MA3205

Geometry of Curves and Surfaces

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1.1   Curves

Definition 1.1. A curve is a continuous map $\gamma \colon \mathbb{R} \to \mathbb{R}^n$.

Definition 1.2. A smooth curve $\gamma \colon \mathbb{R} \to \mathbb{R}^n$ is C^∞, i.e. differentiable arbitrarily times.

Definition 1.3. A closed curve $\gamma \colon \mathbb{R} \to \mathbb{R}^n$ is periodic, i.e there exists some c such that $\gamma(t + c) = \gamma(t)$ for all $t \in \mathbb{R}$.

Example. Alternatively, a closed curve can be thought of as a continuous map $\gamma \colon S^1 \to \mathbb{R}^n$. For instance, given a closed curve $\gamma \colon \mathbb{R} \to \mathbb{R}^n$ with period c, we can define the corresponding map
$$\tilde{\gamma} \colon S^1 \to \mathbb{R}^n, \qquad \tilde{\gamma}(e^{it}) = \gamma(ct/2\pi).$$

Definition 1.4. A simple curve $\gamma \colon \mathbb{R} \to \mathbb{R}^n$ is injective on its period.

Theorem 1.1 (Four Vertex Theorem)**.** *The curvature of a simple, closed, smooth plane curve has at least two local minima and two local maxima.*

Definition 1.5. A knot is a simple closed curve in $\mathbb{R}^3$.

Definition 1.6. The total absolute curvature of a knot K is the integral of the absolute value of the curvature, taken over the curve, i.e. it is the quantity

$$\oint_K |\kappa(s)|\, ds.$$

Example. The total absolute curvature of a circle is always 2π.

Theorem 1.2 (Fáry-Milnor Theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Definition 1.7. An immersed loop γ is such that γ' is never zero.

Definition 1.8. Two loops are isotopic if there exists an interpolating family of loops between them. Two immersed loops are isotopic if we can choose such an interpolating family of immersed loops.

Example. Without the restriction of immersion, any two loops $\gamma, \eta \colon S^1 \to \mathbb{R}^n$ would be isotopic, since we can always construct the linear interpolations

$$H \colon S^1 \times [0,1] \to \mathbb{R}^n, \qquad H(e^{i\theta}, t) = (1-t)\gamma(e^{i\theta}) + t\eta(e^{i\theta}).$$

Theorem 1.3 (Hirsch-Smale Theory)**.**

1. *Any two immersed loops in $\mathbb{R}^2$ are isotopic if and only if their turning numbers match.*

2. *Any two immersed loops in S^2 are isotopic if and only if their turning numbers modulo 2 match.*

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1.2 Whitney's theorem

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a continuous function $f \colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Remark. The converse, i.e. $f^{-1}(0) = C$ implies C is closed, where f is continuous on Ω, is trivial.

Proof. Set f to be the distance function from C, i.e.

$$f(x) = \inf_{y \in C} d(x, y). \qquad \square$$

Theorem 1.5 (Whitney's Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a smooth function $f \colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Proof. Set $V = \Omega \setminus C$, and cover V by a countable collection of open balls,

$$V = \bigcup_{i=1}^{\infty} B(q_k, r_k).$$

This can always be done since V is open, and using the density of $\mathbb{Q}$ in $\mathbb{R}$ to pick only rational q_k, r_k. Now for each open ball $B(q_k, r_k)$, we can construct a smooth bump functions f_k such that $f^{-1}(0) = \mathbb{R}^n \setminus B(q_k, r_k)$, $f^{-1}(1) = \overline{B(q_k, r_k/2)}$, and all derivatives of f_k vanish on $\mathbb{R}^n \setminus B(q_k, r_k)$.
Define the weights

$$c_k = \max_{\substack{|\alpha| \leq k \\ y \in \overline{B(q_k, r_k)}}} \left| \frac{\partial^\alpha f_k}{\partial x^\alpha}(y) \right|.$$

Note that each c_k is well-defined: there are finitely many multi-indices α given k, and each of the partials $\partial^\alpha f / \partial x^\alpha$ is a smooth function over a compact set, hence bounded. Furthermore, each $c_k \geq 1$. Finally, set

$$f = \sum_{k=1}^{\infty} \frac{f_k}{2^k c_k}.$$

It is clear that $f^{-1}(0) = C$. We can show that the partial sums s_n converge; let $\epsilon > 0$ and choose sufficiently large N such that $1/2^N < \epsilon$. Now for $m > n \geq N$, examine

$$|s_m(x) - s_n(x)| = \sum_{k=n+1}^{m} \frac{f_k(x)}{2^k c_k} \leq \sum_{k=n+1}^{m} \frac{1}{2^k} \leq \frac{1}{2^N} < \epsilon$$

Thus, the convergence is uniform, and f is C^0. For higher derivatives, we examine some α partial of the sums, and use the same argument; at each stage, $|\partial^\alpha f / \partial x^\alpha| < c_k$ whenever $|\alpha| < k$. $\qquad \square$

1.3 Parametrized curves

Definition 1.9. A parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma \colon (\alpha, \beta) \to \mathbb{R}^n$ for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.

Remark. Here, we will always implicitly assume that maps are continuous.

Remark. Such a curve is called regular if $\gamma'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Example. The curve defined by

$$\gamma \colon \mathbb{R} \to \mathbb{R}^n, \qquad t \mapsto a + tb$$

is a straight line through the point a, in the direction b.

Example. The curve defined by

$$\gamma \colon \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (\cos t, \sin t)$$

is the unit circle in $\mathbb{R}^2$, counter-clockwise.

Example. The curve defined by

$$\gamma \colon \mathbb{R} \to \mathbb{R}^3, \qquad t \mapsto (t, \cos t, \sin t)$$

is a helix in $\mathbb{R}^3$, wrapped around the x-axis.

Definition 1.10. A diffeomorphism is a smooth map with a smooth inverse.

Example. Suppose that $\gamma \colon (\alpha, \beta) \to \mathbb{R}^n$ is a smooth curve. If we have a diffeomorphism $\varphi \colon (\alpha', \beta') \to (\alpha, \beta)$, then the smooth curve $\eta = \gamma \circ \varphi$ is a reparametrization of γ. Note that
$$\eta'(t) = \gamma'(\varphi(t)) \, \varphi'(t).$$

Lemma 1.6. *If $\varphi \colon (\alpha', \beta') \to (\alpha, \beta)$ is a diffeomorphism, then $\varphi'(t) \neq 0$ for all $t \in (\alpha', \beta')$.*

Definition 1.11. If the diffeomorphism $\varphi' > 0$, we say that it it orientation preserving. If $\varphi' < 0$, we say that it is orientation reversing.

Definition 1.12. The arc length of a differentiable curve $\gamma \colon (\alpha, \beta) \to \mathbb{R}^n$, starting at t_0, is defined as
$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| \, du.$$
We call s the arch length parameter.

Remark. If $\gamma'(t) \neq 0$, then $s'(t) > 0$.

Definition 1.13. A unit speed curve γ is one where $\|\gamma\| = 1$

Lemma 1.7. *Let γ be a regular smooth curve. Then its arc length parameter is a smooth function.*

Proof. Note that γ' and $\langle \cdot, \cdot \rangle$ are smooth functions. Thus,

$$\frac{ds}{dt} = \|\gamma'(t)\| = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} > 0,$$

showing that ds/dt is smooth. $\qquad\square$

Lemma 1.8. *The arc length function s is a diffeomorphism onto its image.*

Proof. This follows from the Inverse Function Theorem, using the smoothness of s. $\qquad\square$

Lemma 1.9. *Let φ denote $s^{-1} \colon (\alpha', \beta') \to (\alpha, \beta)$. Then, $\gamma \circ \varphi$ is a unit speed reparametrization of γ.*

Remark. Any other unit speed reparametrization is related to s by shifts and reflections.

Proof. Note that s is strictly increasing, so $s', \varphi' > 0$. Now,

$$\|(\gamma \circ \varphi)'(t)\| = \|\gamma'(\varphi(t))\| \cdot |\varphi'(t)| = s'(\varphi(t))\varphi'(t) = (s \circ \varphi)'(t) = 1. \qquad\square$$

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