MA 1101 : Mathematics I

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1 Integers

Theorem 1.1. *Define a relation* $\sim_{\mathbb{Z}}$ *on* $\mathbb{N} \times \mathbb{N}$ *as*

$$(m, n) \sim_{\mathbb{Z}} (p, q) \quad \textit{if} \quad m + q = n + p.$$

Then, $\sim_{\mathbb{Z}}$ *is an equivalence relation on* $\mathbb{N} \times \mathbb{N}$.

Proof. For an arbitrary $(m, n) \in \mathbb{N} \times \mathbb{N}$, clearly $(m, n) \sim_{\mathbb{Z}} (m, n)$, hence $\sim_{\mathbb{Z}}$ is reflexive.

Again, for arbitrary $(m, n), (p, q) \in \mathbb{N} \times \mathbb{N}$, if $(m, n) \sim_{\mathbb{Z}} (p, q)$, we have $m + q = n + p$. By the commutativity of addition on natural numbers, $p+n = q+m$, so $(p, q) \sim_{\mathbb{Z}} (m, n)$, hence $\sim_{\mathbb{Z}}$ is symmetric.

For $(m, n), (p, q), (r, s) \in \mathbb{N} \times \mathbb{N}$, if $(m, n) \sim_{\mathbb{Z}} (p, q)$ and $(p, q) \sim_{\mathbb{Z}} (r, s)$, we have $m + q = n + p$ and $p + s = q + r$. Thus, $m + q + p + s = n + p + q + r$, so $m + s = n + r$. Thus, $(m, n) \sim_{\mathbb{Z}} (r, s)$, hence $\sim_{\mathbb{Z}}$ is transitive.

Therefore, $\sim_{\mathbb{Z}}$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. $\square$

Notation. Let us set

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N}) / \sim_{\mathbb{Z}},$$
$$\mathbb{Z}^+ := \{[(n + 1, 1)] : n \in \mathbb{N}\}, \quad \bar{0} := [(1, 1)], \quad \bar{1} := [(2, 1)].$$

Definition (Addition). For $a = [(m, n)],\ b = [(p, q)] \in \mathbb{Z}$, we define

$$a + b := [(m + p, n + q)].$$

Theorem 1.2. *Addition* $(+)$ *is well-defined, associative and commutative.*

Proof. First, we show that $+$ is well-defined. Let $a = [(m, n)] = [(m', n')]$, $b = [(p, q)] = [(p', q')] \in \mathbb{Z}$. We claim that $a + b = [(m + p, n + q)] = [(m' + p', n' + q')]$, i.e. $(m + p, n + q) \sim_{\mathbb{Z}} (m' + p', n' + q')$, i.e $m + p + n' + q' = n + q + m' + p'$. Now, $(m, n) \sim_{\mathbb{Z}} (m', n')$ and $(p, q) \sim_{\mathbb{Z}} (p', q')$, from which we have $m + n' = n + m'$ and $p + q' = q + p'$. Adding these gives the desired result.

For $a, b, c \in \mathbb{Z}$, let $a = [(m, n)], b = [(p, q)], c = [(r, s)]$. From the associativity of addition in $\mathbb{N}$,

$$\begin{aligned} (a + b) + c &= [(m + p, n + q)] + [(r, s)] \\ &= [((m + p) + r, (n + q) + s)] \\ &= [(m + (p + r), n + (q + s))] \\ &= [(m, n)] + [(p + r, q + s)] \\ &= a + (b + c) \end{aligned}$$

Therefore, $+$ is associative.

From the commutativity of addition in $\mathbb{N}$,

$$\begin{aligned} a + b &= [(m + p, n + q)] \\ &= [(p + m, q + n)] \\ &= b + a \end{aligned}$$

Therefore, $+$ is commutative. $\square$

Lemma 1.3. *For all* $m, n, k \in \mathbb{N}$, $[(m, n)] = [(m + k, n + k)] \in \mathbb{Z}$.

Proof. It is sufficient to show that $(m, n) \sim_{\mathbb{Z}} (m+k, n+k)$, i.e. $m+n+k = n+m+k$, which is certainly true. $\square$

Lemma 1.4. *For all $n \in \mathbb{N}$, $[(n,n)] = \bar{0}$.*

Proof. It is sufficient to show that $(n,n) \sim_{\mathbb{Z}} (1,1)$, i.e. $n+1 = n+1$, which is certainly true. $\square$

Theorem 1.5. *For all $a \in \mathbb{Z}$, $\bar{0} + a = a = a + \bar{0}$.*

Proof. Let $a = [(m,n)] \in \mathbb{Z}$.

$$\begin{aligned} a + \bar{0} &= [(m,n)] + [(1,1)] \\ &= [(m+1, n+1)] \\ &= [(m,n)] \\ &= a \\ a + \bar{0} &= a \ = \ \bar{0} + a \end{aligned}$$

$\square$

Theorem 1.6. *For all $a \in \mathbb{Z}$, there exists a unique $x \in \mathbb{Z}$, satisfying $a + x = \bar{0} = x + a$.*

Proof. For $a = [(m,n)] \in \mathbb{Z}$, construct $x = [(n,m)] \in \mathbb{Z}$. Clearly, $a + x = [(m+n, n+m)] = \bar{0}$. From commutativity of $+$, $a + x = \bar{0} = x + a$.

We now show that x is unique. Let $x' \in \mathbb{Z}$, $a + x' = \bar{0} = x' + a$.

$$\begin{aligned} a + x' &= \bar{0} \\ x + (a + x') &= x + \bar{0} \\ (x + a) + x' &= x \\ \bar{0} + x' &= x \\ x' &= x \end{aligned}$$

$\square$

Notation. We denote x as $-a$ and say that $-a$ is the *negative* of a.

Corollary 1.6.1. *If $a = [(m,n)] \in \mathbb{Z}$, then $-a = [(n,m)]$.*

Notation. For $a, b \in \mathbb{Z}$, we write

$$a - b \ := \ a + (-b).$$

Theorem 1.7. *For all $a, b \in \mathbb{Z}$, there exists a unique $x \in \mathbb{Z}$ satisfying $a + x = b$.*

Proof. From the well-defined nature of $+$, there exists a unique $x = b - a = b + (-a) \in \mathbb{Z}$.

$$\begin{aligned} a + x &= a + (b + (-a)) \\ &= a + ((-a) + b) \\ &= (a + (-a)) + b \\ &= \bar{0} + b \\ &= b \end{aligned}$$

Let $x' \in \mathbb{Z}$, $a + x' = b$.

$$\begin{aligned} a + x' &= b \\ x + (a + x') &= x + b \\ (x + a) + x' &= x + b \\ b + x' &= b + x \\ x' &= x \end{aligned}$$

$\square$

Definition (Multiplication). For $a = [(m,n)],\ b = [(p,q)] \in \mathbb{Z}$, we define

$$a \cdot b \ := \ [(mp + nq, mq + np)].$$

Theorem 1.8. *Multiplication $(\cdot)$ is well-defined, associative and commutative.*

Proof. First, we show that $\cdot$ is well-defined. Let $a = [(m,n)] = [(m',n')]$, $b = [(p,q)] = [(p',q')] \in \mathbb{Z}$. We claim that $a \cdot b = [(mp+nq, mq+np)] = [(m'p'+n'q', m'q'+n'p')]$, i.e. $(mp+nq, mq+np) \sim_{\mathbb{Z}} (m'p'+n'q', m'q'+n'p')$.

From $(p,q) \sim_{\mathbb{Z}} (p',q')$,

$$\begin{aligned} p+q' &= q+p' \\ mp+mq' &= mq+mp' \\ np+nq' &= nq+np' \\ mp+nq+mq'+np' &= mq+np+mp'+nq' \\ (mp+nq, mq+np) &\sim_{\mathbb{Z}} (mp'+nq', mq'+np') \end{aligned}$$

From $(m,n) \sim_{\mathbb{Z}} (m',n')$,

$$\begin{aligned} m+n' &= n+m' \\ mp'+n'p' &= np'+m'p' \\ mq'+n'q' &= nq'+m'q' \\ mp'+nq'+m'q'+n'p' &= mq'+np'+m'p'+n'q' \\ (mp'+nq', mq'+np') &\sim_{\mathbb{Z}} (m'p'+n'q', m'q'+n'p') \end{aligned}$$

Transitivity of $\sim_{\mathbb{Z}}$ yields the desired result.

For $a,b,c \in \mathbb{Z}$, let $a = [(m,n)], b = [(p,q)], c = [(r,s)]$.

$$\begin{aligned} (a \cdot b) \cdot c &= [(mp+nq, mq+np)] \cdot [(r,s)] \\ &= [((mp+nq)r + (mq+np)s, (mp+nq)s + (mq+np)r)] \\ &= [(mpr+nqr+mqs+nps, mps+nqs+mqr+npr)] \\ a \cdot (b \cdot c) &= [(m,n)] \cdot [(pr+qs, ps+qr)] \\ &= [(m(pr+qs)+n(ps+qr), m(ps+qr)+n(pr+qs))] \\ &= [(mpr+mqs+nps+nqr, mps+mqr+npr+nqs)] \end{aligned}$$

Therefore, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, i.e. $\cdot$ is associative.

$$\begin{aligned} a \cdot b &= [(mp+nq, mq+np)] \\ &= [(pm+qn, pn+qm)] \\ &= b \cdot a \end{aligned}$$

Therefore, $\cdot$ is commutative. $\square$

Theorem 1.9. *For all $a \in \mathbb{Z}$, $a \cdot \bar{1} = a = \bar{1} \cdot a$.*

Proof. Let $a = [(m,n)] \in \mathbb{Z}$.

$$\begin{aligned} a \cdot \bar{1} &= [(m,n)] \cdot [(2,1)] \\ &= [(2m+n, m+2n)] \\ &= [(m+(m+n), (m+n)+n)] \\ &= [(m,n)] \\ &= a \\ a \cdot \bar{1} &= a = \bar{1} \cdot a \end{aligned}$$

$\square$

Theorem 1.10. *For all $a \in \mathbb{Z}$, $a \cdot \bar{0} = \bar{0} = \bar{0} \cdot a$.*

Proof. Let $a = [(m,n)] \in \mathbb{Z}$.

$$\begin{aligned} a \cdot \bar{0} &= [(m,n)] \cdot [(1,1)] \\ &= [(m+n, m+n)] \\ &= \bar{0} \\ a \cdot \bar{0} &= \bar{0} = \bar{0} \cdot a \end{aligned}$$

$\square$

Theorem 1.11 (Distributivity). *For all $a, b, c \in \mathbb{Z}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.*

Proof. For $a, b, c \in \mathbb{Z}$, let $a = [(m, n)], b = [(p, q)], c = [(r, s)]$.

$$\begin{aligned} a \cdot (b + c) &= [(m, n)] \cdot [(p + r, q + s)] \\ &= [(m(p + r) + n(q + s), m(q + s) + n(p + r))] \\ &= [(mp + mr + nq + ns, mq + ms + np + nr)] \\ &= [(mp + nq, mq + np)] + [(mr + ns, ms + nr)] \\ &= a \cdot b + a \cdot c \end{aligned}$$

□

Theorem 1.12. *For all $a, b \in \mathbb{Z}$, $(-a) \cdot b = -(a \cdot b)$.*

Proof.

$$\begin{aligned} (-a) \cdot b + a \cdot b &= ((-a) + a) \cdot b \\ &= \bar{0} \cdot b \\ &= \bar{0} \\ (-a) \cdot b &= -(a \cdot b) \end{aligned}$$

□

Theorem 1.13. *For all $a, b \in \mathbb{Z}$, $(-a) \cdot (-b) = a \cdot b$.*

Proof.

$$\begin{aligned} (-a) \cdot (-b) + (-(a \cdot b)) &= (-a) \cdot (-b) + (-a) \cdot b \\ &= (-a) \cdot ((-b) + b) \\ &= (-a) \cdot \bar{0} \\ &= \bar{0} \\ (-a) \cdot (-b) &= a \cdot b \end{aligned}$$

□

Lemma 1.14. *If $a = [(m, n)] \in \mathbb{Z}$, $a \neq \bar{0}$, then $m \neq n$.*

Proof. Assume that $m = n$. Then, we have $(m, n) \sim_{\mathbb{Z}} \bar{0}$, contradicting our premise. Hence, we must have $m \neq n$. □

Theorem 1.15 (No zero divisors). *For all $a, b \in \mathbb{Z}$ with $a, b \neq \bar{0}$, we have $a \cdot b \neq \bar{0}$.*

Proof. Let $a = [(m, n)], b = [(p, q)] \in \mathbb{Z}$. Note that $m \neq n$, $p \neq n$, since $a, b \neq \bar{0}$.

Assume that our theorem is false, i.e. $a \cdot b = \bar{0}$. Then $[(mp + nq, mq + np)] = \bar{0} \Rightarrow mp + nq = mq + np$. One of the following must be true.

Case I: If $m > n$, there exists $u \in \mathbb{N}$, such that $m = n + u$. Thus, $(n + u)p + nq = (n + u)q + np \Rightarrow np + up + nq = nq + uq + np$. This implies that $up = uq \Rightarrow p = q$, contradicting $p \neq q$.

Case II: If $n > m$, there exists $v \in \mathbb{N}$, such that $n = m + v$. Thus, $mp + (m + v)q = mq + (m + v)p \Rightarrow mp + mq + vq = mq + mp + vp$. This implies that $vp = vq \Rightarrow p = q$, contradicting $p \neq q$.

Hence, $a \cdot b \neq \bar{0}$. □

Corollary 1.15.1. *For all $a, b \in \mathbb{Z}$, if $a \cdot b = \bar{0}$, then $a = \bar{0}$ or $b = \bar{0}$.*

Theorem 1.16 (Cancellation). *For $a, b, c \in \mathbb{Z}$ with $a \neq \bar{0}$, we have $a \cdot b = a \cdot c \Rightarrow b = c$.*

Proof. For $a, b, c \in \mathbb{Z}$, let $a = [(m, n)], b = [(p, q)], c = [(r, s)]$. We have $m \neq n$.

$$\begin{aligned} a \cdot b &= a \cdot c \\ [(mp + nq, mq + np)] &= [(mr + ns, ms + nr)] \\ mp + nq + ms + nr &= mq + np + mr + ns \\ m(p + s) + n(q + r) &= m(q + r) + n(p + s) \end{aligned}$$

Assume that our theorem is false. Thus, $b \neq c$, i.e. $b + (-c) = [(p + s, q + r)] \neq \bar{0} \Rightarrow p + s \neq q + r$. Without loss of generality, let $p + s > q + r$, i.e. $p + s = q + r + x$ for some $x \in \mathbb{N}$.

Thus, $m(q + r + x) + n(q + r) = m(q + r) + n(q + r + x)$. This implies that $mx = nx \Rightarrow m = n$, which contradicts $m \neq n$.

Hence, $b = c$. □

Definition (Order). For all $a, b \in \mathbb{Z}$, we say that $a > b$ if $a - b \in \mathbb{Z}^+$.

Lemma 1.17. *If $m, n \in \mathbb{N}$, $m > n$, i.e. $m = n + x$ for $x \in \mathbb{N}$, then $a = [(m, n)] \in \mathbb{Z}^+$.*

Proof. We must show that $a = [(n + x, n)] \in \mathbb{Z}^+$, i.e. for some $k \in \mathbb{N}$, $(n + x, n) \sim_{\mathbb{Z}} (k + 1, 1)$, i.e. $n + x + 1 = n + k + 1$. This is clearly true for $k = x$. $\square$

Theorem 1.18. *For all $a, b \in \mathbb{Z}$, we have $a \cdot b > \bar{0}$ if $a, b > \bar{0}$ or $a, b < \bar{0}$.*

Proof. If $a, b > \bar{0}$, then $a, b \in \mathbb{Z}^+$. Thus, $a = [(m + 1, 1)]$ and $b = [(n + 1, 1)]$ for some $m, n \in \mathbb{N}$.

$$\begin{aligned} a \cdot b &= [((m+1)(n+1) + (1)(1), (m+1)1 + 1(n+1))] \\ &= [(mn + m + n + 1 + 1, m + 1 + n + 1)] \\ &= [((m + n + 2) + mn, (m + n + 2))] \in \mathbb{Z}^+ \end{aligned}$$ $\square$

Therefore, $a \cdot b > \bar{0}$.

If $a, b < \bar{0}$, then $\bar{0} - a, \bar{0} - b \in \mathbb{Z}^+$, i.e. $-a, -b > \bar{0}$. Therefore, $(-a) \cdot (-b) > \bar{0} \implies a \cdot b > \bar{0}$

Definition (Identification map). Define $I_{\mathbb{N}} \colon \mathbb{N} \to \mathbb{Z}$ by

$$I_{\mathbb{N}}(n) := [(n + 1, 1)], \quad \text{for all } n \in \mathbb{N}.$$

Theorem 1.19. *$I_{\mathbb{N}}$ is injective.*

Proof. Let $m, n \in \mathbb{N}$.

$$\begin{aligned} I_{\mathbb{N}}(m) &= I_{\mathbb{N}}(n) \\ [(m + 1, 1)] &= [(n + 1, 1)] \\ (m + 1, 1) &\sim_{\mathbb{Z}} (n + 1, 1) \\ m + 1 + 1 &= n + 1 + 1 \\ m &= n \end{aligned}$$

Hence, $I_{\mathbb{N}}$ is injective. $\square$

Theorem 1.20. $I_{\mathbb{N}}(\mathbb{N}) = \mathbb{Z}^+$.

Proof. We first show that $I_{\mathbb{N}}(\mathbb{N}) \subseteq \mathbb{Z}^+$. Let $x \in I_{\mathbb{N}}(\mathbb{N})$. Thus, there exists at least one $k \in \mathbb{N}$ such that $x = I_{\mathbb{N}}(k) = [(k + 1, 1)]$, which implies that $x \in \mathbb{Z}^+$ by definition.

Next, we show that $\mathbb{Z}^+ \subseteq I_{\mathbb{N}}(\mathbb{N})$. Let $x \in \mathbb{Z}^+$. By definition, $x = [(k + 1, 1)]$ for some $k \in \mathbb{N}$. Clearly, $x = I_{\mathbb{N}}(k) \in I_{\mathbb{N}}(\mathbb{N})$.

Hence, we conclude that $I_{\mathbb{N}}(\mathbb{N}) = \mathbb{Z}^+$. $\square$

Theorem 1.21. $I_{\mathbb{N}}(1) = \bar{1}$.

Proof.

$$I_{\mathbb{N}}(1) = [(1 + 1, 1)] = [(2, 1)] = \bar{1}$$ $\square$

Theorem 1.22. *For all $m, n \in \mathbb{N}$, $I_{\mathbb{N}}(m + n) = I_{\mathbb{N}}(m) + I_{\mathbb{N}}(n)$.*

Proof.

$$\begin{aligned} I_{\mathbb{N}}(m) + I_{\mathbb{N}}(n) &= [(m + 1, 1)] + [(n + 1, 1)] \\ &= [(m + 1 + n + 1, 1 + 1)] \\ &= [((m + n) + 1, 1)] \\ &= I_{\mathbb{N}}(m + n) \end{aligned}$$ $\square$

Theorem 1.23. *For all $m, n \in \mathbb{N}$, $I_{\mathbb{N}}(m \cdot n) = I_{\mathbb{N}}(m) \cdot I_{\mathbb{N}}(n)$.*

Proof.

$$\begin{aligned} I_{\mathbb{N}}(m) \cdot I_{\mathbb{N}}(n) &= [(m + 1, 1)] \cdot [(n + 1, 1)] \\ &= [((m+1)(n+1) + (1)(1), (m+1)1 + 1(n+1))] \\ &= [(mn + m + n + 1 + 1, m + n + 1 + 1)] \\ &= [(mn + 1, 1)] \\ &= I_{\mathbb{N}}(m \cdot n) \end{aligned}$$ $\square$

Theorem 1.24. *For all $m, n \in \mathbb{N}$ with $m > n$, $I_{\mathbb{N}}(m) > I_{\mathbb{N}}(n)$.*

Proof.

$$\begin{aligned} I_{\mathbb{N}}(m) - I_{\mathbb{N}}(n) &= [(m+1,1)] + (-[(n+1,1)]) \\ &= [(m+1,1)] + [(1,n+1)] \\ &= [(m+1+1, 1+n+1)] \\ &= [(m,n)]. \end{aligned}$$

From 1.17, $[(m,n)] \in \mathbb{Z}^+$. Therefore, $I_{\mathbb{N}}(m) - I_{\mathbb{N}}(n) \in \mathbb{Z}^+ \implies I_{\mathbb{N}}(m) > I_{\mathbb{N}}(n)$, as desired. □

Identification

For all $n \in \mathbb{N}$, we shall identify $I_{\mathbb{N}}(n)$ with n. With this identification,

$$0 \leftrightarrow \bar{0}$$

$$1 \leftrightarrow \bar{1}$$

$$\mathbb{N} = \mathbb{Z}^+ \subset \mathbb{Z}$$

$$\mathbb{Z} = \{\, n : n \in \mathbb{N} \,\} \cup \{\, -n : n \in \mathbb{N} \,\} \cup \{\, \bar{0} \,\}$$

2 Rationals

Theorem 2.1. *Define a relation $\sim_{\mathbb{Q}}$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))$ as*

$$(m,n) \sim_{\mathbb{Q}} (p,q) \quad \text{if} \quad mq = np.$$

Then, $\sim_{\mathbb{Q}}$ is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))$.

Proof. For an arbitrary $(m,n) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, clearly $(m,n) \sim_{\mathbb{Q}} (m,n)$, hence $\sim_{\mathbb{Q}}$ is reflexive.

Again, for arbitrary $(m,n),(p,q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, if $(m,n) \sim_{\mathbb{Q}} (p,q)$, we have $mq = np$. By the commutativity of multiplication on integers, $pn = qm$, so $(p,q) \sim_{\mathbb{Q}} (m,n)$, hence $\sim_{\mathbb{Q}}$ is symmetric.

For $(m,n),(p,q),(r,s) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, if $(m,n) \sim_{\mathbb{Q}} (p,q)$ and $(p,q) \sim_{\mathbb{Q}} (r,s)$, we have $mq = np$ and $ps = qr$. Thus, $mqps = npqr$, so $ms = nr$. Thus, $(m,n) \sim_{\mathbb{Q}} (r,s)$, hence $\sim_{\mathbb{Q}}$ is transitive.

Therefore, $\sim_{\mathbb{Q}}$ is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))$. □

Notation. Let us set

$$\begin{aligned} \mathbb{Q} &:= (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/\sim_{\mathbb{Q}}, \\ \bar{0} &:= [(0,1)], \quad \bar{1} \;:=\; [(1,1)]. \end{aligned}$$

Definition (Addition). For $a = [(m,n)],\ b = [(p,q)] \in \mathbb{Q}$, we define

$$a + b \;:=\; [(mq + np, nq)].$$

Theorem 2.2. *Addition $(+)$ is well-defined, associative and commutative.*

Proof. First, we show that $+$ is well-defined. Let $a = [(m,n)] = [(m',n')]$, $b = [(p,q)] = [(p',q')] \in \mathbb{Q}$. Now, $(m,n) \sim_{\mathbb{Q}} (m',n')$ and $(p,q) \sim_{\mathbb{Q}} (p',q')$, from which we have $mn' = m'n$ and $pq' = p'q$. We claim

$$\begin{aligned} a + b = [(mq+np, nq)] &= [(m'q' + n'p', n'q')] \\ (mq+np)(n'q') &= (m'q' + n'p')(nq) \\ mn'qq' + nn'pq' &= m'nqq' + nn'p'q \\ qq'(mn' - m'n) &= nn'(p'q - pq') \\ qq'(0) &= nn'(0) \end{aligned}$$

which is clearly true.

For $a, b, c \in \mathbb{Z}$, let $a = [(m, n)], b = [(p, q)], c = [(r, s)]$.

$$\begin{aligned}
(a+b)+c &= [(mq+np, nq)] + [(r,s)] \\
&= [((mq+np)s + nq(r), nqs)] \\
&= [(mqs + nps + nqr, nqs)] \\
&= [(m)qs + n(ps+qr), nqs] \\
&= [(m,n)] + [(ps+qr, qs)] \\
&= a + (b+c)
\end{aligned}$$

Therefore, $+$ is associative.

$$\begin{aligned}
a+b &= [(mq+np, nq)] \\
&= [(pn+qm, qn)] \\
&= b+a
\end{aligned}$$

Therefore, $+$ is commutative. $\square$

Lemma 2.3. *For all $(m, n) \in S$, $k \in \mathbb{Z} \setminus \{0\}$, $[(m, n)] = [(mk, nk)] \in \mathbb{Q}$.*

Proof. It is sufficient to show that $(m, n) \sim_{\mathbb{Q}} (mk, nk)$, i.e. $mnk = nmk$, which is certainly true. $\square$

Lemma 2.4. *For all $n \in \mathbb{Z} \setminus \{0\}$, $[(n, n)] = \bar{1}$.*

Proof. It is sufficient to show that $(n, n) \sim_{\mathbb{Q}} (1, 1)$, i.e. $n \cdot 1 = n \cdot 1$, which is certainly true. $\square$

Theorem 2.5. *For all $a \in \mathbb{Q}$, $\bar{0} + a = a = a + \bar{0}$.*

Proof. Let $a = [(m, n)] \in \mathbb{Q}$.

$$\begin{aligned}
a + \bar{0} &= [(m,n)] + [(0,1)] \\
&= [(m \cdot 1 + n \cdot 0, n \cdot 1)] \\
&= [(m,n)] \\
&= a \\
a + \bar{0} &= a \ = \ \bar{0} + a
\end{aligned}$$

$\square$

Theorem 2.6. *For all $a \in \mathbb{Q}$, there exists a unique $x \in \mathbb{Q}$, satisfying $a + x = \bar{0} = x + a$.*

Proof. For $a = [(m, n)] \in \mathbb{Q}$, construct $x = [(-m, n)] \in \mathbb{Q}$. Clearly, $a + x = [(mn + n(-m), nn)] = \bar{0}$. From commutativity of $+$, $a + x = \bar{0} = x + a$.

We now show that x is unique. Let $x' \in \mathbb{Q}$, $a + x' = \bar{0} = x' + a$.

$$\begin{aligned}
a + x' &= \bar{0} \\
x + (a + x') &= x + \bar{0} \\
(x + a) + x' &= x \\
\bar{0} + x' &= x \\
x' &= x
\end{aligned}$$

$\square$

Notation. We denote x as $-a$ and say that $-a$ is the *negative* of a.

Corollary 2.6.1. *If $a = [(m, n)] \in \mathbb{Q}$, then $-a = [(-m, n)]$.*

Notation. For $a, b \in \mathbb{Q}$, we write

$$a - b \ := \ a + (-b).$$

Theorem 2.7. *For all $a, b \in \mathbb{Q}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a + x = b$.*

Proof. From the well-defined nature of $+$, there exists a unique $x = b - a = b + (-a) \in \mathbb{Q}$.

$$\begin{aligned} a + x &= a + (b + (-a)) \\ &= a + ((-a) + b) \\ &= (a + (-a)) + b \\ &= \bar{0} + b \\ &= b \end{aligned}$$

Let $x' \in \mathbb{Q}$, $a + x' = b$.

$$\begin{aligned} a + x' &= b \\ x + (a + x') &= x + b \\ (x + a) + x' &= x + b \\ b + x' &= b + x \\ -b + (b + x') &= -b + (b + x) \\ (-b + b) + x' &= (-b + b) + x \\ \bar{0} + x' &= \bar{0} + x \\ x' &= x \end{aligned}$$

□

Definition (Multiplication). For $a = [(m, n)],\ b = [(p, q)] \in \mathbb{Q}$, we define

$$a \cdot b \ := \ [(mp, nq)].$$

Theorem 2.8. *Multiplication* $(\cdot)$ *is well-defined, associative and commutative.*

Proof. First, we show that $\cdot$ is well-defined. Let $a = [(m, n)] = [(m', n')]$, $b = [(p, q)] = [(p', q')] \in \mathbb{Q}$. Now, $(m, n) \sim_{\mathbb{Q}} (m', n')$ and $(p, q) \sim_{\mathbb{Q}} (p', q')$, from which we have $mn' = m'n$ and $pq' = p'q$. We claim

$$\begin{aligned} a \cdot b = [(mp, nq)] &= [(m'p', n'q')] \\ (mp)(n'q') &= (nq)(m'p') \\ (mn')(pq') &= (m'n)(p'q) \end{aligned}$$

which is clearly true.

For $a, b, c \in \mathbb{Z}$, let $a = [(m, n)], b = [(p, q)], c = [(r, s)]$.

$$\begin{aligned} (a \cdot b) \cdot c &= [(mp, nq)] \cdot [(r, s)] \\ &= [((mp)r, (nq)s)] \\ &= [(mpr, nqs)] \\ a \cdot (b \cdot c) &= [(m, n)] \cdot [(pr, qs)] \\ &= [(m(pr), n(qs))] \\ &= [(mpr, nqs)] \end{aligned}$$

Therefore, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, i.e. $\cdot$ is associative.

$$\begin{aligned} a \cdot b &= [(mp, nq)] \\ &= [(pm, qn)] \\ &= b \cdot a \end{aligned}$$

Therefore, $\cdot$ is commutative. □

Theorem 2.9. *For all* $a \in \mathbb{Q}$, $a \cdot \bar{1} = a = \bar{1} \cdot a$.

Proof. Let $a = [(m, n)] \in \mathbb{Q}$.

$$\begin{aligned} a \cdot \bar{1} &= [(m, n)] \cdot [(q, 1)] \\ &= [(m \cdot 1, n \cdot 1)] \\ &= [(m, n)] \\ &= a \\ a \cdot \bar{1} &= a \ = \ \bar{1} \cdot a \end{aligned}$$

□

Theorem 2.10. *For all $a \in \mathbb{Z}$, $a \cdot \bar{0} = \bar{0} = \bar{0} \cdot a$.*

Proof. Let $a = [(m, n)] \in \mathbb{Q}$.

$$\begin{aligned} a \cdot \bar{0} &= [(m,n)] \cdot [(0,1)] \\ &= [(m \cdot 0, n)] \\ &= \bar{0} \\ a \cdot \bar{0} &= \bar{0} = \bar{0} \cdot a \end{aligned}$$

□

Theorem 2.11. *For all $a \in \mathbb{Q} \setminus \{\bar{0}\}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a \cdot x = \bar{1} = x \cdot a$.*

Proof. For $a = [(m, n)] \in \mathbb{Q} \setminus \{\bar{0}\}$, construct $x = [(n, m)] \in \mathbb{Q}$. Clearly, $a \cdot x = [(mn, nm)] = \bar{1}$. From commutativity of $\cdot$, $a \cdot x = \bar{1} = x \cdot a$.

We now show that x is unique. Let $x' \in \mathbb{Q}$, $a \cdot x' = \bar{1} = x' \cdot a$.

$$\begin{aligned} a \cdot x' &= \bar{1} \\ x \cdot (a \cdot x') &= x \cdot \bar{1} \\ (x \cdot a) \cdot x' &= x \\ \bar{1} \cdot x' &= x \\ x' &= x \end{aligned}$$

□

Notation. We denote x as a^{-1} and say that a^{-1} is the *inverse* of a.

Theorem 2.12. *For all $a, b \in \mathbb{Q} \setminus \{\bar{0}\}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a \cdot x = b$.*

Proof. From the well-defined nature of $\cdot$, there exists a unique $x = a^{-1} \cdot b \in \mathbb{Q}$.

$$\begin{aligned} a \cdot x &= a \cdot (a^{-1} \cdot b) \\ &= (a \cdot a^{-1}) \cdot b \\ &= \bar{1} \cdot b \\ &= b \end{aligned}$$

Let $x' \in \mathbb{Q}$, $a \cdot x' = b$.

$$\begin{aligned} a \cdot x' &= b \\ x \cdot (a \cdot x') &= x \cdot b \\ (x \cdot a) \cdot x' &= x \cdot b \\ b \cdot x' &= b \cdot x \\ b^{-1} \cdot (b \cdot x') &= b^{-1} \cdot (b \cdot x) \\ (b^{-1} \cdot b) \cdot x' &= (b^{-1} \cdot b) \cdot x \\ \bar{1} \cdot x' &= \bar{1} \cdot x \\ x' &= x \end{aligned}$$

□

Theorem 2.13 (Distributivity). *For all $a, b, c \in \mathbb{Q}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.*

Proof. For $a, b, c \in \mathbb{Q}$, let $a = [(m, n)], b = [(p, q)], c = [(r, s)]$.

$$\begin{aligned} a \cdot (b + c) &= [(m,n)] \cdot [(ps + qr, qs)] \\ &= [(m(ps + qr), nqs)] \\ &= [(mps + nqr, nqs)] \\ a \cdot b + a \cdot c &= [(mp, nq)] + [(mr, ns)] \\ &= [((mp)(ns) + (nq)(mr), (nq)(ns))] \\ &= [(mnps + mnqr, nnqs)] \\ &= [(n(mps + mqr), n(nqs))] \\ &= [(mps + mqr, nqs)] \end{aligned}$$

Hence, $a \cdot (b + c) = a \cdot b + a \cdot c$. □

Theorem 2.14. *For all $a, b \in \mathbb{Q}$, $(-a) \cdot b = -(a \cdot b)$.*

Proof.

$$\begin{aligned} (-a) \cdot b + a \cdot b &= ((-a) + a) \cdot b \\ &= \bar{0} \cdot b \\ &= \bar{0} \\ (-a) \cdot b &= -(a \cdot b) \end{aligned}$$

□

Theorem 2.15. *For all $a, b \in \mathbb{Q}$, $(-a) \cdot (-b) = a \cdot b$.*

Proof.

$$\begin{aligned} (-a) \cdot (-b) + (-(a \cdot b)) &= (-a) \cdot (-b) + (-a) \cdot b \\ &= (-a) \cdot ((-b) + b) \\ &= (-a) \cdot \bar{0} \\ &= \bar{0} \\ (-a) \cdot (-b) &= a \cdot b \end{aligned}$$

□

Lemma 2.16. *If $a = [(m, n)] \in \mathbb{Q}$, $a \neq \bar{0}$, then $m \neq 0$.*

Proof. Assume that $m = 0$. Then, we have $(m, n) \sim_{\mathbb{Q}} \bar{0}$, contradicting our premise. Hence, we must have $m \neq 0$. □

Theorem 2.17 (No zero divisors). *For all $a, b \in \mathbb{Q}$ with $a, b \neq \bar{0}$, we have $a \cdot b \neq \bar{0}$.*

Proof. Let $a = [(m, n)], b = [(p, q)] \in \mathbb{Q}$. Note that $m \neq 0$, $p \neq 0$, since $a, b \neq \bar{0}$.

Assume that our theorem is false, i.e. $a \cdot b = \bar{0}$. Then $[(mp, nq)] = \bar{0} \Rightarrow mp = 0$.

From 1.15.1, $m = 0$ or $p = 0$, which contradicts our premise.

Hence, $a \cdot b \neq \bar{0}$. □

Corollary 2.17.1. *For all $a, b \in \mathbb{Q}$, if $a \cdot b = \bar{0}$, then $a = \bar{0}$ or $b = \bar{0}$.*

Theorem 2.18 (Cancellation). *For $a, b, c \in \mathbb{Q}$ with $a \neq \bar{0}$, we have $a \cdot b = a \cdot c \Rightarrow b = c$.*

Proof.

$$\begin{aligned} a \cdot b &= a \cdot c \\ a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot (a \cdot c) \\ (a^{-1} \cdot a) \cdot b &= (a^{-1} \cdot a) \cdot c \\ b &= c \end{aligned}$$

□

Lemma 2.19. *For all $a = [(m, n)] \in \mathbb{Q}$, $a = [(-m, -n)]$.*

Proof. It is sufficient to show that $(m, n) \sim_{\mathbb{Q}} (-m, -n)$, i.e. $m(-n) = n(-m)$, which is certainly true. □

Definition (Order). For all $a = [(m, n)], b = [(p, q)] \in \mathbb{Q}$, $n, q \in \mathbb{N}$, we say that $a > b$ if $mq > np$.

Theorem 2.20. *For all $a, b \in \mathbb{Q}$, we have $a \cdot b > \bar{0}$ if $a, b > \bar{0}$ or $a, b < \bar{0}$.*

Proof. Let $a = [(m, n)], b = [(p, q)] \in \mathbb{Q}$, $n, q \in \mathbb{N}$. From $n, q \in \mathbb{N} = \mathbb{Z}^+$ we have $n > 0$ and $q > 0$, so $nq > 0 \Rightarrow nq \in \mathbb{N}$.

If $a, b > \bar{0}$, then $m > 0$ and $p > 0$. Thus, $mp > 0$ which gives $a \cdot b = [(mp, nq)] > 0$.

If $a, b < 0$, then $0 > a$ and $0 > b$ so $0 > m$ and $0 > p$. Thus, $-m, -n > 0$, so $(-m)(-n) = mn > 0$, which gives $a \cdot b > 0$. □

Definition (Identification map). Define $I_{\mathbb{Z}} \colon \mathbb{Z} \to \mathbb{Q}$ by

$$I_{\mathbb{Z}}(n) := [(n, 1)], \quad \text{for all } n \in \mathbb{Z}.$$

Theorem 2.21. *$I_{\mathbb{Z}}$ is injective.*

Proof. Let $m, n \in \mathbb{Z}$.

$$\begin{aligned} I_{\mathbb{Z}}(m) &= I_{\mathbb{Z}}(n) \\ [(m,1)] &= [(n,1)] \\ m \cdot 1 &= n \cdot 1 \\ m &= n \end{aligned}$$

Hence, $I_{\mathbb{Z}}$ is injective. □

Theorem 2.22. $I_{\mathbb{Z}}(0) = \bar{0}$.

Proof.

$$I_{\mathbb{Z}}(0) = [(0,1)] = \bar{0}$$

□

Theorem 2.23. $I_{\mathbb{Z}}(1) = \bar{1}$.

Proof.

$$I_{\mathbb{Z}}(1) = [(1,1)] = \bar{1}$$

□

Theorem 2.24. *For all* $m, n \in \mathbb{Z}$, $I_{\mathbb{Z}}(m+n) = I_{\mathbb{Z}}(m) + I_{\mathbb{Z}}(n)$.

Proof.

$$\begin{aligned} I_{\mathbb{Z}}(m) + I_{\mathbb{Z}}(n) &= [(m,1)] + [(n,1)] \\ &= [(m \cdot 1 + 1 \cdot n, 1 \cdot 1)] \\ &= [(m+n, 1)] \\ &= I_{\mathbb{Z}}(m+n) \end{aligned}$$

□

Theorem 2.25. *For all* $m, n \in \mathbb{Z}$, $I_{\mathbb{Z}}(m \cdot n) = I_{\mathbb{Z}}(m) \cdot I_{\mathbb{Z}}(n)$.

Proof.

$$\begin{aligned} I_{\mathbb{Z}}(m) \cdot I_{\mathbb{Z}}(n) &= [(m,1)] \cdot [(n,1)] \\ &= [(m \cdot n, 1 \cdot 1)] \\ &= [(mn, 1)] \\ &= I_{\mathbb{Z}}(m \cdot n) \end{aligned}$$

□

Theorem 2.26. *For all* $m, n \in \mathbb{Z}$ *with* $m > n$, $I_{\mathbb{Z}}(m) > I_{\mathbb{Z}}(n)$.

Proof. We claim $I_{\mathbb{Z}}(m) > I_{\mathbb{Z}}(n)$, i.e. $[(m,1)] > [(n,1)]$. This is equivalent to $m > n$, which is true. □

Identification

For all $n \in \mathbb{Z}$, we shall identify $I_{\mathbb{Z}}(n)$ with n. With this identification,

$$0 \leftrightarrow \bar{0}$$

$$1 \leftrightarrow \bar{1}$$

$$\mathbb{Z} \subset \mathbb{Q}$$