MA2202: PROBABILITY I

Conditional probability

Spring 2021

Satvik Saha
19MS154

*Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.*

Suppose a fair die is rolled and the outcome is even. The probability that the outcome was prime is $1/3$. This is because the only even prime is 2, and we know that we have rolled one of $2, 4$ or 6, all of which are equally likely.

Definition 2.1 (Conditional probability)**.** Suppose a random experiment is performed, and we know that an event B occurred. Let $A \in \mathcal{E}$. The probability of occurrence of A given B is called the conditional probability $P(A \mid B)$.

Remark. Note that the conditional probability A given B is equivalent to the occurrence of $A \cap B$ computed relative to the probability of the occurrence of B.

$$P(A \mid B)\, P(B) = P(A \cap B).$$

Lemma 2.1. *If $A_1, \dots, A_n$ are exhaustive and pairwise mutually exclusive events such that $P(A_i) > 0$, and $B \in \mathcal{E}$, then*

$$P(B) = \sum P(A_i)\, P(B \mid A_i).$$

Theorem 2.2 (Bayes' Theorem)**.** *If $A, B \in \mathcal{E}$ with $P(B) \neq 0$, we have*

$$P(A \mid B) = \frac{P(A)}{P(B)} P(B \mid A).$$

Proof. This follows directly from

$$P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A). \qquad \square$$

Example. Suppose a fair die is rolled and the outcome is even. The probability that the outcome is prime is $1/3$. This is because the only even prime is 2, the available even numbers are $2, 4, 6$, and the available primes are $2, 3, 5$. Thus,

$$P(\text{prime} \mid \text{even}) = \frac{P(\text{even})}{P(\text{prime})} P(\text{even} \mid \text{prime}) = \frac{3/6}{3/6} \cdot \frac{1}{3} = \frac{1}{3}.$$

Corollary 2.2.1. *If* $A_1, \ldots, A_n$ *are exhaustive and mutually exclusive with* $P(A_i) > 0$*, and* $B \in \mathcal{E}$ *such that* $P(B) > 0$*, then*

$$P(A_j \mid B) = \frac{P(A_j)\, P(B \mid A_j)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}.$$

Definition 2.2 (Independent events)**.** If $A, B \in \mathcal{E}$ such that $P(A), P(B) \neq 0$, and we have

$$P(A \mid B) = P(A), \qquad P(B \mid A) = P(B),$$

then we say that A and B are independent.

Equivalently, $P(A \cap B) = P(A)P(B)$.

Definition 2.3 (Pairwise independent events)**.** A set of events $S \subseteq \mathcal{E}$ is called pairwise independent if the events A and B are independent for all $A, B \in S$.

Definition 2.4 (Mutually independent events)**.** The set of events $A_1, \ldots, A_n$ is called mutually independent if

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

for all subsets $I \subseteq \{1, \ldots, n\}$.