MA 1101 : Mathematics I

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Solution 1.

(i) Let $P(n)$ be the statement

$$1 + 2 + \cdots + n \;=\; \frac{1}{2}n(n+1) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{2}1(1+1)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= [1 + 2 + \cdots + k] + (k+1) \\
&= \frac{1}{2}k(k+1) + (k+1) && \text{(From } P(k)\text{)} \\
&= \frac{1}{2}(k+2)(k+1) \\
&= \frac{1}{2}(k+1)((k+1)+1)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(ii) Let $P(n)$ be the statement

$$1^2 + 2^2 + \cdots + n^2 \;=\; \frac{1}{6}n(n+1)(2n+1) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{6}1(1+1)(2+1)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= [1^2 + 2^2 + \cdots + k^2] + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 && \text{(From } P(k)\text{)} \\
&= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6) \\
&= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(iii) Let $P(n)$ be the statement

$$1^2 + 3^2 + \cdots + (2n-1)^2 \;=\; \frac{1}{3}(4n^3 - n) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{3}1(4-3)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^2+3^2+\cdots+(2k-1)^2+(2k+1)^2 &= [1^2+3^2+\cdots+(2k-1)^2]+(2k+1)^2 \\
&= \frac{1}{3}(4k^3-k)+(2k+1)^2 \qquad \text{(From } P(k)\text{)} \\
&= \frac{1}{3}(4k^3-k+12k^2+12k+3) \\
&= \frac{1}{3}(4(k^3+3k^2+3k+1)-k-1)) \\
&= \frac{1}{3}(4(k+1)^3-(k+1))
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(iv) Let $P(n)$ be the statement

$$1^3+2^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2 \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{4}1(1+1)^2$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= [1^3+2^3+\cdots+k^3]+(k+1)^3 \\
&= \frac{1}{4}k^2(k+1)^2+(k+1)^3 \qquad \text{(From } P(k)\text{)} \\
&= \frac{1}{4}(k+1)^2(k^2+4k+4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \frac{1}{4}(k+1)^2((k+1)+1)^2
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(v) Let $P(n)$ be the statement

$$\sum_{r=1}^{n} r(r+1)\dots(r+9) = \frac{1}{11}n(n+1)\dots(n+10) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly,

$$1(1+1)\dots(1+9) = \frac{1}{11}1(1+1)\dots(1+9)(1+10)$$

Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
\sum_{r=1}^{k+1} r(r+1)\dots(r+9) &= \left[\sum_{r=1}^{k} r(r+1)\dots(r+9)\right] + (k+1)(k+2)\dots(k+1+9) \\
&= \frac{1}{11}k(k+1)\dots(k+10) + (k+1)(k+2)\dots(k+1+9) \quad \text{(From } P(k)\text{)} \\
&= \frac{1}{11}(k+1)\dots(k+10)(k+11) \\
&= \frac{1}{11}(k+1)\dots((k+1)+9)((k+1)+10)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Solution 2.

(i) Let $P(n)$ be the statement that for all $n \in \mathbb{N}$,

$$3^n > n^2$$

Base step We establish $P(1)$ and $P(2)$. Clearly, $3^1 > 1^2$. Thus, $P(1)$ is true. Again, $3^2 = 9 > 8 = 2^2$. Thus, $P(2)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$3^{k+1} \;=\; 3 \cdot 3^k \;>\; 3 \cdot k^2$$

We must show $3k^2 > (k+1)^2 \Leftrightarrow 3k^2 - (k+1)^2 > 0$.

$$3k^2 - (k+1)^2 \;=\; 2k^2 - 2k - 1 \;=\; k^2 + (k-1)^2 - 2$$

Clearly, for $k \geq 2$, $k^2 > 2$, so $k^2 + (k-1)^2 > 2$, and we are done.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(ii) Let $P(n)$ be the statement that for all $n \in \mathbb{N}$ and $x > -1$,

$$(1+x)^n \geq 1 + nx. \qquad \text{(Bernoulli's Inequality)}$$

Base Step We establish $P(1)$. Clearly, $(1+x)^1 \geq (1 + 1 \cdot x)$, thus $P(1)$ is true.

Inductive Step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\
&\geq (1+kx) \cdot (1+x) && (x+1>0) \\
&= (1 + x + kx + kx^2) \\
&\geq (1 + (k+1)x) && (k>0 \text{ and } x^2 \geq 0)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(iii) Let $P(n)$ be the statement that for all $n \geq 5$, $n \in \mathbb{N}$,

$$\binom{2n}{n} \;<\; 2^{2n-2}.$$

Base Step We establish $P(5)$. Now, $\binom{2n}{n} = 252$, while $2^{10-2} = 256$. Thus, $P(5)$ is true.

Inductive Step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
\binom{2(k+1)}{k+1} &= \frac{(2k+2)!}{(k+1)!^2} \\
&= \frac{(2k+2)(2k+1)}{(k+1)^2}\binom{2n}{n} \\
&< 2 \cdot \frac{2k+1}{k+1} \cdot 2^{2k-2} \\
&< 2 \cdot \frac{2k+2}{k+1} \cdot 2^{2k-2} \\
&< 2^{2(k+1)-2}
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 5$, $n \in \mathbb{N}$.

Solution 3.

(i) Let $P(n)$ be the statement that every $n \geq 2$, $n \in \mathbb{N}$ has a prime divisor. We prove this using the principle of strong mathematical induction.

Base Step We establish $P(2)$. Clearly, 2 is a prime divisor of itself, so $P(2)$ is true.

Inductive Step We assume that the statements $P(2), P(3), \ldots, P(k-1)$ are all true. We will show that $P(k)$ is true.

If $k \geq 2$ is prime, then we are done, as k is a prime divisor of itself. Otherwise, if k is not prime, then $k = ab$ for some $1 < a, b < k$ and $a, b \in \mathbb{N}$. We see that $a \geq 2$, so by the induction hypothesis, a has a prime divisor $p \in \mathbb{N}$, i.e., $a = pc$ for some $c \in \mathbb{N}$. Thus, $k = (pc)b = p(cb)$, and $cb \in \mathbb{N}$, so p is a prime factor of k. This proves $P(k)$.

Hence, by the principle of strong induction, $P(n)$ is true for all $n \geq 2$, $n \in \mathbb{N}$.

(ii) We define the Fibonacci sequence $(f_n)_{n \geq 0}$ as follows.

$$\begin{aligned} f_0 &:= 0 \\ f_1 &:= 1 \\ f_n &:= f_{n-1} + f_{n-2}, \qquad \text{for all } n \geq 2 \end{aligned}$$

(a) We wish to show that for all $n \in \mathbb{N}$,

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \qquad \text{(Binet's formula)}$$

We prove this using the principle of strong mathematical induction. Let $P(n)$ be the aforementioned statement, and let $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$. Note that φ and ψ both satisfy $x^2 = x + 1$.

$$\left(\frac{1 \pm \sqrt{5}}{2} \right)^2 = \frac{6 \pm 2\sqrt{5}}{4} = \frac{1 \pm \sqrt{5}}{2} + 1$$

Base Step We establish $P(1)$. Clearly, $f_1 = 1 = (\varphi - \psi)/\sqrt{5}$. Thus, $P(1)$ is true.

Inductive Step We assume that the statements $P(2), P(3), \ldots, P(k)$ are all true. We will show that $P(k+1)$ is true.

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \\ &= \frac{1}{\sqrt{5}}(\varphi^n - \psi^n) + \frac{1}{\sqrt{5}}(\varphi^{n-1} + \psi^{n-1}) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n-1}(\varphi + 1) - \psi^{n-1}(\psi + 1)) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n-1}(\varphi^2) - \psi^{n-1}(\psi^2)) \\ &= \frac{1}{\sqrt{5}}(\varphi^{n+1} - \psi^{n+1}) \end{aligned}$$

Hence, by the principle of strong induction, $P(n)$ is true for all $n \in \mathbb{N}$.