Discrete and Algorithmic Geometry.

Sheet 4:

(2) Show that $\int_P f(x)\,dx = \mathrm{Vol}(P).f(P_0)$ where $P_0 = \frac{1}{\mathrm{vol}(P)} \int_P x\,dx$. (bary center of P).

Solution:

Given P a polytope and f a linear function.

$$f(P_0) = f\left(\frac{1}{\mathrm{vol}(P)} \int_P x\,dx\right)$$

$\frac{1}{\mathrm{Vol}(P)}$ is cst. so using linearity of f

we have:
$$= \frac{1}{\mathrm{Vol}(P)} \cdot f\left(\int_P x\,dx\right)$$

$$= \frac{1}{\mathrm{Vol}(P)} \cdot \int_P f(x)\,dx.$$

$$\Rightarrow \mathrm{Vol}(P).f(P_0) = \mathrm{Vol}(P).\frac{1}{\mathrm{Vol}(P)} \cdot \int_P f(x)\,dx$$

$$= \int_P f(x)\,dx.$$

$\square$

(3) Theorem: $\Sigma(P,Q)$ is a polytope of dimension $\dim(P)-\dim(Q)$, whose nonempty faces correspond to the Π-coherent subdivisions of Q, that is, the face lattice of $\Sigma(P,Q)$ is $L(\Sigma(P,Q)) = \{\hat{0}\} \cup \omega_{coh}(P,Q)$.

Proof: • First of all we have that $\Sigma(P,Q)$ is convex using the linearity of the integral as it is a combination of 2 sections.

• $\Sigma(P,Q)$ is contained in the fiber $\pi^{-1}(r_0)$ whose dim is $\dim(P)-\dim(Q)$; so $\dim(\Sigma(P,Q)) \leq \dim(P)-\dim(Q)$.

• We need to show that $\Sigma(P,Q)$ is a polytope. Indeed using the following Lemma we can find that $\Sigma(P,Q)$ is the convex hull of $\frac{1}{Vol(Q)}\int_Q \gamma(x)\,dx$ for γ tight section of $\pi: P \to Q$.

Lemma: Every piecewise linear section that is not tight can be changed locally in 2 opposite directions, and can be written as a convex combination of 2 other sections that have a different integral.

• The vertices of $\Sigma(P,Q)$ are the unique maxima for generic linear functions $c \in (\mathbb{R}^P)^*$.

Since c is generic then every fiber $\pi^{-1}(r)$ for $r \in Q$ has a unique maximal element with respect to c, So c determines a unique tight coherent section γ^c.

The integral over γ^c is the only point of $\Sigma(P,Q)$ which maximizes c.

$\Rightarrow$ The vertices of $\Sigma(P,Q)$ coincide with the tight coherent subdivisions $\pi: P \to Q$.

• Every face of $\Sigma(P,Q) \subseteq \mathbb{R}^P$ is defined by a linear func. on $\mathbb{R}^P$. So it defines a map $\pi^c: P \to Q^c$, so a π-coherent subdivision $\mathcal{F}^c$ of Q.

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$\frac{1}{vol(Q)} \int \gamma dx \in \Sigma(P,Q)$ lies in the face defined by c $\Leftrightarrow$ the image of the section $\gamma: Q \to P$ is contained in $\mathcal{F}^c \subseteq L(P)$.

This proves that there is a bij. bet. the faces of $\Sigma(P,Q)$ + the coherent subdivisions of Q, i.e. $\mathcal{L}(\Sigma(P,Q)) = \{\emptyset\} \cup \omega_{coh}(P,Q)$.

$\square$