MA3203

Analysis IV

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Satvik Saha
19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

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1 Measure theory

1.1 Introduction

Measure theory seeks to generalize the notions of *length*, *area*, *volume* to more general sets: this new notion is called a *measure*. This also allows us to generalize the notion of Riemann integration to a broader class of functions.

Recall that continuous functions, or at least functions with finitely many discontinuities on a closed interval are Riemann integrable. The Dirichlet function, which is discontinuous everywhere, is not.

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This is simply because every non-empty interval contains at least one rational and one irrational number, so the Darboux lower sum is always 0 and the upper sum is always 1 regardless of the choice of partition.

On the other hand, if we had to assign a particular value to this integral, intuition tells us that it ought to be zero. After all, the function f attains a non-zero value only on the countable set $\mathbb{Q} \cap [0, 1]$; it is zero almost everywhere. Formally, we will show that f is non-zero on a set of zero Lebesgue measure, which will allow us to set this Lebesgue integral to zero. We will see that with this new formulation of integration, we end up partitioning the range of f rather than it's domain, and write

$$\int f = 0 \cdot \mu([0, 1] \setminus \mathbb{Q}) + 1 \cdot \mu([0, 1] \cap \mathbb{Q}) = 0.$$

Theorem 1.1 (Lebesgue criterion)**.** *A function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuities has Lebesgue measure zero. This means that the set of discontinuities of f must be coverable by countably many intervals (x_i, y_i) such that the sum of lengths $y_i - x_i$ can be made arbitrarily small.*

1.2 Basic definitions

Definition 1.1. Let X be a set, and let $\mathcal{M}$ be a collection of subsets of X. We say that $\mathcal{M}$ is a σ-algebra over X if it satisfies the following.

1. $\mathcal{M}$ contains X.
2. $\mathcal{M}$ is closed under complementation.
3. $\mathcal{M}$ is closed under countable unions.

Remark. The first condition can be replaced by forcing $\mathcal{M}$ to be non-empty.

Remark. The following properties follow immediately.

1. $\mathcal{M}$ contains $\emptyset$.
2. $\mathcal{M}$ is closed under countable intersections.
3. $\mathcal{M}$ is closed under differences.

Example. Given a set X, its power set forms a σ-algebra over X.

Example. Given a set X, the set $\{\emptyset, X\}$ forms a σ-algebra over X.

Example. Given an uncountable set X, the following set forms the co-countable σ-algebra over X.

$$\{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}.$$

Definition 1.2. Let X be a set, and let $\mathcal{M}$ be a σ-algebra over X. We say that a function $\mu\colon \mathcal{M} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a measure if it satisfies the following.

1. μ is non-negative.
2. $\mu(\emptyset) = 0$.
3. μ is additive over countable unions of disjoint sets, i.e. for any countable collection $\{E_i\}_{i=1}^{\infty}$ such that $E_i \cap E_j = \emptyset$ for all pairs, we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Example. The trivial zero measure sends every set to zero.

Example. In probability theory, we look at the event space $\mathcal{E}$ as a σ-algebra over the sample space Ω. The probability function P is a measure on this event space such that $P(\Omega) = 1$.

Example. Let X be a set, and let $\mathcal{M}$ be its power set as a σ-algebra over X. Fix $x_0 \in X$, and define

$$\mu\colon \mathcal{M} \to [0, \infty], \qquad E \mapsto \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E. \end{cases}$$

This is called the Dirac measure.

Example. Let X be a set, and let $\mathcal{M}$ be its power set as a σ-algebra over X. Define

$$\mu\colon \mathcal{M} \to [0, \infty], \qquad E \mapsto \begin{cases} 0, & \text{if } E = \emptyset, \\ |E|, & \text{if } E \text{ is finite}, \\ \infty, & \text{otherwise}. \end{cases}$$

This is called the counting measure.

Example. Let X be an uncountable set, and let $\mathcal{M}$ be its co-countable sigma algebra. Define

$$\mu\colon \mathcal{M} \to [0, \infty], \qquad E \mapsto \begin{cases} 0, & \text{if } E \text{ is countable}, \\ 1, & \text{if } E^c \text{ is countable}. \end{cases}$$

Definition 1.3. Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. We say that μ is finite if $\mu(E)$ is finite for all $E \in \mathcal{M}$.
2. We say that μ is σ-finite if given $E \in \mathcal{M}$, we can write

$$E = \bigcup_{i=1}^{\infty} E_i$$

for $E_i \in \mathcal{M}$ such that each $\mu(E_i)$ is finite.

1.3 Basic properties

Lemma 1.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then, the following properties hold.*

1. *If $A, B \in \mathcal{M}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
2. *If $A, B \in \mathcal{M}$ such that $A \subseteq B$ and $\mu(A)$ is finite, then $\mu(B - A) = \mu(B) - \mu(A)$.*
3. *If $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \mathcal{M}$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Corollary 1.2.1. *A measure μ is finite if and only if $\mu(X)$ is finite.*

Theorem 1.3 (Continuity from below)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets such that $E_i \subseteq E_j$ for all $i < j$. Then,*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Define $F_i = E_i - E_{i-1}$, denoting $E_0 = \emptyset$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i).$$

Also note that

$$\sum_{i=1}^{n} \mu(F_i) = \mu\left(\bigcup_{i=1}^{\infty} F_n\right) = \mu(E_n).$$

Since the infinite sum in the first part is the limit of partial sums, we have our result. □

Theorem 1.4 (Continuity from above)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets such that $E_i \supseteq E_j$ for all $i < j$. Further assume that $\mu(E_1)$ is finite. Then,*

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Define $F_i = E_1 - E_n$, and note that $F_i \subseteq F_j$ for all $i < j$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{n\to\infty} \mu(F_n).$$

This can be rewritten as

$$\mu\left(E_1 - \bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_1 - E_n).$$

Using the subtractive property and the fact that each $\mu(E_i)$ is finite,

$$\mu(E_1) - \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_1) - \mu(E_n).$$

Pulling the constant $\mu(E_1)$ out from the limit and subtracting from both sides gives our result. □

Example. Consider the counting measure μ on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, and define $E_n = \{n, n+1, \dots\}$. Then,

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = 0, \qquad \lim_{n\to\infty} \mu(E_n) = \infty.$$

1.4 The Borel σ-algebra

Theorem 1.5. *Let X be a set, and let S be a collection of subsets of X. Then, there exists a smallest σ-algebra containing S. This is called the σ algebra generated by S, denoted $\mathcal{M}(S)$.*

Proof. Let Ω be the collection of all σ-algebras on X containing S. Note that $\Omega \neq \emptyset$, since it contains the power set of X. Consider the intersection of all the sigma algebras in Ω,

$$\mathcal{M} = \bigcap_{\mathcal{M}_\lambda \in \Omega} \mathcal{M}_\lambda.$$

We claim that $\mathcal{M}$ is indeed a σ-algebra. To see this, first note that $X \in \mathcal{M}$. Next, pick $E \in \mathcal{M} \subseteq \mathcal{M}_\lambda$, so $E^c \in \mathcal{M}_\lambda$ for all $\mathcal{M}_\lambda \in \Omega$, hence $E^c \in \mathcal{M}$. Finally, pick $\{E_i\}_{i=1}^\infty$ where $E_i \in \mathcal{M} \subseteq \mathcal{M}_\lambda$, which shows that the union of these E_i is in every $\mathcal{M}_\lambda$, hence in $\mathcal{M}$. $\square$

Definition 1.4. Let (X, τ) be a topological space. The σ algebra generated by τ is called the Borel σ-algebra, $\mathcal{B}_X = \mathcal{M}(\tau)$.

Remark. The Borel σ-algebra $\mathcal{B}_X$ contains all open as well as all closed sets in X, as well as their countable unions and intersections.

Theorem 1.6. *Consider the collection β of open intervals in $\mathbb{R}$, and the standard topology τ on $\mathbb{R}$. Then, both β and τ generate the same Borel σ-algebra $\mathcal{B}_\mathbb{R}$.*

Proof. This relies on the fact that every open set $U \subseteq \mathbb{R}$ can be written as a countable union of open intervals. To see this, pick $x \in U$, and an open interval $x \in (a, b) \subset U$. Now pick $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$, hence $x \in (p, q) \subset U$. Now, U is precisely the union of all such intervals (p, q). This collection is countable, due to the countability of the rationals. $\square$

Remark. The same holds if we consider the collection β' of closed intervals in $\mathbb{R}$. This can be shown using the standard trick

$$(a, b) = \bigcup_{n=1}^\infty \left[a + \frac{1}{n}, b - \frac{1}{n}\right].$$

Indeed, we may also consider the collection of intervals of the form $[a, b)$, or the collection of intervals $(a, b]$, or even the collection of intervals (a, ∞), or $(-\infty, b)$, or $[a, \infty)$, or $(-\infty, b]$.

Definition 1.5. A countable union of closed sets is called an F_σ set. A countable intersection of open sets is called a G_δ set.

1.5 Measurable functions

Definition 1.6. Let $(X, \mathcal{M}_X)$, $(Y, \mathcal{M}_Y)$ be measure spaces. We say that a function $f \colon X \to Y$ is $(\mathcal{M}_X, \mathcal{M}_Y)$ measurable if for every $E \in \mathcal{M}_Y$, we have $f^{-1}(E) = \mathcal{M}_X$.

Example. Consider the Borel σ-algebra $\mathcal{B}_\mathbb{R}$ on $\mathbb{R}$, and fix $E \in \mathcal{B}_\mathbb{R}$. Define the characteristic function

$$\chi_E \colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Then, χ_E is measurable. However, we can choose E to be closed and not open, so that χ_E is not continuous.

Lemma 1.7. *Let $f \colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, and let $\mathcal{M}_Y$ be generated by S. Then $f \colon X \to Y$ is measurable if for every $E \in S$, we have $f^{-1}(E) \in \mathcal{M}_X$.*

Proof. Define

$$\mathcal{M} = \{E \subseteq Y : f^{-1}(E) \in \mathcal{M}_X\}.$$

Clearly, $S \subseteq \mathcal{M}$. We now claim that $\mathcal{M}$ is a σ-algebra over Y. First, $Y \in \mathcal{M}$ since $f^{-1}(Y) = X \in \mathcal{M}_X$. Next if $E \in \mathcal{M}$, we have $f^{-1}(E^c) = f^{-1}(E)^c \in \mathcal{M}_X$. Finally, if $\{E_i\}_{i=1}^\infty$ such that each $E_i \in \mathcal{M}$, set E to be their union, whence

$$f^{-1}(E) = \bigcup_{i=1}^\infty f^{-1}(E_i) \in \mathcal{M}.$$

Thus, $\mathcal{M}$ is indeed a σ-algebra. Since S generates $\mathcal{M}_Y$, we have $\mathcal{M}_Y \subseteq \mathcal{M}$, completing the proof. □

Example. A function $f \colon X \to \mathbb{R}$ is measurable if and only if $f^{-1}((a, \infty))$ is measurable for all $a \in \mathbb{R}$.

Theorem 1.8. *Let $f \colon X \to Y$ be continuous. Then, f is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.*

Lemma 1.9. *The composition of measurable functions is measurable. In other words, if $f \colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is surjective and $(\mathcal{M}_X, \mathcal{M}_Y)$ measurable, and $g \colon (Y, \mathcal{M}_Y) \to (Z, \mathcal{M}_Z)$ is $(\mathcal{M}_Y, \mathcal{M}_Z)$ measurable, then $g \circ f$ is $(\mathcal{M}_X, \mathcal{M}_Z)$ measurable.*

Lemma 1.10. *Let $u, v \colon (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ be $(\mathcal{M}, \mathcal{B}_\mathbb{R})$ measurable. Then, $f \colon (X, \mathcal{M}) \to (\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$, defined by $x \mapsto (u(x), v(x))$, is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^2})$ measurable.*

Proof. Basic open sets in $\mathbb{R}^2$ can be chosen as the open rectangles $(a, b) \times (c, d)$. The pre-image of such an open set under f is $u^{-1}((a, b)) \cap v^{-1}((c, d))$, which is clearly a measurable set in X. □

Lemma 1.11. *The maximum and minimum of measurable functions are measurable.*

Corollary 1.11.1. *The positive and negative parts of a measurable function are measurable.*

Corollary 1.11.2. *Let $f, g\colon (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable. Then the sum $f + g$ and the product $u \cdot g$ are also $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable.*

Proof. The maps $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ are continuous, hence measurable. Thus, the composite maps $x \mapsto (f(x), g(x)) \mapsto f(x) + g(x)$ and $x \mapsto (f(x), g(x)) \mapsto f(x)g(x)$ are also measurable. $\square$

Example. Let $(X, \mathcal{M})$ be a measurable space, and let $A_1, \dots, A_n \in \mathcal{M}$. Then the map

$$s\colon X \to \mathbb{R}, \qquad x \mapsto \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$

is measurable. Such functions are called simple functions.

Theorem 1.12. *Let $\{f_n\}_{n=1}^{\infty}$ be a collection of measurable functions $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$, and let $f_n \to f$ pointwise on X. Then, their supremum and infinimum are measurable.*

Theorem 1.13. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f\colon X \to \mathbb{R}$. Then, $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable.*

Theorem 1.14. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f\colon X \to \mathbb{R}$, and let $f_n \to f$ pointwise on X. Then, f is measurable.*

Remark. This is a stronger result than the corresponding one regarding limits of continuous functions.