MA2201: ANALYSIS II

Integration

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Definition 3.1 (Partition)**.** A partition Q of an interval $[a.b]$ is a finite sequence of numbers

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The norm of a partition is defined as

$$\|Q\| = \max |x_{j+1} - x_j|.$$

A tagged partition $\dot{Q}$ is a partition Q together with a set of numbers t_j such that $t_j \in [x_j, x_{j+1}]$.

Definition 3.2 (Riemann sum)**.** The Riemann sum of a function f on an interval $[a, b]$ with respect to a tagged partition $\dot{Q}$ is defined as

$$S(f, \dot{Q}) = \sum_{j=0}^{n-1} f(t_j)(x_{j+1} - x_j).$$

Definition 3.3 (Darboux sums)**.** Given a partition Q of $[a, b]$ and a function f, define

$$m_j = \inf_{t \in [x_j, x_{j+1}]} f(t), \qquad M_j = \sup_{t \in [x_j, x_{j+1}]} f(t).$$

The lower and upper Darboux sums are defined as

$$L(f, Q) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j), \qquad U(f, Q) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j).$$

Definition 3.4 (Darboux integrals)**.** The lower and upper Darboux integrals of a function f are defined as

$$L_f = \inf_Q L(f, Q), \qquad U_f = \sup_Q U(f, Q).$$

Here, the infimum and supremum is taken over all possible partitions Q of $[a, b]$.

If $L_f = U_f$, then the common integral is simply called the Darboux integral,

$$\int_a^b f = L_f = U_f.$$

Such a function f is called Darboux integrable.

Theorem 3.1. *Riemann and Darboux integrals (and integrability) are equivalent.*