MA3104

Linear Algebra II

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1 Linear operators on a vector space

1.1 Preliminaries

We discuss finite dimensional vector spaces V over some field $\mathbb{F}$, along with linear operators $T\colon V \to V$. We also assume that V has the inner product $\langle \cdot, \cdot \rangle$.

Theorem 1.1. *Let $\mathscr{L}(V)$ be the set of all linear operators on the vector space V. Then, $\mathscr{L}(V)$ is a linear algebra over the field $\mathbb{F}$.*

1.2 Ideals in a ring

Definition 1.1. Let $(R, +, \cdot)$ be a ring, where $(R, +)$ is its additive subgroup. A set $I \subseteq R$ is a left ideal of R if $(I, +)$ is a subgroup of $(R, +)$, and $rx \in I$ for every $r \in R$, $x \in I$.

Example. Let $\mathbb{Z}$ be the ring of integers. For some $n \in \mathbb{N}$, the set $n\mathbb{Z}$ is an ideal. In fact, these are the only ideals (along with $\{0\}$).

Definition 1.2. The principal left ideal generated by $x \in R$ is the set

$$I_x = Rx = \{rx : r \in R\}.$$

Example. In the ring of integers $\mathbb{Z}$, every ideal is a principal ideal. This follows directly from the fact that $(\mathbb{Z}, +)$ is a cyclic group, thus any subgroup is cyclic and generated by a single element.

Let $I \subseteq \mathbb{Z}$ be an ideal. If $I = \{0\}$, we are done. Otherwise, let n be the smallest positive integer in I (note that if $a \in I$, then $-a \in I$ which means that I must contain positive integers). This immediately gives $I \supseteq n\mathbb{Z}$. Now for any $m \in I$, use Euclid's Division Lemma to write $m = nq + r$, where $q, r \in \mathbb{Z}$, $0 \leq r < n$. Since I is an ideal, $nq \in I$ hence $m - nq = r \in I$. The minimality of n in I forces $r = 0$, hence $m = nq$ and $I \subseteq n\mathbb{Z}$. This proves $I = n\mathbb{Z}$.

Theorem 1.2. *Let $\mathbb{F}$ be a field and let $\mathbb{F}[x]$ denote the ring of polynomials with coefficients from $\mathbb{F}$. Then, every ideal in $\mathbb{F}[x]$ is a principal ideal.*

Remark. This is analogous to the theorem which states that every subgroup of a cyclic group is cyclic. Both lead to a precise definition of the greatest common divisor.

Corollary 1.2.1. *Let I be a non-trivial ideal in $\mathbb{F}[x]$. Then, there exists a unique monic polynomial $p \in \mathbb{F}[x]$ (leading coefficient 1) such that I is precisely the principal ideal generated by p.*

1.3 Eigenvalues and eigenvectors

Definition 1.3. Let $T \in \mathscr{L}(V)$ and $c \in \mathbb{F}$. We say that c is an eigenvalue or characteristic value of T if $T\boldsymbol{v} = c\boldsymbol{v}$ for some non-zero $\boldsymbol{v} \in V$. The vector $\boldsymbol{v}$ is called an eigenvector of T.

Theorem 1.3. *Let $T \in \mathscr{L}(V)$ and $c \in \mathbb{F}$. The following are equivalent.*

1. *c is an eigenvalue of T.*
2. *$T - cI$ is singular.*
3. *$\det(T - cI) = 0$.*

Definition 1.4. The polynomial $\det(T - xI)$ is called the characteristic polynomial of T.

Definition 1.5. Two linear operators $S, T \in \mathscr{L}(V)$ are similar if there exists an invertible operator $X \in \mathscr{L}(V)$ such that $S = X^{-1}TX$.

Remark. Similarity is an equivalence relation on $\mathscr{L}(V)$, thus partitioning it into similarity classes.

Lemma 1.4. *Similar linear operators have the same characteristic polynomial.*

Proof. Let S, T be similar with $S = X^{-1}TX$. Then,

$$\begin{aligned}\det(S - xI) &= \det(X^{-1}TX - xX^{-1}X) \\ &= \det(X^{-1}) \det(T - xI) \det(X) \\ &= \det(T - xI).\end{aligned}$$ □

Definition 1.6. A linear operator $T \in \mathscr{L}(V)$ is diagonalizable if there is a basis of V consisting of eigenvectors of T.

Remark. The matrix of T with respect to such a basis is diagonal.

1.4 Annihilating polynomials

Definition 1.7. An polynomial p such that $p(T) = 0$ for a given linear operator $T \in \mathscr{L}(V)$ is called an annihilating polynomial of T.

Lemma 1.5. *Every linear operator $T \in \mathscr{L}(V)$, where V is finite dimensional, has a non-trivial annihilating polynomial.*

Proof. Note that the operators $I, T, T^2, \ldots, T^{n^2} \in \mathscr{L}(V)$, of which there are $n^2 + 1$, are linearly dependent, since $\dim \mathscr{L}(V) = n^2$. □

Lemma 1.6. *The annihilating polynomials of T form an ideal in $\mathbb{F}[x]$.*

Definition 1.8. The minimal polynomial of T is the unique monic generator of the annihilating polynomials of T.

Remark. The minimal polynomial of T divides all its annihilating polynomials.

Theorem 1.7. *The minimal polynomial and characteristic polynomial of T share the same roots, except for multiplicities.*

Proof. Let p be the minimal polynomial of T and let f be its characteristic polynomial.

First, let $c \in \mathbb{F}$ be a root of the minimal polynomial, i.e. $p(c) = 0$. The Division Algorithm guarantees

$$p(x) = (x - c)\, q(x)$$

for some monic polynomial q. By the minimality of the degree of p, we have $q(T) \neq 0$, hence there exists non-zero $\boldsymbol{v} \in V$ such that $\boldsymbol{w} = q(T)\,\boldsymbol{v} \neq \boldsymbol{0}$. Thus, $p(T)\,\boldsymbol{v} = \boldsymbol{0}$ gives

$$(T - cI)\,q(T)\,\boldsymbol{v} = \boldsymbol{0}, \qquad T\boldsymbol{w} = c\boldsymbol{w},$$

which shows that c is an eigenvalue, i.e. a root of the characteristic polynomial f.

Next, suppose that c is a root of the characteristic polynomial, i.e. $f(c) = 0$. Thus, c is an eigenvalue of T, hence there exists non-zero $\boldsymbol{v} \in V$ such that $T\boldsymbol{v} = c\boldsymbol{v}$. This gives $p(T)\,\boldsymbol{v} = p(c)\,\boldsymbol{v}$, but $p(T) = 0$ identically, forcing $p(c) = 0$. □

Theorem 1.8 (Cayley-Hamilton)**.** *The characteristic polynomial of T annihilates T.*

Proof. Set $S = \operatorname{adj}(T - xI)$. This is a matrix with polynomial entries, satisfying

$$(T - xI)S = \det(T - xI)I = f(x)I,$$

where f is the characteristic polynomial of T. Now, we can also collect the powers x^n from S and write

$$S = \sum_{k=0}^{n-1} x^k S_k$$

for matrices S_k. Now, calculate

$$\begin{aligned} f(x)I &= (T - xI)S \\ &= (T - xI)\sum_{k=0}^{n-1} x^k S_k \\ &= -x^k S_{k-1} + \sum_{k=1}^{n-1} x^k (TS_k - S_{k-1}) + TS_0. \end{aligned}$$

Compare coefficients with

$$f(x)I = x^n I + a_{n-1}x^{n-1} + \cdots + a_0 I$$

to get

$$S_{n-1} = -I, \qquad TS_0 = a_0 I, \qquad TS_k - S_{k-1} = a_k I \text{ for } 1 \leq k \leq n-1.$$

Thus,

$$\begin{aligned} f(T) &= \sum_{k=0}^{n} a_k T^k \\ &= -T^n S_{n-1} + \sum_{k=1}^{n-1} (TS_k - S_{k-1})T^k + TS_0 \\ &= 0. \end{aligned}$$ □

Corollary 1.8.1. *The minimal polynomial of T divides its characteristic polynomial.*

Corollary 1.8.2. *The minimal polynomial of T in a finite-dimensional vector space V is at most* $\dim V$.