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Equivalence of metric spaces

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We claim that the following metrics on $\mathbb{R}^n$ are equivalent, in that they induce the same topology on $\mathbb{R}^n$.

1. $d_1(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|$, the Euclidean metric.
2. $d_2(\boldsymbol{x}, \boldsymbol{y}) = \max_i\{|x_i - y_i|\}$, the Chebyshev metric.
3. $d_3(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|/(1 + |\boldsymbol{x} - \boldsymbol{y}|)$.

Label the metric spaces $M_i = (\mathbb{R}^n, d_i)$, and their respective collections of open sets τ_i. Denote $B_r^i(\boldsymbol{x})$ to be the open ball in M_i, centred at $\boldsymbol{x}$ with radius r, i.e. the collection of points $\boldsymbol{y} \in M_i$ such that $d_i(\boldsymbol{x}, \boldsymbol{y}) < r$.

($\tau_1 \subseteq \tau_2$) Consider an open ball $B_r^1(\boldsymbol{x}) \subseteq M_1$. Any point $\boldsymbol{y}$ in this ball satisfies $d_1(\boldsymbol{y}, \boldsymbol{x}) = |\boldsymbol{y} - \boldsymbol{x}| < r$. Let $d_1(\boldsymbol{y}, \boldsymbol{x}) = r - \epsilon$ for some $\epsilon > 0$, and set $\epsilon' = \epsilon/\sqrt{n}$. For all $\boldsymbol{z}$ in the open ball $B_{\epsilon'}^2(\boldsymbol{y})$, i.e. such that $d_2(\boldsymbol{z}, \boldsymbol{y}) < \epsilon'$, we have

$$d_1(\boldsymbol{z}, \boldsymbol{y})^2 = \sum_i (z_i - y_i)^2 \leq n \max\{|z_i - y_i|\}^2 = n d_2(\boldsymbol{z}, \boldsymbol{y})^2 < \epsilon^2.$$

The triangle inequality gives

$$d_1(\boldsymbol{z}, \boldsymbol{x}) \leq d_1(\boldsymbol{z}, \boldsymbol{y}) + d_1(\boldsymbol{y}, \boldsymbol{x}) < \epsilon + r - \epsilon = r,$$

hence $\boldsymbol{z} \in B_r^1(\boldsymbol{x})$.

Thus, the open ball $B_y := B_{\epsilon'}^2(\boldsymbol{y}) \subseteq M_2$ is contained within the open ball $B := B_r^1(\boldsymbol{x}) \subseteq M_1$. Take the union of B_y for all $\boldsymbol{y} \in B$, and note that this is precisely equal to B. This is because any element of B is the center of some B_y, and every element in the union is contained within some B_y, which in turn is contained within B. Hence, any open ball in M_1 is open in M_2. Since every open set in a metric space can be written as the union of open balls, we see that every open set in M_1 is an open set in M_2.

($\tau_2 \subseteq \tau_1$) Consider an open ball $B_r^2(\boldsymbol{x}) \subseteq M_2$. Any point $\boldsymbol{y}$ in this ball satisfies $d_2(\boldsymbol{y}, \boldsymbol{x}) = \max_i\{|y_i - x_i|\} < r$. Let $d_2(\boldsymbol{y}, \boldsymbol{x}) = r - \epsilon$ for some $\epsilon > 0$. For all $\boldsymbol{z}$ in the open ball $B_\epsilon^1(\boldsymbol{y})$, we have

$$d_1(\boldsymbol{z}, \boldsymbol{y})^2 = \sum_i (z_i - y_i)^2 < \epsilon^2,$$

hence $|z_i - y_i| < \epsilon$ for all i. Specifically,

$$d_2(\boldsymbol{z}, \boldsymbol{y}) = \max_i\{|z_i - y_i|\} < \epsilon.$$

The triangle inequality further gives

$$d_2(\boldsymbol{z}, \boldsymbol{x}) \leq d_2(\boldsymbol{z}, \boldsymbol{y}) + d_2(\boldsymbol{y}, \boldsymbol{x}) < \epsilon + r - \epsilon = r.$$

Thus, the open ball $B_y := B^1_\epsilon(\boldsymbol{y}) \subseteq M_1$ is contained within the open ball $B := B^2_r(\boldsymbol{x})$. Like before, the union of all such B_y for $\boldsymbol{y} \in B$ yields precisely B, so any open ball in M_2 is an open set in M_1. It follows that any open set in M_2 is an open set in M_1.

($\tau_1 = \tau_3$) Note that for any open ball $B^1_r(\boldsymbol{x}) \subseteq M_1$, any point $\boldsymbol{y}$ in this ball satisfies $d_1(\boldsymbol{y}, \boldsymbol{x}) < r$, which is equivalent to $d_3(\boldsymbol{y}, \boldsymbol{x}) = d_1(\boldsymbol{y}, \boldsymbol{x})/(1 + d_1(\boldsymbol{y}, \boldsymbol{x})) < r/(1+r) := r'$. Thus, $B^1_r(\boldsymbol{x}) = B^3_{r'}(\boldsymbol{x})$.

Similarly, for any open ball $B^3_r(\boldsymbol{x}) \subseteq M_3$, if $r \geq 1$ then $B^3_r(\boldsymbol{x}) = \mathbb{R}^n$ (every point $\boldsymbol{y} \in \mathbb{R}^n$ satisfies $d_3(\boldsymbol{y}, \boldsymbol{x}) < 1$ because $s/(1+s) < 1$ for all non-negative reals s). Otherwise, we can find $r' \geq 0$ such that $r'/(1+r') = r$, specifically choose $r' = r/(1-r)$. Again, this gives $B^3_r(\boldsymbol{x}) = B^1_{r'}(\boldsymbol{x})$.

Thus, every open ball in M_1 is an open ball in M_3, and vice versa. It follows that a set is open in M_1 if and only if it is open in M_3.

Note that we have not used any specific property of d_1 here, merely its relation with d_3. This means that more generally, the open sets in the metric space $(\mathbb{R}^n, d)$ are identical to those in the metric space $(\mathbb{R}^n, d')$, where $d'(x, y) = d(x, y)/(1 + d(x, y))$.

We have used that fact that for all real numbers,

$$0 \leq x < y \iff 0 \leq \frac{x}{1+x} \leq \frac{y}{1+y}.$$

This is equivalent to stating that the function $f\colon [0, \infty) \to [0, 1)$ defined by $x \mapsto x/(1+x)$ is strictly increasing, which is evident by

$$f(x) = \frac{x}{1+x} = \frac{x+1-1}{1+x} = 1 - \frac{1}{1+x}.$$

The map $x \mapsto 1/(1+x)$ is strictly decreasing, hence $x \mapsto 1 - 1/(1+x)$ is strictly increasing.

Together, we have $\tau_1 = \tau_2 = \tau_3$, which means that all three metrics induce the same topology on $\mathbb{R}^n$.