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UNIT-1

* Mathematical Induction →

Is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n

principal of mathematical induction –
consider a statement $P(n)$ were, n is a natural number.
then to determine the validity of $P(n)$ for every n use the following principal :–
Step-1] Check wether the given statement is true for $n=1$
Step-2] Assume that the given statement $P(n)$ is also true for $n=k$, were k is any positive integer
Step-3] Prove that the result is true for $P(k+1)$ for any positive integer k.

– if the above mentioned condition are satisfied, then it can be concluded that $P(n)$ is true for all natural numbers.

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Que-1] Prove – $\sum_{i=1}^{n} i = 1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

$\longrightarrow$ Step 1 – for $n=1$

L.H.S = 1 R.H.S = $\frac{n(n+1)}{2} = 1$.

Step 2 – for $n=k$

$$1+2+3+\cdots+k = \frac{k(k+1)}{2} \quad -②$$

Step 3 – for $n=k+1$

$$\begin{aligned} L.H.S &= 1+2+3+\cdots+(k+1) \\ &= 1+2+3+\cdots+k+(k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \quad -③ \end{aligned}$$

$$R.H.S = \frac{n(n+1)}{2} \quad \{\text{for } n=k+1\}$$

$$= \frac{(k+1)(k+2)}{2} \quad -④$$

hence ③ = ④ {L.H.S = R.H.S}

Que-2] Use mathematical induction and prove :-

$\longrightarrow$ $\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$

Step 1] – for $n=1$

L.H.S = 1 R.H.S = $\frac{1(1+1)(3)}{6} = \frac{\cancel{2}\times\cancel{3}}{\cancel{6}} = 1$

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Step-2] for $n=k$

$$1^2+2^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6} \quad -①$$

Step-3] for $n=k+1$

$$L.H.S = 1^2+2^2+\cdots+(k+1)^2$$

$$= 1^2+2^2+\cdots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)}{6}\left[k(2k+1)+6(k+1)\right]$$

$$= \frac{(k+1)}{6}\left[2k^2+k+6k+6\right] = \frac{(k+1)}{6}\left[2k^2+7k+6\right]$$

$$= \frac{(k+1)}{6}\left[2k^2+4k+3k+6\right] = \frac{(k+1)(k+2)(2k+3)}{6} \quad -②$$

$$R.H.S = \frac{n(n+1)(2n+1)}{6}$$

for $n=k+1$

$$\frac{(k+1)(k+2)(2k+3)}{6} \quad -④$$

hence, $L.H.S = R.H.S$

Que-3] Use mathematical induction and prove:-

$$\rightarrow \sum_{n=1}^{i} i^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

Step-1] for $n=1$

$L.H.S = 1$

$$R.H.S = \frac{1(1+1)^2}{4} = \frac{4}{4} = 1$$

Step-2] for $n=k$

$$1^3+2^3+3^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4} \quad -②$$

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Step-3 for $n = k+1$

$$1^3 + 2^3 + \cdots + (k+1)^3 = \frac{(k+1)^2 (k+2)^2}{4} \quad -(3)$$

L.H.S -

$$1^3 + 2^3 + \cdots + (k+1)^3$$
$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$
$$\frac{k^2 (k+1)^2}{4} + (k+1)^3 \quad -[\because \text{from } 1]$$
$$\frac{k^2 (k+1)^2 + 4(k+1)^3}{4}$$
$$\frac{(k+1)^2}{4} [k^2 + 4(k+1)]$$
$$\frac{(k+1)^2}{4} (k^2 + 4k + 4)$$
$$(k+1)^2 (k^2 + 2k + 2k + 4)$$
$$\frac{(k+1)^2 (k+2)^2}{4} \quad -(4)$$

hence, L.H.S = R.H.S $-[\because$ from 3 & 4$]$

Que-4] Prove the following using MI -

$$1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$$

Step-1] for $n = 1$

$$L.H.S = 3(1) - 2 = 3 - 2 = 1$$
$$R.H.S = \frac{1(3(1)-1)}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$$

Step-2] for $n = k$

$$1 + 4 + 7 + \cdots + (3k-2) = \frac{k(3k-1)}{2} \quad -(1)$$

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(3) Step-3] for $n=k+1$

L.H.S $1+4+7+ \cdots [3(k+1)-2]$

$1+4+7+ \cdots (3k-2)+[3k+3-2]$

$\underline{1+4+7+ \cdots (3k-2)}+(3k+1)$

$\frac{k(3k-1)}{2} + (3k+1)$ — [from 2]

$\frac{k(3k-1)+2(3k+1)}{2}$

$\frac{3k^2-k+6k+2}{2}$

$\frac{3k^2+5k+2}{2}$

$\frac{3k^2+3k+2k+2}{2}$ — $\frac{3k(k+1)+2(k+1)}{2} = \frac{(k+1)(3k+2)}{2}$

R.H.S — for $n=k+1$

$\frac{n(3n-1)}{2} = \frac{(k+1)(3(k+1)-1)}{2} = \frac{(k+1)(3k+2)}{2}$

Ques-5] Prove — $\sum_{i=1}^{n} i$ $1^2+3^2+5^2+ \cdots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

→ Step-1] for $n=1$

L.H.S $=1$ R.H.S $= \frac{1(1)(3)}{3} = 1$

Step-2] for $n=k$

$1^2+3^2+5^2+ \cdots (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

Step-3] for $n=k+1$

$1^2+3^2+5^2+ \cdots (2(k+1)-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

L.H.S —

$1^2+3^2+5^2 \cdots (2k-1)^2+(2k+1)^2$

$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$

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$$\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$\frac{(2k+1)}{3}[k(2k-1)+3(2k+1)]$$

$$\frac{(2k+1)}{3}[2k^2-k+6k+3]$$

$$\frac{(2k+1)}{3}[2k^2+5k+3]$$

$$\frac{(2k+1)}{3}[2k^2+2k+3k+3]$$

$$\frac{(2k+1)}{3}\{2k(k+1)+3(k+1)] = \frac{(2k+1)(k+1)(2k+3)}{3} \quad -2$$

$$R.H.S - \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$= \frac{(k+1)(2k+2-1)(2k+2+1)}{3} = \frac{(k+1)(2k+1)(2k+3)}{3} \quad -3$$

from 2 & 3 → L.H.S = R.H.S — hence proved

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* To find complexity of a loop

ex-1] for (i=1; i<=n; i++)
{ j=0; }

(any statement inside loop constant then we can represent the loop as 1)

$\rightarrow \sum_{i=1}^{n} 1$ = upper limit − (lower limit − 1)
$= n-(1-1) = n \Rightarrow O(n)$

ex-2] for (i=1; i<=n; i++)
{ for (j=1; j<=i; j++)
{
$\underbrace{k=k+1;}_{\text{constant (then we write 1)}}$
}
}

$\rightarrow \sum_{i=1}^{n} \cdot \sum_{j=1}^{i} 1 = \sum_{i=1}^{n} i-(1-1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

(order or complexity will be the highest power of the variable) $\therefore \frac{n^2+n}{2} = O(n^2)$

ex-3] fun(int n)
{ int i, j, k, r;
r=0;
for(i=1; i<=n; i++)
{
for(j=i+1; j<=n; j++)
{
for(k=j; k<=n; k++)
{ r=r+1;
}
}
}
}

* If the upper limit is n and the equation also consist of n then it will be constant take the equation or term outside the bracket

$\rightarrow \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j}^{n} 1$

$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} n-(j-1)$

$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} n+1-j$

$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n+1) - \sum_{j=i+1}^{n} j$

$= \sum_{i=1}^{n} (n+1) \sum_{j=i+1}^{n} 1 - \sum_{j=i+1}^{n} j$

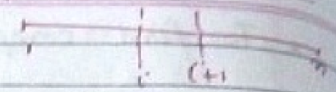
$= \sum_{i=1}^{n} (n+1) \; [n-(i+1-1)] - \sum_{j=i+1}^{n} j$

$= \sum_{i=1}^{n} (n+1)\; (n-i) - \sum_{j=i+1}^{n} j$

$$\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right] \text{ - from MI} \quad \text{Page No.}$$

$$\left[\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right] \text{ - from MI} \quad \text{Date \_\_/\_\_/\_\_}$$

$$\sum_{i=1}^{n} (n+1)\, n - (i + \cancel{r_{n}}) - \sum_{j=i+1}^{n} j$$



$$\sum_{i=1}^{n} (n+1)(n-i) - \left\{ \sum_{j=1}^{n} j - \sum_{j=1}^{i} j \right\} \qquad (1 \text{ to } n) - (1 \text{ to } i) = (\cancel{i+1} \text{ to } n)$$

$$\sum_{i=1}^{n} (n+1)(n-i) - \left\{ \frac{n(n+1)}{2} - \frac{i(i+1)}{2} \right\}$$

$$\sum_{i=1}^{n} (n+1)(n-i) - \frac{n(n+1)}{2} + \frac{i(i+1)}{2}$$

$$\sum_{i=1}^{n} \quad n(n+1) \cancel{-} i(n+1) - \frac{n(n+1)}{2} + \frac{i^2+i}{2}$$

$$\sum_{i=1}^{n} \quad \cancel{n^2}\, n(n+1) - \frac{n(n+1)}{2} - i(n+1) + \frac{i^2+i}{2}$$

$$\sum_{i=1}^{n} \quad \frac{n(n+1)}{2} - i(n+1) + \frac{i^2+i}{2}$$

$$\sum_{i=1}^{n} \frac{n(n+1)}{2} - \sum_{i=1}^{n} i(n+1) + \sum_{i=1}^{n} \frac{i^2+i}{2}$$

$$\frac{n(n+1)}{2} \sum_{i=1}^{n} (1) - (n+1) \sum_{i=1}^{n} i + \frac{1}{2}\left\{ \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \right\}$$

$$\frac{n(n+1)}{2}\; n \cdot (1-1) - (n+1)\frac{n(n+1)}{2} + \frac{1}{2}\left\{ \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right\}$$

$$\frac{n(n+1)}{2}\; n - \frac{n(n+1)^2}{2} + \frac{1}{2}\left\{ \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right\}$$

$$\frac{n^2(n+1)}{2} - \frac{n(n+1)^2}{2} + \frac{1}{2}\left\{ \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{6} \right\}$$

$$\frac{n^3+n^2-n(n^2+2n+1)}{2} + \frac{1}{2}\left\{ \frac{(n^2+n)(2n+1) + 3n^2+3n}{6} \right\}$$

$$\frac{\cancel{n^3}+n^2-\cancel{n^3}-2n^2-n}{2} + \frac{1}{2}\left\{ \frac{2n^3+n^2+2n^2+n+3n^2+3n}{6} \right\}$$

$$\frac{-n^2-n}{2} + \frac{1}{2}\left\{ \frac{2n^3+6n^2+4n}{6} \right\}$$

$$-\frac{n^2-n}{2} + \frac{\cancel{2}}{\cancel{2}}\left\{ \frac{n^3+3n^2+2n}{6} \right\}$$

$$\frac{-n^2-n}{2} + \left\{\frac{n^3+3n^2-n}{6}\right\}$$

$$\frac{-3n^2-3n+n^3+3n^2+4n}{6}$$

$$\frac{n^3-n}{6} = \frac{n^3-n}{6} = O(n^3)$$

[Ex-4]
```
r=0;
for(i=1 to n) do
    for(j=1 to i) do
        for(k=j to i+j) do
            r=r+1;
```

$$\rightarrow \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j}^{i+j} 1$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} i+j-(j+1)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i} i+j-j+1 = \sum_{i=1}^{n}\sum_{j=1}^{i}(i+1)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}(i+1)$$

$$= \sum_{i=1}^{n}(i+1)\sum_{j=1}^{i} 1$$

$$= \sum_{i=1}^{n}(i+1)\quad (i)-(1-1)$$

$$= \sum_{i=1}^{n}(i+1)\quad i\cancel{-1+1}$$

$$= \sum_{i=1}^{n}(i^2+i)$$

$$= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)+3n^2+3n}{6}$$

$$= \frac{(n^2+n)(2n+1)+3n^2+3n}{6}$$

$$= \frac{2n^3+n^2+2n^2+n+3n^2+3n}{6}$$

$$= \frac{2n^3+6n^2+4n}{6}$$

$$= \frac{2(n^3+3n^2+2n)}{6\,3}$$

$$= \frac{n^3+3n^2+2n}{3}$$

$$= O(n^3)$$

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* Asymptotic Notations

Asymptotic notations are mathematical notations used to describe the running time of an algorithm when the input tends towards a particular value or a limiting value.

there are mainly three asymptotic Notations -

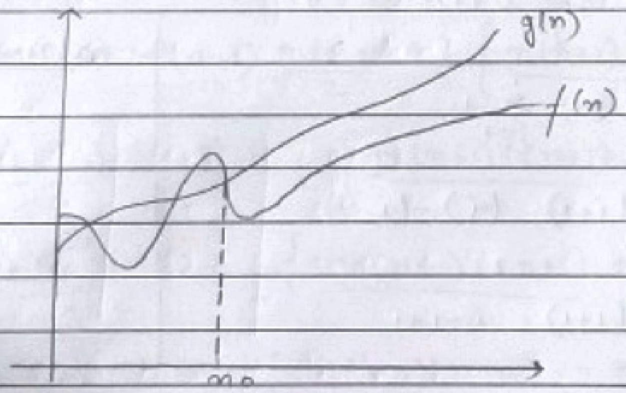
1) Big - o notation
2) omega notation
3) Theta notation

(O)

1) Big - o notation → it represents the upper bond of the running time of an algorithm. Thus, it gives the worst - case complexity of an algorithm.

- Big - O gives the upper bound of a function

$O(g(n)) = \{ f(n) :$ there exist positive constant, c and n_0 such that $0 \leq f(n) \leq c g(n)$ for all $n \geq n_0 \}$



$f(n) = O(g(n))$

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Ex:- find upper bound of the running time of linear function $f(n) = 6n+3$

→ given:- $f(n) = 6n+3$

we know, Big-O → equation is :-

$$0 \le f(n) \le cg(n)$$
$$0 \le 6n+3 \le cg(n)$$

case-1] [g(n) is also function with variable n]

to decide constant value 'c' = constant of first term + constant of second term

i.e $c = 6+3 = 9$

but $n = 1$ → to $f(n) = cg(n)$

case-2]

c = constant of first term + 1

i.e $c = 6+1 = 7$

but $n = 3$ → As $f(n) = c(gn)$

case-1] $c = 6+3 = 9$ $n = 1$ $f(n) = 6n+3$

tabular form:-

$n_0 \rightarrow n$	$f(n)$	$c.g(n)$
1	9	9
2	15	18
3	21	27
		.

hence, $0 \le f(n) \le cg(n)$ is true when $c = 9$ $n = 1$ & $f(n) = 6n+3$

case-2] $c = 6+1 = 9$ $n = 3$ $f(n) = 6n+3$

n	$f(n)$	$cg(n)$
3	21	21
4	27	28
5	33	35

As from 1 & 2 the equation $0 \le f(n) \le cg(n)$ is false
∴ hence, $n = 3$

hence, $0 \le f(n) \le c\,g(n)$ is true when $c = 7$ $n = 3$ & $f(n) = 6n+3$

∴ $f(n) = O(g(n)) = O(n)$ for $c = 9$, $n_0 = 1$
∴ $f(n) = O(g(n)) = O(n)$ for $c = 7$, $n_0 = 3$

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ex-2 find upper bound of running time of quadratic function $f(n) = 3n^2 + 2n + 4$

→ given - $f(n) = 3n^2 + 2n + 4$

we know that, $0 \le f(n) \le c \cdot g(n)$

$0 \le 3n^2 + 2n + 4 \le c \cdot g(n)$

Case-1] $c = 3 + 2 + 4 = 9$. $g(n) = n^2$ - highest power

$0 \le 3n^2 + 2n + 4 \le 9(n^2)$

So, $c = 9$, $g(n) = n^2$ and $n_0 = 1$

Tabular form $3(4) + 2(2) + 4 = 12 + 4 + 4 = 12 + 8$

n	$f(n)$	$c \cdot g(n)$
1	9	9
2	20	36
3	37	81

$3(9) + 2(3) + 4$
$27 + 6 + 4$
$27 + 10$

hence, $c = 9$ & $n_0 = 1$ then $0 \le f(n) \le c \cdot g(n)$ is true.

Case-2] $c = 3 + 1 = 4$ $g(n) = n^2$ $n = 4$

$0 \le 3n^2 + 2n + 4 \le 4n^2$

Tabular form - $3(16) + 8 + 4$, $48 + 8 + 4$ $4(4)^2$ $4 \cdot 16$

n	$f(n)$	$c\,g(n)$
4	60	64
5	89	100

hence, $c = 4$ & $n_0 = 4$ then $0 \le f(n) \le c \cdot g(n)$ is true.

∴ There can be such multiple pair of (c, n_0)

$f(n) = O(g(n)) = O(n^2)$ for $c = 9$, $n_0 = 1$

$f(n) = O(g(n)) = O(n^2)$ for $c = 4$, $n_0 = 4$

and so on.

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ex-3 find upper bound of running time of a cubic function $f(n) = 2n^3 + 4n + 5$.

→ Given:- $f(n) = 2n^3 + 4n + 5$

To find upper bound of $f(n)$, we have to find c and n_0 such that $0 \le f(n) \le c \times g(n)$ for all $n \ge n_0$

i.e. $0 \le f(n) \le c \cdot g(n)$

Case-1] $c = 2 + 4 + 5 = 11$ $\quad g(n) = n^3$ $\quad n = 1$

$0 \le 2n^3 + 4n + 5 \le 11(n^3)$

So, Tabular form -

n	$f(n)$	$c \cdot n^3$
1	11	11
2	29	88
3	71	297

$2(8) + 4(2) + 5$

$16 + 8 + 5 = 13 + 16$

$2(27) + 4(3) + 5 = 54 + 12 + 5$

27×11 $\quad = 66 + 5 = 71$

Case-2] $c = 2 + 1 = 3$ $\quad g(n) = n^3$ $\quad n = 3$

$0 \le 2n^3 + 4n + 5 \le 3(n^3)$

So, tabular form -

n	$f(n)$	$c \cdot n^3$
3		

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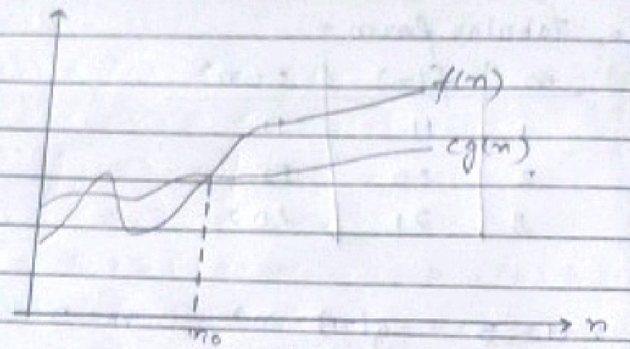
2] Omega Notation (Ω - notation) ⇒
- Omega notation represents the lower bound of the running time of an algorithm. Thus, it provides the best case complexity of an algorithm.

- omega gives the lower bound of a function
- $\Omega(g(n)) = \{f(n)$: there exist positive integer constant c and no such that .



$f(n) = \Omega(g(n))$

Q-1] To find lower bound of f(n) running time of a linear function $f(n) = 6n+3$.

→ Given - $f(n) = 6n+3$

To find lower bound of f(n), we have to find c and no such that $0 \le c \cdot g(n) \le f(n)$ for all $n \ge n_0$

$0 \le c \times g(n) \le f(n)$

$0 \le c \times g(n) \le 6n+3$

$0 \le 6n \le 6n+3$ → true, for all $n \ge n_0$

here we take the c value as constant of n or constant of g(n)

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Q-2] find lower bound of running time of quadratic function $f(n) = 3n^2+2n+4$

→ given - $f(n) = 3n^2+2n+4$

To find lower bound of $f(n)$, we have to find c and n_0 such that $0 \le c \cdot g(n) \le f(n)$ for all $n \ge n_0$

$0 \le c \times g(n) \le f(n)$

$0 \le c \times g(n) \le f(n)$

$0 \le 3n^2 \le 3n^2+2n+4$ → true, for all $n \ge 1$

$0 \le n^2 \le 3n^2+2n+4$, → true, for all $n \ge 1$

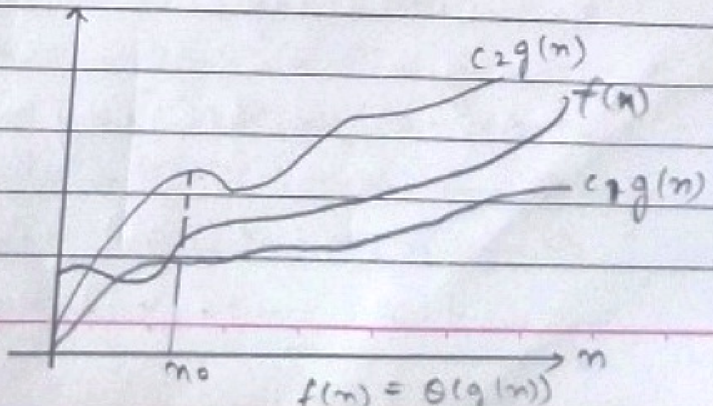
Above both inequalities

3] Theta Notation (Θ-notation) -

- Theta notation encloses the function from above and below.
- Since, it represents the upper and the lower bound of the running time of an algorithm, it is used for analyzing the average - case complexity of an algorithm.
- Thetha bounds the function within constants factors.

for a function $g(n)$, $\Theta(g(n))$ is given by the relation:

$\Theta(g(n)) = \{ f(n)$: there exist positive constant, c_1, c_2 and n_0 such that $0 \le \underbrace{c_1 g(n)}_{\text{Lower Value}} \le f(n) \le \underbrace{c_2 g(n)}_{\text{upper value}}$ for all $n \ge n_0 \}$



$f(n) = \Theta(g(n))$

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ex:- find tight bound of running time of a linear function $f(n) = 6n+3$. To find tight bound of $f(n)$, we have to find c_1, c_2 and n_0 such that, $0 \le c_1 \times g(n) \le f(n) \le c_2 \times g(n)$ for all $n \ge n_0$

$0 \le c_1 \times g(n) \le 6n+3 \le c_2 \times g(n)$

$0 \le 5n \le 6n+3 \le 9n$, for all $n \ge 1$

Above inequality is true and there exists such infinite inequalities.

$f(n) = 6n+3$

$c_1 g(n) \le f(n) \le c_2 g(n)$

$c_1 g(n) \le 6n+3 \le c_2 g(n)$

$\boxed{g(n) = n}$

$c_1 \cdot n \le 6n+3 \le c_2 \cdot n$

if $c_1 = 6$ – take c_1 as coefficient of n

& $c_2 = 9$ – that will sum of $6+3 = 9$.

$\therefore c_1 = 6$, $c_2 = 9$. $g(n) = n$.

$\boxed{\therefore f(n) = \Theta(n)}$ — tight bound.

5M

Que:- Explain asymptotic notation with example of each notation.

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* Recurrence relation

A Recurrence relation is an equation which represents a sequence based on some rule. It helps in finding subsequent term (next term) depending on preceding term (previous term). If we know the previous term in a given series then we can easily determine the next term.

Their are ~~two~~ three methods used to find running time complexity of recursive function. methods are -

i) substitution method.
ii) Changing variable method
iii) Recursion tree method.

i) Substitution method -

In these method we go on expanding the right hand side of recurrence until we get the terminating condition.

Q-1) $T(n) = 2T(n/2) + n \qquad n > 1$ — (A)
$\qquad = 1 \qquad n = 1$

find the value of the function $T(n/2)$ by substituting $n = n/2$ in equation A:

→ $T(n/2)$ by substitution of $n = n/2$ in eq^n (A)

$$T(n/2) = 2T(n/4) + (n/2)$$

put this value in (A)

$$T(n) = 2[2T(n/4) + n/2] + n$$
$$T(n) = 2^2 T(n/2^2) + 2n \quad — ①$$
$$T(n) = 2^2 T(n/4) + 2n \quad — ①$$

functions
e have
$) \le f(n)$
nfinite

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$T(n/4) = 2T(n/8) + n/4$ — put in eqn Ⓑ

$T(n) = 2^2[2T(n/8) + n/4] + 2n$

$T(n) = 2^3 T(n/8) + \cancel{n} + 2n$

$T(n) = 2^3 T(n/8) + 3n$ — ②

∴ generalized equation —

$$\boxed{T(n) = 2^k T(n/2^k) + kn} \quad — Ⓑ$$

- The substitution cannot be carried out infinitely so for a certain value of k, the term $\frac{n}{2^k} = 1$ is called the terminating condition of the recurrence.

- Thus, the ~~equ~~ iteration terminate for the value of k such that $\frac{n}{2^k} = 1$ or $n = 2^k$ or $k = \log_2 n = \lg n$ substituting these value in equation B we get

$$\boxed{T(n) = n\,T(1) + \log_2 n \cdot n}$$

from the initial conditions,

$$T(n) = 1 \text{ for } n = 1$$

$$\boxed{T(n) = n \cdot 1 + \log_2 n \cdot n}$$

big o notation —

$0 \le f(n) \le c_1 \cdot g(n)$

$0 \le n + \log_2 n \cdot n \le c_1 g(n)$

$g(n) = n \log_2 n$

$0 \le n + n\log_2 n \le c_1 \cdot n\log_2 n$

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n_0	$f(n) = n + n\log_2 n$	$c_1 \cdot n\log_2 n$	$c_1 = 2$
2	4	4	
4	12	16	
8	32	48	

worstcase:-

$$f(n) = n + n\log_2 n$$
$$g(n) = n\log_2 n$$
$$c_1 = 2 \ \& \ n_0 = 2$$
$$f(n) = O(g(n))$$
$$f(n) = O(n\lg n)$$

Q- Solve the following releation using substitution method

$$T(n) = 2T(n/2) + n\lg n \quad n>1 \quad -(A)$$
$$= 1 \quad n=1$$

→ $T(n/2)$ by substitution of $n = n/2$ in eq (A)

$$T(n/2) = 2T(n/4) + n/2 \lg n/2 \quad \text{- put in eq (A)}$$

$$T(n) = 2[2T(n/4) + n/2 \lg n/2] + n\lg n$$
$$T(n) = 2^2[T(n/4) + n\lg n/2 + n\lg n \quad -(1)$$

$T(n/4)$ by substitution of $n = n/4$ in eq (A)

$$T(n/4) = 2T(n/8) + n/4 \lg n/4 \quad \text{— put in eq (1)}$$
$$T(n) = 2^2(2T(n/8) + n/4 \lg n/4] + n\lg n/2 + n\lg n$$
$$T(n) = 2^3 T(n/8) + n\lg n/4 + n\lg n/4 + n\lg n.$$

$$T(n) = 2^k T(n/2^k) + \sum_{i=0}^{k-1} n\lg(n/2^i)$$
$$= 2^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} n[\lg n - \lg 2^i]$$
$$= 2^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} n\lg n - \sum_{i=0}^{k-1} n * i$$

$\log_2 2^i = i \log_2 2 = i$

$\sum_{i=0}^{k-1} 1 = [\text{upper limit}] - [\text{lower limit} - 1]$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\quad \sum_{i=0}^{k-1} i = \frac{(k-1)(k-1+1)}{2} = \frac{(k-1)k}{2}$

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$$= 2^k T\left(\frac{n}{2^k}\right) + n\lg n \sum_{i=0}^{k-1} 1 - n \sum_{i=0}^{k-1} i$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n\lg n\, k - n\left[\frac{(k)(k-1)}{2}\right]$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n\lg n k - n\left[\frac{k^2-k}{2}\right]$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + n\lg n k - \frac{nk^2}{2} + \frac{nk}{2} \quad - ③$$

- The terminating condition is –

$\frac{n}{2^k} = 1$ or $n = 2^k$ or $k = \lg n$

After putting the value in eq③

$$T(n) = n\,T(1) + n\lg n \cdot \lg n - \frac{n(\lg n)^2}{2} + \frac{n\lg n}{2}$$

· for intial condition $T(1) = 1$

$$T(n) = n + n(\lg n)^2 - \frac{n(\lg n)^2}{2} + \frac{n\lg n}{2}$$

$$\boxed{T(n) = n + \frac{n(\lg n)^2}{2} + \frac{n\lg n}{2}}$$

big o notation:– $0 \le f(n) \le c_1 \cdot g(n)$

$$f(n) = n + \frac{n(\lg n)^2}{2} + \frac{n\lg n}{2}$$

$$g(n) = n \cdot (\lg n)^2$$

when we take as $n=2$ then

$f(n) = 2 + \frac{2(\log 2)^2}{2} + \frac{2\lg 2}{2}$

$f(n) = 4$

$$0 \le f(n) \le c_1 \cdot g(n)$$

then, $f(n) = O(g(n))$

$= O(n \cdot (\lg n)^2)$

where $n_0 = 2$ & $c_1 = 4$

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* Closed forms :-

1) Geometric series :-

for finite series -

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$

for infinite series -

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

$$\sum_{i=0}^{\infty} i \cdot x^i = \frac{x}{(1-x)^2}$$

2) Arithmetic Series :-

$$\sum_{i=1}^{n} [a+(i-1)d] = na + \frac{n(n-1)}{2} d$$

3) Harmonic series :-

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$$

as $n=2$

$$= \sum_{i=1}^{n} 1/i \approx \ln \cdot x$$

$)^2 + 2 \lg 2$

2

4) Logarithms series :-

if $\log_b^a = c$ then $a = b^c$

- properties :-

i) $\log_b(x \cdot y) = \log_b^x + \log_b^y$

ii) $\log_b(x/y) = \log_b^x - \log_b^y$

iii) $\log_b x^a = a \cdot \log_b x$

iv) $\log_b a = \dfrac{\log_c a}{\log_c b}$

v) $a^{\log_b n} = n^{\log_b a}$

* floor & ceiling :-

  - floor of the real number x is denoted by $\lfloor x \rfloor$ as the largest integer $\leq x$

    for ex:- $\lfloor 2.4 \rfloor = 2$

  - ceiling of the x is denoted as $\lceil x \rceil$ is the smallest integer $\geq x$

    for ex:- $\lceil 2.4 \rceil = 3$

Q- Solve the given recurrence relation using substitution method -

$$T(n) = 4T(n/3) + n \qquad n>1 \quad -(A)$$
$$= 1 \qquad n=1$$

→ $T(n/3)$ by substitution of $n = n/3$ in eq (A) we get

$$T(n/3) = 4T(n/9) + n/3 \quad \text{- put in eq (A)}$$

then,

$$T(n) = 4[4T(n/9) + n/3] + n$$

$$T(n) = 16T(n/9) + \frac{4n}{3} + n$$

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$T(n) = 16\,T(n/9) + \frac{4n}{3} + n$

$T(n) = 4^2\,T(n/9) + \frac{4n}{3} + n$ — ①

put $n/9 = n$ in eq Ⓐ

$T(n/9) = 4\,T(n/27) + n/9$

put $T(n/9)$ in eq ① we get

$$T(n) = 4^2\left[4T(n/27) + n/9\right] + \frac{4n}{3} + n$$

$$T(n) = 4^3\,T(n/27) + \frac{4^2 n}{9} + \frac{4n}{3} + n$$

$$T(n) = 4^k\,T(n/3^k) + \cancel{\frac{4^{k-1}n}{9}} + \sum_{i=0}^{k-1} \frac{4^i n}{3^i}$$

$$\boxed{T(n) = 4^k\,T(n/3^k) + \sum_{i=0}^{k-1} n\left(\frac{4}{3}\right)^i}$$ — generalized term

$$\boxed{\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}} \Rightarrow \sum_{i=0}^{k-1}\left(\frac{4}{3}\right)^i = \frac{(4/3)^{k-1+1} - 1}{\frac{4}{3} - 1}$$

$$= 3\left(\frac{\left(\frac{4}{3}\right)^k - 1}{1}\right) = 3\left[(4/3)^k - 1\right]$$

$$T(n) = 4^k\,T\left(\frac{n}{3^k}\right) + n\,3\left(\left(\frac{4}{3}\right)^k - 1\right)$$

$$= 4^k\,T\left(\frac{n}{3^k}\right) + 3\cdot n\left(\frac{4}{3}\right)^k - 3n$$

when $\log_2$ the $n_0 = 2$ then calculation is easy

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- the terminating condition is -

$\frac{n}{3^k} = 1$ or $n = 3^k$ or $k = \log_3 n$

$T(n) = 4^{\log_3 n}\, T(1) + 3n\left(\frac{4}{3}\right)^{\log_3 n} - 3n$

$= n^{\log_3 4} + 3n\left[\frac{4^{\log_3 n}}{3^{\log_3 n}}\right] - 3n$

$= n^{1.26} + 3n\left[\frac{n^{\log_3 4}}{n^{\log_3 3}}\right] - 3n$

$= n^{1.26} + 3\cancel{n}\left[\frac{n^{1.26}}{\cancel{n}}\right] - 3n$

$= n^{1.26} + 3n^{1.26} - 3n$

$\boxed{T(n) = 4n^{1.26} - 3n}$

big-o-notation: $0 \le f(n) \le c\,g(n)$ $\qquad f(n) = 4n^{1.26} - 3n$

then $g(n) = n^{1.26}$

$0 \le 4n^{1.26} - 3n \le c_1 n^{1.26}$

$n = 1 \qquad c_1 = 4$

$0 \le 4(1)^{1.26} - 3(1) \le 4(1)^{1.26}$

$0 \le 1 \le 4$

$0 \le 4(2)^{1.26} - 3(2) \le 4(2)^{1.26}$

$0 \le 4 * 2.39 - 6 \le 4 * 2.39$

$0 \le 9.56 - 6 \le 9.56$

$0 \le 3.56 \le 9.56$

n	c_1	$f(n)$	$c_1 g(n)$
1	4	1	4
2	4	3.56	9.56

$f(n) = O(n^{1.26})$

where $n_0 = 2$ & $c_1 = \underline{3}$ or $\underline{2}$ or 4

$n_0 = 1$ & $c_1 = \underline{4}$ or 2

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* Changing variable method :—

Q — Solve the following function by substituting values of $n = 2^m$

$$T(n) = 2T(\sqrt{n}) + \lg n \quad n > 2 \quad \text{— (A)}$$
$$= 1 \quad n = 1$$

→ Let $n = 2^m$ or $m = \lg n$

by putting value in equation (A)

$$T(2^m) = 2T(2^{m/2}) + m \quad \text{— (B)}$$

Consider $S(m) = T(2^m)$

So equation B will be :—

$$S(m) = \begin{cases} 2S(m/2) + m & m > 1 \\ 1 & m = 1 \end{cases} \quad \text{— (C)}$$

→ $S(m) = 2S(m/2) + m$ — (C)

by substitution method :—

$S(m/2) = 2S(m/4) + \frac{m}{2}$ — put these value in eq (C)

$$S(m) = 2[2S(m/4) + m/2] + m$$
$$S(m) = 4S(m/4) + m + m$$
$$\boxed{S(m) = 4S(m/4) + 2m} \quad \text{— (i)}$$

$S(m/4) = 2 \cdot S(m/8) + m/4$ — put these value in eq (i)

$$S(m) = 4[2S(m/8) + m/4] + 2m$$
$$\boxed{S(m) = 8S(m/8) + 3m} \quad \text{— (ii)}$$

Generalized equation will be —

$$\boxed{S(m) = 2^k S(m/2^k) + km} \quad \text{— (E)}$$

$\boxed{\lg n = \log_2^n}$

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Terminating condition is –

$S(m)$ $\frac{m}{2^k} = 1$ or $m = 2^k$ or $k = \lg m$

$S(m) = 2^k S(1) + km$

$= 2^k + km$

$\boxed{S(m) = m + \lg m \cdot m}$

big 'O' notation –

$S(m) = O(m \lg m)$

After changing value of m –

$T(n) = O(\lg n \cdot \lg \lg n)$ — $[m = \lg n]$

$0 \leq f(n) \leq c_1 (\lg n \cdot \lg \lg n)$

$n = 2$

$c_1 = 3$

$f(n) = \lg n + \lg n \cdot \lg \lg n$

$n = 4$ $c_1 = 2$

n	$f(n)$	$c_1 g(n)$
2	1	0
3	2.83	3.183

n	$f(n)$	$c_1 g(n)$
4	4	4

$0 \leq \lg n + \lg n \cdot \lg \lg n \leq c_1 (\lg n \cdot \lg \lg n)$

$0 \leq \lg 2 + \lg 2 \cdot \lg \lg 2 \leq c_1 (\lg 2 \cdot \lg \lg 2)$

$0 \leq 1 \leq 0$

for $n = 4$ & $c_1 = 2$ —

$0 \leq f(n) \leq c_1 (\lg n \cdot \lg \lg n)$

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* Recursion Tree method -

In recursion tree each node represents ~~Solve the equation~~ the cost of a single subproblem and we sum the cost within each level of the tree to obtained a set of per level cost the general form of recursion tree is -

$$T(n) = a\,T(n/b) + f(n)$$

It is a complete tree with $n^{\log_b a}$ leaves & $\log_b n$ height

Q - Solve the equation using recurrsion tree method -

$$T(n) = 2T(n/2) + n$$

$$T(n) = a \cdot T(n/b) + f(n)$$

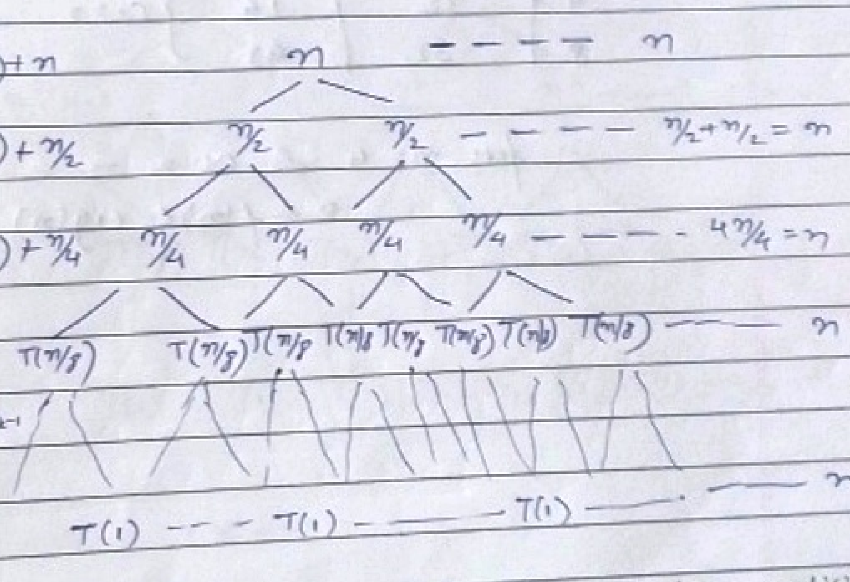
$\longrightarrow T(n) = 2T(n/2) + n$

$T(n) = 2T(n/2^2) + n/2$

$T(n) = 2T(n/2^3) + n/4$

$T\left(\frac{n}{2^{k-1}}\right) = 2T(n/2^k) + n/2^{k-1}$

$T(n/2^k) = 1$



$\{T(n/2^{k-1}) = 2T(n/2^k) + n/2^{k-1}$ - is before condition$\}$

The terminating condition is - $n/2^k = 1$ or $n = 2^k$ or $k = \lg n = \log_2 n$

Total cost $= \underset{[2^k]}{\text{Cost of leaf nodes}} + \text{Cost of internal nodes}$ $= lc + Ic$ - it is at each level that is i.e. kn

$$\text{Total cost} = 2^k + kn$$

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Total cost = $2^k + kn$ — (A)

by applying terminating condition in eq (A)

$$\sum_{i=1}^{k-1} i * n$$

$$n + \log_2 n \, n$$

$$f(n) = n + n \cdot \lg n$$

big o notation – $O[n \lg n]$

$$0 \le f(n) \le c_1 g(n)$$

$$0 \le n + n \cdot \lg n \le c_1 (n \lg n)$$

n	$f(n)$	$c_1 g(n)$	
1	1	0	$c_1 = 1$
2	4	2	
4	12	16	$c_1 = 2$
8	32	48	

$\therefore$ for $n = 4$ & $c_1 = 2$ –

$$0 \le f(n) \le c_1 g(n)$$

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Q- Solve the following using recursion tree method-

$$T(n) = 2T(n/2) + n^2$$

→ Geometric series $r < 1$

$$\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1}$$

$$a = 1, \quad r = \frac{1}{2}$$

$$\frac{a}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = 2 \quad \text{---} \quad (r<1)$$

<u>Solve-</u> $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

$T(n) = 2T(n/2) + n^2$ — n^2 — n^2

$T(n/2) = 2T(n/4) + \frac{n^2}{4}$ — $\frac{n^2}{4}$ $\frac{n^2}{4}$ — $\frac{2n^2}{4} = \frac{1}{2}n^2$

$T(n/4) = 2T(n/8) + \frac{n^2}{16}$ — $\frac{n^2}{16}$ $\frac{n^2}{16}$ $\frac{n^2}{16}$ $\frac{n^2}{16}$ — $\frac{1}{4}n^2$

$T(n/8) = 2T(n/16) + \frac{n^2}{64}$ — $T(n/8)$ $T(n/8)$ $T(n/8)$ $T(n/8)$ $T(n/8)$ $T(n/8)$ $T(n/8)$ $T(n/8)$ — $\frac{1}{8}n^2$

$T\left(\frac{n}{2^k}\right) = 2T\left(\frac{n}{2^{k}}\right) + \frac{n^2}{}$ — $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ — $\left(\frac{1}{2}\right)^{k-1} n^2$

The total cost = $L\cancel{C} + IC$

=

Terminating condition-

$$\frac{n}{2^k} = 1 \quad \text{or} \quad n = 2^k \quad \text{or} \quad k = \log_2 n$$

Total cost = $LC + IC$ (finite)

$$= 2^k + n^2\left[\left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \cdots\right]$$

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$= 2^k + n^2\left[\frac{1}{1-\frac{1}{2}}\right]$

$= 2^k + 2n^2$

$= 2^{\log_2 n} + 2n^2 = n^{\log_2 2} + 2n^2$

$= n + 2n^2 = \underbrace{O(n^2)}$ → complexity

$T(n/4)$

$T(n/16)$

$T\left(\frac{n}{64}\right)$

$T\left(\frac{n}{4^k}\right)$

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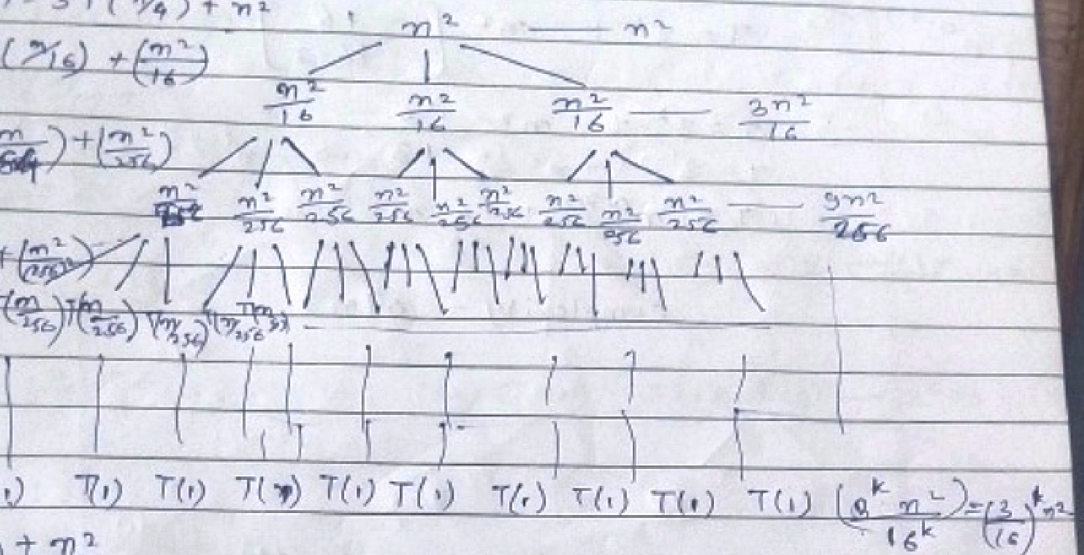
Q— $T(n) = 3T(n/4) + n^2$ Solve by regression tree method—

$\longrightarrow$ $T(n) = 3T(n/4) + n^2$

$T(n/4) = 3T(n/16) + \left(\frac{n^2}{16}\right)$

$T(n/16) = 3T\left(\frac{n}{64}\right) + \left(\frac{n^2}{256}\right)$

$T\left(\frac{n}{64}\right) = 3T\left(\frac{n}{256}\right) + \left(\frac{n^2}{256^2}\right)$



$T\left(\frac{n}{4^k}\right) = 3T\left(\frac{n}{4^k}\right) + n^2$

Terminating condition—

$\left(\frac{n}{4^k}\right) = 1 \qquad n = 4^k \qquad k = \log_4 n$

$LC = 3^k \qquad IC = \sum_{i=0}^{k-1} \left(\frac{3}{16}\right)^k$

(Infinite)

Total cost = LC + IC

$$\boxed{TC = 3^k + n^2\left[\sum_{i=0}^{k-1} \left(\frac{3}{16}\right)^i\right]}$$

$$= 3^{\log_4 n} + \cancel{\sum_{i=0}^{k-1} \left(\frac{3}{16}\right)^i}$$

$$= 3^k + n^2 \sum_{i=0}^{k-1} \left(\frac{3}{16}\right)^i$$

$$= 3^k + n^2 \left[\frac{1}{1-3/16}\right]$$

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$$T = 3^k + n^2 \left[\frac{1}{1 - \frac{3}{16}} \right]$$

$$\left(n^{0.79} + \frac{16}{13} n^2\right)$$

$$T = 3^k + n^2 \left[\frac{16}{16-3} \right]$$

$$= O(n^2)$$

$$T = 3^k + n^2 \frac{16}{13}$$

$$T = 3^{\log_4^n} + n^2 \frac{16}{13} \rightarrow n^{\log_4^3} + n^2 \left(\frac{16}{13}\right)$$

$$T = n^{0.79} + \frac{16}{13} n^2$$

complexity $- O(n^2)$

Q- Solve by recursion tree method-

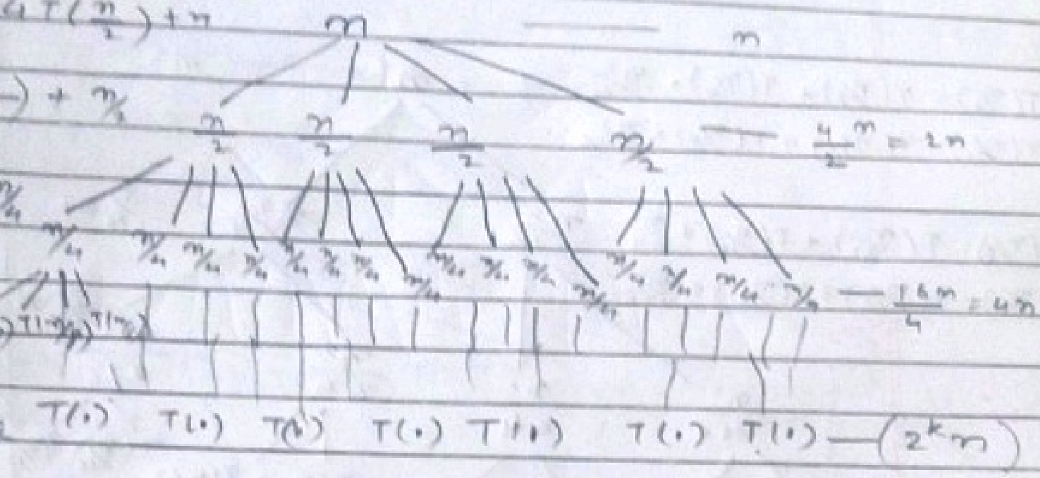
$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$\longrightarrow$ $T(n) = 4T\left(\frac{n}{2}\right) + n$

$T(n/2) = 4T\left(\frac{n}{4}\right) + n/2$

$T(n/4) = 4T\left(\frac{n}{8}\right) + n/4$

$T\left(\frac{n}{2^k}\right) = 4T\left(\frac{n}{2^k}\right) + \frac{n}{2^k}$



$LC = 4^k$

$IC = n\left[\sum_{i=0}^{k-1} 2^i\right]$

Terminating: $\frac{n}{2^k} = 1$ $\quad n = 2^k$ $\quad k = \log_2^n$

$TC = LC + IC$

$= 4^k + n\left[\sum_{i=0}^{k-1} 2^i\right]$ (finite series)

$\left\{\sum_{i=0}^{k-1} 2^i = \frac{2^{k-1+1} - 1}{2-1}\right\}$

$\left\{\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x-1}\right\}$

$TC = 4^k + n\left[\frac{2^{k-1+1} - 1}{2-1}\right]$

$TC = 4^k + n\left[\frac{2^k - 1}{}\right]$

$TC = 4^k + n[n-1]$

$TC = 4^k + n^2 - n$

$TC = 4^{\log_2^n} + n^2 - n$

$TC = n^{\log_2^4} + n^2 - n$

$TC = n^{\log_2^{2^2}} + n^2 - n \longrightarrow TC = n^2 + n^2 - n$

$TC = 2n^2 - n$

Complexity- $O(n^2)$

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Q — Solve the following using recursion tree method.

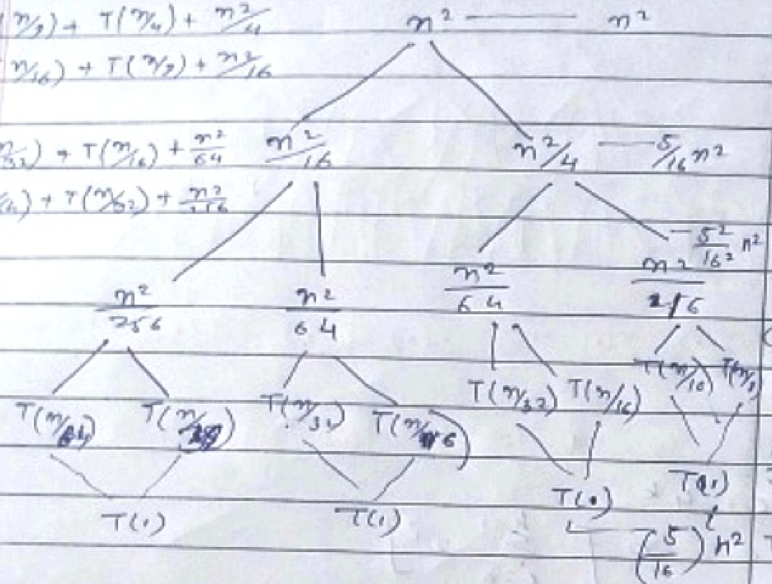
$\rightarrow$ $T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n^2$

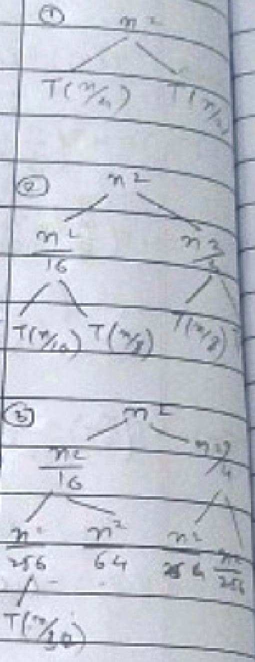
$T(n/2) = T(n/8) + T(n/4) + n^2/4$

$T(n/4) = T(n/16) + T(n/8) + n^2/16$

$T(n/8) = T(n/32) + T(n/16) + \frac{n^2}{64}$

$T(n/16) = T(n/64) + T(n/32) + \frac{n^2}{256}$





The terminating condition consider as —

$\left(\frac{n}{2^k}\right) = 1$

$n = 2^k \quad k = \log_2 n$

$LC = 2^k \qquad IC = n^2\left[\left(\frac{5}{16}\right)^0 + \left(\frac{5}{16}\right)^1 + \dots + \left(\frac{5}{16}\right)^{i-1}\right]$

T(n/4) + T(n/2)

n/4 = 0.2

1/2 = 0.5

1/4 < 1/2 —

∴ Terminating condition will be $\left(\frac{n}{2^k}\right)$

Total cost = LC + IC

$= 2^k + n^2\left[\frac{1}{1 - 5/16}\right]$

$= 2^k + n^2 \frac{16}{11}$

$= 2^{\log_2 n} + n^2 \frac{16}{11}$

$= n^{\log_2 2} + n^2 \frac{16}{11} = n + \frac{n^2\, 16}{11}$

Complexity — $O(n^2)$

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Q — Solve the following using recurrsion tree method —

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

$\frac{1}{3} = 0.33$

$\frac{2}{3} = 0.66$

→

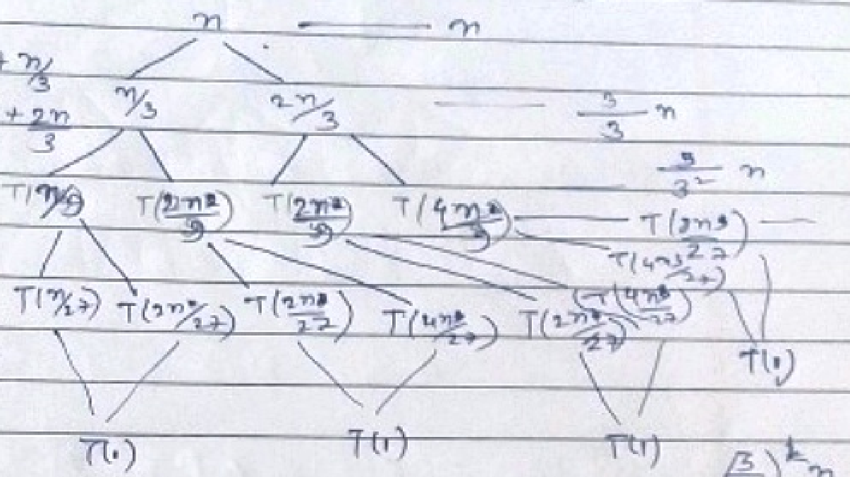
$T\left(\frac{n}{3}\right) = T\left(\frac{n}{9}\right) + T\left(\frac{2n^2}{9}\right) + \frac{n}{3}$

$T\left(\frac{2n}{3}\right) = T\left(\frac{2n^2}{9}\right) + T\left(\frac{4n^2}{9}\right) + \frac{2n}{3}$

$T\left(\frac{n}{9}\right) = T\left(\frac{n}{27}\right) + T\left(\frac{2n^2}{27}\right) + \frac{n}{9}$

$T\left(\frac{2n^2}{9}\right) = T\left(\frac{2n^3}{27}\right) + T\left(\frac{4n^3}{27}\right) + \frac{2n^2}{9}$

$T\left(\frac{4n^2}{9}\right) = T\left(\frac{4n^3}{27}\right) + T\left(\frac{8n^3}{27}\right) + \frac{4n^2}{9}$



Terminating condition —

$TC = \left(\frac{2^k n}{3^k}\right) = 1 \qquad \left(\frac{2}{3}\right)^k n = 1$

$n = \left(\frac{3}{2}\right)^k$

~~$2n = 3^k$~~

~~$n = \frac{3^k}{2}$~~ $\qquad \log_{3/2} n = k$

~~$k = \frac{1}{3} \log_{3/2} n$~~

$TC = \frac{3^k}{2} 2^k \qquad TC = n \cdot \left[1^0 + 1^1 + 1^2 \ldots + 1^{k-1}\right] k$

$TC = \frac{1}{2} 3^k + n \left[\frac{1}{1}\right] k$

$TC = 2^k + nk$

$= 2^{\log_{3/2} n} + n \log_{3/2} n$

$= n^{\log_{3/2} 2} + n \log_{3/2} n$

$= n^{1.70} + n \cdot \log_{3/2} n$

Complexity — $O(n \log_{3/2} n)$

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Case1

Case2 :

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* Master method or theorem

The master method provides a general method for solving the recurrence of the form -

$$T(n) = aT(n/b) + f(n) \quad n > 1$$
$$= \Theta(1) \quad n = 1$$

where, $a \geq 1$, $b \geq 1$ are constants and $f(n)$ is positive function

if the conditions of master method are satisfied then the solution of the recurrence can be found quickly the recurrence relation gives the running time of an algorithm that divides a problem of size n into a subproblem each of size n/b were 'a' & 'b' are positive constants each sub problem recursively in time $T(n/b)$. The cost of dividing the subproblem and combining the result of subproblem is given by function $f(n)$.

The solution to recurrence is obtain as follows :-

Case 1 :- If $f(n) = O(n^{\log_b^a - \epsilon})$ for some constant $\epsilon > 0$ then $T(n) = \Theta(n^{\log_b^a})$

Case 2 :- If $f(n) = \Theta(n^{\log_b^a})$ then $T(n) = \Theta(n^{\log_b^a} \cdot \log_b^n)$

Case 3 :- If $f(n) = \Omega(n^{\log_b^a + \epsilon})$ for some constant $\epsilon > 0$ & if $f(n/b) \leq c f(n)$ for some constant $c < 1$ & all sufficiently large n then. $T(n) = \Theta(f(n))$

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Q- Solve the following recurrences using master theorem -

$T(n) = 9T(n/3) + n$

Consider Case-1]

If $f(n) = O(n^{\log_b^a - \epsilon})$ for some constant $\epsilon > 0$

then $T(n) = \Theta(n^{\log_b^a})$

$T(n) = 9T(n/3) + n$

$a = 9, \quad b = 3, \quad f(n) = n$ — ①

$\therefore f(n) = O(n^{\log_b^a - \epsilon})$

$= O(n^{\log_3^9 - \epsilon})$

$= O(n^{\log_3^{3^2} - \epsilon})$

$= O(n^{2 - \epsilon})$

put $\epsilon > 0$ i.e $\epsilon = 1$

$= O(n)$ — ②

As $f(n) = f(n)$ i.e $O(n)$ - [from 1 & 2]

$\therefore$ Case 1 satisfy -

Complexity will be —

$T(n) = \Theta(n^{\log_b^a})$

$T(n) = \Theta(n^{\log_3^9})$

$\boxed{T(n) = \Theta(n^2)}$

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Q - $T(n) = 2T(n/2) + n$ for $n > 1$
for $n = 1$

for find the complexity using master method -

→ Case 2) If $f(n) = \Theta(n^{\log_b a})$ then
$T(n) = \Theta(n^{\log_b a} \cdot \log_b n)$

$\therefore$ $a = 2$ $b = 2$ $f(n) = n$

$\therefore$ $\Theta(n^{\log_2 2}) = \Theta(n)$

As $f(n) = n$ and $\Theta(n)$

$\therefore$ case-2 satisfy the equation. hence, complexity will be -

$T(n) = \Theta(n^{\log_b a} \cdot \log_b n)$

$\boxed{T(n) = \Theta(n \cdot \log_2 n)}$

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* Homogeneous recurrences –
e.g. $f(n) = f(n-1) + f(n-2)$
$f(n) - f(n-1) - f(n-2) = 0$ —— ①

Que –

– it is in the form of $a_0 f(n) + a_1 f(n-1) + a_2 f(n-2) + \cdots + a_k f(n-k) = 0$ →
the characteristic polynomial
$= a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} \ldots = 0$ —— eq (1)

Step-3 find the root of given polynomial equation as
$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Step-4 Use initial condition to calculate 'c_1' and 'c_2'
- Case-I] If roots are real and distinct, the equation is – $f(n) = c_1 r_1^n + c_2 r_2^n$
- Case-II] If roots are real and equal, the equation is – $f(n) = (c_1 + c_2 n) r^n$
- Case-III] ~~If roots are real and~~ for 3 roots – $f(n) = c_1 r_1^n + c_2 r_2^n + (c_3 n) r_3^n$

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Que - Solve the recurrence for fibonacci series for the homogeneous recurrence -

$$t_n = t_{n-1} + t_{n-2} \qquad n > 1$$
$$= n \qquad \text{if } n=0 \text{ or } n=1$$

$t_n - t_{n-1} - t_{n-2} = 0$

- 1) find characteristic equation -
$t_n - t_{n-1} - t_{n-2} = 0$
- 2) find degree 2 and write degree equation -
$a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2}$
- 3) Write polynomial equation
$x^2 - x - 1 = 0$
- 4) find the root of obtain polynomial -

$r_1 = \dfrac{-b + \sqrt{b^2-4ac}}{2a}$ $\qquad r_2 = \dfrac{-b - \sqrt{b^2-4ac}}{2a}$

$r_1 = \dfrac{1+\sqrt{5}}{2}$ $\qquad r_2 = \dfrac{1-\sqrt{5}}{2}$

- roots are real & distinct here [case-1]

$\therefore f(n) = c_1 r_1^n + c_2 r_2^n$

$t_n = c_1 r_1^n + c_2 r_2^n$ $\qquad t_n = n$ if $n=0$ or $n=1$

$$t_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

if $n=0$

$\boxed{c_1 + c_2 = 0}$ - ①

$c_1 = -c_2$

$c_1 = -\left[\dfrac{-1}{\sqrt{5}}\right]$

$\boxed{c_1 = \dfrac{1}{\sqrt{5}}}$

if $n=1$

$c_1 \left(\dfrac{1+\sqrt{5}}{2}\right)^1 + c_2 \left(\dfrac{1-\sqrt{5}}{2}\right)^1 = 1$

$-c_2 \left(\dfrac{1+\sqrt{5}}{2}\right) + c_2 \left(\dfrac{1-\sqrt{5}}{2}\right) = 1$

$c_2 \left[\dfrac{-1-\sqrt{5}}{2} + \dfrac{1-\sqrt{5}}{2}\right] = 1$

$c_2 \left[\dfrac{\cancel{-1} - \sqrt{5} + \cancel{1} - \sqrt{5}}{2}\right] = 1$

$c_2 \left[-\dfrac{\cancel{2}\sqrt{5}}{\cancel{2}}\right] = 1$ $\qquad \boxed{c_2 = \dfrac{-1}{\sqrt{5}}}$

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$\therefore t_n = c_1 r_1^n + c_2 r_2^n$

$$t_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$t_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Q— $t_n = 0$ if $n=0$

$t_n = 5$ if $n = 1$

$= 3t_{n-1} + 4t_{n-2}$ if $n > 1$

→ 1) find characteristic equation -

$t_n = 3t_{n-1} + 4t_{n-2}$

$t_n - 3t_{n-1} - 4t_{n-2} = 0$ — ①

2) find degree of equation and write in form of degree

degree = 2

$\cancel{x^2 - 3x - 4 = 0}$ $a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} = 0$

3) find the polynomial equation.

$x^2 - 3x - 4 = 0$

4) find the root of polynomial eqn -

$x^2 - 3x - 4 = 0$

$x^2 - 4x + x - 4 = 0$

$x(x-4) + 1(x-4) = 0$

$(x-4)(x+1) = 0$

$x = 4 \quad x = -1$

$\boxed{r_1 = 4}$ $\boxed{r_2 = -1}$

5) root are real and distinct.

$f(n) = c_1 r_1^n + c_2 r_2^n$

$t(n) = c_1 (4)^n + c_2(-1)^n$

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if $t_n = 0$ — if $n = 0$

$\therefore \quad 0 = c_1 (4)^0 + c_2 (-1)^0$

$0 = c_1 + c_2$

$\boxed{c_1 = -c_2}$

if $t_n = 5$ — if $n = 1$

$5 = c_1 (4)^1 + c_2 (-1)^1$

$5 = c_1 4 + c_2 (-1)$

$5 = -c_2 4 + c_2 (-1)$

$5 = -4c_2 - c_2$

$5 = -5c_2$

$\boxed{c_2 = -1}$

$\boxed{\therefore \; c_1 = 1}$

$\therefore \; t_n = c_1 (4)^n + c_2 (-1)^n$

$t_n = (4)^n + (-1)(-1)^n$

$\boxed{t_n = (4)^n - (-1)^n}$

Q- $t_n = n$ if $n = 0, n = 1, n = 2$

$= 5t_{n-1} - 8t_{n-2} + 4t_{n-3}$ otherwise

→

- 1) find characteristic equation →

$t_n = 5t_{n-1} - 8t_{n-2} + 4t_{n-3}$

$t_n - 5t_{n-1} + 8t_{n-2} - 4t_{n-3} = 0$ — ①

- 2) find degree of eq and write in form of degree eq -

degree = 3

$a_0 x^k + a_2 x^{k-1} + a_3 x^{k-2} + a_4 x^{k-3} = 0$

- 3) find the polynomial eq —

$x^3 - 5x^2 + 8x - 4 = 0$

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— 4) find the root of the eqn -

$$x^3 - 5x^2 + 8x - 4 = 0$$
$$x(x^2 - 5x + 8) - 4 = 0$$
$$x(x^2 - 5x + 8) = 4$$
$$x^2 - 5x + 8 = \frac{4}{x}$$

$x \neq 3$	$x_1 = 1$	$x = 2$
$(3)^3 - 5(3)^2 + 8(3) - 4 = 0$	$(1)^3 - 5(1) + 8(1) - 4 = 0$	$8 - 5(4) + 16 - 4 = 0$
$27 - 5(9) + 24 - 4 = 0$	$1 - 5 + 8 - 4 = 0$	$8 - 20 + 16 - 4 = 0$
$27 - 45 + 20 = 0$	$-4 + 8 - 4 = 0$	$8 - 4 - 4 = 0$
$47 - 45 = 0$	$0 = 0$	$0 = 0$

$\boxed{r_1 = 1}$ $\boxed{r_2 = 2}$ $\boxed{r_3 = 2}$

~~eqn :- tn =~~

root are real and equal -

$$t_n = (C_1 + C_2 n) r^n$$

$$t_n = C_1(1)^n + (C_2 + n(C_3)) 2^n$$

if $n = 0$ — $t_n = n$

$$n = C_1(1)^0 + (C_2 + 0(C_3)) 2^{(0)}$$
$$n = C_1 + C_2$$
$$\boxed{C_1 = -C_2}$$

if $n = 1$ — $t_n = n$

$$1 = C_1(1)^1 + (C_2 + C_3) 2^{(1)}$$
$$1 = C_1 + (C_2 + C_3) 2$$
$$1 = C_1 + 2C_2 + 2C_3$$
$$1 = -C_2 + 2C_2 + 2C_3$$
$$\boxed{1 = C_2 + 2C_3}$$
$$C_2 = 1 - 2C_3$$

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if $n = 2$ — $t_n = n$

$2 = c_1(1)^2 + (c_2 + 2c_3)(2)^2$ $\boxed{c_3 = -\frac{1}{2}}$

$2 = c_1 + (c_2 + 2c_3)\,4$ $c_2 = 1 - 2c_3 \Rightarrow 1 - 2(-\frac{1}{2})$

$2 = c_1 + 4c_2 + 8c_3$ $\boxed{c_2 = 2}$

$2 = -c_2 + 4 \cdot c_2 + 8c_3$ $c_1 = -c_2$

$2 = 3c_2 + 8c_3$ $\therefore \boxed{c_1 = -2}$

$2 = 3(1 - 2c_3) + 8c_3$

$2 = 3 - 6c_3 + 8c_3$ $\therefore t_n = (-2)(1)^n + (2 + n(-\frac{1}{2}))2^n$

$2 = 3 + 2c_3$

$-1 = 2c_3$

* Non homogeneous recurrences:-

Non-homogeneous recurrences can be expressed as-

$a_0t_n + a_1t_{n-1} + a_2t_{n-2} + \cdots + a_kt_{n-k} = b^n P(n)$

i.e

Right hand side of recurrence is not equal to zero-

i.e [RHS $\neq 0$]

To find roots of recurrence -

i) LHS root can be computed using quadratic equation.

ii) RHS root always be $(x-b)$ depending upon the degree of $P(n)$

iii) if degree equals to 0 then roots are $(x-b)$

if degree equals to 1 then roots are $(x-b)(x-b)$

if degree equals to k then roots are $(k+1)$ times $(x-b)$

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Q- Solve the following recurrence.

$t_n = 1 \qquad n=0$

$t_n = 4t_{n-1} + 2^n$ otherwise

→ ~~$t_n - 4t_{n-1} = 2^n$~~

$t_n - 4t_{n-1} = 2^n$

$\boxed{x - 4 = 2^n}$

Solve L.H.S

$x - 4 = 0$

$x = 4$

$\boxed{r = 4}$

Solve R.H.S

$(x-2) = 0$

$\boxed{x = 2}$

$\boxed{r_2 = 2}$

roots are real and distinct.

$t_n = c_1 r_1{}^n + c_2 r_2{}^n$

$t_n = c_1(4)^n + c_2(2)^n$

→ $t_n = 1 \quad n = 0$

$1 = c_1(4)^n + c_2(2)^n$

$1 = c_1(4)^0 + c_2(2)^0$

$1 = c_1 + c_2$

$c_1 + c_2 = 1 \qquad c_1 = 1 - c_2$

→ $n = 1 \qquad t_1 = (4)^n c_1 + c_2(2)^n$

$t_1 = 4t_0 + 2$

$t_1 = 4(1) + 2$

$\boxed{t_1 = 6}$

$t_1 = (4^n) c_1 + c_2(2)^n$

$6 = (4) c_1 + c_2(2)$

$6 = 4c_1 + 2c_2$

$6 = 4(1 - c_2) + 2c_2$

$6 = 4 - 4c_2 + 2c_2$

$6 = 4 - 2c_2$

$2 = -2c_2$

$\boxed{c_2 = -1} \qquad c_1 = 1 + 1 = 2 \; \boxed{c_1 = 2}$

$$\boxed{t_n = 2(4)^n + (-1)(2)^n}$$

a – $t_n = 3 \quad n = 0$

$t_n = 2t_{n-1} + 2^n + 5$ otherwise

if the constant is given than degree = 2.

$\rightarrow \quad t_n - 2t_{n-1} - 2^n - 5 = 0$

$t_n - 2t_{n-1} - 5 = 2^n$

$\therefore \; x^2 - 2x - 5 = 2^n$

L.H.S –

$x^2 - 2x - 5$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$r = \dfrac{2 \pm \sqrt{4 - 4(1)(-5)}}{2}$

$r = \dfrac{2 \pm \sqrt{4 + 20}}{2}$

$r = \dfrac{2 \pm \sqrt{24}}{2}$

$r = \dfrac{2[1 \pm \sqrt{6}]}{2}$

$r = 1 \pm \sqrt{6}$

$\boxed{r_1 = 1 + \sqrt{6}}$

$\boxed{r_2 = 1 - \sqrt{6}}$

R.H.S –

$(x - 2) = 0$

$x = 2$

$\boxed{r_3 = 2}$

– roots are real and distinct.

$\therefore t_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$

$t_n = c_1(1+\sqrt{6})^n + c_2(1-\sqrt{6})^n + c_3(2)^n$

$* \; t_n = 3 \quad n = 0$

$3 = c_1 (1+\sqrt{6})^0 + c_2(1-\sqrt{6})^0 + c_3(2)^0$

$3 = c_1 + c_2 + c_3$

$t_1 = 2t_0 + 2^1 + 5$

$t_1 = 2(3) + 2 + 5$

$t_1 = 6 + 2 + 5$

$\boxed{t_1 = 13} \longrightarrow n = 1$

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* $\underline{t_1 = 13}$ $\underline{n=1}$

$\therefore 13 = c_1(1+\sqrt{6})^1 + c_2(1-\sqrt{6})^1 + c_3(2)^1$

$13 = c_1(1+\sqrt{6}) + c_2(1-\sqrt{6}) + 2c_3$

* $\underline{t_2 = 35}$ $\underline{n=2}$

$t_2 = 2t_1 + 2^{(2)} + 5$

$t_2 = 2(13) + 4 + 5$

$t_2 = 26 + \cancel{20}\ 9$

$\boxed{t_2 = 35}$

$\therefore 35 = c_1(1+\sqrt{6})^2 + c_2(1-\sqrt{6})^2 + c_3(2)^2$

$35 = c_1(1+\sqrt{6})^2 + c_2(1-\sqrt{6})^2 + 4c_3$

solve the three simultaneous eq to get c_1, c_2, c_3 -

$\therefore c_1 + c_2 + c_3 = 3$

$c_1(1+\sqrt{6}) + c_2(1-\sqrt{6}) + 2c_3 = 13$

$c_1(1+\sqrt{6})^2 + c_2(1-\sqrt{6})^2 + 4c_3 = 35$

$c_1 = -4.2041$ $c_2 = -3.7959$ $c_3 = 11$

$t_n = (-4.2041)(1+\sqrt{6})^n + (-3.7959)(1-\sqrt{6})^n + 11(2)^n$

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Questions for Assignment - 1

1) Which are different algorithmic design principles?

2) Give the analysis framework of algorithm with examples.

3) Solve the following using substitution method :-

$$T(n) = 9T(n/3) + n^2 \quad n>1$$
$$= 1 \quad n=1$$

4) Solve the recurrence using changing variable method -

$$T(n) = 2T(n/2) + n(\lg n)^2 \quad n>1$$
$$= 1 \quad n=1$$

5) Solve the recurrence using recursion tree method -

$$T(n) = 4T(n/2) + n^3 \quad n>1$$
$$= 1 \quad n=1$$