MA 1101 : Mathematics I

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Solution 1.
Let R be a relation on $\mathbb{R}^2$ such that

$$(x_1, x_2)\, R\, (y_1, y_2) \quad \text{if} \quad x_1 = y_1.$$

(i) For an arbitrary $(x, y) \in \mathbb{R}^2$, $(x, y)\, R\, (x, y)$, since $x = x$. Therefore, R is reflexive.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$, we can write $x_1 = y_1 \Rightarrow y_1 = x_1$. Thus, we have $(y_1, y_2)\, R\, (x_1, x_2)$. Therefore, R is symmetric.

For $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$ and $(y_1, y_2)\, R\, (z_1, z_2)$, we can write $x_1 = y_1$ and $y_1 = z_1$, from which we have $x_1 = z_1 \Rightarrow (x_1, x_2)\, R\, (z_1, z_2)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} [(x_1, x_2)] &= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2)\, R\, (y_1, y_2)\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : x_1 = y_1\} \\ &= \{(x_1, y) : y \in \mathbb{R}\} \end{aligned}$$

Therefore, the quotient set of R is given by

$$\mathbb{R}/R \;=\; \{L_x : x \in \mathbb{R}\},$$

where $L_x = \{(x, y) : y \in \mathbb{R}\}$. Clearly, each equivalence class $L_x \in \mathbb{R}/R$ is a vertical line in the Cartesian plane, passing through $(x, 0)$.

Solution 2.
Let R be a relation on $\mathbb{R}^2$ such that

$$(x_1, x_2)\, R\, (y_1, y_2) \quad \text{if} \quad x_1^2 + x_2^2 = y_1^2 + y_2^2$$

(i) For an arbitrary $(x, y) \in \mathbb{R}^2$, $(x, y)\, R\, (x, y)$, since $x^2 + y^2 = x^2 + y^2$. Therefore, R is reflexive.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$, we can write $x_1^2 + x_2^2 = y_1^2 + y_2^2 \Rightarrow y_1^2 + y_2^2 = x_1^2 + x_2^2$. Thus, we have $(y_1, y_2)\, R\, (x_1, x_2)$. Therefore, R is symmetric.

For $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, if $(x_1, x_2)\, R\, (y_1, y_2)$ and $(y_1, y_2)\, R\, (z_1, z_2)$, we can write $x_1^2 + x_2^2 = y_1^2 + y_2^2$ and $y_1^2 + y_2^2 = z_1^2 + z_2^2$, from which we have $x_1^2 + x_2^2 = z_1^2 + z_2^2 \Rightarrow (x_1, x_2)\, R\, (z_1, z_2)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} [(x_1, x_2)] &= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2)\, R\, (y_1, y_2)\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = y_1^2 + y_2^2\} \end{aligned}$$

Clearly, each equivalence class is a circle of radius $r = \sqrt{x_1^2 + x_2^2}$ centred at the origin. Such a circle can be denoted by $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$. Therefore, the quotient set of R is given by

$$\mathbb{R}/R \;=\; \{C_r : r \geq 0\}.$$

Solution 3.
Let R be a relation on $\mathbb{N}^2$ such that

$$(m, n)\, R\, (p, q) \quad \text{if} \quad m + q = n + p$$

(i) For an arbitrary $(m,n) \in \mathbb{N}^2$, $(m,n)\,R\,(m,n)$, since $m+n = n+m$. Therefore, R is reflexive.

For $(m,n),(p,q) \in \mathbb{N}^2$, if $(m,n)\,R\,(p,q)$, we can write $m+q = n+p \Rightarrow p+n = q+m$. Thus, we have $(p,q)\,R\,(m,n)$. Therefore, R is symmetric.

For $(m,n),(p,q),(r,s) \in \mathbb{N}^2$, note that $m+q = n+p \Leftrightarrow m-n = p-q$. If $(m,n)\,R\,(p,q)$ and $((p,q)\,R\,(r,s)$, we can write $m-n = p-q$ and $p-q = r-s$, from which we have $m-n = r-s \Rightarrow (m,n)\,R\,(r,s)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(m,n) \in \mathbb{N}^2$, we have

$$\begin{aligned}[(m,n)] &= \{(p,q) \in \mathbb{N}^2 : (m,n)\,R\,(p,q)\} \\ &= \{(p,q) \in \mathbb{N}^2 : m+q = n+p\} \\ &= \{(p,q) \in \mathbb{N}^2 : m-n = p-q\}\end{aligned}$$

Clearly, each equivalence class has its elements (p,q) on the line $m-n = x-y$ in the Cartesian plane. Note that $m-n = p-q \Rightarrow n = m+(q-p)$, so for $n \in \mathbb{N}$, we must have $(q-p) \geq 0$. Also note that $(p,q)\,R\,(1, 1+(q-p))$. Thus, we have

$$[(m,n)] = \{(1, 1+(n-m)) : m,n \in \mathbb{N}\}$$

Solution 4.
Let R be a relation on $\mathbb{R}^2 \setminus \{(0,0)\}$ such that

$$(x_1,x_2)\,R\,(y_1,y_2) \quad \text{if} \quad (y_1,y_2) = \alpha(x_1,x_2), \quad \alpha \neq 0$$

(i) Let $x_i \in \mathbb{R} \setminus \{0\}$. Clearly, R is reflexive since $(x_1,x_2) = (1)\cdot(x_1,x_2)$.

Note that $\frac{1}{\alpha} \in \mathbb{R}\setminus\{0\}$, so if $(x_1,x_2)\,R\,(x_3,x_4)$, we have $(x_3,x_4) = \alpha(x_1,x_2) \Rightarrow (x_1,x_2) = \frac{1}{\alpha}(x_3,x_4)$. Therefore, R is symmetric.

If $(x_3,x_4) = \alpha(x_1,x_2)$ and $(x_5,x_6) = \beta(x_3,x_4)$, we have $(x_5,x_6) = (\alpha\beta)\cdot(x_1,x_2)$. Therefore, R is transitive.

Hence, R is an equivalence relation. □

(ii) For $(r,s) \in \mathbb{R} \setminus (0,0)$, we have

$$\begin{aligned}[(r,s)] &= \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : (r,s)\,R\,(x,y)\} \\ &= \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} : (x,y) = \alpha(r,s), \alpha \neq 0\} \\ &= \{(\alpha r, \alpha s) : \alpha, r, s \in \mathbb{R} \setminus \{0\}\}\end{aligned}$$

Clearly, each equivalence class $[(r,s)]$ is a line of slope s/r, through $(1, s/r)$, excluding the origin in the Cartesian plane.

Solution 5.
Let $n \in \mathbb{N}$ and X be a set of n elements. An arbitrary relation R on X is a subset of the Cartesian product $X \times X = X^2$. Note that for $(a,b) \in X^2$, a can be any of the n elements in X, and b can be independently any of the n elements in X. Thus, we have a total of n^2 elements in X^2.

(i) Since R is any subset $R \subseteq X^2$, we can say that a relation on X is any $R \in \mathcal{P}(X^2)$. Thus, the total number of possible relations R is the number of elements in $\mathcal{P}(X^2)$, i.e., 2^{n^2}.

(ii) Let $D = \{(x,x) : x \in X\}$ be the set of the diagonal elements of X^2. Clearly, there are n elements in D. A reflexive relation R must have $D \subseteq R$. Thus, of the n^2 elements of X^2, the n diagonal elements are fixed – the remaining $n^2 - n$ elements can be chosen to be or not to be in R, giving us a total of 2^{n^2-n} such relations.

(iii) Since $xRy \Rightarrow yRx$ if $x = y$, each of the n diagonal elements of X^2 may or may not be present in a symmetric relation R on X. Also, the presence of $(x,y) \in X^2 \setminus D$ in R forces the presence of (y,x) in R. Thus, we have $(n^2-n)/2$ choices for the non-diagonal elements, giving a total of $2^n \cdot 2^{(n^2-n)/2} = 2^{(n^2+n)/2}$ such relations.

(iv) As before, we have $(n^2-n)/2$ choices for non-diagonal elements to fulfil symmetry. The remaining diagonal elements are fixed to fulfil reflexivity, giving a total of $2^{(n^2-n)/2}$ such relations.