MA4202: Ordinary Differential Equations

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Assignment I

Exercise 1 Let $x : [-1, 1] \to \mathbb{R}$ be a continuous function satisfying

$$x(t) = x(0) + \int_0^t x(s)\, ds.$$

Show that

$$x^2(t) = x^2(0) + 2\int_0^t x^2(s)\, ds.$$

Solution Observe that $x'(t) = x(t)$, hence integrating by parts yields

$$\int_0^t x^2(s)\, ds = \int_0^t x(s)x'(s)\, ds = x^2(s)\Big|_0^t - \int_0^t x(s)x'(s)\, ds,$$

whence

$$2\int_0^t x^2(s)\, ds = x^2(t) - x^2(0).$$

Exercise 2 Consider the IVP

$$\dot{x} = x^2 + t^2, \qquad x(0) = 1.$$

Prove that for some $b > 0$, there is a solution defined on $[0, b]$. Also find $c > 0$ such that there is no solution on $[0, c]$.

Solution Fix $d = 1$, $r = 1$. The map $(t, x) \mapsto x^2 + t^2$ is bounded by $M = 5$ on $[t_0 - d, t_0 + d] \times \overline{B_r(x_0)} = [-1, 1] \times [0, 2]$. Thus, Peano's Theorem guarantees a solution on the interval $[0, b]$ with $b = \min(c, r/M) = 1/5$.

Note that for any solution x, we must have

$$x'(t) \geq x^2(t), \qquad -\frac{d}{dt}\left(\frac{1}{x}\right) \geq 1,$$

whence

$$1 - \frac{1}{x(t)} \geq t, \qquad x(t) \geq \frac{1}{1 - t}.$$

Thus, there is no solution on $[0, 1]$.

Exercise 3 Determine the maximal interval of existence for the following IVP.

$$\dot{x} = y\cos^2 x + \sin t \cos y + 1, \qquad \dot{y} = \sin y + x, \qquad x(0) = 0, \quad y(0) = 1.$$

Solution Framing the system of equations as $\dot{\boldsymbol{x}} = f(t, \boldsymbol{x})$ note that

$$|f(t, \boldsymbol{x})| \leq |y\cos^2 y + \sin t \cos y + 1| + |\sin y + x| \leq |y| + |x| + 3 \leq 2|\boldsymbol{x}| + 3.$$

Furthermore, f is C^1; thus the maximal interval of existence for any solution of the given IVP is $\mathbb{R}$.

Exercise 4 Maximize the interval length in the Picard-Lindelöf Theorem for the solution of the IVP

$$\dot{x} = 5 + x^2, \qquad x(0) = 1.$$

Solution For $r > 0$, the maximum value of the map $(t, x) \mapsto 5 + x^2$ on $\mathbb{R} \times \overline{B_r(x_0)} = \mathbb{R} \times [1-r, 1+r]$ is $M = 5 + (1+r)^2$. Also,

$$|f(t,x) - f(t,y)| = |x^2 - y^2| = |x+y||x-y| \leq (2+2r)|x-y|,$$

hence $L = (2 + 2r)$ is the Lipschitz constant for f. Thus, we must choose $h < \min(r/M, 1/L) = \min(r/(6 + 2r + r^2), 1/(2 + 2r))$. This is maximised at $r = \sqrt{6}$.

Exercise 5 Show that the sequence of Picard iterates of the IVP

$$\dot{x} = x^{1/3}, \qquad x(0) = 0$$

converges, but the IVP does not have a unique solution.

Solution It is clear that all Picard iterates of this IVP are identically zero, but we have a family of solutions $\{x_\alpha\}_{\alpha \geq 0}$ described by

$$x_\alpha(t) = \begin{cases} 0, & \text{if } x \in [0, \alpha], \\ k(t-\alpha)^{3/2}, & \text{if } x \in [\alpha, \infty). \end{cases}$$

Exercise 6 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $x\colon I \to \mathbb{R}$ be a solution of $x' = f(x)$ for an interval I. Show that x is a monotone function.

Solution Suppose to the contrary that $x'(a) > 0$ and $x'(b) < 0$ for some $a, b \in I$. Without loss of generality, let $a < b$, $x(a) \leq x(b)$. Pick $\tau \in (a, b)$ such that $x(\tau)$ is maximum, and let σ be the largest number in $[a, \tau]$ such that $x(\sigma) = x(b)$. Then, we must have all $x(t) \geq x(\sigma)$ for $t \in [\sigma, \tau]$, hence $x'(\sigma) \geq 0$. But,

$$0 \leq x'(\sigma) = f(x(\sigma)) = f(x(b)) = x'(b) < 0,$$

a contradiction.

Exercise 7 Let T be a linear operator on $\mathbb{R}^n$ that leaves a subspace $E \subseteq \mathbb{R}^n$ invariant. Show that e^T also leaves E invariant.

Solution Note that for any $x \in \mathbb{R}^n$, we have

$$e^T x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{T^k x}{k!}.$$

Each $T^n x \in E$, so each term in the limit is in E as well. Since linear subspaces of $\mathbb{R}^n$ are closed, the limit $e^T x \in E$.

Exercise 8 Can the Arzela-Ascoli Theorem be applied to the sequence of functions $t \mapsto \sin(nt)$ on $[0, \pi]$?

Solution No; the given family is not equicontinuous. Suppose to the contrary that there exists $\delta > 0$ such that $|\sin(nt) - \sin(ns)| < 1/2$ for all $n \in \mathbb{N}$ whenever $|s - t| < \delta$. Then we can pick $N \in \mathbb{N}$ such that $\pi/2N < \delta$. Thus, $|\pi/2N - 0| < \delta$, but $|\sin(N \cdot \pi/2N) - \sin(0)| = 1 > 1/2$, a contradiction.

Assignment III

Exercise 1 Using appropriate Lyapunov functions, discuss the stability and asymptotic stability of the following systems.

(i)
$$\dot{x} = y, \qquad \dot{y} = -x - y.$$

(ii)
$$y'' + 6y^5 = 0.$$

Solution

(i) This system has an equilibrium point at $(0,0)$. Consider $V(x,y) = x^2 + y^2$; then, $V(0,0) = 0$, $V(x,y) > 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$, and

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = -2y^2.$$

Since $\dot{V}(x,y) \leq 0$, the equilibrium point $(0,0)$ is stable. Furthermore, $(0,0)$ is asymptotically stable.

(ii) Rewrite this system as

$$y_1' = y_2, \qquad y_2' = -6y_1^5.$$

This system has an equilibrium point at $(0,0)$. Consider

$$V(y_1,y_2) = \frac{1}{2}y_2^2 + \int_0^{y_1} 6s^5\,ds = \frac{1}{2}y_2^2 + y_1^6.$$

Then, $V(0,0) = 0$, $V(y_1,y_2) > 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$, and

$$\dot{V} = y_2 \cdot 6y_1^5 - 6y_1^5 \cdot y_2 = 0.$$

Thus, $(0,0)$ is stable. Since every flow line must stay trapped on the surface $V(y_1,y_2) = 0$, the point $(0,0)$ is not asymptotically stable.

Exercise 2 Check whether solutions of the following equations have infinitely many zeros on $(0,\infty)$.

(i) $y'' + (\sin^2 x + 1)y = 0$.

(ii) $y'' - x^2 y = 0$.

(iii) $xy'' + y = 0$.

Solution

(i) Use Sturm's Comparison Theorem against

$$u'' + u = 0,$$

whose solutions have infinitely many zeros on $(0,\infty)$.

(ii) If a non-trivial solution of $y'' - x^2 y = 0$ would have admitted infinitely many zeros on $(0,\infty)$, so would a non-trivial solution of $u'' = 0$, a contradiction.

(iii) On each interval$(0,K^2)$, the given equation $y'' + y/x = 0$ is a Sturm majorant of $u'' + u/K^2 = 0$, which admits non-trivial solutions of such as $u(x) = \sin(x/K)$. This admits zeros at each $nK\pi$, as long as $0 < nK\pi < K^2$, i.e. at least $\lfloor K/\pi \rfloor$ zeros. Thus, non-trivial solutions of $y'' + y/x$ admit at least $\lfloor K/\pi \rfloor + 1$ roots on $(0,K^2)$. Since K is arbitrary, such y admit infinitely many roots.

Exercise 3 Construct a Green's function for the differential operator d^2/dx^2 under the boundary condition $u(0) = u(1) = 0$.

Let f be a continuous function on $[0,1]$. Show that the function

$$u\colon [0,1] \to \mathbb{R}, \qquad x \mapsto \int_0^1 G(x,\xi) f(\xi)\,d\xi$$

satisfies

$$u'' = f, \qquad u(0) = u(1) = 0.$$

Solution Note that $u'' = 0$ admits two linearly independent solutions $y_1(x) = x$, $y_2(x) = 1 - x$ with $B_0 y_1 = 0$, $B_1 y_2 = 0$. Thus, a Green's function for $u'' = 0$ is of the form

$$G(x,\xi) = \begin{cases} \xi(1-x)/c, & \text{if } 0 \le \xi \le x, \\ x(1-\xi)/c, & \text{if } x \le \xi \le 1. \end{cases}$$

Indeed, $c = W'(y_1, y_2) = -1$, so

$$G(x,\xi) = \begin{cases} -\xi(1-x), & \text{if } 0 \le \xi \le x, \\ -x(1-\xi), & \text{if } x \le \xi \le 1. \end{cases}$$

Check that

$$\begin{aligned} u(x) &= \int_0^x -\xi(1-x)f(\xi)\,d\xi + \int_x^1 -x(1-\xi)f(\xi)\,d\xi \\ u'(x) &= \int_0^x \xi f(\xi)\,d\xi - (1-x)xf(x) - \int_x^1 (1-\xi)f(\xi)\,d\xi + x(1-x)f(x) \\ &= \int_0^1 \xi f(\xi)\,d\xi - \int_x^1 f(\xi)\,d\xi \\ u''(x) &= f(x). \end{aligned}$$

Exercise 4 Let λ and a non-zero function u satisfy the following equation.

$$u'' + \sin(x)u + \lambda u = 0, \qquad u(-1) = u(1) = 0.$$

Show that $\lambda \ge -1$.

Solution Suppose that $\lambda < -1$ admits a non-trivial solution u. Then, a Sturm majorant of the given problem is $v'' = 0$, hence any non-trivial solution v ought to admit a zero in $(-1, 1)$. However, one can choose solutions v without roots in $(-1, 1)$, say $v = x + 2$, a contradiction.