MA3101

Analysis III

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Contents

1 Euclidean spaces

We are familiar with the vector space $\mathbb{R}^n$, with the standard inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is defined as

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle = \sum_{k=1}^{n} (x_i - y_i)^2.$$

Exercise 1.1. What are all possible inner products on $\mathbb{R}^n$?

Solution. Note that an inner product is a bilinear, symmetric map such that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$. Thus, an product map on $\mathbb{R}^n$ is completely and uniquely determined by the values $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = a_{ij}$. Let A be the $n \times n$ matrix with entries a_{ij}. Note that A is a real symmetric matrix with positive entries. Now,

$$\langle \boldsymbol{x}, \boldsymbol{e}_j \rangle = x_1 a_{1j} + \cdots + x_n a_{nj} = \boldsymbol{x}^\top \boldsymbol{a}_j,$$

where $\boldsymbol{a}_j$ is the j^{th} column of A. Thus,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{a}_1 y_1 + \cdots + \boldsymbol{x}^\top \boldsymbol{a}_n y_n = \boldsymbol{x}^\top A \boldsymbol{y}.$$

Furthermore, any choice of real symmetric A with positive entries produces an inner product.

Theorem 1.1 (Cauchy-Schwarz)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|.$$

Proof. This is trivial when $\boldsymbol{w} = \boldsymbol{0}$. When $\boldsymbol{w} \neq \boldsymbol{0}$, set $\lambda = \langle \boldsymbol{v}, \boldsymbol{w} \rangle / \|\boldsymbol{w}\|^2$. Thus,

$$0 \leq \|\boldsymbol{v} - \lambda \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \lambda^2 \|\boldsymbol{w}\|^2.$$

Simplifying,

$$0 \leq \|\boldsymbol{v}\|^2 - \frac{|\langle \boldsymbol{v}, \boldsymbol{w} \rangle|^2}{\|\boldsymbol{w}\|^2}.$$

This gives the desired result. Clearly, equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$. □

Theorem 1.2 (Triangle inequality)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

Proof. Write

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\langle \boldsymbol{v}, \boldsymbol{w} \rangle + \|\boldsymbol{w}\|^2 \leq \|\boldsymbol{v}\|^2 + 2|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| + \|\boldsymbol{w}\|^2.$$

Applying Cauchy-Schwarz gives

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2.$$

Equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$ for $\lambda \geq 0$. □

This allows us to define the standard metric on $\mathbb{R}^n$, seen as a point set.

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Definition 1.1. For any $\epsilon > 0$, the set

$$B_\epsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < \epsilon\}$$

is called the open ball centred at $\boldsymbol{x} \in \mathbb{R}^n$ with radius ϵ. This is also called the ϵ neighbourhood of $\boldsymbol{x}$.

Definition 1.2. A set U is open in $\mathbb{R}^n$ if for every $\boldsymbol{x} \in U$, there exists an open ball $B_\epsilon(\boldsymbol{x}) \subset U$.

Remark. Every open ball in $\mathbb{R}^n$ is open.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are open.

Definition 1.3. A set F is closed in $\mathbb{R}^n$ if its complement $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are closed.