Exercise 1 – Probability & Statistics Refresher

1. There are 1000 coins, which look identical. However, 999 of them are “fair” (i.e. when tossing the coin, the probability to get “heads” is 0.5), and one coin is forged (i.e. when tossing the coin, the probability to get “heads” is 0.9). Assuming we selected on random one coin and toss it 10 times, and got “heads” in 9 out of the 10 tosses.
    A. What is the probability that this is the forged coin?
    B. What is the probability that this is one of the “fair” coins?

2. In a faraway country, they have a strange culture where they favor girls over boys. Additionally, they know that boys are trouble, so they settle for one in a family. Thus, each family give birth until their first son is born, and then they stop (they never take the risk of having two boys – double trouble ...). As a result, families in this country are of the following types
    a. Boy
    b. Girl, Boy
    c. Girl, Girl, Boy
    d. Girl, Girl, Girl, Boy
    e. …

    Assume that the probability to have a boy/girl in a given birth is 0.5 and all births are i.i.d. Are there more girls or more boys in this country?

3. Given a random sample $\{x_1, x_2, \dots, x_n\}$ Calculate the Maximum Likelihood Estimator (MLE) for the parameters Θ of the following
    a. Binomial – $\Pr(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$ (calculate the MLE for p)
    b. Poisson – $\Pr(k|\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$ (calculate the MLE for λ)
    c. Normal – $\Pr(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$ (calculate the MLE for both μ and σ^2

4. Given a 2-dimensional random variable $\boldsymbol{x} \sim N_2(\boldsymbol{\mu},\ \Sigma)$

$$\Pr(\boldsymbol{x}|\boldsymbol{\mu},\Sigma) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right]$$

   Where $\boldsymbol{x} = \binom{x_1}{x_2}$, $\boldsymbol{\mu} = \binom{\mu_1}{\mu_2}$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$

   Show that

   a. The marginal distributions are normal, such that

$$x_i \sim N\left(\mu_i, \sigma_i^2\right),\ \ i = 1,2$$

   Hints

   - Consider using the explicit form of the bivariate Normal distribution
     - $f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$

- Recall that $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$

  Before solving the integral, consider changing variable $z_2 = \frac{x_2 - \mu_2}{\sigma_2}$ and complete to square in z_2
- The following integrals may be useful (which one to use depends on your specific solution)
  - $\int_{-\infty}^{\infty} \exp[-\alpha(x+u)^2] dx = \sqrt{\pi/\alpha}$
  - $\int_{-\infty}^{\infty} \exp[-(ax^2+bx+c)] dx = \sqrt{\pi/a} \exp\left(\frac{b^2}{4a} - c\right), \quad a>0$

b. The conditional distribution is normal, such that

$$x_1|x_2 \sim N\left(\mu_1 + \frac{\rho\sigma_1(x_2 - \mu_2)}{\sigma_2}, \sigma_1^2(1-\rho^2)\right)$$

<u>Hints</u>

- Recall that $f_{X_1|X_2=x_2}(x_1) = \frac{f(x_1,x_2)}{f_{X_2}(x_2)}$

  Get to the form $(A^2 - 2AB + B^2) = (A-B)^2$ in the exponent
- Reshape the result such that it will resemble the form of a 1-dim Normal density, and identify the mean and the variance

5. Pearson's Correlation Coefficient is defined as follows

$$\rho = \frac{cov(X,Y)}{\sigma_x \sigma_y} = \frac{E[(x-\mu_x)(y-\mu_y)]}{\sqrt{E[(x-\mu_x)^2]}\sqrt{E\left[(y-\mu_y)^2\right]}}$$

Show that $\quad -1 \le \rho \le 1$ (Hint: use Cauchy–Schwarz inequality)