MA3104

Linear Algebra II

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1 Linear operators on a vector space

1.1 Preliminaries

We discuss finite dimensional vector spaces V over some field $\mathbb{F}$, along with linear operators $\mathrm{T}\colon V \to V$. We also assume that V has the inner product $\langle \cdot, \cdot \rangle$.

Theorem 1.1. *Let $\mathscr{L}(V)$ be the set of all linear operators on the vector space V. Then, $\mathscr{L}(V)$ is a linear algebra over the field $\mathbb{F}$.*

1.2 Field ideals

Definition 1.1. Let $\mathbb{F}$ be a field, and let $\mathbb{F}[x]$ be the ring of polynomials with coefficients from $\mathbb{F}$. An ideal in $\mathbb{F}[x]$ is a subspace I such that $fg \in I$ for all $f \in \mathbb{F}[x]$, $g \in I$.

Definition 1.2. Given $p \in \mathbb{F}[x]$, the set

$$I_p = \mathbb{F}[x]p = \{fp : f \in \mathbb{F}[x]\}$$

is called the principal ideal generated by p.

Theorem 1.2. *Every ideal in $\mathbb{F}[x]$ is a principal ideal.*

Remark. This is analogous to the theorem which states that every subgroup of a cyclic group is cyclic. Both lead to a precise definition of the greatest common divisor.

Corollary 1.2.1. *Let M be a non-trivial ideal in $\mathbb{F}[x]$. Then, there exists a unique monic polynomial $p \in \mathbb{F}[x]$ (leading coefficient 1) such that M is precisely the principal ideal generated by p.*

1.3 Eigenvalues and eigenvectors

Definition 1.3. Let $\mathrm{T} \in \mathscr{L}$ and $\lambda \in \mathbb{F}$. We say that λ is an eigenvalue or characteristic value of T if $\mathrm{T}\,\boldsymbol{v}$ for some non-zero $\boldsymbol{v} \in V$. The vector $\boldsymbol{v}$ is called an eigenvector of T.

Theorem 1.3. *Let $\mathrm{T} \in \mathscr{L}$ and $\lambda \in \mathbb{F}$. The following are equivalent.*

1. *λ is an eigenvalue of T.*
2. *$\mathrm{T} - \lambda\,\mathrm{I}$ is singular.*
3. *$\det(\mathrm{T} - \lambda\,\mathrm{I}) = 0$.*

Definition 1.4. The polynomial $\det(\mathrm{T} - x\,\mathrm{I})$ is called the characteristic polynomial of T.

Definition 1.5. Two linear operators $\mathrm{S}, \mathrm{T} \in \mathscr{L}(V)$ are similar if there exists an invertible operator $\mathrm{X} \in \mathscr{L}(V)$ such that $\mathrm{S} = \mathrm{X}^{-1}\,\mathrm{T}\,\mathrm{X}$.

Remark. Similarity is an equivalence relation on $\mathscr{L}(V)$, thus partitioning it into similarity classes.

Lemma 1.4. *Similar linear operators have the same characteristic polynomial.*

Proof. Let S, T be similar with $\mathrm{S} = \mathrm{X}^{-1}\,\mathrm{T}\,\mathrm{x}$. Then,

$$\begin{aligned}\det(\mathrm{S} - x\,\mathrm{I}) &= \det(\mathrm{X}^{-1}\,\mathrm{T}\,\mathrm{X} - x\,\mathrm{X}^{-1}\,\mathrm{X}) \\ &= \det(\mathrm{X}^{-1})\,\det(\mathrm{T} - x\,\mathrm{I})\,\det(\mathrm{X}) \\ &= \det(\mathrm{T} - x\,\mathrm{I}).\end{aligned}$$

□

Definition 1.6. A linear operator $\mathrm{T} \in \mathscr{L}(V)$ is diagonalizable if there is a basis of V consisting of eigenvectors of T.

Remark. The matrix of T with respect to such a basis is diagonal.

1.4 Annihilating polynomials

Definition 1.7. An polynomial p such that $p(T) = 0$ for a given linear operator $\mathrm{T} \in \mathscr{L}V$ is called an annihilating polynomial of T.

Lemma 1.5. *Every linear operator $T \in \mathscr{L}(V)$, where V is finite dimensional, has a non-trivial annihilating polynomial.*

Proof. Note that the operators $1, T, T^2, \ldots, T^{n^2} \in \mathscr{L}(V)$, of which there are $n^2 + 1$, are linearly dependent, since $\dim \mathscr{L}(V) = n^2$. ☐

Lemma 1.6. *The annihilating polynomials of* T *form an ideal in* $\mathbb{F}[x]$.

Definition 1.8. The minimal polynomial of T is the unique monic generator of the annihilating polynomials of T.

Remark. The minimal polynomial of T divides all its annihilating polynomials.

Theorem 1.7. *The minimal polynomial and characteristic polynomial of* T *share the same roots, except for multiplicities.*

Proof. Let p be the minimal polynomial of T and let f be its characteristic polynomial.

First, let $\lambda \in \mathbb{F}$ be a root of the minimal polynomial, i.e. $p(\lambda) = 0$. The Division Algorithm guarantees

$$p = (x - \lambda)q$$

for some monic polynomial q. By the minimality of the degree of p, we have $q(\mathrm{T}) \neq 0$, hence there exists non-zero $\boldsymbol{v} \in V$ such that $q(\mathrm{T})\boldsymbol{v} \neq \mathbf{0}$. Thus,

$$(\mathrm{T} - \lambda\,\mathrm{I})q(\mathrm{T})\boldsymbol{v} = \mathbf{0}$$

which shows that λ is an eigenvalue, i.e. a root of the characteristic polynomial f.

Next, suppose that λ is a root of the characteristic polynomial, i.e. $f(\lambda) = 0$. Thus, λ is an eigenvalue of T hence there exists non-zero $\boldsymbol{v} \in V$ such that $\mathrm{T}\,\boldsymbol{v} = \lambda\boldsymbol{v}$. This gives $p(\mathrm{T})\boldsymbol{v} = p(\lambda)\boldsymbol{v}$, but $p(\mathrm{T}) = 0$ identically. This forces $p(\lambda) = 0$. ☐

Theorem 1.8 (Cayley-Hamilton)**.** *The characteristic polynomial of* T *annihilates* T.

Corollary 1.8.1. *The minimal polynomial of* T *divides its characteristic polynomial.*

Corollary 1.8.2. *The minimal polynomial of* T *in a finite-dimensional vector space* V *is at most* $\dim V$.