Term presentation
Problem 4

Satvik Saha, 19MS154

November 18, 2020

MA2102: Linear Algebra I

Indian Institute of Science Education and Research, Kolkata

Problem statement

Show that a matrix A is of rank 1 if and only if $A = \boldsymbol{x}\boldsymbol{y}^\top$ for some non-zero column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Problem statement

Show that a matrix A is of rank 1 if and only if $A = \boldsymbol{x}\boldsymbol{y}^\top$ for some non-zero column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The column rank of a matrix A is equal to the dimension of the column space of A.

The row rank of a matrix A is equal to the dimension of the row space of A.

The column rank and row rank of any matrix $A \in M_{m \times n}(F)$ are equal.

The rank of A is equal to the column rank of A.

The column rank of a matrix A is equal to the dimension of the column space of A.

The row rank of a matrix A is equal to the dimension of the row space of A.

The column rank and row rank of any matrix $A \in M_{m \times n}(F)$ are equal.

The rank of A is equal to the column rank of A.

If $A = xy^\top$, then $\operatorname{rank} A = 1$

$$A = \boldsymbol{x}\boldsymbol{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

$$= \begin{bmatrix} y_1\boldsymbol{x} & y_2\boldsymbol{x} & \cdots & y_n\boldsymbol{x} \end{bmatrix}.$$

If $A = \boldsymbol{x}\boldsymbol{y}^\top$, then $\operatorname{rank} A = 1$

$$A = \boldsymbol{x}\boldsymbol{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

$$= \begin{bmatrix} y_1\boldsymbol{x} & y_2\boldsymbol{x} & \cdots & y_n\boldsymbol{x} \end{bmatrix}.$$

If $A = xy^\top$, then $\operatorname{rank} A = 1$

$$A = \boldsymbol{x}\boldsymbol{y}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

$$= \begin{bmatrix} y_1\boldsymbol{x} & y_2\boldsymbol{x} & \cdots & y_n\boldsymbol{x} \end{bmatrix}.$$

If $A = xy^\top$, then $\operatorname{rank} A = 1$

The column space of A consists of all finite linear combinations of the columns of A.

For any element $\boldsymbol{v}$ in the column space of A, we can write

$$\boldsymbol{v} = \lambda_1 y_1 \boldsymbol{x} + \lambda_2 y_2 \boldsymbol{x} + \cdots + \lambda_n y_n \boldsymbol{x} = \lambda \boldsymbol{x}.$$

The column vector $\boldsymbol{x}$ spans the column space of A.
Furthermore, there is some $y_i \boldsymbol{x} \neq \boldsymbol{0}$ in the column space of A.

$$\operatorname{rank} A \leq 1 \leq \operatorname{rank} A \;\Rightarrow\; \operatorname{rank} A = 1.$$

If $A = xy^\top$, then $\operatorname{rank} A = 1$

The column space of A consists of all finite linear combinations of the columns of A.

For any element $\boldsymbol{v}$ in the column space of A, we can write

$$\boldsymbol{v} = \lambda_1 y_1 \boldsymbol{x} + \lambda_2 y_2 \boldsymbol{x} + \cdots + \lambda_n y_n \boldsymbol{x} = \lambda \boldsymbol{x}.$$

The column vector $\boldsymbol{x}$ spans the column space of A.
Furthermore, there is some $y_i \boldsymbol{x} \neq \mathbf{0}$ in the column space of A.

$$\operatorname{rank} A \leq 1 \leq \operatorname{rank} A \;\Rightarrow\; \operatorname{rank} A = 1.$$

If $A = xy^\top$, then $\operatorname{rank} A = 1$

The column space of A consists of all finite linear combinations of the columns of A.

For any element $\boldsymbol{v}$ in the column space of A, we can write

$$\boldsymbol{v} = \lambda_1 y_1 \boldsymbol{x} + \lambda_2 y_2 \boldsymbol{x} + \cdots + \lambda_n y_n \boldsymbol{x} = \lambda \boldsymbol{x}.$$

The column vector $\boldsymbol{x}$ spans the column space of A.
Furthermore, there is some $y_i \boldsymbol{x} \neq \boldsymbol{0}$ in the column space of A.

$$\operatorname{rank} A \leq 1 \leq \operatorname{rank} A \;\Rightarrow\; \operatorname{rank} A = 1.$$

If $\operatorname{rank} A = 1$, then $A = xy^\top$

The column space of A admits a singleton basis. Set $\boldsymbol{x}$ equal to this element.

$$A = \begin{bmatrix} \lambda_1 \boldsymbol{x} & \lambda_2 \boldsymbol{x} & \cdots & \lambda_n \boldsymbol{x} \end{bmatrix}.$$

Set

$$\boldsymbol{y} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

With this choice of column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we have A = $\boldsymbol{x}\boldsymbol{y}^\top$.

If $\operatorname{rank} A = 1$, then $A = \boldsymbol{x}\boldsymbol{y}^\top$

The column space of A admits a singleton basis. Set $\boldsymbol{x}$ equal to this element.

$$A = \begin{bmatrix} \lambda_1 \boldsymbol{x} & \lambda_2 \boldsymbol{x} & \cdots & \lambda_n \boldsymbol{x} \end{bmatrix}.$$

Set

$$\boldsymbol{y} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

With this choice of column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we have A = $\boldsymbol{x}\boldsymbol{y}^\top$.