MA3103

Introduction to Graph Theory and Combinatorics

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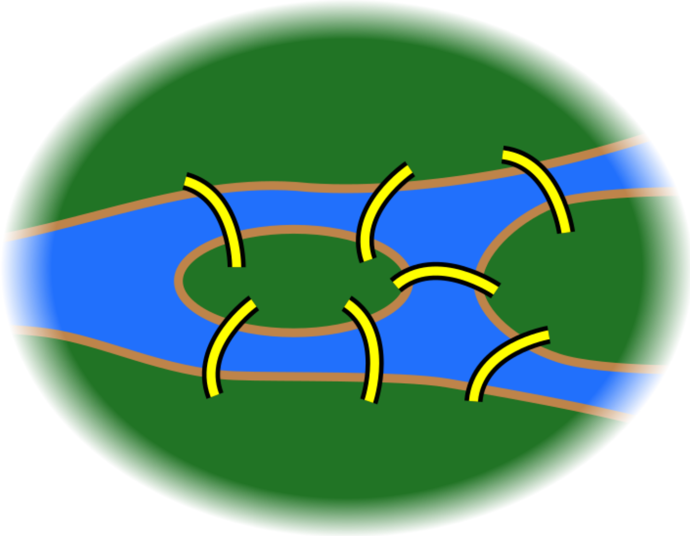
Contents

1 Introduction

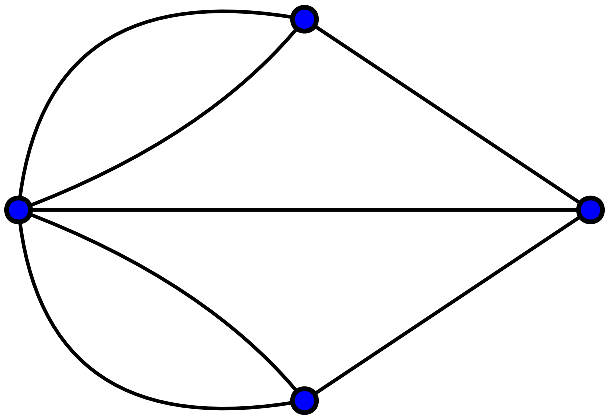
1.1 The Seven Bridges of Königsberg

The diagram below depicts a region in the city of Königsberg, Prussia. There are two islands, connected with the mainland and to each other via seven bridges. The Seven Bridges Problem is posed as follows: is it possible to walk through the entire city, visiting each one of the four landmasses by crossing each of the bridges exactly once?



Leonhard Euler showed that this is impossible; no such walk exists. The techniques he developed in doing so laid the foundations of *graph theory*.

The first thing to note is that the exact shape of the walk/trail is immaterial; all that matters is the sequence of landmasses visited and bridges crossed. Thus, each landmass can be compacted to a single point or *vertex*, and each bridge a line or *edge* connecting two such points. The resulting figure is a graph. Note that the orientations or placements of the points and lines are irrelevant, as long as the connections are undisturbed.



Now, examine a landmass which is on the trail but is neither our starting point, nor our ending point. In order to reach this landmass, we must enter via a bridge; but we cannot stay in the landmass, so we must leave via another a bridge. Thus, for each time we pass through this landmass, we can cross off two bridges joined to it. Once we are done, no bridge may remain unused; this means that we must have started with an even number of bridges joined to this landmass.

However, all four vertices in our graph connect to an odd number of edges. Since we require at least two vertices to act as intermediate points on our path, the desired walk is impossible.

1.2 Basic definitions

Definition 1.1. A graph $G(V, E)$ is an ordered pair of the set of vertices V and the set of edges E.

Definition 1.2. A simple graph is undirected, unweighted, and contains no self-loops or multiple edges joining vertices.

Definition 1.3. For a simple undirected graph, the set of edges E consists of two-element subsets of the set of vertices V.

Remark. For a directed, unweighted graph, the set of edges E consists of ordered pairs of elements from the set of vertices V.

Definition 1.4. A vertex is incident to an edge if that edge joins that vertex.

Definition 1.5. Two vertices are adjacent if there exists an edge connecting them. Two edges are adjacent if they connect to a common vertex.

Definition 1.6. The neighbours of a vertex consist of all vertices adjacent to it. The neighbours of an edge consist of all edges adjacent to it.

The number of neighbours of a vertex is called the degree of that vertex.

Definition 1.7. A complete graph is such that every pair of vertices is connected by an edge. The complete (simple) graph of n vertices is denoted by K_n.

1.3 Some principles

Lemma 1.1 (Pigeonhole Principle)**.** *If $n+1$ objects are placed in n boxes, then we can fin a box containing at least 2 objects.*

Proof. If every box contains at most 1 objects, then the total number of objects falls short. □

Theorem 1.2. *There are no simple graphs where the degrees of all vertices are distinct.*

Proof. Let $G(V, E)$ be a simple graph with n vertices. The degrees of each of these vertices must be an integer among $0, 1, \ldots, n-1$. We now consider two cases.

Case I: There is a vertex of degree 0. Thus, this vertex is adjacent to no other vertex, which means that no vertex can have the full degree $n-1$. This means that the remaining vertices have degrees among $1, 2, \ldots, n-2$, i.e. $n-2$ choices of degree for $n-1$ vertices.

Case II: There is no vertex of degree 0. Thus, the vertices have degrees among $1, 2, \ldots, n-1$, i.e. $n-1$ choices of degree for n vertices.

In either case, the Pigeonhole Principle forces at least two vertices to share the same degree. □

Lemma 1.3 (Strong Pigeonhole Principle)**.** *Let $q_1, q_2, \dots, q_n$ be positive integers. If*

$$N = q_1 + \cdots + q_n - n + 1$$

objects are placed in n boxes, then we can find a box i containing at least q_i objects.

Proof. If every box i contains at most $q_i - 1$ objects, then the total number of objects falls short.

$$N \leq (q_1 - 1) + \cdots + (q_n - 1) = q_1 + \cdots + q_n - n = N - 1$$

□

Theorem 1.4. *The sum of the degrees of all vertices in a simple graph is twice the number of its edges.*

Proof. Let $G((V, E)$ be a simple graph. Define the incidence function $I\colon E \times V \to \{0, 1\}$, such that $I(e, v) = 1$ if e and v are incident, 0 otherwise. We perform the double counting,

$$\sum_{v \in V} \sum_{e \in E} I(e, v) = \sum_{e \in E} \sum_{v \in V} I(e, v).$$

Now, the number of edges incident to a vertex is simply its degree, so $\sum_{e \in E} I(e, v) = d(v)$. Also, every edge is incident to exactly two vertices, so $\sum_{v \in V} I(e, v) = 2$. Thus, we have

$$\sum_{v \in V} d(v) = 2|E|.$$

□

Lemma 1.5 (Inclusion-Exclusion Principle)**.** *For finite sets $A_1, A_2, \dots, A_n$, the number of elements in their union is given by*

$$\sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

Theorem 1.6. *There are $2^{\binom{n}{2}}$ simple graphs with n vertices.*

Exercise 1.1. How many simple graphs are there with n vertices and m edges?

Theorem 1.7. *Let $n, k \in \mathbb{N}$ such that $n > 3$ and $n/2 < k < n$. Let there be n points on a plane such that no three points are collinear. If every point is connected to at least k other points by segments, then there must be at least three segments forming a triangle.*

Proof. Consider a graph $G(V, E)$ with n vertices, such that every vertex has degree at least k. Pick an edge, say $\{x, y\}$, and let A be the neighbours of x apart from y, B be the neighbours of y apart from x. Note that A, B have at least $k-1$ elements each. Suppose that $A \cap B = \emptyset$, i.e. the edge $\{x, y\}$ doe not form a triangle. Thus,

$$|A \cup B| = |A| + |B| - |A \cap B| \geq 2(k-1).$$

However, $|A \cup B| \leq n-2$, hence $n \geq 2k$, or $k \leq n/2$. This is a contradiction. □

Remark. We have shown that *every* segment is part of a triangle. The number of segments here is

$$|E| \geq nk > \frac{n^2}{4}.$$

Exercise 1.2. Is the condition $|E| > n^2/4$ sufficient to ensure the existence of a triangle?

Lemma 1.8 (Cauchy-Schwarz)**.** *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be positive reals. Then,*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Equality holds if and only if every $a_i = \lambda b_i$ for some fixed real λ.

Theorem 1.9 (Mantel)**.** *In a simple graph with n vertices, the condition $|E| > n^2/4$ is sufficient to ensure the existence of a triangle.*

Proof. Let G be a simple graph with n vertices which is triangle-free. Thus, for any edge $\{x, y\} \in E$, the neighbour sets A and B of x and y intersect at no vertex. Thus, we can write

$$d(x) + d(y) = |A \cup B| \leq n.$$

Sum this over all possible edges. On the right, we have $n|E|$. On the left, we have the sum

$$\sum_{x \in V} d(x)^2 \geq \frac{1}{n}\left(\sum_{x \in V} d(x)\right)^2 \geq \frac{1}{n} \cdot 4|E|^2$$

This gives

$$\frac{4|E|^2}{n} \leq n|E|, \qquad |E| \leq \frac{n^2}{4}.$$ □

Example. Consider a circle, with 21 points on its circumference. It follows that among the angles subtended by these points at the center, at most 110 are greater than $2\pi/3$.

Note that there are $\binom{21}{2} = 210$ angles. Furthermore, given any 3 points on the circle (forming a triangle), all three angles subtended by them cannot be greater than $2\pi/3$. Construct a graph with these $n = 21$ points as vertices, such that two vertices are connected by an edge if and only if the angle subtended by them is greater than $2\pi/3$. Now, note that $n^2/4 = 110.25$, thus if there are more than 110 edges, there must exist a triangle of vertices in which all three angles are greater than $2\pi/3$ – a contradiction!

1.4 Bipartite graphs

Definition 1.8. A graph $G(V, E)$ is called bipartite if the vertex set V can be partitioned into 2 parts V_1, V_2 such that every edge in E joins a vertex of V_1 to a vertex of V_2. In other words, there exists a 2- colouring of the vertices such that no edge connects two vertices of the same colour.

Remark. The sum of the degree of the vertices in one part is exactly equal to the number of edges, which in turn is equal to the sum of the degrees of the vertices in the other part.

Definition 1.9. A complete bipartite graph is such that each vertex in one part is connected to every vertex in the other part. Such a graph is denoted by $K_{m,n}$, where the parts have m and n vertices respectively.

Remark. The total number of edges must be the product of the numbers of vertices in each part.

Definition 1.10. A set of vertices (or edges) in a graph is called independent if no two elements in that set are adjacent.

Lemma 1.10. *A bipartite graph is triangle free.*

Corollary 1.10.1. *If we choose even n, we can achieve a triangle free graph with $n^2/4$ edges, namely $K_{n/2,n/2}$. Similarly, if n is odd but $\lfloor n^2/4 \rfloor$ factors into natural numbers $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, then $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ achieves the upper bound again.*

Lemma 1.11. *An r-partite graph is K_{r+1} free.*

Exercise 1.3. Consider a complete r-partite graph on n vertices. What is the maximum number of edges possible?

Solution. Consider $K_{n_1,\ldots,n_r}$, where $n = n_1 + \cdots + n_r$. The number of edges is

$$|E| = \sum_{i<j} n_i n_j.$$

Cauchy-Schwarz gives

$$n^2 = \sum_i n_i^2 + 2\sum_{i<j} n_i n_j \geq \frac{n^2}{r} + 2|E|.$$

Thus,

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Equality is achieved when $n_1 = \cdots = n_r$.

Definition 1.11. The complete r-partite graph $K_{n_1,\ldots,n_r}$ on n vertices, such that $|n_i - n_j| \leq 1$ for all i, j is called Turan's graph, $T_{n,r}$.

Theorem 1.12 (Turan's Theorem)**.** *The number of edges in a K_{r+1} free graph on n vertices is at most*

$$|E(T_{n,r})| = \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Proof. Fix r; we prove the theorem by induction on n. The base case $n = 2$ has already been shown. Suppose that this holds for all K_{r+1} free graphs with less than n vertices. Note that whenever $n \leq r$, the claim is obvious, since

$$|E| \leq \frac{n(n-1)}{2} \leq \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Otherwise, we have $n \geq n + 1$. Let G have the maximum number of edges such that it is K_{r+1} free. We argue that G must contain K_r; if not, there is still scope for adding edges. Call the vertices in this subset A, and the remaining vertices B. Clearly, $|A| = r$ and $|B| = n - r$. Set e_A equal to the number of edges within A, e_B the number of edges within B, and e_{AB} the number of edges between A and B. We must have $|E| = e_A + e_B + e_B$. Now, A has the structure of K_r, so $e_A = r(r-1)/2$. Since the structure of B is K_{r+1} free, we can apply the induction hypothesis on it, giving $e_B \leq (n-r)^2(r-1)/2r$. Finally, no vertex in B can be connected to every vertex in A, so we have $e_{AB} \leq (r-1)(n-r)$. Adding everything together,

$$\begin{aligned}
|E| &\leq \frac{r(r-1)}{2} + \frac{(n-r)^2(r-1)}{2r} + (r-1)(n-r) \\
&= \frac{1}{2}\left[r^2 + (n-r)^2 + 2r(n-r)\right]\frac{r-1}{r} \\
&= \frac{n^2}{2}\left(1 - \frac{1}{r}\right).
\end{aligned}$$

Note that for equality to hold, we require every vertex in B to be connected to $r - 1$ vertices in A. This means that B is the Turan's graph $T_{n-r,r}$, so G is the Turan's graph $T_{n,r}$. □

Example. We give a second proof of Mantel's Theorem. Let G be a triangle free graph on n vertices. Let A be an independent set of G, and let B be the set of remaining vertices. Furthermore, let A be a *largest* independent set of G. Now note that in a triangle free graph, the neighbouring set of any vertex must be an independent set. This gives an upper bound of $|A|$ on the degree of any vertex. Also note that given an arbitrary edge, one of its endpoints must lie in B (both endpoints cannot lie in A since it is an independent set). This forces

$$|E| \leq \sum_{x \in B} d(x) \leq |A||B| \leq \frac{1}{4}(|A| + |B|)^2 = \frac{n^2}{4}.$$

Now, note that for equality in the first case, we require no edges in B, i.e. B must be independent. This forces G to be bipartite. For the second equality, we need every vertex in B to have the full degree $|A|$, so G is a complete bipartite graph. For the third equality, we demand $|A| = |B|$, so $G = T_{n,2}$.

1.5 Subgraphs

Definition 1.12. Let $G(V, E)$ be a graph. We say that $G'(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We write $G' \subseteq G$.

Definition 1.13. The subgraph G' induced by a set of vertices $V' \subseteq V$ is such that G' contains all the edges of G that connect vertices from V'. We write $G' = G[V']$.

Definition 1.14. The subgraph G' spans its parent G if the vertex sets $V' = V$.

Definition 1.15. A k-clique of G is an induced subgraph on k vertices which is complete.

Definition 1.16. The Ramsey number $R(s, t)$ is the least positive integer for which every complete graph on that many vertices, with its edges coloured in red and blue, must contain either a red s-clique or a blue t-clique.

Example. We can see that $R(s, t) = R(t, s)$, $R(1, r) = 1$, $R(2, r) = r$.

Example. We can show that $R(3, 3) = 6$. Indeed, every colouring of K_6 must contain at least *two* monochromatic triangles.

Example. Consider any graph G on 6 vertices. Construct a new graph G' on 6 vertices where two vertices are joined by a red edge if there exists a corresponding edge in G, and blue if not. Thus, G' is a 2-coloured complete graph and hence contains a monochromatic triangle. This means that there are 3 vertices in G where either all of them are connected to each other, or none of them are.

Lemma 1.13. *The number $R(s,t)$ is the smallest positive integer such that any graph on $R(s,t)$ vertices contains either an independent set of size s, or a t-clique.*

Theorem 1.14 (Ramsey Theorem)**.** *The Ramsey number $R(s,t)$ is always finite.*

Proof. We show this for all $s,t \geq 3$ by induction on $s+t$. Note that when $s+t=6$, we know that $R(1,5)$, $R(2,4)$, $R(3,3)$ are all finite. Furthermore, $R(s,1) = R(1,t) = 1$. Suppose that $R(s-1,t)$ and $R(s,t-1)$ are both finite; we claim that $R(s,t) < R(s-1,t)+R(s,t-1)$. Without loss of generality, let $s \geq t$. Consider a complete graph K_n on $R(s-1,t)+R(s,t-1) = n$ vertices. Choose a vertex v, and let V_R be the set of its neighbours connected by red edges, V_B be the neighbours connected by blue edges. Clearly, $n = |V_R|+|V_B|+1$, so either $|V_R| \geq R(s-1,t)$ or $|V_B| \geq R(s,t-1)$. In the first case, the consider the subgraph induced by V_R; either it contains a blue t-clique, or a red $s-1$ clique which means that $V_R \cup \{v\}$ contains a s-clique. In either case, we are done. The case with V_B is analogous. □

Remark. This upper bound can be sharpened to $R(s-1,t)+R(s,t-1)-1$ when both $R(s-1,t)$, $R(s,t-1)$ are even.

Example. Consider $R(4,3)$. We have $R(3,3) = 6$ and $R(4,2) = 4$, hence $R(4,3) \leq 6+4-1 = 9$. We show this is a different way. Note that $R(4,3) = R(3,4)$. In the manner of the previous proof, look at the case $|V_R| \geq |V_B|$. Since $|V_B|+|V_R|+1 = 9$, we have $|V_R| \geq 4$. If V_R contains one red edge, then we have found a red 3-clique. Otherwise, V_R contains a blue 4-clique, so we are done.

Definition 1.17. The Ramsey number $R(n_1,\ldots,n_r)$ is the least positive integer for which every complete graph on that many vertices, with the edges coloured in r different colours, must contain some n_i-clique with colour i.

Theorem 1.15. *The Ramsey number $R(n_1,\ldots,n_r)$ is always finite.*

Proof. Apply induction on the number of colours r. Note that we have already proved the theorem for $r=2$. Suppose that the statement holds for $r-1$ colours. We claim that

$$R(n_1,\ldots,n_r) \leq R(n_1,\ldots,n_{r-2},R(n_{r-1},n_r)).$$

Consider a complete graph K_n on $R(n_1,\ldots,n_{r-2},R(n_{r-1},r)) = n$ vertices, with the edges coloured in $1,\ldots,r$. Thus, the induction hypothesis gurantees that this graph must contain at least one n_i-clique in colour i for $1 \leq i \leq r-2$, or an $R(n_{r-1},n_r)$-clique in colour $r-1$ and r. However, the latter case means that the clique contains either an n_{r-1}-clique in colour $r-1$, or an n_r-clique in colour r. □

Theorem 1.16.

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

Proof. Perform induction on $s+t$. This is true whenever $s+t \leq 5$. Suppose that this holds for all $s+t-1$. Now,

$$R(s,t) \leq R(s,t-1) + R(s-1,t) \leq \binom{s+t-3}{s} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}.$$

□

Lemma 1.17.

$$\binom{2k}{k} \leq 2^{2k}.$$

Corollary 1.17.1.

$$R(s,s) \leq \binom{2s-2}{s-1} \leq 2^{2s-2} < 4^s.$$

Theorem 1.18 (Erdős)**.** *For all $s > 3$, $R(s,s) > \lfloor 2^{s/2} \rfloor$.*

Proof. We show that there exists a way of colouring K_n, $n = \lfloor 2^{s/2} \rfloor$ in red and blue such that there is no monochromatic s-clique.

Let A_R denote the event in which an induced subgraph $K_n[R]$ on $R \subseteq V$ vertices is monochromatic. For any edge, let the probability of it being coloured red or blue be the same, i.e. 1/2. Thus, K_s is red with probability

$$\left(\frac{1}{2}\right)^{\binom{s}{2}}.$$

When $|R| = s$, $P(A_R)$ is twice the above probability. We will show that the probability of the existence of a monochromatic s-clique is strictly less than 1. To see this, apply the Inclusion-Exclusion principle, whence the required probability is

$$\sum_{|R|=s} 2\left(\frac{1}{2}\right)^{\binom{s}{s}} = \binom{n}{s} 2^{1-\binom{s}{2}} \leq \frac{n^s}{s!} \cdot 2^{1+s/2} \cdot 2^{-s^2/2}.$$

Putting the value of n, we see that this is

$$\frac{2^{s^2/2}}{s!} \cdot 2^{1+s/2} \cdot 2^{-s^2/2} = \frac{1}{s!} 2^{1+s/2}.$$

However, it is easy to see that $s! > 2^s$ for all $s \geq 4$, which gives the result. □

Theorem 1.19. *For every $n \geq 1$, there is a lower bound p_0 such that for every prime $p \geq p_0$, the following congruence has a solution.*

$$x^n + y^n \equiv z^n \pmod{p}.$$

Theorem 1.20 (Schur's Theorem)**.** *For any positive integer r, there exists a positive integer $S(r)$ such that for every partition of the integers $\{1, 2, \dots, S(r)\}$ into r parts, there exists one part which contains integers x, y, z where $x + y = z$.*

Remark. This can be rephrased in the following manner. For any r colouring of the integers $1, 2, \dots, S(r)$, one can pick integers x, y, z all of the same colour such that $x + y = z$.

Remark. The integers x, y, z are not necessarily distinct!

Proof. We show this for $r \geq 2$. Let $n = R(3, 3, \dots, 3)$ where there are r colours; we claim that this choice of n satisfies the desired property, i.e. $S(r) \leq n$.

Let $C \colon \{1, 2, \dots, n\} \to \{1, \dots, r\}$ be an arbitrary colouring of the integers. Construct the graph K_n, and colour its edges using the following map.

$$\chi \colon E(K_n) \to \{1, \dots, r\}, \qquad \{v_1, v_2\} \mapsto C(|i - j|).$$

We immediately deduce the existence of a monochromatic triangle, say v_i, v_j, v_j with $i < j < k$. Set $x = j - i$, $y = k - j$, $z = k - i$. Then, $x + y = z$ and $C(x) = C(y) = C(z)$. □

1.6 Degree sequences

Definition 1.18. Let G be a graph on n vertices, labelled $1, \dots, n$. Then, we call the sequence $d(1), \dots, d(n)$ the degree sequence of the graph.

Remark. Recall that given a degree sequence, the sum of the numbers is always twice the number of edges, i.e. the sum is always even. Also, we know that at least two vertices have the same degree.

Theorem 1.21. *Let d_i be a graphic sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then, there is a simple graph with the vertex set $\{x_1, \dots, x_n\}$ such that $d(x_i) = d_i$ and the neighbour set*

$$N(x_1) = \{x_2, x_3, \dots, x_{d_1+1}\}.$$

Proof. Let G be one of the graphs with the degree sequence d_i, $d(x_i) = d_i$. Furthermore, choose G such that the following number is maximised.

$$r_G = |N(x_1) \cap \{x_2, \dots, x_{d_1+1}\}|.$$

If $r = d_1$, we are done. Otherwise, suppose that $r_G < d_1$, in which case one of the vertices x_s, $2 \leq s \leq d_1 + 1$ which is not adjacent to x_1. This also means that there is some vertex x_t, $t > d_1 + 1$ adjacent to x_1. Note that $1 \leq d(x_t) \leq d(x_s)$, so x_s is connected to at least one vertex $x_k \neq x_1$; we can also choose $x_k \neq x_t$, and x_k not connected to x_t. This is simply because $d_s \geq d_t$: every neighbour of x_s cannot be connected to x_t as well. Now, we simply remove the edges $\{x_1, x_t\}$, $\{x_s, x_k\}$, and add the edges $\{x_1, x_s\}$, $\{x_t, x_k\}$ to obtain the graph G'. In doing so, we have not preserved the degrees of every vertex, but observe that $r_{G'} > r_G$, contradicting the maximality of r_G. □

Corollary 1.21.1 (Havel-Hakimi)**.** *A sequence d_i with $d_1 \geq \cdots \geq d_n$ is graphic if and only if the sequence $d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$ is graphic.*

Proof. Simply delete the highest degree vertex in the first case to reach the second, and vice versa. □