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Approximating continuous functions by smooth functions:

The Stone-Weierstrass Theorem

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1 Continuous functions

Definition 1.1. A function $f\colon X \to Y$ is continuous if $f^{-1}(\mathscr{O})$ is open in X for every open set $\mathscr{O}$ in Y.

Lemma 1.1. *If X and Y are metric spaces, a function $f\colon X \to Y$ is continuous if given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ for all $x' \in X$ satisfying $d(x, x') < \delta$.*

Definition 1.2. Let X and Y be metric spaces. A function $f\colon X \to Y$ is uniformly continuous if given $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ for all $x, x' \in X$ satisfying $d(x, x') < \delta$.

Theorem 1.2. *A continuous function on a compact metric space is uniformly continuous.*

2 The Stone-Weierstrass Theorem

Theorem 2.1. *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If A separates points on K and vanishes at no point of K, then the uniform closure of $\mathscr{A}$ consists of all real continuous functions on K.*