MA3103

Introduction to Graph Theory and Combinatorics

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Satvik Saha
19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

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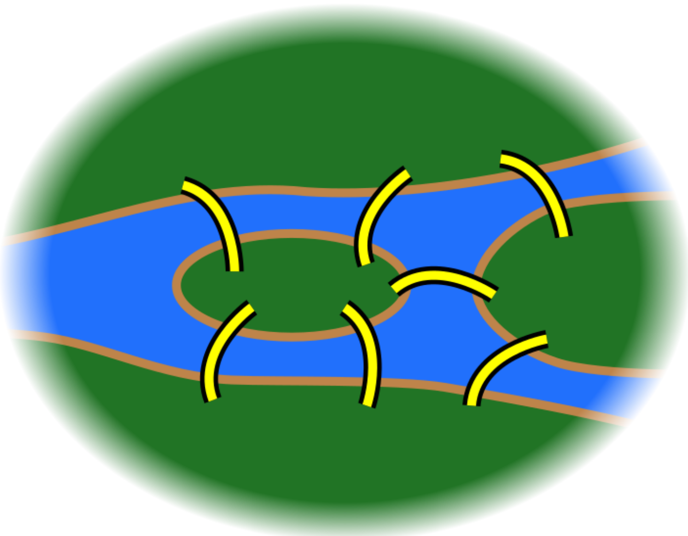
1 Introduction

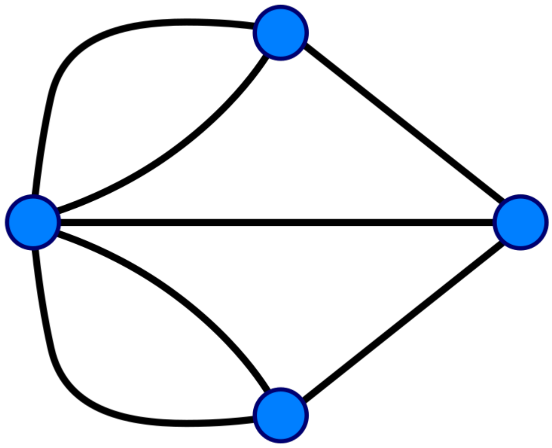
1.1 The Seven Bridges of Königsberg

The diagram below depicts a region in the city of Königsberg, Prussia. There are two islands, connected with the mainland and to each other via seven bridges. The Seven Bridges Problem is posed as follows: is it possible to walk through the entire city, visiting each one of the four landmasses by crossing each of the bridges exactly once?



Leonhard Euler showed that this is impossible; no such walk exists. The techniques he developed in doing so laid the foundations of *graph theory*.

The first thing to note is that the exact shape of the walk/trail is immaterial; all that matters is the sequence of landmasses visited and bridges crossed. Thus, each landmass can be compacted to a single point or *vertex*, and each bridge a line or *edge* connecting two such points. The resulting figure is a graph. Note that the orientations or placements of the points and lines are irrelevant, as long as the connections are undisturbed.



Now, examine a landmass which is on the trail but is neither our starting point, nor our ending point. In order to reach this landmass, we must enter via a bridge; but we cannot stay in the landmass, so we must leave via another a bridge. Thus, for each time we pass through this landmass, we can cross off two bridges joined to it. Once we are done, no bridge may remain unused; this means that we must have started with an even number of bridges joined to this landmass.

However, all four vertices in our graph connect to an odd number of edges. Since we require at least two vertices to act as intermediate points on our path, the desired walk is impossible.

1.2 Basic definitions

Definition 1.1. A graph $G(V, E)$ is an ordered pair of the set of vertices V and the set of edges E.

Definition 1.2. A simple graph is undirected, unweighted, and contains no self-loops or multiple edges joining vertices.

Definition 1.3. For a simple undirected graph, the set of edges E consists of two-element subsets of the set of vertices V.

Remark. For a directed, unweighted graph, the set of edges E consists of ordered pairs of elements from the set of vertices V.

Definition 1.4. A vertex is incident to an edge if that edge joins that vertex.

Definition 1.5. Two vertices are adjacent if there exists an edge connecting them. Two edges are adjacent if they connect to a common vertex.

Definition 1.6. The neighbours of a vertex consist of all vertices adjacent to it. The neighbours of an edge consist of all edges adjacent to it.

The number of neighbours of a vertex is called the degree of that vertex.

Lemma 1.1 (Pigeonhole Principle). *If $n+1$ objects are placed in n boxes, then we can fin a box containing at least 2 objects.*

Proof. If every box contains at most 1 objects, then the total number of objects falls short. □

Theorem 1.2. *There are no simple graphs where the degrees of all vertices are distinct.*

Proof. Let $G(V, E)$ be a simple graph with n vertices. The degrees of each of these vertices must be an integer among $0, 1, \ldots, n-1$. We now consider two cases.

Case I: There is a vertex of degree 0. Thus, this vertex is connected to no other vertex, which means that no vertex can have the full degree $n-1$. This means that the remaining vertices have degrees among $1, 2, \ldots, n-2$, i.e. $n-2$ choices of degree for $n-1$ vertices.

Case II: There is no vertex of degree 0. Thus, the vertices have degrees among $1, 2, \ldots, n-1$, i.e. $n-1$ choices of degree for n vertices.

In either case, the Pigeonhole Principle forces at least two vertices to share the same degree. □

Lemma 1.3 (Strong Pigeonhole Principle). *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$N = q_1 + \cdots + q_n - n + 1$$

objects are placed in n boxes, then we can find a box i containing at least q_i objects.

Proof. If every box i contains at most $q_i - 1$ objects, then the total number of objects falls short.

$$N \leq (q_1 - 1) + \cdots + (q_n - 1) = q_1 + \cdots + q_n - n = N - 1$$ □