MA3203

Analysis IV

Spring 2022

Satvik Saha

19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

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1.1   Introduction

Measure theory seeks to generalize the notions of *length*, *area*, *volume* to more general sets: this new notion is called a *measure*. This also allows us to generalize the notion of Riemann integration to a broader class of functions.

Recall that continuous functions, or at least functions with finitely many discontinuities on a closed interval are Riemann integrable. The Dirichlet function, which is discontinuous everywhere, is not.

$$f \colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This is simply because every non-empty interval contains at least one rational and one irrational number, so the Darboux lower sum is always 0 and the upper sum is always 1 regardless of the choice of partition.

On the other hand, if we had to assign a particular value to this integral, intuition tells us that it ought to be zero. After all, the function f attains a non-zero value only on the countable set $\mathbb{Q} \cap [0, 1]$; it is zero almost everywhere. Formally, we will show that f is non-zero on a set of zero Lebesgue measure, which will allow us to set this Lebesgue integral to zero. We will see that with this new formulation of integration, we end up partitioning the range of f rather than it's domain, and write

$$\int f = 0 \cdot \mu([0,1] \setminus \mathbb{Q}) + 1 \cdot \mu([0,1] \cap \mathbb{Q}) = 0.$$

Theorem 1.1 (Lebesgue criterion). *A function $f \colon [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuities has Lebesgue measure zero. This means that the set of discontinuities of f must be coverable by countably many intervals (x_i, y_i) such that the sum of lengths $y_i - x_i$ can be made arbitrarily small.*

1.2   Basic definitions

Definition 1.1. Let X be a set, and let Σ be a collection of subsets of X. We say that Σ is a σ-algebra over X if it satisfies the following.

1. Σ contains X.

2. Σ is closed under complementation.

3. Σ is closed under countable unions.

Remark. The first condition can be replaced by forcing Σ to be non-empty.

Remark. The following properties follow immediately.

1. Σ contains $\emptyset$.

2. Σ is closed under countable intersections.

3. Σ is closed under differences.

Example. Given a set X, its power set forms a σ-algebra over X.

Example. Given a set X, the set $\{\emptyset, X\}$ forms a σ-algebra over X.

Example. Given an uncountable set X, the following set forms the co-countable σ-algebra over X.
$$\{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}.$$

Definition 1.2. Let X be a set, and let Σ be a σ-algebra over X. We say that a function $\mu \colon \Sigma \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a measure if it satisfies the following.

1. μ is non-negative.

2. $\mu(\emptyset) = 0$.

3. μ is additive over countable unions of disjoint sets, i.e. for any countable collection $\{E_i\}_{i=1}^{\infty}$ such that $E_i \cap E_j = \emptyset$ for all pairs, we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Example. The trivial zero measure sends every set to zero.

Example. In probability theory, we look at the event space $\mathcal{E}$ as a σ-algebra over the sample space Ω. The probability function P is a measure on this event space such that $P(\Omega) = 1$.

Example. Let X be a set, and let Σ be its power set as a σ-algebra over X. Fix $x_0 \in X$, and define

$$\mu \colon \Sigma \to [0, \infty], \qquad E \mapsto \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E. \end{cases}$$

This is called the Dirac measure.

Example. Let X be a set, and let Σ be its power set as a σ-algebra over X. Define

$$\mu \colon \Sigma \to [0, \infty], \qquad E \mapsto \begin{cases} 0, & \text{if } E = \emptyset, \\ |E|, & \text{if } E \text{ is finite}, \\ \infty, & \text{otherwise}. \end{cases}$$

This is called the counting measure.

Example. Let X be an uncountable set, and let Σ be its co-countable sigma algebra. Define

$$\mu \colon \Sigma \to [0, \infty], \qquad E \mapsto \begin{cases} 0, & \text{if } E \text{ is countable}, \\ 1, & \text{if } E^c \text{ is countable}. \end{cases}$$

Lemma 1.2. *Let (X, Σ, μ) be a measure space. Then, the following properties hold.*

1. *If $A, B \in \Sigma$ such that $A \subseteq B$, then $\mu(A) \le \mu(B)$.*
2. *If $A, B \in \Sigma$ such that $A \subseteq B$ and $\mu(A)$ is finite, then $\mu(B - A) = \mu(B) - \mu(A)$.*
3. *If $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \Sigma$, then*

$$\mu \left(\bigcup_{i=1}^{\infty} E_i \right) \le \sum_{i=1}^{\infty} \mu(E_i).$$

Theorem 1.3 (Continuity from below)**.** *Let (X, Σ, μ) be a measure space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets such that $E_i \subseteq E_j$ for all $i < j$. Then,*

$$\mu \left(\bigcup_{i=1}^{\infty} E_i \right) = \lim_{n \to \infty} \mu(E_n).$$

Proof. Define $F_i = E_i - E_{i-1}$, denoting $E_0 = \emptyset$. Thus,

$$\mu \left(\bigcup_{i=1}^{\infty} E_i \right) = \mu \left(\bigcup_{i=1}^{\infty} F_i \right) = \sum_{i=1}^{\infty} \mu(F_i).$$

Also note that

$$\sum_{i=1}^{n} \mu(F_i) = \mu \left(\bigcup_{i=1}^{\infty} F_n \right) = \mu(E_n).$$

Since the infinite sum in the first part is the limit of partial sums, we have our result. $\qquad \square$

Theorem 1.4 (Continuity from above). *Let (X, Σ, μ) be a measure space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets such that $E_i \supseteq E_j$ for all $i < j$. Further assume that $\mu(E_1)$ is finite. Then,*

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Define $F_i = E_1 - E_n$, and note that $F_i \subseteq F_j$ for all $i < j$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{n\to\infty} \mu(F_n).$$

This can be rewritten as

$$\mu\left(E_1 - \bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_1 - E_n).$$

Using the subtractive property and the fact that each $\mu(E_i)$ is finite,

$$\mu(E_1) - \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_1) - \mu(E_n).$$

Pulling the constant $\mu(E_1)$ out from the limit and subtracting from both sides gives our result. $\square$

Example. Consider the counting measure μ on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, and define $E_n = \{n, n+1, \dots\}$. Then,

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = 0, \qquad \lim_{n\to\infty} \mu(E_n) = \infty.$$

Updated on January 13, 2022