PH 1101 : Mechanics I

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We shall show that in polar coordinates, an ellipse is described by the equation

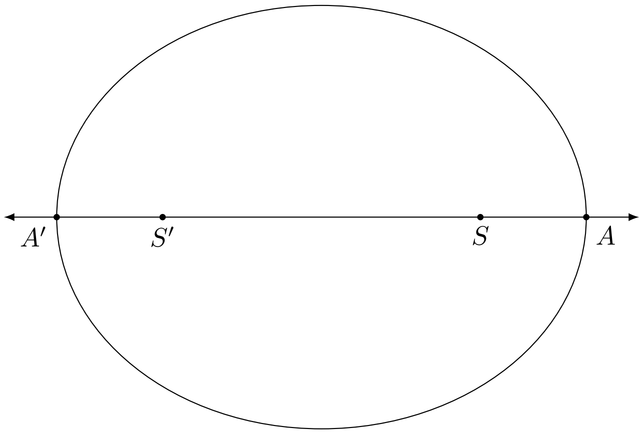
$$r(1 - e\cos\theta) \;=\; l,$$

where the coordinate system is centred at one of the foci of the ellipse.
Here, e is the eccentricity of the ellipse, and l is its semi-latus rectum.

Let the foci of the ellipse be S and S'. We shall define the ellipse as locus of all points P such that

$$SP + S'P \;=\; \text{constant}.$$

Join S and S', and extend it on both sides so that it cuts the ellipse at A and A'.



We must have

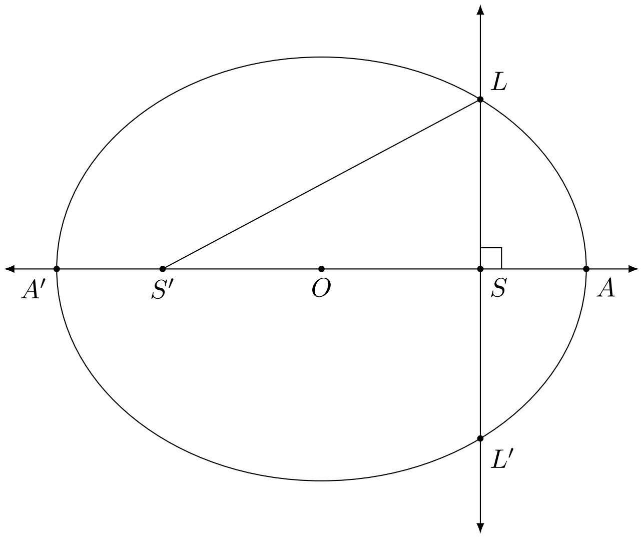
$$\begin{aligned} SA + S'A \;&=\; \text{constant} \\ SA' + S'A' \;&=\; \text{constant}. \end{aligned}$$

Note that $S'A = SS' + SA$, and $SA' = SS' + S'A'$. Thus, we must have $SA = S'A'$. Also,

$$SA + SS' + S'A' \;=\; \text{constant} \;=\; AA'.$$

Define $AA' = 2a$. Clearly, a is the semi-major axis of our ellipse. Let the midpoint of SS' be O. Define $OS = OS' = s$.

Construct a perpendicular to SS' through S, cutting the ellipse at L and L'. Note that SL is the semi-latus rectum l of our ellipse. Join $S'L$.



We have $SL + S'L = AA'$. Applying the Pythagorean theorem on $\triangle SLS'$ gives

$$\begin{aligned} SS'^2 &= SL^2 + S'L^2 \\ (2s)^2 &= (l)^2 + (2a-l)^2 \\ l &= \frac{(a^2 - s^2)}{a} \end{aligned}$$