Digital Communication – CA#3

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For generating raised cosine pulses, I used the same function that was used in the precious CA. For modulated symbols, I used random number generation in the interval (0, 1) and in the intervals mentioned in the question. Next using convolution with raised cosine pulses, and then adding AWGN, 'received_signal' was created. Sampling is done and sampled points of the received signal are saved in samples.

A) ML detection criteria

Prior probabilities are assumed uniform in ML detection

$$P_e = \frac{1}{4}[\Pr\{\hat{a}_m \neq A | a_m = A\} + \Pr\{\hat{a}_m \neq B | a_m = B\} + \Pr\{\hat{a}_m \neq C | a_m = C\} + \Pr\{\hat{a}_m \neq D | a_m = D\}]$$

$$= \frac{1}{4}[\Pr\{Y(t_m) > \Delta_1 | a = A\}$$

$$+ \Pr\{(Y(t_m) < \Delta_1) \cup (Y(t_m) > \Delta_2) | a = B\}$$

$$+ \Pr\{(Y(t_m) < \Delta_2) \cup (Y(t_m) > \Delta_3) | a = C\}$$

$$+ \Pr\{Y(t_m) < \Delta_3 | a = D\}]$$

$$= \frac{1}{4}[\Pr\{n(t_m) > \Delta_1 + 3\}$$

$$+ \Pr\{(n(t_m) < \Delta_1 + 1) \cup (n(t_m) > \Delta_2 + 1)\}$$

$$+ \Pr\{(n(t_m) < \Delta_2 - 1) \cup (n(t_m) > \Delta_3 - 1)\}$$

$$+ \Pr\{n(t_m) < \Delta_3 - 3\}] \ (*)$$

$$\Rightarrow P_e = \frac{1}{4}[Q\left(\frac{\Delta_1 + 3}{N_0}\right) + Q\left(\frac{-\Delta_1 - 1}{N_0}\right) + Q\left(\frac{\Delta_2 + 1}{N_0}\right) + Q\left(\frac{-\Delta_2 + 1}{N_0}\right) + Q\left(\frac{\Delta_3 - 1}{N_0}\right) + Q\left(\frac{-\Delta_3 + 3}{N_0}\right)]$$

Minimizing Error Probability, results in putting the detection criteria, where the probabilities are equal.

$$\overset{(*)}{\Rightarrow} \frac{1}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_1+3)^2}{2N_0}} = \frac{1}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_1+1)^2}{2N_0}} \Rightarrow \Delta_1 = -2$$

$$\frac{1}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_2+1)^2}{2N_0}} = \frac{1}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_2-1)^2}{2N_0}} \Rightarrow \Delta_2 = 0$$

$$\frac{1}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_3-3)^2}{2N_0}} = \frac{1}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_3-1)^2}{2N_0}} \Rightarrow \Delta_3 = 2$$

Substituting results in Error probability formula, minimum error probability is achieved:

$$\Rightarrow P_{e(\min)ML} = \frac{3}{2} Q\left(\frac{1}{N_0}\right)$$

B) MAP detection criteria

$$P_e = \Pr\{\hat{a}_m \neq A | a_m = A\}P_A + \Pr\{\hat{a}_m \neq B | a_m = B\}P_B + \Pr\{\hat{a}_m \neq C | a_m = C\}P_C + \Pr\{\hat{a}_m \neq D | a_m = D\}P_D$$

$$= \Pr\{Y(t_m) > \Delta_1 | a = A\}P_A$$

$$+ \Pr\{(Y(t_m) < \Delta_1) \cup (Y(t_m) > \Delta_2) | a = B\}P_B$$

$$+ \Pr\{(Y(t_m) < \Delta_2) \cup (Y(t_m) > \Delta_3) | a = C\}P_C$$

$$+ \Pr\{Y(t_m) < \Delta_3 | a = D\}P_D$$

$$= \Pr\{n(t_m) > \Delta_1 + 3\}P_A$$

$$+ \Pr\{(n(t_m) < \Delta_1 + 1) \cup (n(t_m) > \Delta_2 + 1)\}P_B$$

$$+ \Pr\{(n(t_m) < \Delta_2 - 1) \cup (n(t_m) > \Delta_3 - 1)\}P_C$$

$$+ \Pr\{n(t_m) < \Delta_3 - 3\}P_D \, (*)$$

$$\Rightarrow P_e = P_A Q\left(\frac{\Delta_1 + 3}{N_0}\right) + P_B[Q\left(\frac{-\Delta_1 - 1}{N_0}\right) + Q\left(\frac{\Delta_2 + 1}{N_0}\right)] + P_C[Q\left(\frac{-\Delta_2 + 1}{N_0}\right) + Q\left(\frac{\Delta_3 - 1}{N_0}\right)] + P_D Q\left(\frac{-\Delta_3 + 3}{N_0}\right)$$

Minimizing Error Probability, results in putting the detection criteria, where the probabilities are equal.

$$\overset{(*)}{\Rightarrow} \frac{P_A}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_1+3)^2}{2N_0}} = \frac{P_B}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_1+1)^2}{2N_0}} \Rightarrow \Delta_1 = -2 - \frac{N_0}{2}\ln\frac{P_B}{P_A}$$

$$\frac{P_B}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_2+1)^2}{2N_0}} = \frac{P_C}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_2-1)^2}{2N_0}} \Rightarrow \Delta_2 = 0$$

$$\frac{P_C}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_3-3)^2}{2N_0}} = \frac{P_D}{\sqrt{2\pi N_0}} e^{\frac{(\Delta_3-1)^2}{2N_0}} \Rightarrow \Delta_3 = 2 + \frac{N_0}{2}\ln\frac{P_C}{P_D}$$

Substituting results in Error probability formula, minimum error probability is achieved:

$$\Rightarrow P_{e(\min)MAP-4PAM} = P_A Q\left(\frac{1 - \frac{N_0}{2}\ln\frac{P_B}{P_A}}{N_0}\right) + P_B[Q\left(\frac{1 + \frac{N_0}{2}\ln\frac{P_B}{P_A}}{N_0}\right) + Q\left(\frac{1}{N_0}\right)] + P_C[Q\left(\frac{1}{N_0}\right) + Q\left(\frac{1 + \frac{N_0}{2}\ln\frac{P_C}{P_D}}{N_0}\right)] + P_D Q\left(\frac{1 - \frac{N_0}{2}\ln\frac{P_C}{P_D}}{N_0}\right)$$

Substituting Prior probabilities, we get:

$$\Delta_1 = -2 - \frac{N_0}{2}\ln 4, \qquad \Delta_2 = 0, \qquad \Delta_3 = 2 + \frac{N_0}{2}\ln 4$$

$$\Rightarrow P_{e(\min)MAP-4PAM} = 0.2Q\left(\frac{1 - \frac{N_0}{2}\ln 4}{N_0}\right) + 0.8[Q\left(\frac{1 + \frac{N_0}{2}\ln 4}{N_0}\right) + Q\left(\frac{1}{N_0}\right)]$$

C) MAP vs. ML – Error Probability Comparison

As we can see MAP receiver has less Error probability in all SNR values and for higher SNR's, and error probability is lower as we expected. The gap between MAP and ML performance shrinks as the prioir probabilities tend towards uniform distribution.