MA3201

Topology

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Contents

1 Introduction

1.1 Topological spaces

Definition 1.1. A topology on some set X is a family τ of subsets of X, satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from τ are in τ.
3. All finite intersections of elements from τ are in τ.

The sets from τ are declared to be open sets in the topological space (X, τ).

Example. Any set X admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

Example. Let X be a set. The cofinite topology on X is the collection of complements of finite sets, along with the empty set. Note that when X is finite, this is simply the discrete topology.

Definition 1.2. Let τ, τ' be two topologies on the set X. We say that τ is finer than τ' if τ has more open sets than τ'. In such a case, we also say that τ' is coarser than τ.

1.2 Topological bases

Definition 1.3. Let (X, τ) be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology τ such that every open set $U \in \tau$ is expressible as a union of elements from β.

Definition 1.4. Let X be a set, and let β be a collection of subsets of X satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then, β generates a topology on X, namely the collection of all unions of elements of β.

Lemma 1.1. *Let τ be a topology on X, and let $\beta \subseteq \tau$ be a collection of open sets. Then, β is a basis of τ, or generates τ, if for every $x \in U \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U$.*

Example. The collection of all open balls in $\mathbb{R}^n$ form a basis of the usual topology.

Lemma 1.2. *Let X be equipped with the topologies τ and τ', and let β and β' be the respective bases of these topologies. Then, τ is finer than τ' if and only if given $x \in B' \in \beta'$, there exists $x \in B \in \beta$ such that $B \subseteq B'$.*

Example. The collections of open balls in $\mathbb{R}^n$ generate the same topology as the collection of all open rectangles in $\mathbb{R}^n$.

Example. Consider the topologies on $\mathbb{R}$ generated by the following bases.

1. $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.
2. $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.
3. $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$ where $K = \{1/n : n \in \mathbb{Z}\}$.

We call the topology generated by β_2 the lower limit topology, denoted $\mathbb{R}_\ell$. The topology generated by β_3 is denoted $\mathbb{R}_K$. Both of these are strictly finer than the standard topology.

Definition 1.5. A sub-basis for some topology on X is a collection ρ of subsets of X whose union is the whole of X. The topology generated by ρ is defined to be the topology generated by the collection of all finite intersections of elements of ρ.

1.3 Product topology

Definition 1.6. Let (X_1, τ_1), (X_2, τ_2) be topological spaces. Then $\tau_1 \times \tau_2$ generates the product topology on $X_1 \times X_2$.

Example. The product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard topology, coincides with the standard topology on $\mathbb{R}^2$.

Lemma 1.3. *If β_1, β_2 are bases of the topologies τ_1, τ_2, then $\beta_1 \times \beta_2$ and $\tau_1 \times \tau_2$ generate the same product topology.*

Proof. Given $(x_1, x_2) \in U$ where $U \subseteq X_1 \times X_2$ is open in the product topology, recall that U can be written as a union of the basic open sets $U_{1i} \times U_{2i}$, where $U_{1i} \in \tau_1$ and $U_{2i} \in \tau_2$. Suppose that $(x_1, x_2) \in U_1 \times U_2$. Thus, we can choose $B_1 \in \beta_1$, $B_2 \in \beta_2$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$. Thus, $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$. $\square$

Definition 1.7. The projection maps are defined as $\pi_i \colon X_1 \times \cdots X_k \to X_i$, $(x_1, \ldots, x_k) \mapsto x_i$.

Lemma 1.4. *The collection of elements of the form $\pi_1^{-1}(U_1)$ or $\pi_2^{-1}(U_2)$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$, forms a sub-basis of the product topology on $X_1 \times X_2$.*

Proof. Note that $\pi_1^{-1}(X_1) = X_1 \times X_2$. Now it is easy to see that finite intersections of elements of the form $U_1 \times X_2$ or $X_1 \times U_2$ where U_1, U_2 are open, are all of the form $U_1 \times U_2$ which is precisely a basis of the product topology. $\square$

Corollary 1.4.1. *We can restrict ourselves to the sub-basis of elements of the form $\pi_1^{-1}(B_1)$ or $\pi_2^{-1}(B_2)$, where $B_1 \in \beta_1$, $B_2 \in \beta_2$ for some bases β_1, β_2 of τ_1, τ_2.*

1.4 Subspace topology

Definition 1.8. Let (X, τ) be a topological space, and let $Y \subset X$. Then the collection $U \cap Y$ for all $U \in \tau$ comprises the subspace topology τ_Y on Y induced by the topology τ on X.

Lemma 1.5. *If β is a basis for the topology on X, and $Y \subset X$, then the collection $B \cap Y$ for all $B \in \beta$ generates the subspace topology on Y.*

Lemma 1.6. *An open set of Y is open in X if Y is open in X.*

Proof. Let $U \subset Y$ be open in Y, then $U = V \cap Y$ for some open set V in X. If additionally Y is open in X, this immediately shows that U is open in X. □

Theorem 1.7. *Let (X, τ_X), (Y, τ_Y) be topological spaces, and let $A \subseteq X$, $B \subseteq Y$. Then, there are two ways of assigning a natural topology on $A \times B$.*

1. *Take the product topology on $X \times Y$, and consider the subspace topology induced by it on $A \times B$.*
2. *Take the subspace topologies on A induced by τ_X, B induced by τ_Y, and consider the product topology generated by them on $A \times B$.*

These two methods generate the same topology on $A \times B$.

Proof. Open sets in 1 look like $(U \times V) \cap (A \times B)$, where $U \in \tau_X$, $V \in \tau_Y$). Open sets in 2 look like $(U' \cap A) \times (V' \cap B)$, where $U' \in \tau_X$, $V' \in \tau_Y$, which can be rewritten as $(U' \times V') \cap (A \times B)$. It is easy to see that these describe precisely the same sets. □

1.5 Order topology

Definition 1.9. Let X be a set with a simple order $<$. Then the collection of sets of the form (a, b), $[a_0, b)$, $(a, b_0]$ where a_0 is the minimal element of X, b_0 is the maximal element of X, generate the order topology on X.

Example. The order topology on $\mathbb{N}$ is precisely the discrete topology.

Definition 1.10. Let X_1, X_2 be simply ordered sets. The dictionary order on $X_1 \times X_2$ is defined as follows: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$.

Example. Consider $X = \{1, 2\} \times \mathbb{N}$, where both $\{1, 2\}$ and $\mathbb{N}$ are endowed with the discrete topology. Note that the product topology on X is the discrete topology.

Now consider the dictionary order on X. Here, $(1, 1)$ is the smallest element, so we can list the elements of X in ascending order. Note that every $(1, m) < (2, n)$, for all $m, n \in \mathbb{N}$. Now, note that all singletons $\{(1, m)\}$ are open in the order topology on X. The same is true for the singletons $\{(1, n)\}$ for all $n > 1$. However, the singleton $\{(2, 1)\}$ is *not* open in the order topology.

Example. Consider $\mathbb{R}$ with the usual topology, and $X = [0, 1) \cup \{2\}$. Then, $\{2\}$ is open in the subspace topology on X, but it is not open in the order topology on X.

Lemma 1.8. *The open rays of the form $(a, +\infty)$ and $(-\infty, a)$ in X form a sub-basis of the order topology on X.*

Proof. Note that $(a, b) = (-\infty, b) \cap (a, +\infty)$, $[a_0, b) = (-\infty, b)$, and $(a, b_0] = (a, +\infty)$. □

Definition 1.11. Let X be a simply ordered set, and $Y \subseteq X$. Then, we say that Y is convex in X if given $a, b \in Y$ such that $a < b$, the interval $(a, b) = \{x \in X : a < x < b\} \subseteq Y$.

Theorem 1.9. *Let Y be convex in X. Then, the subspace topology and the order topology on Y induced from the order topology on X coincide.*

1.6 Continuous maps

Definition 1.12. Let $f : X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in Y must be open in X.

Definition 1.13. Let $f : X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is a homeomorphism if f is continuous, f is invertible, and f^{-1} is continuous. We also say that X and Y are homeomorphic when such a homeomorphism between them exists.