MA3101 : Introduction to Graph Theory and Combinatorics

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Exercise 1 Show that $R(2, k) = k$ for all $k \geq 2$.

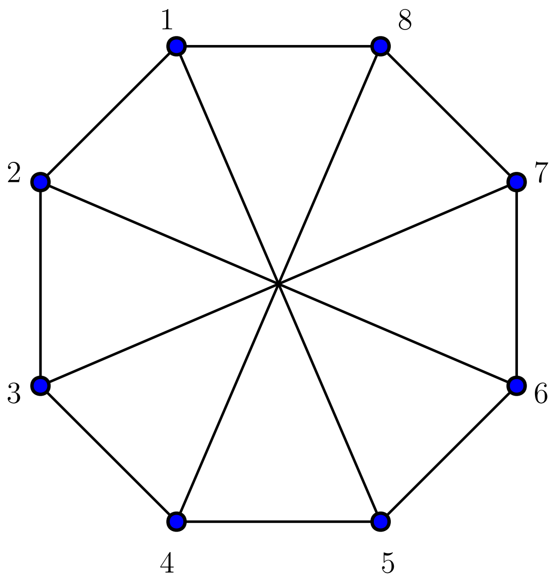
Solution Consider a colouring of the edges of $R(2, k)$ in red and blue. There are two possibilities: there exists a red edge $\{x, y\}$, or all edges are blue. In the former case, x, y form a red K_2; in the latter case, the entire graph is itself a blue K_k.

Exercise 2 Show that $R(3, 3, 3) \leq 17$.

Solution Use Exercise 5 and $R_2(3) = R(3, 3) = 6$ to write $R_3(3) \leq (6 - 1) \cdot 3 + 2 = 17$.

Exercise 3 Draw a regular graph on 8 vertices such that it does not have a K_3, and an independent set of cardinality 4.

Solution



Each vertex has degree 3. It is clear by inspection that this is triangle free (the symmetry of the graph makes this easy to see).

Suppose that this graph contains an independent set of size 4, and without loss of generality suppose that this set contains the vertex 1. This is directly connected to vertices $2, 5, 9$. Thus, the remaining three elements must be chosen from $\{3, 4, 7, 8\}$. However, at most one vertex can be chosen from each of the pairs $\{3, 7\}$ and $\{4, 8\}$ (they form edges), which makes our choice of an independent set of size 4 impossible.

Exercise 4 In case $R(m - 1, n)$ and $R(m, n - 1)$ are both even, prove that $R(m, n) \leq R(m - 1, n) + R(m, n - 1) - 1$.

Exercise 5 The Ramsey number $R_k(3)$ is the minimum number r of vertices such that, if the edges of K_k are coloured with k colours, there is always a monochromatic triangle. Now, show that for $n = (R_{k-1}(3) - 1) \cdot k + 2$, if the edges of K_n are coloured with k colours, there is always a monochromatic triangle.

Solution Consider an r colouring of the edges of K_n, and fix a vertex x. Define the sets of vertices A_i, $1 \leq i \leq k$, such that each A_i consists of the neighbours of x connected to it by an edge of colour i. Clearly, all A_i are disjoint, and together they contain $n - 1 = (R_{k-1}(3) - 1) \cdot k + 1$ vertices. Thus, the Strong Pigeonhole Principle guarantees that there is some colour, say j, such that $|A_j| \geq R_{k-1}(3)$. Now, if the graph induced by A_j contains an edge $\{y, z\}$ of colour j, then we have found a monochromatic triangle (colour j), namely x, y, z. Otherwise, A_j contains edges of only $k-1$ colours, thus $|A_j| \geq R_{k-1}(3)$ guarantees the existence of a monochromatic triangle. In either case, we are done.

Exercise 6 Show that $R(s, t) > (s - 1)(t - 1)$.

Solution Consider a graph G on $(s - 1)(t - 1)$ vertices, which consists of $s - 1$ disconnected copies of K_{t-1}. It is clear that G contains neither an independent set of size s, nor a t-clique. This is because of the Pigeonhole Principle: given any s vertices, at least two of them must belong to the same copy of K_{t-1} (of which there are $s - 1$), hence the set is not independent. Again, given any t vertices, not all of them can belong to the same copy of K_{t-1} (each of which contains only $t - 1$ vertices), hence this is not a t-clique.

Note that in our construction, G is a complete $(t - 1)$-partite graph with $s - 1$ vertices in each part.

Exercise 7 If n is a positive integer satisfying

$$\binom{n}{s} < 2^{\binom{s}{2}-1},$$

then show that $R(s, s) > n$.

Solution We show that there is a 2-colouring of the edges of K_n such that it contains no monochromatic K_s. To do this, we count the number of ways a K_n can contain a monochromatic K_s, and show that this is strictly less than the total number of colourings of K_n; the remaining colourings are thus free of a monochromatic K_s.

First, note that K_n contains $\binom{n}{2}$ edges, hence the total number of 2-colourings of K_n are

$$2^{\binom{n}{2}}.$$

Now, consider a $K_s \subset K_n$, which can be chosen in $\binom{n}{s}$ ways. For this to be monochromatic, there are two ways of colouring its edges (say red or blue). The number of remaining edges is $\binom{n}{2} - \binom{s}{2}$. Note that we have over-counted the total number of colourings with a monochromatic K_s, but this is not an issue: the number of such configurations is at most

$$\binom{n}{s} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{s}{2}}.$$

To show that this falls short of all colourings, it is sufficient to prove

$$\binom{n}{s} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{s}{2}} < 2^{\binom{n}{2}}, \qquad \binom{n}{s} < 2^{\binom{s}{2}-1}.$$

This is exactly what was given, completing the proof.

Exercise 8 $S(r)$ is called Schur's number if it is the smallest integer such that any r colouring of the integers $\{1, 2, \ldots, S(r)\}$ contains three monochromatic integers x, y, z such that $z = x + y$. Now, show that $S(r + 1) \geq 3S(r) - 1$.

Solution Set $n = S(r) - 1$. Thus, there exists an r colouring of $\{1, \ldots, n\}$, say $\mathscr{C}$, such that no three monochromatic integers satisfy $x+y = z$. We shall construct an $r+1$ colouring $\mathscr{C}'$ of $\{1, 2, \ldots, 3n+1\}$ such that the integer sum property is still not satisfied; this is enough to show that $S(r+1) > 3n+1 = 3S(r)-2$.

To do this, colour the first block $\{1, 2, \ldots, n\}$ as in $\mathscr{C}$, i.e. $\mathscr{C}'(i) = \mathscr{C}(i)$ for $1 \leq i \leq n$. Also colour the last block $\{2n + 2, \ldots, 3n + 1\}$ in the say way, i.e. $\mathscr{C}'(2n + 1 + i) = \mathscr{C}(i)$ for $1 \leq i \leq n$. Colour the remaining block $\{n + 1, \ldots, 2n + 1\}$ in the new, $(r + 1)$th colour, i.e. $\mathscr{C}'(n + i) = r + 1$ for $1 \leq i \leq n + 1$.

Suppose that $x, y, z \in \{1, 2, \ldots, 3n+1\}$ have the colour $r+1$. Then, they must be in the middle block, so $x + y \geq n + 1 + n + 1 = 2n + 2$ but $z \leq 2n + 1$. Thus, no $(r+1)$-coloured triple x, y, z can satisfy $x + y = z$.

Thus, if monochromatic x, y, z satisfy $x + y = z$, then they must belong to the extreme blocks and cannot have the $(r+1)$th colour. Without loss of generality, suppose $x \leq y < z$. If both x and y belong to the first block, then $x + y \leq n + n = 2n$, forcing z to belong to the first block as well (the integers $\{n+1, \ldots, 2n\}$ are in the wrong colour). However, this is not possible by the construction of $\mathscr{C}$. Again, if both x and y belong to the last block, then $x + y \geq 2n + 2 + 2n + 2 = 4n + 4 > 3n + 1$, making the choice of z impossible.

Thus, we must have x in the first block and y, z in the last block. Write $y = 2n+1+y'$, $z = 2n+1+z'$ where $1 \leq y', z' \leq n$. Then, $x + y = z$ gives $x + y' = z'$, with $x, y', z' \in \{1, \ldots, n\}$. Furthermore, we have $\mathscr{C}'(y) = \mathscr{C}(y')$, $\mathscr{C}'(z) = \mathscr{C}(z')$ by construction, so x, y', z' are monochromatic. Again, this is not possible because by the construction of $\mathscr{C}$.

Exercise 9 Using the above relation, show that $S(r) \geq (3^r + 1)/2$.

Solution We have seen by inspection that $S(1) = 2$, $S(2) = 5$, hence these base cases hold. Suppose that the desired inequality holds for some $r \geq 2$. Then, using the inequality from the previous exercise with the induction hypothesis,

$$S(r+1) \geq 3S(r) - 1 \geq 3 \cdot \frac{3^r + 1}{2} - 1 = \frac{3^{r+1} + 1}{2}.$$

This proves the desired result by induction.

Exercise 10 For $r \geq 1$, show that there exists a smallest natural number $S'(r)$ such that every r colouring of the integers $\{1, 2, \ldots, N\}$ with $N \geq S'(r)$ will necessarily contain three integers x, y, z (all must not be the same) of the same colour such that $z = xy$.

Solution Let $n = 2^{S(r)}$; we claim that $S'(r) \leq n$. Consider an arbitrary r colouring of the integers $\{1, 2, \ldots, n\}$; this gives an r colouring of the subset $A = \{2^1, 2^2, \ldots, 2^{S(r)}\}$. Again, this gives an r colouring of the integers $B = \{1, 2, \ldots, S(r)\}$, by setting the colour of $k \in B$ to the colour of $2^k \in A$. Thus, we are guaranteed monochromatic $x', y', z' \in B$ such that $x' + y' = z'$. The corresponding elements $x = 2^{x'}$, $y = 2^{y'}$, $z = 2^{z'} \in A$ are monochromatic, with $xy = z$. Furthermore, not all of them can be the same since $z' > x', y'$.

Exercise 11 Can you construct graphs with the following degree sequences? Provide proper justifications.

(i) 5, 5, 4, 3, 2, 2, 2, 1
(ii) 5, 5, 4, 4, 2, 2, 1, 1
(iii) 5, 5, 5, 3, 2, 2, 1, 1
(iv) 5, 5, 5, 4, 2, 1, 1, 1

Solution We employ the Havel-Hakimi scheme, which involves arranging the sequence in descending order, removing d_1 and subtracting 1 from the next d_i numbers, then repeating.

(i)

$$\begin{aligned} &5, 5, 4, 3, 2, 2, 2, 1 \\ &4, 3, 2, 1, 1, 2, 1 \\ &4, 3, 2, 2, 1, 1, 1 && \text{(Reorder)} \\ &2, 1, 1, 0, 1, 1 \\ &2, 1, 1, 1, 1 && \text{(Reorder)} \\ &0, 0, 1, 1 \end{aligned}$$

At this point, the sequence is clearly graphic: consider K_2 with 2 extra isolated points. Thus, the initial sequence is also graphic.

(ii)

$$\begin{aligned} 5, 5, 4, 4, 2, 2, 1, 1 \\ 4, 3, 3, 1, 1, 1, 1 \\ 2, 2, 0, 0, 1, 1 \end{aligned}$$

At this point, the sequence is clearly graphic: consider P_4 (a path of four vertices in a line) with 2 extra isolated points. Thus, the initial sequence is graphic.

(iii)

$$\begin{aligned} 5, 5, 5, 3, 2, 2, 1, 1 \\ 4, 4, 2, 1, 1, 1, 1 \\ 3, 1, 0, 0, 1, 1 \end{aligned}$$

At this point, the sequence is clearly graphic: consider $K_{1,3}$ (a star with 3 points) with 2 extra isolated points. Thus, the initial sequence is graphic.

(iv)

$$\begin{aligned} &5, 5, 5, 4, 2, 1, 1, 1 \\ &4, 4, 3, 1, 0, 1, 1 \\ &4, 4, 3, 1, 1, 1, 0 && \text{(Reorder)} \\ &3, 2, 0, 0, 1, 0 \end{aligned}$$

At this point, the sequence is clearly not graphic: the first vertex has degree 3 but there are only two remaining vertices of positive degree. Thus, the initial sequence is not graphic.

Exercise 12 For each of the sequences of numbers below, explain why there cannot be any graph having that sequence.

(i) 5, 5, 5, 4, 4, 3, 3

(ii) 6, 5, 4, 3, 2, 2, 0

Solution

(i) Here, the sum of the degrees of the vertices is an odd number, which contradicts the fact that the sum ought to be $2|E|$ which is an even number.

(ii) Here, the first vertex has degree 6, but there are only 5 remaining vertices with positive order.

Exercise 13 Can you construct graphs with the following degree sequences? Can you construct bipartite graphs from those graphic sequences?

(i) 5, 3, 3, 3, 2, 2, 2, 2, 2, 2

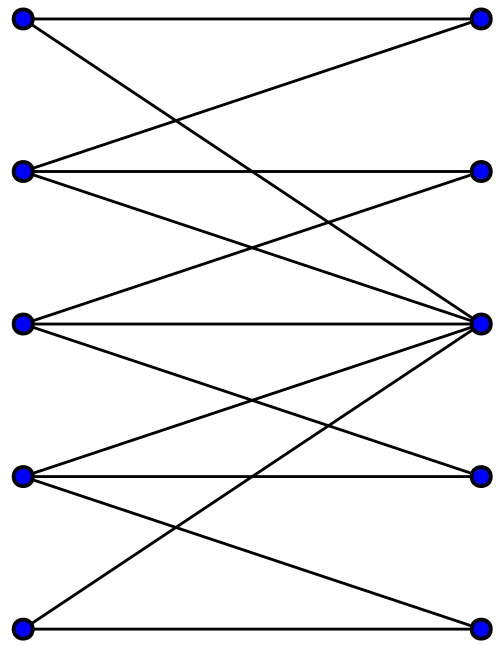
(ii) 6, 6, 6, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4

Solution

(i)

$$\begin{aligned} 5, 3, 3, 3, 2, 2, 2, 2, 2, 2 \\ 2, 2, 2, 1, 1, 2, 2, 2, 2 \end{aligned}$$

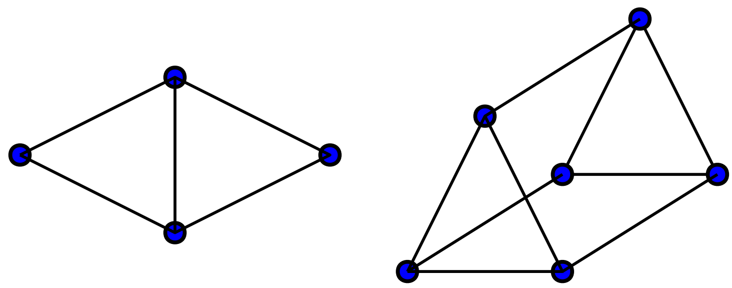
This is clearly graphic: consider P_9 (a path of 9 vertices). A bipartite graph with the given degree sequence is given below.



(ii)

$$
\begin{aligned}
&6,6,6,4,4,4,4,4,4,4,4,4,4 \\
&5,5,3,3,3,3,4,4,4,4,4,4 \\
&5,5,4,4,4,4,4,4,3,3,3,3 && \text{(Reorder)} \\
&4,3,3,3,3,4,4,3,3,3,3 \\
&4,4,4,3,3,3,3,3,3,3,3 && \text{(Reorder)} \\
&3,3,2,2,3,3,3,3,3,3
\end{aligned}
$$

This is graphic. Divide the sequence into the parts $3,3,2,2$ and $3,3,3,3,3,3$.



Suppose that G is a bipartite graph with the given degree sequence. The sum of the degrees is $6 \cdot 3 + 4 \cdot 10 = 58$, hence G has exactly 29 edges. Since G is bipartite, every edge connects a vertex from one part to the other. Thus, the total number of edges leaving one part is also 29, and the sum of degrees of the vertices in that part is also 29. This contradicts the fact that all vertices have even degree.

Exercise 14 Show that a graph that has 17 vertices and 73 edges cannot be bipartite.

Solution Calculate

$$|E| = 73 > \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{17^2}{4} \right\rfloor = 72.$$

Mantel's theorem guarantees the existence of a triangle in any such graph. However, a bipartite graph is always triangle-free.

Exercise 15 If a sequence $(d_i)_1^n$ of non-negative integers such that $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic, then show that for $1 \leq m \leq n$,

$$\sum_{i=1}^{m} d_i \;\leq\; m(m-1) + \sum_{i=m+1}^{n} \min(d_i, m).$$

Solution Given a graph G obeying this scheme, let $A = \{x_1, x_2, \ldots, x_m\}$ be the set of m vertices with the highest degrees, $d_1, d_2, \ldots, d_m$, and let B be the remaining vertices. Let $I(e,x)$ be the incidence function between edges and vertices. Let F be the set of edges which have at both endpoints in A, and let H be the set of edges with one endpoint in A, the other in B. Note that

$$\sum_{i=1}^{m} d_i = \sum_{x \in A} d(x) = \sum_{x \in A} \sum_{e \in F \cup H} I(e,x) = \sum_{e \in F \cup H} \sum_{x \in A} I(e,x) = \sum_{e \in F} 2 + \sum_{e \in H} 1 = 2|F| + |H|.$$

This is because each $e \in F$ contributes 2 to the sum via both endpoints, while each $e \in H$ contributes only 1. We only need to consider edges with at least one endpoint in A.

Now, clearly $|F| \leq \binom{m}{2} = m(m-1)/2$. To count the edges in H, pick a vertex $x \in B$. This can contribute at most $d(x)$ edges to the entire graph; at the same time, there are only m vertices in A. Thus, x contributes at most $\min(d(x), m)$ edges to H. Summing over all vertices in B and putting things together,

$$\sum_{i=1}^{m} d_i \;\leq\; 2|F| + |H| \;\leq\; m(m-1) + \sum_{i=m+1}^{n} \min(d_i, m).$$