

MA3201

Topology

Spring 2022

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1 Introduction

1.1 Topological spaces

Definition 1.1. A topology on some set X is a family τ of subsets of X , satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from τ are in τ .
3. All finite intersections of elements from τ are in τ .

The sets from τ are declared to be open sets in the topological space (X, τ) .

Example. Any set X admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

Example. Let X be a set. The cofinite topology on X is the collection of complements of finite sets, along with the empty set. Note that when X is finite, this is simply the discrete topology.

Definition 1.2. Let τ, τ' be two topologies on the set X . We say that τ is finer than τ' if τ has more open sets than τ' . In such a case, we also say that τ' is coarser than τ .

1.2 Topological bases

Definition 1.3. Let (X, τ) be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology τ such that every open set $U \in \tau$ is expressible as a union of elements from β .

Definition 1.4. Let X be a set, and let β be a collection of subsets of X satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then, β generates a topology on X , namely the collection of all unions of elements of β .

Lemma 1.1. Let τ be a topology on X , and let $\beta \subseteq \tau$ be a collection of open sets. Then, β is a basis of τ , or generates τ , if for every $x \in U \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U$.

Example. The collection of all open balls in \mathbb{R}^n form a basis of the usual topology.

Lemma 1.2. Let X be equipped with the topologies τ and τ' , and let β and β' be the respective bases of these topologies. Then, τ is finer than τ' if and only if given $x \in B' \in \beta'$, there exists $x \in B \in \beta$ such that $B \subseteq B'$.

Example. The collections of open balls in \mathbb{R}^n generate the same topology as the collection of all open rectangles in \mathbb{R}^n .

Example. Consider the topologies on \mathbb{R} generated by the following bases.

1. $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.
2. $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.
3. $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$ where $K = \{1/n : n \in \mathbb{Z}\}$.

We call the topology generated by β_2 the lower limit topology, denoted \mathbb{R}_ℓ . The topology generated by β_3 is denoted \mathbb{R}_K . Both of these are strictly finer than the standard topology.

Definition 1.5. A sub-basis for some topology on X is a collection ρ of subsets of X whose union is the whole of X . The topology generated by ρ is defined to be the topology generated by the collection of all finite intersections of elements of ρ .

1.3 Product topology

Definition 1.6. Let $(X_1, \tau_1), (X_2, \tau_2)$ be topological spaces. Then $\tau_1 \times \tau_2$ generates the product topology on $X_1 \times X_2$.

Example. The product topology on $\mathbb{R} \times \mathbb{R}$, where \mathbb{R} is equipped with the standard topology, coincides with the standard topology on \mathbb{R}^2 .

Lemma 1.3. If β_1, β_2 are bases of the topologies τ_1, τ_2 , then $\beta_1 \times \beta_2$ and $\tau_1 \times \tau_2$ generate the same product topology.

Proof. Given $(x_1, x_2) \in U$ where $U \subseteq X_1 \times X_2$ is open in the product topology, recall that U can be written as a union of the basic open sets $U_{1i} \times U_{2i}$, where $U_{1i} \in \tau_1$ and $U_{2i} \in \tau_2$. Suppose that $(x_1, x_2) \in U_1 \times U_2$. Thus, we can choose $B_1 \in \beta_1, B_2 \in \beta_2$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$. Thus, $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$. \square

Definition 1.7. The projection maps are defined as $\pi_i : X_1 \times \cdots \times X_k \rightarrow X_i, (x_1, \dots, x_k) \mapsto x_i$.

Lemma 1.4. The collection of elements of the form $\pi_1^{-1}(U_1)$ or $\pi_2^{-1}(U_2)$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$, forms a sub-basis of the product topology on $X_1 \times X_2$.

Proof. Note that $\pi_1^{-1}(X_1) = X_1 \times X_2$. Now it is easy to see that finite intersections of elements of the form $U_1 \times X_2$ or $X_1 \times U_2$ where U_1, U_2 are open, are all of the form $U_1 \times U_2$ which is precisely a basis of the product topology. \square

Corollary 1.4.1. We can restrict ourselves to the sub-basis of elements of the form $\pi_1^{-1}(B_1)$ or $\pi_2^{-1}(B_2)$, where $B_1 \in \beta_1, B_2 \in \beta_2$ for some bases β_1, β_2 of τ_1, τ_2 .

1.4 Subspace topology

Definition 1.8. Let (X, τ) be a topological space, and let $Y \subset X$. Then the collection $U \cap Y$ for all $U \in \tau$ comprises the subspace topology τ_Y on Y induced by the topology τ on X .

Lemma 1.5. If β is a basis for the topology on X , and $Y \subset X$, then the collection $B \cap Y$ for all $B \in \beta$ generates the subspace topology on Y .

Lemma 1.6. An open set of Y is open in X if Y is open in X .

Proof. Let $U \subset Y$ be open in Y , then $U = V \cap Y$ for some open set V in X . If additionally Y is open in X , this immediately shows that U is open in X . \square

Theorem 1.7. Let (X, τ_X) , (Y, τ_Y) be topological spaces, and let $A \subseteq X$, $B \subseteq Y$. Then, there are two ways of assigning a natural topology on $A \times B$.

1. Take the product topology on $X \times Y$, and consider the subspace topology induced by it on $A \times B$.
2. Take the subspace topologies on A induced by τ_X , B induced by τ_Y , and consider the product topology generated by them on $A \times B$.

These two methods generate the same topology on $A \times B$.

Proof. Open sets in 1 look like $(U \times V) \cap (A \times B)$, where $U \in \tau_X$, $V \in \tau_Y$. Open sets in 2 look like $(U' \cap A) \times (V' \cap B)$, where $U' \in \tau_X$, $V' \in \tau_Y$, which can be rewritten as $(U' \times V') \cap (A \times B)$. It is easy to see that these describe precisely the same sets. \square

1.5 Order topology

Definition 1.9. Let X be a set with a simple order $<$. Then the collection of sets of the form (a, b) , $[a_0, b)$, $(a, b_0]$ where a_0 is the minimal element of X , b_0 is the maximal element of X , generate the order topology on X .

Example. The order topology on \mathbb{N} is precisely the discrete topology.

Definition 1.10. Let X_1, X_2 be simply ordered sets. The dictionary order on $X_1 \times X_2$ is defined as follows: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$.

Example. Consider $X = \{1, 2\} \times \mathbb{N}$, where both $\{1, 2\}$ and \mathbb{N} are endowed with the discrete topology. Note that the product topology on X is the discrete topology.

Now consider the dictionary order on X . Here, $(1, 1)$ is the smallest element, so we can list the elements of X in ascending order. Note that every $(1, m) < (2, n)$, for all $m, n \in \mathbb{N}$. Now, note that all singletons $\{(1, m)\}$ are open in the order topology on X . The same is true for the singletons $\{(1, n)\}$ for all $n > 1$. However, the singleton $\{(2, 1)\}$ is *not* open in the order topology.

Example. Consider \mathbb{R} with the usual topology, and $X = [0, 1) \cup \{2\}$. Then, $\{2\}$ is open in the subspace topology on X , but it is not open in the order topology on X .

Lemma 1.8. *The open rays of the form $(a, +\infty)$ and $(-\infty, a)$ in X form a sub-basis of the order topology on X .*

Proof. Note that $(a, b) = (-\infty, b) \cap (a, +\infty)$, $[a_0, b) = (-\infty, b)$, and $(a, b_0] = (a, +\infty)$. \square

Definition 1.11. Let X be a simply ordered set, and $Y \subseteq X$. Then, we say that Y is convex in X if given $a, b \in Y$ such that $a < b$, the interval $(a, b) = \{x \in X : a < x < b\} \subseteq Y$.

Theorem 1.9. *Let Y be convex in X . Then, the subspace topology and the order topology on Y induced from the order topology on X coincide.*

1.6 Closed sets

Definition 1.12. Let (X, τ) be a topological space. A set $F \subseteq X$ is said to be closed in X if $F^c = X \setminus F \in \tau$.

Example. The sets \emptyset, X are closed in every topological space (X, τ) .

Example. In a set equipped with the discrete topology, every set is both open and closed.

Lemma 1.10. *Arbitrary intersections, and finite unions of closed sets are closed.*

Theorem 1.11. *Let (X, τ) be a topological space, and let $Y \subset X$ be equipped with the subspace topology. Then, a set $F \subseteq Y$ is closed in Y if and only if $F = Y \cap G$, where G is closed in X .*

Proof. Let $F \subset Y$. Now, F is closed in Y , $Y \setminus F = Y \cap F^c$ is open in Y , $Y \cap F^c = Y \cap U$ where U is open in X , $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$ where U^c is closed. The steps are reversible. \square

Lemma 1.12. *A closed set of Y is closed in X if Y is closed in X .*

1.7 Interiors and closures

Definition 1.13. Let $A \subseteq X$ where (X, τ) is a topological space.

1. The interior of A is defined as the union of all open sets contained in A . This is denoted by A° .
2. The closure of A is defined as the intersection of all closed sets containing A . This is denoted by \overline{A} .

Remark. The interior of a set is open, and the closure of a set is closed.

Lemma 1.13. *Let $Y \subset X$ be topological spaces, and let $A \subseteq Y$. Also let $\overline{A}_X, \overline{A}_Y$ denote the closures of A in X, Y respectively. Then, $\overline{A}_Y = \overline{A}_X \cap Y$.*

Theorem 1.14. *Let $A \subset X$. Then,*

1. $x \in \overline{A}$ if and only if every open set containing x has non-empty intersection with A .
2. $x \in \overline{A}$ if and only if every basic open set containing x has non-empty intersection with A , given that the topology on X is generated by those basic open sets.

Definition 1.14. Let $A \subseteq X$ where (X, τ) is a topological space. We say that $x \in X$ is a limit point of X if for every open set U containing x , the deleted neighbourhood $U \setminus \{x\}$ has non-empty intersection with A . The set of limit points of A is denoted by A' .

Example. Let X be a set endowed with the discrete topology. Then, given any set $A \subseteq X$, we have $A' = \emptyset$.

Lemma 1.15. *A closed set contains all its limit points.*

Proof. Let $F \subseteq X$ be closed in X , and let $x \in F'$. Then given any open set containing x , we have $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$, hence $x \in \overline{F} = F$. \square

Lemma 1.16. *Let $A \subseteq X$ where (X, τ) is a topological space. Then, $\overline{A} = A \cup A'$.*

Proof. It is clear that $\overline{A} \supseteq A \cup A'$. Now pick $x \in \overline{A}$. If $x \notin A$, then we know that given any open neighbourhood U of x , we have non-empty $U \cap A$. Furthermore, this intersection can never contain x , hence $x \in A'$. This proves that $\overline{A} \subseteq A \cup A'$. \square

1.8 Convergence of sequences

Definition 1.15. Let $\{x_n\}_{n=1}^\infty$ be a sequence of points from (X, τ) , and let $x \in X$. We say that this sequence converges to x , denoted $x_n \rightarrow x$, if every open neighbourhood of x contains the tail of this sequence. In other words, given $U \in \tau$ such that $x \in U$, there must exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then, the constant sequence of b 's converges to all three points a, b, c .

Example. Let $X = \mathbb{R}$, and τ be the collection of all intervals $(-a, a)$ together with \emptyset, \mathbb{R} . Then, the constant sequence of 0's converges to every point in \mathbb{R} .

Definition 1.16. Let (X, τ) be a topological space. We say that this topological space is Hausdorff if given any two distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Example. The real numbers under the standard topology is Hausdorff.

Theorem 1.17. *Let (X, τ) be a Hausdorff topological space, and let $\{x_n\}_{n=1}^\infty$ be a sequence of points in X . Then, this sequence can converge to at most one point in X .*

Proof. Suppose that $\{x_n\}_{n=1}^\infty$ converges to distinct points $x, y \in X$. Then there exist disjoint open neighbourhoods U, V such that $x \in U$, $y \in V$. Convergence means that both U and V contain a tail of the sequence, which is a contradiction. \square

Lemma 1.18. *The singleton sets in a Hausdorff space are closed.*

Proof. Let $x \in X$ where (X, τ) is Hausdorff. Pick $y \neq x$, whence there exist $U_y, V_y \in \tau$, such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. In particular, $\{x\} \cap V_y = \emptyset$. We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open. \square

Theorem 1.19. *The topology induced by a metric is Hausdorff.*

Proof. Given a metric space X and distinct points $x, y \in X$, we set $r = |x - y|$, $U = B(x, r/3)$, $V = B(y, r/3)$. \square

2 Continuous maps

Definition 2.1. Let $f: X \rightarrow Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y) . We say that f is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in Y must be open in X .

Lemma 2.1. *A function $f: X \rightarrow Y$ is continuous if and only if given a base β of Y , we have $f^{-1}(U) \in \tau_X$ for every $U \in \beta$.*

Example. The identity function $\text{id}: \mathbb{R}_\ell \rightarrow \mathbb{R}$ is continuous, while the identity function $\text{id}: \mathbb{R} \rightarrow \mathbb{R}_\ell$ is not. This is because the topology on \mathbb{R}_ℓ is strictly finer than that on \mathbb{R} .

Lemma 2.2. *A function $f: X \rightarrow Y$ is continuous if and only if for every closed set $F \subseteq Y$, we have $f^{-1}(F)$ closed in X .*

Lemma 2.3. *A function $f: X \rightarrow Y$ is continuous if and only if given any $x \in X$ and an open set $V \subseteq Y$ such that $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$, $f(U) \subseteq V$.*

Theorem 2.4. *The composition of continuous functions is continuous.*

2.1 Restricting and enlarging the domain

Lemma 2.5. *Let $f: X \rightarrow Y$ be continuous, and let $A \subset X$. Then the restriction of f to A is continuous.*

Theorem 2.6. *Let $f: X \rightarrow Y$, and let X be the union of the collection of open sets $\{A_\lambda\}_{\lambda \in \Lambda}$. If the restrictions of f to each A_λ are continuous, then f is continuous.*

Proof. Pick $x \in X$, hence $x \in A_\lambda$ for some $\lambda \in \Lambda$. Now if $f(x) \in V \subset Y$, where V is open in Y , then the continuity of the restriction of f to A_λ gives us an open set $U \subseteq A_\lambda$ such that $f(U) \subseteq V$. Finally since A_λ is open in X , so is U . \square

Definition 2.2. Let X be the union of the collection of open sets $\{A_\lambda\}_{\lambda \in \Lambda}$. We say that this collection is a locally finite cover of X if given $x \in X$, there exists a neighbourhood U of x such that $U \cap A_\lambda$ is non-empty for only finitely many $\lambda \in \Lambda$.

Theorem 2.7. Let $f: X \rightarrow Y$, and let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a locally finite collection of closed sets covering X . If the restrictions of f to each F_λ are continuous, then f is continuous.

Corollary 2.7.1 (Pasting lemma). Let $X = A \cup B$, with A, B closed in X . Let $f: A \rightarrow Y$, $g: B \rightarrow Y$ be continuous, with $f(x) = g(x)$ on $A \cap B$. Then the function $h: X \rightarrow Y$, defined by $x \mapsto f(x)$ on A and $x \mapsto g(x)$ on B , is continuous.

Definition 2.3. A path is a continuous function $\gamma: [0, 1] \rightarrow X$.

Lemma 2.8. Two paths γ_1, γ_2 can be concatenated when $\gamma_1(1) = \gamma_2(0)$.

2.2 Homeomorphisms

Definition 2.4. Let $f: X \rightarrow Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y) . We say that f is a homeomorphism if f is continuous, f is bijective, and f^{-1} is continuous. We also say that X and Y are homeomorphic when such a homeomorphism between them exists.

Example. The interval $(-1, 1)$ is homeomorphic to \mathbb{R} ; for instance, the map $x \mapsto \tan(\pi x/2)$ on $(-1, 1)$ is a homeomorphism. A simpler construction is the map $x \mapsto x/(1 - x^2)$.

2.3 Projection maps

Theorem 2.9. The projection maps $\pi_i: X_1 \times \cdots \times X_k \rightarrow X_i$ are continuous, when the domain is equipped with the product topology. Furthermore, the product topology is the coarsest topology on the domain for which the projection maps are continuous.

Lemma 2.10. Let $f: A \rightarrow X_1 \times \cdots \times X_k$, where the co-domain is equipped with the product topology. Then, f is continuous if and only if the component functions $f_i = \pi_i \circ f$ are continuous.

Proof. Note that if f is continuous, the compositions $\pi_i \circ f$ are immediately continuous. Conversely suppose that each f_i is continuous, and write

$$f(t) = (f_1(t), \dots, f_k(t)).$$

The sets $U_1 \times \dots \times U_k$, where each $U_i \subseteq X_i$ is open, form a basis of the co-domain. Furthermore, their pre-images under f are $f_1^{-1}(U_1) \cap \dots \cap f_k^{-1}(U_k)$, which are open in A . This shows that f is continuous. \square

Definition 2.5. Let J be an arbitrary index set. A J -tuple of elements in a set X is a function $x: J \rightarrow X$, formally denoted $(x_\alpha)_{\alpha \in J}$. If $\{X_\alpha\}_{\alpha \in J}$ is a family of sets, their Cartesian product is defined as

$$\prod_{\alpha \in J} X_\alpha = \{x: J \rightarrow \bigcup_{\alpha \in J} X_\alpha : x_\alpha \in X_\alpha\}.$$

Remark. The fact that we can choose an element from each set in an uncountable collection relies on the Axiom of Choice.

Definition 2.6. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces. The topology generated by $\prod_{\alpha \in J} U_\alpha$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the box topology on $\prod_{\alpha \in J} X_\alpha$.

Definition 2.7. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces. The topology generated by the sub-basis $\pi_\alpha^{-1}(U_\alpha)$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the product topology on $\prod_{\alpha \in J} X_\alpha$.

Remark. The basic open sets are of the form $\pi_{\alpha \in J} U_\alpha$, where all but finitely many $U_\alpha = X_\alpha$. Thus, this is a coarser topology than the box topology.

Lemma 2.11. Let $\prod_{\alpha \in J} X_\alpha$ be equipped with the box or product topology. Then, $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$, where each $A_\alpha \in X_\alpha$.

Lemma 2.12. Let $f: A \rightarrow \prod_{\alpha \in J} X_\alpha$, where the co-domain is equipped with the product topology. Then, f is continuous if and only if the component functions $f_\alpha = \pi_\alpha \circ f$ are continuous.

Remark. This fails when $\prod_{\alpha \in J}$ is equipped with the box topology. Consider $f: \mathbb{R} \rightarrow \prod_{n=1}^{\infty} \mathbb{R}$, $x \mapsto (x, x, \dots)$. Then, the product $\prod_{n=1}^{\infty} (-1/n, 1/n)$ is open in the box topology, but its pre-image under f is $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$, which is not open in \mathbb{R} .

3 Metric spaces

Definition 3.1. A metric space (X, d) is a set equipped with a metric $d: X \times X \rightarrow \mathbb{R}$, such that

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 3.2. An open ball in a metric spaces is the set of points

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Lemma 3.1. *The collection of open balls in a metric space generates its standard topology.*

Example. Consider a set X , equipped with the metric

$$d: X \times X \rightarrow \mathbb{R}, \quad (x, y) \mapsto \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then, this metric induces the discrete topology on X .

Definition 3.3. A topological space (X, τ) is called metrizable if there exists a metric $d: X \times X \rightarrow \mathbb{R}$ which induces the topology τ on X .

Definition 3.4. Let $A \subseteq X$. The diameter of A is defined to be

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

If $\text{diam}(A)$ is finite, we say that A is bounded.

Example. The metric

$$(x, y) \mapsto \frac{|x - y|}{1 + |x - y|}$$

generates the standard topology on \mathbb{R} . Note that \mathbb{R} is unbounded in the standard metric, but bounded in this one.

Definition 3.5. Let (X, d) be a metric space. Then the standard bounded metric corresponding to d is defined as

$$\bar{d}: X \times X \rightarrow \mathbb{R}, \quad (x, y) \mapsto \min\{d(x, y), 1\}.$$

Lemma 3.2. *Both d and \bar{d} generate the same topology.*