MA4203: Probability II

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Exercise 1 Let $\{X_n\}$ be a sequence of pairwise independent and identically distributed random variables. Show that $X_n/n \to 0$ almost surely if and only if $E(|X_1|) < \infty$.

Solution.

$$\begin{aligned}
X_n/n \xrightarrow{as} 0 \quad &\iff \quad \sum_{n=1}^{\infty} P(|X_n/n| > \epsilon) < \infty \text{ for all } \epsilon > 0 && \text{(Borel-Cantelli)} \\
&\iff \quad \sum_{n=1}^{\infty} P(|X_n/\epsilon| > n) < \infty \text{ for all } \epsilon > 0 \\
&\iff \quad \sum_{n=1}^{\infty} P(|X_1/\epsilon| > n) < \infty \text{ for all } \epsilon > 0 && \text{(Identical distributions)} \\
&\iff \quad E(|X_1/\epsilon|) < \infty \text{ for all } \epsilon > 0 \\
&\iff \quad E(|X_1|).
\end{aligned}$$

Remark. Pairwise independence is only needed for the forward implication in the first step.

Exercise 2 Let $\{X_n\}$ be a sequence of identically distributed variables, and let $M_n = \max\{|X_1|, \dots, |X_n|\}$.

(a) If $E(|X_1|) < \infty$, show that $M_n/n \to 0$ almost surely.

(b) If $E(|X_1|^\alpha) < \infty$ for some $\alpha \in (0, \infty)$, then show that $M_n/n^{1/\alpha} \to 0$ almost surely.

(c) If $\{X_n\}$ is also independent, then show that $M_n/n^{1/\alpha} \to 0$ almost surely implies that $E(|X_1|^\alpha) < \infty$.

Solution.

(a) We have $E(|X_1|) < \infty \implies X_n/n \xrightarrow{as} 0$, hence there exists $A \subseteq \Omega$ with $P(A) = 1$ such that $X_n(\omega)/n \to 0$ for all $\omega \in A$. Set $x_n = X_n(\Omega)$, $m_n = M_n(\omega)$, and let $\epsilon > 0$, whence there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n|/n < \epsilon$. Now, observe that for all $n > N$,

$$\frac{m_n}{n} \leq \frac{\max\{|x_1|, \dots, |x_N|\}}{n} + \max\left\{\frac{|x_{N+1}|}{N+1}, \dots, \frac{|x_n|}{n}\right\}.$$

This is because either $m_n \in \{|x_1|, \dots, |x_N|\}$, or $m_n = |x_k|$ for some $N < k \leq n$, so $m_n/n \leq |x_k|/k$. Note that as $n \to \infty$, the first term vanishes and the second term is always bounded by ϵ. Thus, $m_n/n \to 0$, hence $M_n/n \xrightarrow{as} 0$.

(b) We have $E(|X_1|^\alpha) < \infty \implies M_n^\alpha/n \xrightarrow{as} 0$. By the continuous mapping theorem, $M_n/n^{1/\alpha} \xrightarrow{as} 0$.

(c) We have $M_n/n^{1/\alpha} \xrightarrow{as} 0 \implies |X_n|/n^{1/\alpha} \xrightarrow{as} 0$, as $|X_n| \leq M_n$. By the continuous mapping theorem, $|X_n|^\alpha/n \xrightarrow{as} 0$. Thus, the previous exercise gives $E(|X_1|^\alpha) < \infty$.

Exercise 3 Let $\{A_n\}$ be a sequence of independent events with all $P(A_n) < 1$. Show that

$$P(\limsup_{n\to\infty} A_n) = 1 \iff P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1.$$

Solution. We have

$$P(\limsup_{n\to\infty} A_n) = 1 \iff P\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = 1 \implies P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1.$$

For the reverse implication, it suffices to show that for all $k \geq 1$, we have

$$P\left(\bigcup_{n=k}^{\infty} A_n\right) = 1,$$

which is equivalent to

$$P\left(\bigcap_{n=k}^{\infty} A_n^c\right) = 0 \iff \prod_{n=k}^{\infty} P(A_n^c) = 0.$$

But this follows immediately since it holds for $k = 1$; each $P(A_n^c) > 0$ means that this term can be cancelled from the product yielding the equality for all $k > 1$.

Exercise 4 Let $\{X_n\}$ be a sequence of independent and identically distributed exponential random variables with density $f(x) = e^{-x}\chi_{(0,\infty)}(x)$. Define

$$Y_n = \max_{1\leq i\leq n} X_i.$$

(a) Show that

$$\sum_{n=1}^{\infty} P(X_n > \epsilon \log n)$$

converges if $\epsilon > 1$ and diverges if $0 < \epsilon \leq 1$.

(b) Show that $\limsup X_n/\log n = 1$ almost surely.

(c) Show that $\liminf X_n/\log n = 0$ almost surely.

(d) Show that $[X_n > \epsilon \log n \text{ i.o.}] \iff [Y_n > \epsilon \log n \text{ i.o.}]$, and hence $\limsup Y_n/n = 1$ almost surely.

Solution.

(a) Calculate

$$P(X_n > \epsilon \log n) = \int_{\epsilon \log n}^{\infty} e^{-x}\, dx = \frac{1}{n^{\epsilon}}.$$

Thus

$$\sum_{n=1}^{\infty} P(X_n > \epsilon \log n) = \sum_{n=1}^{\infty} \frac{1}{n^{\epsilon}}$$

converges precisely when $\epsilon > 1$, and diverges when $0 < \epsilon \leq 1$.

(b) Putting $\epsilon = 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_n > \log n) = \infty &\iff P(X_n/\log n > 1 \text{ i.o.}) = 1 && \text{(Borel-Cantelli)}\\
&\implies P(\limsup X_n/\log n \geq 1) = 1\\
&\iff \limsup X_n/\log n = 1 \text{ almost surely.}
\end{aligned}$$

The second implication follows from the fact that there is a probability one set A such that for $\omega \in A$, we have $X_n(\omega)/\log n > 1$ infinitely often, i.e. there is a subsequence $\{n_k\}$ such that all $X_{n_k}(\omega)/\log n_k > 1$.

Next, for $\epsilon > 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_n > \epsilon \log n) < \infty \quad &\iff \quad P(X_n/\log n > \epsilon \text{ i.o.}) = 0 \qquad \text{(Borel-Cantelli)} \\
&\iff \quad P(X_n/\log n \le \epsilon \text{ eventually}) = 1 \\
&\implies \quad P(\limsup X_n/\log n \le \epsilon) = 1 \\
&\iff \quad \limsup X_n/\log n \le \epsilon \text{ almost surely.}
\end{aligned}$$

The third implication follows from the fact that there is a probability one set A_ϵ such that for $\omega \in A_\epsilon$, we have $X_{n_k}(\omega)/\log n_k \le \epsilon$ for all sufficiently large n_k, for all subsequences $\{n_k\}$. Since all $A_{1+1/k}$ have probability one, their intersection has probability one by continuity from above, yielding

$$\limsup X_n/\log n \le 1 \text{ almost surely.}$$

Combining the two parts gives the desired result.

(c) Note that for all $0 < \epsilon < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_n < \epsilon \log n) = \sum_{n=1}^{\infty} 1 - \frac{1}{\epsilon} = \infty \quad &\iff \quad P(X_n/\log n < \epsilon \text{ i.o.}) = 1 \\
&\implies \quad P(\liminf X_n/\log n \le \epsilon) = 1.
\end{aligned}$$

Putting $\epsilon = 1/k \to 0$, continuity from above gives

$$\liminf X_n/\log n = 0.$$

(d) It is equivalent to show that

$$A = [X_n \le \epsilon \log n \text{ eventually}] \quad \iff \quad [Y_n \le \epsilon \log n \text{ eventually}] = B.$$

Since $X_n \le Y_n$, we have $\omega \in B \implies Y_n(\omega) \le \epsilon \log n$ for $n \ge N_\epsilon$, hence $X_n(\omega \le \epsilon \log n)$ for $n \ge N_\epsilon$ $\implies \omega \in A$.

Next, let $\omega \in A$, and let $N \in \mathbb{N}$ such that for all $n > N$, we have $X_n(\omega) \le \epsilon \log n$. Then,

$$\frac{Y_n(\omega)}{\log n} \le \frac{\max\{X_1(\omega), \dots, X_N(\omega)\}}{\log n} + \max\left\{\frac{X_{N+1}(\omega)}{\log(N+1)}, \dots, \frac{X_n(\omega)}{\log n}\right\}.$$

The second term is bounded by ϵ, while the first vanishes as $n \to \infty$. Thus, $Y_n(\omega)/\log n \le 2\epsilon$ eventually, so $\omega \in B$.

We have shown that $A = B$; repeating the proof of (b) with $X_n > \epsilon \log n$ i.o. replaced with $Y_n > \epsilon \log n$ i.o. proves that $\limsup Y_n/\log n = 1$ almost surely.

Exercise 5 Show that for any given sequence of random variables $\{X_n\}$, there exists a (deterministic) real sequence $\{a_n\}$ such that $X_n/a_n \xrightarrow{as} 0$.

Solution. Note that for any fixed X_n, we have $P(|X_n| > M) \to 0$ as $M \to \infty$. Thus, for each $n \in \mathbb{N}$, we can pick numbers M_n such that

$$P(|X_n| > M_n) < \frac{1}{2^n}.$$

Set $a_n = nM_n$. Then for $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n/a_n| > \epsilon) \;=\; \sum_{n=1}^{\infty} P(|X_n| > \epsilon n M_n).$$

Let $N \in \mathbb{N}$ such that $N\epsilon > 1$. Then the tail of the above series is

$$\sum_{n=N}^{\infty} P(|X_n| > \epsilon n M_n) \;\le\; \sum_{n=N}^{\infty} P(|X_n| > M_n) \;=\; \sum_{n=N}^{\infty} \frac{1}{2^n} \;<\; \infty.$$

Exercise 6 Let $\{X_n\}$ be a sequence of independent random variables such that $E(X_n) = 0$, $E(X_n^2) = \sigma^2$. Define $s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \to \infty$. Show that for any $a > 1/2$,

$$Y_n = \frac{1}{s_n (\log s_n^2)^a} \sum_{k=1}^{n} X_k \xrightarrow{as} 0.$$

Solution. Set

$$Z_n = \frac{X_n}{s_n (\log s_n^2)^a}.$$

Without loss of generality, let $s_1 > 1$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} V(Z_n) &= \sum_{n=1}^{\infty} \frac{\sigma_n^2}{s_n^2 (\log s_n^2)^{2a}} \\
&= \sum_{n=1}^{\infty} \frac{s_n^2 - s_{n-1}^2}{s_n^2 (\log s_n^2)^{2a}} \\
&\leq \sum_{n=1}^{\infty} \int_{s_{n-1}^n}^{s_n^2} \frac{dx}{x (\log x)^{2a}} \\
&= \sum_{n=1}^{\infty} - \frac{1}{(2a-1) \log(x)^{2a-1}} \Big|_{s_{n-1}^2}^{s_n^2} \\
&= \frac{1}{2a-1} \sum_{n=1}^{\infty} \frac{1}{(\log s_{n-1}^2)^{2a-1}} - \frac{1}{(\log s_n^2)^{2a-1}} \ < \infty.
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} Z_n$ converges in L_2, hence in probability, hence almost surely by Levy's Theorem. Finally, Kronecker's Lemma gives

$$Y_n = \frac{1}{s_n (\log s_n^2)^a} \sum_{k=1}^{n} Z_k s_k (\log s_k^2)^a \xrightarrow{as} 0.$$

Exercise 7 Let f be a bounded measurable function on $[0, 1]$ that is continuous at $1/2$. Evaluate

$$\lim_{n \to \infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 \, dx_2 \, \cdots \, dx_n.$$

Solution. Let U_n be independent and identically distributed random variables, having uniform distribution on $[0, 1]$. Then, the given integral is simply $E(f(\bar{U}_n))$, where $\bar{U}_n = (U_1 + \cdots + U_n)/n$. Since $E(|U_1|) < \infty$, Kolmogorov's Strong Law of Large Numbers gives $\bar{U}_n \to E(U_1) = 1/2$. Since f is continuous at $1/2$, we have $f(\bar{U}_n) \to f(1/2)$. Finally, f is bounded, so the desired limit is $f(1/2)$ by Lebesgue's dominated convergence theorem.

Exercise 8 Let $\{X_n\}$ be a sequence of independent and identically distributed random variables. Investigate the almost sure convergence/divergence of the series $\sum_{n=1}^{\infty} X_n$.

Solution. Kolmogorov's Three-Series Theorem says that if $\sum_{n=1}^{\infty} X_n$ converges almost surely, then for all $A > 0$,

$$\sum_{n=1}^{\infty} P(|X_1| > A) = \sum_{n=1}^{\infty} P(|X_n| > A) < \infty.$$

This sum is finite precisely when $P(|X_1| > A) = 0$ for all $A > 0$. Thus, the series $\sum_{n=1}^{\infty} X_n$ converges almost surely only when each X_n is a degenerate random variable with $P(X_n = 0) = 1$.

Exercise 9 Let $\{X_n\}$ be a sequence of independent random variables with each $X_n \sim N(\mu_n, \sigma_n^2)$. Show that $\sum_{n=1}^\infty X_n$ converges almost surely if and only if both $\sum_{n=1}^\infty \mu_n$ and $\sum_{n=1}^\infty \sigma_n^2$ converge.

Solution. ($\Longleftarrow$) We have $\sum_{n=1}^\infty V(X_n - \mu_n) = \sum_{n=1}^\infty \sigma_n^2 < \infty$, hence $\sum_{n=1}^\infty (X_n - \mu_n)$ converges almost surely. Using $\sum_{n=1}^\infty \mu_n < \infty$, we have $\sum_{n=1}^\infty X_n < \infty$ almost surely.

($\Longrightarrow$) First, suppose that all $\mu_n = 0$. Since $\sum_{n=1}^\infty X_n$ converges almost surely, Kolmogorov's Three-Series Theorem gives

$$\sum_{n=1}^\infty P(|X_n/\sigma_n| > A/\sigma_n) = \sum_{n=1}^\infty P(|X_n| > A) < \infty$$

for all $A > 0$; fix $A = 1$. Now, $Z_n = X_n/\sigma_n$ are independent and identically distributed standard normal random variables. Thus, $P(|Z_n| > A/\sigma_n) \to 0$ forces $\sigma_n \to 0$. We also have

$$\sum_{n=1}^\infty \sigma_n^2 E(Z_n^2 \chi_{|Z_n| \le A/\sigma_n}) = \sum_{n=1}^\infty E(X_n^2 \chi_{|X_n| \le A}) = \sum_{n=1}^\infty V(X_n \chi_{|X_n| \le A}) < \infty$$

Now, $A/\sigma_n \to \infty$, so there exists $N \in \mathbb{N}$ such that $A/\sigma_n \ge 1$ for all $n \ge N$. Then,

$$E(Z_n^2 \chi_{|Z_n| \le A/\sigma_n}) \ge E(Z_n^2 \chi_{|Z_n| \le 1}) = K > 0.$$

This immediately gives

$$\sum_{n=N}^\infty K\sigma_n^2 \le \sum_{n=N}^\infty \sigma_n^2 E(Z_n^2 \chi_{|Z_n| \ge A/\sigma_n}) < \infty,$$

hence $\sum_{n=1}^\infty \sigma_n^2 < \infty$.

We now return to the general case, where $Z_n = (X_n - \mu_n)/\sigma_n$ are independent and identically distributed standard normal variables. As before, the almost sure convergence of $\sum_{n=1}^\infty X_n$ gives

$$\sum_{n=1}^\infty P(\sigma_n Z_n \notin [-\mu_n - A, -\mu_n + A]) = \sum_{n=1}^\infty P(|X_n| > A) < \infty$$

for all $A > 0$. This implies that each term

$$\Phi\left(\frac{-\mu_n - A}{\sigma_n}\right) + \Phi\left(\frac{\mu_n - A}{\sigma_n}\right) = P\left(Z_n \notin \left[\frac{-\mu_n - A}{\sigma_n}, \frac{-\mu_n + A}{\sigma_n}\right]\right) \to 0.$$

This is possible only when both $(\mu_n + A)/\sigma_n \to \infty$ and $(-\mu_n + A)/\sigma_n \to \infty$. Thus, their sum $2A/\sigma_n \to \infty$, hence $\sigma_n \to 0$. If we denote the above interval by J_n, we have $\chi_{J_n} \to 1$. Also, we see that $\mu_n + A$ and $-\mu_n + A$ become positive eventually, hence $|\mu_n| < A$ eventually. Since $A > 0$ was arbitrary, we have $\mu_n \to 0$.

Next, we have

$$\sum_{n=1}^\infty E((\sigma_n Z_n + \mu_n)\chi_{Z_n \in J_n}) = \sum_{n=1}^\infty E(X_n \chi_{|X_n| \le A}) < \infty,$$

and

$$\sum_{n=1}^\infty E((\sigma_n Z_n + \mu_n)^2 \chi_{Z_n \in J_n}) - E((\sigma_n Z_n + \mu_n)\chi_{Z_n \in J_n})^2 = \sum_{n=1}^\infty V(X_n \chi_{|X_n| \le A}) < \infty.$$

Each term here can be broken down as (denote $\chi_n \equiv \chi_{Z_n \in J_n}$)

$$\begin{aligned} & \sigma_n^2 E(Z_n^2 \chi_n) + 2\mu_n \sigma_n E(Z_n \chi_n) + \mu_n^2 E(\chi_n) - \sigma^2 E(Z_n \chi_n)^2 - 2\mu_n \sigma_n E(Z_n) E(\chi_n) - \mu_n^2 E(\chi_n)^2 \\ = \; & \sigma_n^2 V(Z_n \chi_n) + 2\mu_n \sigma_n E(Z_n \chi_n) + \mu_n^2 V(\chi_n) \end{aligned}$$

Now, observe that $\mu_n E(Z_n \chi_n) \le 0$. To see this, suppose that $\mu_n > 0$; if $-\mu_n + A \le 0$, we clearly have $E(Z_n \chi_{Z_n \in J_n}) \le 0$. If $-\mu_n + A > 0$, note that

$$E(Z_n \chi_{Z_n \in J_n}) = \int_{-\mu_n/\sigma_n - A/\sigma_n}^{-\mu_n/\sigma_n + A/\sigma_n} t \, d\Phi(t) = \int_{-\mu_n/\sigma_n - A/\sigma_n}^{\mu_n/\sigma_n - A/\sigma_n} + \int_{\mu_n/\sigma_n - A/\sigma_n}^{-\mu_n/\sigma_n + A/\sigma_n} t \, d\Phi(t) \le 0.$$

This is because the first integral is negative, and the second is zero by symmetry. The case $\mu_n < 0$ is analogous.