MA 1101 : Mathematics I

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Solution 1.
Let R be a relation on $\mathbb{R}^2$ such that

$$(x_1, x_2)\, R\, (y_1, y_2) \quad \text{if} \quad x_1 = y_1.$$

(i) For an arbitrary $(x, y) \in \mathbb{R}^2$, $(x, y)\, R\, (x, y)$, since $x = x$. Therefore, R is reflexive.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$, we can write $x_1 = y_1 \Rightarrow y_1 = x_1$. Thus, we have $(y_1, y_2)\, R\, (x_1, x_2)$. Therefore, R is symmetric.

For $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$ and $(y_1, y_2)\, R\, (z_1, z_2)$, we can write $x_1 = y_1$ and $y_1 = z_1$, from which we have $x_1 = z_1 \Rightarrow (x_1, x_2)\, R\, (z_1, z_2)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} [(x_1, x_2)] &= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2)\, R\, (y_1, y_2)\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : x_1 = y_1\} \\ &= \{(x_1, y) : y \in \mathbb{R}\} \end{aligned}$$

Therefore, the quotient set of R is given by

$$\mathbb{R}/R \;=\; \{L_x : x \in \mathbb{R}\},$$

where $L_x = \{(x, y) : y \in \mathbb{R}\}$. Clearly, each equivalence class $L_x \in \mathbb{R}/R$ is a vertical line in the Cartesian plane, passing through $(x, 0)$.

Solution 2.
Let R be a relation on $\mathbb{R}^2$ such that

$$(x_1, x_2)\, R\, (y_1, y_2) \quad \text{if} \quad x_1^2 + x_2^2 = y_1^2 + y_2^2$$

(i) For an arbitrary $(x, y) \in \mathbb{R}^2$, $(x, y)\, R\, (x, y)$, since $x^2 + y^2 = x^2 + y^2$. Therefore, R is reflexive.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$, we can write $x_1^2+x_2^2 = y_1^2+y_2^2 \Rightarrow y_1^2+y_2^2 = x_1^2+x_2^2$. Thus, we have $(y_1, y_2)\, R\, (x_1, x_2)$. Therefore, R is symmetric.

For $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$ and $(y_1, y_2)\, R\, (z_1, z_2)$, we can write $x_1^2+x_2^2 = y_1^2+y_2^2$ and $y_1^2+y_2^2 = z_1^2+z_2^2$, from which we have $x_1^2+x_2^2 = z_1^2+z_2^2 \Rightarrow (x_1, x_2)\, R\, (z_1, z_2)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} [(x_1, x_2)] &= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2)\, R\, (y_1, y_2)\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = y_1^2 + y_2^2\} \end{aligned}$$

Clearly, each equivalence class is a circle of radius $r = \sqrt{x_1^2 + x_2^2}$ centred at the origin. Such a circle can be denoted by $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$. Therefore, the quotient set of R is given by

$$\mathbb{R}/R \;=\; \{C_r : r \geq 0\}.$$

Solution 5.
Let $n \in \mathbb{N}$ and X be a set of n elements. An arbitrary relation R on X is a subset of the Cartesian product $X \times X = X^2$. Note that for $(a, b) \in X^2$, a can be any of the n elements in X, and b can be independently any of the n elements in X. Thus, we have a total of n^2 elements in X^2.

(i) Since R is any subset $R \subseteq X^2$, we can say that a relation on X is any $R \in \mathcal{P}(X^2)$. Thus, the total number of possible relations R is the number of elements in $\mathcal{P}(X^2)$, i.e., 2^{n^2}.

(ii) Let $D = \{(x, x) : x \in X\}$ be the set of the diagonal elements of X^2. Clearly, there are n elements in D. A reflexive relation R must have $D \subseteq R$. Thus, of the n^2 elements of X^2, the n diagonal elements are fixed – the remaining $n^2 - n$ elements can be chosen to be or not to be in R, giving us a total of 2^{n^2-n} such relations.

(iii) Since $xRy \Rightarrow yRx$ if $x = y$, each of the n diagonal elements of X^2 may or may not be present in a symmetric relation R on X. Also, the presence of $(x, y) \in X^2 \setminus D$ in R forces the presence of (y, x) in R. Thus, we have $(n^2 - n)/2$ choices for the non-diagonal elements, giving a total of $2^n \cdot 2^{(n^2-n)/2} = 2^{(n^2+n)/2}$ such relations.

(iv) As before, we have $(n^2 - n)/2$ choices for non-diagonal elements to fulfil symmetry. The remaining diagonal elements are fixed to fulfil reflexivity, giving a total of $2^{(n^2-n)/2}$ such relations.