MA3205

Geometry of Curves and Surfaces

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1 Introduction

1.1 Curves

Definition 1.1. A curve is a continuous map $\gamma\colon \mathbb{R} \to \mathbb{R}^n$.

Definition 1.2. A smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is C^∞, i.e. differentiable arbitrarily times.

Definition 1.3. A closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is periodic, i.e there exists some c such that $\gamma(t + c) = \gamma(t)$ for all $t \in \mathbb{R}$.

Example. Alternatively, a closed curve can be thought of as a continuous map $\gamma\colon S^1 \to \mathbb{R}^n$. For instance, given a closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ with period c, we can define the corresponding map

$$\tilde{\gamma}\colon S^1 \to \mathbb{R}^n, \qquad \tilde{\gamma}(e^{it}) = \gamma(ct/2\pi).$$

Definition 1.4. A simple curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is injective on its period.

Theorem 1.1 (Four Vertex Theorem)**.** *The curvature of a simple, closed, smooth plane curve has at least two local minima and two local maxima.*

Definition 1.5. A knot is a simple closed curve in $\mathbb{R}^3$.

Definition 1.6. The total absolute curvature of a knot K is the integral of the absolute value of the curvature, taken over the curve, i.e. it is the quantity

$$\oint_K |\kappa(s)|\, ds.$$

Example. The total absolute curvature of a circle is always 2π.

Theorem 1.2 (Fáry-Milnor Theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Definition 1.7. An immersed loop γ is such that γ' is never zero.

Definition 1.8. Two loops are isotopic if there exists an interpolating family of loops between them. Two immersed loops are isotopic if we can choose such an interpolating family of immersed loops.

Example. Without the restriction of immersion, any two loops $\gamma, \eta \colon S^1 \to \mathbb{R}^n$ would be isotopic, since we can always construct the linear interpolations

$$H \colon S^1 \times [0,1] \to \mathbb{R}^n, \qquad H(e^{i\theta}, t) = (1-t)\gamma(e^{i\theta}) + t\eta(e^{i\theta}).$$

Theorem 1.3 (Hirsch-Smale Theory)**.**

1. *Any two immersed loops in $\mathbb{R}^2$ are isotopic if and only if the winding numbers match.*
2. *Any two immersed loops in S^2 are isotopic if and only if the winding numbers modulo 2 match.*