Term presentation
Problem 6

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MA2102: Linear Algebra I

Indian Institute of Science Education and Research, Kolkata

Problem statement

Find a basis of the quotient space $\mathsf{M}_n(\mathbb{R}) \,/\, \mathsf{Sym}_n(\mathbb{R})$, where $\mathsf{Sym}_n(\mathbb{R})$ is the subspace of symmetric matrices.

Preliminaries

The subspace of symmetric matrices is such that for any $A \in \mathsf{Sym_n}(\mathbb{R})$,

$$A = A^\top \quad \Leftrightarrow \quad a_{ij} = a_{ji}.$$

It can be shown that the skew-symmetric matrices form a subspace such that for any $B \in \mathsf{Skew_n}(\mathbb{R})$,

$$B = -B^\top \quad \Leftrightarrow \quad b_{ij} = -b_{ji}.$$

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Preliminaries

Any matrix $X \in \mathsf{M_n}(\mathbb{R})$ can be written uniquely as the sum of a symmetric and a skew-symmetric matrix.

$$X = \frac{1}{2}(X + X^\top) + \frac{1}{2}(X - X^\top).$$

Furthermore, $\mathsf{Sym_n}(\mathbb{R}) \cap \mathsf{Skew_n}(\mathbb{R}) = \{\mathbf{0}\}$, because if X is both symmetric and skew-symmetric,

$$X^\top = X = -X^\top \quad \Rightarrow \quad X = \mathbf{0}.$$

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Basis of $\mathsf{Skew_n}(\mathbb{R})$

Let $E_{ij} \in \mathsf{M_n}(\mathbb{R})$ be the matrix whose i, j^{th} element is 1, and the remaining elements are 0. The set of β of all such E_{ij} comprises the standard basis of $\mathsf{M_n}(\mathbb{R})$. For any $X \in \mathsf{M_n}(\mathbb{R})$,

$$X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} E_{ij}.$$

Set

$$B_{ij} = E_{ij} - E_{ji}.$$

Thus, B_{ij} is a skew-symmetric matrix with 1 in the i, j^{th} position, and -1 in the j, i^{th} position. Let

$$\gamma = \left\{ B_{ij} : 1 \leq i < j \leq n \right\}.$$

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Basis of $\mathsf{Skew_n}(\mathbb{R})$

The linear independence of γ follows from the linear independence of $\beta = \{E_{ij}\}$.

$$\sum_{i<j} c_{ij}B_{ij} = \sum_{i<j} c_{ij}E_{ij} - c_{ij}E_{ji} = \mathbf{0}.$$

Suppose $B \in \mathsf{Skew_n}(\mathbb{R})$. Then, $b_{ij} = -b_{ji}$ and $b_{ii} = 0$.

$$\begin{aligned} B = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}E_{ij} &= \sum_{i<j} b_{ij}E_{ij} + \sum_{i=j} b_{ij}E_{ij} + \sum_{i>j} b_{ij}E_{ij} \\ &= \sum_{i<j} b_{ij}E_{ij} + \mathbf{0} + \sum_{i<j} b_{ji}E_{ji} \\ &= \sum_{i<j} b_{ij}E_{ij} - b_{ij}E_{ji} = \sum_{i<j} b_{ij}B_{ij}. \end{aligned}$$

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Quotient spaces

Let V be vector space over F and let $W \subseteq V$ be a subspace. The quotient space V/W consists of equivalence classes $[\mathsf{v}]$, where $\mathsf{v} \in V$ and

$$[\mathsf{v}] = \mathsf{v} + W = \{\mathsf{v} + \mathsf{w} \colon \ \mathsf{w} \in W\}.$$

Equivalently, $\mathsf{u} \in [\mathsf{v}]$ if and only if $\mathsf{u} - \mathsf{v} \in W$.

We define

$$[\mathsf{u}] + [\mathsf{v}] = [\mathsf{u} + \mathsf{v}], \qquad \lambda[\mathsf{v}] = [\lambda\,\mathsf{v}].$$

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Quotient spaces

Let $v \in V$ and $w_1, w_2 \in W$. Then

$$[v + w_1] = [v + w_2].$$

Pick $u \in [v + w_1]$. Then $u = v + w_1 + w_1'$ for some $w_1' \in W$. Now, $w_1 + w_1' \in W$, so $(w_1 + w_1') - w_2 \in W$. This means that

$$u = v + w_1 + w_1' = v + w_2 + (w_1 + w_1' - w_2) \in [v + w_2].$$

The reverse inclusion follows by symmetry.

Alternatively, note that since addition in well-defined,

$$[v + w_1] = [v] + [w_1] = [v] + [0] = [v] + [w_2] = [v + w_2].$$

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Basis of $\mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R})$

We claim that the set

$$\gamma' = \{[B_{ij}] \colon B_{ij} \in \gamma\} = \left\{[B_{ij}] \colon 1 \leq i < j \leq n\right\}$$

is a basis of $\mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R})$.

Consider the linear combination

$$\sum_{i<j} c_{ij}[B_{ij}] \;=\; [\mathbf{0}]$$

$$[B] \;=\; [\mathbf{0}]$$

This means that the skew-symmetric matrix $B \in [\mathbf{0}]$, i.e. $B = \mathbf{0} + A = A$ for some $A \in \mathsf{Sym_n}(\mathbb{R})$. This forces $B = \mathbf{0}$, whence $c_{ij} = 0$. Thus, γ' is linearly independent.

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Basis of $\mathsf{M_n}(\mathbb{R}) / \mathsf{Sym_n}(\mathbb{R})$

Pick $[X] \in \mathsf{M_n}(\mathbb{R}) / \mathsf{Sym_n}(\mathbb{R})$. Since $X \in \mathsf{M_n}(\mathbb{R})$, write $X = A + B$ where $A \in \mathsf{Sym_n}(\mathbb{R})$ and $B \in \mathsf{Skew_n}(\mathbb{R})$. Note that

$$[X] = [A + B] = [B].$$

Now, expand B in the basis γ.

$$B = \sum_{i<j} b_{ij} B_{ij}.$$

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Then,

$$[X] \;=\; [B] \;=\; \sum_{i<j} b_{ij} [B_{ij}].$$

Thus, γ' is linearly independent and spans $\mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R})$. This proves that γ' is a basis of $\mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R})$.

Moreover, γ and γ' contain $1 + 2 + \cdots + (n-1) = n(n-1)/2$ elements, so

$$\dim \mathsf{Skew_n}(\mathbb{R}) = \dim \mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R}) = \frac{1}{2} n(n-1).$$

For a linear map $T\colon V \to W$, the map

$$\mathscr{T} : V/\ker T \to \operatorname{im} T, \qquad [\mathsf{v}] \mapsto T(\mathsf{v})$$

is a linear isomorphism. By setting

$$T\colon\ \mathrm{M_n}(\mathbb{R}) \to \mathrm{M_n}(\mathbb{R}), \qquad X \mapsto \frac{1}{2}(X - X^\top),$$

note that $\ker T = \mathrm{Sym_n}(\mathbb{R})$ and $\operatorname{im} T = \mathrm{Skew_n}(\mathbb{R})$.

If $T(X) = B \in \mathrm{Skew_n}(\mathbb{R})$, then $X = [B]$. Since a linear isomorphism sends a basis to a basis, and the inverse of a linear isomorphism is a linear isomorphism, the set $\mathscr{T}^{-1}(\gamma) = \gamma'$ is a basis of $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$.