MA3101

# Analysis III

Autumn 2021

Satvik Saha
19MS154

*Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.*

## Contents



## 1 Euclidean spaces

### 1.1 $\mathbb{R}^n$ as a vector space

We are familiar with the vector space $\mathbb{R}^n$, with the standard inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is defined as

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle = \sum_{k=1}^{n} (x_i - y_i)^2.$$

**Exercise 1.1.** What are all possible inner products on $\mathbb{R}^n$?

*Solution.* Note that an inner product is a bilinear, symmetric map such that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$. Thus, an product map on $\mathbb{R}^n$ is completely and uniquely determined by the values $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = a_{ij}$. Let $A$ be the $n \times n$ matrix with entries $a_{ij}$. Note that $A$ is a real symmetric matrix with positive entries. Now,

$$\langle \boldsymbol{x}, \boldsymbol{e}_j \rangle = x_1 a_{1j} + \cdots + x_n a_{nj} = \boldsymbol{x}^\top \boldsymbol{a}_j,$$

where $\boldsymbol{a}_j$ is the $j^{\text{th}}$ column of $A$. Thus,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{a}_1 y_1 + \cdots + \boldsymbol{x}^\top \boldsymbol{a}_n y_n = \boldsymbol{x}^\top A \boldsymbol{y}.$$

Furthermore, any choice of real symmetric $A$ with positive entries produces an inner product.

**Theorem 1.1** (Cauchy-Schwarz)**.** *Given two vectors* $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$*, we have*

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|.$$

*Proof.* This is trivial when $\boldsymbol{w} = \boldsymbol{0}$. When $\boldsymbol{w} \neq \boldsymbol{0}$, set $\lambda = \langle \boldsymbol{v}, \boldsymbol{w} \rangle / \|\boldsymbol{w}\|^2$. Thus,

$$0 \leq \|\boldsymbol{v} - \lambda \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \lambda^2 \|\boldsymbol{w}\|^2.$$

Simplifying,

$$0 \leq \|\boldsymbol{v}\|^2 - \frac{|\langle \boldsymbol{v}, \boldsymbol{w} \rangle|^2}{\|\boldsymbol{w}\|^2}.$$

This gives the desired result. Clearly, equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$. □

**Theorem 1.2** (Triangle inequality)**.** *Given two vectors* $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$*, we have*

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

*Proof.* Write

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\langle \boldsymbol{v}, \boldsymbol{w} \rangle + \|\boldsymbol{w}\|^2 \leq \|\boldsymbol{v}\|^2 + 2|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| + \|\boldsymbol{w}\|^2.$$

Applying Cauchy-Schwarz gives

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2.$$

Equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$ for $\lambda \geq 0$. □

## 1.2 $\mathbb{R}^n$ as a metric space

Our previous observations allow us to define the standard metric on $\mathbb{R}^n$, seen as a point set.

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

**Definition 1.1.** For any $\delta > 0$, the set

$$B_\delta(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < \delta\}$$

is called the open ball centred at $\boldsymbol{x} \in \mathbb{R}^n$ with radius $\delta$. This is also called the $\delta$ neighbourhood of $\boldsymbol{x}$.

**Definition 1.2.** A set $U$ is open in $\mathbb{R}^n$ if for every $\boldsymbol{x} \in U$, there exists an open ball $B_\delta(\boldsymbol{x}) \subset U$.

*Remark.* Every open ball in $\mathbb{R}^n$ is open.

*Remark.* Both $\emptyset$ and $\mathbb{R}^n$ are open.

**Definition 1.3.** A set $F$ is closed in $\mathbb{R}^n$ if its complement $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.

*Remark.* Both $\emptyset$ and $\mathbb{R}^n$ are closed.

*Remark.* Finite sets in $\mathbb{R}^n$ are closed.

**Theorem 1.3.** *Unions and finite intersections of open sets are open.*

**Corollary 1.3.1.** *Intersections and finite unions of closed sets are closed.*

**Definition 1.4.** An interior point $x$ of a set $S \subseteq \mathbb{R}^n$ is such that there is a neighbourhood of $x$ contained within $S$.

**Definition 1.5.** An exterior point $x$ of a set $S \subseteq \mathbb{R}^n$ is an interior point of the complement $\mathbb{R}^n \setminus S$.

**Definition 1.6.** A boundary point of a set is neither an interior point, nor an exterior point.

**Definition 1.7.** A limit point $x$ of a set $S \subseteq \mathbb{R}^n$ is such that every neighbourhood of $x$ contains a point from $S$.

**Definition 1.8.** The closure of a set $S \subseteq \mathbb{R}^n$ is the union of $S$ and its limit points.

*Remark.* The closure of a set is the smallest closed set containing it.

### 1.3 Continuous maps

**Definition 1.9.** A map $f\colon X \to Y$ is continuous if the pre-image of every open set from $Y$ is open in $X$.

**Lemma 1.4.** *A map $f\colon X \to Y$ is continuous if the pre-image of every closed set from $Y$ is closed in $X$.*

**Theorem 1.5.** *The projection maps* $\pi_i \colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \mapsto x_i$ *are continuous.*

*Proof.* Let $U \subseteq \mathbb{R}$ be open; we claim that $\pi_i^{-1}(U)$ is open. Pick $\boldsymbol{x} \in \pi_i^{-1}(U)$, and note that $\pi_i(\boldsymbol{x}) = x_i \in U$. Thus, there exists $\delta > 0$ such that $(x_i - \delta, x_i + \delta) \subset U$. Now examine $B_\delta(\boldsymbol{x})$; for any point $\boldsymbol{y}$ within this open ball, we have $d(\boldsymbol{x}, \boldsymbol{y}) < \delta$ hence

$$|x_i - y_i|^2 \leq \sum_{k=1}^{n} (x_k - y_k)^2 = d(\boldsymbol{x}, \boldsymbol{y})^2 < \delta^2.$$

In other words, $\pi_i(\boldsymbol{y}) = y_i \in (x_i - \delta, x_i + \delta)$, hence $\pi_i\, B_\delta(\boldsymbol{x}) \subseteq (x_i - \delta, x_i + \delta) \subset U$. Thus, given arbitrary $\boldsymbol{x} \in \pi_i^{-1}(U)$, we have found an open ball $B_\delta(\boldsymbol{x}) \subset \pi_i^{-1}(U)$. ☐

**Lemma 1.6.** *Finite sums, products, and compositions of continuous functions are continuous.*

**Theorem 1.7.** *All polynomial functions of the coordinates in* $\mathbb{R}^n$ *are continuous.*

*Example.* The unit sphere $S^{n-1} \subset \mathbb{R}^n$ is closed. It is by definition the pre-image of the singleton closed set $\{1\}$ under the continuous map

$$\boldsymbol{x} \mapsto x_1^2 + \cdots + x_n^2.$$