MA 2102 : Mathematical Methods II

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Fourier Series and Transforms (M.L. Boas, Chapter 7)

For a function f with period 2ℓ, we write our Fourier expansions in the form

$$f(x) \;=\; \sum_{n=-\infty}^{+\infty} c_n e^{in\pi x/\ell} \;=\; a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}.$$

Here, the coefficients c_n are calculated as

$$c_n = \frac{1}{2\ell}\int_{\alpha}^{\alpha+2\ell} f(x)e^{-in\pi x/\ell}\,dx,$$

for any starting point α. We could calculate a_n and b_n using the standard integrals. However, with knowledge of c_n, we note that

$$f(x) \;=\; c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/\ell} + c_{-n}e^{-in\pi x/\ell} \;=\; c_0 + \sum_{n=1}^{\infty}(c_n + c_{-n})\cos\frac{n\pi x}{\ell} + i(c_n - c_{-n})\sin\frac{n\pi x}{\ell}.$$

Thus, comparing coefficients, we obtain $a_0 = c_0$, $a_n = c_n + c_{-n}$, and $b_n = i(c_n - c_{-n})$.

Section 8. Problem 10.

(a) Sketch several periods of the function f of period 2π which is equal to x on $-\pi < x < \pi$. Expand f in a sine-cosine Fourier series and in a complex exponential Fourier series.

(b) Sketch several periods of the function f of period 2π which is equal to x on $0 < x < 2\pi$. Expand f in a sine-cosine Fourier series and in a complex exponential Fourier series.

Solution. The functions f have period 2π, so $\ell = \pi$.

(a) We directly calculate the Fourier coefficients

$$a_0 = c_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} x\,dx = 0.$$

For $n \neq 0$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} xe^{-inx}dx = -\frac{1}{2\pi in}xe^{-inx}\Big|_{-\pi}^{+\pi} + \frac{1}{2\pi in}\cancel{\int_{-\pi}^{+\pi} e^{-inx}}\,dx = -\frac{1}{2in}(e^{-in\pi} + e^{in\pi}) = \frac{i}{n}\cos n\pi.$$

Thus, for $n > 0$,

$$a_n = \frac{i}{n}(-1)^n - \frac{i}{n}(-1)^{-n} = 0, \qquad b_n = \frac{i^2}{n}(-1)^n + \frac{i^2}{n}(-1)^{-n} = \frac{2}{n}(-1)^{n+1}.$$

This means that

$$f(x) \;=\; \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{i}{n}(-1)^n e^{inx} \;=\; \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx. \qquad (\star)$$

(b) Again,

$$a_0 = c_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\cdot\frac{(2\pi)^2}{2} = \pi.$$

For $n \neq 0$,

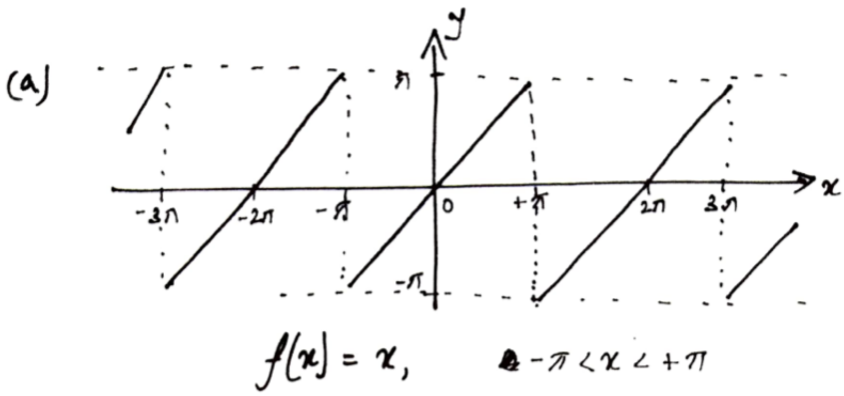
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}\,dx = -\frac{1}{2\pi in}xe^{-inx}\Big|_0^{2\pi} + \cancel{\frac{1}{2\pi in}\int_0^{2\pi} e^{-inx}\,dx} = -\frac{1}{in}e^{-2n\pi i} = \frac{i}{n}.$$

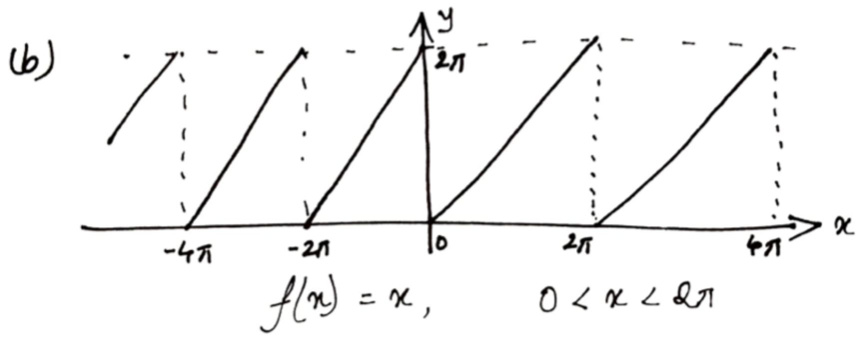
Thus, for $n > 0$,

$$a_n = \cancel{\frac{i}{n} - \frac{i}{n}} = 0, \qquad b_n = \frac{i^2}{n} + \frac{i^2}{n} = -\frac{2}{n}.$$

This means that

$$f(x) \;=\; \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{i}{n}e^{inx} \;=\; \pi - \sum_{n=1}^{\infty}\frac{2}{n}\sin nx. \tag{$\star$}$$





Problem 14. Sketch several periods of the following functions and expand them in a sine-cosine Fourier series and in a complex exponential Fourier series.

(a) $f(x) = \sin \pi x, \quad -\frac{1}{2} < x < \frac{1}{2}$.

(b) $f(x) = \sin \pi x, \quad 0 < x < 1$.

Solution. The functions f have period 1, so $\ell = 1/2$.

(a) We calculate

$$c_0 = \int_{-1/2}^{+1/2} \sin \pi x\,dx = \frac{1}{\pi}\,[1-1] = 0.$$

For $n \neq 0$,

$$c_n = \int_{-1/2}^{+1/2} \sin \pi x\, e^{-2n\pi i x}\,dx = \frac{1}{2i}\int_{-1/2}^{+1/2} e^{(-2n+1)\pi i x} - e^{(-2n-1)\pi i x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{ie^{-n\pi i} + ie^{n\pi}}{2n-1} - \frac{-ie^{-n\pi i} - ie^{n\pi i}}{2n+1}\right] = \frac{i}{2\pi}\cdot 2\cos n\pi \cdot \frac{4n}{4n^2-1} = \frac{4ni(-1)^n}{\pi(4n^2-1)}.$$

Thus, for $n > 0$,

$$a_n = \frac{4ni(-1)^n - 4ni(-1)^{-n}}{\pi(4n^2-1)} = 0, \qquad b_n = \frac{4ni^2(-1)^n + 4ni^2(-1)^{-n}}{\pi(4n^2-1)} = \frac{-8n(-1)^n}{\pi(4n^2-1)}.$$

This means that

$$f(x) \;=\; \sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} \frac{4ni}{\pi(4n^2-1)}(-1)^n e^{2n\pi i x} \;=\; -\sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)}(-1)^n \sin 2n\pi x. \qquad (\star)$$

(b) We calculate

$$c_0 = \int_0^1 \sin \pi x \, dx = \frac{1}{\pi}\left[-\cos\pi + 1\right] = \frac{2}{\pi}.$$

For $n \neq 0$,

$$c_n = \int_0^1 \sin\pi x \, e^{-2n\pi i x}\, dx = \frac{1}{2i}\int_0^1 e^{(-2n+1)\pi i x} - e^{(-2n-1)\pi i x}\, dx$$

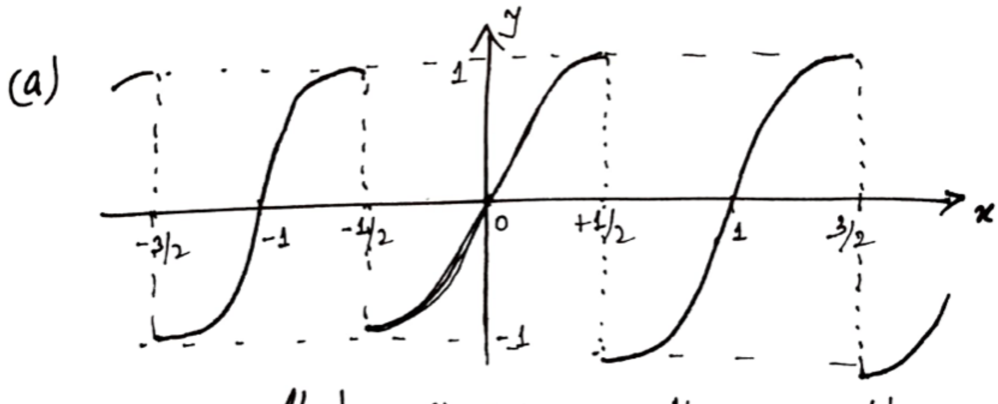
$$= \frac{1}{2\pi}\left[\frac{-1-1}{2n-1} - \frac{-1-1}{2n+1}\right] = \frac{1}{2\pi}\cdot(-2)\cdot\frac{2}{4n^2-1} = \frac{-2}{\pi(4n^2-1)}.$$

Thus, for $n > 0$,

$$a_n = \frac{-2-2}{\pi(4n^2-1)} = \frac{-4}{\pi(4n^2-1)}, \qquad b_n = \frac{-2i+2i}{\pi(4n^2-1)} = 0.$$

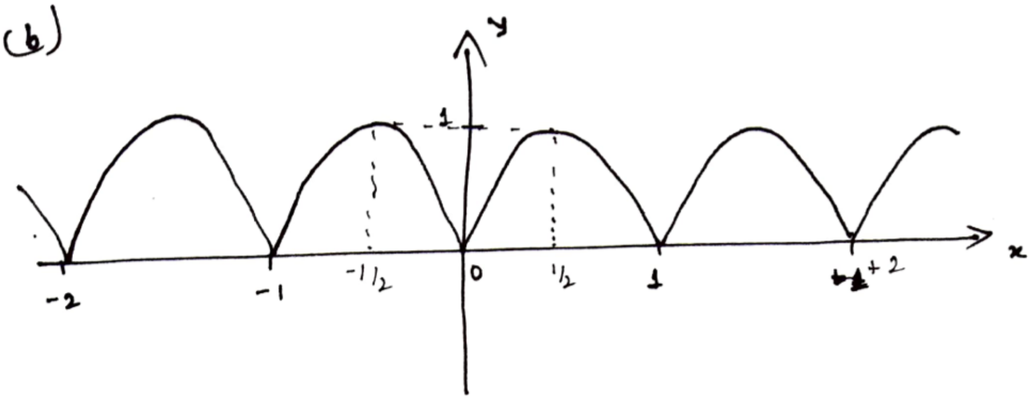
This means that

$$f(x) \;=\; \frac{2}{\pi} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} \frac{2}{\pi(4n^2-1)} e^{2n\pi i x} \;=\; \frac{2}{\pi} - \sum_{n=1}^{\infty}\frac{4}{\pi(4n^2-1)}\cos 2n\pi x. \qquad (\star)$$



$f(x) = \sin \pi x, \quad -1/2 < x < +1/2$

(b)



$f(x) = \sin \pi x, \quad 0 < x < 1$

Problem 15 (c) Sketch (or computer plot) the following function on the interval $(1, 1)$ and expand it in a complex exponential series and in a sine-cosine series.

$$f(x) = \begin{cases} x + x^2, & -1 < x < 0, \\ x - x^2, & 0 < x < 1. \end{cases}$$

Solution. We calculate

$$c_0 = \frac{1}{2} \int_{-1}^{+1} f(x)\, dx = \frac{1}{2} \int_0^1 f(x) + f(-x)\, dx = 0.$$

For $n \neq 0$,

$$c_n = \frac{1}{2} \int_{-1}^{+1} f(x) e^{-in\pi x}\, dx = \frac{1}{2} \int_0^1 (x - x^2) e^{-in\pi x} + (-x + x^2) e^{in\pi x}\, dx$$

Now,

$$\int_0^1 x e^{in\pi x}\, dx = \frac{1}{in\pi} x e^{in\pi x} \Big|_0^1 - \frac{1}{in\pi} \int_0^1 e^{in\pi x}\, dx = \frac{1}{in\pi} e^{in\pi} + \frac{1}{n^2\pi^2}(e^{in\pi} - 1).$$

$$\int_0^1 x^2 e^{in\pi x}\, dx = \frac{1}{in\pi} x^2 e^{in\pi x} \Big|_0^1 - \frac{2}{in\pi} \int_0^1 x e^{in\pi x}\, dx = \frac{1}{in\pi} e^{in\pi} + \frac{2}{n^2\pi^2} e^{in\pi} - \frac{2}{in^3\pi^3}(e^{in\pi} - 1).$$

Thus,

$$\int_0^1 (-x + x^2) e^{in\pi x}\, dx = \frac{1}{n^2\pi^2}(e^{in\pi} + 1) - \frac{2}{in^3\pi^3}(e^{in\pi} - 1),$$
$$\int_0^1 (x - x^2) e^{-in\pi x}\, dx = \frac{1}{n^2\pi^2}(-e^{-in\pi} - 1) - \frac{2}{in^3\pi^3}(e^{-in\pi} - 1).$$

Adding these together and halving,

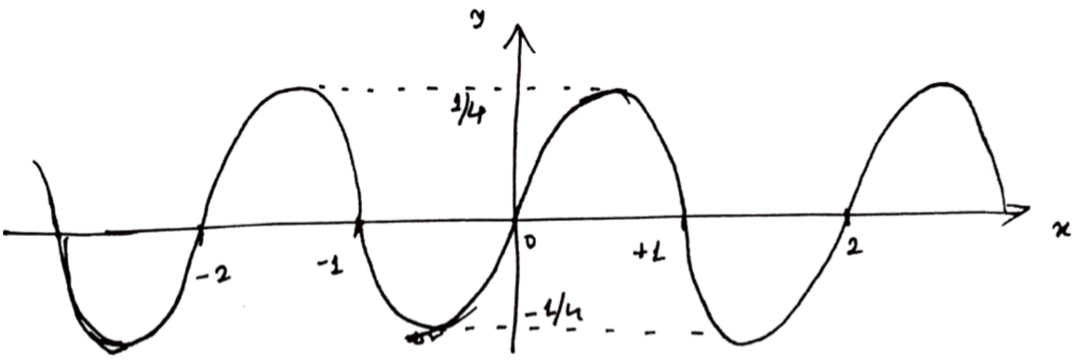
$$c_n = -\frac{2}{in^3\pi^3}(e^{in\pi} - 1).$$

Note that this vanishes for even n. For odd n, we have $c_n = -4i/n^3\pi^3$. Thus, for odd $n > 1$,

$$a_n = \cancel{-\frac{4i}{n^3\pi^3} + \frac{4i}{n^3\pi^3}} = 0, \qquad b_n = -\frac{4i^2}{n^3\pi^3} - \frac{4i^2}{n^3\pi^3} = \frac{8}{n^3\pi^3}.$$

Thus,

$$f(x) \;=\; -\frac{4i}{\pi^3} \sum_{\substack{n=-\infty \\ \text{odd } n}}^{+\infty} \frac{1}{n^3} e^{n\pi i x} \;=\; \frac{8}{\pi^3} \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^3} \sin n\pi x. \qquad (\star)$$



$$f(x) = \begin{cases} x + x^2, & -1 < x < 0 \\ x - x^2, & 0 < x < 1 \end{cases}$$

Problem 17. Sketch several periods of the following periodic function and expand it in a sine-cosine Fourier series and in a complex exponential Fourier series.

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 3. \end{cases}$$

Solution. The function f has period 4, so $\ell = 2$. We calculate

$$c_0 = \frac{1}{4}\int_{-1}^{3} f(x)\,dx = \frac{3}{4}.$$

For $n \neq 0$, we have

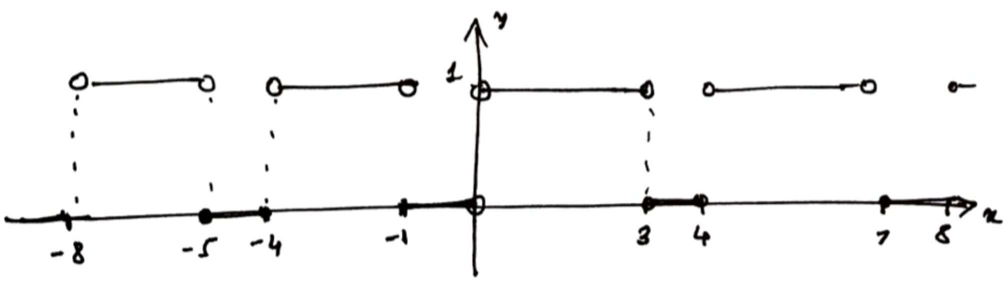
$$c_n = \frac{1}{4}\int_{-1}^{3} f(x)e^{-in\pi x/2}\,dx = \frac{1}{4}\cdot\frac{-2}{in\pi}(e^{-3in\pi/2} - 1) = -\frac{1}{2n\pi i}(e^{-3n\pi i/2} - 1).$$

Thus, for $n > 1$, we have

$$\begin{aligned} a_n &= -\frac{(e^{-3n\pi i/2} - 1) - (e^{3n\pi i/2} - 1)}{2n\pi i} = \frac{1}{n\pi}\sin\frac{3n\pi}{2}, \\ b_n &= -i\frac{(e^{-3n\pi i/2} - 1) + (e^{3n\pi i/2} - 1)}{2n\pi i} = -\frac{1}{n\pi}\cos\frac{3n\pi}{2} + \frac{1}{n\pi}. \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \frac{3}{4} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{2n\pi i}(e^{-3n\pi i/2} - 1)e^{n\pi i x/2} \\ &= \frac{3}{4} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\sin\frac{3n\pi}{2}\cos\frac{n\pi x}{2} + \frac{1}{n\pi}\left(1 - \cos\frac{3n\pi}{2}\right)\sin\frac{n\pi x}{2}. \end{aligned} \qquad (\star)$$



$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 3. \end{cases}$$

Problem 20. Sketch several periods of the following periodic function and expand it in a sine-cosine Fourier series and in a complex exponential Fourier series.

$$f(x) = \begin{cases} x/2, & 0 < x < 2, \\ 1, & 2 < x < 3. \end{cases}$$

Solution. The function f has period 3, so $\ell = 3/2$. We calculate

$$c_0 = \frac{1}{3}\int_0^3 f(x)\,dx = \frac{1}{3}\cdot\frac{x^2}{4}\bigg|_0^2 + \frac{1}{3} = \frac{2}{3}.$$

For $n \neq 0$, we have

$$\begin{aligned} c_n &= \frac{1}{3}\int_0^2 \frac{x}{2}e^{-2in\pi x/3}\,dx + \frac{1}{3}\int_2^3 e^{-2in\pi x/3}\,dx \\ &= -\frac{1}{4n\pi i}xe^{-2in\pi x/3}\bigg|_0^2 + \frac{1}{4n\pi i}\int_0^2 e^{-2in\pi x/3}\,dx - \frac{1}{2n\pi i}e^{-2in\pi x/3}\bigg|_2^3 \\ &= -\frac{1}{2n\pi i}\cancel{e^{-4in\pi/3}} + \frac{3}{8n^2\pi^2}(e^{-4in\pi/3} - 1) - \frac{1}{2n\pi i}(1 - \cancel{e^{-4n\pi i/3}}) \\ &= -\frac{1}{2n\pi i} - \frac{3}{8n^2\pi^2} + \frac{3}{8n^2\pi^2}e^{-4in\pi/3}. \end{aligned}$$

Thus, for $n > 1$,

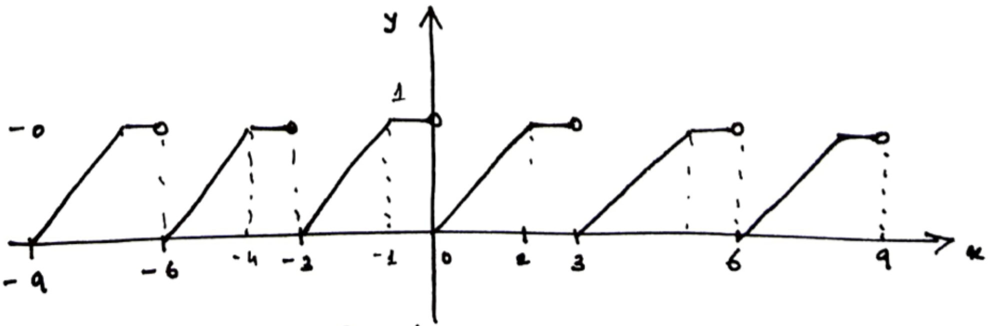
$$\begin{aligned} a_n &= \cancel{-\frac{1}{2n\pi i} + \frac{1}{2n\pi i}} - \frac{3}{8n^2\pi^2} - \frac{3}{8n^2\pi^2} + \frac{3}{8n^2\pi^2}e^{-4in\pi/3} + \frac{3}{8n^2\pi^2}e^{4in\pi/3} \\ &= \frac{3}{4n^2\pi^2}\left(\cos\frac{4n\pi}{3} - 1\right), \\ b_n &= -\frac{1}{2n\pi} - \frac{1}{2n\pi} \cancel{- \frac{3i}{8n^2\pi^2} + \frac{3i}{8n^2\pi^2}} + \frac{3i}{8n^2\pi^2}e^{-4in\pi/3} - \frac{3i}{8n^2\pi^2}e^{4in\pi/3} \\ &= -\frac{1}{n\pi} + \frac{3}{4n^2\pi^2}\sin\frac{4n\pi}{3}. \end{aligned}$$

We can separate these into 3 cases, based on the residue of n modulo 3. Note that $\cos(4(3k)\pi/3) = \sin(4(3k)\pi/3) = 0$, $\cos(4(3k \pm 1)\pi/3) = -1/2$, and $\sin(4(3k \pm 1)\pi/3) = \mp\sqrt{3}/2$. Thus,

n mod 3	0	1	2
a_n	0	$-\frac{9}{8n^2\pi^2}$	$-\frac{9}{8n^2\pi^2}$
b_n	$-\frac{1}{n\pi}$	$-\frac{1}{n\pi} - \frac{3\sqrt{3}}{8n^2\pi^2}$	$-\frac{1}{n\pi} + \frac{3\sqrt{3}}{8n^2\pi^2}$

We thus expand

$$\begin{aligned} f(x) &= \frac{2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left(-\frac{1}{2n\pi i} - \frac{3}{8n^2\pi^2} + \frac{3}{8n^2\pi^2}e^{-4in\pi/3}\right)e^{2in\pi x/3} \\ &= \frac{2}{3} + \sum_{n=1}^{\infty}\frac{3}{4n^2\pi^2}\left(\cos\frac{4n\pi}{3} - 1\right)\cos\frac{2n\pi x}{3} + \left(-\frac{1}{n\pi} + \frac{3}{4n^2\pi^2}\sin\frac{4n\pi}{3}\right)\sin\frac{2n\pi x}{3}. \qquad (\star) \end{aligned}$$



$$f(x) = \begin{cases} x/2, & 0 < x < 2, \\ 1, & 2 < x < 3. \end{cases}$$