

MA2201: ANALYSIS II

# Integration

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**Definition 3.1** (Partition). A partition  $P$  of an interval  $[a, b]$  is a finite sequence of numbers

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The norm of a partition is defined as

$$\|P\| = \max |x_{j+1} - x_j|.$$

**Definition 3.2** (Tagged partition). A tagged partition  $\dot{P}(x_j, \xi_j)$  is a partition  $P$  together with a set of numbers  $\xi_j$  such that  $\xi_j \in [x_j, x_{j+1}]$ .

**Definition 3.3** (Riemann sum). The Riemann sum of a function  $f$  on an interval  $[a, b]$  with respect to a tagged partition  $\dot{P}$  is defined as

$$S(f, \dot{P}) = \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j).$$

**Definition 3.4** (Riemann integral). A function  $f$  is called Riemann integrable on an interval  $[a, b]$  if there is some  $\ell \in \mathbb{R}$  where for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that all tagged partitions  $\dot{P}$  of  $[a, b]$  with  $\|\dot{P}\| < \delta$  satisfy

$$|S(f, \dot{P}) - \ell| < \epsilon.$$

The number  $\ell$  is the value of the Riemann integral,

$$\int_a^b f = \ell.$$

**Theorem 3.1.** *If a function is Riemann integrable on an interval, then the value of the integral is unique.*

*Proof.* Let  $f$  be Riemann integrable on  $[a, b]$ , with integral values  $\ell$  and  $\ell'$ . Then, for every  $\epsilon > 0$ , we find  $\delta > 0$  such that for all tagged partitions  $\dot{P}$  with  $\|\dot{P}\| < \delta$ ,

$$|S(f, \dot{P}) - \ell| < \frac{\epsilon}{2}, \quad |S(f, \dot{P}) - \ell'| < \frac{\epsilon}{2}.$$

Note that such a partition  $\dot{P}$  always exists. Thus,

$$|\ell - \ell'| \leq |S(f, \dot{P}) - \ell| + |S(f, \dot{P}) - \ell'| < \epsilon$$

for all  $\epsilon > 0$ , which forces  $\ell = \ell'$ . □

**Theorem 3.2.** *If  $f$  is Riemann integrable on  $[a, b]$ , then  $f$  is bounded on that interval. Furthermore, if  $M > 0$  is such that  $|f(x)| \leq M$  for all  $x \in [a, b]$ , then*

$$-M(b-a) \leq \int_a^b f \leq M(b-a).$$

*Proof.* Suppose not. Let the Riemann integral of  $f$  on  $[a, b]$  be  $\ell$ . For  $\epsilon = 1$ , we find  $\delta > 0$  such that for all tagged partitions  $\dot{P}$  of  $[a, b]$  with  $\|\dot{P}\| < \delta$ , we have  $|S(f, \dot{P}) - \ell| < 1$ . This means that

$$S(f, \dot{P}) < |\ell| + 1.$$

Let  $Q = \{x_0, \dots, x_n\}$  be such a partition. The unboundedness of  $f$  means that we can find a subinterval  $[x_k, x_{k+1}]$  where  $f$  is unbounded. Now, choose tags  $\xi_j$  creating the tagged partition  $\dot{Q}$ . We choose the tag  $\xi_k \in [x_k, x_{k+1}]$  such that

$$|f(\xi_k)(x_{k+1} - x_k)| > |\ell| + 1 + \left| \sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j) \right|.$$

Now, observe that the triangle inequality demands

$$|S(f, \dot{Q})| \geq |f(\xi_k)(x_{k+1} - x_k)| - \left| \sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j) \right| > |\ell| + 1,$$

which is a contradiction. Thus,  $f$  must be bounded on  $[a, b]$ .

Next, for any tagged partition  $\dot{P}$  of  $[a, b]$ , we have

$$|S(f, \dot{P})| \leq \sum_{j=0}^{n-1} |f(\xi_j)(x_{j+1} - x_j)| \leq M(b-a).$$

Let the Riemann integral of  $f$  be  $\ell$ . Thus, for all  $\epsilon > 0$ , we find  $\delta > 0$  such that for all tagged partitions  $\dot{P}$  with  $\|\dot{P}\| < \delta$ ,

$$||S(f, \dot{P})| - |\ell|| \leq |S(f, \dot{P}) - \ell| < \epsilon.$$

This gives

$$|\ell| < |S(f, \dot{P})| + \epsilon \leq M(b-a) + \epsilon.$$

Since this holds for all  $\epsilon > 0$ , we may write

$$|\ell| \leq M(b-a). \quad \square$$

**Theorem 3.3.** *If  $f$  is Riemann integrable on  $[a, b]$ , and  $\dot{P}_n$  is any sequence of tagged partitions of  $[a, b]$  such that  $\|\dot{P}_n\| \rightarrow 0$ , then*

$$\int_a^b f = \lim_{n \rightarrow \infty} S(f, \dot{P}_n).$$

*Proof.* Let  $\epsilon > 0$ . We find  $\delta > 0$  such that for all tagged partitions  $\dot{P}$  with  $\|\dot{P}\| < \delta$ , we have

$$|S(f, \dot{P}) - \int_a^b f| < \epsilon.$$

Now, since  $\|\dot{P}_n\| \rightarrow 0$ , we can choose  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $\|\dot{P}_n\| < \delta$ . Thus, for all  $n \geq N$ ,

$$|S(f, \dot{P}_n) - \int_a^b f| < \epsilon.$$

In other words,

$$\int_a^b f = \lim_{n \rightarrow \infty} S(f, \dot{P}_n). \quad \square$$

**Definition 3.5** (Refinement). A partition  $\tilde{P}$  is said to be a refinement of a partition  $P$  if  $P \subset \tilde{P}$ .

**Definition 3.6** (Common refinement). Given two partitions  $P_1$  and  $P_2$  of an interval  $[a, b]$ , we say that  $\tilde{P}$  is their common refinement if  $P_1 \cup P_2 \subset \tilde{P}$ .

**Definition 3.7** (Darboux sums). Given a partition  $P$  of  $[a, b]$  and a bounded function  $f$ , define

$$m_j = \inf_{t \in [x_j, x_{j+1}]} f(t), \quad M_j = \sup_{t \in [x_j, x_{j+1}]} f(t).$$

The lower and upper Darboux sums are defined as

$$L(f, P) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j), \quad U(f, P) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j).$$

**Lemma 3.4.** *If  $P$  is a partition of an interval  $[a, b]$ , then*

$$L(f, P) \leq U(f, P).$$

*Proof.* This follows directly from the fact that the infimum is less than or equal to the supremum, i.e.  $m_j \leq M_j$ .  $\square$

**Theorem 3.5.** *Let  $\tilde{P}$  be a refinement of a partition  $P$  of an interval  $[a, b]$ . Then,*

$$L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, P)$$

*Proof.* Suppose that

$$P = \{x_0, \dots, x_k, x_{k+1}, \dots, x_n\},$$

$$\tilde{P} = \{x_0, \dots, x_k, y, x_{k+1}, \dots, x_n\}.$$

Set

$$m_1 = \inf_{t \in [x_k, y]} f(t), \quad m_2 = \inf_{t \in [y, x_{k+1}]} f(t), \quad m = \inf_{t \in [x_k, x_{k+1}]} f(t).$$

Then, observe that

$$L(f, \tilde{P}) - L(f, P) = m_1(y - x_k) + m_2(x_{k+1} - y) - m(x_{k+1} - x_k).$$

Now, from the properties of the infimum,  $m_1 \geq m$  and  $m_2 \geq m$ , so

$$L(f, \tilde{P}) - L(f, P) \geq m(y - x_k + x_{k+1} - y - x_{k+1} + x_k) = 0.$$

This procedure of adding one point can be repeated finitely many times to obtain the same conclusion for any refinement of  $P$ . The case for the upper sum is analogous.  $\square$

**Corollary 3.5.1.** *For any two partitions  $P_1$  and  $P_2$  of  $[a, b]$ ,*

$$L(f, P_1) \leq U(f, P_2).$$

*Proof.* Note that  $P_1 \cup P_2$  is a common refinement of  $P_1$  and  $P_2$ , hence

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2).$$

$\square$

**Corollary 3.5.2.** *If  $\{P_n\}$  is a sequence of refinements of a partition  $P_0$  of  $[a, b]$ , then the following limits exist.*

$$L_{f, P_n} = \lim_{n \rightarrow \infty} L(f, P_n), \quad U_{f, P_n} = \lim_{n \rightarrow \infty} U(f, P_n).$$

*Proof.* This follows from the monotone convergence theorem, together with the fact that  $U(f, P_0)$  and  $L(f, P_0)$  are upper and lower bounds of the two respective sequences.  $\square$

**Corollary 3.5.3.** *The following quantities exist, where the infimum and supremum is taken over all possible partitions  $P$  of  $[a, b]$ .*

$$L_f = \sup L(f, P), \quad U_f = \inf U(f, P).$$

Furthermore, for any partition  $P$ ,

$$L(f, P) \leq L_f \leq U_f \leq U(f, P).$$

*Proof.* First examine the set of all lower Darboux sums,  $\{L(f, P)\}$ . This set is non-empty, since any partition of  $[a, b]$  gives a corresponding lower sum. Note that we have already demanded that  $f$  is bounded! This set is also bounded above, by any upper sum. Thus, the completeness of the reals guaranteed the existence of a supremum. The case for upper sums is analogous.

The outermost inequalities trivially follow from the definitions of the infimum and supremum. The middle inequality follows from the fact that if  $A$  and  $B$  are two subsets of  $\mathbb{R}$  such that  $\alpha \in A, \beta \in B$  implies  $\alpha \leq \beta$ , then  $\sup A \leq \inf B$ .  $\square$

**Definition 3.8** (Darboux integrals). The lower and upper Darboux integrals of a function  $f$  are defined as

$$L_f = \sup L(f, P), \quad U_f = \inf U(f, P).$$

Here, the infimum and supremum is taken over all possible partitions  $P$  of  $[a, b]$ .

If  $L_f = U_f$ , then the common integral is simply called the Darboux integral,

$$\int_a^b f = L_f = U_f.$$

Such a function  $f$  is called Darboux integrable.

**Theorem 3.6.** *Riemann and Darboux integrability are equivalent and assign the same value to the integrals. Specifically, a function  $f$  is Riemann integrable on  $[a, b]$  if for every  $\epsilon > 0$ , there exists a partition  $P$  such that*

$$U(f, P) - L(f, P) < \epsilon.$$