MA3201

Topology

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1 Introduction

1.1 Topological spaces

Definition 1.1. A topology on some set X is a family τ of subsets of X, satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from τ are in τ.
3. All finite intersections of elements from τ are in τ.

The sets from τ are declared to be open sets in the topological space (X, τ).

Example. Any set X admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

Example. Let X be a set. The cofinite topology on X is the collection of complements of finite sets, along with the empty set. Note that when X is finite, this is simply the discrete topology.

Definition 1.2. Let τ, τ' be two topologies on the set X. We say that τ is finer than τ' if τ has more open sets than τ'. In such a case, we also say that τ' is coarser than τ.

Definition 1.3. Let (X, τ) be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology τ such that every open set $U \in \tau$ is expressible as a union of elements from β.

Definition 1.4. Let X be a set, and let β be a collection of subsets of X satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $x \in B \subseteq B_1 \cap B_2$ such that $B \in \beta$.

Then, β generates a topology on X, namely the collection of all unions of elements of β.

1.2 Continuous maps

Definition 1.5. Let $f\colon X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in Y must be open in X.

Definition 1.6. Let $f\colon X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is a homeomorphism if f is continuous, f is invertible, and f^{-1} is continuous. We also say that X and Y are homeomorphic when such a homeomorphism between them exists.