MA3203

Analysis IV

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1 Measure theory

1.1 Introduction

Measure theory seeks to generalize the notions of *length*, *area*, *volume* to more general sets: this new notion is called a *measure*. This also allows us to generalize the notion of Riemann integration to a broader class of functions.

Recall that continuous functions, or at least functions with finitely many discontinuities on a closed interval are Riemann integrable. The Dirichlet function, which is discontinuous everywhere, is not.

$$f\colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This is simply because every non-empty interval contains at least one rational and one irrational number, so the Darboux lower sum is always 0 and the upper sum is always 1 regardless of the choice of partition.

On the other hand, if we had to assign a particular value to this integral, intuition tells us that it ought to be zero. After all, the function f attains a non-zero value only on the countable set $\mathbb{Q} \cap [0, 1]$; it is zero almost everywhere. Formally, we will show that f is non-zero on a set of zero Lebesgue measure, which will allow us to set this Lebesgue integral to zero. We will see that with this new formulation of integration, we end up partitioning the range of f rather than it's domain, and write

$$\int f = 0 \cdot \mu([0, 1] \setminus \mathbb{Q}) + 1 \cdot \mu([0, 1] \cap \mathbb{Q}) = 0.$$

1.2 Basic definitions

Definition 1.1. Let X be a set, and let Σ be a collection of subsets of X. We say that Σ is a σ-algebra over X if it satisfies the following.

1. Σ is closed under complementation.
2. Σ contains X.
3. Σ is closed under countable unions.

Remark. The following properties follow immediately.

1. Σ contains $\emptyset$.
2. Σ is closed under countable intersections.

Definition 1.2. Let X be a set, and let Σ be a σ-algebra over X. We say that a function $\mu\colon \Sigma \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called a measure if it satisfies the following.

1. μ is non-negative.
2. $\mu(\emptyset) = 0$.
3. μ is additive over countable unions, i.e. for any countable collection $\{E_i\}_{i=1}^{\infty}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$