

MA 1101 : Mathematics I

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Solution 1.

Let $a < b$ and let $f: [a, b] \rightarrow \mathbb{R}$ be convex. We claim that

$$\max\{f(a), f(b)\} \geq f(x), \text{ for all } x \in (a, b).$$

To prove this, let $x \in (a, b)$ be given. We set $M = \max\{f(a), f(b)\}$ and $\lambda = (b - x)/(b - a)$. Clearly, $\lambda > 0$ and $1 - \lambda = (x - a)/(b - a) > 0$, thus $\lambda \in [0, 1]$.

By the convexity of f , we have

$$\begin{aligned} f(\lambda a + (1 - \lambda)b) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ f(x) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ &\leq \lambda M + (1 - \lambda)M \\ &= M \end{aligned}$$

This proves the desired statement. □

Solution 2.

Let $a < b$ and let $f: (a, b) \rightarrow \mathbb{R}$ be differentiable. We claim that f is convex if and only if

$$f(y) - f(x) \geq f'(x)(y - x), \text{ for all } x, y \in (a, b) \quad (\star)$$

To prove this, we first assume that f is convex. We will show that (\star) holds.

If $x = y$, the result follows trivially. Let $y > x$. We choose $\alpha \in (a, b)$ such that $y > x > \alpha > a$. Using the Rising Slope Theorem, we have

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(x) - f(\alpha)}{x - \alpha}.$$

Taking the limit as $\alpha \rightarrow x$, we have

$$\frac{f(y) - f(x)}{y - x} \geq f'(x).$$

Again, if $x > y$, we choose $\beta \in (a, b)$ such that $a > \beta > x > y$. Using the Rising Slope Theorem, we have

$$\frac{f(\beta) - f(x)}{\beta - x} \geq \frac{f(x) - f(y)}{x - y}.$$

Taking the limit as $\beta \rightarrow x$, we have

$$f'(x) \geq \frac{f(x) - f(y)}{x - y}.$$

In either case,

$$f(y) - f(x) \geq f'(x)(y - x), \text{ for all } x, y \in (a, b).$$

We now assume that (\star) holds. We will show that f is convex.

Let $x, y, z \in (a, b)$, such that $x > y > z$. Using (\star) , we have

$$f(x) - f(y) \geq f'(y)(x - y)$$

$$f(z) - f(y) \geq f'(y)(z - y)$$

Rearranging,

$$\frac{f(x) - f(y)}{x - y} \geq f'(y)$$

$$\frac{f(z) - f(y)}{z - y} \leq f'(y)$$

This is the same as

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(y) - f(z)}{y - z}.$$

Therefore, using the Rising Slope Theorem, f is convex.

This proves the desired result. \square

Solution 3.

Let $n \in \mathbb{N}$, let $a_i, \lambda_i > 0$ for all $i = 1, \dots, n$, and let $p \geq 1$. We claim that

$$\frac{\sum \lambda_i a_i^p}{\sum \lambda_i} \geq \left(\frac{\sum \lambda_i a_i}{\sum \lambda_i} \right)^p.$$

To prove this, we define $f: (0, \infty) \rightarrow \mathbb{R}$ by $f(x) := x^p$ for all $x \in (0, \infty)$. Note that $f''(x) = p(p-1)x^{p-2} \geq 0$ for all $x \in (0, \infty)$. Hence, f is convex.

Using Jensen's Inequality on a_i , with weights $\lambda_i / \sum \lambda_i$, we have

$$f\left(\frac{\sum \lambda_i a_i}{\sum \lambda_i}\right) \leq \frac{\sum \lambda_i f(a_i)}{\sum \lambda_i},$$

from which the desired statement follows directly. \square

Solution 4.

(i) Let $a > 0$. We claim that for all $x \geq y > 0$,

$$\frac{a^x - 1}{x} \geq \frac{a^y - 1}{y}.$$

To prove this, note that when $x = y$, the inequality follows trivially. Assume $x > y$.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) := a^x$ for all $x \in \mathbb{R}$. Note that $f''(x) = a^x(\log a)^2 \geq 0$ for all $x \in \mathbb{R}$. Hence, f is convex.

We set $\lambda = y/x$. Note that $\lambda > 0$ and $1 - \lambda = (x - y)/x > 0$. Thus, $\lambda \in [0, 1]$.

By the convexity of f , we have

$$\begin{aligned} f(\lambda x + (1 - \lambda)0) &\leq \lambda f(x) + (1 - \lambda)f(0) \\ f(y) &\leq \frac{y}{x}f(x) + \frac{(x - y)}{x}f(0) \\ xa^y &\leq ya^x + (x - y)a^0 \\ xa^y - x &\leq ya^x - y \\ \frac{a^y - 1}{y} &\leq \frac{a^x - 1}{x} \end{aligned}$$

This proves the desired statement. \square

(ii) We claim that for all $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right)^n.$$

To prove this, we apply Bernoulli's inequality on the following.

$$\begin{aligned}
\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} &= \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1} \left(1 + \frac{1}{n}\right) \\
&= \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} \left(1 + \frac{1}{n}\right) \\
&= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \left(1 + \frac{1}{n}\right) \\
&\geq \left(1 - \frac{n+1}{(n+1)^2}\right) \left(1 + \frac{1}{n}\right) \\
&= \frac{n}{n+1} \cdot \frac{n+1}{n} \\
&= 1
\end{aligned}$$

□