MA1101: Mathematics I

Solutions for Problem Sheet II

Problem 1. Denote $X = \{1, 2, 3, 4, 5\}$ and $\Delta = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$, i.e. the *diagonal* of $X \times X$. We list examples of the described relations on X without proof below.

(i) Relations that are reflexive, not symmetric, not transitive.

$$\begin{aligned} R_{11} &= \Delta \cup \{(1, 2), (2, 3)\}, \\ R_{12} &= \Delta \cup \{(2, 3), (3, 4)\}, \\ R_{12} &= \Delta \cup \{(3, 4), (4, 5)\}. \end{aligned}$$

(ii) Relations that are not reflexive, are symmetric, not transitive.

$$\begin{aligned} R_{21} &= \{(1, 2), (2, 1)\}, \\ R_{22} &= \{(2, 3), (3, 2)\}, \\ R_{22} &= \{(3, 4), (4, 3)\}. \end{aligned}$$

(iii) Relations that are not reflexive, not symmetric, are transitive.

$$\begin{aligned} R_{31} &= \{(1, 2)\}, \\ R_{32} &= \{(2, 3)\}, \\ R_{32} &= \{(3, 4)\}. \end{aligned}$$

(iv) Relations that are reflexive, are symmetric, not transitive.

$$\begin{aligned} R_{41} &= \Delta \cup \{(1, 2), (2, 1), (2, 3), (3, 2)\}, \\ R_{42} &= \Delta \cup \{(2, 3), (3, 2), (3, 4), (4, 3)\}, \\ R_{42} &= \Delta \cup \{(3, 4), (4, 3), (4, 5), (5, 4)\}. \end{aligned}$$

(v) Relations that are not reflexive, are symmetric, are transitive.

$$\begin{aligned} R_{51} &= \emptyset, \\ R_{52} &= \{(1, 1)\}, \\ R_{52} &= \{(2, 2)\}. \end{aligned}$$

(vi) Relations that are reflexive, not symmetric, are transitive.

$$\begin{aligned} R_{61} &= \Delta \cup \{(1, 2)\}, \\ R_{62} &= \Delta \cup \{(2, 3)\}, \\ R_{62} &= \Delta \cup \{(3, 4)\}. \end{aligned}$$

(vii) Relations that are not reflexive, not symmetric, not transitive.

$$\begin{aligned} R_{71} &= \{(1, 2), (2, 3)\}, \\ R_{72} &= \{(2, 3), (3, 4)\}, \\ R_{72} &= \{(3, 4), (4, 5)\}. \end{aligned}$$

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(viii) Relations that are reflexive, are symmetric, are transitive.

$$R_{81} = X \times X,$$
$$R_{82} = \Delta,$$
$$R_{82} = \Delta \cup \{(1,2),(2,1)\}.$$

Problem 2. Let $x \in X$ be arbitrary. Using the given property of R, there exists $a \in X$ such that xRa. By symmetry of R, we have aRx. Combining xRa and aRx using the transitivity of R, we have xRx. This proves that R is reflexive. $\square$

Problem 3.

(i) For arbitrary $(x,y) \in \mathbb{R}^2$, we have $(x,y) \sim (x,y)$, since $x = x$. Therefore, $\sim$ is reflexive.

Let $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$. If $(x_1,x_2) \sim (y_1,y_2)$, we can write $x_1 = y_1$ hence $y_1 = x_1$. Thus, we have $(y_1,y_2) \sim (x_1,x_2)$. Therefore, $\sim$ is symmetric.

Let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in \mathbb{R}^2$. If $(x_1,x_2) \sim (y_1,y_2)$ and $(y_1,y_2) \sim (z_1,z_2)$, we can write $x_1 = y_1$ and $y_1 = z_1$. Thus, $x_1 = z_1$, giving $(x_1,x_2) \sim (z_1,z_2)$. Therefore, $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation. $\square$

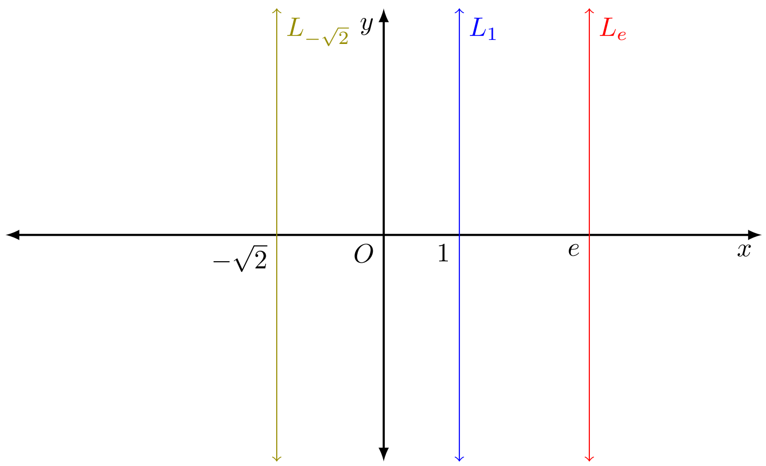
(ii) For $(x_1,x_2) \in \mathbb{R}^2$, its equivalence class is given by

$$\begin{aligned}[(x_1,x_2)] &= \{(y_1,y_2) \in \mathbb{R}^2 : (x_1,x_2) \sim (y_1,y_2)\} \\ &= \{(y_1,y_2) \in \mathbb{R}^2 : x_1 = y_1\} \\ &= \{(x_1,y_2) : y_2 \in \mathbb{R}\} \\ &= \{(x_1,y) : y \in \mathbb{R}\}.\end{aligned}$$

Therefore, the quotient set of R is given by

$$\mathbb{R}/\sim = \{L_x : x \in \mathbb{R}\},$$

where $L_x = \{(x,y) : y \in \mathbb{R}\}$. Clearly, each equivalence class $L_x \in \mathbb{R}/\sim$ is a vertical line in the Cartesian plane, passing through $(x,0)$.



Problem 4.

(i) For an arbitrary $(x,y) \in \mathbb{R}^2$, we have $(x,y) \sim (x,y)$, since $x^2+y^2 = x^2+y^2$. Therefore, $\sim$ is reflexive.

Let $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$. If $(x_1,x_2) \sim (y_1,y_2)$, we can write $x_1^2+x_2^2 = y_1^2+y_2^2$, hence $y_1^2+y_2^2 = x_1^2+x_2^2$. Thus, we have $(y_1,y_2) \sim (x_1,x_2)$. Therefore, $\sim$ is symmetric.

Let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in \mathbb{R}^2$. If $(x_1,x_2) \sim (y_1,y_2)$ and $(y_1,y_2) \sim (z_1,z_2)$, we can write $x_1^2+x_2^2 = y_1^2+y_2^2$ and $y_1^2+y_2^2 = z_1^2+z_2^2$. This gives $x_1^2+x_2^2 = z_1^2+z_2^2$, hence $(x_1,x_2) \sim (z_1,z_2)$. Therefore, $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation. $\square$

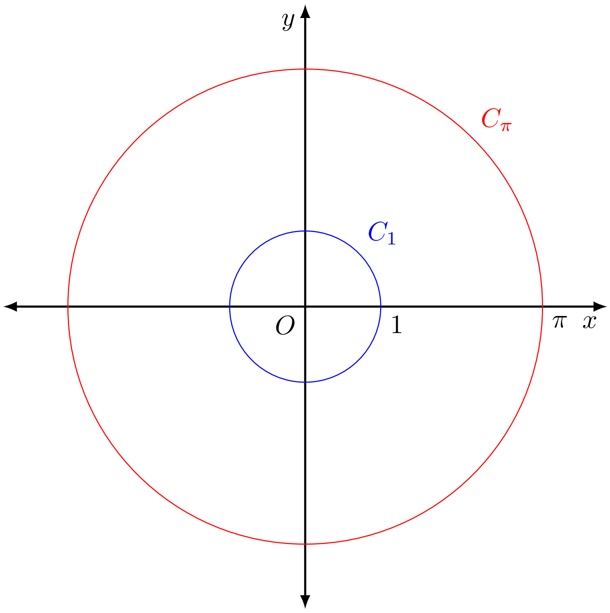
(ii) For $(x_1, x_2) \in \mathbb{R}^2$, its equivalence class is given by

$$\begin{aligned}[(x_1, x_2)] &= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2) \sim (y_1, y_2)\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = y_1^2 + y_2^2\}.\end{aligned}$$

Clearly, this is a circle of radius $r = \sqrt{x_1^2 + x_2^2}$ centred at the origin. Such a circle can be denoted by $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$.

Therefore, the quotient set of $\sim$ is given by

$$\mathbb{R}/\sim = \{C_r : r \geq 0\}.$$



Problem 5.

(i) For an arbitrary $(m, n) \in \mathbb{N}^2$, $(m, n) \sim (m, n)$, since $m + n = n + m$. Therefore, $\sim$ is reflexive.

Let $(m, n), (p, q) \in \mathbb{N}^2$. If $(m, n) \sim (p, q)$, we can write $m + q = n + p$, hence $p + n = q + m$. Thus, we have $(p, q) \sim (m, n)$. Therefore, $\sim$ is symmetric.

Let $(m, n), (p, q), (r, s) \in \mathbb{N}^2$. Note that $m+q = n+p$ is equivalent to $m-n = p-q$. If $(m, n) \sim (p, q)$ and $((p, q) \sim (r, s)$, we can write $m - n = p - q$ and $p - q = r - s$, from which we have $m - n = r - s$, hence $(m, n) \sim (r, s)$. Therefore, $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation. $\square$

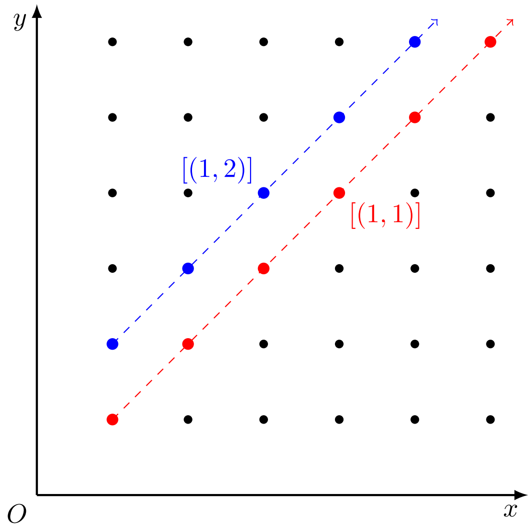
(ii) For $(m, n) \in \mathbb{N}^2$, we its equivalence class is given by

$$\begin{aligned}[(m, n)] &= \{(p, q) \in \mathbb{N}^2 : (m, n) \sim (p, q)\} \\ &= \{(p, q) \in \mathbb{N}^2 : m + q = n + p\} \\ &= \{(p, q) \in \mathbb{N}^2 : m - n = p - q\}.\end{aligned}$$

Clearly, each equivalence class has its elements $(p, q) \in \mathbb{N}^2$ on the line $m-n = x-y$ in the Cartesian plane. Note that $m - n = p - q$ gives $q = p - (m - n)$. Thus, $q \in \mathbb{N}$ forces $p > (m - n)$. This gives

$$[(m, n)] = \{(p, p - (m - n)) : p \in \mathbb{N}, p > (m - n)\}$$

Therefore, the quotient set of $\sim$ consists of 'lines' $L_d = \{(p, p - d) \in \mathbb{N}^2 : p \in \mathbb{N}, p > d\}$ for each $d \in \mathbb{Z}$.



Problem 6.

(i) Let all $x_i \in \mathbb{R} \setminus \{0\}$ in the following discussion.

Clearly, $\sim$ is reflexive since $(x_1, x_2) = 1 \cdot (x_1, x_2)$.

Let $(x_1, x_2) \sim (x_3, x_4)$. Then, $(x_3, x_4) = \alpha(x_1, x_2)$ for some $\alpha \neq 0$, hence $(x_1, x_2) = \frac{1}{\alpha}(x_3, x_4)$. Therefore, $\sim$ is symmetric.

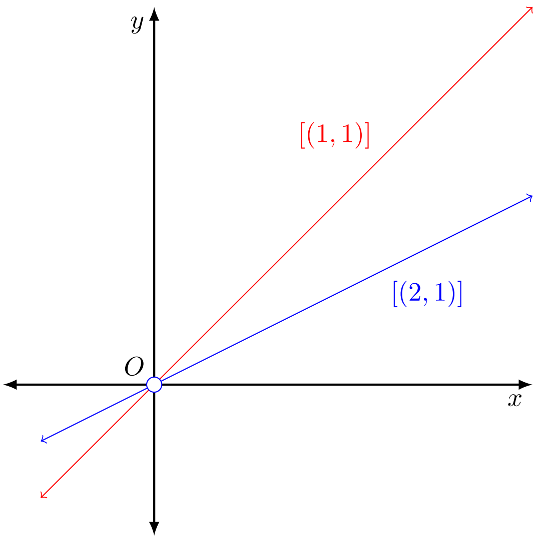
Let $(x_1, x_2) \sim (x_3, x_4)$ and $(x_3, x_4) \sim \beta(x_5, x_6)$. Then, $(x_3, x_4) = \alpha(x_1, x_2)$ and $(x_5, x_6) = \beta(x_3, x_4)$ for some $\alpha, \beta \neq 0$. Thus, $(x_5, x_6) = (\alpha\beta) \cdot (x_1, x_2)$ and $\alpha\beta \neq 0$, so $(x_1, x_2) \sim (x_5, x_6)$. Therefore, $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation. $\square$

(ii) For $(r, s) \in \mathbb{R} \setminus (0, 0)$, its equivalence class is given by

$$\begin{aligned}
[(r,s)] &= \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : (r,s) \sim (x,y)\} \\
&= \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : (x,y) = \alpha(r,s), \alpha \neq 0\} \\
&= \{(\alpha r, \alpha s) : \alpha \in \mathbb{R} \setminus \{0\}\}.
\end{aligned}$$

Clearly, each equivalence class $[(r, s)]$ is a line of slope s/r, through $(1, s/r)$, excluding the origin in the Cartesian plane.



Problem 7.

(i) There are 2^{n^2} relations on X.

This is because every relation on X is simply a subset of the Cartesian product $X \times X$, and vice versa. This means that there are exactly as many relations on X as there are subsets of $X \times X$. Now, there are n^2 elements of $X \times X$, hence 2^{n^2} subsets of $X \times X$. $\square$

(ii) There are 2^{n^2-n} reflexive relations on X.

Denote $\Delta = \{(x, x) \colon x \in X\}$, the *diagonal* of $X \times X$. Clearly, there are n elements in Δ.

With this, note that a relation R on X is reflexive if and only if $\Delta \subseteq R$. This means that there are exactly as many reflexive relations on X as there are subsets of $X \times X$ which contain Δ. Of the n^2 elements of $X \times X$, we have n of them in Δ, leaving $n^2 - n$ elements free to either be or not to be in R. As a result, we have 2^{n^2-n} such subsets of $X \times X$. $\square$

(iii) There are $2^{n(n+1)/2}$ symmetric relations on X.

Enumerate $X = \{x_1, \ldots, x_n\}$. Note that when forming a symmetric relation R on X by choosing a subset of $X \times X$, the choice of any (x_i, x_j) where $i \leq j$ forces the choice of (x_j, x_i). Furthermore, any symmetric relation on X can be formed in this way. This means that there are exactly as many symmetric relations on X are there are subsets of

$$U = \{(x_i, x_j) \in X \times X \colon 1 \leq i \leq j \leq n\}.$$

Clearly U has $1 + 2 + \cdots + n = n(n + 1)/2$ elements, hence $2^{n(n+1)/2}$ subsets. $\square$

(iv) There are $2^{n(n-1)/2}$ reflexive and symmetric relations on X.

Note that when forming a reflexive and symmetric relation R on X by choosing a subset of $X \times X$, the choice of all (x_i, x_i) where $1 \leq i \leq n$ is forced (by reflexivity). Additionally, the choice of any (x_i, x_j) where $i < j$ forces the choice of (x_j, x_i). Finally, any reflexive and symmetric relation on X can be formed in this way. This means that there are exactly as many symmetric relations on X are there are subsets of

$$U' = \{(x_i, x_j) \in X \times X \colon 1 \leq i < j \leq n\}.$$

Clearly U' has $0 + 1 + \cdots + (n - 1) = n(n - 1)/2$ elements, hence $2^{n(n-1)/2}$ subsets. $\square$