MA3205

Geometry of Curves and Surfaces

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Contents

1 Curves

1.1 Introduction

Definition 1.1. A curve is a continuous map $\gamma\colon \mathbb{R} \to \mathbb{R}^n$.

Definition 1.2. A smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is C^∞, i.e. differentiable arbitrarily times.

Definition 1.3. A closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is periodic, i.e there exists some c such that $\gamma(t + c) = \gamma(t)$ for all $t \in \mathbb{R}$.

Example. Alternatively, a closed curve can be thought of as a continuous map $\gamma\colon S^1 \to \mathbb{R}^n$. For instance, given a closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ with period c, we can define the corresponding map

$$\tilde{\gamma}\colon S^1 \to \mathbb{R}^n, \qquad \tilde{\gamma}(e^{it}) = \gamma(ct/2\pi).$$

Definition 1.4. A simple curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is injective on its period.

Theorem 1.1 (Four Vertex Theorem)**.** *The curvature of a simple, closed, smooth plane curve has at least two local minima and two local maxima.*

Definition 1.5. A knot is a simple closed curve in $\mathbb{R}^3$.

Definition 1.6. The total absolute curvature of a knot K is the integral of the absolute value of the curvature, taken over the curve, i.e. it is the quantity

$$\oint_K |\kappa(s)|\, ds.$$

Example. The total absolute curvature of a circle is always 2π.

Theorem 1.2 (Fáry-Milnor Theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Definition 1.7. An immersed loop γ is such that γ' is never zero.

Definition 1.8. Two loops are isotopic if there exists an interpolating family of loops between them. Two immersed loops are isotopic if we can choose such an interpolating family of immersed loops.

Example. Without the restriction of immersion, any two loops $\gamma, \eta\colon S^1 \to \mathbb{R}^n$ would be isotopic, since we can always construct the linear interpolations

$$H\colon S^1 \times [0,1] \to \mathbb{R}^n, \qquad H(e^{i\theta}, t) = (1-t)\gamma(e^{i\theta}) + t\eta(e^{i\theta}).$$

Theorem 1.3 (Hirsch-Smale Theory)**.**

1. *Any two immersed loops in $\mathbb{R}^2$ are isotopic if and only if their turning numbers match.*
2. *Any two immersed loops in S^2 are isotopic if and only if their turning numbers modulo 2 match.*

1.2 Whitney's theorem

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a continuous function $f\colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Remark. The converse, i.e. $f^{-1}(0) = C$ implies C is closed, where f is continuous on Ω, is trivial.

Proof. Set f to be the distance function from C, i.e.

$$f(x) = \inf_{y \in C} d(x, y).$$

□

Theorem 1.5 (Whitney's Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a smooth function $f\colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Proof. Set $V = \Omega \setminus C$, and cover V by a countable collection of open balls,

$$V = \bigcup_{i=1}^{\infty} B(q_k, r_k).$$

This can always be done since V is open, and using the density of $\mathbb{Q}$ in $\mathbb{R}$ to pick only rational q_k, r_k. Now for each open ball $B(q_k, r_k)$, we can construct a smooth bump functions f_k such that $f^{-1}(0) = \mathbb{R}^n \setminus B(q_k, r_k)$, $f^{-1}(1) = \overline{B(q_k, r_k/2)}$, and all derivatives of f_k vanish on $\mathbb{R}^n \setminus B(q_k, r_k)$.

Define the weights

$$c_k = \max_{\substack{|\alpha| \leq k \\ y \in \overline{B(q_k, r_k)}}} \left| \frac{\partial^\alpha f_k}{\partial x^\alpha}(y) \right|.$$

Note that each c_k is well-defined: there are finitely many multi-indices α given k, and each of the partials $\partial^\alpha f / \partial x^\alpha$ is a smooth function over a compact set, hence bounded. Furthermore, each $c_k \geq 1$. Finally, set

$$f = \sum_{k=1}^{\infty} \frac{f_k}{2^k c_k}.$$

It is clear that $f^{-1}(0) = C$. We can show that the partial sums s_n converge; let $\epsilon > 0$ and choose sufficiently large N such that $1/2^N < \epsilon$. Now for $m > n \geq N$, examine

$$|s_m(x) - s_n(x)| = \sum_{k=n+1}^{m} \frac{f_k(x)}{2^k c_k} \leq \sum_{k=n+1}^{m} \frac{1}{2^k} \leq \frac{1}{2^N} < \epsilon$$

Thus, the convergence is uniform, and f is C^0. For higher derivatives, we examine some α partial of the sums, and use the same argument; at each stage, $|\partial^\alpha f / \partial x^\alpha| < c_k$ whenever $|\alpha| < k$. □

1.3 Parametrized curves

Definition 1.9. A parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.

Remark. Here, we will always implicitly assume that maps are continuous.

Remark. Such a curve is called regular if $\gamma'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^n, \qquad t \mapsto a + tb$$

is a straight line through the point a, in the direction b.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (\cos t, \sin t)$$

is the unit circle in $\mathbb{R}^2$, counter-clockwise.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^3, \qquad t \mapsto (t, \cos t, \sin t)$$

is a helix in $\mathbb{R}^3$, wrapped around the x-axis.

Definition 1.10. A diffeomorphism is a smooth map with a smooth inverse.

Example. Suppose that $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ is a smooth curve. If we have a diffeomorphism $\varphi\colon (\alpha', \beta') \to (\alpha, \beta)$, then the smooth curve $\eta = \gamma \circ \varphi$ is a reparametrization of γ. Note that

$$\eta'(t) = \gamma'(\varphi(t))\, \varphi'(t).$$

Lemma 1.6. *If $\varphi\colon (\alpha', \beta') \to (\alpha, \beta)$ is a diffeomorphism, then $\varphi'(t) \neq 0$ for all $t \in (\alpha', \beta')$.*

Definition 1.11. If the diffeomorphism $\varphi' > 0$, we say that it it orientation preserving. If $\varphi' < 0$, we say that it is orientation reversing.

Definition 1.12. The arc length of a differentiable curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$, starting at t_0, is defined as

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\|\, du.$$

We call s the arch length parameter.

Remark. If $\gamma'(t) \neq 0$, then $s'(t) > 0$.

Definition 1.13. A unit speed curve γ is one where $\|\gamma'\| = 1$

Lemma 1.7. *Let γ be a regular smooth curve. Then its arc length parameter is a smooth function.*

Proof. Note that γ' and $\langle\cdot,\cdot\rangle$ are smooth functions. Thus,

$$\frac{ds}{dt} = \|\gamma'(t)\| = \sqrt{\langle\gamma'(t), \gamma'(t)\rangle} > 0,$$

showing that ds/dt is smooth. □

Lemma 1.8. *The arc length function s is a diffeomorphism onto its image.*

Proof. This follows from the Inverse Function Theorem, using the smoothness of s. □

Lemma 1.9. *Let φ denote $s^{-1}\colon (\alpha', \beta') \to (\alpha, \beta)$. Then, $\gamma\circ\varphi$ is a unit speed reparametrization of γ.*

Remark. Any other unit speed reparametrization is related to s by shifts and reflections.

Proof. Note that s is strictly increasing, so $s', \varphi' > 0$. Now,

$$\|(\gamma\circ\varphi)'(t)\| = \|\gamma'(\varphi(t))\| \cdot |\varphi'(t)| = s'(\varphi(t))\varphi'(t) = (s\circ\varphi)'(t) = 1.$$ □

1.4 Curvature

Definition 1.14. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ be a regular curve. Let Δs be the length of the curve from $\gamma(t)$ to $\gamma(t+\Delta t)$, and let $\Delta\theta$ be the angle between these two vectors. Then, the curvature of γ at $\gamma(t)$ is defined as

$$\lim_{\Delta t\to 0} \frac{\Delta\theta}{\Delta s}.$$

Remark. For a unit speed curve, the curvature is precisely $\|\gamma''(s)\|$.

Example. For a straight line $a + bt$, the curvature vanishes identically.

Example. For a circle of radius R, the curvature is $1/R$. Note that we parametrize

$$\gamma(s) = (x_0 + R\cos(t/R), y_0 + R\sin(t/R)).$$

Definition 1.15. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^3$ be a regular C^2 curve. Its curvature is defined as

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

Remark. It is easy to check that the curvature at a point is independent of parametrization.

Definition 1.16. Given a C^2 plane curve γ such that $\ddot{\gamma}(0) \neq 0$, it is said to turn to the right when $\det(\dot{\gamma}(0), \ddot{\gamma}(0))$ is negative.

Definition 1.17. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ be a regular C^2 curve. Its curvature is defined as

$$\kappa(t) = \frac{\|\gamma''\langle\gamma', \gamma'\rangle - \gamma'\langle\gamma', \gamma''\rangle\|}{\|\gamma'(t)\|^4}.$$

Definition 1.18. Consider a regular smooth curve γ, such that $\ddot{\gamma}(s) \neq 0$ at s. Then, $\dot{\gamma}(s)$ and $\ddot{\gamma}(s)$ are perpendicular, and span the osculating plane at $\gamma(s)$.

Theorem 1.10. *Consider a regular smooth curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$, such that $\ddot{\gamma}(s) \neq 0$ at s.*

1. *For s_1, s_2, s_3 sufficiently close to s, the points $\gamma(s_i)$ are not colinear.*
2. *As s_1, s_2, s_3 tend to s, the planes $A(s_1, s_2, s_3)$ tend to the osculating plane at $\gamma(s)$.*
3. *The circumcircle $C(s_1, s_2, s_3)$ associated with these points tend to a circle $C(s)$ lying in the osculating plane which passes through $\gamma(s)$. Furthermore, this has radius $1/\|\ddot{\gamma}(s)\|$.*
4. *For s_1 sufficiently close to s, there is a unique plane containing $\gamma(s_1)$ and the tangent line to γ at s. As s_1 tends to s, these planes converge to the osculating plane at $\gamma(s)$.*

Proof. Let $\{(s_{1n}, s_{2n}, s_{3n})\}_{n=1}^{\infty}$ be a sequence converging to (s, s, s); assume that $s_{1n} < s_{2n} < s_{3n}$. Suppose that $\gamma(s_{1n})$, $\gamma(s_{2n})$, $\gamma(s_{3n})$ line on a line ℓ_n, for every n. Define the planes $V_n = \ell_n^{\perp}$, and look at the functions

$$f_n^v(t) = \langle\gamma(t) - \gamma(s_{1n}), v\rangle, \qquad v \in V_n.$$

Notice that s_{1n}, s_{2n}, s_{3n} are zeroes of f_n^v. Thus, we can choose s_{12n}, s_{23n}, where $s_{1n} \neq s_{12n} \leq s_{2n} \leq s_{23n} \leq s_{3n}$, such that $(f_n^v)'(s_{12n}) = (f_n^v)'(s_{23n}) = 0$. This gives

$$\langle\gamma'(s_{12n}), v\rangle = \langle\gamma'(s_{23n}), v\rangle = 0.$$

Repeating yields a point s_n such that $\langle\gamma''(s_n), v\rangle = 0$. Now, there is a neighbourhood of s on which

$$\|\gamma'(u) - \gamma'(s)\| < \epsilon, \qquad \|\gamma''(u) - \gamma''(s)\| < \epsilon.$$

As a result,

$$\langle\gamma'(s), v\rangle \leq \|v\|\epsilon, \qquad \langle\gamma''(s), v\rangle \leq \|v\|\epsilon.$$

Thus, given a vector in the osculating plane, $w = a\gamma'(s) + b\gamma''(s)$, we have $\|w\|^2 = a^2 + b^2k^2$, and

$$|\langle w, v\rangle| \leq (|a| + |b|)\|v\|\epsilon \leq c\|w\|\|v\|\epsilon.$$

This means that the osculating plane is part of the ϵ-perpendicular region to V_n. $\square$

Lemma 1.11. *Let d be the Euclidean distance between $\gamma(0)$ and $\gamma(s)$, and s be the arc length between these two points. Then,*

$$\lim_{s\to 0} \frac{d}{s} = 1, \qquad \lim_{s\to 0} \frac{d-s}{s^3} = -\frac{1}{24}\|\ddot{\gamma}(0)\|^2$$

1.5 Torsion

Definition 1.19. Let $\gamma\colon (\alpha,\beta) \to \mathbb{R}^3$ be a C^2, regular space curve such that $\ddot{\gamma}(s) \neq 0$. The torsion of $\gamma(s)$ is defined as

$$\tau(s) = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s},$$

where $\Delta\theta$ is the angle between the osculating planes to γ at s and $s + \Delta s$.

Remark. When we talk of the angle between two planes, we examine the normals

$$n(s) = \frac{\dot{\gamma}(s) \times \ddot{\gamma}(s)}{\|\dot{\gamma}(s) \times \ddot{\gamma}(s)\|}.$$

Lemma 1.12. *The torsion at $\gamma(s)$ can be expressed at*

$$\tau(s) = \frac{\|(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)\|}{\|\ddot{\gamma}(s)\|^2}.$$

Lemma 1.13. *The torsion at $\gamma(s)$ can be expressed at*

$$\tau(t) = \frac{\det(\gamma'(t)\ \gamma''(t)\ \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

Definition 1.20. The tangent, principal normal, and binormal at $\gamma(s)$ are

$$t(s) = \dot{\gamma}(s), \qquad n(s) = \frac{\ddot{\gamma}(s)}{\|\ddot{\gamma}(s)\|}, \qquad b(s) = t(s) \times n(s).$$

Theorem 1.14 (Frenet-Seret)**.** *For a unit speed curve with nowhere vanishing curvature, the following holds.*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Corollary 1.14.1. *A regular space curve with non-zero curvature everywhere is planar if and only if* $\tau = 0$.

Corollary 1.14.2. *Under suitable conditions, a space curve is completely determined by* κ, τ. *Two unit speed curves having the same curvature and torsion functions are identical, up to a rotation and a shift.*

Example. Suppose that a unit speed space curve γ lies entirely on a sphere of radius r. Then we have

$$\begin{aligned}
\|\gamma - a\|^2 &= r^2, \\
2\dot{\gamma} \cdot (\gamma - a) &= 0, \\
t \cdot (\gamma - a) &= 0, \\
\dot{t} \cdot (\gamma - a) + t \cdot \dot{\gamma} &= 0, \\
\kappa n \cdot (\gamma - a) + 1 &= 0, \\
n \cdot (\gamma - a) &= -1/\kappa, \\
\dot{n} \cdot (\gamma - a) + n \cdot \dot{\gamma} &= \dot{\kappa}/\kappa^2, \\
-\kappa t \cdot (\gamma - a) + \tau b \cdot (\gamma - a) &= \dot{\kappa}/\kappa^2, \\
b \cdot (\gamma - a) &= \dot{\kappa}/\kappa^2 \tau, \\
\dot{b} \cdot (\gamma - a) + b \cdot \dot{\gamma} &= (\dot{\kappa}/\kappa^2 \tau)', \\
-\tau n \cdot (\gamma - a) &= (\dot{\kappa}/\kappa^2 \tau)', \\
\tau/\kappa &= (\dot{\kappa}/\kappa^2 \tau)'.
\end{aligned}$$

Thus, our curve satisfies

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\kappa^2 \tau}\right).$$

By setting $\rho = 1/\kappa$, $\sigma = 1/\tau$, this reads

$$\rho = -\sigma \frac{d}{ds}(\dot{\rho}\sigma).$$

Conversely, assume that the above holds. Consider the quantity $\rho^2 + (\dot{\rho}\sigma)^2$; differentiating this gives

$$2\rho\dot{\rho} + 2\dot{\rho}\sigma \frac{d}{ds}(\dot{\rho}\sigma) = 2\rho\dot{\rho} + 2\dot{\rho}(-\rho) = 0.$$

Thus, we have

$$\rho^2 + (\dot{\rho}\sigma)^2 = r^2$$

for some positive constant r. Now, define the curve

$$\alpha = \gamma + \rho n + \dot{\rho}\sigma b.$$

Then,

$$\dot{\alpha} = \dot{\gamma} + \dot{\rho} n + \rho \dot{n} + \frac{d}{ds}(\dot{\rho}\sigma) b + (\dot{\rho}\sigma)\dot{b} = t + \dot{\rho} n - \rho\kappa t + \rho\tau b - \rho/\sigma b - \dot{\rho}\sigma\tau n = 0.$$

This means that the curve α is constant, say $\alpha = a$, whence

$$\|\gamma - a\|^2 = \|\rho n + \dot{\rho}\sigma b\|^2 = \rho^2 + (\dot{\rho}\sigma)^2 = r^2,$$

hence γ lies on a sphere.

1.6 Evolutes and involutes

Definition 1.21. The evolute of a curve is the locus of the centres of its osculating circles.

Remark. Consider a curve γ, with normal n and radius of curvature $\rho = 1/\kappa$. Then the equation of its evolute is given by $\gamma + \rho n$.

Example. The evolute of a circle is a constant curve, namely its centre.

Example. The evolute of a cycloid is a shifted copy of itself.

Definition 1.22. The involute of a curve is the locus of a point on a piece of taut string, as the string is wrapped around the curve.

Remark. Consider a curve γ, with tangent t. Then the equations of its involutes are given by $\gamma - t(s - a)$, for different choices of a. All such involutes are merely shifted copies of themselves.

Theorem 1.15. *The evolute of an involute of a curve is the curve itself.*

2 Surfaces

2.1 Introduction

Definition 2.1. A surface $\Sigma \subseteq \mathbb{R}^3$ is a subset satisfying the property that for any $p \in \Sigma$, there exists an open set $W \subseteq \mathbb{R}^3$, an open set $U \subseteq \mathbb{R}^2$, and a homeomorphism $\varphi\colon U \to W \cap \Sigma$.

Remark. The pair $(W \cap \Sigma, \varphi^{-1})$ is called a chart around p.

Remark. Without loss of generality, we can demand that the open set $U \subseteq \mathbb{R}^2$ be the unit disc centred at 0.

Example. Any affine plane in $\mathbb{R}^3$ is a surface.

Example. The unit sphere S^2 is a surface. Note that the north hemisphere is homeomorphic to the unit disc via a projection map; this gives us a chart for $(0, 0, 1)$. By symmetry, we can produce similar charts for any point on the sphere.

Remark. The sphere cannot be realized as a surface using only a single chart. This is because S^2 is compact while the unit disc is not, hence they cannot be homeomorphic.

Example. The cylinder defined by $x^2 + y^2 = 1$ is a surface. Note that we can produce the homeomorphism $(1, \theta, z) \mapsto (e^z, \theta)$, which maps the cylinder to $\mathbb{R}^2 \setminus 0$.

Remark. Note that the cylinder is just $S^1 \times \mathbb{R}$, and the plane minus the origin is just $S^1 \times (0, \infty)$. Thus we need only find a homeomorphism mapping $\mathbb{R} \to (0, \infty)$.

Example. Let $U \subseteq \mathbb{R}^2$ be open, and let $f \colon U \to \mathbb{R}$ be continuous. The graph of f, which is the set $\Gamma_f = \{(x, y, f(x, y)) : (x, y) \in U\}$, is a surface.

Lemma 2.1. *All homeomorphisms $\psi \colon \mathbb{R}^3 \to \mathbb{R}^3$ take surfaces to surfaces.*