MA3101

Analysis III

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Contents

1 Euclidean spaces

1.1 $\mathbb{R}^n$ as a vector space

We are familiar with the vector space $\mathbb{R}^n$, with the standard inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is defined as

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle = \sum_{k=1}^{n} (x_i - y_i)^2.$$

Exercise 1.1. What are all possible inner products on $\mathbb{R}^n$?

Solution. Note that an inner product is a bilinear, symmetric map such that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$. Thus, an product map on $\mathbb{R}^n$ is completely and uniquely determined by the values $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = a_{ij}$. Let A be the $n \times n$ matrix with entries a_{ij}. Note that A is a real symmetric matrix with positive entries. Now,

$$\langle \boldsymbol{x}, \boldsymbol{e}_j \rangle = x_1 a_{1j} + \cdots + x_n a_{nj} = \boldsymbol{x}^\top \boldsymbol{a}_j,$$

where $\boldsymbol{a}_j$ is the j^{th} column of A. Thus,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{a}_1 y_1 + \cdots + \boldsymbol{x}^\top \boldsymbol{a}_n y_n = \boldsymbol{x}^\top A \boldsymbol{y}.$$

Furthermore, any choice of real symmetric A with positive entries produces an inner product.

Theorem 1.1 (Cauchy-Schwarz)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|.$$

Proof. This is trivial when $\boldsymbol{w} = \boldsymbol{0}$. When $\boldsymbol{w} \neq \boldsymbol{0}$, set $\lambda = \langle \boldsymbol{v}, \boldsymbol{w} \rangle / \|\boldsymbol{w}\|^2$. Thus,

$$0 \leq \|\boldsymbol{v} - \lambda \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \lambda^2 \|\boldsymbol{w}\|^2.$$

Simplifying,

$$0 \leq \|\boldsymbol{v}\|^2 - \frac{|\langle \boldsymbol{v}, \boldsymbol{w} \rangle|^2}{\|\boldsymbol{w}\|^2}.$$

This gives the desired result. Clearly, equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$. □

Theorem 1.2 (Triangle inequality)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

Proof. Write

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\langle \boldsymbol{v}, \boldsymbol{w} \rangle + \|\boldsymbol{w}\|^2 \leq \|\boldsymbol{v}\|^2 + 2|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| + \|\boldsymbol{w}\|^2.$$

Applying Cauchy-Schwarz gives

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2.$$

Equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$ for $\lambda \geq 0$. □

1.2 $\mathbb{R}^n$ as a metric space

Our previous observations allow us to define the standard metric on $\mathbb{R}^n$, seen as a point set.

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Definition 1.1. For any $\delta > 0$, the set

$$B_\delta(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < \delta\}$$

is called the open ball centred at $\boldsymbol{x} \in \mathbb{R}^n$ with radius δ. This is also called the δ neighbourhood of $\boldsymbol{x}$.

Definition 1.2. A set U is open in $\mathbb{R}^n$ if for every $\boldsymbol{x} \in U$, there exists an open ball $B_\delta(\boldsymbol{x}) \subset U$.

Remark. Every open ball in $\mathbb{R}^n$ is open.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are open.

Definition 1.3. A set F is closed in $\mathbb{R}^n$ if its complement $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are closed.

Remark. Finite sets in $\mathbb{R}^n$ are closed.

Theorem 1.3. *Unions and finite intersections of open sets are open.*

Corollary 1.3.1. *Intersections and finite unions of closed sets are closed.*

Definition 1.4. An interior point x of a set $S \subseteq \mathbb{R}^n$ is such that there is a neighbourhood of x contained within S.

Example. Every point in an open set is an interior point by definition. The interior of a set is the largest open set contained within it.

Definition 1.5. An exterior point x of a set $S \subseteq \mathbb{R}^n$ is an interior point of the complement $\mathbb{R}^n \setminus S$.

Definition 1.6. A boundary point of a set is neither an interior point, nor an exterior point.

Example. The boundary of the unit open ball $B_0(0) \subset \mathbb{R}^n$ is the sphere S^{n-1}.

Definition 1.7. A limit point x of a set $S \subseteq \mathbb{R}^n$ is such that every neighbourhood of x contains a point from S.

Definition 1.8. The closure of a set $S \subseteq \mathbb{R}^n$ is the union of S and its limit points.

Remark. The closure of a set is the smallest closed set containing it.

1.3 $\mathbb{R}^n$ as a topological space

Definition 1.9. A topology on a set X is a collection of sets τ such that

1. $\emptyset \in \tau$
2. $X \in \tau$
3. Arbitrary union of sets from τ belong to τ.
4. Finite intersections of sets from τ belong to τ.

Sets from τ are called open sets.

Theorem 1.4. *The Euclidean metric d induces the standard topology τ_d on $\mathbb{R}^n$.*

Definition 1.10. Given a topological space (X, τ) and a subset $Y \subseteq X$, the collection of sets $U \cap Y$ where $U \in \tau$ is a topology τ_Y on Y. We call this collection the subspace topology on Y, induced by the topology on X.

Definition 1.11. A set $K \subset X$ in a topological space is compact if every open cover of K has a finite sub-cover. That is, for every collection if $\{U\}_{\alpha \in A}$ of open sets such that K is contained in their union, then there exists a finite sub-collection $U_{\alpha_1}, \ldots, U_{\alpha_k}$ such that K is also contained in their union.

Example. All finite sets are compact.

Example. Given a convergent sequence of real numbers $x_n \to x$, the collection $\{x_n\}_{n \in \mathbb{N}} \cup \{x\}$ is compact.

Example. In $\mathbb{R}^n$, compact sets are precisely those sets which are closed and bounded. This is the Heine-Borel Theorem.

Theorem 1.5. *The closed intervals $[a, b] \subset \mathbb{R}$ are compact.*

Remark. This can be extended to show that any k-cell $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $[a, b]$, and suppose that $I_1 = [a, b]$ has no finite sub-cover. Then, at least one of the intervals $[a, (a + b)/2]$ and $[(a + b)/2, b]$ must not have a finite sub-cover; pick one and call it I_2. Similarly, one of the halves of I_2 must not have a finite sub-cover; call it I_3. In this process, we generate a sequence of closed intervals $I_1 \supset I_2 \supset \ldots$, none of which have a finite sub-cover. The length of each interval is given by

$$|I_n| = 2^{-n+1} \|b - a\| \to 0.$$

Now, pick a sequence of points $\{x_n\}$ where each $x_n \in I_n$. Then, $\{x_n\}$ is a Cauchy sequence. To see this, given any $\epsilon > 0$, we can find sufficiently large n_0 such that $2^{-n_0+1}\|b-a\| < \epsilon$. Thus, $x_n \in I_n \subset I_{n_0}$ for all $n \geq n_0$, which means that for any $m, n \geq n_0$, we have $x_m, x_n \in I_{n_0}$ forcing[1]

$$\|x_m - x_n\| \leq |I_{n_0}| = 2^{-n_0+1}\|b-a\| < \epsilon.$$

From the completeness of $\mathbb{R}$, this sequence must converge in $\mathbb{R}$, specifically in $[a,b]$. Thus, $x_n \to x$ for some $x \in [a,b]$. It can also be seen that the limit $x \in I_n$ for all $n \in \mathbb{N}$; if not, say $x \notin I_{n_0}$, then $x \in [a,b] \setminus I_{n_0}$ which is open, hence there is an open interval such that $(x-\delta, x+\delta) \cap I_{n_0} = \emptyset$. However, I_{n_0} contains all $x_{n \geq n_0}$, thus this δ-neighbourhood of x would miss out a tail of $\{x_n\}$.

Now, pick the open set $U \in \{U_\alpha\}$ which covers the point x. Thus, $x \in U$ so U contains some non-empty open interval $(x-\delta, x+\delta)$ around x. Choose n_0 such that $2^{-n_0+1}\|b-a\| < \delta$; this immediately gives $I_{n_0} \subseteq (x-\delta, x+\delta) \subset U$. This contradicts that fact that I_{n_0} has no finite sub-cover from $\{U_\alpha\}$, completing the proof. □

1.4 Continuous maps

Definition 1.12. A map $f\colon X \to Y$ is continuous if the pre-image of every open set from Y is open in X.

Lemma 1.6. *A map $f\colon X \to Y$ is continuous if the pre-image of every closed set from Y is closed in X.*

Theorem 1.7. *The projection maps $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \mapsto x_i$ are continuous.*

Proof. Let $U \subseteq \mathbb{R}$ be open; we claim that $\pi_i^{-1}(U)$ is open. Pick $\boldsymbol{x} \in \pi_i^{-1}(U)$, and note that $\pi_i(\boldsymbol{x}) = x_i \in U$. Thus, there exists $\delta > 0$ such that $(x_i - \delta, x_i + \delta) \subset U$. Now examine $B_\delta(\boldsymbol{x})$; for any point $\boldsymbol{y}$ within this open ball, we have $d(\boldsymbol{x}, \boldsymbol{y}) < \delta$ hence

$$|x_i - y_i|^2 \leq \sum_{k=1}^{n} (x_k - y_k)^2 = d(\boldsymbol{x}, \boldsymbol{y})^2 < \delta^2.$$

In other words, $\pi_i(\boldsymbol{y}) = y_i \in (x_i - \delta, x_i + \delta)$, hence $\pi_i\, B_\delta(\boldsymbol{x}) \subseteq (x_i - \delta, x_i + \delta) \subset U$. Thus, given arbitrary $\boldsymbol{x} \in \pi_i^{-1}(U)$, we have found an open ball $B_\delta(\boldsymbol{x}) \subset \pi_i^{-1}(U)$. □

Lemma 1.8. *Finite sums, products, and compositions of continuous functions are continuous.*

[1] If $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, note that $a \leq x_1 < x_2 \leq b$, so

$$|x_2 - x_1| = x_2 - x_1 \leq b - a.$$

Theorem 1.9. *All polynomial functions of the coordinates in $\mathbb{R}^n$ are continuous.*

Example. The unit sphere $S^{n-1} \subset \mathbb{R}^n$ is closed. It is by definition the pre-image of the singleton closed set $\{1\}$ under the continuous map

$$\boldsymbol{x} \mapsto x_1^2 + \cdots + x_n^2.$$