MA 2103 : Mathematical Methods II

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Fourier Series and Transforms (M.L. Boas, Chapter 7)

Section 9. Problem 2. Write the following as the sum of an even function and an odd function.

(a) $\ln|1-x|$.

(b) $(1+x)(\sin x + \cos x)$.

Solution. Given a function f, note that we can write it in the form

$$f(x) = \underbrace{\frac{1}{2}(f(x) + f(-x))}_{g(x)} + \underbrace{\frac{1}{2}(f(x) - f(-x))}_{h(x)}.$$

Note that g is even and h is odd, because

$$g(-x) = \frac{1}{2}(f(-x) + f(x)) = g(x), \qquad h(-x) = \frac{1}{2}(f(-x) - f(x)) = -h(x).$$

Thus, $f = g + h$ is the desired decomposition.

(a) We write $f = g + h$, where

$$g(x) = \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|, \qquad h(x) = \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|.$$

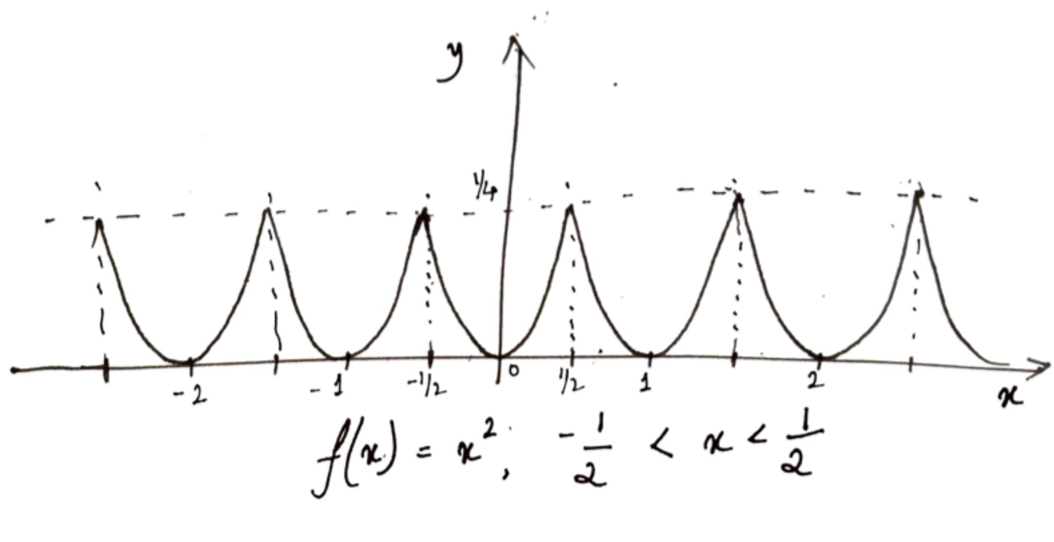
(b) Again, $f = g + h$, where

$$g(x) = \frac{1}{2}(1+x)(\sin x + \cos x) + \frac{1}{2}(1-x)(-\sin x + \cos x) = \cos x + x\sin x,$$
$$h(x) = \frac{1}{2}(1+x)(\sin x + \cos x) - \frac{1}{2}(1-x)(-\sin x + \cos x) = \sin x + x\cos x.$$

Problem 9. Sketch several periods of the following function (given over one period), decide whether its even or odd, and expand it as a Fourier series.

$$f(x) = x^2, \qquad -\frac{1}{2} < x < \frac{1}{2}.$$

Solution.



We note that the given function is even. Thus, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x.$$

We calculate

$$a_0 = 2\int_0^{\frac{1}{2}} x^2 \, dx = 2 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{12}.$$

For $n \geq 1$,

$$\begin{aligned} a_n = 4\int_0^1 x^2 \cos 2n\pi x \, dx &= \cancel{\frac{2}{n\pi} x^2 \sin 2n\pi x\Big|_0^{\frac{1}{2}}} - \frac{2}{n\pi}\int_0^{\frac{1}{2}} 2x \sin 2n\pi x \, dx \\ &= \frac{1}{n^2\pi^2} x \cos 2n\pi x\Big|_0^{\frac{1}{2}} - \cancel{\frac{1}{n^2\pi^2}\int_0^{\frac{1}{2}} \cos 2n\pi x \, dx} \\ &= \frac{1}{n^2\pi^2} \cos n\pi. \end{aligned}$$

Thus,

$$f(x) \;=\; \frac{1}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \cos n\pi \cos 2n\pi x.$$

Problem 14. Give algebraic proofs for even and odd functions that

(a) even times even = even; odd times odd = even; even times odd = odd.

(b) The derivative of an even function is odd; the derivative of an odd function is even.

Solution. Consider arbitrary even and odd functions f_i and g_i respectively. Thus,

$$f_i(x) = f_i(-x), \qquad g_i(x) = -g_i(-x).$$

(a) Firstly, we show that $f_1 f_2$ is even.

$$(f_1f_2)(x) = f_1(x)f_2(x) = f_1(-x)f_2(-x) = (f_1f_2)(-x).$$

Next, we show that g_1g_2 is even.

$$(g_1g_2)(x) = g_1(x)g_2(x) = (-g_1(-x))(-g_2(-x)) = (g_1g_2)(-x).$$

Finally, we show that fg is odd.

$$(fg)(x) = f(x)g(x) = f(-x)(-g(-x)) = -(fg)(-x).$$

These establish the desired properties.

(b) Define $h\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto -x$. Thus, the definitions of even and odd functions imply that

$$f \circ h = f, \qquad g \circ h = -g.$$

Differentiating and applying the chain rule,

$$(f' \circ h)(h') = f', \qquad (g' \circ h)(h') = -g'.$$

However, $h' = -1$, so

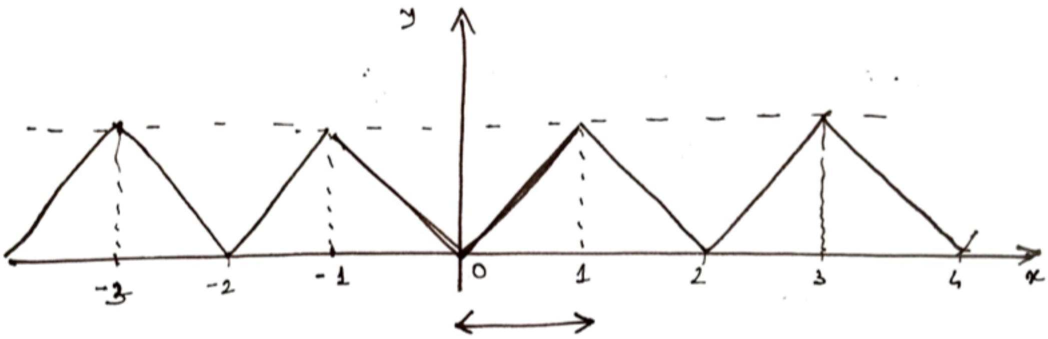
$$f' \circ h = -f', \qquad g' \circ h = g'.$$

This is precisely the definition of f' and g' being *odd* and *even* respectively.

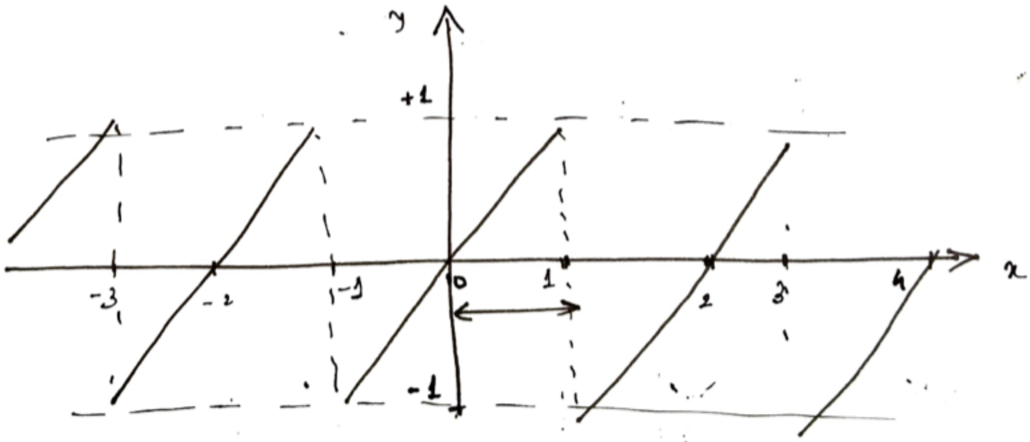
$$f'(-x) = -f'(x), \qquad g'(-x) = g(x).$$

Problem 15. Given $f(x) = x$ for $0 < x < 1$, sketch the even function f_c of period 2 and the odd function f_s of period 2, each of which equals $f(x)$ on $0 < x < 1$. Expand f_c in a cosine series and f_s in a sine series.

Solution.



$$f_c(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$



$$f_s(x) = x, \quad -1 < x < 1$$

We calculate the Fourier coefficients together.

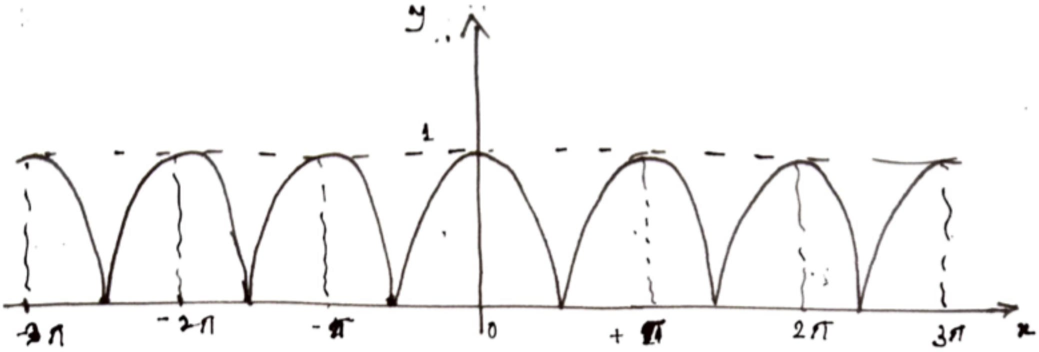
$$a_0 = \int_0^1 x\,dx = \frac{1}{2},$$
$$a_n = 2\int_0^1 x \cos n\pi x\,dx = \cancel{\frac{2}{n\pi} x \sin n\pi x\Big|_0^1} - \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx = \frac{2}{n^2\pi^2}(1 - \cos n\pi),$$
$$b_n = 2\int_0^1 x \sin n\pi x\,dx = -\frac{2}{n\pi} x \cos n\pi x\Big|_0^1 + \frac{2}{n\pi}\cancel{\int_0^1 \cos n\pi x\,dx} = -\frac{2}{n\pi}\cos n\pi.$$

Thus,

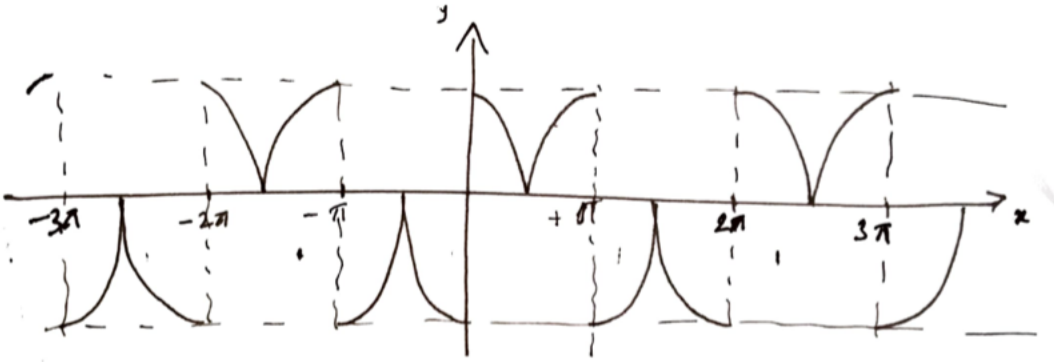
$$f_c = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}(1 - \cos n\pi)\cos n\pi x, \qquad f_s = -\sum_{n=1}^{\infty} \frac{2}{n\pi}\cos n\pi \sin n\pi x.$$

Problem 19. Given $f(x) = |\cos x|$ for $0 < x < \pi$, sketch several periods of the even function f_c and the odd function f_s of periods 2π and the function f_p of period π, each of which equals $f(x)$ on $0 < x < \pi$. Expand each of them in an appropriate Fourier series.

Solution.



$f_c(x) = |\cos x| = f_p(x)$



$$f_s(x) = \begin{cases} -|\cos x|, & -\pi < x < 0, \\ |\cos x|, & 0 < x < \pi \end{cases}$$

Note that $f_c = f_p$. We calculate

$$a_0 = \frac{1}{\pi}\int_0^\pi |\cos x|\,dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}.$$

When $n \neq 1$, we have

$$\begin{aligned} a_n = \frac{2}{\pi}\int_0^\pi |\cos x| \cos nx\,dx &= \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx + \cos x \cos(n\pi - nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx + \cos x \cos nx \cos n\pi + \cancel{\cos x \sin nx \sin n\pi}\,dx \\ &= \frac{2}{\pi}(1+\cos n\pi)\cdot\frac{1}{2}\int_0^{\pi/2} \cos{(n+1)x} + \cos{(n-1)x}\,dx \\ &= \frac{1}{\pi}(1+\cos n\pi)\left[\frac{1}{n+1}\sin{(n+1)x} + \frac{1}{n-1}\sin{(n-1)x}\right]_0^{\pi/2}. \end{aligned}$$

Note that this vanishes when n is odd. Otherwise, note that $\sin(2n+1)\pi/2 = (-1)^n$, so

$$a_{2n} = \frac{2}{\pi}\left[\frac{1}{2n+1}(-1)^n - \frac{1}{2n-1}(-1)^n\right] = -\frac{4}{\pi}\cdot\frac{(-1)^n}{4n^2-1}.$$

We can calculate a_1 separately. Note that the first three lines of the previous process still hold. Thus,

$$a_1 = \frac{2}{\pi}(1+\cos\pi)\int_0^{\pi/2}\cos^2 x\,dx = 0.$$

Again, when $n \neq 1$, we have

$$\begin{aligned}
b_n = \frac{2}{\pi}\int_0^{\pi}|\cos x|\sin nx\,dx &= \frac{2}{\pi}\int_0^{\pi/2}\cos x\sin nx + \cos x\sin(n\pi - nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi/2}\cos x\sin nx + \cancel{\cos x\cos nx}\sin n\pi - \cos x\sin nx\cos n\pi\,dx \\
&= \frac{2}{\pi}(1-\cos n\pi)\cdot\frac{1}{2}\int_0^{\pi/2}\sin(n+1)x + \sin(n-1)x\,dx \\
&= -\frac{1}{\pi}(1-\cos n\pi)\left[\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^{\pi/2}.
\end{aligned}$$

Note that this vanishes when n is even. Otherwise, we calculate

$$\begin{aligned}
b_{4k+1} &= -\frac{2}{\pi}\left[\frac{1}{4k+2}(\cos(2k+1)\pi - 1) + \frac{1}{4k}(\cos 2k\pi - 1)\right] \\
&= \frac{2}{\pi}\cdot\frac{1}{2k+1}, \\
b_{4k-1} &= -\frac{2}{\pi}\left[\frac{1}{4k}(\cos 2k\pi - 1) + \frac{1}{4k-2}(\cos(2k-1)\pi - 1)\right] \\
&= \frac{2}{\pi}\cdot\frac{1}{2k-1}.
\end{aligned}$$

From the first three lines of the previous process,

$$b_1 = \frac{1}{\pi}(1-\cos\pi)\int_0^{\pi/2}\sin 2x\,dx = -\frac{2}{\pi}\cdot\frac{1}{2}\cos 2x\Big|_0^{\pi/2} = \frac{2}{\pi}.$$

Thus,

$$\begin{aligned}
f_c(x) = f_p(x) &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2nx, \\
f_s(x) &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\sin(4n+1)x + \sin(4n+3)x\right].
\end{aligned}$$

Problem 25. Suppose that $f(x)$ and $f'(x)$ are both expanded as Fourier series on $(-\pi, \pi)$, with coefficients a_n, b_n and a_n', b_n'. Show that $b_n' = -na_n$, and obtain a simlar relation between a_n' and b_n using the integral definitions. Show that this is the same result obtained by differentiating the series term by term.

Solution. We start with the integrals

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx.$$

We use integration by parts,

$$\int u\,dv = uv - \int v\,du, \text{ or } \int uv\,dx = u\int v\,dx - \int u'\int v\,dx\,dx.$$

Thus,

$$\begin{aligned} a_n &= +\frac{1}{n\pi}\cancel{f(x)\sin nx\Big|_{-\pi}^{+\pi}} - \frac{1}{n\pi}\int_{-\pi}^{+\pi} f'(x)\sin nx\,dx. \\ b_n &= -\frac{1}{n\pi}f(x)\cos nx\Big|_{-\pi}^{+\pi} + \frac{1}{n\pi}\int_{-\pi}^{+\pi} f'(x)\cos nx\,dx. \end{aligned}$$

We recognize the second terms as $-b'_n/n$ and a'_n/n respectively. Note that the first term in the first sum vanishes because $\sin n\pi = 0$. In the second sum, we use the Dirichlet conditions and the 2π periodicity of f to extrapolate its values at the points $\pm\pi$.

$$f(-\pi) = f(+\pi) = \frac{1}{2}\left[\lim_{x\to-\pi^+} f(x) + \lim_{x\to+\pi^-} f(x)\right].$$

This follows since the periodicty of f guarantees $\lim_{x\to+\pi^+} f(x) = \lim_{x\to-\pi^+} f(x)$, and $\lim_{x\to+\pi^-} f(x) = \lim_{x\to-\pi^-}$. Consequentially, the first term in b_n also vanishes. Thus, multiplying by n, we obtain

$$na_n = -b'_n, \qquad nb_n = a'_n.$$

We can also see this from the Fourier series of f and f', assuming they both exist.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx.$$

Differentiating this once,

$$f'(x) = \sum_{n=1}^{\infty} -na_n\sin nx + nb_n\cos nx = a'_0 + \sum_{n=1}^{\infty} b'_n\sin nx + a'_n\cos nx.$$

The desired relations, $b'_n = -na_n$ and $a'_n = nb_n$ follow directly from comparing terms. Note that $a'_0 = 0$.

Problem 26. Find the Fourier series of the given function.

$$f(x) = \begin{cases} 3x^2 + 2x^3, & -1 < x < 0, \\ 3x^2 - 2x^3, & 0 < x < 1. \end{cases}$$

Differentiate both the series and the function repeatedly until you obtain a discontinuous function. Plot these, along with a few terms of the corresponding Fourier series. Note the number of terms needed for a good fit.

Solution. Note that the given function is even. Thus,

$$\begin{aligned} a_0 &= \int_0^1 3x^2 - 2x^3\,dx = 1 - \frac{1}{2} = \frac{1}{2}, \\ a_n = 2\int_0^1 (3x^2 - 2x^3)\cos n\pi x\,dx &= \frac{2}{n\pi}\cancel{(3x^2 - 2x^3)\sin n\pi x\Big|_0^1} - \frac{2}{n\pi}\int_0^1 6(x - x^2)\sin n\pi x\,dx \\ &= \frac{12}{n^2\pi^2}\cancel{(x - x^2)\cos n\pi x\Big|_0^1} - \frac{12}{n^2\pi^2}\int_0^1 (1 - 2x)\cos n\pi x\,dx \\ &= -\frac{12}{n^3\pi^3}\cancel{(1 - 2x)\sin n\pi x\Big|_0^1} + \frac{12}{n^3\pi^3}\int_0^1 (-2)\sin n\pi x\,dx \\ &= -\frac{24}{n^4\pi^4}\cos n\pi x\Big|_0^1 \\ &= -\frac{24}{n^4\pi^4}(1 - \cos n\pi). \end{aligned}$$

Thus,

$$f(x) = \frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{48}{n^4\pi^4}\cos n\pi x.$$

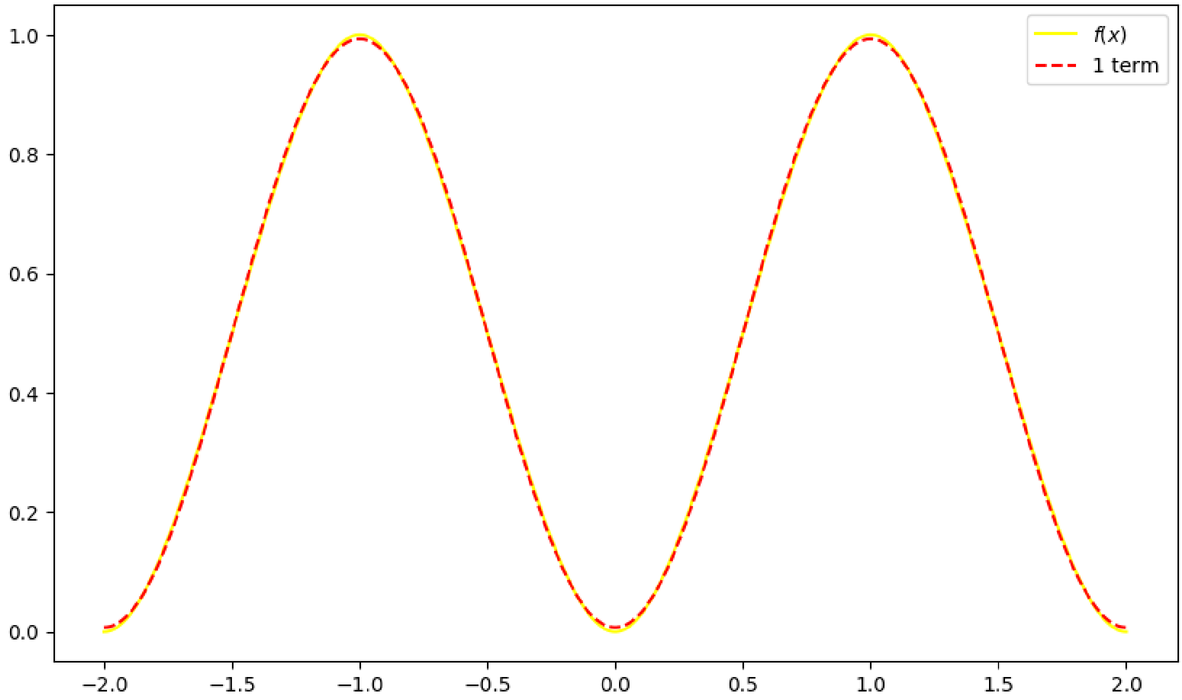


Figure 1: Plot of $f(x)$. A single term of the Fourier series gives a very good fit.

We differentiate the original function to get

$$f'(x) = \begin{cases} 6(x + x^2), & -1 < x < 0, \\ 6(x - x^2), & 0 < x < 1. \end{cases}$$

$$f''(x) = \begin{cases} 6(1 + 2x), & -1 < x < 0, \\ 6(1 - 2x), & 0 < x < 1. \end{cases}$$

$$f'''(x) = \begin{cases} 12, & -1 < x < 0, \\ -12, & 0 < x < 1. \end{cases}$$

Note that $f'''(x)$ is discontinuous. We can similarly differentiate the Fourier series

$$f'(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{48}{n^3\pi^3} \sin n\pi x,$$

$$f''(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{48}{n^2\pi^2} \cos n\pi x,$$

$$f'''(x) = -\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{48}{n\pi} \sin n\pi x.$$

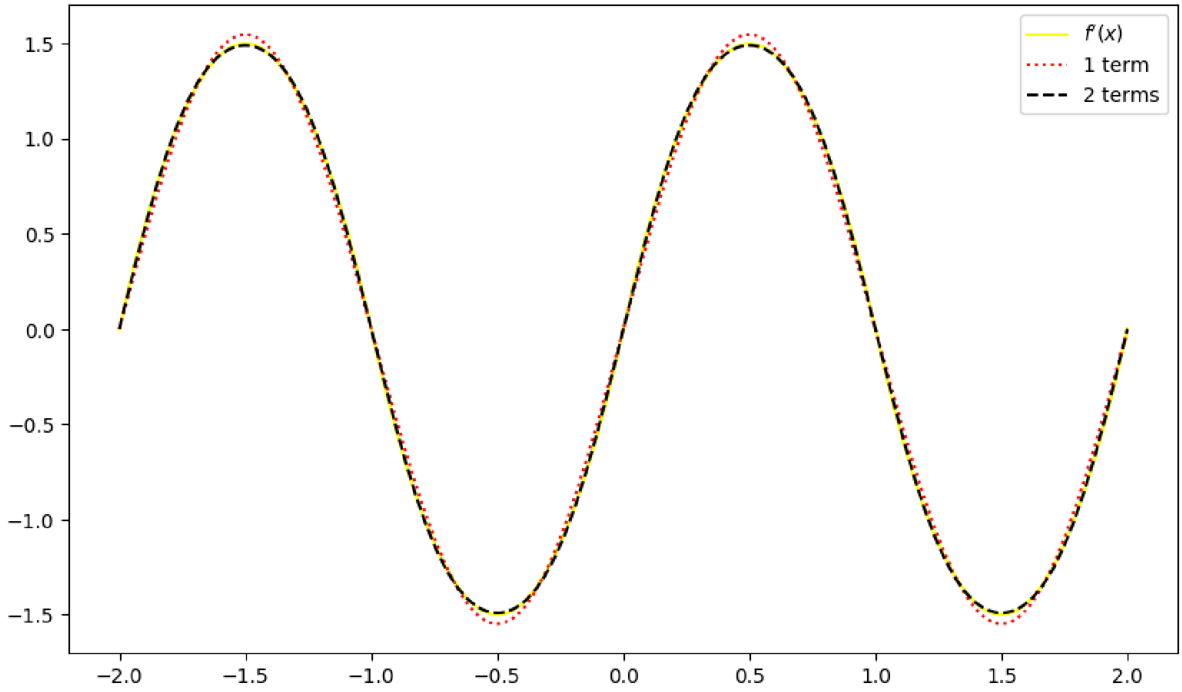


Figure 2: Plot of $f'(x)$. Two terms of the Fourier series give a good fit.

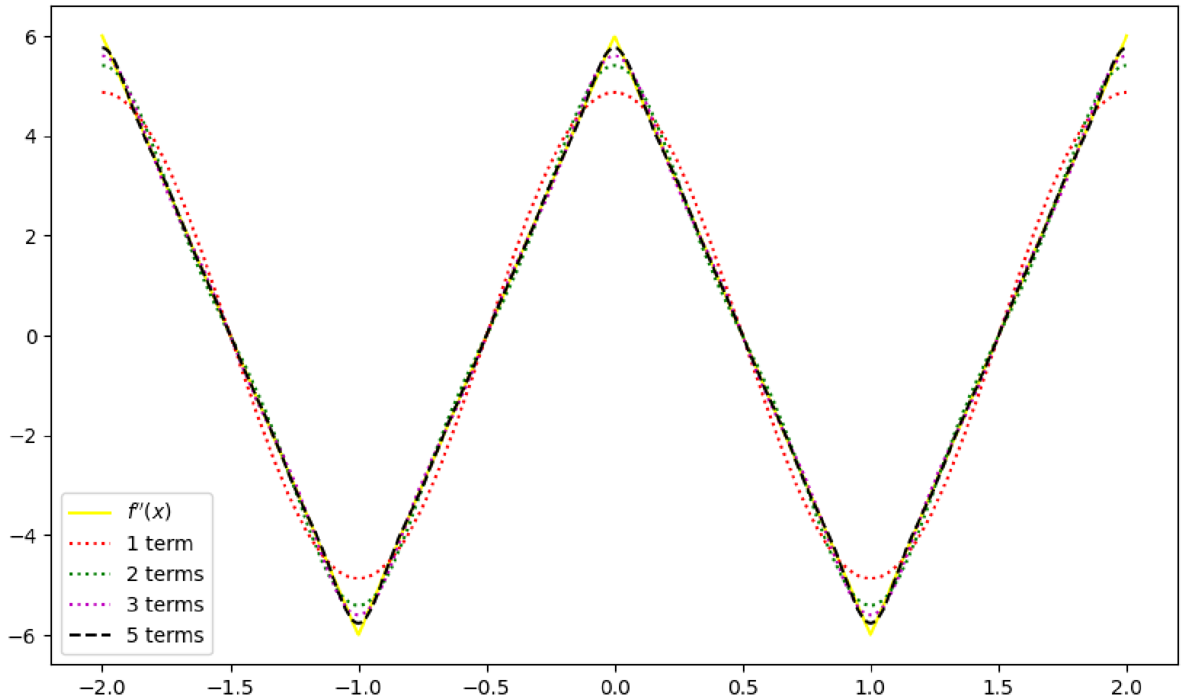


Figure 3: Plot of $f''(x)$. Five terms of the Fourier series give a good fit, although the sharp corners never fit properly.

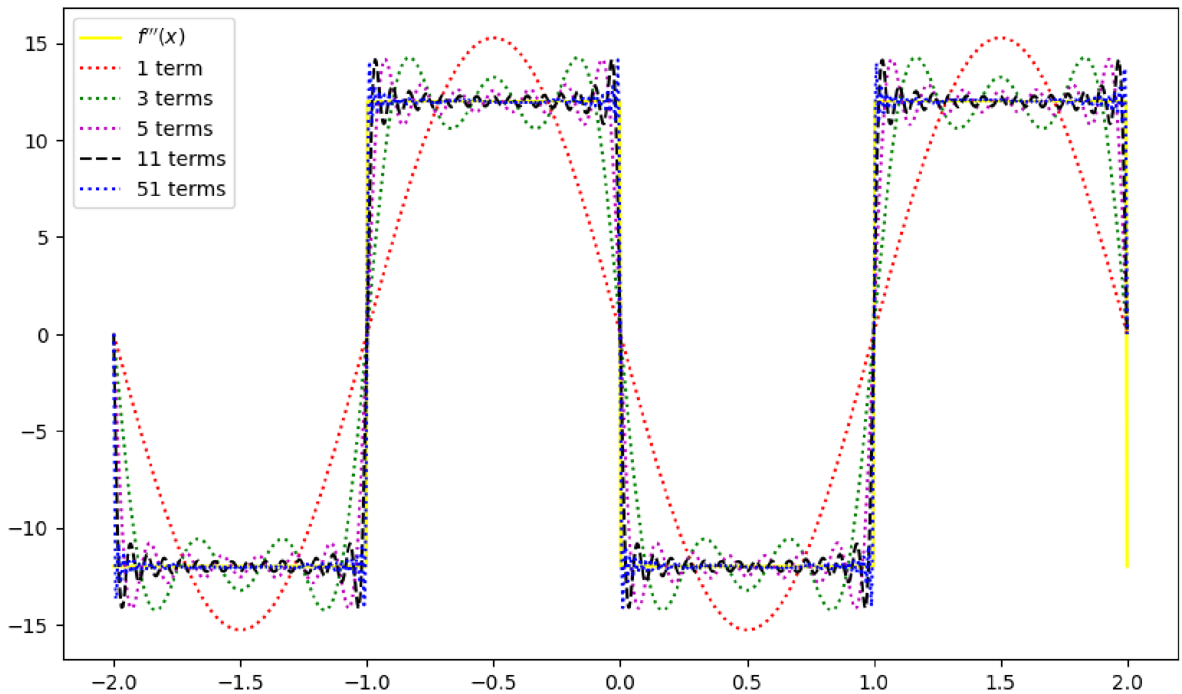


Figure 4: Plot of $f'''(x)$. Fifteen terms of the Fourier series give a good fit, although fringes always remain near the discontinuity, even upon allowing around 50 terms.