

MA2202: PROBABILITY I

Random variables

Spring 2021

Satvik Saha
19MS154

*Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.*

Definition 3.1 (Random variable). Given a probability space (Ω, \mathcal{E}, P) , a function $X: \Omega \rightarrow \mathbb{R}$ is called a random variable if $X^{-1}(r, \infty) \in \mathcal{E}$ for all $r \in \mathbb{R}$.

Remark. For some $S \subseteq \mathbb{R}$, we denote

$$P(X \in S) = P(\{\omega \in \Omega: X(\omega) \in S\}).$$

Definition 3.2 (Discrete random variable). A random variable which can assume only a countably infinite number of values is called a discrete random variable.

Example. Let $X: \Omega \rightarrow \mathbb{R}$ denote the number of heads obtained when a fair coin is tossed thrice. Note that $\Omega = \{0, 1, 2, 3\}$. Thus, $P(X = 0) = P(X = 4) = 1/8$ and $P(X = 1) = P(X = 2) = 3/8$.

Definition 3.3 (Probability distribution). The probability distribution of a random variable X is the set of pairs $(X(A), P(A))$ for all $A \in \mathcal{E}$.

Definition 3.4 (Probability mass function). Let X be a discrete random variable. The probability mass function of X is the function $p_X: \mathbb{R} \rightarrow [0, 1]$,

$$p_X(\alpha) = P(X = \alpha).$$

Remark. Since X is a discrete random variable, the set $S = \{x \in \mathbb{R}: p_X(x) > 0\}$ is countable, and

$$\sum_{x \in S} p_X(x) = 1.$$

Definition 3.5 (Expectation). The expectation of $g(X)$, for $g: \mathbb{R} \rightarrow \mathbb{R}$ and a discrete random variable X is defined as

$$E[g(X)] = \sum_{x \in S} g(x) p_X(x),$$

if the series converges absolutely.

Example. The n^{th} moment of a discrete random variable $E[X^n]$ is defined as

$$E[X^n] = \sum_{x \in S} x^n p_X(x),$$

if the series converges absolutely.

The first moment $\mu = E[X]$ is called the mean. The second moment $\sigma^2 = E[(X - \mu)^2]$ is called the variance. Note that

$$\sigma^2 = \sum (x - \mu)^2 p(x) = \sum x^2 p(x) - 2\mu x p(x) + \mu^2 p(x).$$

Simplifying,

$$\sigma^2 = E[X^2] - E[X]^2.$$

Definition 3.6 (Cumulative distribution function). The cumulative distribution function of a random variable X is defined as the function $F_X: \mathbb{R} \rightarrow [0, 1]$,

$$F_X(\alpha) = P(X \leq \alpha).$$

Definition 3.7 (Continuous random variable). A continuous random variable X is such that its cumulative distribution function F_X is continuous.

Definition 3.8 (Probability density function). Let X be a continuous random variable with a cumulative distribution function F_X . If we write

$$F_X(\alpha) = \int_{-\infty}^{\alpha} f_X(x) dx$$

for all $\alpha \in \mathbb{R}$, then f_X is a probability density function. If f_X is continuous, then the Fundamental Theorem of Calculus guarantees that $f_X(x) = F'_X(x)$.

Remark. Note that we can write

$$P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} f_X(x) dx.$$

We also demand

$$\int_{-\infty}^{+\infty} f_X(x) dx.$$

Example. The uniform distribution on an interval $[a, b] \subset \mathbb{R}$ is defined using the probability density function

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Definition 3.9 (Expectation). The expectation of $g(X)$, for $g: \mathbb{R} \rightarrow \mathbb{R}$ and a continuous random variable X is defined as

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

if the integral converges absolutely.