

MA3205

# Geometry of Curves and Surfaces

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## 1 Introduction

### 1.1 Curves

**Definition 1.1.** A curve is a continuous map  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$ .

**Definition 1.2.** A smooth curve  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$  is  $C^\infty$ , i.e. differentiable arbitrarily times.

**Definition 1.3.** A closed curve  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$  is periodic, i.e there exists some  $c$  such that  $\gamma(t + c) = \gamma(t)$  for all  $t \in \mathbb{R}$ .

*Example.* Alternatively, a closed curve can be thought of as a continuous map  $\gamma: S^1 \rightarrow \mathbb{R}^n$ . For instance, given a closed curve  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$  with period  $c$ , we can define the corresponding map

$$\tilde{\gamma}: S^1 \rightarrow \mathbb{R}^n, \quad \tilde{\gamma}(e^{it}) = \gamma(ct/2\pi).$$

**Definition 1.4.** A simple curve  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$  is injective on its period.

**Theorem 1.1** (Four Vertex Theorem). *The curvature of a simple, closed, smooth plane curve has at least two local minima and two local maxima.*

**Definition 1.5.** A knot is a simple closed curve in  $\mathbb{R}^3$ .

**Definition 1.6.** The total absolute curvature of a knot  $K$  is the integral of the absolute value of the curvature, taken over the curve, i.e. it is the quantity

$$\oint_K |\kappa(s)| ds.$$

*Example.* The total absolute curvature of a circle is always  $2\pi$ .

**Theorem 1.2** (Fáry-Milnor Theorem). *If the total absolute curvature of a knot  $K$  is at most  $4\pi$ , then  $K$  is an unknot.*

**Definition 1.7.** An immersed loop  $\gamma$  is such that  $\gamma'$  is never zero.

**Definition 1.8.** Two loops are isotopic if there exists an interpolating family of loops between them. Two immersed loops are isotopic if we can choose such an interpolating family of immersed loops.

*Example.* Without the restriction of immersion, any two loops  $\gamma, \eta: S^1 \rightarrow \mathbb{R}^n$  would be isotopic, since we can always construct the linear interpolations

$$H: S^1 \times [0, 1] \rightarrow \mathbb{R}^n, \quad H(e^{i\theta}, t) = (1 - t)\gamma(e^{i\theta}) + t\eta(e^{i\theta}).$$

**Theorem 1.3** (Hirsch-Smale Theory).

1. *Any two immersed loops in  $\mathbb{R}^2$  are isotopic if and only if their turning numbers match.*
2. *Any two immersed loops in  $S^2$  are isotopic if and only if their turning numbers modulo 2 match.*

## 1.2 Whitney's theorem

**Lemma 1.4.** *Let  $\Omega \subset \mathbb{R}^n$  be open and let  $C \subseteq \Omega$  be closed. Then there exists a continuous function  $f: \Omega \rightarrow \mathbb{R}$  such that  $f^{-1}(0) = C$ .*

*Remark.* The converse, i.e.  $f^{-1}(0) = C$  implies  $C$  is closed, where  $f$  is continuous on  $\Omega$ , is trivial.

*Proof.* Set  $f$  to be the distance function from  $C$ , i.e.

$$f(x) = \inf_{y \in C} d(x, y). \quad \square$$

**Theorem 1.5** (Whitney's Theorem). *Let  $\Omega \subset \mathbb{R}^n$  be open and let  $C \subseteq \Omega$  be closed. Then there exists a smooth function  $f: \Omega \rightarrow \mathbb{R}$  such that  $f^{-1}(0) = C$ .*

*Proof.* Set  $V = \Omega \setminus C$ , and cover  $V$  by a countable collection of open balls,

$$V = \bigcup_{i=1}^{\infty} B(q_k, r_k).$$

This can always be done since  $V$  is open, and using the density of  $\mathbb{Q}$  in  $\mathbb{R}$  to pick only rational  $q_k, r_k$ . Now for each open ball  $B(q_k, r_k)$ , we can construct a smooth bump functions  $f_k$  such that  $f^{-1}(0) = \mathbb{R}^n \setminus B(q_k, r_k)$ ,  $f^{-1}(1) = \overline{B(q_k, r_k/2)}$ , and all derivatives of  $f_k$  vanish on  $\mathbb{R}^n \setminus B(q_k, r_k)$ .

Define the weights

$$c_k = \max_{\substack{|\alpha| \leq k \\ y \in B(q_k, r_k)}} \left| \frac{\partial^\alpha f_k(y)}{\partial x^\alpha} \right|.$$

Note that each  $c_k$  is well-defined: there are finitely many multi-indices  $\alpha$  given  $k$ , and each of the partials  $\partial^\alpha f / \partial x^\alpha$  is a smooth function over a compact set, hence bounded. Furthermore, each  $c_k \geq 1$ . Finally, set

$$f = \sum_{k=1}^{\infty} \frac{f_k}{2^k c_k}.$$

It is clear that  $f^{-1}(0) = C$ . We can show that the partial sums  $s_n$  converge; let  $\epsilon > 0$  and choose sufficiently large  $N$  such that  $1/2^N < \epsilon$ . Now for  $m > n \geq N$ , examine

$$|s_m(x) - s_n(x)| = \sum_{k=n+1}^m \frac{f_k(x)}{2^k c_k} \leq \sum_{k=n+1}^m \frac{1}{2^k} \leq \frac{1}{2^N} < \epsilon$$

Thus, the convergence is uniform, and  $f$  is  $C^0$ . For higher derivatives, we examine some  $\alpha$  partial of the sums, and use the same argument; at each stage,  $|\partial^\alpha f / \partial x^\alpha| < c_k$  whenever  $|\alpha| < k$ .  $\square$

## 1.3 Parametrized curves

**Definition 1.9.** A parametrized curve in  $\mathbb{R}^n$  is a smooth map  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^n$  for some  $\alpha, \beta$  with  $-\infty \leq \alpha < \beta \leq \infty$ .

*Remark.* Here, we will always implicitly assume that maps are continuous.

*Remark.* Such a curve is called regular if  $\gamma'(t) \neq 0$  for all  $t \in (\alpha, \beta)$ .

*Example.* The curve defined by

$$\gamma: \mathbb{R} \rightarrow \mathbb{R}^n, \quad t \mapsto a + tb$$

is a straight line through the point  $a$ , in the direction  $b$ .

*Example.* The curve defined by

$$\gamma: \mathbb{R} \rightarrow \mathbb{R}^2, \quad t \mapsto (\cos t, \sin t)$$

is the unit circle in  $\mathbb{R}^2$ , counter-clockwise.

*Example.* The curve defined by

$$\gamma: \mathbb{R} \rightarrow \mathbb{R}^3, \quad t \mapsto (t, \cos t, \sin t)$$

is a helix in  $\mathbb{R}^3$ , wrapped around the  $x$ -axis.

**Definition 1.10.** A diffeomorphism is a smooth map with a smooth inverse.

*Example.* Suppose that  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^n$  is a smooth curve. If we have a diffeomorphism  $\varphi: (\alpha', \beta') \rightarrow (\alpha, \beta)$ , then the smooth curve  $\eta = \gamma \circ \varphi$  is a reparametrization of  $\gamma$ . Note that

$$\eta'(t) = \gamma'(\varphi(t)) \varphi'(t).$$

**Lemma 1.6.** If  $\varphi: (\alpha', \beta') \rightarrow (\alpha, \beta)$  is a diffeomorphism, then  $\varphi'(t) \neq 0$  for all  $t \in (\alpha', \beta')$ .

**Definition 1.11.** If the diffeomorphism  $\varphi' > 0$ , we say that it is orientation preserving. If  $\varphi' < 0$ , we say that it is orientation reversing.

**Definition 1.12.** The arc length of a differentiable curve  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^n$ , starting at  $t_0$ , is defined as

$$s(t) = \int_{t_0}^t \|\gamma'(u)\| \, du.$$

We call  $s$  the arch length parameter.

*Remark.* If  $\gamma'(t) \neq 0$ , then  $s'(t) > 0$ .

**Definition 1.13.** A unit speed curve  $\gamma$  is one where  $\|\gamma'\| = 1$

**Lemma 1.7.** *Let  $\gamma$  be a regular smooth curve. Then its arc length parameter is a smooth function.*

*Proof.* Note that  $\gamma'$  and  $\langle \cdot, \cdot \rangle$  are smooth functions. Thus,

$$\frac{ds}{dt} = \|\gamma'(t)\| = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} > 0,$$

showing that  $ds/dt$  is smooth. □

**Lemma 1.8.** *The arc length function  $s$  is a diffeomorphism onto its image.*

*Proof.* This follows from the Inverse Function Theorem, using the smoothness of  $s$ . □

**Lemma 1.9.** *Let  $\varphi$  denote  $s^{-1}: (\alpha', \beta') \rightarrow (\alpha, \beta)$ . Then,  $\gamma \circ \varphi$  is a unit speed reparametrization of  $\gamma$ .*

*Remark.* Any other unit speed reparametrization is related to  $s$  by shifts and reflections.

*Proof.* Note that  $s$  is strictly increasing, so  $s', \varphi' > 0$ . Now,

$$\|(\gamma \circ \varphi)'(t)\| = \|\gamma'(\varphi(t))\| \cdot |\varphi'(t)| = s'(\varphi(t))\varphi'(t) = (s \circ \varphi)'(t) = 1. \quad \square$$

## 1.4 Curvature

**Definition 1.14.** Let  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^n$  be a regular curve. Let  $\Delta s$  be the length of the curve from  $\gamma(t)$  to  $\gamma(t + \Delta t)$ , and let  $\Delta \theta$  be the angle between these two vectors. Then, the curvature of  $\gamma$  at  $\gamma(t)$  is defined as

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta s}.$$

*Remark.* For a unit speed curve, the curvature is precisely  $\|\gamma''(s)\|$ .

*Example.* For a straight line  $a + bt$ , the curvature vanishes identically.

*Example.* For a circle of radius  $R$ , the curvature is  $1/R$ . Note that we parametrize

$$\gamma(s) = (x_0 + R \cos(t/R), y_0 + R \sin(t/R)).$$

**Definition 1.15.** Let  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^3$  be a regular  $C^2$  curve. Its curvature is defined as

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

*Remark.* It is easy to check that the curvature at a point is independent of parametrization.

**Definition 1.16.** Given a  $C^2$  plane curve  $\gamma$  such that  $\ddot{\gamma}(0) \neq 0$ , it is said to turn to the right when  $\det(\dot{\gamma}(0), \ddot{\gamma}(0))$  is negative.

**Definition 1.17.** Let  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^n$  be a regular  $C^2$  curve. Its curvature is defined as

$$\kappa(t) = \frac{\|\gamma'' \langle \gamma', \gamma' \rangle - \gamma' \langle \gamma', \gamma'' \rangle\|}{\|\gamma'(t)\|^4}.$$

**Definition 1.18.** Consider a regular smooth curve  $\gamma$ , such that  $\ddot{\gamma}(s) \neq 0$  at  $s$ . Then,  $\dot{\gamma}(s)$  and  $\ddot{\gamma}(s)$  are perpendicular, and span the osculating plane at  $\gamma(s)$ .

**Theorem 1.10.** Consider a regular smooth curve  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^n$ , such that  $\ddot{\gamma}(s) \neq 0$  at  $s$ .

1. For  $s_1, s_2, s_3$  sufficiently close to  $s$ , the points  $\gamma(s_i)$  are not colinear.
2. As  $s_1, s_2, s_3$  tend to  $s$ , the planes  $A(s_1, s_2, s_3)$  tend to the osculating plane at  $\gamma(s)$ .
3. The circumcircle  $C(s_1, s_2, s_3)$  associated with these points tend to a circle  $C(s)$  lying in the osculating plane which passes through  $\gamma(s)$ . Furthermore, this has radius  $1/\|\ddot{\gamma}(s)\|$ .
4. For  $s_1$  sufficiently close to  $s$ , there is a unique plane containing  $\gamma(s_1)$  and the tangent line to  $\gamma$  at  $s$ . As  $s_1$  tends to  $s$ , these planes converge to the osculating plane at  $\gamma(s)$ .

*Proof.* Let  $\{(s_{1n}, s_{2n}, s_{3n})\}_{n=1}^{\infty}$  be a sequence converging to  $(s, s, s)$ ; assume that  $s_{1n} < s_{2n} < s_{3n}$ . Suppose that  $\gamma(s_{1n}), \gamma(s_{2n}), \gamma(s_{3n})$  line on a line  $\ell_n$ , for every  $n$ . Define the planes  $V_n = \ell_n^\perp$ , and look at the functions

$$f_n^v(t) = \langle \gamma(t) - \gamma(s_{1n}), v \rangle, \quad v \in V_n.$$

Notice that  $s_{1n}, s_{2n}, s_{3n}$  are zeroes of  $f_n^v$ . Thus, we can choose  $s_{12n}, s_{23n}$ , where  $s_{1n} \neq s_{12n} \leq s_{2n} \leq s_{23n} \leq s_{3n}$ , such that  $(f_n^v)'(s_{12n}) = (f_n^v)'(s_{23n}) = 0$ . This gives

$$\langle \gamma'(s_{12n}), v \rangle = \langle \gamma'(s_{23n}), v \rangle = 0.$$

Repeating yields a point  $s_n$  such that  $\langle \gamma''(s_n), v \rangle = 0$ . Now, there is a neighbourhood of  $s$  on which

$$\|\gamma'(u) - \gamma'(s)\| < \epsilon, \quad \|\gamma''(u) - \gamma''(s)\| < \epsilon.$$

As a result,

$$\langle \gamma'(s), v \rangle \leq \|v\|\epsilon, \quad \langle \gamma''(s), v \rangle \leq \|v\|\epsilon.$$

Thus, given a vector in the osculating plane,  $w = a\gamma'(s) + b\gamma''(s)$ , we have  $\|w\|^2 = a^2 + b^2k^2$ , and

$$|\langle w, v \rangle| \leq (|a| + |b|)\|v\|\epsilon \leq c\|w\|\|v\|\epsilon.$$

This means that the osculating plane is part of the  $\epsilon$ -perpendicular region to  $V_n$ .  $\square$

**Lemma 1.11.** *Let  $d$  be the Euclidean distance between  $\gamma(0)$  and  $\gamma(s)$ , and  $s$  be the arc length between these two points. Then,*

$$\lim_{s \rightarrow 0} \frac{d}{s} = 1, \quad \lim_{s \rightarrow 0} \frac{d - s}{s^3} = -\frac{1}{24} \|\ddot{\gamma}(0)\|^2$$

## 1.5 Torsion

**Definition 1.19.** Let  $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^3$  be a  $C^2$ , regular space curve such that  $\ddot{\gamma}(s) \neq 0$ . The torsion of  $\gamma(s)$  is defined as

$$\tau(s) = \lim_{\Delta s \rightarrow 0} \frac{\Delta \theta}{\Delta s},$$

where  $\Delta \theta$  is the angle between the osculating planes to  $\gamma$  at  $s$  and  $s + \Delta s$ .

*Remark.* When we talk of the angle between two planes, we examine the normals

$$n(s) = \frac{\dot{\gamma}(s) \times \ddot{\gamma}(s)}{\|\dot{\gamma}(s) \times \ddot{\gamma}(s)\|}.$$

**Lemma 1.12.** *The torsion at  $\gamma(s)$  can be expressed at*

$$\tau(s) = \frac{\|(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \ddot{\gamma}'(s)\|}{\|\ddot{\gamma}(s)\|^2}.$$

**Lemma 1.13.** *The torsion at  $\gamma(s)$  can be expressed at*

$$\tau(t) = \frac{\det(\gamma'(t) \ \gamma''(t) \ \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

**Definition 1.20.** The tangent, principal normal, and binormal at  $\gamma(s)$  are

$$t(s) = \dot{\gamma}(s), \quad n(s) = \frac{\ddot{\gamma}(s)}{\|\ddot{\gamma}(s)\|}, \quad b(s) = t(s) \times n(s).$$

**Theorem 1.14** (Frennet-Serret). *For a unit speed curve with nowhere vanishing curvature, the following holds.*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

**Corollary 1.14.1.** *A regular space curve with non-zero curvature everywhere is planar if and only if  $\tau = 0$ .*