Computer Project

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"Writing code a computer can understand is science. Writing code other programmers can understand is an art."

— **Jason Gorman**

"Curiosity begins as an act of tearing to pieces, or analysis."

— **Samuel Alexander**

Problem 9 Calculate the *square root* of a given positive number, using only *addition*, *subtraction*, *multiplication* and *division*.

Solution The problem of finding the *square root* of a positive real number k is equivalent to finding a positive root of the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$f(x) \;=\; x^2 - k$$

This problem can be solved using *Newton's method*. *Newton's method* is an iterative process for finding a root of a general function $f : \mathbb{R} \to \mathbb{R}$ by creating an initial guess, then improving upon it.

Let f' denote the derivative of the function f. Thus, the equation of the tangent to the curve $f(x)$, drawn through the point $(x_n, f(x_n))$ is given by the following equation.

$$y \;=\; f'(x_n)(x - x_n) + f(x_n)$$

The idea here is that the *x-intercept* of this tangent will be a better approximation to the root of the function f. Setting $y = 0$, solving for x and renaming it to x_{n+1} yields the following expression.

$$x_{n+1} \;=\; x_n \;-\; \frac{f(x_n)}{f'(x_n)}$$

Plugging in the required function for this problem, we have

$$x_{n+1} \;=\; x_n \;-\; \frac{x_n^2 - k}{2x_n}$$

Simplifying, we arrive at our expression for the term x_{n+1} in our iterative process.

$$x_{n+1} \;=\; \frac{1}{2}\left(x_n \;+\; \frac{k}{x_n}\right)$$

This is the sort of simple expression we have been looking for, involving only one addition and two multiplications per iteration. As n becomes very large, the term x_n approaches the *square root* of k.

"Objects are abstractions of processing. Threads are abstractions of schedule."

— **James O. Coplien**

Problem 10 Let a *fraction* here be restricted to the ratio of two integers, m and n, where $n \neq 0$. Thus, a fraction $\frac{m}{n}$ is said to be reduced its *lowest terms* when m and n are relatively prime.

Implement this model of *fractions*, such that they are *immutable* and reduced to their *lowest terms* by default. Also implement a simple method for adding two *fractions*.

Solution The problem of reducing a fraction $\frac{m}{n}$ to its lowest terms can be solved simply by dividing the numerator and the denominator by their *greatest common divisor*, i.e., $\gcd(m, n)$. This works as $\gcd(p, q) = 1$ if and only if p and q are relatively prime. Fraction addition can also be implemented using the following formula.

$$\frac{a}{b} + \frac{c}{d} \;=\; \frac{ad + bc}{bd}$$

The gcd of two integers can be calculated recursively using *Euclid's algorithm*.

$$\gcd(a, b) \;=\; \gcd(b, a \bmod b)$$

"Dividing one number by another is mere computation; knowing what to divide by what is mathematics."

— **Jordan Ellenberg**

Problem 11 A rational number q can be broken down into a *simple continued fraction* in the form given below.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

This may be represented by the abbreviated notation $[a_0; a_1, a_2, \ldots, a_n]$. For example, $[0; 1, 1, 2, 1, 4, 2]$ is shorthand for the following.

$$\frac{42}{73} = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}}}$$

Calculate the *simple continued fraction* expression for a given, positive fraction.

Solution We can thus solve this problem recursively by noting that the following holds.

$$\frac{p}{q} = \underbrace{\left\lfloor \frac{p}{q} \right\rfloor}_{\text{Integer part}} + \underbrace{\frac{p \bmod q}{q}}_{\text{Fractional part}}$$

Thus, by defining $f(\frac{p}{q})$ as the continued fraction representation of the fraction $\frac{p}{q}$, we can write

$$f\left(\frac{p}{q}\right) = \left\lfloor \frac{p}{q} \right\rfloor + \frac{1}{f\left(\frac{q}{p \bmod q}\right)}$$

"Intelligence is the ability to avoid doing work, yet getting the work done."

— **Linus Torvalds**

Problem 12 The *binomial coefficient*[9] of two integers $n \geq k \geq 0$ is defined as follows.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Here, $n!$ is the *factorial* of n, defined as follows.

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n$$

Compute the binomial coefficient for two given integers.

Solution

[9]They are given this name as they describe the coefficients of the expansion of powers of a binomial, according to the *binomial theorem*.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

— **John von Neumann**

Problem 13 Palindromes can be generated in many ways. One of them involves picking a number, reversing the order of its digits and adding the result to the original. For example, we have

$$135 + 531 = 666$$

Not all numbers will yield a palindrome after one step. Instead, we can repeat the above process, using the sum obtained as as the new number to reverse.

$$\begin{aligned} 963 + 369 &= 1332 \\ 1332 + 2331 &= 3663 \end{aligned}$$

This process is often called the *196-algorithm.* Some numbers seem never to yield a palindrome even after millions of iterations. These are called *Lychrel numbers.* The smallest of these in base 10 is conjectured to be the number 196, although none have been mathematically proven to exist.

Generate the steps and final palindrome of the *196-algorithm*, given a natural number as a *seed* [10].

[10] A *seed* is an initial number, from which subsequent numbers are generated.

"Over thinking leads to problems that doesn't even exist in the first place."

— **Jayson Engay**

Problem 14 Compute the *prime factorization* of a given natural number.

Solution This solution is meant to showcase the drawbacks of using *recursion* in some problems.

Let $f(n)$ denote the expansion of the *prime factorization* of the natural number n. We *could* observe that if we can find naturals p and q such that $n = pq$, we can write

$$f(pq) \ = \ f(p) + f(q)$$

Using this, we can wrap up the iteration over the naturals into a recursive function.

The problem with this approach is that for moderately large numbers, the number of nested calls grows rapidly. For large enough numbers, the default memory allocated for the *call stack* by the *Java Virtual Machine* falls woefully short. As a result, it becomes necessary to manually set the size of the *thread stack size* by passing the `-Xss<size>` option to the *JVM* during program execution.

*"Meaning lies as much
in the mind of the reader
as in the Haiku."*

— **Douglas Hofstadter**

Problem 15 A *codebook* is a document which stores a *lookup table* for coding and decoding text – each word has a different word, phrase or string to replace it. Design a system which, when given a *codebook* written in plaintext, translates a gievn sentence into its encoded form.

Solution

"Hofstadter's Law: It always takes longer than you expect, even when you take into account Hofstadter's Law."

— **Douglas Hofstadter**

Problem 16 Analyse the frequency of each letter in the english alphabet appearing in a given file. Store the results in a different file.

Solution All that has to be done here is reading the contents of a file, counting the occurences of each character, then tabulating the results before writing them to another file. Here, the characters have also been sorted based on their frequencies.

This project was compiled with XƎLaTeX.

All files involved in the making of this project can be found at
`https://github.com/sahasatvik/Computer-Project/tree/master/XI`

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