Notes from a course* on

Representation Theory of Finite Groups

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1. Linear representations of groups

Definition 1.1 (Linear representation): Let G be a finite group, and let V be a vector space. A linear representation (σ, V) of G is a homomorphism

$$\sigma : G \to \mathrm{GL}(V).$$

Example 1.1.1: The *trivial* representation of G is defined by $g \mapsto \mathrm{id}_V$.

Example 1.1.2: Consider a vector space V of dimension $\mathrm{ord}(G)$, and pick a basis $\{e_h\}_{h \in G}$. The *regular* representation $\tau : G \to \mathrm{GL}(V)$ of G is defined as follows: $\tau(g)$ sends each of the basis vectors $e_h \mapsto e_{gh}$.

*MA4204, instructed by Dr. Swarnendu Datta.

The following propositions show that it is possible to define group representations in terms of a special class of group actions of G on the vector space V.

Proposition 1.2: Let G be a finite group, and let V be a vector space. Let $\rho : G \times V \to V$ be a group action of G on V, such that each for each G, the map $v \mapsto \rho(g, v)$ is linear. Then, (σ, V) is a linear representation, where

$$\sigma : G \to \mathrm{GL}(V), \qquad g \mapsto (v \mapsto \rho(g, v)).$$

Proposition 1.3: Let (σ, V) be a linear representation. Then, the map

$$\rho : G \times V \to V, \qquad (g, v) \mapsto \sigma(g)(v)$$

is a group action of G on V, where for each $g \in G$, the map $v \mapsto \rho(g, v)$ is linear.

In this discussion, we will always work with finite groups, as well as finite dimensional vector spaces over a base field K. Typically, we will consider $K = \mathbb{C}$.

We will often abbreviate (σ, V) with V, and $\sigma(g)$ with g when the presence of σ is clear from context.

Definition 1.4: The dimension of a representation (σ, V) is $\dim(V)$.

Example 1.4.1: The only one dimensional representation of S_3 in $\mathbb{C}^\times$ is the sign homomorphism. To see this, consider an arbitrary homomorphism $\sigma : S_3 \to \mathbb{C}^\times$. Note that $\ker(\sigma)$ must be a normal subgroup of S_3, hence must be one of $\{e\}, A_3, S_3$. The third option yields the trivial representation $\sigma = \mathrm{id}_{\mathbb{C}^\times}$, and the first option gives the contradiction $S_3 \cong \mathrm{im}(\sigma) \subset \mathbb{C}^\times$ (the right side is abelian while the left is not). This leaves $\ker(\sigma) = A_3$, i.e. $\sigma(g) = 1$ for all even permutations $g \in S_3$. The remaining elements of S_3 (the odd permutations) must be sent to -1, since for any odd permutation $h \in S_3$, the permutation h^2 is even, so $\sigma(h)^2 = \sigma(h^2) = 1$. The result is precisely the sign homomorphism

$$\sigma : S_3 \to \mathbb{C}^\times, \qquad g \mapsto \begin{cases} +1 \text{ if } g \in A_3 \\ -1 \text{ if } g \notin A_3. \end{cases}$$

Example 1.4.2: Construct an equilateral triangle in $\mathbb{C}^2$ centered at the origin, and consider the natural action of S_3 on it (permuting its vertices v_1, v_2, v_3). This induces a two dimensional representation $\sigma : S_3 \to \mathrm{GL}(\mathbb{C}^2)$. Note that $\{v_1, v_2\}$ forms a basis of $\mathbb{C}^2$; the third vertex can be obtained via $v_3 = -v_1 - v_2$. With this, we can calculate the image of (v_1, v_2) under the action of each $g \in S_3$, and hence the matrices of $\sigma(g)$ in the given basis as follows.

g	$(\sigma(g)(v_1), \sigma(g)(v_2))$	Matrix of $\sigma(g)$
e	(v_1, v_2)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(12)	(v_2, v_1)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(23)	$(v_1, v_3) = (v_1, -v_1 - v_2)$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$
(31)	$(v_3, v_2) = (-v_1 - v_2, v_2)$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(123)	$(v_2, v_3) = (v_2, -v_1 - v_2)$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
(321)	$(v_3, v_1) = (-v_1 - v_2, v_1)$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

Consider the setting $K = \mathbb{C}$. The fact that G is a finite group means that each element $g \in G$ has finite order, hence satisfies $g^m = 1$ for some $m \mid \operatorname{ord}(G)$. This means that $\sigma(g)^m = \mathrm{id}_V$, whence $x^m - 1$ is an annihilating polynomial for $\sigma(g)$. A consequence of this is that the minimal polynomial of $\sigma(g)$ is a factor of $x^m - 1$; but the latter splits into distinct linear factors. Furthermore, all eigenvalues of $\sigma(g)$ are roots of its minimal polynomial. This yields the following result.

Proposition 1.5: Suppose that $K = \mathbb{C}$. Let (σ, V) be a representation of G, and let $g \in G$. Then, $\sigma(g)$ is diagonalizable, and its eigenvalues are roots of unity.

2. Subrepresentations

Definition 2.1 (Stable subspace): Let (σ, V) be a representation of G, and let $W \subseteq V$ be a subspace of V. We say that W is a stable subspace of V if it is invariant under the action of G, i.e. $gw \in W$ for all $g \in G$, $w \in W$.

Example 2.1.1: Let S_3 act on $\mathbb{C}^3$ by permuting the basis vectors $\{e_1, e_2, e_3\}$. Then, the subspace $\operatorname{span}\{e_1 + e_2 + e_3\}$ is stable.

Definition 2.2 (Subrepresentation): Let W be a stable subspace of V. We say that (σ, W) is a subrepresentation of (σ, V).

Theorem 2.3: Suppose that $\operatorname{char}(K) \nmid \operatorname{ord}(G)$. Let W be a stable subspace of V. Then, there exists a stable subspace W' of V such that $V = W \oplus W'$.

When working with the field $K = \mathbb{C}$, Theorem 2.3 admits a simpler form by invoking the orthocomplement of $W \subseteq V$, with respect to a suitable Hermitian form on V. We say that a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is G-invariant if for all $g \in G$, $v, v' \in V$, we have $\langle gv, gv' \rangle = \langle v, v' \rangle$.

Theorem 2.4: Suppose that $K = \mathbb{C}$. If W is a stable subspace of V, then $W^\perp$ is a stable subspace of V, with $V = W \oplus W^\perp$.

Remark: The subspace $W^\perp$ is defined with respect to a non-degenerate G-invariant Hermitian form.

Proof: For all $g \in G$, $w \in W$, $w' \in W^\perp$, observe that $g^{-1}w \in W$, so

$$\langle gw', w\rangle = \langle gw', gg^{-1}w\rangle = \langle w', g^{-1}w\rangle = 0,$$

whence $gw' \in W^\perp$. □

Example 2.4.1: Continuing Example 2.1.1, the subspace $\operatorname{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$ is also stable under the action of S_3. This gives a two dimensional subrepresentation of S_3. In fact, it is easy to check that the matrices describing this representation in the basis $\{2e_1 - e_2 - e_3, 2e_2 - e_3 - e_1\}$ are precisely the same as those in Example 1.4.2, making these two representations identical in some sense.

Remark: Given any Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to C$, we can obtain a G-invariant Hermitian form on V defined by

$$(u, v) \mapsto \sum_{g \in G} \langle gu, gv\rangle.$$

Returning to Theorem 2.3, observe that if π_W is a projection onto the subspace W, then we may write $V = W \oplus \ker(\pi_W)$. With this in mind, we will construct the required subspace W' as the kernel of a suitable projection map π_W. For this, we demand that π_W be G-invariant.

Definition 2.5: A linear map $f : V \to V'$ is called G-invariant if it is compatible with the G-action, i.e. for all $g \in G$, $v \in V$, we have $f(gv) = gf(v)$.

Note that the above definition implicitly deals with *two* representations (σ, V) and (σ', V) of G. The indicated property really looks like $\sigma'(g)(f(v)) = f(\sigma(g)(v))$ when written in full.

Lemma 2.6: Let $f : V \to V'$ be G-invariant. Then,

1. $\ker(f)$ is a stable subspace of V.
2. $\operatorname{im}(f)$ is a stable subspace of V'.

Given any linear map $f : V \to V'$, we can construct a G-invariant linear map via

$$\tilde{f} : V \to V', \qquad v \mapsto \sum_{g \in G} gf(g^{-1}v).$$

With this, we are ready to furnish a proof of our theorem.

Proof of Theorem 2.3: Let $\pi : V \to W$ be any projection onto W. Observe that

$$\pi_W : V \to W, \qquad v \mapsto \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} g\pi(g^{-1}v)$$

is a G-invariant projection onto W. Setting $W' = \ker(\pi_W)$ completes the proof. □

Remark: Note how the assumption that $\operatorname{char}(\mathrm{K}) \nmid \operatorname{ord}(G)$ is crucial for defining the projection π_W.

3. Irreducible representations

Definition 3.1 (Irreducible representations): We way that a representation is irreducible if it admits no proper non-trivial subrepresentations.

In other words, a representation (σ, V) is irreducible *if and only if* the only G-invariant subspaces of V are $\{0\}, V$.

Example 3.1.1: All one dimensional representations are irreducible.

Theorem 3.2 (Maschke's Theorem): Suppose that $\operatorname{char}(K) \nmid \operatorname{ord}(G)$. Then, every representation of G over the field K can be written as a direct sum of irreducible representations of G.

Proof: Follows immediately from Theorem 2.3. □

Example 3.2.1: Combining Examples 2.1.1 and 2.4.1, we have the decomposition

$$\mathbb{C}^3 = \operatorname{span}\{e_1 + e_2 + e_3\} \oplus \operatorname{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$$

into irreducible subrepresentations of S_3.

When we say that two representations (σ, V) and (σ', V') are isomorphic, denoted $V \cong V'$, we mean that there exists a G-invariant linear bijection $f : V \to V'$. The following result offers a very powerful characterization of G-invariant maps between irreducible representations.

Theorem 3.3 (Schur's Lemma): Let V, V' be two irreducible representations of G, and let $f : V \to V'$ be a G-invariant linear map.

1. If $V \ncong V'$, then $f = 0$.
2. If $V = V'$ and K is algebraically closed, then f is a scalar map, i.e. $f = \lambda \ \mathrm{id}_V$ for some $\lambda \in K$.

Proof:

1. Suppose that $f \neq 0$. It suffices to show that f is an isomorphism; to do so, we make extensive use of Lemma 2.6.

   First, $\ker(f) \subseteq V$ is stable, hence must be one of $\{0\}, V$ by the irreducibility of V. The assumption $f \neq 0$ forces $\ker(f) = \{0\}$, whence f is injective.

   Next, $\operatorname{im}(f) \subseteq V'$ is stable, hence must be one of $\{0\}, V'$ by the irreducibility of V'. Again, $f \neq 0$ forces $\operatorname{im}(f) = V'$, whence f is surjective.

2. We have a G-invariant linear bijection $f : V \to V$; suppose that $f \neq 0$. Let λ be an eigenvalue of f, and observe that the map $(f - \lambda \ \mathrm{id}_V)$ is also G-invariant; indeed, for all $g \in G$, $v \in V$,

   $$(f - \lambda)(gv) = f(gv) - \lambda gv = gf(v) - g\lambda v = g(f - \lambda)(v).$$

   Since λ is an eigenvalue of f, we have $\ker(f - \lambda) \neq \{0\}$. Since $\ker(f - \lambda) \subseteq V$ is stable and V is irreducible, we must have $\ker(f - \lambda) = V$, whence $f - \lambda \, \mathrm{id}_V = 0$.

□

Remark: Note how the existence of the scalar $\lambda \in K$ is guaranteed by the fact that K is algebraically closed.

Corollary 3.3.1: All $\mathbb{C}$-linear irreducible representations of finite abelian groups are one dimensional.

Proof: Let (σ, V) be an irreducible representation of a finite abelian group G. Check that for each $g \in G$, the linear map $\sigma(g) : V \to V$ is G-invariant, since it commutes with all $\sigma(h)$ for $h \in G$. From Schur's Lemma (Theorem 3.3), each $\sigma(g)$ is a scalar map. As a result, every one dimensional subspace of V is stable. The result now follows from the irreducibility of V. □

4. Characters

Definition 4.1 (Character): The character χ_V of a representation (σ, V) of G is the function

$$\chi_V : G \to K, \qquad g \mapsto \operatorname{tr}(\sigma(g)).$$

Example 4.1.1: $\chi_V(1) = \dim(V)$.

Observe that $\chi_V(g)$ is precisely the sum of eigenvalues of $\sigma(g)$. The eigenvalues of $\chi_V(g^{-1})$ are simply reciprocals of those of $\chi_V(g)$; in the setting $K = \mathbb{C}$, the following result is immediate from Proposition 1.5.

Proposition 4.2: Suppose that $K = \mathbb{C}$. Then, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

The fact that the trace is invariant under conjugation, i.e. $\operatorname{tr}(tst^{-1}) = \operatorname{tr}(s)$, yields the following result.

Lemma 4.3: χ_V is a class function, i.e. χ_V is constant on conjugacy classes of G.

Lemma 4.4: Isomorphic representations have the same character.

Proof: Let $f : V \to V'$ be an isomorphism of representations (σ, V) and (σ', V') of G. Then for each $g \in G$, we have $f \circ \sigma(g) = \sigma'(g) \circ f$, hence $\sigma(g) = f^{-1} \circ \sigma'(g) \circ f$. Taking the trace of both sides and using the cyclic property gives $\operatorname{tr}(\sigma(g)) = \operatorname{tr}(\sigma'(g))$ as desired. □

4.1. Orthogonality of characters

The space K^G of all maps $G \to K$ forms a vector space over K, with dimension $\operatorname{ord}(G)$. In the setting $K = \mathbb{C}$, we may define the following inner product.

$$\langle \cdot, \cdot \rangle : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}, \qquad (\varphi, \psi) \mapsto \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

Remark: For characters χ, χ', Proposition 4.2 gives

$$\langle \chi, \chi' \rangle = \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} \chi(g)\chi'(g^{-1}).$$

Theorem 4.5 (Orthogonality of characters): Suppose that $K = \mathbb{C}$. Let (σ, V), (σ', V') be two irreducible representations of G.

1. If $V \ncong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 0$.
2. If $V \cong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 1$.

Proof: Let $\{v_1, ..., v_n\}$ be a basis of V, and let $\{v'_1, ..., v'_m\}$ be a basis of V'. Given any linear map $f : V \to V'$, we will denote $\tilde{f} = \sum_{g \in G} \sigma'(g) \circ f \circ \sigma(g)^{-1}$; recall that $\tilde{f}$ is G-invariant.

1. Observe that Schur's Lemma (Theorem 3.3) forces all such $\tilde{f} = 0$. In particular, consider the maps e_{ij} defined for each $1 \leq i \leq n$, $1 \leq j \leq m$ as
$$e_{ij} : V \to V', \qquad \sum_i \alpha_i v_i \mapsto \alpha_i v'_j.$$
These maps $\{e_{ij}\}$ form a basis of $\mathcal{L}(V, V')$. Check that the matrix entries obey
$$\left[a \circ e_{ij} \circ b\right]_{k\ell} = [a]_{ki}[b]_{j\ell},$$
so using $\tilde{e}_{ij} = 0$ gives the relations
$$\left[\tilde{e}_{ij}\right]_{kl} = \sum_{g \in G} \left[\sigma'(g) \circ e_{ij} \circ \sigma(g)^{-1}\right]_{kl} = \sum_{g \in G} [\sigma'(g)]_{ki} \left[\sigma(g)^{-1}\right]_{j\ell} = 0$$
for all $1 \leq i, k \leq n$, $1 \leq j, \ell \leq m$. These hold in particular for $i = k$, $j = \ell$; summing over $1 \leq i \leq n$, $1 \leq j \leq m$, we have
$$\begin{aligned} 0 = \sum_{ij} \sum_{g \in G} [\sigma'(g)]_{ii} \left[\sigma(g)^{-1}\right]_{jj} &= \sum_{g \in G} \left(\left(\sum_i [\sigma'(g)]_{ii} \right) \left(\sum_j \left[\sigma(g)^{-1}\right]_{jj} \right) \right) \\ &= \sum_{g \in G} \chi_V(g) \chi_{V'}(g^{-1}) \\ &= \operatorname{ord}(G) \langle \chi_V, \chi_{V'} \rangle. \end{aligned}$$
2. Schur's Lemma (Theorem 3.3) forces all such $\tilde{f} = \lambda_f \operatorname{id}_V$ for scalars $\lambda_f \in \mathbb{C}$. To extract λ_f, take the trace of both sides to obtain
$$n\lambda_f = \dim(V)\lambda_f = \sum_{g \in G} \operatorname{tr}\left(\sigma'(g) \circ f \circ \sigma(g)^{-1}\right) = \operatorname{ord}(G) \operatorname{tr}(f).$$
With this, each $\tilde{e}_{ij} = \lambda_{ij} \delta_{ij} \operatorname{id}_V$, where $\lambda_{ij} = \operatorname{ord}(G)/n$. Thus, we obtain the relations
$$\sum_{g \in G} [\sigma'(g)]_{ki} \left[\sigma(g)^{-1}\right]_{j\ell} = \frac{1}{n} \operatorname{ord}(G) \delta_{ij} \delta_{kl}$$
for all $1 \leq i, j, k, \ell \leq n$. Following a similar process as before,

$$
\begin{aligned}
\operatorname{ord}(G)\langle \chi_V, \chi_{V'} \rangle &= \sum_{g\in G}\left(\left(\sum_i [\sigma'(g)]_{ii}\right)\left(\sum_j \left[\sigma(g)^{-1}\right]_{jj}\right)\right) \\
&= \sum_{ij}\sum_{g\in G} [\sigma'(g)]_{ii}\left[\sigma(g)^{-1}\right]_{jj} \\
&= \sum_{ij}\frac{1}{n}\operatorname{ord}(G)\delta_{ij} \\
&= \operatorname{ord}(G)
\end{aligned}
$$

This completes the proof. □

With this, the characters of irreducible representations form an orthonormal subset of class functions on G. To check whether a representation V is irreducible or not, it is enough to verify that $\langle \chi_V, \chi_V \rangle = 1$.

Corollary 4.5.1: The number of irreducible representations of G (up to isomorphism) is at most the number of conjugacy classes of G.

Given any representation V of G, we can use Maschke's Theorem (Theorem 3.2) to decompose it as a direct sum of (non-isomorphic) irreducible representations $V_1, \ldots, V_k$, with multiplicities $m_1, \ldots, m_k$. By representing the elements of G as matrices in block diagonal form, we can derive the following result.

Lemma 4.6: Let $V_1, \ldots, V_k$ be irreducible representations of G, and let

$$V \cong m_1 V_1 \oplus \cdots \oplus m_k V_k.$$

Then,

$$\chi_V = m_1 \chi_{V_1} + \cdots + m_k \chi_{V_k}.$$

The multiplicities can be recovered as $m_i = \langle \chi_V, \chi_{V_i} \rangle$.

This immediately tells us that $\chi_V = \chi_{V'}$ if and only if $V \cong V'$. Furthermore, we have the relation

$$\langle \chi_V, \chi_V \rangle = \sum_i m_i^2.$$

4.2. The character table for S_3

We have now established that the trivial representation, the one dimensional representation from Example 1.4.1, and the two dimensional representation from Example 1.4.2 are the only irreducible representations of S_3. Note that S_3 has three conjugacy classes: $\{e\}$, $\{(12), (23), (31)\}$, and $\{(123), (321)\}$. With this, we can construct the *character table* for S_3, with each row containing the characters of the group elements with respect to the given representation.

S_3	e	(12)	(23)	(31)	(123)	(321)
Trivial	1	1	1	1	1	1

Sign	1	−1	−1	−1	1	1
Standard	2	0	0	0	−1	−1

Observe that the rows of this table are orthogonal; indeed, so are the columns!

4.3. The character table for S_4

Let S_4 act on $\mathbb{C}^4$ by permuting the basis vectors $\{e_1, e_2, e_3, e_4\}$, and let (σ, V) denote the induced (natural) representation. Note that each matrix $\sigma(g)$ is a permutation, hence its trace $\chi_V(g)$ is precisely the number of elements of $\{1, 2, 3, 4\}$ fixed by the action of g. With this, we can compute χ_V for each conjugacy class (identified by its cycle type) as follows.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
Trivial	1	1	1	1	1
Sign	1	−1	1	1	−1
V	4	2	0	1	0

Compute

$$\langle \chi_V, \chi_V \rangle = \frac{1}{24}(4^2 + 6 \cdot 2^2 + 8 \cdot 1^2) = 2,$$

whence V is not irreducible. Indeed, we know that $W_1 = \operatorname{span}\{e_1 + e_2 + e_3 + e_4\}$ is a trivial subrepresentation of V of dimension 1. Furthermore, $2 = 1^2 + 1^2$ is the only way of writing 2 as a sum of squares of integers, so V must decompose into precisely two irreducible subrepresentations with both multiplicities 1. This means that $V \cong W_1 \oplus W_3$ for some irreducible representation W_3 of dimension 3. Using $\chi_V = \chi_{W_1} + \chi_{W_3}$, we can compute the character χ_{W_3} and obtain the following.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
W_3	3	1	−1	0	−1

Next, we move on to a different representation of S_4: consider all subsets of size 2 of $\{1, 2, 3, 4\}$ (of which there are 6), and consider the action on this collection induced by the permutations on the set $\{1, 2, 3, 4\}$.

Remark: If we wish to define a transitive action of G on a set X (and thereby examine the vector space $\operatorname{span}\{e_x\}_{x \in X}$ with the action of G defined via $ge_x = e_{gx}$), we may invoke the Orbit-Stabilizer Theorem, along with the fact that there is only one orbit (all of X) to demand that $\operatorname{ord}(X) \mid \operatorname{ord}(G)$.

Let (τ, V') denote the induced representation. Again, $\chi_{V'}(g)$ is the number of 2-subsets fixed by the action of g. For instance, an element $(ab) \in S_4$ will only fix 2-subsets $\{a, b\}, \{c, d\}$, while an element $(abc) \in S_4$ fixes no 2-subset. With this, we have the following.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
V'	6	2	2	0	0

Compute $\langle \chi_{V'}, \chi_{V'} \rangle = 3 = 1^2 + 1^2 + 1^2$. Again, we may compute $\langle \chi_{V'}, \chi_{W_1} \rangle = 1$ and $\langle \chi_{V'}, \chi_{W_3} \rangle = 1$, which tells us that $V' \cong W_1 \oplus W_3 \oplus W_2$ for some irreducible representation W_2 of dimension 2. Using $\chi_{V'} = \chi_{W_1} + \chi_{W_3} + \chi_{W_2}$, we can compute the character χ_{W_2}.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
W_2	2	0	2	−1	0

We now have 4 irreducible characters of S_4; indeed, we may combine W_3 with the sign representation to get another irreducible representation W_3', completing the character table of S_4.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
Trivial	1	1	1	1	1
Sign	1	−1	1	1	−1
W_2	2	0	2	−1	0
W_3	3	1	−1	0	−1
W_3'	3	−1	−1	0	1

The last trick uses the following proposition.

Proposition 4.7: Let (σ, V) and $(\tau, \mathbb{C}^\times)$ be representations of G. Then, $(\tau\sigma, V)$ is also a representation of G, where $(\tau\sigma)(g) = \tau(g)\sigma(g)$. Furthermore, $\chi_{\tau\sigma} = \chi_\tau \chi_\sigma$.

Remark: The above proposition is a special case of Proposition 4.12.

4.4. The character of the regular representation

We focus our attention once again to the regular representation, as defined in Example 1.1.2. Note that when G acts on itself by left multiplication, only the identity element 1 fixes all $\operatorname{ord}(G)$ elements of G, while the remaining elements have no fixed points at all. With this, we have the following proposition.

Proposition 4.8: Let (τ, V_G) be the regular representation of G, and let χ_τ denote its character. Then,

$$\chi_\tau(g) = \begin{cases} \operatorname{ord}(G) & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\chi_\tau = d_1 \chi_1 + \dots + d_k \chi_k$$

where $\chi_1, \dots, \chi_k$ are the irreducible characters of G, and each $d_i = \chi_i(1)$ is the dimension of the corresponding irreducible representation.

By simply evaluating $\chi_\tau(1)$, we have the following result.

Corollary 4.8.1: Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G, and let each $d_i = \chi_i(1)$. Then,

$$\sum_i d_i^2 = \operatorname{ord}(G).$$

Proposition 4.9: Let $f : G \to \mathbb{C}$ be a class function, and let (σ, V) be a representation of G. Define

$$f_\sigma : V \to V, \qquad v \mapsto \sum_{g \in G} f(g)\, \sigma(g)(v).$$

Then, f_σ is G-invariant. Furthermore, if (σ, V) is irreducible, then Schur's Lemma (Theorem 3.3) gives $f_\sigma = \lambda\, \mathrm{id}_V$, where $\lambda = \operatorname{ord}(\mathrm{G})\ \langle f, \overline{\chi_\sigma} \rangle / \dim(V)$.

With this construction, we can improve upon Corollary 4.5.1 and precisely count the number of irreducible representations of a group G (up to isomorphism). However, we still do not have any simple way of calculating these representations explicitly.

Theorem 4.10: The number of irreducible representations of G (up to isomorphism) is precisely the number of conjugacy classes of G.

Proof: Let $\mathcal{C}$ be the space of class functions on G, with dimension equal to the number of conjugacy classes of G. Let $\mathcal{X}$ be the subspace of $\mathcal{C}$ spanned by the irreducible characters $\{\chi_i\}$ of G. We claim that $\mathcal{X} = \mathcal{C}$. It is enough to show that the orthocomplement of $\mathcal{X}$ in $\mathcal{C}$ is trivial. For this, pick $f \in \mathcal{C}$ such that all $\langle f, \overline{\chi_i} \rangle = 0$. Let (τ, V_G) be the regular representation of G, and use Proposition 4.8 to write

$$V_G \cong d_1 V_1 \oplus \ldots \oplus d_k V_k$$

where $\{(\sigma_i, V_i)\}$ are the irreducible representations corresponding to the characters $\{\chi_i\}$. Using Proposition 4.9, each $f_{\sigma_i} = 0$, hence $f_\tau = 0$. Evaluating f_τ at the element $e_1 \in V_G$, we have

$$\sum_{g \in G} f(g)\, \sigma(g)(e_1) = \sum_{g \in G} f(g)\, e_g = 0.$$

Since $\{e_g\}_{g \in G}$ forms a basis of V_G, we must have $f = 0$. □

Corollary 4.10.1: The irreducible characters of G form a basis of the space of all class functions on G.

4.5. The tensor product of representations

The construction used in Proposition 4.7 generalizes nicely to tensor products of representations, as follows.

Definition 4.11: Let V, V' be two representations of G. Then, the tensor product $V \otimes V'$ is a representation of G induced by the action defined by

$$g(v \otimes v') = (gv) \otimes (gv').$$

Recall that if $\{v_i\}$ is a basis of V and $\{v_j'\}$ is a basis of V', then $\{v_i \otimes v_j'\}$ forms a basis of $V \otimes V'$. Using this, the next proposition follows.

Proposition 4.12: Let V, V' be two representations of G. Then, $\chi_{V \otimes V'} = \chi_V \chi_{V'}$.

Let's focus on the tensor product $V \otimes V$ and examine the involution defined by

$$\iota : V \otimes V \to V \otimes V, \qquad v \otimes v' \mapsto v' \otimes v.$$

This map has two eigenspaces, corresponding to the eigenvalues 1 and -1, which we define as $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ respectively. Furthermore, it is clear that these eigenspaces are stable under the action of G, since the action of G commutes with ι. This gives us the following decomposition of $V \otimes V$.

Proposition 4.13: Let V be a representation of G. Then,

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Observe that $\mathrm{Sym}^2(V)$ is spanned by elements of the form $v \otimes v' + v' \otimes v$, while $\mathrm{Alt}^2(V)$ is spanned by elements of the form $v \otimes v' - v' \otimes v$. This, together with $\dim(V \otimes V) = n^2$, tells us that

$$\dim(\mathrm{Sym}^2(V)) = \binom{n}{2} + n, \qquad \dim(\mathrm{Alt}^2(V)) = \binom{n}{2}.$$

To compute the characters of these representations, first note that

$$\chi_V^2 = \chi_{\mathrm{Sym}^2(V)} + \chi_{\mathrm{Alt}^2(V)}.$$

Fix $g \in G$ and choose a basis $\{v_i\}$ of V such that the action of g is diagonalized, i.e. $gv_i = \lambda_i v_i$; recall that this is always possible via Proposition 1.5 when we are working over the field $K = \mathbb{C}$. We need only check the action of g on the basis elements of $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$. To this end, compute

$$g(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i \lambda_j (v_i \otimes v_j + v_j \otimes v_i),$$

whence

$$\chi_{\mathrm{Sym}^2(V)}(g) = \sum_{i \leq j} \lambda_i \lambda_j = \sum_{i<j} \lambda_i \lambda_j + \sum_i \lambda_i^2 = \frac{1}{2} \left(\sum_i \lambda_i \right)^2 + \frac{1}{2} \sum_i \lambda_i^2.$$

However, $\sum_i \lambda_i$ and $\sum_i \lambda_i^2$ are precisely $\chi_V(g)$ and $\chi_V(g^2)$. Thus, we have the following result.

Proposition 4.14:

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right), \qquad \chi_{\mathrm{Alt}^2(V)}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

4.6. The character table for S_5

We proceed much in the same manner as in the computation for S_4; the character of the natural representation V of S_5 (induced by the action of S_5 on 5 objects) can be computed by counting the number of fixed points of the corresponding conjugacy class.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
	$\times 1$	$\times 10$	$\times 15$	$\times 20$	$\times 20$	$\times 30$	$\times 24$
Trivial	1	1	1	1	1	1	1
Sign	1	−1	1	1	−1	−1	1
V	5	3	1	2	0	1	0

Since the inner product of the first and last rows is non-zero, V contains a copy of the trivial representation, which we subtract from its character to obtain the representation W_4. This happens to be irreducible (verified by checking $\langle \chi_W, \chi_W \rangle = 1$), and we obtain another irreducible representation W_4' by taking the tensor product with the sign representation.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
W_4	4	2	0	1	−1	0	−1
W_4'	4	−2	0	1	1	0	−1

We now compute the characters of $\mathrm{Sym}^2(W_4)$ and $\mathrm{Alt}^2(W_4)$ using Proposition 4.14. To do so, we compute $\chi_{W_4}(g)^2$ and $\chi_{W_4}(g^2)$ for each conjugacy class; the latter can be computed by examining what happens to each cycle type after squaring.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
$\chi_{W_4}^2$	16	4	0	1	1	0	1
$\chi_{W_4}(g^2)$	4	4	4	1	1	0	−1
$\mathrm{Alt}^2(W_4)$	6	0	−2	0	0	0	1
$\mathrm{Sym}^2(W_4)$	10	4	2	1	1	0	0

Note that $\mathrm{Alt}^2(W_4)$ is irreducible, but $\mathrm{Sym}^2(W_4)$ is not. Indeed, the latter clearly has a positive inner product with the trivial representation, hence contains a copy of it which we take away giving us W_9.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
W_9	9	3	1	0	0	−1	−1

Looking back, we have found five irreducible representations of S_5, and hence are left to discover two more. The sum of squares of their dimensions is 70; Corollary 4.8.1 tells us that the sum of squares of the dimensions of the remaining two must be $\operatorname{ord}(S_5) - 70 = 50$. Now, $50 = 1^2 + 7^2 = 5^2 + 5^2$, but the first decomposition is not possible since the trivial and sign representations are the only ones of dimension 1. Thus, both remaining representations are of dimension 5.

Observe that $\langle \chi_{W_9}, \chi_{W_9} \rangle = 2 = 1^2 + 1^2$, hence W_9 must be composed of two irreducible representations. The only way to write 9 as the sum of two dimensions of irreducible representations (which we know are $1, 1, 4, 4, 5, 5, 6$) is $4 + 5$. Indeed, $\langle \chi_{W_9}, \chi_{W_4} \rangle = 1$, so we take W_4 away leaving us with an irreducible representation W_5. Taking the tensor product with the sign representation yields our final irreducible representation $W_{5'}$. Thus, the complete character table of S_5 is as follows.

S_5	e $\times 1$	(ab) $\times 10$	$(ab)(cd)$ $\times 15$	(abc) $\times 20$	$(ab)(cdf)$ $\times 20$	$(abcd)$ $\times 30$	$(abcdf)$ $\times 24$
Trivial	1	1	1	1	1	1	1
Sign	1	−1	1	1	−1	−1	1
W_4	4	2	0	1	−1	0	−1
W_4'	4	−2	0	1	1	0	−1
W_5	5	1	1	−1	1	−1	0
W_5'	5	−1	1	−1	−1	1	0
$\operatorname{Alt}^2(W_4)$	6	0	−2	0	0	0	1

Remark: Observe that the columns of this character table are pairwise orthogonal! Indeed, we already know from Proposition 4.8 that the first column of any character table must be orthogonal to the rest. Furthermore, this character table strongly suggests a relation of the form

$$\sum_i |\chi_i(g)|^2 = \frac{\operatorname{ord}(G)}{\operatorname{ord}(\operatorname{Cl}(g))},$$

where $\chi_i, \ldots, \chi_k$ are the irreducible characters of G, and $\operatorname{Cl}(g)$ denotes the conjugacy class of $g \in G$.

4.7. Orthogonality revisited

Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be the conjugacy classes of G, let $n_1, \ldots, n_k$ be their sizes, and let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Furthermore, let's work with the field $K = \mathbb{C}$. Set $n = \operatorname{ord}(G)$. We know from Theorem 4.5 that (with a little abuse of notation)

$$\sum_\ell n_\ell \, \chi_i(\mathcal{O}_\ell) \overline{\chi_j(\mathcal{O}_\ell)} = \delta_{ij} n.$$

We construct the matrix $A \in M_k(\mathbb{C})$ via $A_{i\ell} = \sqrt{n_\ell / n}\, \chi_i(\mathcal{O}_\ell)$ and note that $AA^\dagger = \mathbb{I}_n$. Thus, A is a unitary matrix with orthonormal rows. As a result, the columns of A must also be orthonormal, giving

$$\delta_{ij} = \sum_\ell A_{\ell i} \overline{A_{\ell j}} = \frac{\sqrt{n_i n_j}}{n} \sum_\ell \chi_\ell(\mathcal{O}_i) \overline{\chi_\ell(\mathcal{O}_j)}.$$

In other words,

$$\sum_{\ell} \chi_\ell(\mathcal{O}_i)\overline{\chi_\ell(\mathcal{O}_j)} = \begin{cases} n/n_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the columns of the character table of G also obey an orthonormality relation. We supply a different proof below.

Theorem 4.15 (Column orthogonality of characters): Suppose that $K = \mathbb{C}$. Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G, and let $g, g' \in G$. Denote $g \sim g'$ if g and g' are conjugates. Then,

$$\sum_i \chi_i(g)\overline{\chi_i(g')} = \begin{cases} \operatorname{ord}(G)/\operatorname{ord}(\operatorname{Cl}(g)) & \text{if } g \sim g' \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Define the class function

$$f_g : G \to \mathbb{C}, \qquad h \mapsto \begin{cases} 1 & \text{if } g \sim h \\ 0 & \text{otherwise.} \end{cases}$$

Using Corollary 4.10.1, we can expand f_g in terms of the irreducible characters as

$$f_g = a_1\chi_1 + \ldots + a_k\chi_k,$$

and compute each $a_i = \langle f_g, \chi_i \rangle = \operatorname{ord}(\operatorname{Cl}(g))\,\overline{\chi_i(g)}/\operatorname{ord}(G)$. Thus,

$$\frac{\operatorname{ord}(\operatorname{Cl}(g))}{\operatorname{ord}(G)} \sum_i \chi_i(g)\overline{\chi_i(h)} = \overline{f_g(h)} = \begin{cases} 1 & \text{if } g \sim h \\ 0 & \text{otherwise} \end{cases}$$

as desired. □

4.8. The character table for A_5

Note that A_5 has five conjugacy classes; the 5-cycles split into two conjugacy classes in A_5. As usual, we start with the natural representation V of A_5, then take away the copy of the trivial representation yielding an irreducible representation W_4.

A_5	e $\times 1$	$(12)(34)$ $\times 15$	(123) $\times 20$	(12345) $\times 12$	(13524) $\times 12$
Trivial	1	1	1	1	1
V	5	1	2	0	0
W_4	4	0	1	−1	−1

We compute $\operatorname{Sym}^2(W_4)$ and $\operatorname{Alt}^2(W_4)$ as usual.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
$\chi_{W_4}^2$	16	0	1	1	1
$\chi_{W_4}(g^2)$	4	4	1	−1	−1

$\mathrm{Alt}^2(W_4)$	6	−2	0	1	1
$\mathrm{Sym}^2(W_4)$	10	2	1	0	0

Now, $\mathrm{Sym}^2(W_4)$ contains copies of both the trivial representation and W_4; taking them away yields the irreducible representation W_5.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
W_5	5	1	−1	0	0

Looking back, we have found three out of five irreducible representations. The sum of squares of their dimensions is 42, hence the sum of squares dimensions of the remaining two representations must be $\mathrm{ord}(A_5) - 42 = 18$ by Corollary 4.8.1. The only way to write this as a sum of two squares is $18 = 3^2 + 3^2$, hence the remaining two irreducible representations must both have dimension 3.

We can now fill in most of the character table for A_5 as follows.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
	$\times 1$	$\times 15$	$\times 20$	$\times 12$	$\times 12$
Trivial	1	1	1	1	1
W_4	4	0	1	−1	−1
W_3	3	a	b	c	d
W_3'	3	a'	b'	c'	d'
W_5	5	1	−1	0	0

Column orthogonality (Theorem 4.15) gives the relations $1 + 3a + 3a' + 5 = 0$ and $1^2 + a^2 + {a'}^2 + 1^2 = 60/15 = 4$, which simplifies to $a + a' = -2$ and $a^2 + {a'}^2 = 2$. This is only possible if $a = a' = -1$. Again, we have the relations $1 + 4 + 3b + 3b' - 5 = 0$ and $1^2 + 1^2 + b^2 + {b'}^2 + 1^2 = 60/20 = 3$, which simplify to $b + b' = 0$ and $b^2 + {b'}^2 = 0$. This is only possible if $b = b' = 0$.

Similar computations give $c + c' = d + d' = 1$, and $c^2 + {c'}^2 = d^2 + {d'}^2 = 3$. This gives $cc' = dd' = -1$. Thus, c, c' and d, d' are the roots of $x^2 - x - 1$, which we denote as $\varphi = \left(1 + \sqrt{5}\right)/2$ and $\psi = \left(1 - \sqrt{5}\right)/2$. Without loss of generality, set $c = \varphi$, $c' = \psi$. Using the column orthogonality $1 + 1 + cd + c'd' = 0$, we discard the case $d = \varphi$, $d' = \psi$, leaving $d = \psi$, $d' = \varphi$. This completes our computation of the character table of A_5.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
	$\times 1$	$\times 15$	$\times 20$	$\times 12$	$\times 12$
Trivial	1	1	1	1	1
W_4	4	0	1	−1	−1
W_3	3	−1	0	φ	ψ
W_3'	3	−1	0	ψ	φ
W_5	5	1	−1	0	0