MA3203

Analysis IV

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Contents

1 Measure theory

1.1 Introduction

Measure theory seeks to generalize the notions of *length*, *area*, *volume* to more general sets: this new notion is called a *measure*. This also allows us to generalize the notion of Riemann integration to a broader class of functions.

Recall that continuous functions, or at least functions with finitely many discontinuities on a closed interval are Riemann integrable. The Dirichlet function, which is discontinuous everywhere, is not.

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This is simply because every non-empty interval contains at least one rational and one irrational number, so the Darboux lower sum is always 0 and the upper sum is always 1 regardless of the choice of partition.

On the other hand, if we had to assign a particular value to this integral, intuition tells us that it ought to be zero. After all, the function f attains a non-zero value only on the countable set $\mathbb{Q} \cap [0, 1]$; it is zero almost everywhere. Formally, we will show that f is non-zero on a set of zero Lebesgue measure, which will allow us to set this Lebesgue integral to zero. We will see that with this new formulation of integration, we end up partitioning the range of f rather than it's domain, and write

$$\int f = 0 \cdot \mu([0,1] \setminus \mathbb{Q}) + 1 \cdot \mu([0,1] \cap \mathbb{Q}) = 0.$$

Theorem 1.1 (Lebesgue criterion)**.** *A function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuities has Lebesgue measure zero. This means that the set of discontinuities of f must be coverable by countably many intervals (x_i, y_i) such that the sum of lengths $y_i - x_i$ can be made arbitrarily small.*

1.2 Basic definitions

Definition 1.1. Let X be a set, and let $\mathcal{M}$ be a collection of subsets of X. We say that $\mathcal{M}$ is a σ-algebra over X if it satisfies the following.

1. $\mathcal{M}$ contains X.
2. $\mathcal{M}$ is closed under complementation.
3. $\mathcal{M}$ is closed under countable unions.

Remark. The first condition can be replaced by forcing $\mathcal{M}$ to be non-empty.

Remark. The following properties follow immediately.

1. $\mathcal{M}$ contains $\emptyset$.
2. $\mathcal{M}$ is closed under countable intersections.
3. $\mathcal{M}$ is closed under differences.

Example. Given a set X, its power set forms a σ-algebra over X.

Example. Given a set X, the set $\{\emptyset, X\}$ forms a σ-algebra over X.

Example. Given an uncountable set X, the following set forms the co-countable σ-algebra over X.

$$\{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}.$$

Definition 1.2. Let X be a set, and let $\mathcal{M}$ be a σ-algebra over X. We say that a function $\mu\colon \mathcal{M} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a measure if it satisfies the following.

1. μ is non-negative.
2. $\mu(\emptyset) = 0$.
3. μ is additive over countable unions of disjoint sets, i.e. for any countable collection $\{E_i\}_{i=1}^{\infty}$ such that $E_i \cap E_j = \emptyset$ for all pairs, we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Example. The trivial zero measure sends every set to zero.

Example. In probability theory, we look at the event space $\mathcal{E}$ as a σ-algebra over the sample space Ω. The probability function P is a measure on this event space such that $P(\Omega) = 1$.

Example. Let X be a set, and let $\mathcal{M}$ be its power set as a σ-algebra over X. Fix $x_0 \in X$, and define

$$\mu\colon \mathcal{M} \to [0,\infty], \qquad E \mapsto \begin{cases} 1, & \text{if } x_0 \in E, \\ 0, & \text{if } x_0 \notin E. \end{cases}$$

This is called the Dirac measure.

Example. Let X be a set, and let $\mathcal{M}$ be its power set as a σ-algebra over X. Define

$$\mu\colon \mathcal{M} \to [0,\infty], \qquad E \mapsto \begin{cases} 0, & \text{if } E = \emptyset, \\ |E|, & \text{if } E \text{ is finite}, \\ \infty, & \text{otherwise}. \end{cases}$$

This is called the counting measure.

Example. Let X be an uncountable set, and let $\mathcal{M}$ be its co-countable sigma algebra. Define

$$\mu\colon \mathcal{M} \to [0,\infty], \qquad E \mapsto \begin{cases} 0, & \text{if } E \text{ is countable}, \\ 1, & \text{if } E^c \text{ is countable}. \end{cases}$$

Definition 1.3. Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. We say that μ is finite if $\mu(E)$ is finite for all $E \in \mathcal{M}$.
2. We say that μ is σ-finite if given $E \in \mathcal{M}$, we can write

$$E = \bigcup_{i=1}^{\infty} E_i$$

for $E_i \in \mathcal{M}$ such that each $\mu(E_i)$ is finite.

1.3 Basic properties

Lemma 1.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then, the following properties hold.*

1. *If $A, B \in \mathcal{M}$ such that $A \subseteq B$, then $\mu(A) \le \mu(B)$.*
2. *If $A, B \in \mathcal{M}$ such that $A \subseteq B$ and $\mu(A)$ is finite, then $\mu(B - A) = \mu(B) - \mu(A)$.*
3. *If $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \mathcal{M}$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu(E_i).$$

Corollary 1.2.1. *A measure μ is finite if and only if $\mu(X)$ is finite.*

Theorem 1.3 (Continuity from below)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets such that $E_i \subseteq E_j$ for all $i < j$. Then,*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Define $F_i = E_i - E_{i-1}$, denoting $E_0 = \emptyset$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i).$$

Also note that

$$\sum_{i=1}^{n} \mu(F_i) = \mu\left(\bigcup_{i=1}^{\infty} F_n\right) = \mu(E_n).$$

Since the infinite sum in the first part is the limit of partial sums, we have our result. □

Theorem 1.4 (Continuity from above)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets such that $E_i \supseteq E_j$ for all $i < j$. Further assume that $\mu(E_1)$ is finite. Then,*

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Define $F_i = E_1 - E_n$, and note that $F_i \subseteq F_j$ for all $i < j$. Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{n\to\infty} \mu(F_n).$$

This can be rewritten as

$$\mu\left(E_1 - \bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_1 - E_n).$$

Using the subtractive property and the fact that each $\mu(E_i)$ is finite,

$$\mu(E_1) - \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_1) - \mu(E_n).$$

Pulling the constant $\mu(E_1)$ out from the limit and subtracting from both sides gives our result. □

Example. Consider the counting measure μ on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, and define $E_n = \{n, n+1, \ldots\}$. Then,

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = 0, \qquad \lim_{n\to\infty} \mu(E_n) = \infty.$$

1.4 The Borel σ-algebra

Theorem 1.5. *Let X be a set, and let S be a collection of subsets of X. Then, there exists a smallest σ-algebra containing S. This is called the σ algebra generated by S, denoted $\mathcal{M}(S)$.*

Proof. Let Ω be the collection of all σ-algebras on X containing S. Note that $\Omega \neq \emptyset$, since it contains the power set of X. Consider the intersection of all the sigma algebras in Ω,

$$\mathcal{M} = \bigcap_{\mathcal{M}_\lambda \in \Omega} \mathcal{M}_\lambda.$$

We claim that $\mathcal{M}$ is indeed a σ-algebra. To see this, first note that $X \in \mathcal{M}$. Next, pick $E \in \mathcal{M} \subseteq \mathcal{M}_\lambda$, so $E^c \in \mathcal{M}_\lambda$ for all $\mathcal{M}_\lambda \in \Omega$, hence $E^c \in \mathcal{M}$. Finally, pick $\{E_i\}_{i=1}^\infty$ where $E_i \in \mathcal{M} \subseteq \mathcal{M}_\lambda$, which shows that the union of these E_i is in every $\mathcal{M}_\lambda$, hence in $\mathcal{M}$. □

Definition 1.4. Let (X, τ) be a topological space. The σ algebra generated by τ is called the Borel σ-algebra, $\mathcal{B}_X = \mathcal{M}(\tau)$.

Remark. The Borel σ-algebra $\mathcal{B}_X$ contains all open as well as all closed sets in X, as well as their countable unions and intersections.

Theorem 1.6. *Consider the collection β of open intervals in $\mathbb{R}$, and the standard topology τ on $\mathbb{R}$. Then, both β and τ generate the same Borel σ-algebra $\mathcal{B}_\mathbb{R}$.*

Proof. This relies on the fact that every open set $U \subseteq \mathbb{R}$ can be written as a countable union of open intervals. To see this, pick $x \in U$, and an open interval $x \in (a, b) \subset U$. Now pick $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$, hence $x \in (p, q) \subset U$. Now, U is precisely the union of all such intervals (p, q). This collection is countable, due to the countability of the rationals. □

Remark. The same holds if we consider the collection β' of closed intervals in $\mathbb{R}$. This can be shown using the standard trick

$$(a, b) = \bigcup_{n=1}^\infty \left[a + \frac{1}{n}, b - \frac{1}{n}\right].$$

Indeed, we may also consider the collection of intervals of the form $[a, b)$, or the collection of intervals $(a, b]$, or even the collection of intervals (a, ∞), or $(-\infty, b)$, or $[a, \infty)$, or $(-\infty, b]$.

Definition 1.5. A countable union of closed sets is called an F_σ set. A countable intersection of open sets is called a G_δ set.

1.5 Measurable functions

Definition 1.6. Let $(X, \mathcal{M}_X)$, $(Y, \mathcal{M}_Y)$ be measure spaces. We say that a function $f \colon X \to Y$ is $(\mathcal{M}_X, \mathcal{M}_Y)$ measurable if for every $E \in \mathcal{M}_Y$, we have $f^{-1}(E) = \mathcal{M}_X$.

Example. Consider the Borel σ-algebra $\mathcal{B}_\mathbb{R}$ on $\mathbb{R}$, and fix $E \in \mathcal{B}_\mathbb{R}$. Define the characteristic function

$$\chi_E \colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Then, χ_E is measurable. However, we can choose E to be closed and not open, so that χ_E is not continuous.

Lemma 1.7. *Let $f \colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, and let $\mathcal{M}_Y$ be generated by S. Then $f \colon X \to Y$ is measurable if for every $E \in S$, we have $f^{-1}(E) \in \mathcal{M}_X$.*

Proof. Define

$$\mathcal{M} = \{E \subseteq Y : f^{-1}(E) \in \mathcal{M}_X\}.$$

Clearly, $S \subseteq \mathcal{M}$. We now claim that $\mathcal{M}$ is a σ-algebra over Y. First, $Y \in \mathcal{M}$ since $f^{-1}(Y) = X \in \mathcal{M}_X$. Next if $E \in \mathcal{M}$, we have $f^{-1}(E^c) = f^{-1}(E)^c \in \mathcal{M}_X$. Finally, if $\{E_i\}_{i=1}^\infty$ such that each $E_i \in \mathcal{M}$, set E to be their union, whence

$$f^{-1}(E) = \bigcup_{i=1}^\infty f^{-1}(E_i) \in \mathcal{M}.$$

Thus, $\mathcal{M}$ is indeed a σ-algebra. Since S generates $\mathcal{M}_Y$, we have $\mathcal{M}_Y \subseteq \mathcal{M}$, completing the proof. $\square$

Example. A function $f \colon X \to \mathbb{R}$ is measurable if and only if $f^{-1}((a, \infty))$ is measurable for all $a \in \mathbb{R}$.

Theorem 1.8. *Let $f \colon X \to Y$ be continuous. Then, f is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.*

Lemma 1.9. *The composition of measurable functions is measurable. In other words, if $f \colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is surjective and $(\mathcal{M}_X, \mathcal{M}_Y)$ measurable, and $g \colon (Y, \mathcal{M}_Y) \to (Z, \mathcal{M}_Z)$ is $(\mathcal{M}_Y, \mathcal{M}_Z)$ measurable, then $g \circ f$ is $(\mathcal{M}_X, \mathcal{M}_Z)$ measurable.*

Lemma 1.10. *Let $u, v \colon (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ be $(\mathcal{M}, \mathcal{B}_\mathbb{R})$ measurable. Then, $f \colon (X, \mathcal{M}) \to (\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$, defined by $x \mapsto (u(x), v(x))$, is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^2})$ measurable.*

Proof. Basic open sets in $\mathbb{R}^2$ can be chosen as the open rectangles $(a, b) \times (c, d)$. The pre-image of such an open set under f is $u^{-1}((a, b)) \cap v^{-1}((c, d))$, which is clearly a measurable set in X. $\square$

Corollary 1.10.1. *Let $f, g\colon (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable. Then the sum $f+g$ and the product $f \cdot g$ are also $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable.*

Proof. The maps $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ are continuous, hence measurable. Thus, the composite maps $x \mapsto (f(x), g(x)) \mapsto f(x) + g(x)$ and $x \mapsto (f(x), g(x)) \mapsto f(x)g(x)$ are also measurable. $\square$

Example. Let $(X, \mathcal{M})$ be a measurable space, and let $A_1, \ldots, A_n \in \mathcal{M}$. Then the map

$$s\colon X \to \mathbb{R}, \qquad x \mapsto \sum_{i=1}^{n} c_i \chi_{A_i}(x)$$

is measurable. Such functions are called simple functions.

Lemma 1.11. *The maximum and minimum of measurable functions are measurable.*

Corollary 1.11.1. *The positive and negative parts of a measurable function are measurable.*

Remark. Recall that

$$f^{+} = \max\{f, 0\}, \qquad f^{-1} = -\min\{f, 0\}, \qquad f = f^{+} - f^{-}.$$

Thus, in order to show that a result holds for all measurable functions, it suffices to show the result only for all non-negative measurable functions.

Theorem 1.12. *Let $\{f_n\}_{n=1}^{\infty}$ be a collection of measurable functions $f_n\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then, their supremum and infimum are measurable.*

Theorem 1.13. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f_n\colon X \to \mathbb{R}$. Then, $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable.*

Theorem 1.14. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f_n\colon X \to \mathbb{R}$, and let $f_n \to f$ pointwise on X. Then, f is measurable.*

Remark. This is a stronger result than the corresponding one regarding limits of continuous functions.

1.6 Lebesgue integration

Theorem 1.15. *Let $f\colon X \to [0,\infty]$ be measurable. Then, there exists a sequence of simple functions $s_n\colon X \to [0,\infty)$ such that*

$$0 \le s_1 \le s_2 \le \cdots \le s_n \le f$$

for all $n \in \mathbb{N}$, and $s_n \to f$.

Corollary 1.15.1. *Any measurable function $f\colon X \to [-\infty,\infty]$ can be written as the limit of a sequence of simple functions $\{s_n\}_{n=1}^{\infty}$, with $s_n \to f$.*

Definition 1.7. When dealing with the extended reals in measure theory, we use the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

Remark. We want to have

$$\int_0^\infty 0 = 0 \cdot \mu(f^{-1}(0)) = 0 \cdot \mu([0,\infty)) = 0.$$

Definition 1.8. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $s\colon X \to [0,\infty)$ be a simple, measurable function, of the form

$$s = \sum_{i=1}^{n} c_i \chi_{A_i}, \qquad A_i \in \mathcal{M}$$

where $c_1, \dots, c_n$ are distinct values of s. Then, the Lebesgue integral of s on $E \in \mathcal{M}$ is defined as

$$\int_E s\, d\mu = \sum_{i=1}^{n} c_i \cdot \mu(E \cap A_i).$$

Example. The Dirichlet function is the simple function $\chi_{\mathbb{Q}}$. Thus, upon assigning a σ-algebra and a measure μ on $\mathbb{R}$, we will be able to assign its Lebesgue integral on $\mathbb{R}$ as the value $\mu(\mathbb{Q})$.

Definition 1.9. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f\colon X \to [0,\infty)$ be a measurable function. Then, the Lebesgue integral of f on $E \in \mathcal{M}$ is defined as

$$\int_E f\, d\mu = \sup\left\{\int_E s\, d\mu,\ \text{for all simple functions } s, \text{where } 0 \le s \le f\right\}.$$

Theorem 1.16. *Let $f, g\colon X \to [0, \infty]$ be measurable functions.*

1. *If $0 \le f \le g$, then*
$$\int_E f\, d\mu \le \int_E g\, d\mu.$$
2. *If $A \subset B$, then*
$$\int_A f\, d\mu \le \int_B f\, d\mu.$$
3. *For $c \in \mathbb{R}$,*
$$\int_E cf\, d\mu = c\int_E f\, d\mu.$$
4. *If $f = 0$, then*
$$\int_E f\, d\mu = 0.$$
5. *If $\mu(E) = 0$, then*
$$\int_E f\, d\mu = 0.$$

Definition 1.10. We say that a statement is true *almost everywhere* on X if it is true everywhere on $X \setminus E$ for a measure zero set E.

Lemma 1.17. *Let $f\colon X \to [0, \infty]$ be a measurable function. If*
$$\int_X f\, d\mu = 0,$$
then $f = 0$ almost everywhere.

Proof. We wish to show that the set $E = f^{-1}(0, \infty]$ has measure zero. Now, note that this is the union of the measurable sets $E_n = f^{-1}(1/n, \infty]$. Since $E_n \subset E \subseteq X$, we have
$$\int_{E_n} \frac{1}{n}\, d\mu \le \int_{E_n} f\, d\mu \le \int_X f\, d\mu.$$
However, this is just
$$0 \le \frac{1}{n}\mu(E_n) \le \int_{E_n} f\, d\mu \le 0,$$
hence each $\mu(E_n) = 0$. Continuity from below gives $\mu(E) = 0$. $\square$

Lemma 1.18. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $s\colon X \to \mathbb{R}$ be a non-negative simple function. Define*
$$\nu(E) = \int_E s\, d\mu, \qquad E \in \mathcal{M}.$$
Then, ν is a measure on $\mathcal{M}$.

1.6.1 Monotone convergence

Theorem 1.19 (Monotone convergence)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions $f_n\colon X \to \mathbb{R}$, such that $f_n \leq f_{n+1}$, and $f_n \to f$ pointwise. Then,*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Lemma 1.20. *Let $f, g\colon X \to \mathbb{R}$ be measurable functions. Then, $f + g$ is measurable.*

Theorem 1.21. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions $f_n\colon X \to \mathbb{R}$. Define*

$$f\colon X \to \mathbb{R}, \qquad f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then, f is measurable and

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Theorem 1.22 (Fatou)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions $f_n\colon X \to \mathbb{R}$. Then,*

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Corollary 1.22.1. *If $f_n \to f$, then*

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Corollary 1.22.2. *If $f_n \to f$ and each $0 \leq f_n \leq f$, then*

$$\int_X f \, d\mu = \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Example. Consider the functions $f_n = \chi_{[n,n+1)}$. Now, $f_n \to 0$ pointwise, but the Lebesgue integrals of f_n are all 1.

Lemma 1.23. *Let $\{a_{ij}\}_{i,j\in\mathbb{N}}$ be sequences of non-negative terms. Then,*

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}.$$

Proof. Define $f_i(j) = a_{ij}$, and $f = \sum_i f_i$. Using the counting measure,

$$\int_{\mathbb{N}} f_i \, d\mu = \sum_j a_{ij}.$$

Also,

$$f(j) = \sum_i a_{ij}, \qquad \sum_j \sum_i a_{ij} = \sum_j f(j) = \int_{\mathbb{N}} f \, d\mu = \sum_i \int_{\mathbb{N}} f_i \, d\mu = \sum_i \sum_j a_{ij}.$$ □

Theorem 1.24. *Let $f\colon X \to [0,\infty]$ be a non-negative measurable function. Define*

$$\nu(E) = \int_E f \, d\mu, \qquad E \in \mathcal{M}.$$

Then, ν is a measure on $\mathcal{M}$. Furthermore, if $g\colon X \to [0,\infty]$ is measurable, then

$$\int_X g \, d\nu = \int_X g \, f \, d\mu.$$

1.6.2 Dominated convergence

Definition 1.11. For a measurable function $f\colon X \to [-\infty,\infty]$, we may define

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

as long as at least one of these terms is finite.

Definition 1.12. Let

$$L^1(\mu) = \{f\colon X \to \mathbb{C} : f \text{ is measurable, and } \int_X |f| \, d\mu \text{ is finite}\}.$$

For $f \in L^1(\mu)$, we may write $f = u + iv$ where u, v are real valued measurable functions and define

$$\int_X f \, d\mu = \int_X u \, d\mu + i \int_X v \, d\mu.$$

Lemma 1.25. *When $f, g \in L^1(\mu)$, $\alpha, \beta \in \mathbb{C}$, we have*

$$\int_X \alpha f + \beta g \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Theorem 1.26. *The space of functions $L^1(\mu)$ is a vector space over $\mathbb{C}$. The map*

$$T \colon L^1(\mu) \to \mathbb{C}, \qquad f \mapsto \int_X f \, d\mu$$

is a linear map. Furthermore, $L^1(\mu)$ is a metric space, with

$$d(f, g) = \int_X |f - g| \, d\mu.$$

Remark. Observe that

$$|T(f) - T(g)| \leq d(f, g),$$

making T a Lipschitz continuous map.

Theorem 1.27 (Dominated convergence)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex measurable functions $f_n \colon X \to \mathbb{C}$, such that $f_n \to f$ pointwise on X. Furthermore, let $g \colon X \to [0, \infty)$, $g \in L^1(\mu)$ such that $|f_n| \leq g$. Then, $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0, \qquad \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Corollary 1.27.1 (Bounded convergence)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex measurable functions $f_n \colon X \to \mathbb{C}$, such that $f_n \to f$ pointwise on X. Furthermore let $|f_n| \leq M$ for some $M \in \mathbb{R}$, and let $\mu(X) < \infty$. Then, $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0, \qquad \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Example. Consider

$$f_n \colon [0, 1] \to \mathbb{R}, \qquad x \mapsto \frac{nx}{1 + n^2 x^2}.$$

Then, it can be shown that $f_n \to 0$ pointwise on $[0, 1]$. Furthermore, each $|f_n| < 1$ and $\mu([0, 1]) = 1$, hence

$$\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1 + n^2 x^2} \, d\mu = 0.$$

Theorem 1.28 (Fundamental theorem)**.** *Let $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ where $f \in L^1(\mu)$, $|f| < c$. Then,*

$$\int_{[a,b]} f \, d\mu = F(b) - F(a).$$

Proof. Define

$$f_n\colon [a,b] \to \mathbb{R}, \qquad x \mapsto n(F(x+1/n) - F(x)),$$

and note that $f_n \to f$. □

Theorem 1.29. *Let $f\colon X \times [a,b] \to \mathbb{C}$, where $-\infty < a < b < \infty$, and let each $f(\cdot, t) = f_t \in L^1(\mu, X)$. Define*

$$F(t) = \int_X f(x,t) \, d\mu(x).$$

Suppose that $\partial f/\partial t$ exists, with $g \in L^1(\mu, X)$ such that

$$\left| \frac{\partial f}{\partial t}(x,t) \right| \le g(x).$$

Then, F is differentiable, with

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x,t) \, d\mu(x).$$

Theorem 1.30. *Let $f\colon X \to \mathbb{C}$, where $f \in L^1(\mu)$. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $E \in \mathcal{M}$ with $\mu(E) < \delta$, we have*

$$\int_E |f| \, d\mu < \epsilon.$$

In other words,

$$\lim_{\mu(E) \to 0} \int_E |f| \, d\mu = 0.$$

Proof. Define

$$f_n\colon X \to \mathbb{C}, \qquad x \mapsto \begin{cases} |f(x)|, & \text{if } |f(x)| \le n, \\ n, & \text{if } |f(x)| > n. \end{cases}$$

Then each $|f_n| \le n$, $f_n \to |f|$ monotonically. Thus, the monotone convergence theorem will give

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X |f| \, d\mu.$$

For sufficiently large N, we have

$$\int_X |f| \, d\mu - \int_X f_N \, d\mu < \frac{\epsilon}{2}.$$

By restricting the domain of integration to $E \subseteq X$,

$$\int_E |f| \, d\mu - \int_E f_N \, d\mu < \frac{\epsilon}{2}.$$

Finally, set $\delta = \epsilon/2N$, so that for $\mu(E) < \delta$, we have

$$\int_E |f| \, d\mu < \int_E f_N \, d\mu + \frac{\epsilon}{2} < N\mu(E) + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Theorem 1.31. *Let $f\colon X \to \mathbb{C}$, where $f \in L^1(\mu)$. Suppose that for all $E \in \mathcal{M}$, we have*

$$\int_E f \, d\mu = 0.$$

Then, $f = 0$ almost everywhere on X.

Theorem 1.32. *Let $f\colon [a, b] \to \mathbb{R}$ be bounded on the compact interval $[a, b]$. If f is Riemann integrable, then $f \in L^1(\mu)$ with*

$$\int_a^b f = \int_{[a,b]} f \, d\mu.$$

Remark. The converse fails, since $\chi_{\mathbb{Q}}$ is not Riemann integrable, but it is Lebesgue integrable.

Example. Suppose that we wish to compute the integral

$$\int_{(0,1)} \frac{1}{\sqrt{x}} \, d\mu.$$

Note that the corresponding Riemann integral is improper. Thus, we define the functions

$$f_n\colon (0, 1) \to \mathbb{R}, \qquad x \mapsto \frac{1}{\sqrt{x}} \chi_{[1/n,1)}(x),$$

and note that $f_n(x) \to 1/\sqrt{x}$ on $(0, 1)$ monotonically. Thus, the monotone convergence theorem guarantees that

$$\int_{(0,1)} \frac{1}{\sqrt{x}} \, d\mu = \lim_{n\to\infty} \int_{1/n}^1 \frac{1}{\sqrt{x}} \, dx = \lim_{n\to\infty} 2\left(1 - \frac{1}{\sqrt{n}}\right) = 2.$$

Example. Suppose that we wish to compute

$$\int_{(1,\infty)} \frac{1}{x^2} \, d\mu.$$

Again, define

$$f_n\colon (1, \infty) \to \mathbb{R}, \qquad x \mapsto \frac{1}{x^2} \chi_{(1,n)}(x).$$

Then $f_n(x) \to 1/x^2$ on $(1, \infty)$ monotonically, so

$$\int_{(0,1)} \frac{1}{x^2} \, d\mu = \lim_{n\to\infty} \int_1^n \frac{1}{x^2} \, dx = \lim_{n\to\infty} 1 - \frac{1}{n} = 1.$$

Example. Suppose that we wish to compute

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{-2x}\,dx.$$

By setting

$$f_n\colon [0,\infty)\to\mathbb{R}, \qquad x\mapsto \left(1-\frac{x}{n}\right)^n e^{-2x}\chi_{[0,n]}(x),$$

we have

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{-2x}\,dx = \lim_{n\to\infty}\int_{[0,\infty)} f_n\,d\mu.$$

Furthermore, $|f_n(x)|\le e^{-2x}$ and the latter is in $L^1(\mu)$. Thus, the dominated convergence theorem guarantees that this limit is

$$\int_{[0,\infty)}\lim_{n\to\infty} f_n\,d\mu = \int_0^\infty e^{-3x}\,dx = \frac{1}{3}.$$

1.7 L^p spaces

Definition 1.13. Let $(X,\mathcal{M},\mu)$ be a measure space. For $1\le p<\infty$, we define

$$L^p(\mu) = \{f\colon X\to\mathbb{C} : f \text{ is measurable, and } \int_X |f|^p\,d\mu \text{ is finite}\}.$$

Remark. We denote

$$\|f\|_p = \left(\int_X |f|^p\,d\mu\right).$$

Definition 1.14. Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f\colon X\to\mathbb{C}$ be measurable. We define the essential supremum of f as

$$\|f\|_\infty = \inf\{M : \mu\{x\in X : |f(x)|>M\}=0\}.$$

Remark. We have $f\le\|f\|_\infty$ almost everywhere in X.

Example. Note that for

$$f\colon (0,1)\to\mathbb{R}, \qquad x\mapsto 1/x,$$

we have $\|f\|_\infty=\infty$. By convention, $\inf\emptyset=\infty$.

Definition 1.15. Let $(X,\mathcal{M},\mu)$ be a measure space. We define the space of essentially bounded functions as

$$L^\infty(\mu) = \{f\colon X\to\mathbb{C} : f \text{ is measurable, and } \|f\|_\infty \text{ is finite}\}.$$

Lemma 1.33 (Young)**.** *For $a, b \geq 0$, $p, q > 1$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Using Jensen's inequality on the logarithm,

$$\log(ta^p + (1-t)b^q) \geq t\log(a^p) + (1-t)\log(b^q).$$

Putting $t = 1/p$ and exponentiating immediately gives the result. □

Lemma 1.34 (Hölder)**.** *For $1 \leq p, q \leq \infty$, f, g measurable,*

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. This is trivial when either $f, g = 0$, or $p, q = 1$. Otherwise, applying Young's inequality and integrating gives

$$\int_X |f(x)g(x)| \, d\mu \leq \frac{1}{p} \int_X |f(x)|^p \, d\mu + \frac{1}{q} \int_X |g(x)|^q \, d\mu.$$

When $\|f\|_p = \|g\|_q = 1$, this immediately gives the desired inequality since the right hand side is 1. Otherwise, define $F = f/\|f\|_p$, $G = g/\|g\|_q$, upon which $\|F\|_p = \|G\|_q = 1$, hence

$$\|fg\|_1 = \int_X |f(x)g(x)| \, d\mu \leq \|f\|_p \|g\|_q.$$ □

Lemma 1.35 (Minkowski)**.** *For $1 \leq p \leq \infty$, f, g measurable,*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. This is trivial when $p = 1, \infty$. Otherwise, for $1 < p < \infty$, note that

$$|f(x) + g(x)|^p \leq (|f(x) + g(x))^p \leq (2\max(f(x), g(x)))^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right).$$

This shows that when $f, g \in L^p(\mu)$, we have $f + g \in L^p(\mu)$. Set $F = |f+g|^{p-1}$, when the triangle inequality followed by Hölder's inequality gives

$$\|f+g\|_p^p \leq \int_X F(x)|f(x)| \, d\mu + \int_X F(x)|g(x)| \, d\mu \leq \|F(x)\|_q \|f\|_p + \|F(x)\|_q \|g\|_p,$$

where $q = 1 - 1/p$. Using $(p-1)q = p$,

$$\|F(x)\|_q = \left(\int_X |f(x) + g(x)|^{(p-1)^q} \, d\mu\right)^{1/q} = \left(\int_X |f(x) + g(x)|^p \, d\mu\right)^{1-1/p} = \frac{\|f+g\|_p^p}{\|f+g\|_p}.$$

This immediately gives the result. □

Theorem 1.36. *The spaces of functions $L^p(\mu)$ and $L^\infty(\mu)$ are complete metric spaces.*

Remark. Two functions in such a space are identified if they are equal almost everywhere.

Corollary 1.36.1. *Let $f_n \to f$ in $L^p(\mu)$, where each $f_n \in L^p(\mu)$, and $1 \leq p \leq \infty$. Then, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ pointwise in $\mathbb{C}$, almost everywhere in X. Moreover, there exists $h \in L^p(\mu)$ such that $|f_{n_k}| \leq h$ almost everywhere in X.*

Theorem 1.37. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $1 \leq p < q \leq \infty$. Then, $L^p(\mu) \supseteq L^q(\mu)$.*

Proof. Set $u = q/p > 1$, $v = 1 - 1/u$, whence

$$\||f|^p\|_1 \leq \||f|^p\|_u \|1\|_v = \|1\|_v \left(\int_X |f|^q \, d\mu\right)^{p/q} = \mu(X)^{1/v} \|f\|_q^p.$$

Thus,

$$\|f\|_p = \||f|^p\|_1^{1/q} \leq \mu(X)^{1/p-1/q} \|f\|_q.$$

□

Example. Note that the map $x \mapsto 1/\sqrt{x}$ is in $L^1(0,1)$, but not in $L^2(0,1)$.

Theorem 1.38. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Then, $L^\infty(\mu) \subseteq L^p(\mu)$ for all $1 \leq p \leq \infty$, and*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Theorem 1.39. *Let S be the set of all simple, measurable, complex valued functions which are non-zero on a set of finite measure. Then, the closure of S in $L^p(\mu)$ is the whole of $L^p(\mu)$, for $1 \leq p < \infty$.*

Theorem 1.40. *The set of all continuous functions on $\mathbb{R}^n$ with compact support is dense in $L^p(\mu)$, for $1 \leq p < \infty$.*

Theorem 1.41. *The set of all smooth functions on $\mathbb{R}^n$ with compact support is dense in $L^p(\mu)$, for $1 \leq p < \infty$.*