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# Solutions to exercises from M.A. Armstrong's *Groups and Symmetry*

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# Chapter 1

# Symmetries of the Tetrahedron

**Exercise 1.1** Glue two copies of a regular tetrahedron together so that they have a triangular face in common, and work out all the rotational symmetries of this new solid.

*Solution.* The resultant bi-pyramid has five vertices; label the ones furthest apart as 1 and 2, and then label the remaining ones on the equator as 3, 4, 5. The rotational symmetries are those which permute these five vertices such that 1 and 2 never leave the long axis, and the cyclicity of the vertices 1, 2, 3, 4 is preserved. Thus, we have 3 rotations about the long axis (by 0, $2\pi/3$, and $4\pi/3$) which cycle the vertices 3, 4, 5, then three rotations, each about an axis through one of the vertices 3, 4, 5 and the midpoint of the opposite edge (by 0, $\pi$) which swap the positions of 1 and 2 and reverse the cyclicity of 3, 4, 5. This gives a total of $2 \times 3 = 6$ symmetries.

**Exercise 1.2** Find all rotational symmetries of a cube.

*Solution.* First consider the four rotations (by 0, $\pi/2$, $\pi$, $3\pi/2$) about an axis which passes through the centres of opposite faces; there are three such axes giving $3 \times 3 = 9$ such rotational symmetries (excluding the identity symmetry, which we will add on at the end). Next, consider the two rotations (by 0, $\pi$) about an axis passing through the centres of opposite edges; there are six such axes, giving $1 \times 6 = 6$ such rotational symmetries. Next, consider the three rotations (by 0, $2\pi/3$, $4\pi/3$) about an axis passing through opposite vertices; there are four such axes, giving $2 \times 4 = 8$ such rotational symmetries. Adding these up, we have $9 + 6 + 8 = 23$ rotational symmetries. The identity symmetry brings the total to 24 rotational symmetries of the cube.

**Exercise 1.3** Adopt the notation of Figure 1.4. Show that the axis of the composite rotation $srs$ passes through the vertex 4, and that the axis of $rsrr$ is determined by the midpoints of edges 12 and 34.

*Solution.* Let the original state of the tetrahedron be represented by the tuple $(1, 2, 3, 4)$, indicating the labels on the four vertices. Applying $s$ permutes this to $(4, 3, 2, 1)$, and applying $r$ permutes (the original) to $(1, 4, 2, 3)$. Thus, the action $srs$ is the permutation $(1, 2, 3, 4) \to (4, 3, 2, 1) \to (4, 1, 3, 2) \to (2, 3, 1, 4)$. Note that the vertex labelled 4 is a fixed point, hence the axis of this composite rotation must have passed through this vertex.

Similarly, the action $rsrr$ maps $(1, 2, 3, 4) \to (1, 4, 2, 3) \to (1, 3, 4, 2) \to (2, 4, 3, 1) \to (2, 1, 4, 3)$. There are no fixed points, hence this is not a rotation through a vertex. Instead, the first pair and last pair of vertices have swapped, which indicates a rotation about an axis through the centres of the edges 12 and 34.

**Exercise 1.4** Having completed the previous exercise, express each of the twelve rotational symmetries of the tetrahedron in terms of $r$ and $s$.

*Solution.* The twelve rotational symmetries of the tetrahedron are $e$ (the identity), $r$, $r^2$, $s$, $rs$, $r^2s$, $srs$, $rsrs$, $r^2srs$, $sr^2s$, $rsr^2s$, $r^2sr^2s$. See Exercise 1.7 for their actions on the $(1,2,3,4)$ state.

**Exercise 1.5** Again with the notation of Figure 1.4, check that $r^{-1} = rr$, $s^{-1} = s$, $(rs)^{-1} = srr$ and $(sr)^{-1} = rrs$.

*Solution.* Note that $r^3$ maps $(1,2,3,4) \to (1,4,2,3) \to (1,3,4,2) \to (1,2,3,4)$, so $r^3 = e$. Thus, $(rr)r = e = r(rr)$, so $r^{-1} = rr$.

Next, note that $ss$ maps $(1,2,3,4) \to (4,3,2,1) \to (1,2,3,4)$, so $ss = e$. Thus, $(s)s = e = e(s)$, so $s^{-1} = s$.

Next, note that $(rs)(srr) = r(ss)(rr) = r(e)(rr) = rrr = e$, and $(srr)(rs) = s(rrr)s = s(e)s = ss = e$, so $(rs)^{-1} = srr$.

Finally, note that $(sr)(rrs) = (srr)(rs) = e$ and $(rrs)(sr) = (rr)(ss)r = rrr = e$, so $(sr)^{-1} = rrs$.

**Exercise 1.6** Show that a regular tetrahedron has a total of twenty-four symmetries if reflections and products of reflections are allowed. Identify a symmetry which is not a rotation and not a reflection. Check that this symmetry is a product of three reflections.

*Solution.* Note that a reflection about the plane passing through an edge and the centroid swaps the remaining two vertices. Thus, by representing the vertex configuration of the tetrahedron as a tuple $(1,2,3,4)$, this can be mapped to any of the $4! = 24$ permutations by employing suitable reflections (for example, see bubble sort).

Consider the action which takes the tetrahedron $(1,2,3,4)$ to the state $(4,1,2,3)$. This is not a rotation about a vertex, nor a reflection about a plane through an edge because there are no fixed points. This is not a rotation about an axis through the centres of opposite sides either, since those must swap the labels on two pairs of adjacent vertices. However, this can be reached via the reflections $(1,2,3,4) \to (4,2,3,1) \to (4,1,3,2) \to (4,1,2,3)$; these were reflections about planes passing through the edges 23, 13, 12.

**Exercise 1.7** Let $q$ denote reflection of a regular tetrahedron in the plane determined by its centroid and one of its edges. Show that the rotational symmetries, together with those of the form $uq$, where $u$ is a rotation, give all twenty-four symmetries of the tetrahedron.

*Solution.* We let $q$ mean the reflection in the plane through the centroid and the side 12; note that this maps $(1,2,3,4) \to (1,2,4,3)$. Below, we list all 24 permutations of the tuple $(1,2,3,4)$, which represent all 24 symmetries of the tetrahedron.

| Symmetry | State | Symmetry | State |
|---|---|---|---|
| $e$ | 1, 2, 3, 4 | $q$ | 1, 2, 4, 3 |
| $r$ | 1, 4, 2, 3 | $rq$ | 1, 3, 2, 4 |
| $r^2$ | 1, 3, 4, 2 | $r^2q$ | 1, 4, 3, 2 |
| $s$ | 4, 3, 2, 1 | $sq$ | 3, 4, 2, 1 |
| $rs$ | 4, 1, 3, 2 | $rsq$ | 3, 1, 4, 2 |
| $r^2s$ | 4, 2, 1, 3 | $r^2sq$ | 3, 2, 1, 4 |
| $srs$ | 2, 3, 1, 4 | $srsq$ | 2, 4, 1, 3 |
| $rsrs$ | 2, 4, 3, 1 | $rsrsq$ | 2, 3, 4, 1 |
| $r^2srs$ | 2, 1, 4, 3 | $r^2srsq$ | 2, 1, 3, 4 |
| $sr^2s$ | 3, 1, 2, 4 | $sr^2sq$ | 4, 1, 2, 3 |
| $rsr^2s$ | 3, 4, 1, 2 | $rsr^2sq$ | 4, 3, 1, 2 |
| $r^2sr^2s$ | 3, 2, 4, 1 | $r^2sr^2sq$ | 4, 2, 3, 1 |

**Exercise 1.8** Find all plane symmetries (rotations and reflections) of a regular pentagon and of a regular hexagon.

*Solution.* Any symmetry of a regular $n$-gon is one which preserves adjacent vertices, i.e. two labelled vertices must remain adjacent before and after the symmetry action. Thus, we have $n$ rotations (by $2k\pi/n$), along with $n$ rotations followed by a reflection (these are mirror images of the previous $n$ symmetries). Thus, a regular $n$-gon has $2n$ plane symmetries; ten for a pentagon and twelve for a hexagon.

**Exercise 1.9** Show that the hexagonal plate of Figure 1.2 has twenty-four symmetries in all. Identify those symmetries which commute with all the others.

*Solution.* By representing the vertices of the hexagon as the tuple $(1, 2, 3, 4, 5, 6)$, we see symmetries of the plate are precisely those actions which permute these elements, preserving the adjacency but allowing a reversal in order. There are six tuples of the form $(1+n, 2+n, \ldots, 6+n)$ and six more of the form $(6+n, 5+n, \ldots, 1+n)$. Because we are dealing with a hexagonal plate, there is another symmetry which is a reflection passing through a plane parallel to the hexagonal face. This does not change the labels on the vertices on the hexagonal face in any way, but exchanges the top and bottom faces. Thus, this symmetry commutes with all other symmetries, and the combination of any one of the twelve previously shown symmetries with this reflection symmetry produces a new symmetry, bringing our total to twenty-four.

**Exercise 1.10** Make models of the octahedron, dodecahedron, and icosahedron. Try to spot as many symmetries of these solids as you can.