MA2201: Analysis II

Sequences of functions

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Satvik Saha
19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

Pointwise convergence

Definition 1.1 (Sequences of functions)**.** Let the functions $f_n \colon X \to Y$ be defined for all $n \in \mathbb{N}$ and let the sequences $\{f_n(x)\}$ converge for all $x \in X$. Define the function $f \colon X \to Y$ as

$$f(x) = \lim_{n \to \infty} f_n(x)$$

for all $x \in X$. We call f the limit of $\{f_n\}$, or say that $\{f_n\}$ converges to f pointwise on X.

Example. Consider the functions $f_n \colon [0, 1] \to \mathbb{R}$, $x \mapsto x^n$. It can be shown that $x^n \to 0$ when $x \in [0, 1)$ and $x^n \to 1$ when $x = 1$. Thus, $f = \lim_{n\to\infty} f_n$ is well defined.

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}.$$

Note that while each f_n is continuous in this example, the limit f is not.

Example. Consider the functions $f_n \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x/n$. We see that $f_n \to 0$. Note that 0 here denotes the zero function.

Example. Consider the functions $f_n \colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} x^2, & \text{if } |x| \le n \\ +n, & \text{if } x > +n \\ -n, & \text{if } x < -n \end{cases}.$$

This converges pointwise to $f \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Note that for any $x \in \mathbb{R}$, we can find sufficiently large $N \in \mathbb{N}$ such that $|x| \le N$. This means that the tail of the sequence $\{f_n(x)\}$ becomes a constant sequence $\{x^2\}$ from the N^{th} term onwards, so $f_n(x) \to x^2$ for all $x \in \mathbb{R}$.

Example. Consider the functions $f_n\colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \lim_{m\to\infty} (\cos n!\pi x)^{2m}.$$

We observe that $f_n(x) = 1$ only when $n!x$ is an integer. Now, if x is rational, $n!x$ will become an integer for sufficiently large n, whereas if x is irrational, $n!x$ can never be an integer. Thus, we see that $f_n \to f$, where $f\colon \mathbb{R} \to \mathbb{R}$ is the Dirichlet function defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Note that f is discontinuous everywhere!

Exercise 1.1. Show that if a sequence of functions $\{f_n\}$ converges on X, then the sequence of functions $\{|f_n|\}$ also converges on X.

Solution. Suppose that $f_n \to f$. Then given $\epsilon > 0$, $x \in X$, we find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)| < \epsilon.$$

This gives $|f_n| \to |f|$ on X.

Definition 1.2 (Series of functions)**.** Let the functions $f_n\colon X \to Y$ be defined for all $n \in \mathbb{N}$ and let the series $\sum f_n(x)$ converge for all $x \in X$. Define the function $f\colon X \to Y$ as

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in X$. We call f the sum of the series $\sum f_n$.

Example. Consider the functions $f_n\colon (0,1) \to \mathbb{R}$, $x \mapsto x^n$. Note that the sum

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

does indeed converge for all $x \in (0,1)$. Thus, the sum $f = \sum f_n$ is well defined.

$$f(x) = \frac{x}{1-x}.$$

Uniform convergence

Definition 1.3 (Uniform convergence)**.** Let the functions $f_n\colon X \to Y$ be defined for all $n \in \mathbb{N}$. We say that the sequence $\{f_n\}$ converges uniformly on X to f if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in X$, we have

$$|f_n(x) - f(x)| < \epsilon.$$

Remark. Note that for convergence $f_n \to f$, we need only find N depending on ϵ and x. Uniform convergence requires N depending on ϵ which ensures the inequality for *all* $x \in X$.

Example. Consider $f_n\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x + 1/n$. We see that $f_n \to f$ uniformly on $\mathbb{R}$, where $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x$. Note that given $\epsilon > 0$, we find $N \in \mathbb{N}$ such that $N\epsilon > 1$ using the Archimedean property. Thus, for all $n \geq N$ and $x \in \mathbb{R}$ we have

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Lemma 1.1. *The sequence of functions $\{f_n\}$ does not converge uniformly on X to its pointwise limit f if there exists some $\epsilon_0 > 0$, some subsequence $\{f_{n_k}\}$ and some sequence $\{x_k\}$ in X such that for all $k \in \mathbb{N}$,*

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

Example. The sequence of functions $\{f_n\}$ where $f_n\colon [0,1] \to \mathbb{R}$, $x \mapsto x^n$ does not converge uniformly on $[0,1]$. We have already described $f = \lim_{n\to\infty} f_n$. Set $\epsilon_0 = 1/2$, $x_k = (1/2)^{1/k}$ and $n_k = k$. Thus,

$$|f_{n_k}(x_k) - f(x_k)| = \frac{1}{2} \geq \epsilon_0.$$

Note that $x_k \to 1$, which is the point of discontinuity of f.

Example. The sequence of functions $\{f_n\}$ where $f_n\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x/n$ does not converge uniformly on $\mathbb{R}$. Recall that $f_n \to 0$, but when $\epsilon_0 = 1$, $n_k = x_k = k$, we have

$$|f_{n_k}(x_k) - f(x_k)| = 1 \geq \epsilon_0.$$

Theorem 1.2 (Cauchy criterion for uniform convergence)**.** *The sequence of real-valued functions $\{f_n\}$ converges uniformly on X if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and $x \in X$, we have*

$$|f_n(x) - f_m(x)| < \epsilon.$$

Remark. We require the functions f_n to be real or complex valued, since Cauchy sequences are precisely the convergent sequences in a complete metric space.

Proof. First suppose that $\{f_n\}$ converges uniformly on X, and $f_n \to f$. This means that given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all $n \geq N$, $x \in X$. Thus, for all $m, n \geq N$ and $x \in X$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

Now suppose that the Cauchy criterion holds. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and $x \in X$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Recall that the Cauchy criterion for real sequences guarantees that the sequence $\{f_n(x)\}$ converges, thus the function $f = \lim_{n\to\infty} f_n$ is well defined. To show that the convergence of $f_n \to f$ is uniform, fix n and let $m \to \infty$, so $f_m(x) \to f(x)$. Thus for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| < \epsilon,$$

as desired. □

Theorem 1.3. *Let $f_n \colon X \to Y$ and let $f_n \to f$. Set*

$$M_n = \sup_{x \in X} |f_n(x) - f(x)|.$$

Then, $\{f_n\}$ converges uniformly on X to f if and only if $M_n \to 0$.

Proof. Suppose that $f_n \to f$ uniformly on X. Let $\epsilon > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

This means that for all $n \geq N$,

$$M_n = \sup |f_n(x_n) - f(x_n)| \leq \frac{\epsilon}{2} < \epsilon.$$

Also note that all $M_n \geq 0$, since they are the supremums of non-negative quantities. This means that $M_n \to 0$, as desired.

Now suppose that $M_n \to 0$. This means that for arbitrary $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|M_n| = \sup |f_n(x) - f(x)| < \epsilon.$$

Now, from the properties of the supremum, we see that for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| \leq \sup |f_n(x) - f(x)| < \epsilon.$$

This proves that $f_n \to f$ uniformly. □

Example. Consider $f_n \colon [0, 1/2] \to \mathbb{R}$, $x \mapsto x^n$. We see that $f_n \to 0$, and that

$$M_n = \sup |f_n(x) - f(x)| = \frac{1}{2^n} \to 0.$$

Thus, $\{f_n\}$ converges uniformly on $[0, 1/2]$ to 0.

Theorem 1.4 (Weierstrass M-test). *Let $f_n\colon X \to Y$ and suppose that for all $n \in \mathbb{N}$ and $x \in X$,*

$$|f_n(x)| \leq M_n.$$

Then the series $\sum f_n$ converges uniformly on X if $\sum M_n$ converges.

Proof. Let $\epsilon > 0$. Since $\sum M_n$ converges, we can use the Cauchy criterion for the convergence of real series to choose $N \in \mathbb{N}$ such that for all $m \geq n \geq N$,

$$\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} M_k \leq \epsilon$$

for all $x \in X$. Note that the left hand side is simply $|s_m(x) - s_{n-1}(x)|$ where $s_k(x)$ is the k^{th} partial sum of the series $\sum f_n(x)$. Thus, the Cauchy criterion gives the uniform convergence of $\{s_n\}$, hence the uniform convergence of the series $\sum f_n$. □

Remark. The converse is not true. Simply setting $f_n = 0$, we observe that the series $\sum f_n$ converges uniformly on $\mathbb{R}$ to 0. On the other hand, $|f_n(x)| \leq 1$ for all $x \in \mathbb{R}$, and the series $\sum 1$ diverges to ∞.

Example. Consider the functions

$$f_n\colon [-A, +A] \to \mathbb{R}, \qquad f_n(x) = \frac{x^n}{n!}.$$

Note that $|f_n(x)| \leq A^n/n!$, and

$$\sum_{k=0}^{n-1} \frac{A^k}{k!} \to e^A.$$

Thus, the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges uniformly on $[-A, +A]$.

Theorem 1.5. *Let the functions $f_n\colon X \to Y$ be continuous, and suppose that $f_n \to f$ uniformly on X in a metric space. Then, f is continuous on X.*

Proof. Let $\epsilon > 0$. We wish to show that f is continuous at arbitrary $x_0 \in X$.

Since $f_n \to f$ uniformly on X, we find $N \in \mathbb{N}$ such that for all $x \in X$ and $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

In particular, this holds for $n = N$, and $x = x_0$.

The continuity of each f_n on X means f_N is continuous on X in particular, so we can find $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}.$$

Putting these together, for every $x \in X$ such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

This means that f is continuous at x_0 for arbitrary $x_0 \in X$, i.e. f is continuous on X. $\square$

Corollary 1.5.1. *Let the functions $f_n \colon X \to Y$ be continuous, and let $f_n \to f$ pointwise on X. If f is not continuous on X, then the sequence of functions $\{f_n\}$ does not converge uniformly on X.*

Proof. This is simply the contrapositive of the given theorem. $\square$

Example. The functions $f_n \colon [0,1] \to \mathbb{R}$, $x \mapsto x^n$ do not converge uniformly on $[0,1]$ because each f_n is continuous, but their limit $\lim_{n\to\infty} f_n$ is discontinuous at $x = 1$.

Theorem 1.6. *Let K be compact, and suppose that*

1. *$\{f_n\}$ is a sequence of continuous functions on K.*
2. *$\{f_n\}$ converges pointwise to a continuous function f on K.*
3. *$f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$.*

The, $f_n \to f$ uniformly on K.

Proof. Set $g_n = f_n - f$, and note that each g_n is also decreasing and continuous, with $g_n \to 0$. Also note that $g_n \geq 0$. We claim that $g_n \to 0$ uniformly on K.

Let $\epsilon > 0$. Set

$$K_n = \{x \in K \colon g_n(x) \geq \epsilon\}.$$

Now, note that $K_n \supseteq K_{n+1}$ since g_n is decreasing, $K_n = g_n^{-1}[\epsilon, \infty)$ is closed since g_n is continuous, and thus $K_n \subseteq K$ is compact. If $K_n \neq \emptyset$ for all $n \in \mathbb{N}$, recall that

$$\mathcal{K} = \bigcap_{n \in \mathbb{N}} K_n \neq \emptyset.$$

Selecting $x_0 \in \mathcal{K}$, we have $g_n(x_0) \geq \epsilon$ for all $n \in \mathbb{N}$. This contradicts the fact that $g_n \to 0$ pointwise on K. Thus, there must exist $N \in \mathbb{N}$ such that $K_{n \geq N} = \emptyset$. Thus, we have

$$0 \leq g_n(x) < \epsilon$$

for all $n \geq N$, all $x \in K$, as desired. $\square$

Definition 1.4. Let X be a metric space and denote $\mathscr{C}(X)$ as the set of all complex-valued, continuous, bounded functions with domain X. Define the supremum norm on each $f \in \mathscr{C}(X)$ as

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Then, $\mathscr{C}(X)$ is a metric space.

Remark. Note that the supremum norm $\|\cdot\|$ satisfies symmetry, positivity, and the triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|$$

for all $f, g \in \mathscr{C}(X)$.

Theorem 1.7. *The metric space $\mathscr{C}(X)$ is complete.*

Proof. Suppose that $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$. This means that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $\|f_n - f_m\| < \epsilon$. Theorem 1.2 shows that $\{f_n\}$ converges uniformly to some function f on X. Theorem 1.5 guarantees that f is continuous. Note that f is bounded, since there exists $n \in \mathbb{N}$ such that $\|f - f_n\| < 1$ and f_n is bounded.

Thus, $f \in \mathscr{C}(X)$, where $f_n \to f$ uniformly on X. It follows from Theorem 1.3 that $\|f - f_n\| \to 0$. □

Definition 1.5 (Equicontinuity)**.** A sequence of functions $\{f_n\}$ on a set X is called equicontinuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have

$$|f_n(x) - f_n(y)| < \epsilon.$$