

MA3201

# Topology

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Satvik Saha  
19MS154

*Indian Institute of Science Education and Research, Kolkata,  
Mohanpur, West Bengal, 741246, India.*

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# 1 Introduction

## 1.1 Topological spaces

**Definition 1.1.** A topology on some set  $X$  is a family  $\tau$  of subsets of  $X$ , satisfying the following.

1.  $\emptyset, X \in \tau$ .
2. All unions of elements from  $\tau$  are in  $\tau$ .
3. All finite intersections of elements from  $\tau$  are in  $\tau$ .

The sets from  $\tau$  are declared to be open sets in the topological space  $(X, \tau)$ .

*Example.* Any set  $X$  admits the indiscrete topology  $\tau_{id} = \{\emptyset, X\}$ , as well as the discrete topology  $\tau_d = \mathcal{P}(X)$ . Both of these are trivial examples.

*Example.* Let  $X$  be a set. The cofinite topology on  $X$  is the collection of complements of finite sets, along with the empty set. Note that when  $X$  is finite, this is simply the discrete topology.

**Definition 1.2.** Let  $\tau, \tau'$  be two topologies on the set  $X$ . We say that  $\tau$  is finer than  $\tau'$  if  $\tau$  has more open sets than  $\tau'$ . In such a case, we also say that  $\tau'$  is coarser than  $\tau$ .

## 1.2 Topological bases

**Definition 1.3.** Let  $(X, \tau)$  be a topological space. We say that  $\beta \subseteq \tau$  is a base of the topology  $\tau$  such that every open set  $U \in \tau$  is expressible as a union of elements from  $\beta$ .

**Definition 1.4.** Let  $X$  be a set, and let  $\beta$  be a collection of subsets of  $X$  satisfying the following.

1. For every  $x \in X$ , there exists  $x \in B \in \beta$ .
2. For every  $x \in X$  such that  $x \in B_1 \cap B_2$ ,  $B_1, B_2 \in \beta$ , there exists  $B \in \beta$  such that  $x \in B \subseteq B_1 \cap B_2$ .

Then,  $\beta$  generates a topology on  $X$ , namely the collection of all unions of elements of  $\beta$ .

**Lemma 1.1.** Let  $\tau$  be a topology on  $X$ , and let  $\beta \subseteq \tau$  be a collection of open sets. Then,  $\beta$  is a basis of  $\tau$ , or generates  $\tau$ , if for every  $x \in U \in \tau$ , there exists  $B \in \beta$  such that  $x \in B \subseteq U$ .

*Example.* The collection of all open balls in  $\mathbb{R}^n$  form a basis of the usual topology.

**Lemma 1.2.** *Let  $X$  be equipped with the topologies  $\tau$  and  $\tau'$ , and let  $\beta$  and  $\beta'$  be the respective bases of these topologies. Then,  $\tau$  is finer than  $\tau'$  if and only if given  $x \in B' \in \beta'$ , there exists  $x \in B \in \beta$  such that  $B \subseteq B'$ .*

*Example.* The collections of open balls in  $\mathbb{R}^n$  generate the same topology as the collection of all open rectangles in  $\mathbb{R}^n$ .

*Example.* Consider the topologies on  $\mathbb{R}$  generated by the following bases.

1.  $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ .
2.  $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ .
3.  $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$  where  $K = \{1/n : n \in \mathbb{Z}\}$ .

We call the topology generated by  $\beta_2$  the lower limit topology, denoted  $\mathbb{R}_\ell$ . The topology generated by  $\beta_3$  is denoted  $\mathbb{R}_K$ . Both of these are strictly finer than the standard topology.

**Definition 1.5.** A sub-basis for some topology on  $X$  is a collection  $\rho$  of subsets of  $X$  whose union is the whole of  $X$ . The topology generated by  $\rho$  is defined to be the topology generated by the collection of all finite intersections of elements of  $\rho$ .

### 1.3 Product topology

**Definition 1.6.** Let  $(X_1, \tau_1), (X_2, \tau_2)$  be topological spaces. Then  $\tau_1 \times \tau_2$  generates the product topology on  $X_1 \times X_2$ .

*Example.* The product topology on  $\mathbb{R} \times \mathbb{R}$ , where  $\mathbb{R}$  is equipped with the standard topology, coincides with the standard topology on  $\mathbb{R}^2$ .

**Lemma 1.3.** *If  $\beta_1, \beta_2$  are bases of the topologies  $\tau_1, \tau_2$ , then  $\beta_1 \times \beta_2$  and  $\tau_1 \times \tau_2$  generate the same product topology.*

*Proof.* Given  $(x_1, x_2) \in U$  where  $U \subseteq X_1 \times X_2$  is open in the product topology, recall that  $U$  can be written as a union of the basic open sets  $U_{1i} \times U_{2i}$ , where  $U_{1i} \in \tau_1$  and  $U_{2i} \in \tau_2$ . Suppose that  $(x_1, x_2) \in U_{1i} \times U_{2i}$ . Thus, we can choose  $B_1 \in \beta_1, B_2 \in \beta_2$  such that  $x_1 \in B_1 \subseteq U_{1i}$  and  $x_2 \in B_2 \subseteq U_{2i}$ . Thus,  $(x_1, x_2) \in B_1 \times B_2 \subseteq U_{1i} \times U_{2i} \subseteq U$ .  $\square$

**Definition 1.7.** The projection maps are defined as  $\pi_i: X_1 \times \cdots \times X_k \rightarrow X_i, (x_1, \dots, x_k) \mapsto x_i$ .

**Lemma 1.4.** *The collection of elements of the form  $\pi_1^{-1}(U_1)$  or  $\pi_2^{-1}(U_2)$ , where  $U_1 \in \tau_1$  and  $U_2 \in \tau_2$ , forms a sub-basis of the product topology on  $X_1 \times X_2$ .*

*Proof.* Note that  $\pi_1^{-1}(X_1) = X_1 \times X_2$ . Now it is easy to see that finite intersections of elements of the form  $U_1 \times X_2$  or  $X_1 \times U_2$  where  $U_1, U_2$  are open, are all of the form  $U_1 \times U_2$  which is precisely a basis of the product topology.  $\square$

**Corollary 1.4.1.** *We can restrict ourselves to the sub-basis of elements of the form  $\pi_1^{-1}(B_1)$  or  $\pi_2^{-1}(B_2)$ , where  $B_1 \in \beta_1$ ,  $B_2 \in \beta_2$  for some bases  $\beta_1, \beta_2$  of  $\tau_1, \tau_2$ .*

## 1.4 Subspace topology

**Definition 1.8.** Let  $(X, \tau)$  be a topological space, and let  $Y \subset X$ . Then the collection  $U \cap Y$  for all  $U \in \tau$  comprises the subspace topology  $\tau_Y$  on  $Y$  induced by the topology  $\tau$  on  $X$ .

**Lemma 1.5.** *If  $\beta$  is a basis for the topology on  $X$ , and  $Y \subset X$ , then the collection  $B \cap Y$  for all  $B \in \beta$  generates the subspace topology on  $Y$ .*

**Lemma 1.6.** *An open set of  $Y$  is open in  $X$  if  $Y$  is open in  $X$ .*

*Proof.* Let  $U \subset Y$  be open in  $Y$ , then  $U = V \cap Y$  for some open set  $V$  in  $X$ . If additionally  $Y$  is open in  $X$ , this immediately shows that  $U$  is open in  $X$ .  $\square$

**Theorem 1.7.** *Let  $(X, \tau_X), (Y, \tau_Y)$  be topological spaces, and let  $A \subseteq X, B \subseteq Y$ . Then, there are two ways of assigning a natural topology on  $A \times B$ .*

1. *Take the product topology on  $X \times Y$ , and consider the subspace topology induced by it on  $A \times B$ .*
2. *Take the subspace topologies on  $A$  induced by  $\tau_X$ ,  $B$  induced by  $\tau_Y$ , and consider the product topology generated by them on  $A \times B$ .*

*These two methods generate the same topology on  $A \times B$ .*

*Proof.* Open sets in 1 look like  $(U \times V) \cap (A \times B)$ , where  $U \in \tau_X, V \in \tau_Y$ . Open sets in 2 look like  $(U' \cap A) \times (V' \cap B)$ , where  $U' \in \tau_X, V' \in \tau_Y$ , which can be rewritten as  $(U' \times V') \cap (A \times B)$ . It is easy to see that these describe precisely the same sets.  $\square$

## 1.5 Order topology

**Definition 1.9.** Let  $X$  be a set with a simple order  $<$ . Then the collection of sets of the form  $(a, b)$ ,  $[a_0, b)$ ,  $(a, b_0]$  where  $a_0$  is the minimal element of  $X$ ,  $b_0$  is the maximal element of  $X$ , generate the order topology on  $X$ .

*Example.* The order topology on  $\mathbb{N}$  is precisely the discrete topology.

**Definition 1.10.** Let  $X_1, X_2$  be simply ordered sets. The dictionary order on  $X_1 \times X_2$  is defined as follows:  $(x_1, x_2) < (y_1, y_2)$  if  $x_1 < y_1$ , or if  $x_1 = y_1$  and  $x_2 < y_2$ .

*Example.* Consider  $X = \{1, 2\} \times \mathbb{N}$ , where both  $\{1, 2\}$  and  $\mathbb{N}$  are endowed with the discrete topology. Note that the product topology on  $X$  is the discrete topology.

Now consider the dictionary order on  $X$ . Here,  $(1, 1)$  is the smallest element, so we can list the elements of  $X$  in ascending order. Note that every  $(1, m) < (2, n)$ , for all  $m, n \in \mathbb{N}$ . Now, note that all singletons  $\{(1, m)\}$  are open in the order topology on  $X$ . The same is true for the singletons  $\{(1, n)\}$  for all  $n > 1$ . However, the singleton  $\{(2, 1)\}$  is *not* open in the order topology.

*Example.* Consider  $\mathbb{R}$  with the usual topology, and  $X = [0, 1) \cup \{2\}$ . Then,  $\{2\}$  is open in the subspace topology on  $X$ , but it is not open in the order topology on  $X$ .

**Lemma 1.8.** The open rays of the form  $(a, +\infty)$  and  $(-\infty, a)$  in  $X$  form a sub-basis of the order topology on  $X$ .

*Proof.* Note that  $(a, b) = (-\infty, b) \cap (a, +\infty)$ ,  $[a_0, b) = (-\infty, b)$ , and  $(a, b_0] = (a, +\infty)$ . □

**Definition 1.11.** Let  $X$  be a simply ordered set, and  $Y \subseteq X$ . Then, we say that  $Y$  is convex in  $X$  if given  $a, b \in Y$  such that  $a < b$ , the interval  $(a, b) = \{x \in X : a < x < b\} \subseteq Y$ .

**Theorem 1.9.** Let  $Y$  be convex in  $X$ . Then, the subspace topology and the order topology on  $Y$  induced from the order topology on  $X$  coincide.

## 1.6 Closed sets

**Definition 1.12.** Let  $(X, \tau)$  be a topological space. A set  $F \subseteq X$  is said to be closed in  $X$  if  $F^c = X \setminus F \in \tau$ .

*Example.* The sets  $\emptyset, X$  are closed in every topological space  $(X, \tau)$ .

*Example.* In a set equipped with the discrete topology, every set is both open and closed.

**Lemma 1.10.** *Arbitrary intersections, and finite unions of closed sets are closed.*

**Theorem 1.11.** *Let  $(X, \tau)$  be a topological space, and let  $Y \subset X$  be equipped with the subspace topology. Then, a set  $F \subseteq Y$  is closed in  $Y$  if and only if  $F = Y \cap G$ , where  $G$  is closed in  $X$ .*

*Proof.* Let  $F \subset Y$ . Now,  $F$  is closed in  $Y$ ,  $Y \setminus F = Y \cap F^c$  is open in  $Y$ ,  $Y \cap F^c = Y \cap U$  where  $U$  is open in  $X$ ,  $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$  where  $U^c$  is closed. The steps are reversible.  $\square$

**Lemma 1.12.** *A closed set of  $Y$  is closed in  $X$  if  $Y$  is closed in  $X$ .*

## 1.7 Interiors and closures

**Definition 1.13.** Let  $A \subseteq X$  where  $(X, \tau)$  is a topological space.

1. The interior of  $A$  is defined as the union of all open sets contained in  $A$ . This is denoted by  $A^\circ$ .
2. The closure of  $A$  is defined as the intersection of all closed sets containing  $A$ . This is denoted by  $\overline{A}$ .

*Remark.* The interior of a set is open, and the closure of a set is closed.

**Lemma 1.13.** *Let  $Y \subset X$  be topological spaces, and let  $A \subseteq Y$ . Also let  $\overline{A}_X, \overline{A}_Y$  denote the closures of  $A$  in  $X, Y$  respectively. Then,  $\overline{A}_Y = \overline{A}_X \cap Y$ .*

**Theorem 1.14.** *Let  $A \subset X$ . Then,*

1.  $x \in \overline{A}$  if and only if every open set containing  $x$  has non-empty intersection with  $A$ .
2.  $x \in \overline{A}$  if and only if every basic open set containing  $x$  has non-empty intersection with  $A$ , given that the topology on  $X$  is generated by those basic open sets.

**Definition 1.14.** Let  $A \subseteq X$  where  $(X, \tau)$  is a topological space. We say that  $x \in X$  is a limit point of  $X$  if for every open set  $U$  containing  $x$ , the deleted neighbourhood  $U \setminus \{x\}$  has non-empty intersection with  $A$ . The set of limit points of  $A$  is denoted by  $A'$ .

*Example.* Let  $X$  be a set endowed with the discrete topology. Then, given any set  $A \subseteq X$ , we have  $A' = \emptyset$ .

**Lemma 1.15.** *A closed set contains all its limit points.*

*Proof.* Let  $F \subseteq X$  be closed in  $X$ , and let  $x \in F'$ . Then given any open set containing  $x$ , we have  $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$ , hence  $x \in \overline{F} = F$ .  $\square$

**Lemma 1.16.** *Let  $A \subseteq X$  where  $(X, \tau)$  is a topological space. Then,  $\overline{A} = A \cup A'$ .*

*Proof.* It is clear that  $\overline{A} \supseteq A \cup A'$ . Now pick  $x \in \overline{A}$ . If  $x \notin A$ , then we know that given any open neighbourhood  $U$  of  $x$ , we have non-empty  $U \cap A$ . Furthermore, this intersection can never contain  $x$ , hence  $x \in A'$ . This proves that  $\overline{A} \subseteq A \cup A'$ .  $\square$

## 1.8 Convergence of sequences

**Definition 1.15.** Let  $\{x_n\}_{n=1}^\infty$  be a sequence of points from  $(X, \tau)$ , and let  $x \in X$ . We say that this sequence converges to  $x$ , denoted  $x_n \rightarrow x$ , if every open neighbourhood of  $x$  contains the tail of this sequence. In other words, given  $U \in \tau$  such that  $x \in U$ , there must exist  $N \in \mathbb{N}$  such that  $x_n \in U$  for all  $n \geq N$ .

*Example.* Let  $X = \{a, b, c\}$ , and  $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ . Then, the constant sequence of  $b$ 's converges to all three points  $a, b, c$ .

*Example.* Let  $X = \mathbb{R}$ , and  $\tau$  be the collection of all intervals  $(-a, a)$  together with  $\emptyset, \mathbb{R}$ . Then, the constant sequence of 0's converges to every point in  $\mathbb{R}$ .

**Definition 1.16.** Let  $(X, \tau)$  be a topological space. We say that this topological space is Hausdorff if given any two distinct points  $x, y \in X$ , there exist open sets  $U, V \in \tau$  such that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ .

*Example.* The real numbers under the standard topology is Hausdorff.

**Theorem 1.17.** *Let  $(X, \tau)$  be a Hausdorff topological space, and let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of points in  $X$ . Then, this sequence can converge to at most one point in  $X$ .*

*Proof.* Suppose that  $\{x_n\}_{n=1}^{\infty}$  converges to distinct points  $x, y \in X$ . Then there exist disjoint open neighbourhoods  $U, V$  such that  $x \in U, y \in V$ . Convergence means that both  $U$  and  $V$  contain a tail of the sequence, which is a contradiction.  $\square$

**Lemma 1.18.** *The singleton sets in a Hausdorff space are closed.*

*Proof.* Let  $x \in X$  where  $(X, \tau)$  is Hausdorff. Pick  $y \neq x$ , whence there exist  $U_y, V_y \in \tau$ , such that  $x \in U_y, y \in V_y$ , and  $U_y \cap V_y = \emptyset$ . In particular,  $\{x\} \cap V_y = \emptyset$ . We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open.  $\square$

**Theorem 1.19.** *The topology induced by a metric is Hausdorff.*

*Proof.* Given a metric space  $X$  and distinct points  $x, y \in X$ , we set  $r = |x - y|$ ,  $U = B(x, r/3)$ ,  $V = B(y, r/3)$ .  $\square$

## 2 Continuous maps

**Definition 2.1.** Let  $f: X \rightarrow Y$  be a function between the topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ . We say that  $f$  is continuous if for every  $U \in \tau_Y$ , we have  $f^{-1}(U) \in \tau_X$ . In other words, the pre-image of every open set in  $Y$  must be open in  $X$ .

**Lemma 2.1.** *A function  $f: X \rightarrow Y$  is continuous if and only if given a base  $\beta$  of  $Y$ , we have  $f^{-1}(U) \in \tau_X$  for every  $U \in \beta$ .*

*Example.* The identity function  $\text{id}: \mathbb{R}_\ell \rightarrow \mathbb{R}$  is continuous, while the identity function  $\text{id}: \mathbb{R} \rightarrow \mathbb{R}_\ell$  is not. This is because the topology on  $\mathbb{R}_\ell$  is strictly finer than that on  $\mathbb{R}$ .

**Lemma 2.2.** *A function  $f: X \rightarrow Y$  is continuous if and only if for every closed set  $F \subseteq Y$ , we have  $f^{-1}(F)$  closed in  $X$ .*

**Lemma 2.3.** *A function  $f: X \rightarrow Y$  is continuous if and only if given any  $x \in X$  and an open set  $V \subseteq Y$  such that  $f(x) \in V$ , there exists an open set  $U \subseteq X$  such that  $x \in U$ ,  $f(U) \subseteq V$ .*



**Theorem 2.4.** *The composition of continuous functions is continuous.*

## 2.1 Restricting and enlarging the domain

**Lemma 2.5.** *Let  $f: X \rightarrow Y$  be continuous, and let  $A \subset X$ . Then the restriction of  $f$  to  $A$  is continuous.*

**Theorem 2.6.** *Let  $f: X \rightarrow Y$ , and let  $X$  be the union of the collection of open sets  $\{A_\lambda\}_{\lambda \in \Lambda}$ . If the restrictions of  $f$  to each  $A_\lambda$  are continuous, then  $f$  is continuous.*

*Proof.* Pick  $x \in X$ , hence  $x \in A_\lambda$  for some  $\lambda \in \Lambda$ . Now if  $f(x) \in V \subset Y$ , where  $V$  is open in  $Y$ , then the continuity of the restriction of  $f$  to  $A_\lambda$  gives us an open set  $U \subseteq A_\lambda$  such that  $f(U) \subseteq V$ . Finally since  $A_\lambda$  is open in  $X$ , so is  $U$ .  $\square$

**Definition 2.2.** Let  $X$  be the union of the collection of open sets  $\{A_\lambda\}_{\lambda \in \Lambda}$ . We say that this collection is a locally finite cover of  $X$  if given  $x \in X$ , there exists a neighbourhood  $U$  of  $x$  such that  $U \cap A_\lambda$  is non-empty for only finitely many  $\lambda \in \Lambda$ .

**Theorem 2.7.** *Let  $f: X \rightarrow Y$ , and let  $\{F_\lambda\}_{\lambda \in \Lambda}$  be a locally finite collection of closed sets covering  $X$ . If the restrictions of  $f$  to each  $F_\lambda$  are continuous, then  $f$  is continuous.*

**Corollary 2.7.1** (Pasting lemma). *Let  $X = A \cup B$ , with  $A, B$  closed in  $X$ . Let  $f: A \rightarrow Y$ ,  $g: B \rightarrow Y$  be continuous, with  $f(x) = g(x)$  on  $A \cap B$ . Then the function  $h: X \rightarrow Y$ , defined by  $x \mapsto f(x)$  on  $A$  and  $x \mapsto g(x)$  on  $B$ , is continuous.*

**Definition 2.3.** A path is a continuous function  $\gamma: [0, 1] \rightarrow X$ .

**Lemma 2.8.** *Two paths  $\gamma_1, \gamma_2$  can be concatenated when  $\gamma_1(1) = \gamma_2(0)$ .*

## 2.2 Homeomorphisms

**Definition 2.4.** Let  $f: X \rightarrow Y$  be a function between the topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ . We say that  $f$  is a homeomorphism if  $f$  is continuous,  $f$  is bijective, and  $f^{-1}$  is continuous. We also say that  $X$  and  $Y$  are homeomorphic when such a homeomorphism between them exists.

*Example.* The interval  $(-1, 1)$  is homeomorphic to  $\mathbb{R}$ ; for instance, the map  $x \mapsto \tan(\pi x/2)$  on  $(-1, 1)$  is a homeomorphism. A simpler construction is the map  $x \mapsto x/(1 - x^2)$ .

### 2.3 Projection maps

**Theorem 2.9.** *The projection maps  $\pi_i: X_1 \times \cdots \times X_k \rightarrow X_i$  are continuous, when the domain is equipped with the product topology. Furthermore, the product topology is the coarsest topology on the domain for which the projection maps are continuous.*

**Lemma 2.10.** *Let  $f: A \rightarrow X_1 \times \cdots \times X_k$ , where the co-domain is equipped with the product topology. Then,  $f$  is continuous if and only if the component functions  $f_i = \pi_i \circ f$  are continuous.*

*Proof.* Note that if  $f$  is continuous, the compositions  $\pi_i \circ f$  are immediately continuous. Conversely suppose that each  $f_i$  is continuous, and write

$$f(t) = (f_1(t), \dots, f_k(t)).$$

The sets  $U_1 \times \cdots \times U_k$ , where each  $U_i \subseteq X_i$  is open, form a basis of the co-domain. Furthermore, their pre-images under  $f$  are  $f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k)$ , which are open in  $A$ . This shows that  $f$  is continuous.  $\square$

**Definition 2.5.** Let  $J$  be an arbitrary index set. A  $J$ -tuple of elements in a set  $X$  is a function  $x: J \rightarrow X$ , formally denoted  $(x_\alpha)_{\alpha \in J}$ . If  $\{X_\alpha\}_{\alpha \in J}$  is a family of sets, their Cartesian product is defined as

$$\prod_{\alpha \in J} X_\alpha = \{x: J \rightarrow \bigcup_{\alpha \in J} X_\alpha : x_\alpha \in X_\alpha\}.$$

*Remark.* The fact that we can choose an element from each set in an uncountable collection relies on the Axiom of Choice.

**Definition 2.6.** Let  $\{X_\alpha\}_{\alpha \in J}$  be a collection of topological spaces. The topology generated by  $\prod_{\alpha \in J} U_\alpha$ , where each  $U_\alpha \subseteq X_\alpha$  is open, is called the box topology on  $\prod_{\alpha \in J} X_\alpha$ .

**Definition 2.7.** Let  $\{X_\alpha\}_{\alpha \in J}$  be a collection of topological spaces. The topology generated by the sub-basis  $\pi_\alpha^{-1}(U_\alpha)$ , where each  $U_\alpha \subseteq X_\alpha$  is open, is called the product topology on  $\prod_{\alpha \in J} X_\alpha$ .

*Remark.* The basic open sets are of the form  $\pi_{\alpha \in J} U_\alpha$ , where all but finitely many  $U_\alpha = X_\alpha$ . Thus, this is a coarser topology than the box topology.

**Lemma 2.11.** *Let  $\prod_{\alpha \in J} X_\alpha$  be equipped with the box or product topology. Then,  $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$ , where each  $A_\alpha \subseteq X_\alpha$ .*

**Lemma 2.12.** *Let  $f: A \rightarrow \prod_{\alpha \in J} X_\alpha$ , where the co-domain is equipped with the product topology. Then,  $f$  is continuous if and only if the component functions  $f_\alpha = \pi_\alpha \circ f$  are continuous.*

*Remark.* This fails when  $\prod_{\alpha \in J}$  is equipped with the box topology. Consider  $f: \mathbb{R} \rightarrow \prod_{n=1}^\infty \mathbb{R}$ ,  $x \mapsto (x, x, \dots)$ . Then, the product  $\prod_{n=1}^\infty (-1/n, 1/n)$  is open in the box topology, but its pre-image under  $f$  is  $\bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}$ , which is not open in  $\mathbb{R}$ .

### 3 Metric spaces

**Definition 3.1.** A metric space  $(X, d)$  is a set equipped with a metric  $d: X \times X \rightarrow \mathbb{R}$ , such that

1.  $d(x, y) = 0$  if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Definition 3.2.** An open ball in a metric spaces is the set of points

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

**Lemma 3.1.** *The collection of open balls in a metric space generates its standard topology.*

*Example.* Consider a set  $X$ , equipped with the metric

$$d: X \times X \rightarrow \mathbb{R}, \quad (x, y) \mapsto \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then, this metric induces the discrete topology on  $X$ .

#### 3.1 Metrizable spaces

**Definition 3.3.** A topological space  $(X, \tau)$  is called metrizable if there exists a metric  $d: X \times X \rightarrow \mathbb{R}$  which induces the topology  $\tau$  on  $X$ .

**Definition 3.4.** Let  $A \subseteq X$ . The diameter of  $A$  is defined to be

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

If  $\text{diam}(A)$  is finite, we say that  $A$  is bounded.

*Example.* The metric

$$(x, y) \mapsto \frac{|x - y|}{1 + |x - y|}$$

generates the standard topology on  $\mathbb{R}$ . Note that  $\mathbb{R}$  is unbounded in the standard metric, but bounded in this one.

**Definition 3.5.** Let  $(X, d)$  be a metric space. Then the standard bounded metric corresponding to  $d$  is defined as

$$\bar{d}: X \times X \rightarrow \mathbb{R}, \quad (x, y) \mapsto \min\{d(x, y), 1\}.$$

**Lemma 3.2.** Both  $d$  and  $\bar{d}$  generate the same topology.

**Theorem 3.3.** The product topology on  $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \dots$  is metrizable, using the metric

$$D(x, y) = \sup_n \left\{ \frac{1}{n} \bar{d}(x, y) \right\}.$$

**Lemma 3.4** (Sequence lemma). Let  $A \subseteq X$ , let  $x \in X$ , and let the sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \in A$  converge with  $x_n \rightarrow x$ . Then,  $x \in \bar{A}$ .

*Remark.* The converse holds if  $X$  is metrizable.

*Example.* Consider  $X = \mathbb{R}^\omega$  equipped with the box topology. Choose  $A = \{(x_1, x_2, \dots) : x_i > 0\}$ . Then,  $0 = (0, 0, \dots) \in \bar{A}$ ; this is clear from the fact that any open set around 0 contains the basic open set  $\prod_i (a_i, b_i)$  with  $a_i < 0 < b_i$ . However, there is no sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \in A$ , such that  $x_n \rightarrow 0$ . Note that if this were the case, then each  $x_n = (x_{n1}, x_{n2}, \dots)$ . Now,  $B = \prod_i (-x_{ii}, x_{ii})$  contains none of the points  $x_n$ , since the  $n$ th coordinate of  $B$  eliminates the point  $n$ .

**Corollary 3.4.1.**  $\mathbb{R}^\omega$  equipped with the box topology is not metrizable.

## 4 Compactness

**Definition 4.1.** Let  $X$  be a topological space. We say that  $X$  is compact if every open cover of  $X$  has a finite subcover.

**Lemma 4.1.** Let  $Y \subseteq X$ . Then,  $Y$  is compact if and only if every open cover of  $Y$  by open sets in  $X$  has a finite subcover.

## 4.1 Compact subspaces

**Lemma 4.2.** *All compact sets in a metric space are bounded.*

*Proof.* Let  $K \subseteq X$  be compact. Then,  $K$  admits an open cover of open balls  $B(0, n)$  from which we can extract a finite subcover; however, this can be reduced to just one open ball  $B(0, N)$  for some  $N$ . Thus  $K \subset B(0, N)$  is bounded.  $\square$

**Lemma 4.3.** *A closed subset of a compact space is compact.*

*Proof.* Let  $K$  be compact, and  $F \subseteq K$  be closed. Consider an open cover  $\{U_\alpha\}_{\alpha \in J}$  of  $F$ . By adding  $K \setminus F$  to this collection, we have an open cover of  $K$ , from which we can extract a finite subcover  $U_{i_1}, U_{i_2}, \dots, U_{i_k}, K \setminus F$ . By discarding the latter, we have found a finite subcover of  $F$ .  $\square$

**Lemma 4.4.** *In a Hausdorff space, every compact set is closed.*

*Proof.* Let  $X$  be Hausdorff, and  $K \subseteq X$  be compact. Fix  $x_0 \in X \setminus K$ , and note that given any  $y \in K$ , there exist open neighbourhoods  $U_y, V_y$  such that  $x_0 \in U_y$ ,  $y \in V_y$ ,  $U_y \cap V_y = \emptyset$ . Thus, the collection of all such  $\{V_y\}_{y \in K}$  is an open cover of  $K$ , from which we can extract a finite subcover  $V_{y_1}, \dots, V_{y_k}$ . Corresponding to this,  $x_0 \in U_{y_1} \cap \dots \cap U_{y_k} \subseteq X \setminus K$ . Thus,  $x_0$  lies in the interior of  $X \setminus K$ . This shows that  $X \setminus K$  is open, hence  $K$  is closed.  $\square$

**Theorem 4.5.** *The image of a compact space under a continuous map is compact.*

**Lemma 4.6.** *Let  $f: X \rightarrow Y$  be a continuous bijection. If  $X$  is compact and  $Y$  is Hausdorff, then  $f$  is a homeomorphism.*

*Proof.* We need only show that  $f$  is a closed map; now every closed set  $F \subseteq X$  is compact because  $X$  is compact, hence  $f(F) \subseteq Y$  is compact. Since  $Y$  is Hausdorff, the compact set  $f(F)$  is closed.  $\square$

## 4.2 Products of compact spaces

**Lemma 4.7** (Tube lemma). *Let  $X, Y$  be topological spaces, and let  $Y$  be compact. Let  $x_0 \in X$ , and let  $\{x_0\} \times Y \subset N \subseteq X \times Y$  where  $N$  is open. Then, there exists an open set  $W \subseteq X$  such that  $\{x_0\} \times Y \subseteq W \times Y \subseteq N$ .*

*Proof.* Note that  $\{x_0\} \times Y$  is compact, being homeomorphic to  $Y$ . Thus, it can be covered with basic open sets  $U_1 \times V_1, \dots, U_k \times V_k$  such that each  $U_i \times V_i \subset N$ . Simply set  $W = U_1 \cap \dots \cap U_k$ .  $\square$

**Theorem 4.8.** *Let  $X, Y$  be compact topological spaces. Then,  $X \times Y$  is compact.*

*Proof.* Let  $\{U_\alpha\}_{\alpha \in J}$  be an open cover of  $X \times Y$ . Pick  $x \in X$ , whence  $\{x\} \times Y$  is compact and admits a finite subcover  $U_{x i_1}, \dots, U_{x i_k}$ . Denote their union by  $U_x$ ; the tube lemma guarantees an open set  $W_x \subseteq X$  such that  $\{x\} \times Y \subseteq W_x \times Y \subseteq U_x$ . Now, the collection  $\{W_x\}_{x \in X}$  is an open cover of  $X$ , hence admits a finite subcover  $W_{x_1}, \dots, W_{x_n}$ . This also means that  $W_{x_1} \times Y, \dots, W_{x_n} \times Y$  is a finite cover of  $Y$ . However, each  $W_{x_i} \times Y \subseteq U_{x_i}$  can be covered by finitely many  $U_\alpha$ , which means that we have a finite subcover of  $X \times Y$ .  $\square$

### 4.3 Euclidean spaces

**Lemma 4.9.** *Let  $X$  be a simply ordered set with the least upper bound property. Then, the intervals  $[a, b]$  are compact.*

**Theorem 4.10** (Heine-Borel). *Compact sets of  $\mathbb{R}^n$  are precisely those which are closed and bounded.*

### 4.4 Limit point compactness

**Definition 4.2.** Let  $X$  be a topological space. We say that  $X$  is limit point compact if every infinite subset of  $X$  has a limit point.

**Lemma 4.11.** *A compact space is limit point compact.*

*Proof.* Let  $X$  be compact, and let  $A \subseteq X$  have no limit points. Then,  $A = A \cup A' = \bar{A}$  is closed in  $X$ , hence compact. Now given any  $a \in A$ , we know that  $a$  is not a limit point of  $A$ , hence we can choose an open neighbourhood  $U_a$  such that  $U_a \cap A = \{a\}$ . The collection  $\{U_a\}_{a \in A}$  is now an open cover of  $A$ , and hence admits a finite subcover  $U_{a_1}, \dots, U_{a_k}$ . Let  $U$  denote their union, whence  $A = A \cap U = \{a_1, \dots, a_k\}$  is finite.  $\square$

*Example.* Let  $X = \mathbb{N} \times \{0, 1\}$ , where  $\mathbb{N}$  has the discrete topology, and  $\{0, 1\}$  has the indiscrete topology. Then, every subset of  $X$  has a limit point; indeed, given any  $\{(n, b)\}$ , we have a limit point  $(n, 1 - b)$ . However,  $X$  is clearly not compact, since the open cover of sets  $\{n\} \times \{0, 1\}$  does not admit any finite subcover.

**Theorem 4.12.** *Let  $X$  be a metrizable space. Then,  $X$  is limit point compact if and only if it is compact.*

## 5 Connectedness

**Definition 5.1.** Let  $X$  be a topological space, and let  $U, V \subseteq X$  be open, non-empty, disjoint, with  $U \cup V = X$ . We say that  $U, V$  form a separation of  $X$ .

**Definition 5.2.** A topological space  $X$  is said to be connected if it admits no separation.

**Lemma 5.1.** *A topological space  $X$  is connected if and only if the only subsets that are both open and closed in it are  $\emptyset, X$ .*

**Lemma 5.2.** *Let  $X$  be a topological space, and let  $Y \subseteq X$  be a subspace. Then, a separation of  $Y$  is a pair of open sets  $A, B \subseteq X$  such that  $\overline{A} \cap B = \emptyset$ ,  $A \cap \overline{B} = \emptyset$ .*

**Lemma 5.3.** *Let  $C, D$  form a separation of  $X$ , and let  $Y \subseteq X$  be a connected subspace. Then, either  $Y \subseteq C$ ,  $Y \subseteq D$ .*

**Lemma 5.4.** *The union of a collection of connected spaces with a common point is connected.*

*Proof.* Let  $\{X_\alpha\}_{\alpha \in J}$  be a collection of connected spaces, with the common point  $x_0$ , and let  $X$  be their union. Suppose that  $U, V$  is a separation of  $X$ ; then each of the connected  $X_\alpha$  must be contained in one of  $U, V$ . However, since all  $X_\alpha$  share the common point  $x_0$ , they must all lie in the same half, say  $U$ , forcing  $V = \emptyset$ , a contradiction.  $\square$

**Lemma 5.5.** *Let  $A \subseteq X$  be connected, and let  $A \subseteq B \subseteq \overline{A}$ . Then,  $B$  is connected.*

**Theorem 5.6.** *The image of a connected space under a continuous maps is connected.*

**Theorem 5.7.** *A finite Cartesian product of connected spaces is connected.*

*Proof.* Let  $X, Y$  be connected spaces. Fix  $(a, b) \in X \times Y$ . Now,  $X \times \{b\}$  is connected, being homeomorphic to  $X$ . Furthermore, each  $\{x\} \times Y$  is connected, for each  $x \in Y$ . Now, the set  $T_x = \{x\} \times Y \cup X \times \{b\}$  is connected, being the union of connected spaces with the common point  $(x, b)$ . Finally, the union of all such  $T_x$  is connected, being the union of connected spaces with the common point  $(a, b)$ . This union is just  $X \times Y$ , which is thus connected.  $\square$

*Example.* The countable product  $\mathbb{R}^\omega$  with the box topology is disconnected. Consider

$$A = \text{set of all bounded sequences}, \quad B = \text{set of all unbounded sequences}.$$

Now,  $A \cap B = \emptyset$ ,  $A \cup B = \mathbb{R}^\omega$ ,  $A, B \neq \emptyset$ . It can also be shown that  $A, B$  are open.

*Example.* The countable product  $\mathbb{R}^\omega$  with the product topology is connected. To show this, define

$$\tilde{R}^n = \{(x_1, \dots, x_n, 0, 0, \dots) : x_i \in \mathbb{R}\}.$$

Then, set  $X = \bigcup_{n=1}^{\infty} \tilde{R}^n$ , and note that since each  $\tilde{R}^n \cong \mathbb{R}^n$  is connected with all of them sharing the common point  $(0, 0, \dots)$ ,  $X$  must be connected. We now show that  $\overline{X} = \mathbb{R}^\omega$ . Indeed, given  $x = (x_1, x_2, \dots) \in \mathbb{R}^\omega$ , an open neighbourhood of  $x$  looks like  $U = U_1 \times U_2 \times \dots$ , where all but finitely many  $I_i = \mathbb{R}$ . In other words, there exists sufficiently large  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $U_n = \mathbb{R}$ . Thus, the point  $(x_1, x_2, \dots, x_n, 0, 0, \dots) \in U \cap \tilde{R}^N$ .

**Lemma 5.8.** *The closed intervals  $[a, b] \subset \mathbb{R}$  are connected.*

## 5.1 Path connectedness

**Definition 5.3.** A topological space  $X$  is said to be path connected if there exists a path joining any two points in  $X$ . In other words, given  $a, b \in X$ , there always exists a continuous map  $\gamma: [0, 1] \rightarrow X$  such that  $\gamma(0) = a$ ,  $\gamma(1) = b$ .

**Lemma 5.9.** *All path connected spaces are connected.*

*Proof.* Note that if  $X = U \cup V$  is a separation of the path connected space  $X$ , then  $[0, 1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$  is a separation of the connected interval  $[0, 1]$ , a contradiction.  $\square$

**Lemma 5.10.** *The image of a path connected space under a continuous map is path connected.*

*Example.* The unit sphere  $S^{n-1}$  is path connected. Note that the map

$$f: \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}, \quad x \mapsto x/\|x\|$$

is continuous and surjective. Thus, it maps the path connected set  $\mathbb{R}^n \setminus \{0\}$  to  $S^{n-1}$ , which must be path connected.



*Example.* The set  $\overline{S}$ , called the topologist's sine curve, is connected but not path connected.

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

Note that  $S$  is the continuous image of the connected interval  $(0, 1]$ , hence connected. This further shows that  $\overline{S}$  is connected. Now,

$$\overline{S} = S \cup \{(0, y) : -1 \leq y \leq 1\}.$$

However,  $\overline{S}$  is not path connected, since there exists no path joining  $(0, 0)$  and  $(1/\pi, 0)$ . Indeed, given any path  $\gamma: [0, 1] \rightarrow \overline{S}$  starting at  $(0, 0)$ , it cannot escape  $\{0\} \times [-1, 1]$ . To see this, write  $\gamma = (\gamma_1, \gamma_2)$ ,  $\gamma_2(0) = 0$ . By continuity of  $\gamma_2$ , we can choose  $\delta > 0$  such that  $|\gamma_2(t)| < 1/2$  for all  $0 \leq t \leq \delta$ . Suppose that  $\gamma_1(t^*) = \tau > 0$  for some  $0 \leq t \leq \delta$ . By the intermediate value theorem,  $\gamma_1$  takes all the values between 0 and  $\tau$  in the interval  $[0, t^*]$ . Choose  $N$  such that  $2/\pi(2N+1) < \tau$ . Again, there must exist some  $0 < t_0 < t^*$  such that  $\gamma_1(t_0) = 2/\pi(2N+1)$ . Now,  $\gamma_2(t_0) = \sin(1/\gamma_1(t_0)) = 1 > 1/2$ , a contradiction. This means that  $\gamma_1(t) = 0$  for all  $t \in [0, \delta]$ .

## 6 Quotient topology

**Definition 6.1.** Let  $X$  be a topological space, and let  $\sim$  be an equivalence relation on  $X$ . Then  $X/\sim$  denotes the set of all equivalence classes with respect to  $\sim$ . Its elements are of the form  $[x] = \{y \in X : x \sim y\}$ , for  $x \in X$ . Define the map

$$\pi: X \rightarrow X/\sim, \quad x \mapsto [x].$$

The quotient topology on  $X/\sim$  is the finest topology such that  $\pi$  is continuous. In other words,  $U \subseteq X/\sim$  is open if  $\pi^{-1}(U)$  is open in  $X$ .

**Lemma 6.1.** Let  $f: X \rightarrow Y$  be a continuous surjection, with  $X$  compact and  $Y$  Hausdorff. Define an equivalence relation  $\sim$  on  $X$  such that  $x \sim x' \Leftrightarrow f(x) = f(x')$ . Then,  $g: X/\sim \rightarrow Y$ ,  $[x] \mapsto f(x)$  is a homeomorphism.

*Example.* Consider the interval  $[0, 1]$ , with the equivalence relation  $\sim$  which identifies  $0 \sim 1$ , and leaves all other points undisturbed. Then, the quotient space  $[0, 1]/\sim$  is homeomorphic to the circle  $S^1$ .

Note that the quotient map on  $[0, 1]$  is not open, since the image of the open set  $[0, 1/2)$  is not open in  $[0, 1]/\sim$ .

*Example.* Let  $X = \mathbb{R}^{n+1} \setminus \{0\}$ , and define an equivalence relation on  $X$  which identifies points on the same line through the origin together. Then, the resulting quotient space is called the real projective space, denoted  $\mathbb{RP}^n$ .

*Example.* Let  $S^n$  denote the  $n$ -sphere in  $\mathbb{R}^{n+1}$ , and define an equivalence relation on  $S^n$  which identifies antipodal points. Then, the resulting quotient space is also  $\mathbb{RP}^n$ . The quotient map here is an open map.

**Lemma 6.2.** *Let  $f: X \rightarrow Y$  be an open, continuous, surjective map. Define an equivalence relation  $\sim$  on  $X$  such that  $x \sim x' \Leftrightarrow f(x) = f(x')$ . Then,  $g: X/\sim \rightarrow Y$ ,  $[x] \mapsto f(x)$  is a homeomorphism.*

*Example.* By defining  $f$  as a composition of maps  $\mathbb{R}^n \setminus \{0\} \rightarrow S^n \rightarrow S^n/\sim$ , it can be shown that  $\mathbb{RP}^n$  is compact.

## 6.1 One-point compactification

**Definition 6.2.** Let  $X$  be a compact topological space, and let  $A \subset X$  be closed. The one-point compactification of  $X \setminus A$  is defined by

$$Y = (X \setminus A) \cup \{\infty\},$$

with the topology

$$\tau_Y = \{U \subseteq X \setminus A \text{ is open}\} \cup \{Y \setminus C \text{ where } C \text{ is compact in } X \setminus A\}.$$

**Lemma 6.3.** *If  $X$  is compact, Hausdorff, then so is the one-point compactification  $Y$  of  $X \setminus A$ .*

**Lemma 6.4.** *Let  $X$  be a compact, Hausdorff space and  $A \subseteq X$  be a closed set. Define  $\sim$  on  $X$  by identifying  $x \sim x'$  whenever  $x, x' \in A$  and leaving the remaining points undisturbed. Then,  $X/\sim$  is homeomorphic to the one-point compactification  $Y = X \setminus A \cup \{\infty\}$ .*

## 7 Countability axioms

### 7.1 First countability

**Definition 7.1.** Let  $X$  be a topological space. A countable basis at a point  $x \in X$  is a countable collection  $\beta$  of neighbourhoods of  $x$  such that for any neighbourhood  $U$  of  $x$ , there is a basis element  $B \in \beta$  such that  $x \in B \subseteq U$ .

**Definition 7.2.** A topological space  $X$  in which every element  $x \in X$  admits a countable basis is called a first countable space.

*Example.* All metrizable spaces are first countable. Given an element  $x$ , the collection of all open balls centred at  $x$  with rational radii forms a countable basis.

*Example.* The space  $\mathbb{R}_\ell$  is a non-metrizable space which is first countable.

**Lemma 7.1.** *The sequence lemma holds for first countable spaces, i.e. if  $X$  is first countable,  $A \subseteq X$ , and  $x \in \overline{A}$ , then there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \in A$ , such that  $x_n \rightarrow x$ .*

*Example.* The space  $\mathbb{R}^\omega$  with the box topology is not first countable.

## 7.2 Second countability

**Definition 7.3.** A topological space  $X$  which admits a countable basis is called a second countable space.

*Example.* The Euclidean spaces  $\mathbb{R}^n$  are second countable. The collection of all open balls with rational radii, centred at rational points, forms a countable basis.

*Example.* The space  $\mathbb{R}^\omega$  with the product topology is second countable.

**Lemma 7.2.** *If a topological space  $X$  is second countable, then any discrete subspace  $A \subseteq X$  must be countable.*

*Proof.* Let  $\beta = \{B_n\}_{n \in \mathbb{N}}$  be a countable basis of  $X$ . For each  $a \in A$ , note that  $\{a\}$  is open in subspace topology  $A$ , hence there exists a basis element  $B_a \in \beta$  such that  $B_a \cap A = \{a\}$ . Furthermore, this assignment  $A \rightarrow \beta$ ,  $a \mapsto B_a$  is injective, hence  $A$  must be countable.  $\square$

*Example.* The space  $\mathbb{R}^\omega$  with the uniform topology is metrizable hence first countable, but not second countable. This topology is induced by the metric

$$\rho(x, y) = \sup_{i \in \mathbb{N}} \bar{d}(x_i, y_i) = \sup_{i \in \mathbb{N}} \min\{|x_i - y_i|, 1\}.$$

Consider the subspace  $A \subset \mathbb{R}^\omega$ , consisting of all binary sequences. This is clearly an uncountable set. However, for any two distinct members  $x, y \in A$ , we have  $d(x, y) = 1$ . This precisely describes the discrete topology on  $A$ . The contrapositive of the above lemma now shows that  $\mathbb{R}^\omega$  with the uniform topology cannot be second countable.

**Lemma 7.3.** *Let  $X$  be a second countable space. Then, every open cover of  $X$  admits a countable subcover.*

*Remark.* A topological space in which every open cover admits a countable subcover is called a Lindelöf space.

**Lemma 7.4.** *Let  $X$  be a second countable space. Then, there exists a countable subset which is dense in  $X$ .*

*Remark.* A topological space in which there exists a dense countable subset is called a separable space.

*Proof.* Let  $\{B_n\}_{n \in \mathbb{N}}$  be a countable basis of  $X$ . Pick one element  $x_n \in B_n$  for each  $n \in \mathbb{N}$ , whence the set  $\{x_n\}_{n \in \mathbb{N}}$  is countable and dense in  $X$ .  $\square$