MA2201: Analysis II

Integration

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Definition 3.1 (Partition)**.** A partition Q of an interval $[a.b]$ is a finite sequence of numbers

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The norm of a partition is defined as

$$\|Q\| = \max |x_{j+1} - x_j|.$$

A tagged partition $\dot{Q}$ is a partition Q together with a set of numbers t_j such that $t_j \in [x_j, x_{j+1}]$.

Definition 3.2 (Riemann sum)**.** The Riemann sum of a function f on an interval $[a, b]$ with respect to a tagged partition $\dot{Q}$ is defined as

$$S(f, \dot{Q}) = \sum_{j=0}^{n-1} f(t_j)(x_{j+1} - x_j).$$

Definition 3.3 (Riemann integral)**.** A function f is called Riemann integrable on an interval $[a, b]$ if there is some $\ell \in \mathbb{R}$ where for every $\epsilon > 0$, there exists $\delta > 0$ such that all tagged partitions $\dot{Q}$ of $[a, b]$ with $\|\dot{Q}\| < \delta$ satisfy

$$|S(f, \dot{Q}) - \ell| < \epsilon.$$

The number ℓ is the value of the Riemann integral,

$$\int_a^b f = \ell.$$

Theorem 3.1. *If a function is Riemann integrable on an interval, then the value of the integral is unique.*

Proof. Let f be Riemann integrable on $[a,b]$, with integral values ℓ and ℓ'. Then, for every $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions $\dot{Q}$ with $\|\dot{Q}\| < \delta$,

$$|S(f,\dot{Q}) - \ell| < \epsilon, \qquad |S(f,\dot{Q}) - \ell'| < \epsilon.$$

Note that such a partition $\dot{Q}$ always exists. Thus,

$$|\ell - \ell'| \leq |S(f,\dot{Q}) - \ell| + |S(f,\dot{Q}) - \ell'| < 2\epsilon$$

for all $\epsilon > 0$, which forces $\ell = \ell'$. $\square$

Definition 3.4 (Darboux sums)**.** Given a partition Q of $[a,b]$ and a function f, define

$$m_j = \inf_{t\in[x_j,x_{j+1}]} f(t), \qquad M_j = \sup_{t\in[x_j,x_{j+1}]} f(t).$$

The lower and upper Darboux sums are defined as

$$L(f,Q) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j), \qquad U(f,Q) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j).$$

Definition 3.5 (Darboux integrals)**.** The lower and upper Darboux integrals of a function f are defined as

$$L_f = \inf_Q L(f,Q), \qquad U_f = \sup_Q U(f,Q).$$

Here, the infimum and supremum is taken over all possible partitions Q of $[a,b]$.

If $L_f = U_f$, then the common integral is simply called the Darboux integral,

$$\int_a^b f = L_f = U_f.$$

Such a function f is called Darboux integrable.

Theorem 3.2. *Riemann and Darboux integrability are equivalent and assign the same value to the integrals.*