MA1101: Mathematics I

Some Solutions for Problem Sheet I

Problem 1.

(i) We prove $A \cup B = B \cup A$ by first showing that $A \cup B \subseteq B \cup A$, then showing that $B \cup A \subseteq A \cup B$.

Consider

$$\begin{aligned} x \in A \cup B &\implies (x \in A) \text{ or } (x \in B) \\ &\implies (x \in B) \text{ or } (x \in A) \\ &\implies x \in B \cup A. \end{aligned}$$

This proves that $A \cup B \subseteq B \cup A$. Similarly, consider

$$\begin{aligned} x \in B \cup A &\implies (x \in B) \text{ or } (x \in A) \\ &\implies (x \in A) \text{ or } (x \in B) \\ &\implies x \in A \cup B. \end{aligned}$$

This proves that $B \cup A \subseteq A \cup B$. □

(ii) We prove that $(A \cup B) \cup C = A \cup (B \cup C)$ by first showing that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$, then showing that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Consider

$$\begin{aligned} x \in (A \cup B) \cup C &\implies (x \in A \cup B) \text{ or } (x \in C) \\ &\implies ((x \in A) \text{ or } (x \in B)) \text{ or } (x \in C) \\ &\implies (x \in A) \text{ or } (x \in B) \text{ or } (x \in C) \\ &\implies (x \in A) \text{ or } ((x \in B) \text{ or } (x \in C)) \\ &\implies (x \in A) \text{ or } (x \in B \cup C) \\ &\implies x \in A \cup (B \cup C). \end{aligned}$$

This proves that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Similarly, consider

$$\begin{aligned} x \in A \cup (B \cup C) &\implies (x \in A) \text{ or } (x \in B \cup C) \\ &\implies (x \in A) \text{ or } ((x \in B) \text{ or } (x \in C)) \\ &\implies (x \in A) \text{ or } (x \in B) \text{ or } (x \in C) \\ &\implies ((x \in A) \text{ or } (x \in B)) \text{ or } (x \in C) \\ &\implies (x \in A \cup B) \text{ or } (x \in C) \\ &\implies x \in (A \cup B) \cup C. \end{aligned}$$

This proves that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. □

(iii) ($\Rightarrow$) Suppose that $A \subseteq B$. To show that $A \cup B = B$, we first show that $A \cup B \subseteq B$, then show that $B \subseteq A \cup B$.

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Consider

$$\begin{aligned} x \in (A \cup B) &\implies (x \in A) \text{ or } (x \in B) \\ &\implies (x \in B) \text{ or } (x \in B) && \text{(Using } A \subseteq B\text{)} \\ &\implies x \in B. \end{aligned}$$

This proves that $A \cup B \subseteq B$. Similarly, consider

$$\begin{aligned} x \in B &\implies (x \in A) \text{ or } (x \in B) \\ &\implies x \in A \cup B. \end{aligned}$$

This proves that $B \subseteq A \cup B$.

($\Leftarrow$) Suppose that $A \cup B = B$. Consider

$$\begin{aligned} x \in A &\implies (x \in A) \text{ or } (x \in B) \\ &\implies x \in A \cup B \\ &\implies x \in B. && \text{(Using } A \cup B = B\text{)} \end{aligned}$$

This proves that $A \subseteq B$.

(vii) Recall that $X \setminus Y = X \cap Y^{\mathsf{c}}$. We compute

$$\begin{aligned} A \setminus (B \cup C) &= A \cap (B \cup C)^{\mathsf{c}} \\ &= A \cap (B^{\mathsf{c}} \cap C^{\mathsf{c}}) && \text{(De Morgan's Law)} \\ &= (A \cap A) \cap (B^{\mathsf{c}} \cap C^{\mathsf{c}}) \\ &= (A \cap B^{\mathsf{c}}) \cap (A \cap C^{\mathsf{c}}) \\ &= (A \setminus B) \cap (A \setminus C). && \square \end{aligned}$$

(xi) First, note that $X \Delta Y = Y \Delta X$, since

$$\begin{aligned} X \Delta Y &= (X \setminus Y) \cup (Y \setminus X) \\ &= (Y \setminus X) \cup (X \setminus Y) && \text{(Using (i))} \\ &= Y \Delta X. \end{aligned}$$

Next, observe that

$$\begin{aligned} X \Delta Y &= (X \setminus Y) \cup (Y \setminus X) \\ &= (X \cap Y^{\mathsf{c}}) \cup (Y \cap X^{\mathsf{c}}). \end{aligned}$$

With this, we compute

$$\begin{aligned} A\Delta(B\Delta C) &= A\Delta((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C)) \\ &= (A \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))^{\mathsf{c}}) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) \\ &= (A \cap ((B \cap C^{\mathsf{c}})^{\mathsf{c}} \cap (B^{\mathsf{c}} \cap C)^{\mathsf{c}})) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) && \text{(De Morgan's Law)} \\ &= (A \cap ((B^{\mathsf{c}} \cup C) \cap (B \cup C^{\mathsf{c}}))) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) && \text{(De Morgan's Law)} \\ &= (A \cap (((B^{\mathsf{c}} \cup C) \cap B) \cup ((B^{\mathsf{c}} \cup C) \cap C^{\mathsf{c}}))) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) \\ & && \text{(Distributive Law)} \\ &= (A \cap (((B^{\mathsf{c}} \cap B) \cup (C \cap B)) \cup ((B^{\mathsf{c}} \cap C^{\mathsf{c}}) \cup (C \cap C^{\mathsf{c}})))) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) \\ & && \text{(Distributive Law)} \\ &= (A \cap ((\emptyset \cup (B \cap C)) \cup ((B^{\mathsf{c}} \cap C^{\mathsf{c}}) \cup \emptyset))) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) \\ &= (A \cap ((B \cap C)) \cup (B^{\mathsf{c}} \cap C^{\mathsf{c}}))) \cup (A^{\mathsf{c}} \cap ((B \cap C^{\mathsf{c}}) \cup (B^{\mathsf{c}} \cap C))) \\ &= (A \cap B \cap C) \cup (A \cap B^{\mathsf{c}} \cap C^{\mathsf{c}}) \cup (A^{\mathsf{c}} \cap B \cap C^{\mathsf{c}}) \cup (A^{\mathsf{c}} \cap B^{\mathsf{c}} \cap C) && \text{(Distributive Law)} \end{aligned}$$

Interchanging the roles of A and C in the previous argument, we obtain

$$\begin{aligned}
C\Delta(B\Delta A) &= (C\cap B\cap A)\cup(C\cap B^{\mathsf{c}}\cap A^{\mathsf{c}})\cup(C^{\mathsf{c}}\cap B\cap A^{\mathsf{c}})\cup(C^{\mathsf{c}}\cap B^{\mathsf{c}}\cap A)\\
&= (A\cap B\cap C)\cup(A^{\mathsf{c}}\cap B^{\mathsf{c}}\cap C)\cup(A^{\mathsf{c}}\cap B\cap C^{\mathsf{c}})\cup(A\cap B^{\mathsf{c}}\cap C^{\mathsf{c}})\\
&= (A\cap B\cap C)\cup(A\cap B^{\mathsf{c}}\cap C^{\mathsf{c}})\cup(A^{\mathsf{c}}\cap B\cap C^{\mathsf{c}})\cup(A^{\mathsf{c}}\cap B^{\mathsf{c}}\cap C)\\
&= A\Delta(B\Delta C).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
A\Delta(B\Delta C) &= C\Delta(B\Delta A)\\
&= (B\Delta A)\Delta C && \text{(Using } X\Delta Y = Y\Delta X\text{)}\\
&= (A\Delta B)\Delta C. && \text{(Using } X\Delta Y = Y\Delta X\text{)}
\end{aligned}$$

Problem 2.

(iii) We prove $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ by showing that each side is a subset of the other. Consider

$$\begin{aligned}
(x,y) \in A \times (B \setminus C) &\implies (x \in A) \text{ and } (y \in B \setminus C) \\
&\implies (x \in A) \text{ and } (y \in B \cap C^{\mathsf{c}}) \\
&\implies (x \in A) \text{ and } ((y \in B) \text{ and } (y \in C^{\mathsf{c}})) \\
&\implies (x \in A) \text{ and } (y \in B) \text{ and } (y \in C^{\mathsf{c}}) \\
&\implies ((x \in A) \text{ and } (y \in B)) \text{ and } (y \notin C) \\
&\implies ((x,y) \in A \times B) \text{ and } ((x,y) \notin A \times C) \\
&\implies (x,y) \in (A \times B) \cap (A \times C)^{\mathsf{c}} \\
&\implies (x,y) \in (A \times B) \setminus (A \times C).
\end{aligned}$$

This proves that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$. Similarly, consider

$$\begin{aligned}
(x,y) \in (A \times B) \setminus (A \times C) &\implies ((x,y) \in A \times B) \text{ and } ((x,y) \notin A \times C) \\
&\implies ((x \in A) \text{ and } (y \in B)) \text{ and } ((x \notin A) \text{ or } (y \notin C)) \\
&\implies ((x \in A) \text{ and } (y \in B) \text{ and } (x \notin A)) \text{ or } ((x \in A) \text{ and } (y \in B) \text{ and } (y \notin C)) \\
&\implies (x \in A) \text{ and } ((y \in B) \text{ and } (y \notin C)) \\
&\implies (x \in A) \text{ and } (y \in B \setminus C) \\
&\implies (x,y) \in A \times (B \setminus C).
\end{aligned}$$

This proves that $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$. $\square$

(iv) No. Consider the following counterexample.
Let $A = \{0\}, B = \{1\}$. Then,

$$\begin{aligned}
A \times B &= \{(0,1)\}, \\
\mathcal{P}(A \times B) &= \{\emptyset, \{(0,1)\}\}, \\
\mathcal{P}(A) &= \{\emptyset, \{0\}\}, \\
\mathcal{P}(B) &= \{\emptyset, \{1\}\}, \\
\mathcal{P}(A) \times \mathcal{P}(B) &= \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\{0\}, \emptyset), (\{0\}, \{1\})\}.
\end{aligned}$$

(v) Yes. Consider

$$\begin{aligned}
(x,y) \in (A \cap C) \times (B \cap D) &\iff (x \in A \cap C) \text{ and } (y \in B \cap D) \\
&\iff ((x \in A) \text{ and } (x \in C)) \text{ and } ((y \in B) \text{ and } (y \in D)) \\
&\iff ((x \in A) \text{ and } (y \in B)) \text{ and } ((x \in C) \text{ and } (y \in D)) \\
&\iff ((x,y) \in A \times B) \text{ and } ((x,y) \in C \times D) \\
&\iff (x,y) \in (A \times B) \cap (C \times D).
\end{aligned}$$

$\square$

(vi) No. Consider the following counterexample.
Let $A = \{0\}, B = \{1\}, C = \{2\}, D = \{3\}$. Then,

$$\begin{aligned}
A \cup C &= \{0,2\}, \\
B \cup D &= \{1,3\}, \\
(A \cup C) \times (B \cup D) &= \{(0,1), (0,3), (2,1), (2,3)\}, \\
A \times B &= \{(0,1)\}, \\
C \times D &= \{(2,3)\}, \\
(A \times B) \cup (C \times D) &= \{(0,1), (2,3)\}.
\end{aligned}$$

Problem 3.

(i) The number of subsets of X is 2^n.

To prove this, note that for each $x \in X$, we can either choose it or leave it aside when forming a subset of X. In other words, each of the n elements in X presents us with 2 choices, giving us a total of 2^n ways of forming subsets of X. Moreover, every subset of X can be formed in this manner.

(ii) There are $2^n - 1$ non-empty subsets of X.

There is precisely one empty subset out of the 2^n subsets of X.

(iii) There are $(3^n + 1)/2$ ways of choosing two disjoint subsets of X.

For each $x \in X$, we can either place it in one subset, a second subset, or leave it aside. This gives us a total of 3^n ways of forming an ordered pair (A, B) of disjoint subsets A, B of X. However, we are looking for the number of unordered pairs of disjoint subsets. Thus, we have double-counted all cases where $A \neq B$, of which there are $3^n - 1$; the only case where $A = B$ is when $A = B = \emptyset$. This leaves us with $3^n - (3^n - 1)/2 = (3^n + 1)/2$ ways.

(iv) There are $(3^n - 2^{n+1} + 1)/2$ ways of choosing two non-empty disjoint subsets of X.

Of the $(3^n + 1)/2$ ways of choosing two disjoint subsets of X, consider the case where one of them is empty. This means that the other subset is simply an arbitrary subset of X, of which there are 2^n. Removing these from our count leaves precisely all disjoint non-empty pairs of subsets of X. Thus, we have $(3^n + 1)/2 - 2^n = (3^n - 2^{n+1} + 1)/2$ ways.