MA3104

Linear Algebra II

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Satvik Saha

19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

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1.1   Preliminaries

We discuss finite dimensional vector spaces V over some field $\mathbb{F}$, along with linear operators $T \colon V \to V$. We also assume that V has the inner product $\langle \cdot, \cdot \rangle$.

Theorem 1.1. *Let $\mathscr{L}(V)$ be the set of all linear operators on the vector space V. Then, $\mathscr{L}(V)$ is a linear algebra over the field $\mathbb{F}$.*

1.2   Field ideals

Definition 1.1. Let $\mathbb{F}$ be a field, and let $\mathbb{F}[x]$ be the ring of polynomials with coefficients from $\mathbb{F}$. An ideal in $\mathbb{F}[x]$ is a subspace I such that $fg \in I$ for all $f \in \mathbb{F}[x]$, $g \in I$.

Definition 1.2. Given $p \in \mathbb{F}[x]$, the set

$$I_p = \mathbb{F}[x]p = \{ fp : f \in \mathbb{F}[x] \}$$

is called the principal ideal generated by p.

Theorem 1.2. *Every ideal in $\mathbb{F}[x]$ is a principal ideal.*

Remark. This is analogous to the theorem which states that every subgroup of a cyclic group is cyclic. Both lead to a precise definition of the greatest common divisor.

Corollary 1.2.1. *Let M be a non-trivial ideal in $\mathbb{F}[x]$. Then, there exists a unique monic polynomial $p \in \mathbb{F}[x]$ (leading coefficient 1) such that M is precisely the principal ideal generated by p.*

1.3   Eigenvalues and eigenvectors

Definition 1.3. Let $T \in \mathcal{L}(V)$ and $c \in \mathbb{F}$. We say that c is an eigenvalue or characteristic value of T if $Tv = cv$ for some non-zero $v \in V$. The vector v is called an eigenvector of T.

Theorem 1.3. *Let $T \in \mathcal{L}(V)$ and $c \in \mathbb{F}$. The following are equivalent.*

  1. c is an eigenvalue of T.

  2. $T - cI$ is singular.

  3. $\det(T - cI) = 0$.

Definition 1.4. The polynomial $\det(T - xI)$ is called the characteristic polynomial of T.

Definition 1.5. Two linear operators $S, T \in \mathcal{L}(V)$ are similar if there exists an invertible operator $X \in \mathcal{L}(V)$ such that $S = X^{-1}TX$.

Remark. Similarity is an equivalence relation on $\mathcal{L}(V)$, thus partitioning it into similarity classes.

Lemma 1.4. *Similar linear operators have the same characteristic polynomial.*

Proof. Let S, T be similar with $S = X^{-1}TX$. Then,

$$\begin{aligned}
\det(S - xI) &= \det(X^{-1}TX - xX^{-1}X) \\
&= \det(X^{-1}) \det(T - xI) \det(X) \\
&= \det(T - xI).
\end{aligned}$$
$\square$

Definition 1.6. A linear operator $T \in \mathcal{L}(V)$ is diagonalizable if there is a basis of V consisting of eigenvectors of T.

Remark. The matrix of T with respect to such a basis is diagonal.

1.4   Annihilating polynomials

Definition 1.7. An polynomial p such that $p(T) = 0$ for a given linear operator $T \in \mathcal{L}(V)$ is called an annihilating polynomial of T.

Lemma 1.5. *Every linear operator $T \in \mathcal{L}(V)$, where V is finite dimensional, has a nontrivial annihilating polynomial.*

Proof. Note that the operators $I, T, T^2, \ldots, T^{n^2} \in \mathcal{L}(V)$, of which there are $n^2 + 1$, are linearly dependent, since $\dim \mathcal{L}(V) = n^2$. $\qquad\square$

Lemma 1.6. *The annihilating polynomials of T form an ideal in $\mathbb{F}[x]$.*

Definition 1.8. The minimal polynomial of T is the unique monic generator of the annihilating polynomials of T.

Remark. The minimal polynomial of T divides all its annihilating polynomials.

Theorem 1.7. *The minimal polynomial and characteristic polynomial of T share the same roots, except for multiplicities.*

Proof. Let p be the minimal polynomial of T and let f be its characteristic polynomial.

First, let $c \in \mathbb{F}$ be a root of the minimal polynomial, i.e. $p(c) = 0$. The Division Algorithm guarantees

$$p(x) = (x - c)\, q(x)$$

for some monic polynomial q. By the minimality of the degree of p, we have $q(T) \neq 0$, hence there exists non-zero $\boldsymbol{v} \in V$ such that $\boldsymbol{w} = q(T)\, \boldsymbol{v} \neq \boldsymbol{0}$. Thus, $p(T)\, \boldsymbol{v} = \boldsymbol{0}$ gives

$$(T - cI)\, q(T)\, \boldsymbol{v} = \boldsymbol{0}, \qquad T\boldsymbol{w} = c\boldsymbol{w},$$

which shows that c is an eigenvalue, i.e. a root of the characteristic polynomial f.

Next, suppose that c is a root of the characteristic polynomial, i.e. $f(c) = 0$. Thus, c is an eigenvalue of T, hence there exists non-zero $\boldsymbol{v} \in V$ such that $T\boldsymbol{v} = c\boldsymbol{v}$. This gives $p(T)\, \boldsymbol{v} = p(c)\, \boldsymbol{v}$, but $p(T) = 0$ identically, forcing $p(c) = 0$. $\qquad\square$

Theorem 1.8 (Cayley-Hamilton)**.** *The characteristic polynomial of T annihilates T.*

Corollary 1.8.1. *The minimal polynomial of T divides its characteristic polynomial.*

Corollary 1.8.2. *The minimal polynomial of T in a finite-dimensional vector space V is at most $\dim V$.*