MA 1101 : Mathematics I

Satvik Saha, `19MS154` October 20, 2019

Solution 1.

(i) Let $S \subseteq \mathbb{R}$ be a finite set with $n \in \mathbb{N}$ elements. We claim that S has no limit points. We enumerate the elements of S as $x_1, x_2, \ldots, x_n$. Let $a \in \mathbb{R}$.

(a) If $a \notin S$, let us choose $|x_i - a| > \epsilon_i > 0$, for all $i = 1, 2, \ldots, n$. We set $A_i = (a - \epsilon_i, a + \epsilon_i)$ to be the ϵ_i neighbourhood of a. If $x_i > a$, we have $x_i = a + (x_i - a) > a + \epsilon_i$, and if $x_i < a$, we have $x_i = a - (a - x_i) < a - \epsilon_i$. Thus, $x_i \notin A_i$.

We set $A = \bigcap A_i$. Since A is the intersection of a finite number of non-empty open intervals, A is also a non-empty open interval.

Thus, $x_i \notin A$ for all $x_i \in S$, i.e. $S \cap A = \emptyset$. Thus, there is no $x \in S$ within the $\epsilon = \min \epsilon_i > 0$ neighbourhood of a. Hence, a is not a limit point.

(b) If $a \in S$, without loss of generality, we set $a = x_1$. We again choose $|x_i - a| > \epsilon_i > 0$, for all $i = 2, 3, \ldots, n$. We set $A_i = (a - \epsilon_i, a + \epsilon_i)$ to be the ϵ_i neighbourhood of a. Clearly, $a = x_1 \in A_1$. Arguing as before, $x_i \notin A_i$ for $i = 2, 3, \ldots, n$.

We set $A = \bigcap A_i$. Thus, $a = x_1 \in A$ and $x_i \notin A$ for $i \neq 1$, i.e. $S \cap A = \{a\}$ Thus, the only $x \in S$ within the $\epsilon = \min \epsilon_i$ neighbourhood of a is a. Hence, a is not a limit point.

Therefore, any finite set S has no limit points. $\square$

(ii) Let $S = (0, \infty) \subseteq \mathbb{R}$. We claim that $[0, \infty)$ is the set of all limit points of S. Let $a \in \mathbb{R}$.

(a) If $a \in [0, \infty)$, let $\epsilon > 0$ be given. Thus, $a \geq 0 \Rightarrow a + \epsilon/2 > 0$, and $a - \epsilon < a + \epsilon/2 < a + \epsilon$. Hence, we have found $x = a + \epsilon/2 \in S$ such that $x \in (a - \epsilon, a + \epsilon)$ and $x \neq a$. Thus, a is a limit point.

(b) If $a \notin [0, \infty)$, i.e. $a < 0$, we choose $\epsilon = -a$. Hence, $(a - \epsilon, a + \epsilon) \cap S = (2a, 0) \cap (0, \infty) = \emptyset$. Thus, a is not a limit point.

This proves our claim. $\square$

(iii) Let $S = [1, 2) \cup \{3\}$. We claim that $[1, 2]$ is the set of all limit points of S. Let $a \in \mathbb{R}$.

(a) If $a \in [1, 2]$, let $\epsilon > 0$ be given. We set $\epsilon' = \min\{\epsilon, a - 1, 2 - a\}$, and $x = a + \epsilon'/2$. Thus, $x > a \geq 1$ and $x < a + \epsilon' \leq a + (2 - a) = 2$. Also, $-\epsilon < \epsilon'/2 < \epsilon$. Hence, we have found $x \in [1, 2] \subset S$ such that $x \in (a - \epsilon, a + \epsilon)$ and $x \neq a$. Thus, a is a limit point.

(b) If $a \in \{3\}$, i.e. $a = 3$, we choose $\epsilon = 1/2 > 0$. Hence, $(a - \epsilon, a + \epsilon) \cap S = (2.5, 3.5) \cap ([1, 2) \cup \{3\}) = \{3\}$. Hence, $x \in S$ and $x \in (a - \epsilon, a + \epsilon)$ forces $x = a$. Thus, a is not a limit point.

(c) If $a < 1$, we choose $\epsilon = 1 - a$. Hence, $(a - \epsilon, a + \epsilon) \cap S = (2a - 1, 1) \cap ([1, 2) \cup \{3\}) = \emptyset$. Thus, a is not a limit point.

(d) If $2 < a < 3$, we choose $\epsilon = \frac{1}{2} \min\{a - 2, 3 - a\}$. Thus, $a - \epsilon > a - 2\epsilon \geq a - (a - 2) = 2$ and $a + \epsilon < a + 2\epsilon \leq a + (3 - a) = 3$. Therefore, $(a - \epsilon, a + \epsilon) \subset (2, 3)$. Hence, $(a - \epsilon, a + \epsilon) \cap S = \emptyset$. Thus, a is not a limit point.

(e) If $a > 3$, we choose $\epsilon = a - 3$. Hence, $(a - \epsilon, a + \epsilon) \cap S = (3, 2a - 3) \cap S = \emptyset$. Thus, a is not a limit point.

This proves our claim. $\square$

(iv) Let $S = [1, 2) \cup (2, 3)$. We claim that $[1, 3]$ is the set of all limit points of S. Let $a \in \mathbb{R}$.

(a) If $a \in [1,3]$, let $\epsilon > 0$ be given. We set $\epsilon' = \min\{\epsilon, a-1, 3-a\}$, and $x_- = a-\epsilon'/2$, $x^+ = a+\epsilon'/2$. Thus,

$$x_- > a - \epsilon' \geq a - (a-1) = 1,$$
$$x_- < a \leq 3,$$
$$x_+ > a \geq 1,$$
$$x_+ < a + \epsilon' \leq a + (3-a) = 3.$$

Thus, $x_-, x_+ \in (1,3)$. Since $x_- < x_+$, at least one of them is $x \neq 2$. Also, $-\epsilon < -\epsilon'/2 < \epsilon'/2 < \epsilon$. Hence, we have found $x \in (1,3) \setminus \{2\} \subset S$ such that $x \in (a-\epsilon, a+\epsilon)$ and $x \neq a$. Thus, a is a limit point.

(b) If $a < 1$, we choose $\epsilon = 1 - a$. Hence, $(a-\epsilon, a+\epsilon) \cap S = (2a-1, 1) \cap S = \emptyset$. Thus, a is not a limit point.

(c) If $a > 3$, we choose $\epsilon = a - 3$. Hence, $(a-\epsilon, a+\epsilon) \cap S = (3, 2a-3) \cap S = \emptyset$. Thus, a is not a limit point.

This proves our claim. □

(v) Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. We claim that 0 is the only limit point of S. Let $a \in \mathbb{R}$.

(a) If $a = 0$, let $\epsilon > 0$ be given. By the *Archimedean Property* of the reals, we choose $n \in \mathbb{N}$ such that $n\epsilon > 1$. Thus, $\frac{1}{n} \in S$ and $\frac{1}{n} \in (0-\epsilon, 0+\epsilon)$. Thus, 0 is a limit point.

(b) If $a \geq 1$, we choose $\epsilon = a - 1$. Thus, $(a-\epsilon, a+\epsilon) \cap S = (1, 2a-1) \cap S = \emptyset$, since $S \subset (0,1]$. Thus, a is not a limit point.

(c) If $a \in S \setminus \{1\}$, we find $n \in \mathbb{N}$ such that $a = \frac{1}{n}$. We choose $\frac{1}{n} - \frac{1}{n+1} > \epsilon > 0$. Thus, $a - \epsilon > \frac{1}{n+1}$ and $a + \epsilon = \frac{2}{n} - \frac{1}{n+1} < \frac{1}{n-1}$, since $n^2 - 1 < n^2$. Hence, $S \cap (a-\epsilon, a+\epsilon) = \{a\}$. Thus, a is not a limit point of S.

(d) If $a \in (0,1] \setminus S$, we find $n \in \mathbb{N}$ such that $\frac{1}{n+1} < a < \frac{1}{n}$. We choose $\min\{\frac{1}{\frac{1}{n}-a, a-\frac{1}{n+1}}\} > \epsilon > 0$. Thus, $a - \epsilon > a - (a - \frac{1}{n+1}) = \frac{1}{n+1}$ and $a + \epsilon < a + (\frac{1}{n} - a) = \frac{1}{n}$. Hence, $S \cap (a-\epsilon, a+\epsilon) = \emptyset$. Thus, a is not a limit point.

(e) If $a < 0$, we choose $\epsilon = -a$. Hence, $S \cap (a-\epsilon, a+\epsilon) = S \cap (2a, 0) = \emptyset$.

Thus proves our claim. □

(vi) Let $S = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$. We claim that $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is the set of all limit points of S. Let $a \in \mathbb{R}$, $S' = \{\frac{1}{n} : n \in \mathbb{N}\}$.

(a) If $a = 0$, let $\epsilon > 0$ be given. We choose $n \in \mathbb{N}$ such that $n\epsilon > 2$. Thus, $\frac{2}{n} = \frac{1}{n} + \frac{1}{n} \in S$ and $\frac{1}{n} + \frac{1}{n} \in (0-\epsilon, 0+\epsilon)$. Thus, 0 is a limit point.

(b) If $a \in S'$, let $\epsilon > 0$ be given. We find $n \in \mathbb{N}$ such that $a = \frac{1}{n}$. We choose $k \in \mathbb{N}$ such that $k\epsilon > 1$. Thus, $\frac{1}{n} + \frac{1}{k} \in S$ and $a < \frac{1}{n} + \frac{1}{k} < \frac{1}{n} + \epsilon$, so $\frac{1}{n} + \frac{1}{k} \in (a-\epsilon, a+\epsilon)$. Thus, a is a limit point.

(c) If $a \notin S', a > 0$, we choose an $\epsilon > 0$ such that $S' \cap (a-\epsilon, a+\epsilon) = \emptyset$. We can do so since a is not a limit point of S'. Also, minimize ϵ such that $a - \epsilon > 0$.

Consider the elements $x = \frac{1}{m} + \frac{1}{n} \in S \cap (a-\epsilon/2, a+\epsilon/2)$, where $m, n \in \mathbb{N}$. Without loss of generality, let $m \leq n$. Thus,

$$a - \frac{\epsilon}{2} < \frac{1}{n} + \frac{1}{m} < a + \frac{\epsilon}{2}$$

Since $(a-\epsilon, a+\epsilon)$ has no element of the from $\frac{1}{k}$ where $k \in \mathbb{N}$,

$$\frac{1}{n} \leq \frac{1}{m} \leq a - \epsilon$$

Also,

$$a - \frac{\epsilon}{2} < \frac{1}{n} + \frac{1}{m} \leq \frac{2}{m}$$

Thus,

$$\frac{1}{m} > \frac{1}{2}(a - \frac{\epsilon}{2})$$

This means that there are only a finite number of m. Also,

$$a - \frac{\epsilon}{2} < \frac{1}{n} + \frac{1}{m} < \frac{1}{n} + a - \epsilon$$

$$\frac{1}{n} > \frac{\epsilon}{2}$$

Thus, there are only a finite number of n. This means that there are a finite number of x. Hence, $S \cap (a - \epsilon/2, a + \epsilon/2)$ is a finite set. Hence, a is not a limit point.

(d) If $a < 0$, we choose $\epsilon = -a$. Hence, $S \cap (a - \epsilon, a + \epsilon) = S \cap (2a, 0) = \emptyset$.

This proves our claim. $\square$

Solution 2. Note that for any $x \in \mathbb{R}$, x is trivially a limit point of $\mathbb{R}$, since every $\epsilon > 0$ neighbourhood of $\mathbb{R}$ contains infinitely many real numbers other than x. In addition, removing a finite number of points from $\mathbb{R}$ means that x is still a limit point of $\mathbb{R}$.

(i) We have $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := \lfloor x \rfloor$. We claim that $\lim_{x \to 0} f(x)$ does not exist.

Suppose not, i.e. $\lim_{x \to 0} f(x) = L$. We find δ such that

$$0 < |x - 0| < \delta \implies |f(x) - L| < \frac{1}{4}$$

We choose $0 < x_0 < \min\{1, \delta\}$. Thus, $f(x_0) - f(-x_0) = 1$. Now,

$$\begin{aligned} 1 &= |f(x_0) - f(-x_0)| \\ &= |(f(x_0) - L) - (f(-x_0) - L)| \\ &\leq |f(x_0) - L| + |f(-x_0) - L| \\ &< \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

This is a contradiction, thus proving our claim. $\square$

(ii) We have $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := \lfloor x \rfloor - \lfloor x/3 \rfloor$. We claim that $\lim_{x \to 0} f(x) = 0$.

Let $\epsilon > 0$ be given. We set $\delta = \min\{1, \epsilon\}$.

Then, for all $x \in \mathbb{R}$ satisfying $0 < |x - 0| < \delta$, we have $|\lfloor x \rfloor - \lfloor x/3 \rfloor - 0| = 0 < \epsilon$ This proves our claim. $\square$

(iii) We have $f\colon \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) = \frac{x^3 - 8}{x - 2}$. We claim that $\lim_{x \to 2} f(x) = 12$.

Let $\epsilon > 0$ be given. We set $\delta = \min\{1, \epsilon/7\}$.

Then, for all $x \in \mathbb{R} \setminus \{2\}$ satisfying $0 < |x - 2| < \delta$, we have

$$\begin{aligned} \left| \frac{x^3 - 8}{x - 2} - 12 \right| &= \left| x^2 + 2x + 4 - 12 \right| \\ &= \left| x^2 + 2x - 8 \right| \\ &= |(x - 2)(x + 4)| \\ &= |x - 2| \, |x - 2 + 6| \\ &\leq |x - 2| \, (|x - 2| + 6) \\ &< \delta(\delta + 6) \\ &\leq \frac{\epsilon}{7}(1 + 6) \\ &= \epsilon \end{aligned}$$

This proves our claim. $\square$

(iv) We have $f\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x) := x\sin\frac{1}{x}$. We claim that $\lim_{x\to 0} f(x) = 0$.

Let $\epsilon > 0$ be given. We set $\delta = \epsilon$.

Then, for all $x \in \mathbb{R}\setminus\{0\}$ satisfying $0 < |x-0| < \delta$, we have $\left|x\sin\frac{1}{x}\right| \le |x| < \epsilon$ This proves our claim. □

(v) We have $f\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x) := x/|x|$. We claim that $\lim_{x\to 0} f(x)$ does not exist.

Suppose not, i.e. $\lim_{x\to 0} f(x) = L$. We find δ such that

$$0 < |x-0| < \delta \implies |f(x)-L| < \frac{1}{2}$$

Note that $f(x) - f(-x) = 2$. Thus,

$$\begin{aligned} 2 &= |f(\delta/2) - f(-\delta/2)| \\ &= |(f(\delta/2) - L) - (f(-\delta/2) - L)| \\ &\le |f(\delta/2) - L| + |f(-\delta/2) - L| \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$

This is a contradiction, thus proving our claim. □

Solution 2. Let $\emptyset \ne D \subseteq \mathbb{R}$, $f, g\colon D \to R$ and let a be a limit point of D. Let $\lim_{x\to a}$ and $\lim_{x\to a} g(x)$ exist. We write

$$\lim_{x\to a} f(x) := L, \qquad \lim_{x\to a} g(x) := M.$$

(i) We claim that $\lim_{x\to a}(f(x) + g(x)) = L + M$.

Let $\epsilon > 0$ be given. We find δ_f, δ_g such that for all $x \in D$,

$$0 < |x-a| < \delta_f \implies |f(x) - L| < \epsilon/2,$$

$$0 < |x-a| < \delta_g \implies |g(x) - M| < \epsilon/2.$$

We set $\delta = \min\{\delta_f, \delta_g\}$. Then, for all $x \in D$ satisfying $0 < |x-a| < \delta$, we have

$$\begin{aligned} |(f(x)+g(x)) - (L+M)| &= |(f(x)-L) + (g(x)-M)| \\ &\le |f(x)-L| + |g(x)-M| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

This proves our claim. □

(ii) We claim that for all $\alpha \in \mathbb{R}$, $\lim_{x\to a}(\alpha f(x)) = \alpha L$.

Let $\epsilon > 0$ be given. If $\alpha \ne 0$, we find δ_f such that for all $x \in D$,

$$0 < |x-a| < \delta_f \implies |f(x) - L| < \epsilon/|\alpha|.$$

We set $\delta = \delta_f$. Then, for all $x \in D$ satisfying $0 < |x-a| < \delta$, we have

$$\begin{aligned} |\alpha f(x) - \alpha L| &= |\alpha||f(x) - L| \\ &< |\alpha|\frac{\epsilon}{|\alpha|} \\ &= \epsilon \end{aligned}$$

If $\alpha = 0$, we trivially have

$$0 < |x-a| < \delta = \epsilon \implies |\alpha f(x) - \alpha L| = 0 < \epsilon.$$

This proves our claim. □

(iii) We claim that $\lim_{x\to a} f(x)g(x) = LM$. To prove this, we first show that $\lim_{x\to a}(f(x) - L)(g(x) - M) = 0$.

Let $\epsilon > 0$ be given. We find δ_f, δ_g such that for all $x \in D$,

$$0 < |x - a| < \delta_f \implies |f(x) - L| < \sqrt{\epsilon},$$

$$0 < |x - a| < \delta_g \implies |g(x) - M| < \sqrt{\epsilon}.$$

We set $\delta = \min\{\delta_f, \delta_g\}$. Then, for all $x \in D$ satisfying $0 < |x - a| < \delta$, we have

$$\begin{aligned} |(f(x) - L)(g(x) - M) - 0| &= |f(x) - L||g(x) - M| \\ &< \sqrt{\epsilon}\sqrt{\epsilon} \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{x\to a}(f(x) - L)(g(x) - M) = 0$.

We now show that for a constant function $h\colon D \to \mathbb{R}$, $h(x) = k$, we have $\lim_{x\to a} h(x) = k$.

Let $\epsilon > 0$ be given. We set $\delta = \epsilon$. Then, for all $x \in D$ satisfying $0 < |x - a| < \delta$, we have $|h(x) - k| \; = \; 0 < \epsilon$.

Therefore,

$$\begin{aligned} 0 &= \lim_{x\to a}(f(x) - L)(g(x) - M) \\ &= \lim_{x\to a}(f(x)g(x) - Lg(x) - Mf(x) + LM) \\ &= \lim_{x\to a} f(x)g(x) - \lim_{x\to a} Lg(x) - \lim_{x\to a} Mf(x) + \lim_{x\to a} LM \\ &= \lim_{x\to a} f(x)g(x) - L\lim_{x\to a} g(x) - M\lim_{x\to a} f(x) + LM \\ &= \lim_{x\to a} f(x)g(x) - LM - ML + LM \\ &= \lim_{x\to a} f(x)g(x) - LM \end{aligned}$$

$$\lim_{x\to a} f(x)g(x) \;=\; LM$$

$\square$

(iv) We claim that if $M \neq 0$, $\lim_{x\to a} f(x)/g(x) = L/M$. To prove this, we first show that $\lim_{x\to a} 1/g(x) = 1/M$.

Let $\epsilon > 0$ be given. We find δ_1, δ_2 such that for all $x \in D$,

$$0 < |x - a| < \delta_1 \implies |g(x) - M| < \frac{1}{2}|M|,$$

$$0 < |x - a| < \delta_2 \implies |g(x) - M| < \frac{1}{2}\epsilon|M|^2.$$

We set $\delta = \min\{\delta_1, \delta_2\}$. Then, for all $x \in D$ satisfying $0 < |x - a| < \delta$, we have

$$\begin{aligned}
\frac{1}{2}|M| &> |g(x) - M| \\
&\geq ||g(x)| - |M|| \\
&\geq |M| - |g(x)| \\
|g(x)| &> \frac{1}{2}|M| > 0 \\
\frac{1}{|g(x)|} &< \frac{2}{|M|} \\
\left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \frac{|g(x) - M|}{|Mg(x)|} \\
&= |g(x) - M|\frac{1}{|M||g(x)|} \\
&< \frac{1}{2}\epsilon|M|^2\frac{2}{|M|^2} \\
&= \epsilon
\end{aligned}$$

Thus, $\lim_{x\to a} 1/g(x) = 1/M$. Therefore,

$$\begin{aligned}
\lim_{x\to a} f(x)g(x) &= \lim_{x\to a} f(x)\frac{1}{g(x)} \\
&= \lim_{x\to a} f(x) \lim_{x\to a} \frac{1}{g(x)} \\
&= \frac{L}{M}
\end{aligned}$$

$\square$