MA3201

# Topology

Spring 2022


Satvik Saha

19MS154

*Indian Institute of Science Education and Research, Kolkata,*
*Mohanpur, West Bengal, 741246, India.*


## Contents



# 1 Introduction

## 1.1 Topological spaces

**Definition 1.1.** A topology on some set $X$ is a family $\tau$ of subsets of $X$, satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from $\tau$ are in $\tau$.
3. All finite intersections of elements from $\tau$ are in $\tau$.

The sets from $\tau$ are declared to be open sets in the topological space $(X, \tau)$.

*Example.* Any set $X$ admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

*Example.* Let $X$ be a set. The cofinite topology on $X$ is the collection of complements of finite sets, along with the empty set. Note that when $X$ is finite, this is simply the discrete topology.

**Definition 1.2.** Let $\tau, \tau'$ be two topologies on the set $X$. We say that $\tau$ is finer than $\tau'$ if $\tau$ has more open sets than $\tau'$. In such a case, we also say that $\tau'$ is coarser than $\tau$.

## 1.2 Topological bases

**Definition 1.3.** Let $(X, \tau)$ be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology $\tau$ such that every open set $U \in \tau$ is expressible as a union of elements from $\beta$.

**Definition 1.4.** Let $X$ be a set, and let $\beta$ be a collection of subsets of $X$ satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then, $\beta$ generates a topology on $X$, namely the collection of all unions of elements of $\beta$.

**Lemma 1.1.** *Let $\tau$ be a topology on $X$, and let $\beta \subseteq \tau$ be a collection of open sets. Then, $\beta$ is a basis of $\tau$, or generates $\tau$, if for every $x \in U \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U$.*

*Example.* The collection of all open balls in $\mathbb{R}^n$ form a basis of the usual topology.

**Lemma 1.2.** *Let $X$ be equipped with the topologies $\tau$ and $\tau'$, and let $\beta$ and $\beta'$ be the respective bases of these topologies. Then, $\tau$ is finer than $\tau'$ if and only if given $x \in B' \in \beta'$, there exists $x \in B \in \beta$ such that $B \subseteq B'$.*

*Example.* The collections of open balls in $\mathbb{R}^n$ generate the same topology as the collection of all open rectangles in $\mathbb{R}^n$.

*Example.* Consider the topologies on $\mathbb{R}$ generated by the following bases.

1. $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.
2. $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.
3. $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$ where $K = \{1/n : n \in \mathbb{Z}\}$.

We call the topology generated by $\beta_2$ the lower limit topology, denoted $\mathbb{R}_\ell$. The topology generated by $\beta_3$ is denoted $\mathbb{R}_K$. Both of these are strictly finer than the standard topology.

**Definition 1.5.** A sub-basis for some topology on $X$ is a collection $\rho$ of subsets of $X$ whose union is the whole of $X$. The topology generated by $\rho$ is defined to be the topology generated by the collection of all finite intersections of elements of $\rho$.

## 1.3 Product topology

**Definition 1.6.** Let $(X_1, \tau_1)$, $(X_2, \tau_2)$ be topological spaces. Then $\tau_1 \times \tau_2$ generates the product topology on $X_1 \times X_2$.

*Example.* The product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard topology, coincides with the standard topology on $\mathbb{R}^2$.

**Lemma 1.3.** *If $\beta_1, \beta_2$ are bases of the topologies $\tau_1, \tau_2$, then $\beta_1 \times \beta_2$ and $\tau_1 \times \tau_2$ generate the same product topology.*

*Proof.* Given $(x_1, x_2) \in U$ where $U \subseteq X_1 \times X_2$ is open in the product topology, recall that $U$ can be written as a union of the basic open sets $U_{1i} \times U_{2i}$, where $U_{1i} \in \tau_1$ and $U_{2i} \in \tau_2$. Suppose that $(x_1, x_2) \in U_1 \times U_2$. Thus, we can choose $B_1 \in \beta_1$, $B_2 \in \beta_2$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$. Thus, $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$. □

**Definition 1.7.** The projection maps are defined as $\pi_i \colon X_1 \times \cdots X_k \to X_i$, $(x_1, \ldots, x_k) \mapsto x_i$.

**Lemma 1.4.** *The collection of elements of the form $\pi_1^{-1}(U_1)$ or $\pi_2^{-1}(U_2)$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$, forms a sub-basis of the product topology on $X_1 \times X_2$.*

*Proof.* Note that $\pi_1^{-1}(X_1) = X_1 \times X_2$. Now it is easy to see that finite intersections of elements of the form $U_1 \times X_2$ or $X_1 \times U_2$ where $U_1, U_2$ are open, are all of the form $U_1 \times U_2$ which is precisely a basis of the product topology. □

**Corollary 1.4.1.** *We can restrict ourselves to the sub-basis of elements of the form $\pi_1^{-1}(B_1)$ or $\pi_2^{-1}(B_2)$, where $B_1 \in \beta_1$, $B_2 \in \beta_2$ for some bases $\beta_1$, $\beta_2$ of $\tau_1, \tau_2$.*

## 1.4 Subspace topology

**Definition 1.8.** Let $(X, \tau)$ be a topological space, and let $Y \subset X$. Then the collection $U \cap Y$ for all $U \in \tau$ comprises the subspace topology $\tau_Y$ on $Y$ induced by the topology $\tau$ on $X$.

**Lemma 1.5.** *If $\beta$ is a basis for the topology on $X$, and $Y \subset X$, then the collection $B \cap Y$ for all $B \in \beta$ generates the subspace topology on $Y$.*

**Lemma 1.6.** *An open set of $Y$ is open in $X$ if $Y$ is open in $X$.*

*Proof.* Let $U \subset Y$ be open in $Y$, then $U = V \cap Y$ for some open set $V$ in $X$. If additionally $Y$ is open in $X$, this immediately shows that $U$ is open in $X$. □

**Theorem 1.7.** *Let $(X, \tau_X)$, $(Y, \tau_Y)$ be topological spaces, and let $A \subseteq X$, $B \subseteq Y$. Then, there are two ways of assigning a natural topology on $A \times B$.*

1. *Take the product topology on $X \times Y$, and consider the subspace topology induced by it on $A \times B$.*
2. *Take the subspace topologies on $A$ induced by $\tau_X$, $B$ induced by $\tau_Y$, and consider the product topology generated by them on $A \times B$.*

*These two methods generate the same topology on $A \times B$.*

*Proof.* Open sets in 1 look like $(U \times V) \cap (A \times B)$, where $U \in \tau_X$, $V \in \tau_Y$). Open sets in 2 look like $(U' \cap A) \times (V' \cap B)$, where $U' \in \tau_X$, $V' \in \tau_Y$, which can be rewritten as $(U' \times V') \cap (A \times B)$. It is easy to see that these describe precisely the same sets. □

## 1.5 Order topology

**Definition 1.9.** Let $X$ be a set with a simple order $<$. Then the collection of sets of the form $(a, b)$, $[a_0, b)$, $(a, b_0]$ where $a_0$ is the minimal element of $X$, $b_0$ is the maximal element of $X$, generate the order topology on $X$.

*Example.* The order topology on $\mathbb{N}$ is precisely the discrete topology.

**Definition 1.10.** Let $X_1, X_2$ be simply ordered sets. The dictionary order on $X_1 \times X_2$ is defined as follows: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$.

*Example.* Consider $X = \{1, 2\} \times \mathbb{N}$, where both $\{1, 2\}$ and $\mathbb{N}$ are endowed with the discrete topology. Note that the product topology on $X$ is the discrete topology.

Now consider the dictionary order on $X$. Here, $(1, 1)$ is the smallest element, so we can list the elements of $X$ in ascending order. Note that every $(1, m) < (2, n)$, for all $m, n \in \mathbb{N}$. Now, note that all singletons $\{(1, m)\}$ are open in the order topology on $X$. The same is true for the singletons $\{(1, n)\}$ for all $n > 1$. However, the singleton $\{(2, 1)\}$ is *not* open in the order topology.

*Example.* Consider $\mathbb{R}$ with the usual topology, and $X = [0,1) \cup \{2\}$. Then, $\{2\}$ is open in the subspace topology on $X$, but it is not open in the order topology on $X$.

**Lemma 1.8.** *The open rays of the form $(a, +\infty)$ and $(-\infty, a)$ in $X$ form a sub-basis of the order topology on $X$.*

*Proof.* Note that $(a,b) = (-\infty, b) \cap (a, +\infty)$, $[a_0, b) = (-\infty, b)$, and $(a, b_0] = (a, +\infty)$. □

**Definition 1.11.** Let $X$ be a simply ordered set, and $Y \subseteq X$. Then, we say that $Y$ is convex in $X$ if given $a, b \in Y$ such that $a < b$, the interval $(a,b) = \{x \in X : a < x < b\} \subseteq Y$.

**Theorem 1.9.** *Let $Y$ be convex in $X$. Then, the subspace topology and the order topology on $Y$ induced from the order topology on $X$ coincide.*

### 1.6 Closed sets

**Definition 1.12.** Let $(X, \tau)$ be a topological space. A set $F \subseteq X$ is said to be closed in $X$ if $F^c = X \setminus F \in \tau$.

*Example.* The sets $\emptyset, X$ are closed in every topological space $(X, \tau)$.

*Example.* In a set equipped with the discrete topology, every set is both open and closed.

**Lemma 1.10.** *Arbitrary intersections, and finite unions of closed sets are closed.*

**Theorem 1.11.** *Let $(X, \tau)$ be a topological space, and let $Y \subset X$ be equipped with the subspace topology. Then, a set $F \subseteq Y$ is closed in $Y$ if and only if $F = Y \cap G$, where $G$ is closed in $X$.*

*Proof.* Let $F \subset Y$. Now, $F$ is closed in $Y$, $Y \setminus F = Y \cap F^c$ is open in $Y$, $Y \cap F^c = Y \cap U$ where $U$ is open in $X$, $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$ where $U^c$ is closed. The steps are reversible. □

**Lemma 1.12.** *A closed set of $Y$ is closed in $X$ if $Y$ is closed in $X$.*

### 1.7 Interiors and closures

**Definition 1.13.** Let $A \subseteq X$ where $(X, \tau)$ is a topological space.

1. The interior of $A$ is defined as the union of all open sets contained in $A$. This is denoted by $A^\circ$.
2. The closure of $A$ is defined as the intersection of all closed sets containing $A$. This is denoted by $\overline{A}$.

*Remark.* The interior of a set is open, and the closure of a set is closed.

**Lemma 1.13.** *Let $Y \subset X$ be topological spaces, and let $A \subseteq Y$. Also let $\overline{A}_X$, $\overline{A}_Y$ denote the closures of $A$ in $X$, $Y$ respectively. Then, $\overline{A}_Y = \overline{A}_X \cap Y$.*

**Theorem 1.14.** *Let $A \subset X$. Then,*

1. *$x \in \overline{A}$ if and only if every open set containing $x$ has non-empty intersection with $A$.*
2. *$x \in \overline{A}$ if and only if every basic open set containing $x$ has non-empty intersection with $A$, given that the topology on $X$ is generated by those basic open sets.*

**Definition 1.14.** Let $A \subseteq X$ where $(X, \tau)$ is a topological space. We say that $x \in X$ is a limit point of $X$ if for every open set $U$ containing $x$, the deleted neighbourhood $U \setminus \{x\}$ has non-empty intersection with $A$. The set of limit points of $A$ is denoted by $A'$.

*Example.* Let $X$ be a set endowed with the discrete topology. Then, given any set $A \subseteq X$, we have $A' = \emptyset$.

**Lemma 1.15.** *A closed set contains all its limit points.*

*Proof.* Let $F \subseteq X$ be closed in $X$, and let $x \in F'$. Then given any open set containing $x$, we have $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$, hence $x \in \overline{F} = F$. □

**Lemma 1.16.** *Let $A \subseteq X$ where $(X, \tau)$ is a topological space. Then, $\overline{A} = A \cup A'$.*

*Proof.* It is clear that $\overline{A} \supseteq A \cup A'$. Now pick $x \in \overline{A}$. If $x \notin A$, then we know that given any open neighbourhood $U$ of $x$, we have non-empty $U \cap A$. Furthermore, this intersection can never contain $x$, hence $x \in A'$. This proves that $\overline{A} \subseteq A \cup A'$. □

### 1.8 Convergence of sequences

**Definition 1.15.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points from $(X, \tau)$, and let $x \in X$. We say that this sequence converges to $x$, denoted $x_n \to x$, if every open neighbourhood of $x$ contains the tail of this sequence. In other words, given $U \in \tau$ such that $x \in U$, there must exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

*Example.* Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then, the constant sequence of $b$'s converges to all three points $a, b, c$.

*Example.* Let $X = \mathbb{R}$, and $\tau$ be the collection of all intervals $(-a, a)$ together with $\emptyset, \mathbb{R}$. Then, the constant sequence of 0's converges to every point in $\mathbb{R}$.

**Definition 1.16.** Let $(X, \tau)$ be a topological space. We say that this topological space is Hausdorff if given any two distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

*Example.* The real numbers under the standard topology is Hausdorff.

**Theorem 1.17.** *Let $(X, \tau)$ be a Hausdorff topological space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in $X$. Then, this sequence can converge to at most one point in $X$.*

*Proof.* Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to distinct points $x, y \in X$. Then there exist disjoint open neighbourhoods $U, V$ such that $x \in U$, $y \in V$. Convergence means that both $U$ and $V$ contain a tail of the sequence, which is a contradiction. $\square$

**Lemma 1.18.** *The singleton sets in a Hausdorff space are closed.*

*Proof.* Let $x \in X$ where $(X, \tau)$ is Hausdorff. Pick $y \neq x$, whence there exist $U_y, V_y \in \tau$, such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. In particular, $\{x\} \cap V_y = \emptyset$. We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open. $\square$

**Theorem 1.19.** *The topology induced by a metric is Hausdorff.*

*Proof.* Given a metric space $X$ and distinct points $x, y \in X$, we set $r = |x - y|$, $U = B(x, r/3)$, $V = B(y, r/3)$. $\square$

# 2 Continuous maps

**Definition 2.1.** Let $f\colon X \to Y$ be a function between the topological spaces $(X, \tau_X)$ and $(Y, \tau_Y)$. We say that $f$ is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in $Y$ must be open in $X$.

**Lemma 2.1.** *A function $f\colon X \to Y$ is continuous if and only if given a base $\beta$ of $Y$, we have $f^{-1}(U) \in \tau_X$ for every $U \in \beta$.*

*Example.* The identity function $\mathrm{id}\colon \mathbb{R}_\ell \to \mathbb{R}$ is continuous, while the identity function $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}_\ell$ is not. This is because the topology on $\mathbb{R}_\ell$ is strictly finer than that on $\mathbb{R}$.

**Lemma 2.2.** *A function $f\colon X \to Y$ is continuous if and only if for every closed set $F \subseteq Y$, we have $f^{-1}(F)$ closed in $X$.*

**Lemma 2.3.** *A function $f\colon X \to Y$ is continuous if and only if given any $x \in X$ and an open set $V \subseteq Y$ such that $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$, $f(U) \subseteq V$.*

**Theorem 2.4.** *The composition of continuous functions is continuous.*

## 2.1 Restricting and enlarging the domain

**Lemma 2.5.** *Let $f\colon X \to Y$ be continuous, and let $A \subset X$. Then the restriction of $f$ to $A$ is continuous.*

**Theorem 2.6.** *Let $f\colon X \to Y$, and let $X$ be the union of the collection of open sets $\{A_\lambda\}_{\lambda \in \Lambda}$. If the restrictions of $f$ to each $A_\lambda$ are continuous, then $f$ is continuous.*

*Proof.* Pick $x \in X$, hence $x \in A_\lambda$ for some $\lambda \in \Lambda$. Now if $f(x) \in V \subset Y$, where $V$ is open in $Y$, then the continuity of the restriction of $f$ to $A_\lambda$ gives us an open set $U \subseteq A_\lambda$ such that $f(U) \subseteq V$. Finally since $A_\lambda$ is open in $X$, so is $U$. $\square$

**Definition 2.2.** Let $X$ be the union of the collection of open sets $\{A_\lambda\}_{\lambda \in \Lambda}$. We say that this collection is a locally finite cover of $X$ if given $x \in X$, there exists a neighbourhood $U$ of $x$ such that $U \cap A_\lambda$ is non-empty for only finitely many $\lambda \in \Lambda$.

**Theorem 2.7.** *Let $f\colon X \to Y$, and let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a locally finite collection of closed sets covering $X$. If the restrictions of $f$ to each $F_\lambda$ are continuous, then $f$ is continuous.*

**Corollary 2.7.1** (Pasting lemma)**.** *Let $X = A \cup B$, with $A, B$ closed in $X$. Let $f\colon A \to Y$, $g\colon B \to Y$ be continuous, with $f(x) = g(x)$ on $A \cap B$. Then the function $h\colon X \to Y$, defined by $x \mapsto f(x)$ on $A$ and $x \mapsto g(x)$ on $B$, is continuous.*

**Definition 2.3.** A path is a continuous function $\gamma\colon [0, 1] \to X$.

**Lemma 2.8.** *Two paths $\gamma_1, \gamma_2$ can be concatenated when $\gamma_1(1) = \gamma_2(0)$.*

## 2.2 Projection maps

**Theorem 2.9.** *The projection maps $\pi_i\colon X_1 \times \cdots \times X_k \to X_i$ are continuous, when the domain is equipped with the product topology. Furthermore, the product topology is the coarsest topology on the domain for which the projection maps are continuous.*

**Lemma 2.10.** *Let $f\colon A \to X_1 \times \cdots \times X_k$, where the co-domain is equipped with the product topology. Then, $f$ is continuous if and only if the component functions $f_i = \pi_i \circ f$ are continuous.*

*Proof.* Note that if $f$ is continuous, the compositions $\pi_i \circ f$ are immediately continuous. Conversely suppose that each $f_i$ is continuous, and write

$$f(t) = (f_1(t), \ldots, f_k(t)).$$

The sets $U_1 \times \cdots \times U_k$, where each $U_i \subseteq X_i$ is open, form a basis of the co-domain. Furthermore, their pre-images under $f$ are $f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k)$, which are open in $A$. This shows that $f$ is continuous. □

**Definition 2.4.** Let $J$ be an arbitrary index set. A $J$-tuple of elements in a set $X$ is a function $x\colon J \to X$, formally denoted $(x_\alpha)_{\alpha\in J}$. If $\{X_\alpha\}_{\alpha\in J}$ is a family of sets, their Cartesian product is defined as

$$\prod_{\alpha\in J} X_\alpha = \{x\colon J \to \bigcup_{\alpha\in J} X_\alpha \colon x_\alpha \in X_\alpha\}.$$

*Remark.* The fact that we can choose an element from each set in an uncountable collection relies on the Axiom of Choice.

**Definition 2.5.** Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The topology generated by $\prod_{\alpha\in J} U_\alpha$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the box topology on $\prod_{\alpha\in J} X_\alpha$.

### 2.3 Homeomorphisms

**Definition 2.6.** Let $f \colon X \to Y$ be a function between the topological spaces $(X, \tau_X)$ and $(Y, \tau_Y)$. We say that $f$ is a homeomorphism if $f$ is continuous, $f$ is bijective, and $f^{-1}$ is continuous. We also say that $X$ and $Y$ are homeomorphic when such a homeomorphism between them exists.

*Example.* The interval $(-1, 1)$ is homeomorphic to $\mathbb{R}$; for instance, the map $x \mapsto \tan(\pi x/2)$ on $(-1, 1)$ is a homeomorphism. A simpler construction is the map $x \mapsto x/(1 - x^2)$.