MA4104: Algebraic Topology

The fundamental group of $\mathbb{C} \times \mathbb{C} \setminus \Delta$.

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Consider the space $\mathbb{C} \times \mathbb{C} \setminus \Delta$, where $\Delta = \{(z, z) : z \in \mathbb{C}\}$. We may identify $\mathbb{R}^2 \cong \mathbb{C}$ via the usual map $(x, y) \mapsto x + iy$; with this, the space under question may be identified with

$$\mathbb{R}^4 \setminus \{(x, y, x, y) : x, y \in \mathbb{R}\} \;=\; \mathbb{R}^4 \setminus \operatorname{span}\{e_1 + e_3, e_2 + e_4\}.$$

The homeomorphism

$$\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4, \qquad (x, y, z, w) \mapsto (x + z, y + w, x - z, y - w)$$

restricts to the homeomorphism

$$\mathbb{R}^4 \setminus \operatorname{span}\{e_1, e_2\} \cong \mathbb{R}^4 \setminus \operatorname{span}\{e_1 + e_3, e_2 + e_4\}.$$

In other words, our original space is homeomorphic to

$$\mathbb{R}^4 \setminus (\mathbb{R}^2 \times \{0\} \times \{0\}).$$

However, this can be identified again with

$$(\mathbb{C} \times \mathbb{C}) \setminus (\mathbb{C} \times \{0\}) \;=\; \mathbb{C} \times (\mathbb{C} \setminus \{0\}).$$

Now, $\mathbb{C}$ is contractible, and $\mathbb{C} \setminus \{0\}$ deformation retracts to the unit circle S^1. With this, we have established the homotopy equivalence

$$\mathbb{C} \times \mathbb{C} \setminus \Delta \;\sim\; \{0\} \times S^1 \cong S^1.$$

In particular, this demonstrates that the space which we have been examining is path connected. As a result, we can safely discuss its first fundamental group without reference to a particular basepoint (if insisted upon, pick $(1, -1) \in \mathbb{C} \times \mathbb{C} \setminus \Delta$, which gets mapped to $e_3 \in \mathbb{R}^4 \setminus (\mathbb{R}^2 \times \{0\} \times \{0\})$, hence $(0, 1) \in \{0\} \times S^1$). Thus,

$$\pi_1(\mathbb{C} \times \mathbb{C} \setminus \Delta) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

The following sections elaborate on certain details from the above discussion.

Identification of $\mathbb{C}$ with $\mathbb{R}^2$

We use the standard identification

$$\phi\colon \mathbb{R}^2 \to \mathbb{C}, \qquad (x, y) \mapsto x + iy$$

and its inverse to interchangeably talk about $\mathbb{C}$ and $\mathbb{R}^2$ here.

Homeomorphism of $\mathbb{R}^4$

The map $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ described earlier can be written in the following manner.

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The 4×4 matrix here is of full rank; all four columns are orthogonal, hence linearly independent. As a result, φ is a bijective linear map, hence a homeomorphism.

By restricting φ to $\mathbb{R}^4 \setminus \operatorname{span}\{e_1, e_2\}$, we obtain a homeomorphism onto its image. Since $\varphi(e_1) = e_1 + e_3$ and $\varphi(e_2) = e_2 + e_4$, we have removed precisely $\operatorname{span}\{e_1 + e_3, e_2 + e_4\}$ from the image of φ. Thus,

$$\mathbb{R}^4 \setminus \operatorname{span}\{e_1, e_2\} \cong \mathbb{R}^4 \setminus \operatorname{span}\{e_1 + e_3, e_2 + e_4\}$$

via φ, as desired.

Deformation retracts

The deformation retract of $\mathbb{C}$ onto the point 0 looks like

$$h_1\colon [0,1] \times \mathbb{C} \to \mathbb{C}, \qquad (t, z) \mapsto (1-t)z.$$

Similarly, the deformation retract of $\mathbb{C} \setminus \{0\}$ onto the circle S^1 looks like

$$h_2\colon [0,1] \times \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \qquad (t, z) \mapsto (1-t)z + tz/|z|.$$

Note that $(1-t)z + tz/|z| \neq 0$; if it were, then $|z| = t/(t-1) < 0$, a contradiction.

These can be performed on $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$ one after another on the corresponding slots, or in one go as

$$h\colon [0,1] \times \mathbb{C} \times \mathbb{C} \setminus \{0\} \to \mathbb{C} \times \mathbb{C} \setminus \{0\}, \qquad (t, z_1, z_2) \mapsto (t, (1-t)z_1, (1-t)z_2 + tz_2/|z_2|).$$

Note that $h(0, \cdot, \cdot) = \mathrm{id}_{\mathbb{C} \times \mathbb{C} \setminus \{0\}}$ and $h(1, z_1, z_2) = (0, z_2/|z_2|) \in \{0\} \times S^1$. Also, it is clear that each $h(t, \cdot, \cdot)$ fixes $\{0\} \times S^1$; when $z_2 \in S^1$, we have

$$(1-t)z_2 + tz_2/|z_2| = (1-t)z_2 + tz_2 = z_2.$$

Thus, h describes a deformation retraction of $\mathbb{C} \times \mathbb{C} \setminus \{0\}$ onto $\{0\} \times S^1$, which is homeomorphic to just S^1.

Path connectedness of a space and its deformation retract

Suppose that $h\colon I \times X \to X$ is a deformation retract of X onto $A \subseteq X$. Then, given $x \in X$, we have a path $h(\cdot, x)\colon I \to X$ joining $h(0, x) = x$ with $h(1, x) \in A$. If in addition we know that A is path connected, then given any $x, x' \in X$, we can pick a path γ joining $h(1, x)$ and $h(1, x')$ in A. Thus, $h(\cdot, x) * \gamma * \overline{h(\cdot, x')}$ describes a path joining x and x', proving that X is path connected.

* Analogy with $\mathbb{R}^3 \setminus (\mathbb{R} \times \{0\} \times \{0\})$

Here, we have shown that removing a 2-plane from $\mathbb{R}^4$ keeps it path connected. However, this is a bit difficult to visualize. An analogous construction involves removing a line from $\mathbb{R}^3$. Now, it is clear that this space is path connected; indeed, it deformation retracts to a circle once again. Additionally, it is easy to see that each based homotopy class of loops is completely determined by the number of times it winds around the line that has been removed.

Indeed, the standard deformation retraction of $\mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ to the unit sphere S^3 restricts to a deformation retraction

$$\mathbb{C} \times \mathbb{C} \setminus \Delta \;\to\; S^3 \setminus \{(e^{it}, e^{it}) : t \in \mathbb{R}\}.$$

The space on the right consists of S^3 minus a copy of S^1. Performing stereographic projection about a point on this (removed) circle, say $(1,1)$, gives a homeomorphism to $\mathbb{R}^3$ minus a straight line.

* Interpretation of $\mathbb{C} \times \mathbb{C} \setminus \Delta$ as a configuration space of particles

A point in $\mathbb{C}$ can be thought of as the coordinate of a particle on a plane; thus, points in $\mathbb{C} \times \mathbb{C}$ can be used to represent the joint positions of two particles on a plane. Removing the diagonal Δ imposes the restriction that the two particles cannot occupy the same position on the plane. Thus, $\mathbb{C} \times \mathbb{C} \setminus \Delta$ represents the configuration space of two particles on a plane.

It is now easy to see that $\mathbb{C} \times \mathbb{C} \setminus \Delta$ is path connected. Finding a path in this space, i.e. a path between two configurations (z_1, z_2) and (z_1', z_2') reduces to finding paths for the particles individually, one from z_1 to z_1' and another from z_2 to z_2'. The only restriction here is that the paths cannot intersect at the same 'time' parameter t; but we can do better and have them *never* intersect. This is possible since $\mathbb{C} \setminus \{0\}$ is path connected; we simply fix the first particle and move the other into place, then fix the second and move the first. There are a few edge cases where the destination for one particle is the initial position for the other, but these can be easily dealt with by noticing that the two particles can be interchanged by simultaneously moving them clockwise along a circle passing through their initial positions.