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Approximating continuous functions by smooth functions:

The Stone-Weierstrass Theorem

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1 Sequences of functions

Definition 1.1. Let $\{f_n\}$ be a sequence of functions on a set E. We say that $\{f_n\}$ converges *pointwise* to a function f on E if the sequences $f_n(x) \to f(x)$ for every $x \in E$. We write $f_n \to f$ pointwise on E.

Definition 1.2. Let $\{f_n\}$ be a sequence of functions on a set E. We say that $\{f_n\}$ converges *uniformly* to a function f on E if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \epsilon$ for all $x \in E$, $n \geq n_0$. We write $f_n \to f$ uniformly on E.

Lemma 1.1. *Let $\mathscr{G}$ be a collection of functions on a set E, and let f be a function on E with the following property: given $\epsilon > 0$, there exists $g \in \mathscr{G}$ such that $|g(x) - f(x)| < \epsilon$ for all $x \in E$. Then, f is the uniform limit of functions in $\mathscr{G}$.*

Proof. For all $n \in \mathbb{N}$, let $g_n \in \mathscr{G}$ be the function such that $|g_n(x) - f(x)| < 1/n$ for all $x \in E$. Then, $g_n \to f$ uniformly on E. To prove this, let $\epsilon > 0$. Using the Archimedean property of the reals, pick $n_0 \in \mathbb{N}$ such that $n_0\epsilon > 1$. Thus, for all $x \in E$ and $n \geq n_0$, we have

$$|g_n(x) - f(x)| < \frac{1}{n} \leq \frac{1}{n_0} < \epsilon.$$

□

Theorem 1.2 (Cauchy criterion)**.** *Let $\{f_n\}$ be a sequence of real valued functions on a set E. This sequence of functions converges uniformly on E if and only if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$, $m, n \geq n_0$.

Proof. First, suppose that the sequence of real valued functions $\{f_n\}$ converges uniformly on E, with $f_n \to f$ uniformly. Given $\epsilon > 0$, we choose $n_0 \in \mathbb{N}$ such that for all $x \in E$ and $n \geq n_0$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Now, for all $x \in E$ and $m, n \geq n_0$, we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n(x) - f(x)) - (f_m(x) - f(x))| \\ &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, $\{f_n\}$ is a Cauchy sequence.

Next, suppose that $\{f_n\}$ is a Cauchy sequence. Given $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ such that for all $x \in E$ and $m, n \geq 0$, we have

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2}.$$

Now for each point $x \in E$, the Cauchy criterion for convergence of a sequence of real numbers guarantees that $\lim_{n\to\infty} f_n(x)$ exists. Thus, we can define the function f on E such that $f(x) = \lim_{n\to\infty} f_n(x)$, hence $f_n \to f$ pointwise. Fix $x_0 \in E$, and pick $n_0' \in \mathbb{N}$ such that for all $m \geq n_0'$, we have $|f_m(x_0) - f(x_0)| < \epsilon/2$. Choose $m = \max(n_0, n_0')$, whence for all $n \geq n_0$, we have

$$\begin{aligned}
|f_n(x_0) - f(x_0)| &= |(f_n(x_0) - f_m(x_0)) + (f_m(x_0) - f(x_0))| \\
&\leq |f_n(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Note that x_0 was arbitrary, with n_0 chosen independently of x_0. Thus, $\{f_n\}$ converges uniformly on E. □

Theorem 1.3. *Let $\{f_n\}$ be a sequence of real valued functions on a set E, and let f be a real valued function on X such that $f_n \to f$ pointwise. For all $n \in \mathbb{N}$, set*

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|$$

Then, $f_n \to f$ uniformly on E if and only if $M_n \to 0$.

Proof. First, suppose that $M_n \to 0$. This means that given $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$M_n = \sup_{n\in X} |f_n(x) - f(x)| < \epsilon.$$

This directly gives

$$|f_n(x) - f(x)| \leq \sup_{x\in X} |f_n(x) - f(x)| < \epsilon$$

for all $x \in E$ and $n \geq n_0$, hence $f_n \to f$ uniformly on E.

Next, suppose that $f_n \to f$ uniformly on E. Let $\epsilon > 0$ and pick $n_0 \in \mathbb{N}$ such that for all $x \in E$ and $n \geq n_0$, we have $|f_n(x) - f(x)| < \epsilon/2$. Taking supremums gives

$$M_n = \sup_{x\in E} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon,$$

hence $M_n \to 0$. □

Theorem 1.4. *Let $\{f_n\}$ be a sequence of real valued bounded, functions on a set E, and let f be a function on E such that $f_n \to f$ uniformly. Then, f is bounded on E.*

Proof. Using the uniform convergence of $\{f_n\}$, choose $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1$$

for all $x \in E$ and $n \geq n_0$. Specifically, this holds for $n = n_0$ so for all $x \in E$, we have

$$f_n(x) - 1 < f(x) < f_n(x) + 1.$$

However, f_n is bounded so there exists $M > 0$ such that $|f_n(x)| < M$ for all $x \in E$, hence

$$-M - 1 < f(x) < M + 1$$

or $|f(x)| < M + 1$ for all $x \in E$. □

Theorem 1.5 (Uniform limit theorem). *Let $\{f_n\}$ be a sequence of continuous functions on a metric space X, and let f be a function on X such that $f_n \to f$ uniformly. Then, f is continuous on X.*

Proof. Fix $x_0 \in X$, and let $\epsilon > 0$. Use the uniform convergence of $\{f_n\}$ to pick $n_0 \in \mathbb{N}$ such that for all $x \in X$ and $n \geq n_0$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

Note that the above also holds specifically at $x = x_0$. Use the continuity of each f_n to choose $\delta > 0$ such that for all $x \in X$ satisfying $|x - x_0| < \delta$, we have

$$|f_n(x_0) - f_n(x)| < \frac{\epsilon}{3}.$$

Set $n = n_0$, whence for all $x \in X$ satisfying $|x - x_0| < \delta$, we have

$$\begin{aligned}|f(x_0) - f(x)| &= |(f(x_0) - f_n(x_0)) + (f_n(x_0) - f_n(x)) + (f_n(x) - f(x))| \\ &\leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Thus, f is continuous at x_0. Since x_0 was chosen arbitrarily, f is continuous on X. □

Theorem 1.6 (Dini's theorem). *Let $\{f_n\}$ be a sequence of continuous real valued functions on a compact metric space K such that $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$, and let f be a continuous function on K such that $f_n \to f$ pointwise. Then, $f_n \to f$ uniformly on K.*

Proof. Let $\epsilon > 0$. For each $n \in \mathbb{N}$, set $g_n = f_n - f$ and note that the each g_n is continuous, with $g_n \geq g_{n+1}$ and $g_n \to 0$ pointwise on K. Since $\{g_n\}$ is a decreasing sequence, we must have $g_n \geq 0$ for all $n \in \mathbb{N}$. It is sufficient to show that $g_n \to 0$ uniformly on K, i.e. there exists $n_0 \in \mathbb{N}$ such that for all $x \in K$ and $n \geq n_0$, we have $g_n(x) < \epsilon$.

Define the sets

$$G_n = g_n^{-1}[\epsilon, \infty) = \{x \in K : g_n(x) \geq \epsilon\}.$$

Since each g_n is continuous, the sets G_n which are the pre-images of closed sets in $\mathbb{R}$ are closed. Furthermore, G_n is the intersection of the closed set G_n and the compact set K, hence each G_n is compact. Note that if $x \in G_{n+1}$ for some $n \in \mathbb{N}$, then $g_{n+1}(x) \geq \epsilon \implies g_n(x) \geq g_{n+1}(x) \geq \epsilon$ so $x \in G_n$; this means that $G_n \supseteq G_{n+1}$ for all $n \in \mathbb{N}$. Thus, if any G_{n_0} happened to be empty, then all subsequent $G_{n \geq n_0} = \emptyset$ as well.

Suppose that all G_n are non-empty. Then the countable intersection of nested compact sets $G = \bigcap_{n \in \mathbb{N}} G_n$ must also be non-empty. Pick $x_0 \in G$, and note that $x_0 \in G_n$ for all $n \in \mathbb{N}$, which means $g_n(x_0) \geq \epsilon$ for all $n \in \mathbb{N}$. This contradicts the fact that $g_n(x_0) \to 0$ pointwise. Thus, there must be some $G_{n_0} = \emptyset$, hence $G_{n \geq n_0} = \emptyset$. In other words, for all $n \geq n_0$, there is no $x \in K$ such that $g_n(x) \geq \epsilon$. This completes the proof. □

Remark. Note that the continuity of f, the compactness of K, and the monotonicity of $\{f_n\}$ are all essential.

(a) Consider $f_n : [0, 1] \to \mathbb{R}$, $x \mapsto x^n$, and note that $x^n \geq x^{n+1}$ on $[0, 1]$. We have $f_n \to f$ pointwise on the compact interval $[0, 1]$, where

$$f : [0,1] \to \mathbb{R}, \qquad x \mapsto \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

However, f is not continuous, and indeed $f_n \not\to f$ uniformly on $[0, 1]$ by the contrapositive of Theorem 1.5.

(b) Consider $f_n\colon (0,1) \to \mathbb{R}$, $x \mapsto x^n$, with $x^n \geq x^{n+1}$. We have $f_n \to 0$ pointwise on the open interval $(0,1)$. However, $(0,1)$ is not compact, and indeed $f_n \not\to 0$ uniformly on $(0,1)$. Note that $0 < 2^{-1/n} < 1$ for all $n \in \mathbb{N}$, and $f_n(2^{-1/n}) = 1/2$. Thus, there is no $n_0 \in \mathbb{N}$ such that $|f_n(x)| < 1/4$ for all $n \geq n_0$.

(c) Consider the triangular spike functions

$$f_n\colon [0,1] \to \mathbb{R}, \qquad x \mapsto \begin{cases} nx, & \text{if } 0 \leq x \leq 1/2n, \\ 1 - nx, & \text{if } 1/2n < x \leq 1/n, \\ 0, & \text{if } 1/n < x \leq 1. \end{cases}$$

Note that $f_n \to 0$ pointwise on the compact interval $[0,1]$, because given any $x_0 \in (0,1]$, we can choose sufficiently large $n_0 \in \mathbb{N}$ such that $n_o x_0 > 1$, hence $f_n(x_0) = 0$ for all $n \geq n_0$; if $x_0 = 0$, then $f_n(0) = 0$ for all $n \in \mathbb{N}$ anyway. However, the sequence $\{f_n\}$ is not monotonic, and indeed this convergence is not uniform on $[0,1]$. Note that $f_n(1/2n) = 1/2$ for all $n \in \mathbb{N}$, hence $\sup |f_n(x)| \geq 1/2 \not\to 0$.

2 The Weierstrass Approximation Theorem

Definition 2.1 (Bernstein polynomials)**.** The Bernstein polynomial $B_n^k(x)$ for integers $0 \leq k \leq n$ is defined as

$$B_n^k(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Remark. Each polynomial B_n^k on the interval $[0,1]$ peaks at $x = k/n$.

Definition 2.2 (Bernstein expansions)**.** Let f be a real valued function on $[0,1]$. The Bernstein polynomial expansion of f is defined as

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Lemma 2.1. *The following identities hold.*

$$B_n(1,x) = 1, \qquad B_n(x,x) = x, \qquad B_n(x^2,x) = \frac{x}{n} + \frac{n-1}{n}x^2.$$

Proof. The Binomial Theorem gives the expansion

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Taking a partial derivative with respect to x and multiplying by x/n, we have

$$\begin{aligned} n(x+y)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}, \\ x(x+y)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \left(\frac{k}{n}\right). \end{aligned}$$

Repeating the same procedure, we have

$$\begin{aligned} (x+y)^{n-1} + (n-1)x(x+y)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k} \left(\frac{k}{n}\right), \\ \frac{x}{n}(x+y)^{n-1} + \frac{n-1}{n}x^2(x+y)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \left(\frac{k}{n}\right)^2. \end{aligned}$$

Finally, set $y = 1 - x$ upon which all $x + y$ terms become 1 and the right hand sides become the Bernstein expansions of 1, x, and x^2. This establishes the desired identities. $\square$

Theorem 2.2. *Let f be a real valued continuous function on the closed interval $[0,1]$, and let $\epsilon > 0$. There exists a polynomial p such that $|p(x) - f(x)| < 0$ on $x \in [0,1]$.*

Proof. We claim that a Bernstein polynomial expansion $B_n(f,x)$ of sufficiently high order satisfies the given conditions. Let $\epsilon > 0$. Now, the continuity of f on the compact interval $[0,1]$ implies that it is uniformly continuous and bounded. Thus, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ for $x, x_0 \in [0,1]$, we have

$$|f(x) - f(x_0)| < \frac{\epsilon}{2}.$$

Fix $x_0 \in [0,1]$. Observe that $M = \sup_{x\in[0,1]} |f(x)|$ is finite. Thus, in those cases where $|x - x_0| \geq \delta$, use $|x - x_0|/\delta \geq 1$ and the triangle inequality to write

$$|f(x) - f(x_0)| \leq 2M \leq 2M \left(\frac{x - x_0}{\delta}\right)^2.$$

Thus, for all $x \in [0,1]$ we have

$$|f(x) - f(x_0)| \leq 2M \left(\frac{x - x_0}{\delta}\right)^2 + \frac{\epsilon}{2}.$$

Write

$$\begin{aligned}
B_n(f,x) - f(x_0) &= B_n(f,x) - B_n(1,x)\, f(x_0) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) - \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(x_0) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left[f\left(\frac{k}{n}\right) - f(x_0) \right].
\end{aligned}$$

The triangle inequality, followed by our estimate of $|f(x) - f(x_0)|$, gives

$$\begin{aligned}
|B_n(f,x) - f(x_0)| &\leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left| f\left(\frac{k}{n}\right) - f(x_0) \right| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left[2M \left(\frac{k/n - x_0}{\delta}\right)^2 + \frac{\epsilon}{2} \right] \\
&= \frac{2M}{\delta^2} B_n((x - x_0)^2, x) + \frac{\epsilon}{2} \\
&= \frac{2M}{\delta^2} \left[B_n(x^2, x) - 2x_0 B_n(x,x) + x_0^2 \right] + \frac{\epsilon}{2} \\
&= \frac{2M}{\delta^2} \left[\frac{x}{n} + x^2 - \frac{x^2}{n} - 2x x_0 + x_0^2 \right] + \frac{\epsilon}{2}.
\end{aligned}$$

The term in square brackets can be rearranged as

$$(x - x_0)^2 + \frac{1}{n}(x - x^2).$$

Use $(x - 1/2)^2 \geq 0$ to conclude that $x - x^2 \leq 1/4$, and evaluate the expression at $x = x_0$ to write

$$|B_n(f,x_0) - f(x_0)| \leq \frac{M}{2n\delta^2} + \frac{\epsilon}{2}.$$

Therefore, setting $n_0 > M/2\epsilon\delta^2$, we see that

$$|B_{n_0}(f, x_0) - f(x_0)| < \epsilon.$$

Since $x_0 \in [0, 1]$ was arbitrary, this concludes the proof. □

Corollary 2.2.1. *Given any real valued continuous function f on $[0, 1]$, there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[0, 1]$.*

Corollary 2.2.2. *The same holds for any real valued continuous function on some closed interval $[a, b]$.*

Proof. Consider the continuous bijection

$$\varphi\colon [0, 1] \to [a, b], \qquad x \mapsto (b - a)x + a.$$

For an arbitrary real valued continuous function $f\colon [a, b] \to \mathbb{R}$, note that the composition $g = f \circ \varphi$ is also continuous with domain $[0, 1]$. Given $\epsilon > 0$, we find a polynomial p such that

$$|p(x) - f(\varphi(x))| = |p(x) - g(x)| < \epsilon$$

on $[0, 1]$, which means that

$$|p(\varphi^{-1}(x)) - f(x)| < \epsilon$$

on $[a, b]$. Now, $\varphi^{-1}(x) = (x - a)/(b - a)$, hence $p \circ \varphi^{-1}$ is also a polynomial, as desired. □

3 Metric spaces of continuous functions

Theorem 3.1. *Let X be a metric space and let $\mathscr{C}(X)$ denote the set of all real valued, continuous, bounded functions on X. Define the distance function*

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

for all $f, g \in \mathscr{C}(X)$. Then, $\mathscr{C}(X)$ is a metric space.

Proof. Let $f, g, h \in \mathscr{C}(X)$ be arbitrary. The non-negativity of $d(f, g)$ is evident since it is the supremum of non-negative quantities. Furthermore, f and g are bounded so $d(f, g) = \sup |f - g| \leq \sup |f| + \sup |g|$ is finite. We clearly have $d(f, f) = 0$; conversely, if $d(f, g) = 0$, then $0 \leq \sup |f - g| = 0$ forcing $|f(x) - g(x)| = 0$ on X, hence $f = g$. Symmetry of d is evident from the fact that $|f(x) - g(x)| = |g(x) - f(x)|$ everywhere, hence $d(f, g) = d(g, f)$. Finally, the triangle inequality gives

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|,$$

whence taking supremums immediately gives $d(f, h) \leq d(f, g) + d(g, h)$. □

Theorem 3.2. *The metric space $\mathscr{C}(X)$ is complete.*

Proof. We claim that every Cauchy sequences in $\mathscr{C}(X)$ converges in $\mathscr{C}(X)$

Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(x)$. Given any $\epsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, $d(f_n, f_m) < \epsilon$. Thus,

$$|f_n(x) - f_m(x)| \leq \sup_{x \in X} |f_n(x) - f_m(x)| = d(f_n, f_m) < \epsilon$$

for all $x \in X$ and $m, n \geq n_0$, hence Theorem 1.2 says that $f_n \to f$ uniformly on X. Theorem 1.3 says that $d(f_n, f) \to 0$. Since each f_n is bounded and continuous, Theorems 1.4 and 1.5 guarantee that f is also bounded and continuous. Thus, $f \in \mathscr{C}(X)$, hence the Cauchy sequence $\{f_n\}$ converges in $\mathscr{C}(X)$. □

4 Algebras of functions

Definition 4.1. A family $\mathscr{A}$ of real valued functions on a set E is called an algebra if $f + g$ inA, $fg \in \mathscr{A}$, and $cf \in \mathscr{A}$ for all $f, g \in \mathscr{A}$ and $c \in \mathbb{R}$.

Definition 4.2. An algebra A is uniformly closed if given any sequence of functions $\{f_n\}$ in $\mathscr{A}$ such that $f_n \to f$ uniformly, we have $f \in \mathscr{A}$.

Definition 4.3. The uniform closure $\mathscr{B}$ of an algebra $\mathscr{A}$ is the set of all functions which are limits of uniformly convergent sequences of functions in $\mathscr{A}$.

Theorem 4.1. *The uniform closure $\mathscr{B}$ of an algebra $\mathscr{A}$ of bounded functions on a set E is a uniformly closed algebra.*

Proof. Let $f, g \in \mathscr{B}$. By construction, we can choose sequences $\{f_n\}$ and $\{g_n\}$ in $\mathscr{A}$ such that $f_n \to f$ and $g_n \to g$ uniformly. Since each f_n is bounded, we see that f is also bounded by Theorem 1.4, and the same applies for g. In order to prove that $\mathscr{B}$ is an algebra, we show that $f_n + g_n \to f + g$, $f_n g_n \to fg$, and $cf_n \to cf$ uniformly for all $c \in \mathbb{R}$.

(a) Let $\epsilon > 0$, and let $n_1, n_2 \in \mathbb{N}$ such that for all $x \in E$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \qquad \text{for all } n \geq n_1,$$

$$|g_n(x) - g(x)| < \frac{\epsilon}{2}, \qquad \text{for all } n \geq n_2.$$

Thus, for all $x \in E$ and $n \geq \max(n_1, n_2)$, we have

$$|(f_n(x) - g_n(x)) - (f(x) + g(x))| < |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon.$$

(b) Let $\epsilon > 0$. Note that

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &\leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)|. \end{aligned}$$

Thus, let $n_0 \in \mathbb{N}$ be such that for all $x \in E$ and $n \geq n_0$, we have $|f_n(x) - f(x)| < 1$, hence $|f_n(x)| < |f(x)| + 1$. Since f is bounded, this means that we can choose $M_1 > 0$ such that $|f_n(x)| < M_1$ for all $x \in E$, $n \geq n_0$. Similarly, pick $M_2 > 0$ such that $|g(x)| < M$ for all $x \in E$. Finally, pick $n_1, n_2 \in \mathbb{N}$ such that for all $x \in E$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2M_2}, \qquad \text{for all } n \geq n_1,$$

$$|g_n(x) - g(x)| < \frac{\epsilon}{2M_1}, \qquad \text{for all } n \geq n_2.$$

It immediately follows that for all $x \in E$ and $n \geq\geq \max(n_0, n_1, n_2)$, we have

$$|f_n(x)g_n(x) - f(x)g(x)| < M_1 \frac{\epsilon}{2M_1} + \frac{\epsilon}{2M_2} M_2 = \epsilon.$$

Remark. Without the requirement of boundedness, we see that $x + 1/n \to x$ uniformly on $\mathbb{R}$, but $(x + 1/n)^2 = x^2 + 2x/n + 1/n^2 \to x^2$ only pointwise on $\mathbb{R}$, not uniformly.

(c) Let $\epsilon > 0$ and $c \in \mathbb{R}$. If $c = 0$, we trivially have $0 \to 0$ uniformly for the constant zero functions. Otherwise, pick $n_0 \in \mathbb{N}$ such that for all $x \in E$ and $n \geq n_0$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{|c|}.$$

This immediately shows that for all $x \in E$ and $n \geq n_0$,

$$|cf_n(x) - cf(x)| = |c||f_n(x) - f(x)| < \epsilon.$$

To prove that $\mathscr{B}$ is uniformly closed, we must show that it contains all its uniform limits. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}$ such that $h_n \to h$ uniformly for some function on E. Now, for each $h_n \in \mathscr{B}$, there exists a sequence $\{h_{ni}\}_{i\in\mathbb{N}}$ in $\mathscr{A}$ such that $h_{ni} \to h_n$ uniformly.

Let $\epsilon > 0$, and pick $n_0 \in \mathbb{N}$ such that for all $x \in E$ and $n \geq n_0$, we have

$$|h_n(x) - h(x)| < \frac{\epsilon}{2}.$$

Next, for each such $n \geq n_0$, pick $i_n \in \mathbb{N}$ such that for all $x \in E$ and $i \geq i_n$, we have

$$|h_{ni_n}(x) - h_n(x)| < \frac{\epsilon}{2}.$$

Now, for all $x \in E$ and $n \geq n_0$, observe that

$$\begin{aligned}
|h_{ni_n}(x) - h(x)| &= |(h_{ni_n}(x) - h_n(x)) + (h_n(x) - h(x))| \\
&\leq |h_{ni_n}(x) - h_n(x)| + |h_n(x) - h(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, the sequence $\{h_{ni_n}\}_{n\in\mathbb{N}}$ in $\mathscr{A}$ converges uniformly to h, so $h \in \mathscr{B}$. This proves that $\mathscr{B}$ is uniformly closed. □

Theorem 4.2. *Let $\mathscr{A}$ be an algebra of real valued, bounded functions on a set E and let $\mathscr{B}$ be its uniform closure. If $f, g \in \mathscr{B}$, then the functions $|f| \in \mathscr{B}$, $\max(f, g) \in \mathscr{B}$, $\min(f, g) \in \mathscr{B}$.*

Proof. Let $\epsilon > 0$. Since f is bounded, there exists $M > 0$ such that $|f(x)| < M$ for all $x \in E$. Using Theorem 2 and its corollaries, pick a polynomial p such that for all $x \in [-M, +M]$,

$$|p(x) - |x|| < \epsilon.$$

Now, let $g = p \circ f$, which is a polynomial of f. Since $\mathscr{B}$ is an algebra, it contains all natural powers $f^n \in \mathscr{B}$, the scalar multiples $cf^n \in \mathscr{B}$, and the finite linear combinations $\sum c_n f^n$ hence we have $g \in \mathscr{B}$ Thus, for all $x \in E$ we have $f(x) \in [-M, +M]$, so

$$|g(x) - f(x)| = |p(f(x)) - |f(x)|| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary and $\mathscr{B}$ is uniformly closed, we have shown that $|f| \in \mathscr{B}$.

To show that $\max(f, g) \in B$ and $\min(f, g) \in \mathscr{B}$, simply observe that

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|,$$

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.$$

Note that these denote the pointwise maximum and minimum. □

Definition 4.4. A family $\mathscr{A}$ of functions on a set E is said to separate points on E if given distinct points $x_1, x_2 \in E$, there exists a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

Definition 4.5. A family $\mathscr{A}$ of functions on a set E is said to vanish at no point of E if given $x \in E$, there exists a function $f \in \mathscr{A}$ such that $f(x) \neq 0$.

Theorem 4.3. *Let $\mathscr{A}$ be an algebra of real valued functions on E which separates points on E and vanishes at no point of E. Let $x_1, x_2 \in E$ be distinct, and let $c_1, c_2 \in \mathbb{R}$. Then there exists a function $f \in \mathscr{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. Since $\mathscr{A}$ vanishes at no point of E, choose $f_1, f_2 \in \mathscr{A}$ such that $f_1(x_1) \neq 0$ and $f_2(x_2) \neq 0$. Since $\mathscr{A}$ separates points on E, pick the function $g \in \mathscr{A}$ such that $g(x_1) \neq g(x_2)$. Now, define the functions h_1, h_2 on E as

$$h_1(x) = \frac{g(x) - g(x_2)}{g(x_1) - g(x_2)} \frac{f_1(x)}{f_1(x_1)}, \qquad h_2(x) = \frac{g(x) - g(x_1)}{g(x_2) - g(x_1)} \frac{f_2(x)}{f_2(x_2)}.$$

Observe that $h_1, h_2 \in \mathscr{A}$ (use $(g - g(x_2))f_1 = gf_1 - g(x_2)f_1 \in \mathscr{A}$ and the analogous relation), with $h_1(x_1) = h_2(x_2) = 1$ and $h_1(x_2) = h_2(x_1) = 0$ (essentially, $h_i(x_j) = \delta_{ij}$). Finally, define the function f on E as

$$f(x) = c_1 h_1(x) + c_2 h_2(x).$$

Clearly, $f \in \mathscr{A}$ with $f_1(x_1) = x_1$ and $f_2(x_2) = c_2$ as desired. $\square$

Remark. This is essentially the process of Lagrange interpolation. In order to interpolate distinct $x_1, \dots, x_n$ with $c_1, \dots, c_n$, use the above theorem to choose the functions h_{ij} for distinct i, j such that $h_{ij}(x_i) = 1$, $h_{ij}(x_j) = 0$ for each pair i, j. Thus, the function

$$h_i = \prod_{i \neq j} h_{ij}$$

satisfies $h_i(x_i) = 1$ and $h_i(x_{j \neq i}) = 0$. The desired interpolating function is thus

$$f = \sum_{i=1}^{n} c_i h_i.$$

5 The Stone-Weierstrass Theorem

Theorem 5.1. *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact metric space K. If $\mathscr{A}$ separates points on K and vanishes at no point of K, then the uniform closure of $\mathscr{A}$ consists of all real valued, continuous functions on K.*

Proof. Let $\mathscr{B}$ be the uniform closure of $\mathscr{A}$. Since $\mathscr{A}$ consists of real valued, continuous functions on a compact interval, they are all uniformly continuous and bounded, with their uniform limits being continuous as well. Thus, $\mathscr{B}$ is an algebra of real valued, uniformly continuous and bounded functions. To show that $\mathscr{B}$ is precisely $\mathscr{C}(K)$, we fix $f \in \mathscr{C}(K)$ and show that f is the uniform limit of functions from $\mathscr{B}$. Since $\mathscr{B}$ is uniformly closed, this would imply $f \in \mathscr{B}$, thus completing the proof.

Let $\epsilon > 0$, and let $s, t \in K$. Using Theorem 4.3, find the functions $g_{st} \in \mathscr{B}$ such that $g(s) = f(s)$ and $g(t) = f(t)$. Fix s, and note that for each $t \in K$, the continuity of g_{st} means that there exists an open set $U_{st} \subseteq K$ such that

$$g_{st}(x) > f(x) - \epsilon$$

for all $x \in U_{st}$. Now, the collection of open sets $\{U_{st}\}_{t \in K}$ clearly covers the compact set K, hence we can choose a finite sub-cover $\{U_{st}\}_{t \in T}$ where $T \subset K$ is finite. Define the function g_s on K as

$$g_s = \max_{t \in T} g_{st}.$$

By finitely many applications of Theorem 4.2, we have $g_s \in \mathscr{B}$. Furthermore, given $x \in K$, we can choose $t \in T$ such that $x \in U_{st}$, hence $g_s(x) \geq g_{st}(x) > f(x) - \epsilon$. Thus, for all $x \in K$, we have

$$g_s(x) > f(x) - \epsilon.$$

We repeat this process again, this time to obtain an upper bound. For each $s \in K$, the continuity of g_s means that there exists an open set $U_s \subseteq K$ such that

$$g_s(x) < f(x) + \epsilon$$

for all $x \in U_s$. Now, the collection of open sets $\{U_s\}_{s\in K}$ covers the compact set K, hence we choose a finite sub-cover $\{U_s\}_{s\in S}$ where $S \subset K$ is finite. Define the function g on K as

$$g = \min_{s\in S} g_s.$$

Again, Theorem 4.2 gives $g \in \mathscr{B}$, and given $x \in K$, we can choose $s \in S$ such that $x \in U_s$, hence $g(x) \leq g_s(x) < f(x) + \epsilon$. Furthermore, every g_s obeys $g_s(x) > f(x) - \epsilon$ everywhere; since g is the minimum of finitely many functions, given $x \in K$ we find $s \in S$ such that $g(x) = g_s(x) > f(x) - \epsilon$. This shows that for all $x \in K$,

$$f(x) - \epsilon < g(x) < f(x) + \epsilon,$$

or $|g(x) - f(x)| < \epsilon$. Thus, f is the uniform limit of functions in $\mathscr{B}$, proving that the uniform closure of $\mathscr{A}$ is the set $\mathscr{C}(K)$. □

Corollary 5.1.1. *Let K be a compact subset of the Euclidean metric space $\mathbb{R}^n$, and let $\mathscr{P}$ be the algebra of polynomials in n variables on K. Then, given any real valued, continuous function f on K, there exists a sequence of polynomials $\{p_n\} \subset \mathscr{P}$ such that $p_n \to f$ uniformly on K.*

Proof. We need only check that $\mathscr{P}$ is an algebra of real continuous functions which separates points on K and vanishes at no point of K, after which the result follows directly from the above theorem.

Note that every polynomial function $p \in \mathscr{P}$ is of the form

$$p(x) = \sum_{i_1,i_2,\ldots,i_n} c_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}.$$

Each term in the finite sum is the product of projection maps $(x_1, \ldots, x_n) \mapsto x_i$, which are continuous. Thus, the polynomial p is indeed real valued and continuous, with the coefficients $c_{i_1\ldots i_n} \in \mathbb{R}$. It is evident that given a scalar $c \in \mathbb{R}$, we have $cp \in \mathscr{P}$ since the result is of the same form. Given $p, q \in \mathscr{P}$, it is also evident that $p+q \in \mathscr{P}$; term by term multiplication shows that $pq \in \mathscr{P}$ as well. Thus, $\mathscr{P}$ is an algebra.

To show that $\mathscr{P}$ vanishes nowhere on K, note that the constant polynomial $p(x_1, \ldots, x_n) = 1 \neq 0$ on any $K \subseteq \mathbb{R}^n$. To show that $\mathscr{P}$ separates points on K, pick $y, w \in K$ where $y \neq w$. Then $y_j \neq w_j$ for at least one index j, so the polynomial $p(x_1, \ldots, x_n) = x_j$ separates w and y. Applying the Stone-Weierstrass Theorem completes the proof. □

Corollary 5.1.2. *Let $\mathscr{F}$ be the algebra of functions on $[0, \pi]$ of the form*

$$f(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx),$$

for real coefficients a_i, b_i. Then, given any real valued, continuous function f on $[0, \pi]$, there exists a sequence of functions $\{f_n\} \subset \mathscr{F}$ such that $f_n \to f$ uniformly on $[0, \pi]$.

Proof. Like before, we need only check that $\mathscr{F}$ satisfies the requirements of the Stone-Weierstrass Theorem.

It is clear that $\mathscr{F}$ is closed under sums and scalar multiples. To show that it is closed under products, we supply the following identities.

$$\sin(nx)\sin(mn) = \frac{1}{2}\left[\cos((n-m)x) - \cos((n+m)x)\right],$$

$$
\begin{aligned}
\sin(nx)\cos(mn) &= \frac{1}{2}\left[\sin((n+m)x) + \sin((n-m)x)\right], \\
\cos(nx)\cos(mn) &= \frac{1}{2}\left[\cos((n+m)x) + \cos((n-m)x)\right].
\end{aligned}
$$

Thus, $\mathscr{F}$ is indeed an algebra. Now, $\mathscr{F}$ contains the constant function $x \mapsto 1$, hence $\mathscr{F}$ vanishes nowhere on $[0, \pi]$. Furthermore, given distinct $x_1, x_2 \in [0, \pi]$, we must have $\cos(x_1) \neq \cos(x_2)$, because the map $x \mapsto \cos(x)$ is strictly decreasing on $[0, \pi]$, and hence is injective. Thus, $\mathscr{F}$ separates points on $[0, \pi]$.

□

Remark. Note that $\mathscr{F}$ does not separate the points 0 and 2π, due to the periodicity of the cosine and sine functions. Naturally, in order to extend the domain to $[0, \pi\alpha]$, we may redefine $\mathscr{F}$ to consist of functions of the form

$$
f(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx/\alpha) + b_n \sin(nx/\alpha).
$$

Theorem 5.2. *Let $\mathscr{A}$ be an algebra on a compact metric space K which satisfies the requirements of the Stone-Weierstrass Theorem. Then, given $f \in \mathscr{C}(K)$, there exists a monotonically decreasing sequence of functions from $\mathscr{A}$ which converge uniformly to f on K.*

Proof. For all $n \in \mathbb{N}$, define the functions $f_n = f + 2/3^n$ and use the Stone-Weierstrass theorem to select functions $g_n \in \mathscr{A}$ such that

$$
|g_n(x) - f_n(x)| < \frac{1}{3^n}
$$

everywhere on K. As a result, each g_n satisfies

$$
f + \frac{1}{3^n} < g_n < f + \frac{1}{3^{n-1}}
$$

on K. This immediately gives $g_n > g_{n+1}$ for all $n \in \mathbb{N}$. Furthermore, for all $x \in K$, we have

$$
|g_n(x) - f(x)| < \frac{1}{3^{n-1}} \to 0
$$

which establishes $g_n \to f$ uniformly on K by Theorem 1.3. □