MA 1201 : Mathematics II

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Solution 1.

(i) We claim that
$$\lim_{n\to\infty} \frac{n}{n^2+1} \;=\; 0.$$
To prove this, let $\epsilon > 0$. We seek $k(\epsilon) \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,
$$\left|\frac{n}{n^2+1}\right| \;<\; \epsilon.$$
Now, since $n^2 + 1 > n^2$,
$$\frac{n}{n^2+1} \;<\; \frac{n}{n^2} \;=\; \frac{1}{n}.$$
Thus, setting $k(\epsilon) = \lfloor 1/\epsilon \rfloor + 1 > 1/\epsilon$, for all $n \geq k$,
$$\frac{n}{n^2+1} \;<\; \frac{1}{n} \;\leq\; \frac{1}{k} < \epsilon.$$
This completes the proof. □

(ii) We claim that
$$\lim_{n\to\infty} \frac{2n}{n+1} \;=\; 2.$$
To prove this, let $\epsilon > 0$. We seek $k(\epsilon) \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,
$$\left|\frac{2n}{n+1} - 2\right| \;=\; \frac{2}{n+1} \;<\; \epsilon.$$
Now,
$$\frac{2}{n+1} \;<\; \frac{2}{n}.$$
Thus, setting $k(\epsilon) = \lfloor 2/\epsilon \rfloor + 1 > 2/\epsilon$ completes the proof. □

(iii) We claim that
$$\lim_{n\to\infty} \frac{3n+1}{2n+5} \;=\; \frac{3}{2}.$$
To prove this, let $\epsilon > 0$. We seek $k(\epsilon) \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,
$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| \;=\; \frac{13/2}{2n+5} \;<\; \epsilon.$$
Now,
$$\frac{13/2}{2n+5} \;<\; \frac{13}{4n}.$$
Thus, setting $k(\epsilon) = \lfloor 13/4\epsilon \rfloor + 1 > 13/4\epsilon$ completes the proof. □

(iv) We claim that

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} \;=\; \frac{1}{2}.$$

To prove this, let $\epsilon > 0$. We seek $k(\epsilon) \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| \;=\; \frac{5/2}{2n^2+3} \;<\; \epsilon.$$

Now,

$$\frac{5/2}{2n^2+3} \;<\; \frac{5}{4n^2} \;\leq\; \frac{5}{4n}.$$

Thus, setting $k(\epsilon) = \lfloor 5/4\epsilon \rfloor + 1 > 5/4\epsilon$ completes the proof. □

Solution 2. Let $x_n \geq 0$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = L$. We claim that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}$.

To prove this, let $\epsilon > 0$ be given.
Note that since $x_n \geq 0$, we must have $L \geq 0$.[†]

If $L = 0$, then we find $k' \in \mathbb{N}$ such that for all $n \geq k'$, $n \in \mathbb{N}$, $|x_n| < \epsilon^2$. Thus, we have $|\sqrt{x_n}| < \epsilon$ for all $n \geq k'$, as desired.
Otherwise, $L > 0$. Since $\{x_n\}_n$ converges to L, we find $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,

$$|x_n - L| < \sqrt{L}\,\epsilon.$$

Now, for all $n \geq k$, $n \in \mathbb{N}$,

$$|\sqrt{x_n} - \sqrt{L}| \;=\; \frac{|x_n - L|}{|\sqrt{x_n} + \sqrt{L}|} \;<\; \frac{\sqrt{L}\,\epsilon}{\sqrt{x_n} + \sqrt{L}} \;\leq\; \epsilon.$$

This proves our claim. □

Solution 3. Let $\lim_{n\to\infty} x_n = L$. We claim that $\lim_{n\to\infty} |x_n| = |L|$.
To prove this, let $\epsilon > 0$. We find $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,

$$|x_n - L| < \epsilon.$$

Now, for all $n \geq k$, $n \in \mathbb{N}$,

$$||x_n| - |L|| \leq |x_n - L| < \epsilon.^{\ddagger}$$

This proves our claim. □

The converse of the given statement is false. We supply the counterexample $x_n = (-1)^n$ for all $n \in \mathbb{N}$. The sequence $\{|x_n|\}_n = \{1\}_n$ clearly converges to 1, yet $\{(-1)^n\}_n$ diverges.

[†] If $L < 0$, we find $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$, $|x_n - L| < -L$. This implies that $L - (-L) < x_n < L + (-L)$, i.e. $2L < x_n < 0$, a contradiction.

[‡] The Triangle Inequality gives

$$|x_n| = |(x_n - L) + L| \leq |x_n - L| + |L|,$$
$$|L| = |(L - x_n) + x_n \leq |x_n - L| + |x_n|.$$

Thus,

$$-|x_n - L| \leq |x_n| - |L| \leq |x_n - L|.$$

Solution 4. Let $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = L$. We claim that $\lim_{n\to\infty} z_n = L$, where $\{z_n\}_n$ is the sequence defined by

$$\begin{aligned} z_{2n-1} &= x_n \\ z_{2n} &= y_n \end{aligned}$$

for all $n \in \mathbb{N}$.
To prove this, let $\epsilon > 0$. We find $k_1, k_2 \in \mathbb{N}$ such that

$$|x_n - L| < \epsilon, \quad \text{for all } n \geq k_1, n \in \mathbb{N},$$

$$|y_n - L| < \epsilon, \quad \text{for all } n \geq k_2, n \in \mathbb{N}.$$

Thus, for all $n \geq \max\{2k_1 - 1, 2k_2\}$, $n \in \mathbb{N}$,

$$\begin{aligned} |z_n - L| &= |z_{2m-1} - L| = |x_m - L| < \epsilon, \quad && \text{if } n \text{ is odd}, \\ |z_n - L| &= |z_{2m} - L| = |y_m - L| < \epsilon, && \text{if } n \text{ is even}. \end{aligned}$$

This proves our claim. $\square$

Solution 5.

(i) We claim that

$$\lim_{n\to\infty} (2^n + 3^n)^{\frac{1}{n}} = 3.$$

To prove this, we observe that for all $n \in \mathbb{N}$,

$$(0 + 3^n)^{\frac{1}{n}} < (2^n + 3^n)^{\frac{1}{n}} < (3^n + 3^n)^{\frac{1}{n}}.$$

Taking limits as $n \to \infty$, $(3^n)^{\frac{1}{n}} \to 3$ and $(2 \cdot 3^n)^{\frac{1}{n}} \to 1 \cdot 3 = 3$. Thus, using the Sandwich Theorem, we conclude that $(2^n + 3^n)^{\frac{1}{n}} \to 3$. $\square$

(ii) We claim that

$$\lim_{n\to\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = 0.$$

To prove this, we set

$$x_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \prod_{k=1}^{n} \frac{2k-1}{2k}.$$

Now, $(n+1)^2 = n^2 + 2n + 1 > n^2 + 2n = n(n+1)$, for all $n \in \mathbb{N}$. Thus, $\frac{n}{n+1} < \frac{n+1}{n+2}$. Therefore,

$$x_n^2 = \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{2k-1}{2k} < \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{2k}{2k+1} = \frac{1}{2n+1}.$$

Using $x_n > 0$, for all $n \in \mathbb{N}$, we have

$$0 < x_n < \frac{1}{\sqrt{2n+1}}.$$

Taking limits as $n \to \infty$, $\frac{1}{\sqrt{2n+1}} \to 0$. Hence, using the Sandwich Theorem, we conclude that $x_n \to 0$. $\square$

Remark. We can obtain slightly tighter bounds on x_n by observing that for all $k \in \mathbb{N}$,

$$\frac{4k-3}{4k+1} \leq \left(\frac{2k-1}{2k}\right)^2 \leq \frac{3k-2}{3k+1}.$$

This gives us

$$\prod_{k=1}^{n} \frac{4k-3}{4k+1} \leq \prod_{k=1}^{n} \left(\frac{2k-1}{2k}\right)^2 \leq \prod_{k=1}^{n} \frac{3k-2}{3k+1}.$$

$$\frac{1}{\sqrt{4n+1}} \leq x_n \leq \frac{1}{\sqrt{3n+1}}.$$

Solution 6. Let $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}_n$ be a bounded sequence. We claim that $\lim_{n\to\infty} x_n y_n = 0$. To prove this, let $\epsilon > 0$. Since $\{y_n\}_n$ is bounded, we find $M \in \mathbb{R}$ such that $|y_n| < M$ for all $n \in \mathbb{N}$. Again, since $\{x_n\}_n$ converges to 0, we find $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,

$$|x_n| < \frac{\epsilon}{|M|}.$$

Hence, for all $n \geq k$, $n \in \mathbb{N}$, we have

$$|x_n y_n| < |x_n||M| < \epsilon.$$

This proves our claim. $\square$

To compute $\lim_{n\to\infty}(-1)^n n/(n^2+1)$, we note that the sequence $n/(n^2+1) \to 0$ and $(-1)^n$ is bounded. Hence,

$$\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0.$$

Solution 7.

(i) We wish to compute $\lim_{n\to\infty} n^{\frac{1}{n^2}}$. We observe that for all $n \in \mathbb{N}$,

$$1 \leq n < 1 + n \leq \left(1 + \frac{1}{n}\right)^{n^2}.$$

The last inequality follows from the Binomial Theorem. Thus,

$$1 \leq n^{\frac{1}{n^2}} < 1 + \frac{1}{n}.$$

Taking limits as $n \to \infty$, $\frac{1}{n} \to 0$. Hence, using the Sandwich Theorem, we conclude that $n^{\frac{1}{n^2}} \to 1$.

(ii) We wsh to compute $\lim_{n\to\infty}(n!)^{\frac{1}{n^2}}$. We observe that for all $n \in \mathbb{N}$,

$$1 \leq n! \leq n^n,$$

$$1 \leq (n!)^{\frac{1}{n^2}} \leq n^{\frac{1}{n}}.$$

Taking limits as $n \to \infty$, $n^{\frac{1}{n}} \to 1$, Hence, using the Sandwich Theorem, we conclude that $(n!)^{\frac{1}{n^2}} \to 1$.

Solution 8. We claim that the sequence defined by $x_n = \sin(\frac{n\pi}{2})$, for all $n \in \mathbb{N}$, diverges.
Suppose not, i.e. the given sequence converges to L. Then, we find $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,

$$|x_n - L| < \frac{1}{4}.$$

Observe that $x_{4k} = 0$ and $x_{4k+1} = 1$. Thus,

$$1 = |x_{4k} - x_{4k+1}| \leq |x_{4k} - L| + |x_{4k+1} - L| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

This is a contradiction, thus proving our claim. $\square$

Solution 9.

(i) We show that $\lim_{n\to\infty}(2n)^{\frac{1}{n}} = 1$. Note that as $n \to \infty$, the sequences $2^{\frac{1}{n}} \to 1$ and $n^{\frac{1}{n}} \to 1$. Hence, their product also converges to 1. $\square$

(ii) We show that $\lim_{n\to\infty} n^2/n! = 0$. Note that for all $n \geq 6$, $n \in \mathbb{N}$, we have $n! > n^3$. This is easily shown by induction, since $6! > 6^3$, and if $k! > k^3$, then $(k+1)! = (k+1)\cdot k! > (k+1)k^3 > (k+1)^3$. The last inequality holds since $k > 5 \implies k^3 > 5k^2 > k^2 + 2k^2 + k^2 > k^2 + 2k + 1$. Hence, for all $n \geq 6$, $n \in \mathbb{N}$, we have

$$0 < \frac{n^2}{n!} < \frac{1}{n}.$$

Taking limits as $n \to \infty$, $\frac{1}{n} \to 0$. Applying the Sandwich Theorem yields the desired result. $\square$

(iii) We show that $\lim_{n\to\infty} 2^n/n! = 0$. Note that for all $n \geq 6$, $n \in \mathbb{N}$, we have $(n-1)! > 2^n$. This is easily shown by induction, since $5! > 2^6$, and if $(k-1)! > 2^k$, then $k! = k \cdot (k-1)! > k \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$. The last inequality holds since $k \geq 6$. Hence, for all $n \geq 6$, $n \in \mathbb{N}$, we have

$$0 < \frac{2^n}{n!} < \frac{1}{n}.$$

Taking limits as $n \to \infty$, $\frac{1}{n} \to 0$. Applying the Sandwich Theorem yields the desired result. $\square$