MA2102 : Linear Algebra I

Satvik Saha, 19MS154 November 26, 2020 Let $a \in \mathbb{R}$. Consider the set

$$S_a^n = \left\{1,\, (x-a),\, (x-a)^2,\, \ldots,\, (x-a)^n\right\}.$$

Show that S_a^n is a basis for $P_n(\mathbb{R})$, the space of polynomials of degree at most n.

We notate the superscript S_a^n to denote the highest degree term $(x-a)^n$.

Solution We first calculate the dimension of $P_n(\mathbb{R})$ and exhibit the standard basis.

Lemma 1. *The set $S_0^n = \{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n(\mathbb{R})$.*

Proof. Note that any polynomial in $P_n(\mathbb{R})$ is of the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

This means that $P_n(\mathbb{R}) \subseteq \operatorname{span} S_0^n$. We now show linear independence by considering the linear combination

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \;=\; \mathbf{0}.$$

Because the polynomial on the left is equated with the zero polynomial on the right, it must identically evaluate to 0 no matter the choice of $x \in \mathbb{R}$. We choose $n+1$ distinct reals x, which we exhibit as $n+1$ roots of the polynomial on the left. However, the degree of this polynomial is at most n. We conclude that the polynomial on the left is the zero polynomial, so $c_0 = c_1 = \cdots = c_n$.
This proves that the set S_0^n is a basis of $P_n(\mathbb{R})$. □

We use the fact that $\dim P_n(\mathbb{R}) = n + 1$, from Lemma 1. Thus, we need only show that S_a^n is linearly independent for $a \neq 0$. This is sufficient to prove that S_a^n is a basis of $P_n(\mathbb{R})$, because the Replacement Theorem guarantees that any linearly independent set of size $\dim P_n(\mathbb{R})$ will be a basis.

Lemma 2. *The polynomial $(x-a)^n - x^n$ has degree at most $n-1$, for $n \in \mathbb{N}$.*

Proof. We expand $(x-a)^n$ using the Binomial Theorem to obtain

$$(x-a)^n = x^n - nax^{n-1} + \binom{n}{2} a^2 x^{n-2} + \cdots + (-1)^n a^n.$$

Subtracting x^n from both sides leaves terms of degree at most $n-1$ on the right. Thus,

$$(x-a)^n - x^n \;\in\; P_{n-1}(\mathbb{R}) \;\subset\; P_n(\mathbb{R}).$$

The coefficients of this $n-1$ degree polynomial are the binomial coefficients with alternating sign, as seen above. □

With this, we prove that S_a^n for $a \neq 0$ is a basis of $P_n(\mathbb{R})$ by induction. For $n = 0$, the claim is trivial, since we have $S_a^0 = S_0^0 = \{1\}$, which is a linearly independent set in $P_0(\mathbb{R})$, hence a basis of $P_0(\mathbb{R})$.

For $n = 1$, consider the linear combination of elements from $S_a^1 = \{1,\, (x-a)\}$

$$c_0 + c_1(x-a) \;=\; \mathbf{0},$$

for arbitrary $c_0, c_1 \in \mathbb{R}$. Successively set $x = a$ and $x = 0$. Thus, $c_0 = 0$ and $c_0 - c_1 a = 0$, whence $c_0 = c_1 = 0$. This shows that S_a^1 is linearly independent in $P_1(\mathbb{R})$, hence a basis of $P_1(\mathbb{R})$.

Suppose that for $n = k$, the set $S_a^k = \left\{1,\, (x-a),\, \ldots, (x-a)^k\right\}$ is a basis of $P_k(\mathbb{R})$. Consider the linear combination of elements from S_a^{k+1},

$$c_0 + c_1(x-a) + \cdots + c_k(x-a)^k + c_{k+1}(x-a)^{k+1} \;=\; \mathbf{0}.$$

Subtract and add $c_{k+1}x^{k+1}$.

$$\Big[c_0 + c_1(x-a) + \cdots + c_k(x-a)^k + c_{k+1}\left((x-a)^{k+1} - x^{k+1}\right)\Big] + c_{k+1}x^{k+1} \;=\; \mathbf{0}.$$

Note that the portion in square brackets is a polynomial of degree at most k. This is because $(x-a)^{k+1} - x^{k+1}$ has degree at most k by Lemma 2, and the remaining terms also have degree at most k. Thus, we expand this bracketed polynomial in the basis S_a^k.

$$p_k(x) \;=\; a_0 + a_1x + \cdots + a_kx^k \in P_k(\mathbb{R}).$$

Replacing this in the previous equation,

$$a_0 + a_1x + \cdots + a_kx^k + c_{k+1}x^{k+1} \;=\; \mathbf{0}.$$

The linear independence of $S_0^{k+1} = \{1, x, \ldots, x^{k+1}\}$ from Lemma 1 gives $a_0 = a_1 = \cdots = a_k = c_{k+1} = 0$. Substituting this back into the original linear combination with the c_j coefficients, we have

$$c_0 + c_1(x-a) + \cdots + c_k(x-a)^k \;=\; \mathbf{0}.$$

The induction hypothesis, whereby S_a^k is linearly independent, gives $c_0 = c_1 = \cdots = c_k = 0 = x_{k+1}$. This shows that S_a^{k+1} is linearly independent in $P_{k+1}(\mathbb{R})$, which means it is a basis of $P_{k+1}(\mathbb{R})$.

Thus, by the principle of mathematical induction, the set S_a^n is a basis of $P_n(\mathbb{R})$ for all integers $n \geq 0$. This completes the proof.