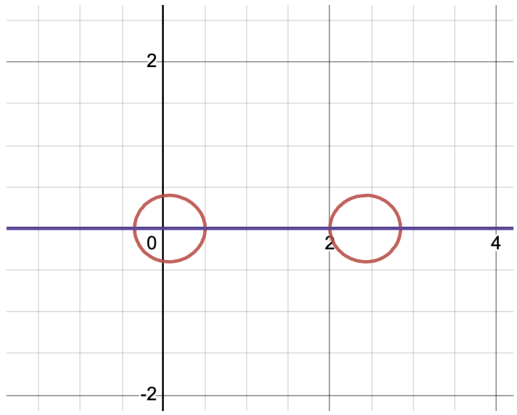

Figure 8: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-2.5)^2+y^2}=1$ and $y=0$

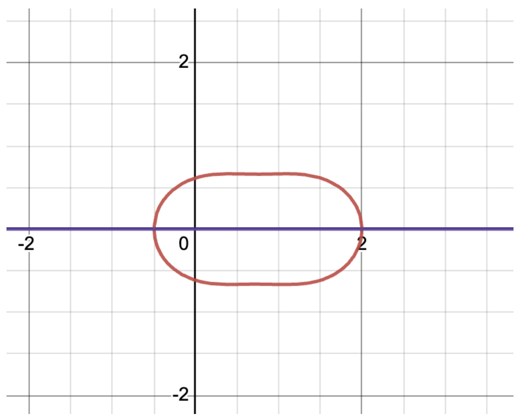

Figure 9: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x-1.5)^2+y^2}=1$ and $y=0$

We can see that when $a>2$ in this equation, the graph is no longer of one closed shape. And when $a<2$ (excluding negative values of a) in this equation, the graph is still of one closed shape, but with only 2 intersections. This means we can assume that 2 is the greatest positive value of a in the equation $\sqrt{x^2+y^2}\cdot\sqrt{(x-a)^2+y^2}=1$ that will result in the graph being a closed figure.

6 Constraints on a and b

If 2 is the greatest possible value of a in this situation, we have already found some constraint on a and b that will ensure the graph will be a closed shape. To find more specific values of a and b, we can take a closer look at the middle intersection between $\sqrt{x^2+y^2}\cdot\sqrt{(x-a)^2+y^2}=1$ and $y=\frac{b}{a}x$. We can recall that the middle intersection point of this graph and the line $y=\frac{b}{a}x$ is $(1,0)$ when $a=2$. This time, we can take $a=-2$ instead, then graph:

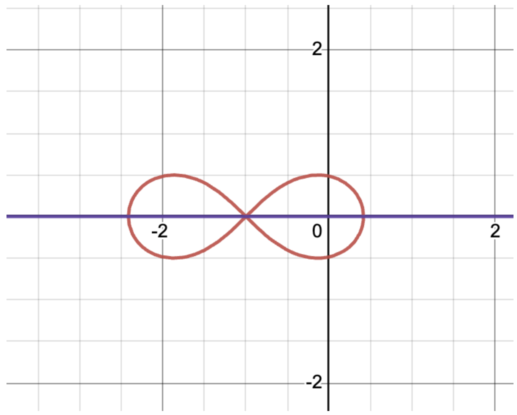

Figure 10: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{(x+2)^2+y^2}=1$ and $y=0$

It appears that the middle intersection point in this graph occurs at $(-1,0)$. To check this, we can repeat the same process as earlier by plugging in 0 for y in the equation:

$$\sqrt{x^2+0^2}\cdot\sqrt{(x+2)^2+0^2}=1$$

Simplifying:

$$|x|\cdot|x+2|=1$$
$$x=-1+\sqrt{2},\ x=-1-\sqrt{2},\ x=-1$$

Again, because one of the roots is at the point $x=-1$, $(-1,0)$ is in fact the middle intersection of this graph.

We can look at similar examples, except with slightly different equations. For example, we can take the graph of $\sqrt{x^2+y^2}\cdot\sqrt{x^2+(y-b)^2}=1$. We can start by substituting 2 for b and then graphing:

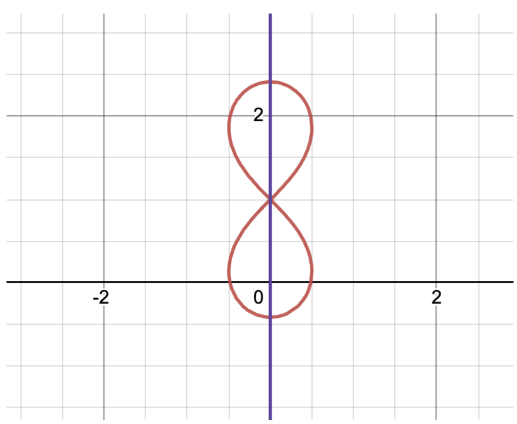

Figure 11: Graph of $\sqrt{x^2+y^2}\cdot\sqrt{x^2+(y-2)^2}=1$ and $x=0$

Again, in this equation, there appears to be 3 intersections along the line $x=0$, one of which is $(0,1)$. To confirm that this is true, we can substitute 0 for x into $\sqrt{x^2+y^2}\cdot\sqrt{x^2+(y-2)^2}=1$:

$$\sqrt{0^2+y^2}\cdot\sqrt{0^2+(y-2)^2}=1$$

Simplifying:

$$|y|\cdot|y-2|=1$$
$$y=1+\sqrt{2},\ y=1-\sqrt{2},\ y=1$$

We got 3 solutions, one of which is $y=1$, meaning that there are in fact 3 intersections and the middle one is $(0,1)$.

7 An Equation for the Intersections

By now, we can start to notice a pattern in the middle intersections of these graphs. Let's take the example of the first equation, $\sqrt{x^2+y^2}\cdot\sqrt{(x-2)^2+y^2}=1$. In this scenario, $a=2$ and the middle intersection occurs at $x=1$. Now, we can take the equation $\sqrt{x^2+y^2}\cdot\sqrt{(x+2)^2+y^2}=1$. Here, $a=-2$ and the middle intersection occurs at $x=-1$. In the equation $\sqrt{x^2+y^2}\cdot\sqrt{x^2+(y-2)^2}=1$, $a=0$ and the middle intersection occurs when $x=0$. The common theme among these three examples is that the x-coordinate of the middle intersection is always half of a.

Now, we need to use the information we have to find concrete constraints for a and b for the resulting figure to be a closed shape. We need to find an equation that represents the intersections of $\sqrt{x^2+y^2}\cdot\sqrt{(x-a)^2+(y-b)^2}=1$ with $y=\frac{b}{a}x$. To do this, we can substitute $\frac{b}{a}x$ for y into the equation:

$$\sqrt{x^2+\left(\frac{b}{a}x\right)^2}\cdot\sqrt{(x-a)^2+(\frac{b}{a}x-b)^2}-1=0$$

All the "middle intersections" we found are the intersections in the middle of the two outer ones with $y=\frac{b}{a}x$ when the value of a in the equation is such that there are exactly 3 total intersections. We can use the information we found earlier—that the x-coordinate of the middle intersection is half of the value of a. This means we can express this x-coordinate as $\frac{a}{2}$. The equation

$$\sqrt{x^2+\left(\frac{b}{a}x\right)^2}\cdot\sqrt{(x-a)^2+(\frac{b}{a}x-b)^2}-1=0$$

represents all of the intersections with the line $y=\frac{b}{a}x$, and $\frac{a}{2}$ is one of the possible intersections with this line. The roots of our new equation must be all of the intersections with the line, as all of its factors would each represent a root (the whole equation is all of these factors multiplied together).

Given this, dividing the new equation by possible factors will allow us to find an equation in terms of a and b that represents the boundary of the region that (a,b) can reside within. We already know that one of the roots of this equation is $\frac{a}{2}$, which means that one of the factors is $x-\frac{a}{2}$. If $x-\frac{a}{2}$ is one of the factors of this equation, then dividing this equation by it should result in a remainder of 0.

8 Dividing

We can use polynomial long division to divide

$$\sqrt{x^2+\left(\frac{b}{a}x\right)^2}\cdot\sqrt{(x-a)^2+(\frac{b}{a}x-b)^2}-1=0$$

by $x-\frac{a}{2}$.

First, we can simplify our dividend:

$$\left(x^2+\left(\frac{b}{a}x\right)^2\right)\cdot\left((x-a)^2+(\frac{b}{a}x-b)^2\right)-1=0$$

Simplifying even further:

$$0=x^4\left(\frac{a^4+2a^2b^2+b^4}{a^4}\right)-x^3\left(\frac{2a^4+4b^2a^2+2b^4}{a^3}\right)+x^2\left(\frac{a^4+2b^2a^2+b^4}{a^2}\right)-1$$

Now, we can divide the polynomial

$$x^4\left(\frac{a^4+2a^2b^2+b^4}{a^4}\right)-x^3\left(\frac{2a^4+4b^2a^2+2b^4}{a^3}\right)+x^2\left(\frac{a^4+2b^2a^2+b^4}{a^2}\right)-1$$

by $x - \frac{a}{2}$.

In our first step of division, we can see that the amount of times $x - \frac{a}{2}$ fits into $x^4\left(\frac{a^4+2a^2b^2+b^4}{a^4}\right) - x^3\left(\frac{2a^4+4b^2a^2+2b^4}{a^3}\right)$ is $x^4\left(\frac{a^4+2a^2b^2+b^4}{a^4}\right) - x^3\left(\frac{a^4+2b^2a^2+b^4}{2a^3}\right)$. Subtracting the second value from the first, we will get a remainder of $-x^3\left(\frac{3a^4+6a^2b^2+3b^4}{2a^3}\right)$. Next, we can see the amount of times that $x - \frac{a}{2}$ fits into $-x^3\left(\frac{3a^4+6a^2b^2+3b^4}{2a^3}\right) + x^2\left(\frac{a^4+2b^2a^2+b^4}{a^2}\right)$ is $-x^3\left(\frac{3a^4+6a^2b^2+3b^4}{2a^3}\right) + x^2\left(\frac{3a^4+6b^2a^2+3b^4}{4a^2}\right)$. Subtracting again, our remainder is now $x^2\left(\frac{a^4+2a^2b^2+b^4}{4a^2}\right)$. The amount of times that $x - \frac{a}{2}$ fits into $x^2\left(\frac{a^4+2a^2b^2+b^4}{4a^2}\right) - 1$ is $x^2\left(\frac{a^4+2a^2b^2+b^4}{4a^2}\right) - x\left(\frac{a^4+2a^2b^2+b^4}{8a}\right)$. Subtracting again, our remainder is $x\left(\frac{a^4+2a^2b^2+b^4}{8a}\right) - 1$. The amount of times $x - \frac{a}{2}$ fits into this value is $x\left(\frac{a^4+2a^2b^2+b^4}{8a}\right) - \frac{a^4+2a^2b^2+b^4}{16}$. Subtracting for the last time, our final remainder is $\frac{a^4+2a^2b^2+b^4}{16} - 1$.

We divided this polynomial by $x - \frac{a}{2}$, which we know is a factor of it, as $\frac{a}{2}$ was a root of the equation. Because this is a factor, we should end up with a remainder of 0. This means that the remainder $\frac{a^4+2a^2b^2+b^4}{16} - 1 = 0$. We can now simplify this equation:

$$\frac{a^4 + 2a^2b^2 + b^4}{16} - 1 = 0$$
$$a^4 + 2a^2b^2 + b^4 - 16 = 0$$
$$\frac{a^2}{4} + \frac{b^2}{4} + \frac{a^2b^2}{2} = 4$$
$$\frac{a^4 + a^2b^2}{4} + \frac{b^4 + a^2b^2}{4} = 4$$
$$\frac{a^2}{4}\left(a^2 + b^2\right) + \frac{b^2}{4}\left(a^2 + b^2\right) = 4$$
$$\left(a^2 + b^2\right)\left(a^2 + b^2\right) = 16$$
$$\left(a^2 + b^2\right)^2 = 16$$
$$a^2 + b^2 = \pm 4$$

We can see that there are two versions of the simplified equation. If 4 is positive, then the resulting figure is a circle. If 4 is negative, there will be no real solutions for a or b, as the sum of two squares is always positive. This means the solution we are looking for is $a^2 + b^2 = 4$.

9 Analysis

We can recall that we were searching for values of a and b that would result in our figure being one closed shape. As we stated earlier, the equation that we got $\left(a^2 + b^2 = 4\right)$ must represent the boundary of the region that (a, b) can reside within. Therefore, (a, b) must be a point inside the region bounded by the curve $a^2 + b^2 = 4$. Or, (a, b) must satisfy the inequality $a^2 + b^2 \leq 4$ so that the resulting figure of the equation $\sqrt{x^2 + y^2} \cdot \sqrt{(x-a)^2 + (y-b)^2} = 1$ is a closed shape. If we were to graph this in the ab coordinate plane, (a, b) must reside within this region:

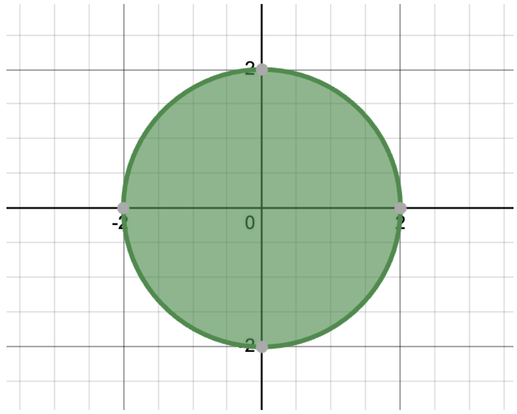

Figure 12: Graph of $a^2 + b^2 \leq 4$

10 Further Inquiry

In this paper, the two points that we have considered are the origin and a random (a, b). Based on our solution and analysis, we know that to meet the requirements of our question, (a, b) resides within a certain region (in this case, a circle). However, it is natural to ask: would our conclusion would change if we were to change the fact that one of the points we considered was the origin? What if, instead of using the origin and (a, b), we used any random point along the x-axis and (a, b)? How would our conclusion change if it were to be any random point along the y-axis?

Perhaps the most general question we could consider is what if the point was not along any axis, but any random point just like (a, b)? All three of these questions would generalize the question even further, as we would definitely have to consider a wider range of situations, and maybe even the possiblity that we cannot find a solution at all (as we could run into too many arbitrary values). All three of these directions for further inquiry would explore the nature/qualities of closed and not-closed figures in even more depth.