Binomial Theorem
Lecture 1

- Introductions
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- Middle term in the Binomial Expansion

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Introductions

$2+\sqrt{3}$

$x+y$

$a+\frac{1}{a}$

$2+\frac{1}{3}$ ×

Binomial

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In general, we have to find terms of $(a+b)^n$, where n is a +ive integer

Use of P&C in Binomial Theorem

factorial $n \in W$

$\Rightarrow n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$\Rightarrow \boxed{n! = n\,(n-1)!} \rightarrow \dfrac{1}{(n-1)!} = \dfrac{n}{n!} \Rightarrow$ put $n=0 \Rightarrow \boxed{\dfrac{1}{(-1)!} = 0}$

ie if $n \in N$ $\boxed{\dfrac{1}{(-n)!} = 0}$

Put $n=1 \Rightarrow 1! = 1\times 0! \Rightarrow \boxed{0! = 1}$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, ...

Combination $\quad {}^{n}C_{r} = \dfrac{n!}{r!\times(n-r)!}$

$\rightarrow {}^{n}C_{0} = 1$, ${}^{n}C_{1} = n$, ${}^{n}C_{2} = \dfrac{n(n-1)}{2!}$

$\rightarrow {}^{n}C_{n} = 1$, ${}^{n}C_{n-1} = n$, - - -

Properties of ${}^nC_r = \dfrac{n!}{r! \times (n-r)!}$

* ${}^nC_r = {}^nC_{n-r}$ eg: ${}^9C_3 = {}^9C_6$, ${}^5C_4 = {}^5C_1$

* ${}^nC_p = {}^nC_q \Rightarrow p = q$ OR $p + q = n$

* ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ OR ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

eg: ${}^5C_2 + {}^5C_3 = {}^6C_3$

* $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r} \longrightarrow$ * ${}^nC_r = \dfrac{n}{r}\left({}^{n-1}C_{r-1}\right)$

Binomial Theorem for a positive integral index

- The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **Binomial Theorem.**
- If $x, y \in R$ and $n \in N$, then ;
- $(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 + \ldots\ldots + {}^nC_r\, x^{n-r} y^r + \ldots\ldots + {}^nC_n\, y^n$
-
- $= \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r.$
-

$(2x - y)^5 = {}^5C_0 (2x)^5 + {}^5C_1 (2x)^4 (-y)^1 + {}^5C_2 (2x)^3 (-y)^2 + {}^5C_3 (2x)^2 (-y)^3 + {}^5C_4 (2x)(-y)^4 + {}^5C_5 (-y)^5$

Observations

- The number of terms in the expansion is $(n + 1)$ i.e. one or more than the index.
- The sum of the indices of x & y in each term is n.
- The binomial coefficients of the terms ${}^nC_0$, ${}^nC_1$ .... **equidistant from the beginning and the end are equal.**

$(x+y)^n \longrightarrow$ Number of terms $= \underline{n+1}$

$\left(2x-\frac{1}{x}\right)^{14} \longrightarrow$ Number of terms $=15$

Some Particular cases & Examples

$$(x+y)^n+(x-y)^n=2\{{}^nC_0x^n+{}^nC_2x^{n-2}y^2+\ldots\}$$

$$(x+y)^n = {}^nC_0x^n+{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2+\ldots$$

$$(x+y)^n-(x-y)^n=2\{{}^nC_1x^{n-1}y+{}^nC_3x^{n-3}y^3+\ldots\}$$

$$(x-y)^n = {}^nC_0x^n-{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2-\ldots$$

$(x+y)^{100}+(x-y)^{100}$

No. of terms $=51$

- **Problems:**
- Find the Integral part of $(\sqrt{2}+1)^6$
- Using Binomial theorem, show that 9^n-8n-1 is always divisible by 64

$$\left[(\sqrt{2}+1)^6\right] = ?$$

Now, $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2\left\{{}^6C_0(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 + {}^6C_4(\sqrt{2})^2 + {}^6C_6(\sqrt{2})^0\right\}$

$$= 2\left\{1\times 8 + 15\times 4 + 15\times 2 + 1\times 1\right\}$$

$$= 2\times 99 = 198$$

Now, as $0 < \sqrt{2}-1 < 1$

$\Rightarrow 0 < (\sqrt{2}-1)^6 < 1$

ie $(\sqrt{2}+1)^6 = 198 - \underbrace{(\sqrt{2}-1)^6}_{(0\ \text{to}\ 1)}$

$\Rightarrow \left[(\sqrt{2}+1)^6\right] = 197$

$9^n - 8n - 1 \,||\, 64$

As, $9^n - 8n - 1$

$= (1+8)^n - 8n - 1$

$= \{ {}^nC_0 + {}^nC_1 8 + {}^nC_2 8^2 + \cdots \} - 8n - 1$

$= \{ \cancel{1} + \cancel{n(8)} + {}^nC_2 64 + \cdots \} - \cancel{8n} - \cancel{1}$

$= 64 \{ {}^nC_2 + {}^nC_3 8 + {}^nC_4 8^2 + \cdots \}$ is divisible by 64

$${}^nC_r = \frac{n!}{r! \times (n-r)!}$$

$${}^9C_2 = {}^9C_7 = \frac{9\times 8}{2} = 36$$

$${}^{10}C_3 = \frac{10 \times \cancel{9}^3 \times \cancel{8}^4}{\cancel{3} \times \cancel{2} \times 1} = 120$$

$${}^2C_3 = 0 = \frac{2!}{3!(-1)!}$$

Important Terms In The Binomial Expansion:

- General term
- Term Independent of x
- Middle Term
- Numerically Greatest Term

General term & Examples

- The general term OR $(r + 1)^{th}$ term in the expansion of $(x + y)^n$ is given by:

  $T_{r+1} = {}^nC_r \, x^{n-r} \, . \, y^r$

$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots$

$t_1 \quad t_2 \quad t_3$

Problems:

$\left(x - \frac{1}{x}\right)^8$

$t_4 = {}^8C_3 \, x^5 \left(-\frac{1}{x}\right)^3$

$= \boxed{-56x^2}$

$\left\{ t_{r+1} = {}^nC_r \, x^{n-r} \, y^r \rightarrow (r+1)\text{th term from beginning} \right.$

Problems

- Find the coefficients of :
-
- (i) x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ (ii) x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$
-
- (iii) Find the relation between a & b, so that these coefficients are equal.

(i) $t_{r+1} = {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r$

$= {}^{11}C_r a^{11-r} \left(\frac{1}{b}\right)^r x^{22-3r}$

for x^7, we have $22-3r=7$

$\Rightarrow \boxed{r=5}$

So, coeff of $x^7 = {}^{11}C_5 a^6 b^{-5}$

$x^{-7} \longrightarrow \left(ax - \frac{1}{bx^2}\right)^{11}$

$\Downarrow$

$$t_{r+1} = {}^{11}C_r (ax)^{11-r} \left(-\frac{1}{bx^2}\right)^r$$

$$= {}^{11}C_r\, a^{11-r} \left(-\frac{1}{b}\right)^r x^{11-3r}$$

for, coeff of x^{-7}, we have

$$11 - 3r = -7 \Rightarrow \boxed{r = 6}$$

So, coeff of $x^{-7} = \underline{{}^{11}C_6\, a^5 \left(\frac{1}{b}\right)^6}$

(iii) as coefficients are equal

$$\Rightarrow \cancel{{}^{11}C_5}\, a^6\, b^{-5} = \cancel{{}^{11}C_6}\, a^5\, b^{-6}$$

$$\Rightarrow \boxed{a\cdot b = 1}$$

Problems

- Find the term independent of x in the
- expansion of $\left[\frac{1}{2}x^{1/3} + x^{-1/5}\right]^8$

x^0

$$t_{r+1} = {}^8C_r\left(\frac{1}{2}x^{1/3}\right)^{8-r}\left(x^{-1/5}\right)^r$$

$$= {}^8C_r\left(\frac{1}{2}\right)^{8-r} x^{\frac{8-r}{3}-\frac{r}{5}}$$

for x^0, we have $\frac{8-r}{3}-\frac{r}{5}=0$

$$\Rightarrow 40-5r-3r=0 \Rightarrow \boxed{r=5}$$

i.e. term independent of x

$$= {}^8C_5\left(\frac{1}{2}\right)^3$$

$$= 56\cdot\frac{1}{2\times2\times2} = 7$$

Middle term & Examples

- The middle term(s) is the expansion of $(x + y)^n$ is (are):
- **(a) If n is even , there is only one middle term which is given by ;**

$r = \frac{n}{2}$

- $$T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$$
- **(b) If n is odd , there are two middle terms which are:**
- $$T_{(n+1)/2} \quad \& \quad T_{[(n+1)/2]+1}$$

$r = \frac{n-1}{2}$ $\qquad$ $r = \frac{n+1}{2}$

$$\left(\frac{a}{x}+\frac{x}{a}\right)^9$$

Middle terms →

As there are 10 terms

$\Rightarrow$ middle terms are t_5 & t_6

where $t_5 = {}^9C_4\left(\frac{a}{x}\right)^5\left(\frac{x}{a}\right)^4 = {}^9C_4\left(\frac{a}{x}\right) = \frac{9\times\cancel{8}^{2}\times7\times\cancel{6}}{\cancel{4}\times\cancel{3}\times\cancel{2}}\left(\frac{a}{x}\right) = 126\left(\frac{a}{x}\right)$

& $t_6 = {}^9C_5\left(\frac{a}{x}\right)^4\left(\frac{x}{a}\right)^5$

$= {}^9C_4\left(\frac{x}{a}\right)$

$= 126\left(\frac{x}{a}\right)$

Conceptual Problems

- In the binomial $\left(2^{\frac{1}{3}}+3^{-\frac{1}{3}}\right)^{n}$, if the ratio of the seventh term from beginning of the expansion to the seventh term from its end is 1/6, then n is equal to-

t_7 from beginning $= {}^{n}C_{6}\left(2^{1/3}\right)^{n-6}\left(3^{-1/3}\right)^{6}$

t_7 from end in $\left(2^{1/3}+3^{-1/3}\right)^{n} = t_7$ from beginning in $\left(3^{-1/3}+2^{1/3}\right)^{n}$

$$\frac{\cancel{{}^{n}C_{6}}\left(2^{1/3}\right)^{n-6}\left(3^{-1/3}\right)^{6}}{\cancel{{}^{n}C_{6}}\left(3^{-1/3}\right)^{n-6}\left(2^{1/3}\right)^{6}} = \frac{1}{6}$$

find n

$$\Rightarrow \left(2^{1/3}\right)^{n-12}\left(3^{-1/3}\right)^{6-n+6} = \frac{1}{6}$$

$$\Rightarrow 2^{\frac{n-12}{3}}\, 3^{\frac{n-12}{3}} = 6^{-1}$$

$$\Rightarrow (6)^{\frac{n-12}{3}} = 6^{-1} \Rightarrow \frac{n-12}{3} = -1 \Rightarrow \boxed{n = 9}$$