MA4104: Algebraic Topology

The fundamental group of $\mathbb{C} \times \mathbb{C} \setminus \Delta$.

Satvik Saha
19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

Consider the space $\mathbb{C} \times \mathbb{C} \setminus \Delta$, where $\Delta = \{(z, z) : z \in \mathbb{C}\}$. We may identify $\mathbb{R}^2 \cong \mathbb{C}$ via the usual map $(x, y) \mapsto z + iy$; with this, the space under question may be identified with

$$\mathbb{R}^4 \setminus \{(x, y, x, y) : x, y \in \mathbb{R}\} \;=\; \mathbb{R}^4 \setminus \operatorname{span}\{e_1 + e_3, e_2 + e_4\}.$$

The homeomorphism

$$\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4, \qquad (x, y, z, w) \mapsto (x + z, y + w, x - z, y - w)$$

restricts to the homeomorphism

$$\mathbb{R}^4 \setminus \operatorname{span}\{e_1, e_2\} \cong \mathbb{R}^4 \setminus \operatorname{span}\{e_1 + e_3, e_2 + e_4\}.$$

In other words, our original space is homeomorphic to

$$\mathbb{R}^4 \setminus (\mathbb{R}^2 \times \{0\} \times \{0\}).$$

However, this can be identified again with

$$(\mathbb{C} \times \mathbb{C}) \setminus (\mathbb{C} \times \{0\}) \;=\; \mathbb{C} \times (\mathbb{C} \setminus \{0\}).$$

Now, $\mathbb{C}$ is contractible, and $\mathbb{C} \setminus \{0\}$ deformation retracts to the unit circle S^1. With this, we have established the homotopy equivalence

$$\mathbb{C} \times \mathbb{C} \setminus \Delta \;\sim\; \{0\} \times S^1 \cong S^1.$$

In particular, this demonstrates that the space which we have been examining is path connected. As a result, we can safely discuss its first fundamental group without reference to a particular basepoint (if insisted upon, pick $(1, -1) \in \mathbb{C} \times \mathbb{C} \setminus \Delta$, which gets mapped to $e_3 \in \mathbb{R}^4 \setminus (\mathbb{R}^2 \times \{0\} \times \{0\})$, hence $(0, 1) \in \{0\} \times S^1$). Thus,

$$\pi_1(\mathbb{C} \times \mathbb{C} \setminus \Delta) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Identification of $\mathbb{C}$ with $\mathbb{R}^2$

We use the standard identification

$$\phi\colon \mathbb{R}^2 \to \mathbb{C}, \qquad x + iy \mapsto x + iy$$

and its inverse to interchangeably talk about $\mathbb{C}$ and $\mathbb{R}^2$ here.

Homeomorphism of $\mathbb{R}^4$

The map $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ described earlier can be written in the following manner.

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The 4×4 matrix here is of full rank; all four columns are orthogonal, hence linearly independent. As a result, φ is a bijective linear map, hence a homeomorphism.

By restricting φ to $\mathbb{R}^4 \setminus \operatorname{span}\{e_1, e_2\}$, we obtain a homeomorphism onto its image. Since $\varphi(e_1) = e_1 + e_3$ and $\varphi(e_2) = e_2 + e_4$, we have removed precisely $\operatorname{span}\{e_1 + e_3, e_2 + e_4\}$ from the image of φ. Thus,

$$\mathbb{R}^4 \setminus \operatorname{span}\{e_1, e_2\} \cong \mathbb{R}^4 \setminus \operatorname{span}\{e_1 + e_3, e_2 + e_4\}$$

via φ, as desired.

Deformation retracts

The deformation retract of $\mathbb{C}$ onto the point 0 looks like

$$h_1\colon [0, 1] \times \mathbb{C} \to \mathbb{C}, \qquad (t, z) \mapsto (1 - t)z.$$

Similarly, the deformation retract of $\mathbb{C} \setminus \{0\}$ onto the circle S^1 looks like

$$h_2\colon [0, 1] \times \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \qquad (t, z) \mapsto (1 - t)z + tz/|z|.$$

Note that $(1 - t)z + tz/|z| \neq 0$; if it were, then $|z| = t/(t - 1) < 0$, a contradiction.

These can be performed on $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$ one after another on the corresponding slots, or in one go as

$$h\colon [0, 1] \times \mathbb{C} \times \mathbb{C} \setminus \{0\} \to \mathbb{C} \times \mathbb{C} \setminus \{0\}, \qquad (t, z_1, z_2) \mapsto (t, (1 - t)z_1, (1 - t)z_2 + tz_2/|z_2|).$$

Note that $h(0, \cdot, \cdot) = \mathrm{id}_{\mathbb{C} \times \mathbb{C} \setminus \{0\}}$ and $h(1, z_1, z_2) = (0, z_2/|z_2|) \in \{0\} \times S^1$. Also, it is clear that each $h(t, \cdot, \cdot)$ fixes $\{0\} \times S^1$; when $z_2 \in S^1$, we have

$$(1 - t)z_2 + tz_2/|z_2| = (1 - t)z_2 + tz_2 = z_2.$$

Thus, h describes a deformation retraction of $\mathbb{C} \times \mathbb{C} \setminus \{0\}$ onto $\{0\} \times S^1$, which is homeomorphic to just S^1.

Path connectedness of a space and its deformation retract

Suppose that $h\colon I \times X \to X$ is a deformation retract of X onto $A \subseteq X$. Then, given $x \in X$, we have a path $h(\cdot, x)\colon I \to X$ joining $h(0, x) = x$ with $h(1, x) \in A$. If in addition we know that A is path connected, then given any $x, x' \in X$, we can pick a path γ joining $h(1, x)$ and $h(1, x')$ in A. Thus, $h(\cdot, x) * \gamma * \overline{h(\cdot, x')}$ describes a path joining x and x', proving that X is path connected.

* Analogy with $\mathbb{R}^3 \setminus (\mathbb{R} \times \{0\} \times \{0\})$

Here, we have shown that removing a 2-plane from $\mathbb{R}^4$ keeps it path connected. However, this is a bit difficult to visualize. An analogous construction involves removing a line from $\mathbb{R}^3$. Now, it is clear that this space is path connected; indeed, it deformation retracts to a circle once again. Additionally, it is easy to see that each based homotopy class of loops is completely determined by the number of times it winds around the line that has been removed.