MA3205

Geometry of Curves and Surfaces

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1 Introduction

1.1 Some results

Definition 1.1. A curve is a continuous map $\gamma\colon \mathbb{R} \to \mathbb{R}^n$.

Definition 1.2. A smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is C^∞, i.e. differentiable arbitrarily times.

Definition 1.3. A closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is periodic, i.e there exists some c such that $\gamma(t + c) = \gamma(t)$ for all $t \in \mathbb{R}$.

Example. Alternatively, a closed curve can be thought of as a continuous map $\gamma\colon S^1 \to \mathbb{R}^n$. For instance, given a closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ with period c, we can define the corresponding map

$$\tilde{\gamma}\colon S^1 \to \mathbb{R}^n, \qquad \tilde{\gamma}(e^{it}) = \gamma(ct/2\pi).$$

Definition 1.4. A simple curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is injective on its period.

Theorem 1.1 (Four Vertex Theorem)**.** *The curvature of a simple, closed, smooth plane curve has at least two local minima and two local maxima.*

Definition 1.5. A knot is a simple closed curve in $\mathbb{R}^3$.

Definition 1.6. The total absolute curvature of a knot K is the integral of the absolute value of the curvature, taken over the curve, i.e. it is the quantity

$$\oint_K |\kappa(s)|\, ds.$$

Example. The total absolute curvature of a circle is always 2π.

Theorem 1.2 (Fáry-Milnor Theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Definition 1.7. An immersed loop γ is such that γ' is never zero.

Definition 1.8. Two loops are isotopic if there exists an interpolating family of loops between them. Two immersed loops are isotopic if we can choose such an interpolating family of immersed loops.

Example. Without the restriction of immersion, any two loops $\gamma, \eta\colon S^1 \to \mathbb{R}^n$ would be isotopic, since we can always construct the linear interpolations

$$H\colon S^1 \times [0,1] \to \mathbb{R}^n, \qquad H(e^{i\theta}, t) = (1-t)\gamma(e^{i\theta}) + t\eta(e^{i\theta}).$$

Theorem 1.3 (Hirsch-Smale Theory)**.**

1. *Any two immersed loops in $\mathbb{R}^2$ are isotopic if and only if their turning numbers match.*
2. *Any two immersed loops in S^2 are isotopic if and only if their turning numbers modulo 2 match.*

1.2 Whitney's theorem

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a continuous function $f\colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Remark. The converse, i.e. $f^{-1}(0) = C$ implies C is closed, where f is continuous on Ω, is trivial.

Proof. Set f to be the distance function from C, i.e.

$$f(x) = \inf_{y \in C} d(x, y).$$ □

Theorem 1.5 (Whitney's Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a smooth function $f\colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Definition 1.9. A parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.

Remark. Here, we will always implicitly assume that maps are continuous.

Remark. Such a curve is called regular if $\gamma'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^n, \qquad t \mapsto a + tb$$

is a straight line through the point a, in the direction b.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (\cos t, \sin t)$$

is the unit circle in $\mathbb{R}^2$, counter-clockwise.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^3, \qquad t \mapsto (t, \cos t, \sin t)$$

is a helix in $\mathbb{R}^3$, wrapped around the x-axis.