MA 1101 : Mathematics I

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Solution 1.

(i) Let $P(n)$ be the statement

$$1 + 2 + \cdots + n \;=\; \frac{1}{2}n(n+1) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{2}1(1+1)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= [1 + 2 + \cdots + k] + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) && \text{(From } P(k)\text{)} \\ &= \frac{1}{2}(k+2)(k+1) \\ &= \frac{1}{2}(k+1)((k+1)+1) \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(ii) Let $P(n)$ be the statement

$$1^2 + 2^2 + \cdots + n^2 \;=\; \frac{1}{6}n(n+1)(2n+1) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{6}1(1+1)(2+1)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= [1^2 + 2^2 + \cdots + k^2] + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 && \text{(From } P(k)\text{)} \\ &= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6) \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(iii) Let $P(n)$ be the statement

$$1^2 + 3^2 + \cdots + (2n-1)^2 \;=\; \frac{1}{3}(4n^3 - n) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{3}1(4-3)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^2+3^2+\cdots+(2k-1)^2+(2k+1)^2 &= [1^2+3^2+\cdots+(2k-1)^2]+(2k+1)^2 \\
&= \frac{1}{3}(4k^3-k)+(2k+1)^2 \qquad \text{(From } P(k)\text{)} \\
&= \frac{1}{3}(4k^3-k+12k^2+12k+3) \\
&= \frac{1}{3}(4(k^3+3k^2+3k+1)-k-1)) \\
&= \frac{1}{3}(4(k+1)^3-(k+1))
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(iv) Let $P(n)$ be the statement

$$1^3+2^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2 \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{4}1(1+1)^2$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= [1^3+2^3+\cdots+k^3]+(k+1)^3 \\
&= \frac{1}{4}k^2(k+1)^2+(k+1)^3 \qquad \text{(From } P(k)\text{)} \\
&= \frac{1}{4}(k+1)^2(k^2+4k+4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \frac{1}{4}(k+1)^2((k+1)+1)^2
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(v) Let $P(n)$ be the statement

$$\sum_{r=1}^{n} r(r+1)\ldots(r+9) = \frac{1}{11}n(n+1)\ldots(n+10) \qquad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly,

$$1(1+1)\ldots(1+9) = \frac{1}{11}1(1+1)\ldots(1+9)(1+10)$$

Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
\sum_{r=1}^{k+1} r(r+1)\ldots(r+9) &= \left[\sum_{r=1}^{k} r(r+1)\ldots(r+9)\right] + (k+1)(k+2)\ldots(k+1+9) \\
&= \frac{1}{11}k(k+1)\ldots(k+10) + (k+1)(k+2)\ldots(k+1+9) \quad \text{(From } P(k)\text{)} \\
&= \frac{1}{11}(k+1)\ldots(k+10)(k+11) \\
&= \frac{1}{11}(k+1)\ldots((k+1)+9)((k+1)+10)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Solution 2.

(i) Let $P(n)$ be the statement that for all $n \in \mathbb{N}$,

$$3^n > n^2$$

Base step We establish $P(1)$ and $P(2)$. Clearly, $3^1 > 1^2$. Thus, $P(1)$ is true. Again, $3^2 = 9 > 8 = 2^2$. Thus, $P(2)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$3^{k+1} \;=\; 3 \cdot 3^k \;>\; 3 \cdot k^2$$

We must show $3k^2 > (k+1)^2 \Leftrightarrow 3k^2 - (k+1)^2 > 0$.

$$3k^2 - (k+1)^2 \;=\; 2k^2 - 2k - 1 \;=\; k^2 + (k-1)^2 - 2$$

Clearly, for $k \geq 2$, $k^2 > 2$, so $k^2 + (k-1)^2 > 2$, and we are done.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(ii) Let $P(n)$ be the statement that for all $n \in \mathbb{N}$ and $x > -1$,

$$(1+x)^n \geq 1 + nx. \qquad \text{(Bernoulli's Inequality)}$$

Base Step We establish $P(1)$. Clearly, $(1+x)^1 \geq (1 + 1 \cdot x)$, thus $P(1)$ is true.

Inductive Step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\
&\geq (1+kx) \cdot (1+x) && (x+1>0) \\
&= (1 + x + kx + kx^2) \\
&\geq (1 + (k+1)x) && (k > 0 \text{ and } x^2 \geq 0)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(iii) Let $P(n)$ be the statement that for all $n \geq 5$, $n \in \mathbb{N}$,

$$\binom{2n}{n} \;<\; 2^{2n-2}.$$

Base Step We establish $P(5)$. Now, $\binom{2n}{n} = 252$, while $2^{10-2} = 256$. Thus, $P(5)$ is true.

Inductive Step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
\binom{2(k+1)}{k+1} &= \frac{(2k+2)!}{(k+1)!^2} \\
&= \frac{(2k+2)(2k+1)}{(k+1)^2}\binom{2n}{n} \\
&< 2 \cdot \frac{2k+1}{k+1} \cdot 2^{2k-2} \\
&< 2 \cdot \frac{2k+2}{k+1} \cdot 2^{2k-2} \\
&< 2^{2(k+1)-2}
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.