MA2201: Analysis II

Differentiation

Spring 2021

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The origins of differential calculus lie in our attempts to approximate various functions using linear ones. Suppose that we have been given a curve described by the function f, and we want to *locally* approximate the function around a point x using a straight line. In other words, for a small shift h, we want to write

$$f(x+h) \approx f(x) + kh.$$

Here, k is the slope of the straight line. In order to obtain k, we can rearrange the above to get

$$k \approx \frac{f(x+h) - f(x)}{h}.$$

As we pick smaller and smaller neighbourhoods of x, we want our approximation to get better and better. Thus, if such an approximation is possible, then the value of k must stabilize. This means that the limit

$$k = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

must exist. Note that this immediately forces the continuity of f, since

$$\lim_{h \to 0} f(x+h) - f(x) = \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0k = 0,$$

whereby $\lim_{x \to a} f(x) = f(a)$. Splitting the limit is justified because the individual limits exist. If such a limit k exists, we call it the derivative of f at x, denoted $f'(x)$. We are now able to write

$$f(x+h) \approx f(x) + f'(x)h.$$

Definition 2.1 (Derivative)**.** The derivative of a function $f\colon [a, b] \to \mathbb{R}$ at a point $x \in [a, b]$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if such a limit exists. Note that we only demand a one-sided limit if x is an endpoint. If the derivative of f exists at every point in $[a, b]$, we say that f is differentiable on $[a, b]$.

Example. Consider the map $x \mapsto x^n$, where $n \in \mathbb{N}$. Using the binomial theorem, we can write

$$(x+h)^n = x^n + nx^{n-1}h + \cdots + h^n,$$

which means that

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{1}{h}\left[(x+h)^n - x^n\right] = \lim_{h\to 0}\left[nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right] = nx^{n-1}.$$

Note that the process of differentiation we described can be generalised to multivariable functions. The idea is to locally approximate a function with an affine function.

Theorem 2.1. *If $f\colon (a,b) \to \mathbb{R}$ is differentiable on (a,b), then it is also continuous on (a,b).*

Theorem 2.2. *Let $f\colon I \to \mathbb{R}$ be a continuous function. Then,*

1. *f maps compact sets to compact sets.*
2. *f maps connected sets to connected sets.*

Corollary 2.2.1. *A continuous function $f\colon I \to \mathbb{R}$ maps intervals to intervals.*

Corollary 2.2.2. *A continuous function $f\colon [a,b] \to \mathbb{R}$ attains its minimum and maximum on $[a,b]$.*

Definition 2.2. Given $f\colon (a,b) \to \mathbb{R}$, a point $c \in (a,b)$ is said to be a point of local maximum if there exists a neighbourhood I_c of c such that

$$f(c) > f(x),$$

for all $x \in I_c \setminus \{c\}$. There is an analogous definition for a local minimum.

Theorem 2.3. *If $f\colon (a,b) \to \mathbb{R}$ is differentiable and $c \in (a,b)$ is a point of local minimum or maximum, then $f'(c) = 0$.*

Remark. The converse is not true. Note that the derivative of $x \mapsto x^3$ vanishes at $x = 0$, but that is not a local minimum or maximum.

Proof. Let c be a local minimum or maximum of f, but suppose that $f'(c) \neq 0$. Define the function

$$g\colon (a,b) \to \mathbb{R}, \qquad g(x) = \begin{cases} (f(x) - f(c))/(x-c), & \text{if } x \neq c \\ f'(c), & \text{if } x = c \end{cases}$$

We note that g is continuous. Also, $f'(c) = g(c) \neq 0$. If $g(c) > 0$, there exists a neighbourhood $I_\delta = (c-\delta, c+\delta)$ such that for all $x \in I_\delta$, $g(x) > 0$, from the continuity of g. This means that on I_c,

$$\frac{f(x) - f(c)}{x - c} > 0,$$

which gives $f(x) > f(c)$ on $(c, c+\delta)$ and $f(x) < f(c)$ on $(c-\delta, c)$. This means that c cannot be a local minimum, nor a local maximum. There is an analogous case assuming $g(c) < 0$, which leads to the same contradiction. Thus, we must have $f'(c) = g(c) = 0$. □

Theorem 2.4. *If $f\colon (a,b) \to \mathbb{R}$ is twice differentiable, and $c \in (a,b)$ is such that $f'(c) = 0$ and $f''(c) < 0$, then c is a point of local maximum. If $f'(c) = 0$ and $f''(c) > 0$, then c is a point of local minimum.*

Theorem 2.5 (Rolle's Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b), with $f(a) = f(b)$. Then, there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Set $f(a) = f(b) = \kappa$. From the continuity of f, note that the image of the closed interval $[a,b]$ is another closed interval $[\alpha,\beta]$. This means that $\alpha \leq \kappa \leq \beta$. Note that if $\alpha = \beta = \kappa$, then the function f is identically equal to the constant κ, hence $f'(x) = 0$ everywhere on $[a,b]$. By the continuity of f, it must attain its maximum and minimum on $[a,b]$. If $\beta > \kappa$, then the maximum is al least β and is hence not attained at the endpoints, which means that the point of maximum lies in (a,b). If $\alpha < \kappa$, then the same argument shows that f attains a minimum in (a,b). Thus, in either case, we have found $c \in (a,b)$ which is either a maximum or minimum of f, i.e. $f'(c) = 0$. □

Theorem 2.6 (Rolle's Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Then, there exists $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)\,(b-a).$$

Proof. Apply Rolle's Theorem on the function defined as

$$g\colon [a,b] \to \mathbb{R}, \qquad g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a} \cdot (x-a).$$

Note that g is continuous on $[a,b]$, differentiable on (a,b), and $g(a) = g(b) = 0$. □