Summary: "Matrix Method for Field Problems" by R. F. Harrington

Abstract

The main goal of this article is to use method of moments in order to find the current distribution at any given point of a certain object(wire), using matrix equations, where the wire object is represented by an admittance matrix, the excitation by a voltage matrix and the current of the wire is given by the product of the admittance matrix with the voltage matrix. Actually, any method where the functional operator can be reduced to a matrix based one, can be interpreted as a method of moments.

Introduction

The method of moments which is the underlying mathematical technique of the matrix method, is often considered an approximate solution, which is a misnomer as it can converge to the exact answer is some cases, Matrix method reduces the original functional equation to a matrix equation. What differs among approximate solutions is the computational time for obtaining the solution with a given accuracy.

The paper focuses on only the equations from the homogenous type:

$$L(f) = g$$

L is a linear operator; g is the excitation or source and f is the field or response. 2. G is known and there are two types of problems according to whether f or L wants to be determined:

1. Problem of **Analysis**: L is known and f (response) is unknown.
2. Problem of **Synthesis**: f is known (desired) and we should design L (object topology) to satisfy the equation.

Problem Formulation

Inner Products, Norm and Distance

- By definition any given operator satisfying the following equations can be considered as an inner product:

$$\langle f, g \rangle = \langle g, f \rangle \tag{2}$$

$$\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle \tag{3}$$

$$\langle f^*, f \rangle > 0, \quad \text{if } f \neq 0 \\ = 0, \quad \text{if } f = 0 \tag{4}$$

- And the norm of a function can therefore be described in terms of this inner product:

$$\| f \| = \sqrt{\langle f, f^* \rangle}. \tag{5}$$

- The Euclidean distance between 2 vectors:

$$d(f, g) = \| f - g \| \tag{6}$$

Operator properties:

- The adjoint operator associated to an operator follows this definition:

$$\langle Lf, g\rangle = \langle f, L^a g\rangle$$

- An operator is called self-adjoint if the operator and adjoint operator are equal and have the same domain

- Positive definition, Semi-positive definition and Negative definition of an operator can be expressed by the sign of the following expression:

$$\langle f^*, Lf\rangle >$$

- If L is a one-to-one operator, the inverse of it can be defined as follows:

$$f = L^{-1}(g).$$

Method of moments:

Before discussing the method of moments, the fact that arbitrary weight function can enter the definition of inner product may also be considered. The role of this weight function is to simplify the equations if it's chosen properly.

The core idea behind the method of moments is based on the expansion of the response function in to multiple finite or infinite basis(expansion) functions:

$$f = \sum_n \alpha_n f_n$$

By substituting this expansion in the main equation, we obtain:

$$\sum_n \alpha_n L(f_n) = g.$$

By defining a set of weight functions and taking m copies of inner weighted products from the equation above, we obtain m equations each described as:

$$\sum_n \alpha_n \langle w_m, Lf_n\rangle = \langle w_m, g\rangle$$

Here's where the **matrix form of equations** is obtained by placing all m equations beside one another row-wise:

$$[l_{mn}][\alpha_n] = [g_m] \tag{26}$$

where

$$[l_{mn}] = \begin{bmatrix} \langle w_1, Lf_1\rangle & \langle w_1, Lf_2\rangle & \cdots \\ \langle w_2, Lf_1\rangle & \langle w_2, Lf_2\rangle & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{27}$$

$$[\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} \qquad [g_m] = \begin{bmatrix} \langle w_1, g\rangle \\ \langle w_2, g\rangle \\ \vdots \end{bmatrix}. \tag{28}$$

- We should note that the matrix I_{mn}, itself contains integrals but the integrals are easier to evaluate compared to the main integral equation.

- If the matrix I_{mn} is invertible (has a non-zero determinant), the constants associated to the expansion functions will be obtained by applying the inverse matrix ($I_{mn}{}^{-1}$) to the equation from left:

$$[\alpha_n] = [l_{nm}^{-1}][g_m] \qquad (29)$$

- It should be noted that if the matrix I_{mn} is of infinite order ($n = \infty$)(modeling zero error approximation of the field function), it only is invertible in special cases, thus **the method of moments can converge to the accurate solution but only under certain circumstances:**

  - One special case is to **consider expansion functions as eigenvectors** of the main equation that leads to a **diagonal** (and thus invertible) matrix.

  - Choosing f_n and w_n properly are crucial to deal with a sufficiently low level of computation complexity.

    - f_n, w_n should be linearly independent as an example.
    - Setting $f_n = w_n$, is a choice that's known as **"Galerkin's Method"**

- Eventually the response function can be obtained by multiplying the transpose of the constant vector in the expansion functions vector:

$$[\tilde{f}] = [f_1 \quad f_2 \quad f_3 \quad \cdots] \qquad (30)$$

and write

$$f = [\tilde{f}_n][\alpha_n] = [\tilde{f}_n][l_{nm}^{-1}][g_m]. \qquad (31)$$

- A common choice for f_n is $f_n = x^n - x$, we should ensure that the boundary conditions are also satisfied, considering our choice.

Special Techniques:

- As long as the operator equation is simple, application of method of moments and choosing proper expansion and weighting functions leads to a straight-forward solution, but usually this is not the case so some other special methods are introduced to simplify the process of method of moments:

1. Point Matching:

This method is based on an approximation of the expanded equation at certain discrete points of the region of interest. This is equivalent to using the Delta Dirac functions as testing (weight) functions.

2. *Sub-sectional Basis:*

   We define each of the basis(expansion) functions only over a corresponding subsection of the region of interest so that the weighted sum of them reduces to a simple multiplication in that specific region.

3. *Extended Operators:*

   The domain of the operator can be extended in order to apply to new functions that are easier to evaluate, as long as the operation itself remains the same.

4. *Approximate Operators:*

   Sometimes it is helpful to also approximate the operator for complicated problems. The differential operators can be approximated by finite-difference and the integral operators can be approximated by means approximating the integral's kernel.

5. *Perturbation Solutions:*

   This approach tackles the problem by indirectly solving a perturbed version of it that has as easier solution and finally applying the inverse perturbations to the solution.

Variational Interpretation of Method of Moments:

This interpretation describes the method of moments by terms related to linear algebra. We consider the range of L, as $\mathcal{S}(Lf)$, the spanned space of Lf_n, as $\mathcal{S}(Lf_n)$, and the space spanned by the weighting function w_n, as $\mathcal{S}(w_n)$.

The method of moments actually e**quates the projection of Lf and Lf_n, on the space spanned by each weighting function $\mathcal{S}(w_n)$.**

In other words, the method of moments suggests that **approximate and actual operator yield exact components in the subspace of $\mathcal{S}(w_n)$, after applied on the response function.**

Applications in Electrostatics (Laplacian Operator)

We know that the electric field intensity has the following relationship with the electric potential:

$$\boldsymbol{E} = -\nabla\phi$$

And by substituting the third maxwell equation in the equation above, we obtain the Laplacian equation for a given electric charge density:

$$-\varepsilon\nabla^2\phi = \rho$$

As we can see we have reached an equation containing an operator, a field (response) function and a source(excitation) function.

$$L = -\varepsilon\nabla^2$$

We should note that for obtaining unique solutions the boundary conditions may also be satisfied. A well-known solution for this problem is:

$$\phi(x, y, z) = \iiint \frac{\rho(x', y', z')}{4\pi\varepsilon R}\, dx'dy'dz' \tag{46}$$

Thus, the inverse operator is as follows:

$$L^{-1} = \iiint dx'dy'dz' \frac{1}{4\pi\varepsilon R}\cdot$$

The fact that the inverse operator is directly dependent to the boundary conditions should be considered.

Proof of Adjoint Operator:

Forming the dot product according to an arbitrary weighting function we reach:

$$\langle L\phi, \psi\rangle = \iiint (-\varepsilon\nabla^2\phi)\psi\, d\tau \tag{49}$$

where $d\tau = dxdydz$. Green's identity is

$$\iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, d\tau = \oiint_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) ds \tag{50}$$

Evaluating the equivalent integral in the right side when $r \to \infty$, we obtain zero. So:

$$\iiint \psi\nabla^2\phi\, d\tau = \iiint \phi\nabla^2\psi\, d\tau \tag{51}$$

from which it is evident that the adjoint operator L^a is

$$L^a = L = -\varepsilon\nabla^2. \tag{52}$$

As both operators have identical domains, the operator L is a self-adjoint operator.

Proof of Positive Definite Operator:

Similar to the previous part, using vector identity, divergence theorem and an infinite sphere ($r \to \infty$):

$$\langle \phi^*, L\phi\rangle = \iiint \varepsilon|\nabla\phi|^2\, d\tau$$

Which implies that L is a positive definite operator

Charged Conducting Plates:

The electrostatic potential for a square plate on $z = 0$ plane and centered at origin, is as follows:

$$\phi(x, y, z) = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{\sigma(x', y')}{4\pi\varepsilon R}$$

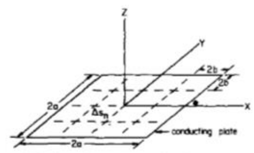

Fig. 1. A square conducting plate.

According to the fact that potential is uniformly spread across a PEC:

$$V = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{\sigma(x', y')}{4\pi\varepsilon\sqrt{(x - x')^2 + (y - y')^2}}$$

For approximating the capacitance of this conducting plane given a charge distribution:

$$VC = q$$

Capacitance acts as the response function which is willing to be determined. By dividing this plane into N square subsections (width Δy_i , length Δx_i) and using Point Matching approximate method:

We can expand the charge density into basis functions and the capacitance expression simplifies:

$$\sigma(x, y) \approx \sum_{n=1}^{N} \alpha_n f_n.$$

$$f_n = \begin{cases} 1 & \text{on } \Delta s_n \\ 0 & \text{on all other } \Delta s_m \end{cases}$$

Point matching implies:

$$V = \sum_{n=1}^{N} l_{mn}\alpha_n, \qquad m = 1, 2, \cdots, N \qquad (61)$$

where

$$l_{mn} = \int_{\Delta x_n} dx' \int_{\Delta y_n} dy' \frac{1}{4\pi\varepsilon\sqrt{(x_m - x')^2 + (y_m - y')^2}}. \qquad (62)$$

By choosing 2D Delta Dirac function as weighting functions and applying method of moments:

$$l_{nn} = \int_{-b}^{b} dx \int_{-b}^{b} dy \frac{1}{4\pi\varepsilon\sqrt{x^2 + y^2}}$$
$$= \frac{2b}{\pi\varepsilon} \ln(1 + \sqrt{2}) = \frac{2b}{\pi\varepsilon}(0.8814). \qquad (71)$$

$$l_{mn} \approx \frac{\Delta s_n}{4\pi\varepsilon R_{mn}} = \frac{b^2}{\pi\varepsilon\sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}} \quad m \neq n. \quad (72)$$

The accuracy of results is shown in the table below. The approximate charge density is also shown here:

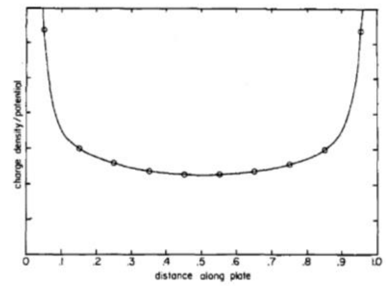

Fig. 2. Approximate charge density on subareas closest to the centerline of a square plate.

TABLE I

CAPACITANCE OF A SQUARE PLATE (PICOFARADS PER METER)

No. of subareas	$C/2a$ approx. l_{mn}	$C/2a$ exact l_{mn}
1	31.5	31.5
9	37.3	36.8
16	38.2	37.7
36	39.2	38.7
100		39.5