MA2202: Probability I

Random variables

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Definition 3.1 (Random variable)**.** Given a probability space $(\Omega, \mathcal{E}, P)$, a function $X\colon \Omega \to \mathbb{R}$ is called a random variable if $X^{-1}(r, \infty) \in \mathcal{E}$ for all $r \in \mathbb{R}$.

Remark. For some $S \subseteq \mathbb{R}$, we denote

$$P(X \in S) = P(\{\omega \in \Omega\colon X(\omega) \in S\}).$$

Definition 3.2 (Discrete random variable)**.** A random variable which can assume only a countably infinite number of values is called a discrete random variable.

Example. Let $X\colon \Omega \to \mathbb{R}$ denote the number of heads obtained when a fair coin is tossed thrice. Note that $\Omega = \{0, 1, 2, 3\}$. Thus, $P(X = 0) = P(X = 4) = 1/8$ and $P(X = 1) = P(X = 2) = 3/8$.

Definition 3.3 (Probability distribution)**.** The probability distribution of a random variable X is the set of pairs $(X(A), P(A))$ for all $A \in \mathcal{E}$.

Definition 3.4 (Probability mass function)**.** Let X be a discrete random variable. The probability mass function of X is the function $p_X\colon \mathbb{R} \to [0, 1]$,

$$p_X(\alpha) = P(X = \alpha).$$

Remark. Since X is a discrete random variable, the set $S = \{x \in \mathbb{R}\colon p_X(\alpha) > 0\}$ is countable, and

$$\sum_{x \in S} p_X(x) = 1.$$

Definition 3.5 (Expectation)**.** The expectation of $g(X)$, for $g\colon \mathbb{R} \to \mathbb{R}$ and a discrete random variable X is defined as

$$E[g(X)] = \sum_{x \in S} g(x)\, p_X(x),$$

if the series converges absolutely.

Example. The n^{th} moment of a discrete random variable $E[X^n]$ is defined as

$$E[X^n] = \sum_{x \in S} x^n\, p_X(x),$$

if the series converges absolutely.

The first moment $\mu = E[X]$ is called the mean. The second moment $\sigma^2 = E[(X-\mu)^2]$ is called the variance. Note that

$$\sigma^2 = \sum (x-\mu)^2\, p(x) = \sum x^2\, p(x) - 2\mu x\, p(x) + \mu^2 p(x).$$

Simplifying,

$$\sigma^2 = E[X^2] - E[X]^2.$$

Definition 3.6 (Cumulative distribution function)**.** The cumulative distribution function of a random variable X is defined as the function $F_X\colon \mathbb{R} \to [0,1]$,

$$F_X(\alpha) = P(X \leq \alpha).$$

Definition 3.7 (Continuous random variable)**.** A continuous random variable X is such that its cumulative distribution function F_X is continuous.

Definition 3.8 (Probability density function)**.** Let X be a continuous random variable with a cumulative distribution function F_X. If we write

$$F_X(\alpha) = \int_{-\infty}^{\alpha} f_X(x)\, dx$$

for all $\alpha \in \mathbb{R}$, then f_X is a probability density function. If f_X is continuous, then the Fundamental Theorem of Calculus guarantees that $f_X(x) = F_X'(x)$.

Remark. Note that we can write

$$P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} f_X(x)\, dx.$$

We also demand

$$\int_{-\infty}^{+\infty} f_X(x)\, dx.$$

Example. The uniform distribution on an interval $[a, b] \subset \mathbb{R}$ is defined using the probability density function

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Definition 3.9 (Expectation)**.** The expectation of $g(X)$, for $g\colon \mathbb{R} \to \mathbb{R}$ and a continuous random variable X is defined as

$$E[g(X)] = \int_{-\infty}^{+\infty} g(X)\, f_X(x)\, dx$$

if the integral converges absolutely.

Definition 3.10 (Mixed random variable)**.** A random variable whose cumulative distribution function F_X is discontinuous at countably many points, with F_X being continuous and strictly increasing in at least one interval is called a mixed random variable.

Definition 3.11 (Conditional probability distributions)**.** Let X and Y be two random variables, and let $A, B \subseteq \mathbb{R}$. If $P(Y \in B) > 0$, then

$$P(X \in A \,|\, Y \in B)\, P(Y \in B) = P(X \in A, Y \in B).$$

Definition 3.12 (Independent random variables)**.** We say that X and Y are independent if

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

for all $A, B \subseteq \mathbb{R}$.

Example. Consider an experiment where a fair die is rolled until a 6 is obtained, and let X be the random variable denoting the number of throws. Also, let Y be a random variable which is 1 if the first even outcome is a 6, and 0 otherwise. Observe that $P(X = 1) = 1/6$ and $P(Y = 1) = 1/3$. Also, $P(X = 1, Y = 1) = 1/6 \neq P(X = 1)\, P(Y = 1)$, hence X and Y are not independent random variables.

It can be shown that

$$P(X = n \,|\, Y = 1) = \frac{1}{2^n}.$$

Bernoulli distribution

Consider an experiment with several identical trials. In each trial, there are two possible outcomes; the probability of a success if given by p and the probability of a failure is $q = 1 - p$. We could let X_i be a random variable denoting the outcome of the i^{th} outcome, so we demand that $\{X_i\}$ are independent and identically distributed.