K-SVD: An Algorithm for Designing Overcomplete Dictionaries for Sparse Representation

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Summary: The paper deals with the exploration of previous sparse representation literature and the development of a novel algorithm - "K-SVD" which is utilized to obtain sparse representation from an over complete dictionary through an iterative method of learning and updating dictionary atoms. Their algorithm is said to have accelerated convergence and is backed with experimental results.

Related work: The author focuses on the prior work in two categories. (i) Sparse Coding - The paper explore multiple pursuit algorithms such as matching pursuit [7], orthogonal matching pursuit [1][12] which are used to pick dictionary atoms. Different approaches such a Basis Pursuit [2] and FOCUSS [6] [11] for convexification provided motivation as well. (ii) Design of dictionaries: Some methods previously looked into were the Maximum likelihood [10] [3] [4], MOD [5], Maximum Aposteriori Probability [8] and Unions of Orthonormal Bases [9]

Approach:

A. K-Means for Vector Quantization - A codebook is constructed though the nearest neighbour algorithm, giving rise to K codewords for compressed representations of signals in space. The objective to to find a codebok **C** that minimizes the MSE **E**, such that - $min_{c,x}||Y - CX||_F^2$ $\forall i$,x$_i$ $= e_k$, and it is done through iterations of updating the codebook and sparse coding to evaluate **X** until convergence.

B. Generalizing K-Means: In this method we have a linear combination of codewords - dictionary elements. The minimization objective of this involves the best sparse representation of the example set **Y** such that $min_{DX}||Y - DX||_F^2$ $||x_i||_0 \leq T_0$. The main difference in the algorithm is how the dictionary is learnt, as opposed to finding a good representation of **X**, **D** is efficiently constructed for d_k by reduces the MSE across all k. Each column is updated one by one - a method based on SVD and the implementation of updating the coefficients in conjunction with the columns leads to faster convergence.

C. K-SVD: Based on the assumption that **D** and **X** except for the d_k^{th} column and x_T^k row, the objective function can be decomposed into a sum of K rank 1 matrices - $||E_k - d_k x_T^k||_F^2$ where E_k is the error matrix. But a problem is that SVD cannot be applied because sparsity was not a constraint on coefficients. To tackle this we define a matrix Ω_k containing 1's with respect to the signal's corresponding atom d_k. The zero entries are discarded. Minimization is achieved through SVD by decomposing the error matrix E_k^R = U $\triangle$ V_T. The first column of the U matrix is the updated dictionary column and the 1st column of V gives the updated coefficient value. Convergence of local minimum is guaranteed.

Datasets, Experiments and Results:

(i) Synthetic: K-SVD performs better than MOD and MAP algorithm for three synthetically generated dictionary D_{20x50} form iid uniform distribution of values for signals $[y_i]_{i=1}^{1500}$ of dim = 20 and in addition of Gaussian noise = 10, 20, 40 dB. The comparison was done by finding the distance generated dictionary and computed dictionary while sweeping across the columns.

(ii) Natural Images -: 11000 natural images of 8x8 pixel's of 500 blocks sorted according to the variances are taken. Except the first column DC, the remaining are set to 0 mean. Next, K-SVD of training dictionary of size 64x441 is applied to the dataset for 2 tasks - Filling in missing pixels and compression. In both the scenarios K-SVD faired better in comparison to overcomplete Haar and DCT.

Strengths: The paper provides a novel algorithm in the sense of sparsity of signal processing. The convergence rate of this algorithm is high. The paper presents the usefulness of having a sparse representation of an overcomplete dictionary, this allows for many applications such as compression, regularization, feature extraction, texture separation etc.,. The K-SVD algorithm is also generalized to work with any pursuit algorithm thus allowing us to use it for generalized applications instead of specific ones. This method would ideally outperform predefined dictionaries as it involves **learning** a good representation for the specific signal.

Weaknesses: K-SVD is computationally expensive, this becomes most prominent when the size of the dictionaries or training samples are exceedingly large. Another problem associated with this approach is finding a good representation of the signal through the dictionary which is essentially a non-convex problem and solving non-convex problems is more difficult. Since the K-SVD involves repeated updation of the dictionary and learning coefficients, we are not ensured a globally optimal solution for the entire dataset.

Reflections: The author provides a holistic approach to the algorithm by first providing the motivation behind sparse representation and dictionary learning for signals. The paper seemingly presents an intuitive flow from the approximation of K-means clustering to application for which the K-SVD algorithm will fair best.

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