MA 1101 : Mathematics I

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Solution 1.
Let R be a relation on $\mathbb{R}^2$ such that

$$(x_1, x_2)\, R\, (y_1, y_2) \quad \text{if} \quad x_1 = y_1.$$

(i) For an arbitrary $(x, y) \in \mathbb{R}^2$, $(x, y)\, R\, (x, y)$, since $x = x$. Therefore, R is reflexive.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$, we can write $x_1 = y_1 \Rightarrow y_1 = x_1$. Thus, we have $(y_1, y_2)\, R\, (x_1, x_2)$. Therefore, R is symmetric.

For $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$ and $(y_1, y_2)\, R\, (z_1, z_2)$, we can write $x_1 = y_1$ and $y_1 = z_1$, from which we have $x_1 = z_1 \Rightarrow (x_1, x_2)\, R\, (z_1, z_2)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} [(x_1, x_2)] &= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2)\, R\, (y_1, y_2)\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : x_1 = y_1\} \end{aligned}$$

$$[(x_1, x_2)] = \{(x_1, y) : y \in \mathbb{R}\}$$

Therefore, the quotient set of R is given by

$$\mathbb{R}/R = \{L_x : x \in \mathbb{R}\},$$

where $L_x = \{(x, y) : y \in \mathbb{R}\}$. Clearly, each equivalence class $L_x \in \mathbb{R}/R$ is a vertical line in the Cartesian plane, passing through $(x, 0)$.

Solution 2.
Let R be a relation on $\mathbb{R}^2$ such that

$$(x_1, x_2)\, R\, (y_1, y_2) \quad \text{if} \quad x_1^2 + x_2^2 = y_1^2 + y_2^2$$