Q6) (Bonus Paper)

Summary: "Basic Array Theory" by W.H.Kummer & L.Fellow, IEEE

Introduction

The paper is mainly about **Patten Analysis and Synthesis for periodic linear, planar, phased and conformal arrays**. It also includes computer-based techniques for pattern synthesis such as: arbitrary side lobe control, beam shaping, phase-only null steering. Random and Quantization Errors are also discussed.

- ### Classification:

- By Topology:

Linear Arrays: equally spaced radiative elements laid out in a straight line.

2-D Planar Arrays: radiative elements oriented on a geometric grid in a plane.

- **Rectangular Arrays:** can also be thought as a set of linear arrays placed next to each other, equally spaced, forming a 2-D array
- **Circular/ Cylindrical Arrays:** can be thought as linear arrays wrapped around a curved surface

- By Beam Design:

Fixed Beam: pattern remains fixed as the parameters that affect it are fixed as well.

Scanning Beam: beam is rapidly positioned in space by means of electromechanical or electronical actuated connected in the feed lines behind the array

Analyzed parameters in linear arrays consist of: Main beam, Side lobes, Grating lobes

Basis of Array Theory relies on Electromagnetic equations in source and measurement regions

Array pattern formulation

General Formulation (Element factor/Array Factor):

Electric field of a single element:

$$\overline{E}_n(x,y,z) = (e^{-jkr}/r)\overline{A}_{e_n}(x',y',z')I_n\exp(jkr'_n\cos\xi_n).$$

The term $\bar{A}_{e_n}$, shows the behavior (pattern) of an individual element in terms of polarization and orientation of the electric field. The rest of the equation shows the effect of amplitude and phase in far field region. $\bar{A}_{e_n}$ is often called the **element factor** and shown with $\overline{EF}_n$ as well.

Due to the super position law the total electric field caused by this arbitrary array is a sum over all elemental fields:

$$\overline{E}(x,y,z) = (e^{-jkr}/r)\sum \overline{A}_{e_n}I_n\exp(jkr'_n\cos\xi_n). \quad (2b)$$

For linear arrays as the elements are identical, $\bar{A}_{e_n}$ can come out of the summation.

$$\overline{E}(x,y,z) = (e^{-jkr}/r)\overline{A}_e\sum I_n\exp(jkr'_n\cos\xi_n)$$

The first term of the equation describes measurement point position effect, the second one describes the element effect as we said, therefore the last one describes the geometrical affect, which is also known as the array factor and is shown by A_a:

$$A_a = \sum I_n\exp(jkr'_n\cos\xi_n).$$

- Side note: Patterns are <u>normalized plots</u> of amplitude of electric field or antenna power.
- From now on, the focus is on array factors, as the pattern is meant to be obtained by designing a proper topology of certain elements.

Linear Arrays:

By forming a set of identical elements along a certain axis a linear array is constructed. We usually choose z-axis for pattern analysis because it results in no φ variation of the measured fields, and therefore the number of coordinate variables are reduced.

For a linear array along z-axis including N elements, with an element separation of d, the array factor general equation is simplified to the following:

$$A_a = \sum_{n=1}^{N} I_n \exp(j(n-1)kd\cos\theta).$$

By setting element currents to a uniform normalized value of 1 we obtain the array factor of a <u>uniform linear array:</u>

$$A_a = \sum_{n=1}^{N} \exp(j(n-1)kd\cos\theta).$$

The expression above**, is maximized** when the exponent becomes zero (as a result all the elements uniformly add up together) which happens in one of the following conditions:

1. $\theta = 90°, 270°$
   The pattern is maximized in these directions, they are called the **main lobes**

2. $kdcos(\theta) = 2\pi$, that results in:

$$\cos(\theta) = \frac{\lambda}{d}$$

That implies separations larger than the wavelength.

- Gratings are usually an undesired phenomenon, thus a limit on the separation can prevent them.

$$Grating\ prevention\ condition: \quad \frac{\lambda}{d} < 1$$

To simplify pattern analysis:

$$\psi = \ kdcos(\theta)$$

Which result in phasor description of the array factor in the complex plane, assuming $e^{j\psi}$ as a phasor for each element. Now the array factor equation shows the vector sum of multiple phasors in a complex plane, each consisting a phase and an amplitude (which was 1 for a uniform linear array). As the angle ψ is constant for each of the elements across the array the vector summations form a regular polygon:

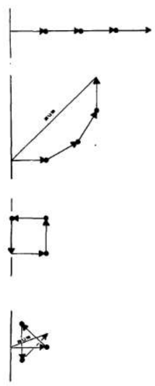


Fig. 3. Phasor presentation of pattern summation. Each phasor is equal to one unit. (a) $\psi = 0°$, sum = 4. (b) $\psi = 30°$, sum = 3.3. (c) $\psi = 90°$, sum = 0. (d) $\psi = 135°$, sum = 1.1.

Conclusions from the phasor summation expression of an AF:

- The number of sides of this polygon is equal to the number of elements forming the array.

- Each external angle shows the phasor angle relative the previous phasor's direction.

- The shape simplifies to a line if $\psi = 0$

- AF will cancel out to zero creatin nullities in the pattern if ψ satisfies this formula:

$$\psi = \frac{2\pi k}{N}; k = 1, 2, \dots, N - 1$$

- Note that, $k = 0$ $(\psi = 0)$, indicates the main radiation direction but nullities occur on the other natural values up to $N - 1$.

  Which are the nullities of the array factor. There's no radiation in these directions
- For example, the figure above shows a 4-element array which has its nullities in $(\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4})$ directions.
- All the equations described up to here, didn't consider initial phases for the currents which stimulate radiation (broadside) . By considering this fact we should actually add an initial phase and redefine ψ, as the following:

$$\psi = kdcos(\theta) + \delta$$

Schelkunoff Pattern Analysis:

In this method of analysis phasors are treated as an angular variable ω $(or\ z)$, in order to pinpoint nullities on a unit circle in the complex plane.

$$\omega = e^{j\psi}$$

By rewriting the array factor expression with this substitution, we obtain:

$$A_a(w) = \sum_{n=1}^{N} \frac{I_n}{I_1} w^{n-1}. \qquad A_a(w) = \left|\frac{I_N}{I_1}\right| \cdot \left|w^{N-1} + \frac{I_{N-1}}{I_N} w^{N-2} + \cdots + \frac{I_1}{I_N}\right|.$$

By normalizing the array factor, we can write it in a polynomial form and obtain its roots:

$$f(w) = |w - w_1| \cdot |w - w_2| \cdots |w - w_{N-1}| = \prod_{n=1}^{N-1} |w - w_n|.$$

For a uniformly illuminated array:

$$f(w) = \sum_{n=1}^{N} w^{n-1}$$

By writing the geometric progression summation formula we obtain the roots:

$$f(w) = \frac{1 - w^N}{1 - w} \qquad w_p = e^{j2\pi p/N}, \quad p = 1, \cdots, N-1.$$

Also, the magnitude of the array factor can be obtained as following:

$$u = \frac{d}{\lambda}$$

$$\left|\frac{\sin N\pi u}{\sin \pi u}\right|$$

For example; the pattern and root plot of a 4-element array is shown in the figures below:

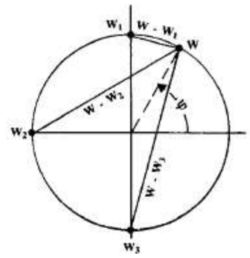
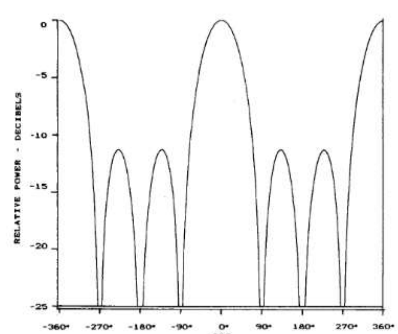


Fig. 5. Schelkunoff unit circle-four element array.



Extension to scanning arrays:

For formulating scanning arrays, a variable phase should be considered that's shown as α:

By rewriting the array factor expression:

$$A_a = \sum_{n=1}^{N} I_{0n} \exp(j(n-1)[kd\cos\theta - \alpha])$$

The main beam is now oriented in a specific direction θ_0, which varies over time. It can be obtained by applying $\psi = kdcos(\theta) - \alpha = 0$:

$$A_a = \sum_{n=1}^{N} I_{0n} \exp(j(n-1)[kd(\cos\theta - \cos\theta_0)])$$

- Note that all formulations are done for the broadside ($\delta = 0$) radiation in this paper and generally ψ obeys the following expression when initial phases are non-zero:

$$\psi = kdcos(\theta) + \delta - \alpha = 0$$

- For a scanning array the conditions to avoid grating lobes are also altered a little bit:

$$d/\lambda \leq \frac{1}{1 + |\cos\theta_0|}.$$

- High peaks on grating lobes can be partially suppressed by using such an element that has a low Element Factor (EF) magnitude for the corresponding angles, to

- Typical spacing for scanning arrays is about: $\frac{d}{\lambda} = 0.5$

SLL optimization and Dolph-Chebyshev

SLL is the difference between main lobe peak and the nearest peak to it.

For a uniformly linear array a SLL of 13dB is shown in the figure below. This is not enough for many applications. Thus, the amplitudes shouldn't be uniformly distributed. Dolph addressed this problem by using Chebyshev polynomials that have an oscillatory range between -1 and +1:

Chebyshev polynomials for an odd number of elements(2N+1) are as follows (Order: 2N+2):

$$
\begin{aligned}
T_{2N}(u) &= \cos(2N \arccos u), \qquad -1 \le u \le 1 \\
&= \cosh(2N \operatorname{arccosh} u), \qquad u \ge 1. \qquad (14)
\end{aligned}
$$

Where u is defined as follows:

$$
\begin{aligned}
\psi &= kd(\cos\theta - \cos\theta_0) \\
u &= \cos(\psi/2) \\
|\text{sidelobe level}|\text{in dB.} &= 20\log\eta, \qquad T_{2N}(u_0) = \eta.
\end{aligned}
$$

As mentioned previously we can also write it in a polynomial form and the roots are obtained as follows:

$$
T_{2N}(u) = cf(w) = \prod_{p=1}^{N}(w - w_p) \qquad \cos(\psi_p/2) = (1/u_0)(\cos(2p-1)\pi/4N), \quad p = 1, \cdots, 2N.
$$

Assuming a symmetrical distribution the pattern reforms in to the following expression:

$$
f(w) = w^N 4^N \prod_{p=1}^{N} \sin\left(\frac{\psi - \psi_p}{2}\right)\sin\left(\frac{\psi + \psi_p}{2}\right)
$$

This is the Chebyshev pattern for 2N+1 elements. Synthesizing such amplitude values is nearly impractical due to the high ration between center magnitude and others.

Taylor design is based on combining uniform and Chebyshev design methods:

$$
f(w) = \exp(jN\psi/2)\left(\frac{\sin((2N+1)/2)\psi}{\sin(\psi/2)}\right)
\left(\frac{\prod_{n=1}^{\bar{n}-1}\sin((\psi - \psi'_n)/2)\sin((\psi + \psi'_n)/2)}{\prod_{n=1}^{\bar{n}-1}\sin((\psi - \psi'_{0n})/2)\sin((\psi + \psi'_{0n})/2)}\right) \quad (16)
$$

Array Directivity:

The general formula for directivity is reduced to the following expression for linear half-wavelength spaced isotropic arrays:

$$D = \left(\sum I_n\right)^2 \Big/ \left(\sum I_n^2\right)$$

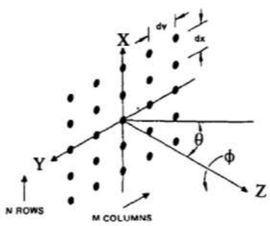
2-D arrays:

Two dimensional arrays can be thought of as set of linear arrays placed along each other with uniform spacing. As a result, the general pattern for a broadside 2-D array is obtained through the following expression:

$$A_a(\theta, \phi) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} I_{mn} \exp\,(jk \cdot \sin\theta(md_x \cos\phi + nd_y \sin\phi)).$$

d_x and d_y, respectively show the interelement spacings along the x and y axes.

Centering a 2-D array on the origin, the array factor for a 2N+1, 2M+1 array is obtained:



$$A_a(\theta, \phi) = \left(\sum_{-M}^{M} I_m \exp(jmkd_x \sin\theta \cos\phi)\right) \cdot \left(\sum_{-N}^{N} I_n \exp(jnkd_y \sin\theta \sin\phi)\right).$$

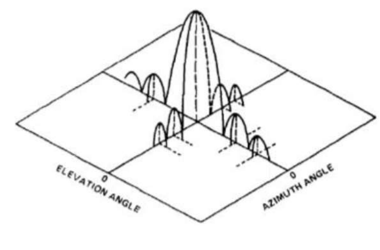


Fig. 10. Planar array pattern-separable distribution.

- One of the main challenges in designing 2-D arrays is a high side lobe level on the principal planes
- Tseng-Cheng distributions are suggested as an analogous to Dolph-Chebyshev distributions
- Array factor for scanned arrays is obtained similar to the 1-D case:

$$A_a = \left(\sum_{-M}^{M} I_m \exp(jm(kd_x \sin\theta \cos\phi - \alpha_x))\right) \cdot \left(\sum_{-N}^{N} I_n \exp(jn(kd_y \sin\theta \sin\phi - \alpha_y))\right)$$

α_x, α_y, are beam angles of the scanning array

- Grating prevention conditions are also obtained similarly:

$$(d_{x,y}/\lambda) \leq 1/\left(1 + \left|\sin\theta_{0_{x,y}}\right|\right)$$

Array Errors

Array errors are divided into 2 major sets:

1. Fabrication errors (due to symmetric bending, or random errors)
2. Quantization errors

- For random errors the formulation of array factor is revised. Errors have a Gaussian distribution with a zero mean:

$$A_a = \sum I_n(1+\Delta_n)\exp(j(n\psi+\delta_n)) \qquad A_a^2 = A_0^2 + \frac{\sigma^2}{D}$$

Δ_n, δ_n, show amplitude and phase errors respectively. A_0^2 shows the previous formulation of error-less array factors. D is also the directivity

- Phase Quantization errors are due to digital phase shifters that use a limited number of bits. For a P-bit phase shifter the quantization error can be derived according to the least phase incrementation possible:

$$Minimum\ phase\ incerement = \frac{360}{2^P}$$

$$Peak\ phase\ error = \frac{Minimum\ phase\ incerement}{2} = \frac{\pm 180}{2^P}$$

This also leads to a relative gain error:

$$(\Delta G/G_0) = \pi^2/(3 \times 2^{2P}).$$

Conformal Arrays:

- **Motivation**: linear and planar arrays are limited to scan angles up to 70° due to gain reduction and aperture mismatch loss.

- Applications such as missiles and high-speed aircraft need 360-degree radar scanning arrays that suggest an alternative of placing elements around circular and cylindrical surfaces, spheres and cones.

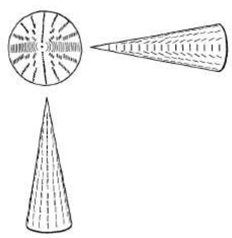


Fig. 18. Conical array, slots oriented for linear polarization in axial direction.

- Practical problems:

1. Element Spacing
2. Polarization Matching
3. Feeding systems
4. Beam Steering

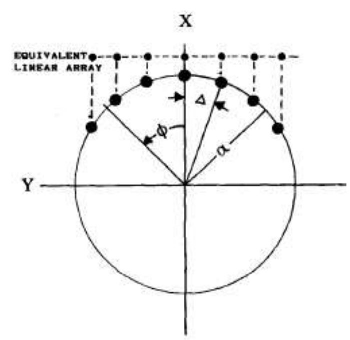


Fig. 19. Geometry for arc array.

- EF and AF terms can't be separated in conformal arrays. Because each element's beam is pointed out in a different direction. Therefore, we are unable to generalize methods discussed in linear and planar arrays to conformal arrays and new methods should be developed.

- Analysis of conformal arrays can be divided in to 3 sub categories:
    - Analysis of active element patterns
    - Pattern Analysis and Synthesis based on surface excitations
    - Design of radiators and feed networks