Term presentation
Problem 1

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MA2102: Linear Algebra I

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Problem statement

Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Preliminaries

An eigenvector of a matrix A is a vector $\boldsymbol{v}$ such that

$$A\boldsymbol{v} = \lambda \boldsymbol{v},$$

for some scalar λ, which is called the eigenvalue of the eigenvector $\boldsymbol{v}$.

The eigenvalues of a matrix A are precisely the roots of the characteristic polynomial

$$\det(A - \lambda I_n) = 0.$$

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Preliminaries

Let $P \in M_n(F)$ be a matrix whose columns are eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of A. Then, $AP = PD$, where D is the diagonal matrix of the corresponding eigenvalues.

$$\begin{aligned} AP &= \begin{bmatrix} A\boldsymbol{v}_1 & A\boldsymbol{v}_2 & \cdots & A\boldsymbol{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\boldsymbol{v}_1 & \lambda_2\boldsymbol{v}_2 & \cdots & \lambda_n\boldsymbol{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \end{aligned}$$

If the eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent, the matrix P is invertible. Then, we can write $P^{-1}AP = D$.

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Computing eigenvalues

We first write the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$.

$$\det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^3 + 2\cdot 8 - 3\cdot 4(1-\lambda)$$

$$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 + 16 - 12 + 12\lambda$$

$$= 5 + 9\lambda + 3\lambda^2 - \lambda^3.$$

By inspection, $p(5) = 0$, so 5 is a root of p. Synthetic division gives

$$p(\lambda) = (5-\lambda)(1 + 2\lambda + \lambda^2) = (5-\lambda)(1+\lambda)^2.$$

Thus, the eigenvalues of A are -1 and 5.

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Computing eigenvectors

We seek $\boldsymbol{v}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$, i.e. $(A - \lambda I_n)\boldsymbol{v} = \mathbf{0}$.

When $\lambda = 5$,

$$\begin{bmatrix} 1-5 & 2 & 2 \\ 2 & 1-5 & 2 \\ 2 & 2 & 1-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4v_1 + 2v_2 + 2v_3 \\ 2v_1 - 4v_2 + 2v_3 \\ 2v_1 + 2v_2 - 4v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This forces $v_1 = v_2 = v_3$. We choose $v_1 = v_2 = v_3 = 1$.

Computing eigenvectors

When $\lambda = -1$,

$$\begin{bmatrix} 1+1 & 2 & 2 \\ 2 & 1+1 & 2 \\ 2 & 2 & 1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 + 2v_2 + 2v_3 \\ 2v_1 + 2v_2 + 2v_3 \\ 2v_1 + 2v_2 + 2v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This only imposes $v_1 + v_2 + v_3 = 0$. The set of solutions $[-v_2 - v_3 \quad v_2 \quad v_3]^\top$ form a two dimensional subspace of $\mathbb{R}^3$. We choose two linearly independent vectors from this subspace by setting $v_2 = 0, v_3 = 1$ in the first case and $v_2 = 1, v_3 = 0$ in the second.

Thus, the eigenvalues and corresponding eigenvectors of A are as follows.

$$\lambda = 5, \qquad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\lambda = -1, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Diagonalization

We perform Gauss Jordan elimination on the matrix $P = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]$, whose columns are the eigenvectors of A.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right].$$

Diagonalization

$$
\begin{aligned}
P^{-1}AP &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 & 1 \\ 5 & 0 & -1 \\ 5 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$