MA2201: Analysis II

Differentiation

Spring 2021

Satvik Saha
19MS154

*Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.*

The origins of differential calculus lie in our attempts to approximate various functions using linear ones. Suppose that we have been given a curve described by the function f, and we want to *locally* approximate the function around a point x using a straight line. In other words, for a small shift h, we want to write

$$f(x+h) \approx f(x) + kh.$$

Here, k is the slope of the straight line. In order to obtain k, we can rearrange the above to get

$$k \approx \frac{f(x+h) - f(x)}{h}.$$

As we pick smaller and smaller neighbourhoods of x, we want our approximation to get better and better. Thus, if such an approximation is possible, then the value of k must stabilize. This means that the limit

$$k = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

must exist. Note that this immediately forces the continuity of f, since

$$\lim_{h \to 0} f(x+h) - f(x) = \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0k = 0,$$

whereby $\lim_{x \to a} f(x) = f(a)$. Splitting the limit is justified because the individual limits exist. If such a limit k exists, we call it the derivative of f at x, denoted $f'(x)$. We are now able to write

$$f(x+h) \approx f(x) + f'(x)h.$$

Definition 2.1 (Derivative)**.** The derivative of a function $f\colon [a, b] \to \mathbb{R}$ at a point $x \in [a, b]$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if such a limit exists. Note that we only demand a one-sided limit if x is an endpoint. If the derivative of f exists at every point in $[a, b]$, we say that f is differentiable on $[a, b]$.

Note that the process we described can be generalised to multivariable functions.