MA4103: Analysis V

Selected notes

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The Fourier Transform

For $f \in L^1(\mathbb{T})$, we define its Fourier transform as the map

$$\hat{f} \colon \mathbb{Z} \to \mathbb{C}, \qquad n \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

Its Fourier series at some $\theta \in [-\pi, \pi]$ is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta} \equiv \lim_{N \to \infty} \underbrace{\sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}}_{S_N f(\theta)}.$$

Thus, we have an association

$$L^1(\mathbb{T}) \to \mathscr{F}(\mathbb{Z}), \qquad f \mapsto \hat{f}.$$

Lemma. *The map $f \mapsto \hat{f}$ is a homomorphism of algebras $(L^1(\mathbb{T}), +, *)$ and $(\mathscr{F}(\mathbb{Z}), +, \cdot)$. In other words,*

$$\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n), \qquad \widehat{(\alpha f)}(n) = \alpha \hat{f}(n), \qquad \widehat{(f * g)}(n) = \hat{f}(n)\, \hat{g}(n)$$

for all $f, g \in L^1(\mathbb{T}), \alpha \in \mathbb{C}, n \in \mathbb{Z}$.

Trigonometric polynomials

We say that g is a trigonometric polynomial if it is of the form

$$g(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}.$$

For example, the family of polynomials

$$Q_n(\theta) = c_n \left(\frac{1+\cos\theta}{2}\right)^n$$

can be shown to be trigonometric polynomials simply by substituting $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and expanding.

Theorem. *The collection of all trigonometric polynomials is dense in $L^1(\mathbb{T})$.*

Sketch of proof. Since $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, it is enough to show that any $f \in C(\mathbb{T})$ can be written as the limit of trigonometric polynomials. Recall that we already have $f * Q_n \to f$ in L^∞, hence in L^1; this is because the family $\{Q_n\}$ forms a summability kernel, hence an approximate identity, and the L^1 norm on the compact space of finite measure $\mathbb{T}$ is dominated by the L^∞ norm. Again, this means that it is enough to show that all $f * Q_n$ are trigonometric polynomials.

Denote $\varphi_n(t) = e^{int}$. Then,

$$\hat{\varphi}_n(k) = \int_{\mathbb{T}} e^{int} e^{-ikt}\, dt = \delta_{nk}.$$

Thus, for any trigonometric polynomial

$$g = \sum_{n=-N}^{N} a_n \varphi_n,$$

we have

$$\hat{g}(k) = \sum_{n=-N}^{N} a_n \delta_{nk} = \begin{cases} a_k, & |k| \le N, \\ 0, & \text{otherwise.} \end{cases}$$

Now compute,

$$(f * \varphi_n)(t) = \frac{1}{2\pi} \int f(t-x)\, e^{inx}\, dx = \frac{1}{2\pi} \int f(t-x)\, e^{-in(t-x)} e^{int}\, dx = \hat{f}(n)\varphi_n(t).$$

Thus, write

$$(f * g)(t) = \sum_{n=-N}^{N} a_n (f * \varphi_n)(t) = \sum_{n=-N}^{N} a_n \hat{f}(n) \varphi_n(t),$$

which immediately shows that all convolutions of f with trigonometric polynomials are trigonometric polynomials too. □

Corollary. *The set $\{e^{int}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$.*

Proof. The given set is orthonormal, and the collection of all its finite linear combinations (i.e. trigonometric polynomials) is dense in $L^2(\mathbb{T})$; the latter follows from the previous proof. □

Lemma (Riemann-Lebesgue)**.** *Let $f \in L^1(\mathbb{T})$. Then,*

$$|\hat{f}(n)| \leq \|f\|_1, \qquad \lim_{|n|\to\infty} |\hat{f}(n)| = 0.$$

As a result, the map $L^1(\mathbb{T}) \to \mathscr{F}(\mathbb{Z})$ described earlier is a bounded linear operator mapping into the subspace $c_0 \subseteq \mathscr{F}(\mathbb{Z})$.

Remark. This map is injective, but not surjective.

Sketch of proof. Use the density of trigonometric polynomials to conclude that it is enough to show that the above formulae hold for all g of the form

$$g(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}.$$

However, we already know that

$$\hat{g}(k) = \sum_{n=-N}^{N} a_n \delta_{nk} = \begin{cases} a_k, & |k| \leq N, \\ 0, & \text{otherwise}, \end{cases}$$

from which it is immediate that $\lim_{|k|\to\infty} |\hat{g}(k)| = 0$.

To show that the map $L^1(\mathbb{T}) \to c_0$ is injective, pick $f \in L^1(\mathbb{T})$ such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then, each $f * Q_n = 0$; but $f * Q_n \to f$ in L^1 forces $f = 0$ almost everywhere. ∎

Convergence of Fourier series

Theorem. *Let $f \in L^1(\mathbb{T})$, and suppose that*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then,

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(\theta) e^{in\theta} \qquad a.e.$$

If we further suppose that $f \in C(\mathbb{T})$, then equality holds everywhere.

Sketch of proof. Setting

$$g(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(\theta) e^{in\theta}, \qquad g_N(\theta) = \sum_{n=-N}^{N} \hat{f}(\theta) e^{in\theta},$$

we claim that g is continuous. Indeed, as $N \to \infty$, we have

$$\|g - g_N\|_\infty \leq \sum_{|n|>N} |\hat{f}(n)| \to 0$$

since the tail of the convergent series $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|$ vanishes. Thus, g being the uniform limit of continuous functions must also be continuous. This in turn gives all $\hat{g}(n) = \hat{f}(n)$, hence $g = f$ almost everywhere (we have $\widehat{(g - f)} = 0$ and know that the Fourier transform is injective). $\square$

Lemma. *Let $f \in C^1(\mathbb{T})$. Then, for all $n \in \mathbb{Z}$, we have*

$$\widehat{(f')}(n) = in\,\hat{f}(n).$$

If we further suppose that $f \in C^2(\mathbb{T})$, then there exists $C > 0$ for which the following holds for all $n \in \mathbb{Z} \setminus \{0\}$.

$$|\hat{f}(n)| \leq \frac{C}{n^2}.$$

Remark. The second condition immediately gives $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, guaranteeing the convergence and equality of the Fourier series of f everywhere.