MA2201

Analysis II

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1 Sequences and series of functions

Definition 1.1. Let the functions $f_n \colon X \to Y$ be defined for all $n \in \mathbb{N}$ and let the sequences $\{f_n(x)\}$ converge for all $x \in X$. We define the function $f \colon X \to Y$ as

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in X$, and call f the limit of $\{f_n\}$. We also say that $\{f_n\}$ converges to f pointwise on X.

Example. Consider the functions $f_n \colon [0,1] \to \mathbb{R}$, $x \mapsto x^n$. It can be shown that $x^n \to 0$ when $x \in [0,1)$ and $x^n \to 1$ and $x^n \to 1$ when $x = 1$. Thus, $f = \lim_{n\to\infty} f_n$ is well defined.

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}.$$

Remark. Note that while each f_n is continuous in this example, the limit f is not.

Example. Consider the functions $f_n \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x/n$. We see that $f_n \to 0$. Note that 0 here denotes the zero function.

Definition 1.2. Let the functions $f_n \colon X \to Y$ be defined for all $n \in \mathbb{N}$ and let the sums $\{\sum f_n(x)\}$ converge for all $x \in X$. We define the function $f \colon X \to Y$ as

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in X$, and call f the sum of the series $\sum f_n$.

Example. Consider the functions $f_n \colon (0,1) \to \mathbb{R}$, $x \mapsto x^n$. Note that the sum

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

does indeed converge for all $x \in (0,1)$. Thus, the sum $f = \sum f_n$ is well defined.

$$f(x) = \frac{x}{1-x}.$$