MA2202: Probability I

Markov chains

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Definition 5.1 (Markov chain)**.** Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of random variables taking values in $\{0, 1, \ldots, N\}$ such that

$$P(X_{n+1} = s_{n+1} | X_n = s_n, \ldots, X_0 = s_0) = P(X_{n+1} = s_{n+1} | X_n = s_n).$$

Then, the sequence $\{X_n\}$ is a Markov chain.

Definition 5.2 (Transition probabilities)**.** We define

$$p_{ij} = P(X_{n+1} = j \mid X_n = i).$$

If p_{ij} does not depend on n, then it is called a stationary transition probability from the i^{th} state to the j^{th} state.

Remark. Note that

$$\sum_{j=0}^{N} P(X_{n+1} = j \mid X_n = i) = \sum_{j=0}^{N} p_{ij} = 1.$$

Definition 5.3 (Stochastic matrix)**.** The stochastic matrix or transition probability matrix $\mathbb{P}$ is defined such that $\mathbb{P}_{ij} = p_{ij}$, where $0 \leq i, j \leq N$.

Remark. Note that setting $\mathbf{1} = (1 \ldots 1)^\top$, we have $\mathbb{P}\mathbf{1} = \mathbf{1}$.

Lemma 5.1. *For a Markov chain with stationary transition probabilities,*

$$P(X_n = s_n, \ldots, X_0 = s_0) = p_{s_{n-1}s_n} p_{s_{n-2}s_{n-1}} \ldots p_{s_0 s_1} P(X_0 = s_0).$$

Definition 5.4 (m-stage probability matrix)**.** We define $\mathbb{P}^{(m)}$ such that

$$\mathbb{P}_{ij}^{(m)} = P(X_m = j \mid X_0 = i).$$