Computer Project

(2017-2019)

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"Writing code a computer can understand is science. Writing code other programmers can understand is an art."

— **Jason Gorman**

"Curiosity begins as an act of tearing to pieces, or analysis."

— **Samuel Alexander**

Problem 9 Calculate the *square root* of a given positive number, using only *addition*, *subtraction*, *multiplication* and *division*.

Solution The problem of finding the *square root* of a positive real number k is equivalent to finding a positive root of the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$f(x) \;=\; x^2 - k$$

This problem can be solved using *Newton's method*. *Newton's method* is an iterative process for finding a root of a general function $f : \mathbb{R} \to \mathbb{R}$ by creating an initial guess, then improving upon it.

Let f' denote the derivative of the function f. Thus, the equation of the tangent to the curve $f(x)$, drawn through the point $(x_n, f(x_n))$ is given by the following equation.

$$y \;=\; f'(x_n)(x - x_n) + f(x_n)$$

The idea here is that the *x-intercept* of this tangent will be a better approximation to the root of the function f. Setting $y = 0$, solving for x and renaming it to x_{n+1} yields the following expression.

$$x_{n+1} \;=\; x_n \;-\; \frac{f(x_n)}{f'(x_n)}$$

Plugging in the required function for this problem, we have

$$x_{n+1} \;=\; x_n \;-\; \frac{x_n^2 - k}{2x_n}$$

Simplifying, we arrive at our expression for the term x_{n+1} in our iterative process.

$$x_{n+1} \;=\; \frac{1}{2}\left(x_n \;+\; \frac{k}{x_n}\right)$$

This is the sort of simple expression we have been looking for, involving only one addition and two multiplications per iteration. As n becomes very large, the term x_n approaches the *square root* of k.

"Objects are abstractions of processing. Threads are abstractions of schedule."

— **James O. Coplien**

Problem 10 Let a *fraction* here be restricted to the ratio of two integers, m and n, where $n \neq 0$. Thus, a fraction $\frac{m}{n}$ is said to be reduced its *lowest terms* when m and n are relatively prime.

Implement this model of *fractions*, such that they are *immutable* and reduced to their *lowest terms* by default. Also implement a simple method for adding two *fractions*.

Solution The problem of reducing a fraction $\frac{m}{n}$ to its lowest terms can be solved simply by dividing the numerator and the denominator by their *greatest common divisor*, i.e., $\gcd(m, n)$. This works as $\gcd(p, q) = 1$ if and only if p and q are relatively prime. Fraction addition can also be implemented using the following formula.

$$\frac{a}{b} + \frac{c}{d} \; = \; \frac{ad + bc}{bd}$$

The gcd of two integers can be calculated recursively using *Euclid's algorithm.*

$$\gcd(a, b) \; = \; \gcd(b, a \bmod b)$$

"Dividing one number by another is mere computation; knowing what to divide by what is mathematics."

— **Jordan Ellenberg**

Problem 11 A rational number q can be broken down into a *simple continued fraction* in the form given below.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

This may be represented by the abbreviated notation $[a_0; a_1, a_2, \ldots, a_n]$. For example, $[0; 1, 1, 2, 1, 4, 2]$ is shorthand for the following.

$$\frac{42}{73} = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}}}$$

Calculate the *simple continued fraction* expression for a given, positive fraction.

Solution We can thus solve this problem recursively by noting that the following holds.

$$\frac{p}{q} = \underbrace{\left\lfloor \frac{p}{q} \right\rfloor}_{\text{Integer part}} + \underbrace{\frac{p \bmod q}{q}}_{\text{Fractional part}}$$

Thus, by defining $f(\frac{p}{q})$ as the continued fraction representation of the fraction $\frac{p}{q}$, we can write

$$f\left(\frac{p}{q}\right) = \left\lfloor \frac{p}{q} \right\rfloor + \frac{1}{f\left(\frac{q}{p \bmod q}\right)}$$

"Intelligence is the ability to avoid doing work, yet getting the work done."

— **Linus Torvalds**

Problem 12 The *binomial coefficient* [9] of two integers $n \geq k \geq 0$ is defined as follows.

$$\binom{n}{k} \;=\; \frac{n!}{k!(n-k)!}$$

Here, $n!$ is the *factorial* of n, defined as follows.

$$n! \;=\; 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n$$

Compute the binomial coefficient for two given integers.

Solution

[9] They are given this name as they describe the coefficients of the expansion of powers of a binomial, according to the *binomial theorem*.

$$(x+y)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

— **John von Neumann**

Problem 13 Palindromes can be generated in many ways. One of them involves picking a number, reversing the order of its digits and adding the result to the original. For example, we have

$$135 + 531 = 666$$

Not all numbers will yield a palindrome after one step. Instead, we can repeat the above process, using the sum obtained as as the new number to reverse.

$$\begin{aligned} 963 + 369 &= 1332 \\ 1332 + 2331 &= 3663 \end{aligned}$$

This process is often called the *196-algorithm*. Some numbers seem never to yield a palindrome even after millions of iterations. These are called *Lychrel numbers*. The smallest of these in base 10 is conjectured to be the number 196, although none have been mathematically proven to exist.

Generate the steps and final palindrome of the *196-algorithm*, given a natural number as a *seed* [10].

[10] A *seed* is an initial number, from which subsequent numbers are generated.

"Over thinking leads to problems that doesn't even exist in the first place."

— **Jayson Engay**

Problem 14 Compute the *prime factorization* of a given natural number.

Solution This solution is meant to showcase the drawbacks of using *recursion* in some problems.

Let $f(n)$ denote the expansion of the *prime factorization* of the natural number n. We *could* observe that if we can find naturals p and q such that $n = pq$, we can write

$$f(pq) \;=\; f(p) + f(q)$$

Using this, we can wrap up the iteration over the naturals into a recursive function.

The problem with this approach is that for moderately large numbers, the number of nested calls grows rapidly. For large enough numbers, the default memory allocated for the *call stack* by the *Java Virtual Machine* falls woefully short. As a result, it becomes necessary to manually set the size of the *thread stack size* by passing the `-Xss<size>` option to the *JVM* during program execution.

This project was compiled with XƎLaTeX.

All files involved in the making of this project can be found at
https://github.com/sahasatvik/Computer-Project/tree/master/XI

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