

MA2102 : Linear Algebra I

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Let $a \in \mathbb{R}$. Consider the set

$$S_a^n = \{1, (x-a), (x-a)^2, \dots, (x-a)^n\}.$$

Show that S_a^n is a basis for $P_n(\mathbb{R})$, the space of polynomials of degree at most n .

We notate the superscript S_a^n to denote the highest degree term $(x-a)^n$.

Solution We first calculate the dimension of $P_n(\mathbb{R})$ and exhibit the standard basis.

Lemma 1. *The set $S_0^n = \{1, x, x^2, \dots, x^n\}$ is a basis of $P_n(\mathbb{R})$.*

Proof. Note that any polynomial in $P_n(\mathbb{R})$ is of the form

$$p(x) = a_0 + a_1x + \dots + a_nx^n.$$

This means that $P_n(\mathbb{R}) \subseteq \text{span } S_0^n$. We now show linear independence by considering the linear combination

$$c_0 + c_1x + c_2x^2 + \dots + c_nx^n = \mathbf{0}.$$

Because the polynomial on the left is equated with the zero polynomial on the right, it must identically evaluate to 0 no matter the choice of $x \in \mathbb{R}$. We choose $n+1$ distinct reals x , which we exhibit as $n+1$ roots of the polynomial on the left. However, the degree of this polynomial is at most n . We conclude that the polynomial on the left is the zero polynomial, so $c_0 = c_1 = \dots = c_n$.

This proves that the set S_0^n is a basis of $P_n(\mathbb{R})$. \square

We use the fact that $\dim P_n(\mathbb{R}) = n+1$, from Lemma 1. Thus, we need only show that S_a^n is linearly independent for $a \neq 0$. This is sufficient to prove that S_a^n is a basis of $P_n(\mathbb{R})$, because the Replacement Theorem guarantees that any linearly independent set of size $\dim P_n(\mathbb{R})$ will be a basis.

Lemma 2. *The polynomial $(x-a)^n - x^n$ has degree at most $n-1$, for $n \in \mathbb{N}$.*

Proof. We expand $(x-a)^n$ using the Binomial Theorem to obtain

$$(x-a)^n = x^n - nax^{n-1} + \binom{n}{2}a^2x^{n-2} + \dots + (-1)^na^n.$$

Subtracting x^n from both sides leaves terms of degree at most $n-1$ on the right. Thus,

$$(x-a)^n - x^n \in P_{n-1}(\mathbb{R}) \subset P_n(\mathbb{R}).$$

The coefficients of this $n-1$ degree polynomial are the binomial coefficients with alternating sign, as seen above. \square

With this, we prove that S_a^n for $a \neq 0$ is a basis of $P_n(\mathbb{R})$ by induction. For $n=0$, the claim is trivial, since we have $S_a^0 = S_0^0 = \{1\}$, which is a linearly independent set in $P_0(\mathbb{R})$, hence a basis of $P_0(\mathbb{R})$.

For $n=1$, consider the linear combination of elements from $S_a^1 = \{1, (x-a)\}$

$$c_0 + c_1(x-a) = \mathbf{0},$$

for arbitrary $c_0, c_1 \in \mathbb{R}$. Successively set $x=a$ and $x=0$. Thus, $c_0 = 0$ and $c_0 - c_1a = 0$, whence $c_0 = c_1 = 0$. This shows that S_a^1 is linearly independent in $P_1(\mathbb{R})$, hence a basis of $P_1(\mathbb{R})$.

Suppose that for $n = k$, the set $S_a^k = \{1, (x - a), \dots, (x - a)^k\}$ is a basis of $P_k(\mathbb{R})$. Consider the linear combination of elements from S_a^{k+1} ,

$$c_0 + c_1(x - a) + \dots + c_k(x - a)^k + c_{k+1}(x - a)^{k+1} = \mathbf{0}.$$

Subtract and add $c_{k+1}x^{k+1}$.

$$\left[c_0 + c_1(x - a) + \dots + c_k(x - a)^k + c_{k+1}((x - a)^{k+1} - x^{k+1}) \right] + c_{k+1}x^{k+1} = \mathbf{0}.$$

Note that the portion in square brackets is a polynomial of degree at most k . This is because $(x - a)^{k+1} - x^{k+1}$ has degree at most k by Lemma 2, and the remaining terms also have degree at most k . Thus, we expand this bracketed polynomial in the basis S_a^k .

$$p_k(x) = a_0 + a_1x + \dots + a_kx^k \in P_k(\mathbb{R}).$$

Replacing this in the previous equation,

$$a_0 + a_1x + \dots + a_kx^k + c_{k+1}x^{k+1} = \mathbf{0}.$$

The linear independence of $S_0^{k+1} = \{1, x, \dots, x^{k+1}\}$ from Lemma 1 gives $a_0 = a_1 = \dots = a_k = c_{k+1} = 0$. Substituting this back into the original linear combination with the c_j coefficients, we have

$$c_0 + c_1(x - a) + \dots + c_k(x - a)^k = \mathbf{0}.$$

The induction hypothesis, whereby S_a^k is linearly independent, gives $c_0 = c_1 = \dots = c_k = 0 = c_{k+1}$. This shows that S_a^{k+1} is linearly independent in $P_{k+1}(\mathbb{R})$, which means it is a basis of $P_{k+1}(\mathbb{R})$.

Thus, by the principle of mathematical induction, the set S_a^n is a basis of $P_n(\mathbb{R})$ for all integers $n \geq 0$. This completes the proof.