MA5102: Partial Differential Equations

Final Assignment

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Contents

Problem 1

Let u be a solution of

$$a(x, y)u_x + b(x, y)u_y = -u$$

and of class C^1 in the closed unit disk Ω in the xy-plane. Furthermore, let

$$a(x, y)x + b(x, y)y > 0$$

on the boundary of Ω. Prove that u vanishes identically on Ω.

Solution

Note that by setting $v = u^2 \geq 0$, we have

$$a(x, y)v_x + b(x, y)v_y = 2uu_x a + 2uu_y b = -2u^2 = -2v.$$

Since v is of class C^1 on Ω, it must attain its maximum $M \geq 0$ at some point $z_0 = (x_0, y_0) \in \Omega$. Now, if z_0 is in the interior of Ω, then we must have $v_x(z_0) = v_y(z_0) = 0$, hence $M = v(z_0) = -\frac{1}{2}(av_x + bv_y)(z_0) = 0$. This forces $v = 0$, hence $u = 0$ on Ω as desired.

This leaves the case where $z_0 \in \partial\Omega$. Consider an initial curve along $\partial\Omega$, and set up the characteristic equations

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b, \quad \frac{dv}{ds} = -2v.$$

Thus, $v(\theta, s) = \varphi(\theta)e^{-2s}$. Let θ_0, s_0 be such that $(x(\theta_0, s_0), y(\theta_0, s_0)) = (x_0, y_0) = z_0$. Then, $\varphi(\theta_0) = v(\theta_0, s_0)e^{2s_0} = Me^{2s_0}$. Note that if $M = 0$, we are done. Otherwise, $M > 0$. Now, at z_0, the tangent (a, b) to the characteristic curve $s \mapsto (x(\theta_0, s), y(\theta_0, s))$ points outwards from Ω, since $(a, b) \cdot (x_0, y_0) > 0$ there. This means that for some $\delta > 0$, the characteristic curve $s \mapsto (x(\theta_0, s), y(\theta_0, s))$ must lie in the interior of Ω for $s \in (s_0 - \delta, s_0)$. However, along this curve, $v(\theta_0, s) = Me^{2(s_0 - s)}$ is decreasing with s! This means that $v(\theta_0, s_0 - \delta/2) = Me^{\delta} > M$ in Ω, contradicting the maximality of M.

In all cases, $\max_\Omega u^2 = M = 0$, hence $u = 0$ identically on Ω.

Problem 2

1. Let (r, θ, ϕ) be spherical coordinates in $\mathbb{R}^3$, i.e.

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta.$$

   Prove that the Laplace operator Δ can be expressed by

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2}.$$

2. Classify homogeneous harmonic polynomials in $\mathbb{R}^3$ by the following steps. Suppose that u is a homogeneous harmonic polynomial of degree m in $\mathbb{R}^3$. Set $u = r^m Q_m(\theta, \phi)$ for some function Q_m defined on S^2.

   i. Prove that Q_m satisfies

$$m(m+1)Q_m + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Q_m}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Q_m}{\partial\phi^2} = 0.$$

   ii. Prove that if Q_m is of the form $f(\theta)g(\phi)$, then

$$Q_m(\theta, \phi) = (A\cos(k\phi) + B\sin(k\phi)) f_{m,k}(\cos\theta),$$

   where

$$f_{m,k}(\mu) = (1 - \mu^2)^{\frac{k}{2}} \frac{d^{m+k}}{d\mu^{m+k}} (1 - \mu^2)^m$$

   for $\mu \in [-1, 1]$ and $k = 0, 1, \ldots, m$.

Solution

1. Note that

$$r^2 = x^2 + y^2 + z^2, \quad r^2 \sin^2\theta = x^2 + y^2, \quad \tan\phi = y/x.$$

   With this,

$$r_x = \frac{x}{r} = \sin\theta\cos\phi, \quad r_y = \frac{y}{r} = \sin\theta\sin\phi, \quad r_z = \frac{z}{r} = \cos\theta.$$

   Next, $x\sin^2(\theta) + r^2\sin(\theta)\cos(\theta)\,\theta_x = x$, hence

$$\theta_x = \frac{x\cos\theta}{r^2\sin\theta} = \frac{1}{r}\cos\theta\cos\phi.$$

Similarly,

$$\theta_y = \frac{y\cos\theta}{r^2\sin\theta} = \frac{1}{r}\cos\theta\sin\phi, \qquad \theta_z = -\frac{z\sin\theta}{r^2\cos\theta} = -\frac{1}{r}\sin\theta.$$

Finally, $\sec^2(\phi)\,\phi_x = -y/x^2$, hence

$$\phi_x = -\frac{y\cos^2\phi}{x^2} = -\frac{1}{r}\frac{\sin\phi}{\sin\theta}.$$

Similarly,

$$\phi_y = -\frac{\cos^2\phi}{x} = -\frac{1}{r}\frac{\cos\phi}{\sin\theta}, \qquad \phi_z = 0.$$

In summary,

$$\boxed{\begin{array}{lll} r_x = \sin\theta\cos\phi, & r_y = \sin\theta\sin\phi, & r_z = \cos\theta \\ \theta_x = \frac{1}{r}\cos\theta\cos\phi, & \theta_y = \frac{1}{r}\cos\theta\sin\phi, & \theta_z = -\frac{1}{r}\sin\theta \\ \phi_x = -\frac{1}{r}\frac{\sin\phi}{\sin\theta}, & \phi_y = -\frac{1}{r}\frac{\cos\phi}{\sin\theta}, & \phi_z = 0. \end{array}}$$

Now, recall that

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j},$$

hence

$$\frac{\partial^2}{\partial x_i^2} = \sum_j \frac{\partial}{\partial x_i}\left[\frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}\right] = \sum_j \frac{\partial^2 y_j}{\partial x_i^2}\frac{\partial}{\partial y_j} + \sum_{jk}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\frac{\partial^2}{\partial y_j\partial y_k}.$$

Thus,

$$\begin{aligned} \Delta = \sum_i \frac{\partial^2}{\partial x_i^2} &= \sum_{ij}\frac{\partial^2 y_j}{\partial x_i^2}\frac{\partial}{\partial y_j} + \sum_{ijk}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\frac{\partial^2}{\partial y_j\partial y_k} \\ &= \sum_{ij}\left(\frac{\partial^2 y_j}{\partial x_i^2}\right)\frac{\partial}{\partial y_j} + \sum_{ij}\left(\frac{\partial y_j}{\partial x_i}\right)^2\frac{\partial^2}{\partial y_j^2} + 2\sum_{\substack{i\\ j<k}}\left(\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\right)\frac{\partial^2}{\partial y_j\partial y_k}. \end{aligned}$$

In our case, first note that

$$r_x\theta_x + r_y\theta_y + r_z\theta_z = \theta_x\phi_x + \theta_y\phi_y + \theta_z\phi_z = \phi_x r_x + \phi_y r_y + \phi_z r_z = 0,$$

so the $\partial^2/\partial r\,\partial\theta$, $\partial^2/\partial\theta\,\partial\phi$, and $\partial^2/\partial\phi\,\partial r$ terms vanish. This leaves

$$\Delta = \sum_{ij} \left[\left(\frac{\partial^2 y_j}{\partial x_i^2} \right) \frac{\partial}{\partial y_j} + \left(\frac{\partial y_j}{\partial x_i} \right)^2 \frac{\partial^2}{\partial y_j^2} \right].$$

Next, note that

$$r_x^2 + r_y^2 + r_z^2 = 1, \quad \theta_x^2 + \theta_y^2 + \theta_z^2 = \frac{1}{r^2}, \quad \phi_x^2 + \phi_y^2 + \phi_z^2 = \frac{1}{r^2 \sin^2 \theta},$$

which are the coefficients of $\partial^2/\partial r^2$, $\partial^2/\partial \theta^2$, and $\partial^2/\partial \phi^2$ respectively. Thus,

$$\begin{aligned} \Delta &= \left[\sum_{ij} \left(\frac{\partial^2 y_j}{\partial x_i^2} \right) \frac{\partial}{\partial y_j} \right] + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \\ &= \left[\sum_j \omega_j \frac{\partial}{\partial y_j} \right] + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \end{aligned}$$

To determine the coefficients ω_j, first observe that $\Delta(x^2 + y^2 + z^2) = 6$. This is also precisely $\Delta r^2 = 2r\omega_r + 2$, hence $\omega_r = 2/r$. Next, observe that $\Delta(x^2 + y^2) = 4$, and this is also precisely

$$\begin{aligned} \Delta(r^2 \sin^2 \theta) &= 4 \sin^2 \theta + 2r^2 \omega_\theta \sin \theta \cos \theta + 2 \sin^2 \theta + 4(1 - 2 \sin^2 \theta) \\ &= -2 \sin^2 \theta + 2r^2 \omega_\theta \sin \theta \cos \theta + 4, \end{aligned}$$

hence $\omega_\theta = \cos \theta / r^2 \sin \theta$. Finally, observe that

$$\Delta \left(\frac{y}{x} \right) = \frac{2y}{x^3} = \frac{2 \sec^2 \phi \tan \phi}{r^2 \sin^2 \theta},$$

which is also precisely

$$\Delta \tan \phi = \omega_\phi \sec^2 \phi + \frac{2 \sec^2 \phi \tan \phi}{r^2 \sin^2 \theta},$$

hence $\omega_\phi = 0$. Putting everything together,

$$\begin{aligned} \Delta &= \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \end{aligned}$$

Remark: We have avoided calculating the nine terms r_{xx}, r_{yy}, r_{zz}, $\theta_{xx}, \theta_{yy}, \theta_{zz}$, $\phi_{xx}, \phi_{yy}, \phi_{zz}$!

2. i. Calculate

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}r^m Q_m(\theta,\phi)\right) = m(m+1)r^{m-2}Q_m,$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q_m}{\partial\theta}r^m\right) = \frac{r^{m-2}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q_m}{\partial\theta}\right),$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2 Q_m}{\partial\phi^2}r^m = \frac{r^{m-2}}{\sin^2\theta}\frac{\partial^2 Q_m}{\partial\phi^2}.$$

Adding everything together,

$$\Delta u = r^{m-2}\left[m(m+1)Q_m + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q_m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Q_m}{\partial\phi^2} = 0\right].$$

Setting this to zero, we must have

$$m(m+1)Q_m + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q_m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Q_m}{\partial\phi^2} = 0.$$

ii. Putting $Q_m = f(\theta)g(\phi)$ in the previous equation, we have

$$m(m+1)fg + \frac{1}{\sin\theta}(\sin\theta f')'g + \frac{1}{\sin^2\theta}fg'' = 0.$$

Rearranging,

$$\left[\frac{m(m+1)f + (\sin\theta f')'/\sin\theta}{f}\right]\sin^2\theta = -\frac{g''}{g} = k^2.$$

The last step equating both sides to a constant k^2 follows since they are independent functions of θ and ϕ respectively. Furthermore, g must be a periodic function of ϕ, which is possible only if the constant is positive, yielding periodic solutions

$$g(\phi) = A\cos(k\phi) + B\sin(k\phi).$$

Additionally, k must be an integer so that g is 2π periodic. Now,

$$\left[m(m+1) - \frac{k^2}{\sin^2\theta}\right]f + \frac{1}{\sin\theta}(\sin\theta f')' = 0.$$

Putting $\mu = \cos\theta$ and $h(\mu) = f(\theta)$, we have

$$\frac{df}{d\theta} = \frac{dh}{d\mu}\frac{d\mu}{d\theta} = -h'(\mu)\sin\theta = -h'(\mu)\sqrt{1-\mu^2},$$

hence

$$\frac{d}{d\theta}(\sin(\theta)f'(\theta)) = -\frac{d}{d\theta}((1-\mu^2)h'(\mu)) = -[2\mu h'(\mu) - (1-\mu^2)h''(\mu)]\sin(\theta).$$

Thus, our differential equation reduces to

$$\left[m(m+1) - \frac{k^2}{1-\mu^2}\right] h - 2\mu h' + (1-\mu^2)h'' = 0.$$

Rewrite this as the Sturm-Liouville problem

$$m(m+1)h + \left((1-\mu^2)h'\right)' = \frac{k^2}{1-\mu^2} h,$$

with $h(-1) = h(1)$

Problem 3

Solve the following Cauchy problem.

$$\begin{aligned} u_{tt} - u_{xx} &= 0, && 0 < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), && 0 \le x < \infty, \\ u_t(x,0) &= g(x), && 0 \le x < \infty, \end{aligned}$$

where $f \in C^2[0,\infty)$ and $g \in C^1[0,\infty)$ satisfy the compatibility conditions $f(0) = f''(0) = g(0) = 0$.

Solution

Extend f and g as odd functions on $\mathbb{R}$. Then, d'Alembert's formula yields

$$u(x,t) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds.$$

Note that the compatibility conditions $f(0) = g(0) = 0$ ensure that the extended versions of f and g are in $C^1(\mathbb{R})$, and the fact that $f''(0) = 0$ ensures that the extended version of f is in $C^2(\mathbb{R})$. Thus, u as obtained above is a classical solution of the given Cauchy problem.

Problem 4

We say that $v \in C^2(\bar{U})$ is subharmonic if $-\Delta v \le 0$ in U.

1. Prove for subharmonic v that for all $B(x,r) \subset U$,

$$v(x) \le \frac{1}{m(B(x,r))} \int_{B(x,r)} v.$$

2. Prove that therefore,

$$\max_{\bar{U}} v = \max_{\partial U} v.$$

3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and $v = \phi(u)$. Prove that v is subharmonic.

4. Prove that $v = |Du|^2$ is subharmonic when u is harmonic.

Solution

1. Define

$$\varphi(r) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} v = \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} v(x + r\omega)\, d\sigma(\omega).$$

Then, calculate

$$\begin{aligned}
\varphi'(r) &= \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} Dv(x + r\omega) \cdot \omega \, d\sigma(\omega) \\
&= \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} Dv \cdot \nu \, d\sigma \\
&= \frac{1}{\sigma(\partial B(x,r))} \int_{B(x,r)} \Delta v \, dy
\end{aligned}$$

The last step follows from Gauss's Divergence Theorem. Now, $\Delta v \geq 0$, hence $\varphi'(r) \geq 0$. Furthermore, $\varphi(r) \to v(x)$ as $r \to 0$, since

$$\min_{\partial B(x,r)} v \leq \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} v \leq \max_{\partial B(x,r)} v,$$

and both $\min_{\partial B(x,r)} v$, $\max_{\partial B(x,r)} v \to v(x)$ as $r \to 0$ by continuity. Thus, $\varphi(r) \geq v(x)$ for all $r > 0$. With this, for r such that $B(x,r) \subset U$, we have

$$\begin{aligned}
\frac{1}{m(B(x,r))} \int_{B(x,r)} v &= \frac{1}{m(B(x,r))} \int_0^r \int_{S^{n-1}} v(x + t\omega) t^{n-1} \, d\sigma(\omega)\, dt \\
&= \frac{\sigma(S^{n-1})}{\omega(n) r^n} \int_0^r \varphi(t)\, t^{n-1}\, dt \\
&\geq \frac{n}{r^n} \int_0^r v(x)\, t^{n-1}\, dt \\
&= v(x).
\end{aligned}$$

2. Suppose that v attains its maximum M at a point x_0 in the interior of U. Then, for all $x \in U$ such that $v(x) = M$, we have $v(x) - v(y) \geq 0$ for all $y \in \bar{U}$, hence

$$0 \leq \frac{1}{m(B(x,r))} \int_{B(x,r)} v(x) - v(y)\, dy = v(x) - \frac{1}{m(B(x,r))} \int_{B(x,r)} v \leq 0.$$

The last inequality follows from the previous result. This forces

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} v(x) - v(y)\, dy = 0$$

where $B(x,r) \subset U$, hence $v = v(x) = M$ on $B(x,r)$. Thus, we have shown that the set $v^{-1}(M)$ is both open (previous argument) and closed (continuity of v) in the connected component of U containing x_0. Since it is non-empty (the point $x_0 \in v^{-1}(M)$), $v^{-1}(M)$

must be the entirety of that connected component of U. Thus, by continuity, v must attain M at some boundary point of U as well.

3. Calculate

$$\Delta v = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\phi'(u)\frac{\partial u}{\partial x_i}\right) = \sum_{i=1}^{n}\left[\phi''(u)\left(\frac{\partial u}{\partial x_i}\right)^2 + \phi'(u)\frac{\partial^2 u}{\partial x_i^2}\right].$$

Using $\Delta u = 0$ and $\phi'' \geq 0$ via convexity, we have

$$\Delta v = \phi''(u)\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 + \phi'(u)\cancel{\Delta u} \geq 0,$$

whence $-\Delta v \leq 0$. Thus, v is subharmonic.

4. Note that since u is harmonic, so are the functions $u_i := \partial u/\partial x_i$. Now,

$$v = |Du|^2 = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 = \sum_{i=1}^{n} u_i^2,$$

so

$$\begin{aligned}
\Delta v &= \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}\left(\sum_{i=1}^{n} u_i^2\right) \\
&= \sum_{ij}\frac{\partial^2 u_i^2}{\partial x_j^2} \\
&= \sum_{ij}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i^2}{\partial x_j}\right) \\
&= \sum_{ij}\frac{\partial}{\partial x_j}\left(2u_i\frac{\partial u_i}{\partial x_j}\right) \\
&= 2\sum_{ij}\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2 + u_i\frac{\partial^2 u_i}{\partial x_j^2}\right] \\
&= 2\sum_{ij}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + 2\sum_{i=1}^{n} u_i\left[\sum_{j=1}^{n}\frac{\partial^2 u_i}{\partial x_j^2}\right] \\
&= 2\sum_{ij}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + 2\sum_{i=1}^{n} u_i\cancel{\Delta u_i} \\
&\geq 0.
\end{aligned}$$

Thus, $-\Delta v \leq 0$, so v is subharmonic.

Problem 5

Let u be the solution of

$$\begin{aligned} \Delta u &= 0, && \text{in } \mathbb{R}^n_+, \\ u &= g, && \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

given by Poisson's formula for the half-space. Assume that g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show that Du is not bounded near $x = 0$.

Solution

Using Poisson's formula, write

$$\begin{aligned} u(x) &= \frac{2x_n}{n\omega(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{\|x-y\|^n}\, dy \\ &= \frac{2x_n}{n\omega(n)} \int_{\partial\mathbb{R}^{n-1}} \frac{\tilde{g}(z)}{\left(\|\tilde{x}-z\|^2 + x_n^2\right)^{n/2}}\, dz. \end{aligned}$$

Here, we denote $\tilde{x} = (x_1, \ldots, x_{n-1})$ and $\tilde{g}(z) = g(z_1, \ldots, z_{n-1}, 0)$. Thus, $u(0) = 0$, and

$$u(\lambda e_n) = \frac{2\lambda}{n\omega(n)} \int_{\partial\mathbb{R}^{n-1}} \frac{\tilde{g}(z)}{\left(\|z\|^2 + \lambda^2\right)^{n/2}}\, dz.$$

With this, we estimate

$$\begin{aligned} \frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{2}{n\omega(n)} \int_{\partial\mathbb{R}^{n-1}} \frac{\tilde{g}(z)}{\left(\|z\|^2 + \lambda^2\right)^{n/2}}\, dz \\ &= \frac{2}{n\omega(n)} \int_{S^{n-2}} \int_0^\infty \frac{\tilde{g}(r\sigma)}{(r^2+\lambda^2)^{n/2}} r^{n-2}\, dr\, d\sigma \\ &= \frac{2}{n\omega(n)} \int_{S^{n-2}} \left(\int_0^1 + \int_1^\infty \right) \frac{\tilde{g}(r\sigma)}{(r^2+\lambda^2)^{n/2}} r^{n-2}\, dr\, d\sigma \end{aligned}$$

Using the boundedness of g, let $|g| < M$. Note that

$$\left| \int_1^\infty \frac{\tilde{g}(r\sigma)}{(r^2+\lambda^2)^{n/2}}\, dr \right| < M \int_1^\infty \frac{dr}{r^2} < \infty,$$

so

$$\frac{2}{n\omega(n)} \int_{S^{n-2}} \int_1^\infty \frac{\tilde{g}(r\sigma)}{(r^2+\lambda^2)^{n/2}} r^{n-2}\, dr\, d\sigma < \infty.$$

For the remaining piece, use $g = r$ when $r \leq 1$ to write

$$\frac{2}{n\omega(n)}\int_{S^{n-2}}\int_0^1 \frac{\tilde{g}(r\sigma)}{(r^2+\lambda^2)^{n/2}} r^{n-2}\,dr\,d\sigma = \frac{2}{n\omega(n)}\int_{S^{n-2}}\int_0^1 \frac{r^{n-1}}{(r^2+\lambda^2)^{n/2}}\,dr\,d\sigma$$
$$= \frac{2(n-1)\omega(n-1)}{n\omega(n)}\int_0^1 \frac{r^{n-1}}{(r^2+\lambda^2)^{n/2}}\,dr.$$

Now, calculate

$$\frac{\partial}{\partial\lambda}\frac{r^{n-1}}{(r^2+\lambda^2)^{n/2}} = -\frac{r^{n-1}}{(r^2+\lambda^2)^{n}}\cdot\frac{n}{2}(r^2+\lambda^2)^{n/2-1}\cdot 2\lambda < 0$$

when $\lambda > 0$. Thus, the functions $r \mapsto r^{n-1}/(r^2+\lambda^2)^{n/2}$ are pointwise monotonically increasing on $(0,1)$ as λ decreases to 0. Therefore, the Monotone Convergence Theorem gives

$$\lim_{\lambda\to 0}\int_0^1 \frac{r^{n-1}}{(r^2+\lambda^2)^{n/2}}\,dr = \int_0^1 \lim_{\lambda\to 0}\frac{r^{n-1}}{(r^2+\lambda^2)^{n/2}}\,dr = \int_0^1 \frac{dr}{r} = \infty.$$

Thus, we obtain

$$\frac{\partial u}{\partial x_n}(0) = \lim_{\lambda\to 0}\frac{u(\lambda e_n)-u(0)}{\lambda} = \infty,$$

from which Du must be unbounded near 0.