MA2201: Analysis II

Integration

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Definition 3.1 (Partition)**.** A partition P of an interval $[a, b]$ is a finite sequence of numbers

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The norm of a partition is defined as

$$\|P\| = \max |x_{j+1} - x_j|.$$

Definition 3.2 (Tagged partition)**.** A tagged partition $\dot{P}(x_j, \xi_j)$ is a partition P together with a set of numbers ξ_j such that $\xi_j \in [x_j, x_{j+1}]$.

Definition 3.3 (Riemann sum)**.** The Riemann sum of a function f on an interval $[a, b]$ with respect to a tagged partition $\dot{P}$ is defined as

$$S(f, \dot{P}) = \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j).$$

Definition 3.4 (Riemann integral)**.** A function f is called Riemann integrable on an interval $[a, b]$ if there is some $\ell \in \mathbb{R}$ where for every $\epsilon > 0$, there exists $\delta > 0$ such that all tagged partitions $\dot{P}$ of $[a, b]$ with $\|\dot{P}\| < \delta$ satisfy

$$|S(f, \dot{P}) - \ell| < \epsilon.$$

The number ℓ is the value of the Riemann integral,

$$\int_a^b f = \ell.$$

Theorem 3.1. *If a function is Riemann integrable on an interval, then the value of the integral is unique.*

Proof. Let f be Riemann integrable on $[a,b]$, with integral values ℓ and ℓ'. Then, for every $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$|S(f,\dot{P}) - \ell| < \frac{\epsilon}{2}, \qquad |S(f,\dot{P}) - \ell'| < \frac{\epsilon}{2}.$$

Note that such a partition $\dot{P}$ always exists. Thus,

$$|\ell - \ell'| \leq |S(f,\dot{P}) - \ell| + |S(f,\dot{P}) - \ell'| < \epsilon$$

for all $\epsilon > 0$, which forces $\ell = \ell'$. □

Theorem 3.2. *If f is Riemann integrable on $[a,b]$, then f is bounded on that interval. Furthermore, if $M > 0$ is such that $|f(x)| \leq M$ for all $x \in [a,b]$, then*

$$-M(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Suppose not. Let the Riemann integral of f on $[a,b]$ be ℓ. For $\epsilon = 1$, we find $\delta > 0$ such that for all tagged partitions $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta$, we have $|S(f,\dot{P}) - \ell| < 1$. This means that

$$S(f,\dot{P}) < |\ell| + 1.$$

Let $Q = \{x_0, \dots, x_n\}$ be such a partition. The unboundedness of f means that we can find a subinterval $[x_k, x_{k+1}]$ where f is unbounded. Now, choose tags ξ_j creating the tagged partition $\dot{Q}$. We choose the tag $\xi_k \in [x_k, x_{k+1}]$ such that

$$|f(\xi_k)(x_{k+1} - x_k)| > |\ell| + 1 + |\sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j)|.$$

Now, observe that the triangle inequality demands

$$|S(f,\dot{Q})| \geq |f(\xi_k)(x_{k+1} - x_k)| - |\sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j)| > |\ell| + 1,$$

which is a contradiction. Thus, f must be bounded on $[a,b]$.

Next, for any tagged partition $\dot{P}$ of $[a,b]$, we have

$$|S(f,\dot{P})| \leq \sum_{j=0}^{n-1} |f(\xi_j)(x_{j+1} - x_j) \leq M(b-a).$$

Let the Riemann integral of f be ℓ. Thus, for all $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$||S(f,\dot{P})| - |\ell|| \leq |S(f,\dot{P}) - \ell| < \epsilon.$$

This gives

$$|\ell| < |S(f,\dot{P})| + \epsilon \leq M(b-a) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we may write

$$|\ell| \leq M(b-a).$$ □

Theorem 3.3. *If f is Riemann integrable on $[a, b]$, and $\dot{P}_n$ is any sequence of tagged partitions of $[a, b]$ such that $\|\dot{P}_n\| \to 0$, then*

$$\int_a^b f = \lim_{n\to\infty} S(f, \dot{P}_n).$$

Proof. Let $\epsilon > 0$. We find $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$, we have

$$|S(f, \dot{P}) - \int_a^b f| < \epsilon.$$

Now, since $\|\dot{P}_n\| \to 0$, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$, $\|\dot{P}_n\| < \delta$. Thus, for all $n \geq N$,

$$|S(f, \dot{P}_n) - \int_a^b f| < \epsilon.$$

In other words,

$$\int_a^b f = \lim_{n\to\infty} S(f, \dot{P}_n).$$

□

Definition 3.5 (Refinement)**.** A partition $\tilde{P}$ is said to be a refinement of a partition P if $P \subset \tilde{P}$.

Definition 3.6 (Common refinement)**.** Given two partitions P_1 and P_2 of an interval $[a, b]$, we say that $\tilde{P}$ is their common refinement if $P_1 \cup P_2 \subset \tilde{P}$.

Definition 3.7 (Darboux sums)**.** Given a partition P of $[a, b]$ and a bounded function f, define

$$m_j = \inf_{t\in[x_j, x_{j+1}]} f(t), \qquad M_j = \sup_{t\in[x_j, x_{j+1}]} f(t).$$

The lower and upper Darboux sums are defined as

$$L(f, P) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j), \qquad U(f, P) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j).$$

Lemma 3.4. *If P is a partition of an interval $[a, b]$, then*

$$L(f, P) \leq U(f, P).$$

Proof. This follows directly from the fact that the infimum is less than or equal to the supremum, i.e. $m_j \leq M_j$.

□

Theorem 3.5. *Let $\tilde{P}$ be a refinement of a partition P of an interval $[a, b]$. Then,*

$$L(f, P) \le L(f, \tilde{P}) \le U(f, \tilde{P}) \le U(f, P)$$

Proof. Suppose that

$$P = \{x_0, \dots, x_k, x_{k+1}, \dots, x_n\},$$
$$\tilde{P} = \{x_0, \dots, x_k, y, x_{k+1}, \dots, x_n\}.$$

Set

$$m_1 = \inf_{t \in [x_k, y]} f(t), \qquad m_2 = \inf_{t \in [y, x_{k+1}]} f(t), \qquad m = \inf_{t \in [x_k, x_{k+1}]} f(t).$$

Then, observe that

$$L(f, \tilde{P}) - L(f, P) = m_1(y - x_k) + m_2(x_{k+1} - y) - m(x_{k+1} - x_k).$$

Now, from the properties of the infimum, $m_1 \ge m$ and $m_2 \ge m$, so

$$L(f, \tilde{P}) - L(f, P) \ge m(y - x_k + x_{k+1} - y - x_{k+1} + x_k) = 0.$$

This procedure of adding one point can be repeated finitely many times to obtain the same conclusion for any refinement of P. The case for the upper sum is analogous. □

Corollary 3.5.1. *For any two partitions P_1 and P_2 of $[a, b]$,*

$$L(f, P_1) \le U(f, P_2).$$

Proof. Note that $P_1 \cup P_2$ is a common refinement of P_1 and P_2, hence

$$L(f, P_1) \le L(f, P_1 \cup P_2) \le U(f, P_1 \cup P_2) \le U(f, P_2).$$ □

Corollary 3.5.2. *If $\{P_n\}$ is a sequence of refinements of a partition P_0 of $[a, b]$, then the following limits exist.*

$$L_{f,P_n} = \lim_{n\to\infty} L(f, P_n), \qquad U_{f,P_n} = \lim_{n\to\infty} U(f, P_n).$$

Proof. This follows from the monotone convergence theorem, together with the fact that $U(f, P_0)$ and $L(f, P_0)$ are upper and lower bounds of the two respective sequences. □

Corollary 3.5.3. *The following quantities exist, where the infimum and supremum is taken over all possible partitions P of $[a, b]$.*

$$L_f = \sup L(f, P), \qquad U_f = \inf U(f, P).$$

Furthermore, for any partition P,

$$L(f, P) \le L_f \le U_f \le U(f, P).$$

Proof. First examine the set of all lower Darboux sums, $\{L(f,P)\}$. This set is non-empty, since any partition of $[a,b]$ gives a corresponding lower sum. Note that we have already demanded that f is bounded! This set is also bounded above, by any upper sum. Thus, the completeness of the reals guaranteed the existence of a supremum. The case for upper sums is analogous.

The outermost inequalities trivially follow from the definitions of the infimum and supremum. The middle inequality follows from the fact that if A and B are two subsets of $\mathbb{R}$ such that $\alpha \in A, \beta \in B$ implies $\alpha \leq \beta$, then $\sup A \leq \inf B$. □

Definition 3.8 (Darboux integrals)**.** The lower and upper Darboux integrals of a function f are defined as

$$L_f = \sup L(f,P), \qquad U_f = \inf U(f,P).$$

Here, the infimum and supremum is taken over all possible partitions P of $[a,b]$.

If $L_f = U_f$, then the common integral is simply called the Darboux integral,

$$\int_a^b f = L_f = U_f.$$

Such a function f is called Darboux integrable.

Theorem 3.6. *A function f is Darboux integrable on $[a,b]$ if and only if for every $\epsilon > 0$, there exists a partition P such that*

$$U(f,P) - L(f,P) < \epsilon.$$

Proof. First, assume that given $\epsilon > 0$, there exists a partition P such that

$$U(f,P) - L(f,P) < \epsilon.$$

By the previous corollary,

$$U_f - L_f \leq U(f,P) - L(f,P) < \epsilon$$

for all > 0, so $U_f = L_f$ giving Darboux integrability.

Now, suppose that f is Darboux integrable on $[a,b]$. This means that $U_f = L_f$. Using the definitions of supremum and infimum, for $\epsilon > 0$, there exists a partition P_1 such that $U(f,P_1) - U_f < \epsilon/2$ and a partition P_2 such that $L_f - L(f,P_2) < \epsilon/2$. Adding,

$$U(f,P_1) - L(f,P_2) < \epsilon.$$

Now, setting $P = P_1 \cup P_2$ as a common refinement of P_1 and P_2, we have

$$U(f,P) - L(f,P) < U(f,P_1) - L(f,P_2) < \epsilon.$$

□

Theorem 3.7. *Riemann and Darboux integrability are equivalent and assign the same value to the integrals.*

Theorem 3.8. *Any real continuous function on $[a, b]$ is Riemann integrable.*

Proof. Note that any continuous function on a compact interval is uniformly continuous. Thus, for $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [a, b]$, we have

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Now, construct a partition of $[a, b]$ which divides the interval into equal subintervals of length $(b-a)/n$, where n is chosen such that $\|P\| < \delta$. This immediately gives

$$U(f, P) - L(f, P) = \sum_{j=0}^{n} (M_j - m_j) \cdot \frac{1}{n}(b-a) \leq n \cdot \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} = \epsilon. \qquad \square$$