STAT6301: Probability Theory I

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Homework 1

Problem 1

1. To show that $f(\mathcal{A})$ is indeed a field, let $\mathscr{F}_0(\mathcal{A})$ be the collection of fields containing $\mathcal{A}$, and write $f(\mathcal{A}) = \bigcap_{\mathcal{F}_0 \in \mathscr{F}_0(A)} \mathcal{F}$.

   (a) $\Omega \in \mathcal{F}$ for all $\mathcal{F} \in \mathscr{F}_0(\mathcal{A})$, hence $\Omega \in f(\mathcal{A})$.

   (b) For $A \in f(\mathcal{A})$, we have $A \in \mathcal{F}$ for arbitrary $\mathcal{F} \in \mathscr{F}_0(\mathcal{A})$, hence $A^c \in \mathcal{F}$ for all $\mathcal{F} \in \mathscr{F}_0(\mathcal{A})$. Thus, $A^c \in f(\mathcal{A})$.

   (c) For $A_1, \dots, A_n \in f(\mathcal{A})$, we have $A_1, \dots, A_n \in \mathcal{F}$ hence $\bigcup_{i=1}^n A_i \in \mathcal{F}$ for all $\mathcal{F} \in \mathscr{F}_0(\mathcal{A})$. Thus, $\bigcup_{i=1}^n A_i \in f(\mathcal{A})$.

   With this, suppose that $\mathcal{G}$ is another field with $\mathcal{A} \subseteq \mathcal{G}$. Then, $\mathcal{G} \in \mathscr{F}_0(\mathcal{A})$ hence $f(\mathcal{A}) = \bigcap_{\mathcal{F} \in \mathscr{F}_0(\mathcal{A})} \mathcal{F} \subseteq \mathcal{G}$ by construction.

2. We first make the following observation: for any finite class $\mathcal{E} = \{E_1, \dots, E_n\}$ of sets in Ω, and any expression involving sets from $\mathcal{E}$ composed of finitely many finite intersections, unions, and complements, the resulting set B can be expressed (uniquely) in the form

   $$B = \bigcup_{d \in D} \bigcap_{i=1}^n E_i^{d_i} := \bigcup_{d \in D} E^d, \tag{1.2.1}$$

   for some $D \subseteq \{0,1\}^n$. Here, we denote $X^0 := X$, $X^1 := X^c$. Note that this is a union of pairwise disjoint sets of the form $E^d := \bigcap_{i=1}^n E_i^{d_i}$, and $\{E^d\}_{d \in \{0,1\}^n}$ forms a partition of Ω. As a result,

   $$B^c = \bigcup_{d \in \{0,1\}^n \setminus D} \bigcap_{i=1}^n E_i^{d_i} = \bigcup_{d \notin D} E^d, \tag{1.2.2}$$

   which is crucially of the 'same form' as B

   Remark: This is inspired by the observation that in boolean algebra, any finite boolean expression over finitely many variables can be written in a canonical 'sum of products' form (as well as a 'product of sums' form). This follows immediately because boolean functions of the form $\{0,1\}^n \to \{0,1\}$ are spanned by basis functions δ_d of the form $x \mapsto x_1^{d_1} \cdots x_n^{d_n}$ which take the value 1 if $x = d$ and 0 otherwise. Again, we have notated $x^0 := x$, $x^1 := \overline{x} = x^c = 1 - x$.

   Remark: This shows that there are precisely 2^n possible sets that can be built (using complements, finite intersections, finite unions, finitely many operations) using n (distinct) sets.

   In general, this is true because for $x \in \Omega$, answering the question $x \in B$? depends only on the answers to the questions $x \in E_i$? (the expression forming B can be broken down into this form), which is completely determined by the answer to $x \in E^d$?.

For our purposes, it is enough to only consider expressions of the form

$$B = \bigcup_{i=1}^{n} \bigcap_{j \in J_i} F_{ij}$$

where all J_i are finite and for each F_{ij}, at least one of $F_{ij} \in \mathcal{E}$ or $F_{ij}^c \in \mathcal{E}$. Note that for each i, we can relabel $\{F_{ij}\}_{j \in J_i} = \{E_k\}_{k \in K_i}$ for indices $K_i \subseteq \{1, \dots, n\}$, and rewrite

$$\bigcap_{j \in J_i} F_{ij} = \bigcap_{k \in K_i} E_k = \left(\bigcap_{k \in K_i} E_k \right) \cap \left(\bigcap_{k \notin K_i} (E_k \cup E_k^c) \right),$$

which will split into a (pairwise disjoint) union of sets (at most $2^{n-|K_i|}$ many) of the form E^d by distributing over the unions. Thus, the expression for B again reduces to a union of sets of the form E^d, which (after removing duplicates) are pairwise disjoint.

With this, denote

$$\begin{aligned} g(\mathcal{A}) = \Bigg\{ B \subseteq \Omega \colon B = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij}, \text{ where } A_{ij} \in \mathcal{A} \text{ or } A_{ij}^c \in \mathcal{A}, \\ \text{and } \left(\bigcap_{j=1}^{n_i} A_{ij} \right) \cap \left(\bigcap_{j=1}^{n_k} A_{kj} \right) = \emptyset \text{ for } k \neq i \Bigg\}. \end{aligned} \qquad (\star)$$

Note that $\mathcal{A} \subseteq g(\mathcal{A}) \subseteq f(\mathcal{A})$; every element $A \in \mathcal{A}$ is trivially of the form demanded in $(\star)$, and $f(\mathcal{A})$ being a field containing $\mathcal{A}$ must contain all finite unions of finite intersections of elements (or their complements) from $\mathcal{A}$. Thus, it suffices to show that $g(\mathcal{A})$ is a field, whence $f(\mathcal{A}) \subseteq g(\mathcal{A})$ via minimality of $f(\mathcal{A})$ forces $f(\mathcal{A}) = g(\mathcal{A})$.

Putting aside the case where $\mathcal{A} = \emptyset$ (in which case one might argue that $g(\mathcal{A}) = \{\emptyset, \Omega\}$, using conventions such as empty unions being $\emptyset$, and empty intersections being Ω; this is certainly a field):

(a) Fix $A_0 \in \mathcal{A}$. Then, $\Omega = A_0 \cup A_0^c \in g(\mathcal{A})$.

(b) Suppose that $B \in g(\mathcal{A})$, hence is of the form demanded in $(\star)$. Use our first observation to express B in the form (1.2.1), then use (1.2.2) to see that B^c can still be expressed in the form demanded by $(\star)$, hence $B^c \in g(\mathcal{A})$.

(c) Suppose that $B_1, \dots, B_n \in g(\mathcal{A})$; again, express them in the form $B_i = \bigcup_{d \in D_i} A^d$, where $A_1, \dots, A_m$ (satisfying $A_j \in \mathcal{A}$ or $A_j^c \in \mathcal{A}$) are all of the sets used in the expressions used to write $B_1, \dots, B_n$ in the form from $(\star)$. Then, $\bigcup_{i=1}^n B_i = \bigcup_{i \in D} A^d$ where $D = \bigcup_{i=1}^n D_i$, which is in the form demanded by $(\star)$, hence $\bigcup_{i=1}^n B_i \in g(\mathcal{A})$.

This shows that $g(\mathcal{A})$ is indeed a field, hence we are done.

Problem 2

1. Given that $\mathcal{A} = \{\{x\} \colon x \in \Omega\}$, set

$$g(\mathcal{A}) = \{A \subseteq \Omega \colon |A| < \infty \text{ or } |A^c| < \infty\}.$$

Again, note that $\mathcal{A} \subseteq g(\mathcal{A}) \subseteq f(\mathcal{A})$. First, singletons have finite cardinality. Next, finite sets are finite unions of singletons hence mush be in $f(\mathcal{A})$; co-finite sets have finite complements which must be in $f(\mathcal{A})$, hence they too must be in $f(\mathcal{A})$. Thus, it suffices to show that $g(\mathcal{A})$ is a field for $f(\mathcal{A}) \subseteq g(\mathcal{A})$, yielding $f(\mathcal{A}) = g(\mathcal{A})$.

(a) Note that $\Omega^c = \emptyset$ has zero cardinality, so $\Omega \in g(\mathcal{A})$.

(b) If $A \in g(\mathcal{A})$, at least one of $|A| < \infty$ or $|A^c| < \infty$, which also guarantees that $A^c \in g(\mathcal{A})$ (recall that $(A^c)^c = A$).

(c) Suppose that $A_1, \dots, A_n \in \mathcal{A}$. If all of them are finite, then their union $A = \bigcup_{i=1}^n A_i$ is also finite, hence $A \in g(\mathcal{A})$. Otherwise, one of them is cofinite; say $|A_1^c| < \infty$ without loss of generality. Then, $A^c = \bigcap_{i=1}^n A_i^c \subseteq A_1^c$ is finite, hence $A \in g(\mathcal{A})$.

Thus, $g(\mathcal{A})$ is a field, and we are done.

2. Note that $\sigma(\mathcal{A})$ is the smallest σ-field containing $\mathcal{A}$; specifically it is also a field containing $\mathcal{A}$, hence we must have $f(\mathcal{A}) \subseteq \sigma(\mathcal{A})$ by minimality of $f(\mathcal{A})$.

   Suppose that $\mathcal{A}$ is finite. We will show that $f(\mathcal{A})$ is a σ-field, from which $\sigma(\mathcal{A}) \subseteq f(\mathcal{A})$ by minimality of $\sigma(\mathcal{A})$, forcing $f(\mathcal{A}) = \sigma(\mathcal{A})$. Indeed, $f(\mathcal{A})$ must be finite, via the characterization in Problem 1.2; there are only finitely many possible sets of the form $\bigcap_{i=1}^n A_i$ where $A_i \in \mathcal{A}$ or $A_i^c \in \mathcal{A}$, hence finitely many possible unions involving them. With this, countable unions of sets from $f(\mathcal{A})$ are actually finite unions, hence belong to $f(\mathcal{A})$. Thus, $f(\mathcal{A})$ is a σ-field, and we are done.

   Note that $\mathcal{A} \subseteq f(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. Thus, $\sigma(\mathcal{A}) \subseteq \sigma(f(\mathcal{A})) \subseteq \sigma(\mathcal{A})$, forcing $\sigma(f(\mathcal{A})) = \sigma(\mathcal{A})$.

   Remark: In general, $\mathcal{A} \subseteq \mathcal{B}$ gives $\mathcal{A} \subseteq \mathcal{B} \subseteq \sigma(\mathcal{B})$ by construction of $\sigma(\mathcal{B})$, hence $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{B})$ by minimality of $\sigma(\mathcal{A})$. Also, we trivially have $\sigma(\sigma(\mathcal{A})) = \sigma(\mathcal{A})$.

3. If $\mathcal{A}$ is countable, the characterization in Problem 1.2 shows that $f(\mathcal{A})$ must be countable. To see this, note that every set $B \in f(\mathcal{A})$ can be written in the form (1.2.1), say $B = \bigcup_{d \in D} E^d$ where $E_1, \dots, E_n \in \mathcal{A}$. Enumerate $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$, and let $\{p_n\}_{n \in \mathbb{N}}$ be an enumeration of the primes (in order). Then, the following map is a surjection.

$$h \colon \mathbb{N} \times \mathbb{N} \to f(\mathcal{A}), \qquad (p_{n_1} \cdots p_{n_k}, p_{d_1+1} \cdots p_{d_m+1}) \mapsto \bigcup_{i=1}^{m} \bigcap_{j=1}^{k} A_{n_j}^{d_{ij}}.$$

   Here, d_{ij} is the j-th binary digit of d_i (which is interpreted as a binary number of length k). Elements of $\mathbb{N} \times \mathbb{N}$ not of the given form are mapped to $\emptyset$. Since $\mathbb{N} \times \mathbb{N}$ is countable, we have $f(\mathcal{A})$ countable.

4. Let $\mathcal{F}_1, \mathcal{F}_2$ be fields in Ω. Define

$$g(\mathcal{F}_1, \mathcal{F}_2) = \left\{ B \subseteq \Omega \colon B = \bigcup_{j=1}^{m} A_{1j} \cap A_{2j},\ A_{ij} \in \mathcal{F}_{ij}, \right.$$
$$\left. \text{and } (A_{ik} \cap A_{2k}) \cap (A_{1j} \cap A_{2k}) = \emptyset \text{ for all } k \neq j \right\}. \qquad (\star\star)$$

   Again, it suffices to show that $g(\mathcal{F}_1, \mathcal{F}_2)$ is a field, since clearly $g(\mathcal{F}_1, \mathcal{F}_2) \subseteq f(\mathcal{F}_1 \cup \mathcal{F}_2)$, so equality will follow from minimality of $f(\mathcal{F}_1 \cup \mathcal{F}_2)$.

   (a) $\Omega \in \mathcal{F}_2 \cap \mathcal{F}_2$, so $\Omega \in g(\mathcal{F}_1, \mathcal{F}_2)$.

   (b) Suppose that $B \in g(\mathcal{F}_1, \mathcal{F}_2)$, hence is of the form described in $(\star\star)$. Rewrite B in the form (1.2.1), hence B^c in the form (1.2.2), and note that the sets $E_1, \dots, E_k$ used in the expression can be separated into those from $\mathcal{F}_1$ and those from $\mathcal{F}_2$. Thus, B^c is also in the form demanded by $(\star\star)$, hence $B^c \in g(\mathcal{F}_1, \mathcal{F}_2)$.

(c) Suppose that $B_1, \dots, B_n \in g(\mathcal{F}_1, \mathcal{F}_2)$; again, express them in the form $B_i = \bigcup_{d \in D_i} A^d$, where $A_1, \dots, A_n$ are all of the sets used to write $B_1, \dots, B_n$ in the form from $(\star\star)$. Then, $B = \bigcup_{i=1}^n B_i = \bigcup_{d \in D} A^d$ where $D = \bigcup_{i=1}^n D_i$. Since we can separate the A_i into those from $\mathcal{F}_1$ and those from $\mathcal{F}_2$, we have expressed B in the form demanded by $g(\mathcal{F}_1, \mathcal{F}_2)$, hence $B \in g(\mathcal{F}_1, \mathcal{F}_2)$.

This shows that $g(\mathcal{F}_1, \mathcal{F}_2)$ is indeed a field, and we are done.

Problem 3

1. Recall that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is simply $\sigma(\tau)$, where τ is the standard Euclidean topology on $\mathbb{R}$ generated by the basis of open intervals. We claim that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{H})$, where
$$\mathcal{H} = \{(\infty, x] \colon x \in \mathbb{Q}\},$$
whence $\mathcal{B}(\mathbb{R})$ is countably generated. Indeed, note that for $x \in \mathbb{Q}$, we have $(\infty, x]^c = (x, \infty) \in \tau \subset \mathcal{B}(\mathbb{R})$, so $\mathcal{H} \subseteq \mathcal{B}(\mathbb{R})$. Thus, $\sigma(\mathcal{H}) \subseteq \mathcal{B}(\mathbb{R})$. Now, sets of the form $(x, y] = (\infty, y] \cap (-\infty, x]^c \in \sigma(\mathcal{H})$ for $x, y \in \mathbb{Q}$. Then, for $x, y \in \mathbb{Q}$, find $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $y_n \uparrow y$, so that sets of the form $(x, y) = \bigcup_{n \in \mathbb{N}} (x, y_n] \in \sigma(\mathcal{H})$. Setting
$$\mathcal{U} = \{(x, y) \colon x, y \in \mathbb{Q}\} \subseteq \sigma(\mathcal{H}),$$
note that $\mathcal{U}$ is countable, and the topology τ is generated by $\mathcal{U}$. In other words, any $U \in \tau$ can be written as a countable union $U = \bigcup_{n \in \mathbb{N}} U_n$ for $\{U_n\}_{n \in \mathbb{N}} \subset \mathcal{U}$. This means that $\tau \subseteq \sigma(\mathcal{H})$, from which $\mathcal{B}(\mathbb{R}) = \sigma(\tau) \subseteq \sigma(\mathcal{H}) \subseteq \mathcal{B}(\mathbb{R})$, and we are done.

   Remark: In general, showing that $\mathcal{A} \subseteq \sigma(\mathcal{B})$ and $\mathcal{B} \subseteq \sigma(\mathcal{A})$ yields $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.

2. Consider the σ-field
$$\mathcal{F} = \{A \subseteq \Omega \colon A \text{ is countable, or } A^c \text{ is countable}\}.$$
When Ω is countable, every $A \subseteq \Omega$ is countable, hence $A = \bigcup_{x \in A} \{x\}$ is a countable union. Thus, $\mathcal{F} = \sigma(\{\{x\} \colon x \in \Omega\})$ is countably generated.

   Suppose that Ω is uncountable, and that $\mathcal{A}$ is a countable generating set for $\mathcal{F}$, i.e. that $\mathcal{F} = \sigma(\mathcal{A})$. Without loss of generality, suppose that every $A \in \mathcal{A}$ is countable; note that if A is uncountable, then A^c must be countable, so we can replace A with A^c in $\mathcal{A}$. Now, observe that $\Omega_0 = \bigcup_{A \in \mathcal{A}} A$, being a countable union of countable sets, is countable. Thus, $\Omega_0 \neq \Omega$; fix $\omega \in \Omega \setminus \Omega_0$. Then, $\{\omega\} \in \mathcal{F}$, whence $\{\omega\}$ belongs to every σ-field containing $\mathcal{A}$. Set
$$\mathcal{F}_0 = \{A \subseteq \Omega \colon A \subseteq \Omega_0 \text{ or } A^c \subseteq \Omega_0\}.$$
Then, $\Omega^c = \emptyset \subseteq \Omega_0$, so $\Omega \in \mathcal{F}_0$. Next, $A \in \mathcal{F}_0$ means that one of A, A^c is a subset of Ω_0, whence $A^c \in \mathcal{F}_0$. Finally, suppose that $A = \bigcup_{n \in \mathbb{N}} A_n$ where $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0$. If all $A_n \subseteq \Omega_0$, we have $A \subseteq \Omega_0$, whence $A \in \mathcal{F}_0$. Otherwise, some $A_k^c \subseteq \Omega_0$, so $A^c = \bigcap_{n \in \mathbb{N}} A_n^c \subseteq A_k^c \subseteq \Omega_0$, whence $A \in \mathcal{F}_0$. This shows that $\mathcal{F}_0$ is a σ-field in Ω. Furthermore, $A \subseteq \Omega_0$ for $A \in \mathcal{A}$, so $\mathcal{A} \subseteq \mathcal{F}_0$, hence $\mathcal{F} \subseteq \mathcal{F}_0$. However, the uncountable set $\{\omega\}^c \notin \mathcal{F}_0$, yet $\{\omega\}^c \in \mathcal{F}$, a contradiction. This proves that Ω must be countable.

3. Set $\mathcal{F}_2 = \mathcal{B}(\mathbb{R})$, and
$$\mathcal{F}_1 = \{A \subseteq \mathbb{R} \colon A \text{ is countable, or } A^c \text{ is countable}\}.$$
Then, $\mathcal{F}_1$ is not countably generated as $\mathbb{R}$ is uncountable, whereas $\mathcal{F}_2$ is countably generated.

   Note that singletons $\{x\}$ for $x \in \mathbb{R}$, being closed in $\mathbb{R}$, are Borel sets. Thus, if $A \subseteq \mathbb{R}$ is countable, then $A = \bigcup_{x \in A} \{x\} \in \mathcal{B}(\mathbb{R})$. Otherwise, if A^c is countable, then $A^c = \bigcup_{x \in A^c} \{x\} \in \mathcal{B}(\mathbb{R})$, hence $A \in \mathcal{B}(\mathbb{R})$. It follows that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Problem 4

1. We have infinite Ω, a field

$$\mathcal{F} = \{A \subseteq \Omega \colon |A| < \infty \text{ or } |A^c| < \infty\},$$

and the map

$$\mathbb{P} \colon \mathcal{F} \to [0, \infty], \qquad A \mapsto \begin{cases} 0, & \text{if } |A| < \infty, \\ 1, & \text{if } |A^c| < \infty. \end{cases}$$

Suppose that $A = \bigcup_{i=1}^n A_i \in \mathcal{F}$ for pairwise disjoint $A_1, \dots, A_n \in \mathcal{F}$. If all $|A_i| < \infty$, then $|A| < \infty$, so

$$\mathbb{P}(A) = 0 = \sum_{i=1}^n 0 = \sum_{i=1}^n \mathbb{P}(A_i).$$

Otherwise, at least one $|A_k| = \infty$ with $|A_k^c| < \infty$. Now, for $i \neq k$, we must have $A_i \subseteq A_k^c$ by pairwise disjointness, hence $|A_i| < \infty$. Since $A_k \subseteq A \in \mathcal{F}$, we must have $|A| = \infty$, forcing $|A^c| < \infty$. Together,

$$\mathbb{P}(A) = 1 = \mathbb{P}(A_k) + \sum_{i \neq k} \mathbb{P}(A_i) = \sum_{i=1}^n \mathbb{P}(A_i).$$

Thus, $\mathbb{P}$ is finitely additive.

2. Enumerate $\Omega = \{\omega_i\}_{i \in \mathbb{N}}$, so

$$\mathbb{P}(\Omega) = 1 \neq 0 = \sum_{i=1}^n \mathbb{P}(\{\omega_i\}).$$

3. Let Ω be uncountable, and suppose that $A = \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$ for pairwise disjoint $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}$. Again, if all $|A_i| < \infty$, then A must be countable. Now, $|A^c| < \infty$ would mean that $A = \Omega \setminus A^c$ is uncountable (since Ω is uncountable), a contradiction. Thus, we must have $|A| < \infty$, whence

$$\mathbb{P}(A) = 0 = \sum_{i=1}^\infty 0 = \sum_{i=1}^\infty \mathbb{P}(A_i).$$

Otherwise, suppose that at least one $|A_k| = \infty$. Then, $|A_k^c| < \infty$, so $A^c \subseteq A_k^c$ gives $|A^c| < \infty$. Again, for $i \neq k$, pairwise disjointness forces $A_i \subseteq A_k^c$ hence $|A_i| < \infty$. Together,

$$\mathbb{P}(A) = 1 = \mathbb{P}(A_k) + \sum_{i \neq k} \mathbb{P}(A_i) = \sum_{i=1}^\infty \mathbb{P}(A_i).$$

Thus, $\mathbb{P}$ is countably additive.

4. We have uncountable Ω, a σ-field

$$\mathcal{G} = \{A \subseteq \Omega \colon A \text{ is countable, or } A^c \text{ is countable}\},$$

and the map

$$\mathbb{P} \colon \mathcal{G} \to [0, \infty], \qquad A \mapsto \begin{cases} 0, & \text{if } A \text{ is countable}, \\ 1, & \text{if } A^c \text{ is countable}. \end{cases}$$

Let $A = \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{G}$ for pairwise disjoint $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{G}$. If all A_i are countable, so is their union A, hence $P(A) = \sum_{n \in \mathbb{N}} P(A_i) = 0$. Otherwise, some A_k is uncountable, hence so is A; $A_k \in \mathcal{G}$ forces A_k^c countable, hence $A^c \subseteq A_k^c$ countable. Now, for $i \neq k$, pairwise disjointness gives $A_i \subseteq A_k^c$ countable. Together,

$$\mathbb{P}(A) = 1 = \mathbb{P}(A_k) + \sum_{i \neq k} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Thus, $\mathbb{P}$ is countably additive.

Problem 5

Given non-negative, finite, finitely additive μ on $\mathbb{R}^k$ (we assume on $\mathcal{B}(\mathbb{R})$, or a larger σ-field, since we need to talk about $\mu(K)$ for compact K), satisfying

$$\mu(A) = \sup\{\mu(K) \colon K \subseteq A \text{ is compact}\}.$$

To show that μ is countably additive, it suffices to prove continuity at $\emptyset$, i.e. for $\mu(B_i) \to 0$ whenever $B_i \downarrow \emptyset$. With this, for pairwise disjoint $\{A_i\}_{i \in \mathbb{N}}$ with $A = \bigcup_{i \in \mathbb{N}} A_i$, we may set $B_n = A \setminus \bigcup_{i=1}^n A_i = \bigcup_{i>n} A_i$. Then, each $B_{n+1} \subseteq B_n$, and $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$, i.e. $B_n \downarrow \emptyset$. It follows that $\mu(B_n) \to 0$; now, use countably additivity to write

$$\mu(A) = \mu\left(\bigcup_{i=1}^n A_i\right) + \mu(B_n) = \sum_{i=1}^n \mu(A_i) + \mu(B_n),$$

and take limits $n \to \infty$ so see that

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i),$$

whence μ is countably additive.

It remains to show that μ is continuous at $\emptyset$. Given $B_i \downarrow \emptyset$, let $\epsilon > 0$ be arbitrary, and choose $\{K_i\}$ such that each $K_i \subseteq B_i$ is compact, with $\mu(B_i) < \mu(K_i) + \epsilon/2^i$. The latter can be written as $\mu(B_i \setminus K_i) \leq \epsilon/2^i$, since μ is finite. Then, $\bigcap_{i \in \mathbb{N}} K_i \subseteq \bigcap_{i \in \mathbb{N}} B_i = \emptyset$. It follows from the properties of compact sets that there exists $N \in \mathbb{N}$ for which $\bigcap_{i=1}^N K_i = \emptyset$. Thus,

$$\mu(B_N) = \mu\left(B_N \setminus \bigcap_{i=1}^N K_i\right) = \mu\left(\bigcup_{i=1}^N B_N \setminus K_i\right) \leq \mu\left(\bigcup_{i=1}^N B_i \setminus K_i\right) < \epsilon.$$

Indeed, $\mu(B_n) < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, we must have $\mu(B_n) \to 0$, completing the proof.

Remark: Recall that if $\{K_n\}_{n \in \mathbb{N}}$ are compact sets in $\mathbb{R}^k$, with $K^N = \bigcap_{n=1}^N K_n \neq \emptyset$ for all $N \in \mathbb{N}$, then $K = \bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$. This is because one can construct a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^k$ where each $x_N \in K^N$. For each $N \in \mathbb{N}$, note that the N-tail of this sequence is contained in K^N, hence there exists a convergent subsequence $x^N_{n_N(m)} \to x^N \in K^N$ as $m \to \infty$. Again, $\{x^n\}$ is a sequence contained in the compact set K_1, and thus must contain a convergent subsequence $x^{n_m} \to x$ as $m \to \infty$. We claim that $x \in K$; indeed, every K^N contains a tail of the sequence x^{n_m}, hence must contain the limit x. This immediately gives $x \in \bigcap_{N \in \mathbb{N}} K^N = K$, whence $K \neq \emptyset$.