MA2202: Probability I

Random vectors

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Definition 4.1 (Random vector)**.** A random vector $\boldsymbol{X} \colon \Omega \to \mathbb{R}^n$ is a tuple of random variables $X_i \colon \Omega \to \mathbb{R}$.

Definition 4.2 (Joint cumulative distribution function)**.** The joint cumulative distribution function of a random vector $\boldsymbol{X}$ is the map $F_{\boldsymbol{X}} \colon \mathbb{R}^n \to [0, 1]$, given as

$$F_{\boldsymbol{X}}(\boldsymbol{s}) = P(X_1 \leq s_1, \dots, X_n \leq s_n).$$

Definition 4.3 (Joint probability mass function)**.** If X_i are discrete random variables, their joint probability mass function is the map $p_{\boldsymbol{X}} \colon \mathbb{R}^n \to [0, 1]$,

$$p_{\boldsymbol{X}}(\boldsymbol{s}) = P(X_1 = s_1, \dots, X_n = s_n).$$

Definition 4.4 (Joint probability density function)**.** Suppose that

$$F_{\boldsymbol{X}}(\boldsymbol{s}) = \int_{-\infty}^{s_n} \cdots \int_{-\infty}^{s_1} f_{\boldsymbol{X}}(t_1, \dots, t_n)\, dt_1 \dots\, dt_n,$$

then $f_{\boldsymbol{X}} \colon \mathbb{R}^n \to [0, 1]$ is the probability density function corresponding to the joint cumulative distribution function $F_{\boldsymbol{X}}$.

Remark. If $f_{\boldsymbol{X}}$ is continuous, then

$$f_{\boldsymbol{X}} = \frac{\partial F_{\boldsymbol{X}}(t_1, \dots, t_n)}{\partial t_1 \dots \partial t_n}.$$

Definition 4.5 (Joint moment generating function)**.** Let $\boldsymbol{X}$ be a random vector. Then, its joint moment generating function is defined as

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = E\left[e^{\boldsymbol{t}^\top \boldsymbol{X}}\right] = E\left[e^{t_1 X_1 + \cdots + t_n X_n}\right].$$

Remark. If $X_1, \ldots, X_n$ are independent, then

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = \prod M_{X_i}(t_i).$$

Theorem 4.1. *If X and Y are independent continuous random variables, then the probability density function of their sum is the convolution $f_{X+Y} = f_X * f_Y$,*

$$f_{X+Y}(x) = \int_{\mathbb{R}} f_X(x-t)\, f_Y(t)\, dt.$$

Example. When X and Y are identical and uniform on $[0,1]$, then

$$f_{X+Y}(x) = \int_0^1 f(x-t)\, dt = \begin{cases} x, & \text{if } a \in [0,1], \\ 2-x, & \text{if } a \in [1,2], \\ 0, & \text{otherwise }. \end{cases}$$

Also,

$$M_{X+Y}(t) = (M(t))^2 = \frac{1}{t^2}(e^t - 1)^2.$$

Definition 4.6 (Conditional distribution)**.** Let X and Y be two discrete random variables. We write

$$P(X = s \,|\, Y = t) = \frac{P(X = s, Y = t)}{P(Y = t)}$$

for $P(Y = t) > 0$. We also have

$$P(X \leq s \,|\, Y = t) = \sum_{r \leq s} P(X = r \,|\, Y = t).$$