Analysis-I

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Literature used in the preparation of these notes:

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W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976
T. Tao, *Analysis* I *&* II, Hindustan Book Agency, 2014
`https://en.wikipedia.org/wiki/Real_analysis`

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CHAPTER 1

Real Numbers

1. Ordered Sets

By *ordered sets*, we mean *linearly ordered sets,* i.e. the sets in which you can "compare" any two elements.[†] Here's the formal definition:

DEFINITION 1 (ORDERED SET). *Let S be a set. An order on S is a relation $(<)$ with the following two properties*:

TRICHOTOMY: *For $x, y \in S$, exactly one of the following holds: $x < y,\ x = y,\ y < x$.*

TRANSITIVITY: *If $x, y, z \in S$ with $x < y$ and $y < z$, then $x < z$.*

A set on which an order is defined is called an ordered set.

NOTATION 1. *Both $x < y$ and $y > x$ have the same meaning.*
By $x \leq y$ (resp. $x \geq y$), we denote that $x \not> y$ (resp. $x \not< y$).

DEFINITION 2 (UPPER AND LOWER BOUNDS). *Let S be an ordered set and let $E \subseteq S$. If there exists an element $u \in S$ (resp. $\ell \in S$) such that $x \leq u$ (resp. $x \geq \ell$) for all $x \in E$, then we say that E is bounded above (resp. bounded below) and call u (resp. ℓ) an upper bound (resp. a lower bound) of E. A subset $E \subseteq S$ which is bounded both from above and below in S is called a bounded subset of S.*

In particular, we would be interested in the least upper bound and the greatest lower bound of a set. Let's define them formally:

DEFINITION 3 (SUPREMUM AND INFIMUM). *Let S be an ordered set and let $E \subseteq S$ be bounded above (resp. bounded below). Suppose, there exists an upper bound $\alpha \in S$ (resp. a lower bound $\lambda \in S$) of E such that for every upper bound u (resp. every lower bound ℓ) of E, we have $u \geq \alpha$ (resp. $\ell \leq \lambda$). Then α (resp. λ) is called the least upper bound or the supremum (resp. the greatest lower bound or the infimum) of E and we write $\alpha = \sup E$ (resp. $\lambda = \inf E$).*

DEFINITION 4 (MAXIMUM AND MINIMUM). *If the least upper bound of a set S belongs to itself, we call it a maximum and write* $\max S = \sup S$. *If the greatest lower bound of a set S belongs to itself, we call it a minimum and write* $\min S = \inf S$.

[†]Of course, there also exist sets with "laws of comparison" where you can not compare all possible pairs; those are called *partially ordered sets* or *posets.* If you have not encountered them already, you will see them later in Algebra (in particular, when you study Rings and Modules). But we would restrict ourselves only to totally ordered sets.

Definition 5 (LUB and GLB Properties). *An ordered set S has the least upper bound property (resp. greatest lower bound property) if* $\sup E$ *(resp.* $\inf E$*) exists for every nonempty subset $E \subseteq S$ which is bounded above (resp. bounded below).*

In fact, the least upper bound property and the greatest lower bound property are equivalent:

Theorem 1 (Equivalence of LUB and GLB Properties). *An ordered set with the least upper bound property must also have the greatest lower bound property and vice-versa.*

Proof. Let an ordered set S have the least upper bound property. Let $E \subseteq S$ be a set which is bounded below. Denote by L the set of all lower bounds of E. Then $L \neq \emptyset$ and every element of E is an upper bound of L. Since S has the least upper bound property, $\sup L$ exists in S. Now, the definitions of Infimum and Supremum imply that

$$\inf E = \sup L.$$

The converse implication also follows similarly. □

Definition 6 (Completeness). *An ordered set is complete if it has the least upper bound property.*

2. Groups, Rings and Fields

Definition 7 (Binary operation). *A binary operation f on a set S is a mapping from the Cartesian product $S \times S$ to S:*

$$f : S \times S \longrightarrow S.$$

Definition 8 (Group). *A group G is a set with a binary operation $*$, (called the "law of composition") which satisfies the group axioms:*

Associativity: $(a * b) * c = a * (b * c)$ *for all* $a, b, c \in G$.
Identity: *There exists an element* $i_G \in G$ *such that* $a * i_G = a$ *for all* $a \in G$.
Inverse: *For every* $a \in G$*, there exists and element b such that* $a * b = b * a = i_G$.

The sets with a binary operation which only partially fulfill the group axioms, have differnt names:

Definition 9 (Semigroup and Monoid). *If $(G, *)$ only satisfies the* Associativity *axiom, we call G a semigroup. If a semigroup G has an* identity *element, we call G a monoid.*

Definition 10 (Abelian group). *A group G with a commutative law of composition $*$ (i.e. which satisfies the following axiom) is called a commutative or abelian* group:*

Commutativity: $a * b = b * a$ *for all* $a, b \in G$.

*Named after Niels Henrik Abel, a Norwegian mathematician who made several pioneering contributions to Mathematics within his short life (1802 – 1829)

DEFINITION 11 (RING). *A ring is a set R equipped with two binary operations $+$ and $*$, called "addition" and "multiplication", such that $(R, +)$ is an abelian group, $(R, *)$ is a monoid and the operation $*$ is ditributive with respect to $+$:*

DISTRIBUTIVITY: $a * (b + c) = a * b + b * c$ *and* $(b + c) * a = b * a + c * a$ *for all* $a, b, c \in G$.

By 0_R and 1_R, we denote the additive and the multipplicative identities of R, respectively. If $*$ is commutative, then we call R a commutative ring.

DEFINITION 12 (FIELD). *A field is a commutative ring F such that $F^* := F \smallsetminus \{0\}$ is also an abelian group under multiplication.*

3. Ordered field

DEFINITION 13 (ORDERED FIELD). *A field F which is also an ordered set such that for $a, b, c \in F$,*

(*i*) *If $a > b$, then $a + c > b + c$.*
(*ii*) *If $a > 0$ and $b > 0$ then $ab > 0$.*

4. Properties of the real numbers

THEOREM 2 (COMPLETENESS OF THE REALS NUMBERS). *There exists a complete ordered field $\mathbb{R}$ which contains $\mathbb{Q}$ as a subfield.*

The proof of the above theorem will follow from the construction of the real numbers which we postpone to a later chapter.

COROLLARY 1 (ARCHIMEDEAN PROPERTY). *$\mathbb{Z}$ is unbounded in $\mathbb{R}$.*

PROOF. If the claim is false, then by the above theorem and by the symmetry about zero, $\mathbb{Z}$ would have a least upper bound $\alpha \in \mathbb{R}$. In particular, since $\alpha - 1$ is not an upper bound of $\mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $\alpha - 1 \leq n$. That implies $\alpha < n + 1 \in \mathbb{Z}$, which is impossible, since α is an upper bound of $\mathbb{Z}$. $\square$

COROLLARY 2 (ARCHIMEDEAN PROPERTY). *Let $x, y \in \mathbb{R}$ with $x \neq 0$. Then there is an integer n such that $nx > y$.*

PROOF. If the claim is false, then y/x would be an upper bound of $\mathbb{Z}$ which contradicts the above corollary! $\square$

COROLLARY 3. *Between any two distinct real numbers, there is a rational number.*

PROOF. Let $x, y \in \mathbb{R}$ with $x < y$. From the last corollary, we know that there is an integer n such that

$$n(y - x) > 1. \tag{4.1}$$

In particular, $n \neq 0$. Let m be the smallest integer such that $nx < m$. The existence of m follows from the Archimedean property and the least upper bound property of $\mathbb{R}$. Then we have

$$m - 1 \leq nx < m. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$nx < m \leq 1 + nx < ny.$$

Since $n \neq 0$, it follows that

$$x < \frac{m}{n} < y.$$

$\square$

Next we prove the existence of nth roots of positive reals.

THEOREM 3 (EXISTENCE AND UNIQUENESS OF POSITIVE nTH ROOTS). *Let n be a positive integer. Then for every positive real number x, there exists a unique positive real number y such that $y^n = x$.*

PROOF. The uniqueness of y follows from the fact that $0 < y_1 < y_2$ implies $y_1^n < y_2^n$. Let

$$E := \{t \mid t^n < x\}$$

and let t_1 and t_2 be two positive real numbers such that $t_1 < \min\{1, x\}$ and let $t_2 > \max\{1, x\}$. Then we have $t_1^n < t_1 < x$, whereas $t_2^n > t_2 > x$. Since $t_1 \in E$, the set E is nonempty. Since $t_2 \notin E$, it follows that E is bounded above by t_2. From the completeness of the real numbers, we conclude that E has a least upper bound $y \in \mathbb{R}$. Unless $y^n = x$, we must have either $y^n < x$ or $y^n > x$.

Suppose, $y^n < x$. Then we shall provide an $h > 0$ such that $(y + h)^n < x$, thereby contradicting the fact that y is an upper bound of E. Let

$$h = \min\left\{1, \frac{x - y^n}{n(y+1)^n}\right\}.$$

From the identity

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}), \tag{4.3}$$

it follows that

$$(y + h)^n - y^n < hn(y + h)^{n-1} \leq hn(y + 1)^{n-1} = x - y^n.$$

Thus, we obtain $(y + h)^n < x$ which leads to a contradiction.

Suppose, $y^n > x$. Then we shall provide a $k > 0$ such that $(y-k)^n > x$, thereby contradicting the fact that y is the least upper bound of E. Let

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Clearly, $0 < k < y$. Again from (4.3), it follows that

$$y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Thus, we obtain $(y-k)^n > x$ which leads to a contradiction.

Hence, we conclude that $y^n = x$. □

The number y, written as $\sqrt[n]{x}$ or $x^{1/n}$, is called the positive nth root of x.

Corollary 4. *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

Proof. Let $\alpha = a^{1/n}$ and $\beta = b^{1/n}$. Since multiplication is commutative, we have

$$ab = \alpha^n\beta^n = (\alpha\beta)^n$$

. The claim now follows from the uniqueness of the positive nth roots. □

Note that now we can define $x^{m/n}$ for all $m, n \in \mathbb{Z}$ with $n \neq 0$.

5. The extended real number system

Definition 14 (Infinity). *The extended real number system consists of the real field $\mathbb{R}$ and two symbols ∞ and $-\infty$. We preserve the original order in $\mathbb{R}$ and define*

$$-\infty < x < \infty$$

for all $x \in \mathbb{R}$.

We introduce the following conventions on the extended real number system: Let $x \in \mathbb{R}$. Then

(a) $x + \infty = \infty$, $x - \infty = -\infty$, $\dfrac{x}{\pm\infty} = 0$.
(b) If $x > 0$, then $x \cdot \infty = \infty$.
(c) If $x < 0$, then $x \cdot \infty = -\infty$.

6. Decimal approximations of the reals

Recall that the floor of a real number x is the supremum of the set of integers smaller than x and the ceiling of a function is the infimum of the set of integers larger than x.

Theorem 4. *A real number r can be approximated to any arbitrary precision by rational numbers with finite decimal representations.*

Proof. For all n, we have

$$\lfloor 10^n r \rfloor \leq 10^n r < \lfloor 10^n r \rfloor + 1.$$

Since every integer has a finite decimal representation, by dividing all the terms in the above inequality with 10^n, we see that r lies between two rational numbers with finite decimal representations whose difference is $1/10^n$. Since we can choose n to be as large as we wish, the claim follows. □

CHAPTER 2

Point-set Topology of $\mathbb{R}$

1. Intervals and cells

DEFINITION 15 (OPEN INTERVAL). *Let $a, b \in \mathbb{R}$ with $a < b$. Then the open interval (a, b) is the set of all real numbers x such that*

$$a < x < b.$$

DEFINITION 16 (CLOSED INTERVAL). *Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed interval $[a, b]$ is the set of all real numbers x such that*

$$a \leq x \leq b.$$

Intervals of the type $(a, b]$ or $[a, b)$ are called *half-open intervals.*

DEFINITION 17 (n-CELL). *Cartesian product of n closed intervals.*

2. Metric spaces

DEFINITION 18 (METRIC SPACE). *A set M with a* metric *or a* distance function *$d : M \times M \to \mathbb{R}$ such that for all $x, y, z \in M$ the following holds:*

SYMMETRY $d(x, y) = d(y, x)$.

NONNEGATIVITY: $d(x, y) \geq 0$, *where equality holds if and only if $x = y$.*

TRIANGLE INEQUALITY: $d(x, z) \leq d(x, y) + d(y, z)$.

EXAMPLE 1. *The Euclidean space $\mathbb{R}^n$ is a metric space with the Euclidean metric*

$$d(x, y) = \|x - y\| := \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

DEFINITION 19 (OPEN BALLS /NEIGHBOURHOODS AND CLOSED BALLS). *The open (resp. closed) ball of radius r in a metric space (M, d) centered at a point x is defined to be the set of all y such that $d(x, y) < r$ (resp. $d(x, y) \leq r$).*

An open ball $B_r(x)$ of radius r centered at a point x is also called a *neighbourhood* of x and denoted also by $N_r(x)$.

DEFINITION 20 (CONVEX SETS). *A set $E \subseteq \mathbb{R}$ is convex if for any pair of points $x, y \in E$, we have*

$$tx + (1 - t)y \in E$$

for all $t \in [0, 1]$.

EXAMPLE 2. *Balls (both open and closed) and n-cells in $\mathbb{R}^n$ are convex.*

DEFINITION 21 (BOUNDED SET). *Let (M, d) be a metric space and let $S \subseteq M$. The set S is bounded if there exists an $x \in M$ such that for some radius $r > 0$, S is contained in the neighbourhood $B_r(x)$.*

DEFINITION 22 (CLOSURE POINT / ADHERENT POINT). *Let (M, d) be a metric space and let $S \subseteq M$. A point ℓ in M is an closure point of S if every neighbourhood of ℓ contains at least a point of S.*

DEFINITION 23 (CLOSURE). *The set of all the closure points of a set S is called the closure of S and denoted by $\overline{S}$ or $\mathrm{Cl}(S)$.*

DEFINITION 24 (CLOSED SET). *A subset S of a metric space is closed if $\overline{S} = S$.*

DEFINITION 25 (DENSE SUBSET). *A subset S of a metric space M is dense in M if $\overline{S} = M$.*

DEFINITION 26 (LIMIT POINT / ACCUMULATION POINT). *Let (M, d) be a metric space and let $S \subseteq M$. A point $\ell \in M$ is a limit point of S if every neighbourhood of ℓ contains at least a point of S which is different from ℓ itself.*

DEFINITION 27 (ISOLATED POINT). *Every closure point of S which is not a limit point of S is called an isolated point of S.*

DEFINITION 28 (INTERIOR POINT). *Let (M, d) be a metric space and let $S \subseteq M$. A point $p \in S$ is an interior point of S if for some $r > 0$, there is a neighbourhood $B_r(p) \subseteq S$.*

DEFINITION 29 (INTERIOR). *The set of all the interior points of a set S is called the interior of S and denoted by S^0 or $\mathrm{Int}(S)$.*

DEFINITION 30 (OPEN SET). *A subset S of a metric space is open if $S^0 = S$.*

DEFINITION 31 (BOUNDARY). *The boundary ∂S of a set S is the set of points in its closure which are not in its interior, i.e. $\partial S := \overline{S} \setminus S^0$.*

THEOREM 5. *Every neighbourhood is an open set.*

PROOF. Let $y \in B_r(x)$ and let $h = d(x, y) < r$. We claim that $B_{r-h}(y) \subseteq B_r(x)$. Let $s \in B_{r-h}(y)$. Then

$$d(s, x) \leq d(s, y) + d(y, x) < r - h + h < r.$$

Hence, $s \in B_r(x)$. Thus, the claim follows. So, $B_r(x)$ is an open set. $\square$

THEOREM 6. *Let (M, d) be a metric space and let $S \subseteq M$. Then every neighbourhood of each limit point of S contains infinitely many points of S.*

PROOF. Let ℓ be a limit point of S and suppose there is a neighbourhood N of ℓ such that $N \cap S \setminus \{\ell\} = \{s_1, s_2, \ldots, s_n\}$. Let

$$r = \min_{1 \leq i \leq n} d(\ell, s_i) > 0.$$

Then the neighbourhood $B_r(\ell)$ contains no point $s \in S$ such that $\ell \neq s$. That implies, ℓ is not a limit point of S. Thus, we get a contradiction!. $\square$

COROLLARY 5. *A finite subset of a metric space has no limit points.* $\square$

THEOREM 7. *A set S is open if and only if its complement is closed.*

PROOF. Suppose, S^c is closed and let $x \in S$. Since x does not belong to the closed set S^c, x is not a limit point of S^c. That implies the existence of a neighbourhood N of x such that $N \cap S^c = \emptyset$. Hence, $N \subseteq S$ and therefore, $x \in S^0$. It follows that $S \subseteq S^0$, i.e. every point in S is an interior point. So, S is open.

Next, suppose that S is open and let $x \in S$ be a limit point of S^c. Then every neighbourhood of x contains a point of S^c. So, x is not an interior point of S which contradicts the fact that $S = S^0$ (as S is open). $\square$

COROLLARY 6. *A set T is closed if and only if its complement is open.* $\square$

THEOREM 8. *Let S be a set and let $\{E_j\}_{j \in J}$ be a collection of subsets of S for some indexing set J. Then*

$$\left(\bigcup_{j \in J} E_j \right)^c = \bigcap_{j \in J} E_j^c.$$

PROOF. If $s \in \left(\bigcup_{j \in J} E_j \right)^c$ then $s \notin E_j$ for any $j \in J$ and hence, $s \in E_j^c$ for all $j \in J$. Therefore, $s \in \bigcap_{j \in J} E_j^c$.

Conversely, if $s \in \bigcap_{j \in J} E_j^c$, then $s \in E_j^c$ for all $j \in J$ and hence, $s \notin E_j$ for any $j \in J$. So, $s \notin \bigcup_{j \in J} E_j$. Therefore, $s \in \left(\bigcup_{j \in J} E_j \right)^c$.

Thus, we see that the sets $\left(\bigcup_{j \in J} E_j \right)^c$ and $\bigcap_{j \in J} E_j^c$ contain each other as subsets. Hence, they are equal. $\square$

THEOREM 9. (*a*) *For any collection $\{O_j\}$ of open sets, $\cup_j O_j$ is open.*
(*b*) *For any collection $\{C_j\}$ of closed sets, $\cap_j C_j$ is closed.*
(*c*) *For any finite collection $\{O_1, O_2, \ldots, O_n\}$ of open sets, $\cap_{j=1}^n O_j$ is open.*
(*d*) *For any finite collection $\{C_1, C_2, \ldots, C_n\}$ of closed sets, $\cup_{j=1}^n C_j$ is closed.*

PROOF. (a) Let $S = \cup_j O_j$. If $s \in S$, then $s \in O_j$ for some j. Since O_j is open, $O_j^0 = O_j$. In particular, s is an interior point of O_j, i.e. a neighbourhood of s is contained in $O_j \subset S$. So, s is an ineterior point of S. Hence, $S \subseteq S^0$ which implies that $S^0 = S$ as $S^0 \subseteq S$ by definition. Therefore, S is open.

(b) Theorem 8 implies that

$$(\cap_j C_j)^c = \cup_j C_j^c.$$

Theorem 7 implies that C_j^c is open for all j. Now, Part (a) implies that $\cup_j C_j^c$ is open and hence, from the above equation we get that $(\cap_j C_j)^c$ is open. Therefore, again by Theorem 7, $\cap_j C_j$ is closed.

(c) For any finite collection $\{O_1, O_2, \ldots, O_n\}$ of open sets, Let $S = \cap_{j=1}^n O_j$. For any $s \in S$, there exist open balls $B_{r_j}(s) \subseteq O_j$ for all $j \in \{1, 2, \ldots, n\}$. Let

$$r = \min(r_1, r_2, \ldots, r_n).$$

Then $B_r(s) \subseteq O_j$ for all $j \in \{1, 2, \ldots, n\}$. That implies $B_r(s) \subseteq S$. Hence, s is an interior point of S. Therefore, $S \subseteq S^0$. Hence, we conclude that S is an open set.

(d) From Theorem 8, we know that

$$\left(\bigcup_{j=1}^n C_j\right)^c = \bigcap_{j=1}^n C_j^c.$$

Since from Theorem 7, we know that C_j^c are closed, the claim follows from Part (c). □

REMARK 1. In the last two assertions of the above theorem, finiteness of the collections is essntial. For example, for $n \in \mathbb{N}$, consider the open interval

$$\mathcal{O}_n := \Big(-\frac{1}{n}, \frac{1}{n}\Big).$$

From the Archimedean property of $\mathbb{R}$, it follows that

$$\bigcap_{n \in \mathbb{N}} \mathcal{O}_n = \{0\},$$

which is not open. It follows from Theorem 7 and Theorem 8 that the union of the closed sets

$$\mathcal{C}_n := \mathbb{R} \setminus \mathcal{O}_n$$

over $n \in \mathbb{N}$ is $\mathbb{R} \setminus \{0\}$, which is not closed.

THEOREM 10. (a) *A nonempty closed set in $\mathbb{R}$ which is bounded above, has a maximum.*
(b) *A nonempty closed set in $\mathbb{R}$ which is bounded below, has a minimum.*

Proof. (a) Let S be nonempty subset of $\mathbb{R}$ which is and bounded above. Since $\mathbb{R}$ is complete, $\sup S$ exists in $\mathbb{R}$. If $\sup S \in S$, then the claim holds trivially. Now, suppose, $\sup S \notin S$. Then $\sup S$ is a limit point of S and since S is closed, $\sup S$ belongs to S. Thus, we get a contradiction! Hence we conclude, $\max S = \sup S \in S$.

(b) Let S be nonempty subset of $\mathbb{R}$ which is bounded below. Since $\mathbb{R}$ is complete, $\inf S$ exists in $\mathbb{R}$. If $\inf S \in S$, then the claim holds trivially. Now, suppose, $\inf S \notin S$. Then $\inf S$ is a limit point of S and since S is closed, $\inf S$ belongs to S. Thus, we get a contradiction! Hence we conclude, $\min S = \inf S \in S$. □

3. Subspace topology

Theorem 11. *Let (M, d) be a metric space and let $S \subset M$. Then $T \subseteq S$ is open in S if and only if $T = S \cap A$ for some open subset A of M.*

Proof. It suffices to prove that every open ball in S is intersection of an open ball in M with S and vice-versa. Let $s \in S$, let $B_r(s)$ be an open ball of radius r around s in M and let $B'_r(s)$ be an open ball of radius r around s in S. Then we have

$$S \cap B_r(s) = S \cap \{x \in M \mid d(s,x) < r\} = \{x \in S \mid d(s,x) < r\} = B'_r(s).$$

□

In particular, if $S \subset M$ is open, then Theorem 9.(c) implies that for any open set $A \subseteq M$, the set $S \cap A$ is open in M.

However, for an arbitrary subset $S \subset M$ and an open set $A \subseteq M$, though Theorem 11 implies that $S \cap A$ is open in S, but the set $S \cap A$ may not be open in M, e.g. open intervals in $\mathbb{R}$ are not open in the Euclidean plane $\mathbb{R}^2$.

4. Compact sets

Definition 32. *An open cover of a subset S of a metric space (M, d) is a collection $\{\mathcal{O}_r\}$ of open subsets of M such that $S \subseteq \cup_r \mathcal{O}_r$.*

Definition 33. *Let (M, d) be a metric space. We call a subset $K \subseteq M$ compact if every open cover of K contains a finite subcover.*

For example, suppose $\{\mathcal{O}_k\}$ be an open cover of a compact set K. Then there are finitely many indices $k_1, k_2, \dots, k_n$ such that

$$K \subseteq \mathcal{O}_{k_1} \cup \cdots \cup \mathcal{O}_{k_n}.$$

In particular, every finite set is compact.

Theorem 12. *Let (M, d) be a metric space and let $S \subset M$. A set $K \subseteq S$ is compact in S if and only if K is compact in M.*

Proof. Suppose, K is compact in M and let $\{\mathcal{O}_m\}$ be an open cover of K in S. Then Theorem 11 implies that there exists open sets $B_m \subseteq M$ such that $\mathcal{O}_m = S \cap B_m$. Since K is compact in M, there exists a finite subcover $\{B_{m_1}, B_{m_2}, \dots, B_{m_n}\}$ of K in M. Since $K \subseteq S$,

$$\begin{aligned} K &\subseteq (S \cap B_{m_1}) \cup (S \cap B_{m_2}) \cup \cdots \cup (S \cap B_{m_n}) \\ &= \mathcal{O}_{m_1} \cup \mathcal{O}_{m_2} \cup \cdots \cup \mathcal{O}_{m_n}. \end{aligned}$$

Hence, K is compact in S.

Conversely, suppose K is compact in S. Let $\{B_m\}$ be an open cover of K in M. Since $K \subseteq S$, it follows that $\{S \cap B_m\}$ is an open cover of K. Hence, there exists a finite subcover $\{S \cap B_{m_1}, S \cap B_{m_2}, \dots, S \cap B_{m_n}\}$ of K in S. That implies

$$\begin{aligned} K &\subseteq (S \cap B_{m_1}) \cup (S \cap B_{m_2}) \cup \cdots \cup (S \cap B_{m_n}) \\ &\subseteq B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_n}. \end{aligned}$$

□

Theorem 13. *Compact subsets of a metric space are closed.*

Proof. Let K be a compact subset of a metric space (M, d). Let $x \in K^c$. It suffices to find a neighbourhood N around x such that $N \subseteq K^c$, i.e. $N \cap K = \emptyset$.

For each $k \in K$, let $r_k := d(x, k)/2$. Then for all $k \in K$,

$$B_{r_k}(k) \cap B_{r_k}(x) = \emptyset.$$

Since K is compact, the open cover $\{B_{r_k}(k)\}_{k \in K}$ of K has a finite subcover. Let $k_1, k_2, \dots, k_n \in K$ be such that

$$K \subseteq B_{r_{k_1}}(k_1) \cup \ldots \cup B_{r_{k_n}}(k_n).$$

Let

$$N := B_{r_{k_1}}(x) \cap \ldots \cap B_{r_{k_n}}(x).$$

Then we have $N \cap K = \emptyset$. □

Theorem 14. *Closed subsets of compact spaces are compact.*

Proof. Let (M, d) be a metric space, let $K \subseteq M$ be compact and let $\mathcal{C} \subset K$ be a closed set in M. Then $\mathcal{C}^c = M \setminus \mathcal{C}$ is open. Since $K \subseteq M$, we have

$$K \setminus \mathcal{C} \subseteq M \setminus \mathcal{C} = \mathcal{C}^c.$$

Let $\{N_j\}$ be an open cover of $\mathcal{C}$. Then $\mathcal{C} \subseteq \cup_j N_j$. Hence,

$$K = (K \setminus \mathcal{C}) \cap \mathcal{C} \subseteq \mathcal{C}^c \cup \Big(\cup_j N_j \Big).$$

Since K is compact, there exists $j_1, j_2, \dots, j_n$ such that

$$K \subseteq \mathcal{C}^c \cup \Big(\cup_{i=1}^n N_{j_i} \Big).$$

Since $\mathcal{C} \subset K$, the above inclusion implies that $\mathcal{C} \subseteq \cup_{i=1}^n N_{j_i}$. Hence $\mathcal{C}$ is compact. □

COROLLARY 7. *The intersection of a closed and a compact set is compact.*

PROOF. Since compact sets are closed (Theorem 13), and since intersection of two closed sets is closed (Theorem 9.(c)), the intersection of a compact set and a closed set is closed. Since closed subsets of a compact set are compact (Thereom 14), the claim follows. □

THEOREM 15. *Let (M, d) be a metric space. If $\{K_j\}$ is a collection of compact subsets of M such that the intersection of every finite subcollection of $\{K_j\}$ is nonempty, then*

$$\bigcap_j K_j \neq \emptyset.$$

PROOF. Let $L_j := K_j^c = M \setminus K_j$ for all j. Since compact sets are closed (Theorem 13) and since the complement of any closed set is open (Theorem 7), the complement L_j of the compact set K_j is open for all j.

Suppose, $\bigcap_j K_j = \emptyset$. In particular, that implies, no point of K_1 belongs to every K_j. In other words, for every $\alpha \in K_1$, there exists a j_α such that $\alpha \notin K_{j_\alpha}$. Then we have $\alpha \in K_{j_\alpha}^c = L_{j_\alpha}$. Hence the collection $\{L_j\}$ forms an open cover of K_1. Since K_1 is compact, there is a finite subcover $\{L_{j_i}\}_{i=1}^n$ such that

$$K_1 \subset L_{j_i} \cup L_{j_2} \cup \ldots \cup L_{j_n} = K_{j_i}^c \cup K_{j_2}^c \cup \ldots \cup K_{j_n}^c.$$

It follows that

$$K_1 \cap K_{j_i} \cap K_{j_2} \cap \ldots \cap K_{j_n} = \emptyset,$$

which contradicts the given fact that the intersection of every finite subcollection of $\{K_j\}$ is nonempty. □

COROLLARY 8. *If $\{K_n\}_{n\in\mathbb{N}}$ is a collection of nonempty compact sets such that $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$, then*

$$\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset.$$

□

THEOREM 16. *Every infinite subset of a compact set K has a limit point in K.*

PROOF. Let $E \subseteq K$ be an infinite subset of K and suppose, no point in K is a limit point of E. Then for each $k \in K$, there exists a neighbourhood N_k which contains no point of E except possibly k, if $k \in E$. The collection $\{N_k\}_{k\in K}$ is an open cover of K such that each N_k contains at most one point of E. Since E is an infinite set, no finite subcollection of $\{N_k\}_{k\in K}$ can cover $E \subseteq K$, which contradicts the compactness of K. □

THEOREM 17. *If $\{I_j\}$ is a collection of closed intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, then*

$$\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset.$$

PROOF. Let $I_n = [a_n, b_n]$ and let E be the set of all a_n. Then E is nonempty and bounded above by b_1 in $\mathbb{R}$. Since $\mathbb{R}$ is complete, E has a supremum in $\mathbb{R}$. Let $s = \sup E$. If $m, n \in \mathbb{N}$, then

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_n.$$

So, for all m, every b_m is an uppder bound of E. Hence, $s \leq b_m$ for all m. Since $s = \sup E$, we have $a_m \leq s$ for all $m \in \mathbb{N}$. Thus, we see that $s \in I_m$ for all $m \in \mathbb{N}$, which implies

$$s \in \bigcap_{n\in\mathbb{N}} I_n.$$

Hence, we conclude that the above intersection is nonempty. □

THEOREM 18. *Let $n \in \mathbb{N}$. If $\{I_j\}$ is a collection of n-cells such that $I_{m+1} \subseteq I_m$ for all $m \in \mathbb{N}$, then*

$$\bigcap_{m\in\mathbb{N}} I_m \neq \emptyset.$$

PROOF. For all $m \in \mathbb{N}$, let $I_{m,1}, I_{m,2}, \ldots I_{m,n}$ be closed intervals in $\mathbb{R}$ such that

$$I_m = I_{m,1} \times I_{m,2} \times \cdots \times I_{m,n}. \tag{4.1}$$

Since the n-cells I_m form a descending chain ordered by inclusion, so do the intervals $I_{m,i}$ for each $i \in \{1, 2, \ldots, n\}$, i.e.

$$I_{m+1,i} \subseteq I_{m,i}$$

for every integer $i \in [1, n]$. Hence, Theorem 17 implies that for every $i \in \{1, 2, \ldots, n\}$, there is a point

$$s_i \in \bigcap_{m\in\mathbb{N}} I_{m,i}. \tag{4.2}$$

Let $\widetilde{s} := (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$. Then it follows from (4.2) and from (4.1) that

$$\widetilde{s} \in \bigcap_{m\in\mathbb{N}} I_m.$$

In particular, the above intersection is nonempty. □

THEOREM 19. *Every n-cell is compact in the Euclidean space $\mathbb{R}^n$.*

PROOF. Let I_1 be an n-cell and let $J_1 = [a_1, b_1], J_2 = [a_2, b_2] \dots J_n = [a_n, b_n]$ be closed intervals in $\mathbb{R}$ such that

$$I_1 = J_1 \times J_2 \times \cdots \times J_n$$

Let $a = (a_1, a_2, \dots, a_n)$ and let $b = (b_1, b_2, \dots, b_n)$. Since for all $x_i, y_i \in [a_i, b_i]$ where $i \in \{1, 2, \dots, n\}$, we have $|x_i - y_i| \leq |a_i - b_i|$, it follows that

$$\sup\{\|x - y\| \mid x, y \in I_1\} = \|a - b\|. \tag{4.3}$$

Suppose, there exists an open cover $\{\mathcal{O}_j\}$ of I_1 which does not contain any finite subcover of I_1. Let $c_i = (a_i + b_i)/2$ for all $i \in \{1, 2, \dots, n\}$, let $J_i' = [a_i, c_i]$ and let $J_i'' = [c_i, b_i]$. Then the cartesian products of all the sets $\{J_i', J_i''\}$ for $i \in \{1, 2, \dots, n\}$ produces 2^n sub-n-cells* of I_1, whose union is I_1. Since the open cover $\{\mathcal{O}_j\}$ of I_1 contains no finite subcover of I_1, at least one of these sub-n-cells, say, $I_2 \subset I_1$ can not be covered by any finite subcollection of $\{\mathcal{O}_j\}$. We define a descending chain of n-cells

$$I_1 \supset I_2 \supset I_3 \supset \cdots \tag{4.4}$$

none of which can not be covered by any finite subcollection of $\{\mathcal{O}_j\}$, iteratively as above by bisecting all the closed intervals of which the n-cell under consideration (say, I_k) is the cartesian product. The cartesian product of all these closed subintervals produces 2^n sub-n-cells of I_k whose union is I_k. Given that the open cover $\{\mathcal{O}_j\}$ of I_k contains no finite subcover of I_k, we conclude that at least one of these sub-n-cells, say, $I_{k+1} \subset I_k$ can not be covered by any finite subcollection of $\{\mathcal{O}_j\}$.

Since for all $k \in \mathbb{N}$, the n-cell I_{k+1} is a cartesian product of closed intervals whose lengths are half of the lengths of the corresponding intervals of which I_k is a cartesian product, it follows by induction from (4.3) that

$$\sup\{\|x - y\| \mid x, y \in I_k\} = \frac{\|a - b\|}{2^{k-1}}. \tag{4.5}$$

Theorem 18 implies that there exists a point $\widetilde{s} \in \mathbb{R}^n$ which lies in every I_k. Since $\{\mathcal{O}_j\}$ is an open cover of I_1, we have $\widetilde{s} \in \mathcal{O}_{j_0}$, for some j_0. Since $\mathcal{O}_{j_0}$ is open, there exists an open ball $B_r(\widetilde{s})$ of radius $r > 0$ around $\widetilde{s}$ such that $B_r(\widetilde{s}) \subseteq \mathcal{O}_{j_0}$.

Since the set $\{2^m \mid m \in \mathbb{N}\}$ has no upper bound in $\mathbb{R}$, there exists some $k_0 \in \mathbb{N}$ such that $2^{k_0 - 1} > \|a - b\|/r$. It follows from (4.5) that

$$\sup\{\|x - y\| \mid x, y \in I_{k_0}\} < r.$$

Since $\widetilde{s} \in I_{k_0}$, it follows that

$$I_{k_0} \subseteq B_r(\widetilde{s}) \subseteq \mathcal{O}_{j_0},$$

*i.e. n-cells which are contained in the n-cell I_1

which is absurd, since we constructed the descending chain of n-cells (4.4) so that none of the n-cells in this chain can be covered by any finite subcollection of $\{\mathcal{O}_j\}$. $\square$

THEOREM 20 (Bolzano-Weierstrass, 1817). *Every bounded infinite subset of the Euclidean Space $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.*

PROOF. Let S be a bounded infinite subset of $\mathbb{R}^n$. Since S is bounded, there exists an n-cell in $I \subset \mathbb{R}^n$ such that $S \subseteq I$. Theorem 19 implies that I is compact. Since S is an infinite subset of the compact set I, Theorem 16 implies that S has a limit point. $\square$

THEOREM 21 (Cantor, 1869). *If $\{S_m\}_{m\in\mathbb{N}}$ is a collection of bounded nonempty closed subsets of the Euclidean Space $\mathbb{R}^n$ such that $S_{m+1} \subseteq S_m$ for all $m \in \mathbb{N}$, then*

$$\bigcap_{m\in\mathbb{N}} S_m \neq \emptyset.$$

PROOF. There are two possibilities: Either each S_m contains infinitely many points or there exists an $m_0 \in \mathbb{N}$ such that the set S_{m_0} is finite.

Case 1 (S_{m_0} is finite for some $m_0 \in \mathbb{N}$)
Let $S_{m_0} = \{s_1, s_2, \dots, s_n\}$. Since $\emptyset \neq S_{m+1} \subseteq S_m$ for all $m \in \mathbb{N}$ and since S_{m_0} has only n elements, it follows that there are at most $n-1$ integers $m \geq m_0$ such that $S_{m+1} \subsetneq S_m$. Let

$$m_1 = \max\{m \mid m \in \mathbb{N},\ S_{m+1} \subsetneq S_m\}.$$

Since $\emptyset \neq S_{m_1} \subset S_{m_0}$, there exists some $j \in \{1, 2, \dots, n\}$ such that $s_j \in S_{m_1} = S_m$ for all $m \geq m_1$. Thus, we obtain that

$$s_j \in \bigcap_{m\in\mathbb{N}} S_m.$$

Case 2 (S_m contains infinitely many points for all $m \in \mathbb{N}$) Then we construct an infinite set of points $A := \{a_1, a_2, a_3, \dots\}$ such that $a_i \in S_i$ for all $i \in \mathbb{N}$. Since A is a subset of the bounded set S_1, the set A is bounded. Hence, by Bolzano-Weierstrass theorem, there exists a limit point ℓ of A in $\mathbb{R}$. For each $n \in \mathbb{N}$, since $A \setminus (A \cap S_n)$ is finite, it follows that ℓ is also a limit point of $A \cap S_n$. That implies, ℓ is a limit point of the closed set S_n. Hence, $\ell \in S_n$ for all $n \in \mathbb{N}$. Thus, we conclude that

$$\ell \in \bigcap_{m\in\mathbb{N}} S_m.$$

$\square$

THEOREM 22 (Heine-Borel, 1895). *For every subset S of the Euclidean space $\mathbb{R}^n$, the following are equivalent:*

(*a*) *S is closed and bounded.*

(*b*) *S is compact.*
(*c*) *Every infinite subset of S has a limit point in S.*

PROOF. $(a) \Rightarrow (b)$:
If S is bounded, then $S \subseteq I$ for some n-cell I, which is compact by Theorem 19. Since S is a closed subset of the compact set I, Theorem 14 implies that S is compact.

$(b) \Rightarrow (c)$:
From Theorem 16, it follows that every infinite subset of the compact set S has a limit point.

$(c) \Rightarrow (a)$:
If S is not bounded, then it is not contained in any open ball of finite radius in $\mathbb{R}^n$. In particular, $S \nsubseteq B_n(0, n)$ for all $n \in \mathbb{N}$. In other words, for all $n \in \mathbb{N}$, there exists a point $\widetilde{s}_n \in S$ such that $\widetilde{s}_n \notin B_n(0, n)$, i.e.

$$\|\widetilde{s}_n\| > n$$

for all $n \in \mathbb{N}$. The set $A = \{\widetilde{s}_1, \widetilde{s}_2, \dots\}$ is an infinite subset of S. Hence, (c) implies that A has a limit point ℓ in S. Let $m \in \mathbb{N}$ be such that $m > \|\ell\| + 1$. Then using the triangle inequality, we obtain

$$\|\widetilde{s}_m - \ell\| \geq \|\widetilde{s}_m\| - \|\ell\| > m - \|\ell\| > 1,$$

which leads to a contradiction to the fact that ℓ is a limit point of A. Hence, we conclude that S is bounded.

Suppose, S is not closed. Then there is a limit point $\ell_1 \in \mathbb{R}^n$ of S such that $\ell_1 \notin S$. Since ℓ_1 is a limit point of S, each of its neighbourhoods $B_{1/n}(\ell_1)$ contains at least one point $\widetilde{t}_n \in S$. The set $T = \{\widetilde{t}_1, \widetilde{t}_2, \dots\}$ is an infinite subset of S. Hence, (c) implies that T has a limit point ℓ_2 in S. Since $\ell_1 \notin S$ and $\ell_2 \in S$, we have $\ell_1 \neq \ell_2$. From the triangle inequality, it follows that

$$\|\ell_1 - \ell_2\| \leq \|\ell_1 - \widetilde{t}_n\| + \|\widetilde{t}_n - \ell_2\| < \frac{1}{n} + \|\widetilde{t}_n - \ell_2\|$$

for all $\widetilde{t}_n \in T$. The above inequality implies that for all integers $n > \frac{2}{\|\ell_1 - \ell_2\|}$, we have

$$\frac{\|\ell_1 - \ell_2\|}{2} \leq \|\widetilde{t}_n - \ell_2\|.$$

Since $\|\ell_1 - \ell_2\|$ is a positive constant, the above inequality implies that ℓ_2 is not a limit point of T. Thus, we obtain a contradiction! Hence, we conclude that S is bounded. □

THEOREM 23 (Cantor Intersection Theorem, 1869: The general case). *If $\{S_m\}_{m \in \mathbb{N}}$ is a collection of nonempty closed and compact subsets of* *a metric space* (M, d) *such that* $S_{m+1} \subseteq S_m$ *for all* $m \in \mathbb{N}$, *then*

$$\bigcap_{m \in \mathbb{N}} S_m \neq \emptyset.$$

PROOF. Either each S_m is infinite or some S_{m_0} is finite.

Case 1 (S_{m_0} is finite for some $m_0 \in \mathbb{N}$)
Let $S_{m_0} = \{s_1, s_2, \ldots, s_n\}$. Since $\emptyset \neq S_{m+1} \subseteq S_m$ for all $m \in \mathbb{N}$ and since S_{m_0} has only n elements, it follows that there are at most $n-1$ integers $m \geq m_0$ such that $S_{m+1} \subsetneq S_m$. Let $m_1 = \max\{m \mid m \in \mathbb{N},\ S_{m+1} \subsetneq S_m\}$. Since $\emptyset \neq S_{m_1} \subset S_{m_0}$, there exists some $j \in \{1, 2, \ldots, n\}$ such that $s_j \in S_{m_1} = S_m$ for all $m \geq m_1$. Thus, we obtain that

$$s_j \in \bigcap_{m \in \mathbb{N}} S_m.$$

Case 2 (S_m contains infinitely many points for all $m \in \mathbb{N}$)
Then we construct an infinite set of points $A := \{a_1, a_2, a_3, \ldots\}$ such that $a_i \in S_i$ for all $i \in \mathbb{N}$. Since A is an infinite subset of a compact set S_1, there exists a limit point ℓ of A in M. For each $n \in \mathbb{N}$, since $A \smallsetminus (A \cap S_n)$ is finite, it follows that ℓ is also a limit point of $A \cap S_n$. That implies, ℓ is a limit point of the closed set S_n. Hence, $\ell \in S_n$ for all $n \in \mathbb{N}$. Thus, we conclude that

$$\ell \in \bigcap_{m \in \mathbb{N}} S_m.$$

□

CHAPTER 3

Sequences and series

DEFINITION 34 (SEQUENCE). *A sequence is a function whose domain is the set of the natural numbers. The values of the function are called the terms of the sequence. Instead of writing $\alpha(n)$ for the terms of a sequence*

$$\alpha : \mathbb{N} \to S, \tag{0.1}$$

it is customary to write α_n and denote the whole sequence by $\{\alpha_n\}$ or by the list $\alpha_1, \alpha_2, \alpha_3, \ldots$.

EXAMPLE 3. *In (0.1), if $S = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or finite, we call $\{\alpha_n\}$ a rational, real, complex or finite sequence, respectively. The set S can also be finite. In particular, if S has only one element, then $\{\alpha_n\}$ is a constant sequence. If α is surjective on $S = \{0, 1\}$, then $\{\alpha_n\}$ is a binary sequence.*

DEFINITION 35 (BOUNDED SEQUENCE). *If the image of $\alpha : \mathbb{N} \to \mathbb{R}$ is bounded below (resp. above), we say that the sequence $\{\alpha_n\}$ is bounded below (resp. above). If a sequence is bounded both from above and below, we call it a* bounded sequence.

DEFINITION 36 (INCREASING AND DECREASING SEQUENCES). *If a real sequence $\{\alpha_n\}$ is such that $\alpha_{j+1} > \alpha_j$ (resp. $\alpha_{j+1} \geq \alpha_j$) for all $j \in \mathbb{N}$, we call $\{\alpha_n\}$ an increasing (resp. nondecreasing) sequence. On the other hand, if $\alpha_{j+1} < \alpha_j$ (resp. $\alpha_{j+1} \leq \alpha_j$) for all $j \in \mathbb{N}$, we call $\{\alpha_n\}$ a decreasing (resp. nonincreasing) sequence.*

EXAMPLE 4. *Let's consider the following sequences:*

- (*i*) $\alpha_n = n$. *The image of α is infinite and $\{\alpha_n\}$ is an increasing and unbounded sequence.*
- (*ii*) $\alpha_n = (-1)^n$. *The image of α is finite and $\{\alpha_n\}$ is a bounded sequence.*
- (*iii*) $\alpha_n = 1/n$. *The image of α is infinite and $\{\alpha_n\}$ is a bounded decreasing sequence.*

1. Convergent sequences

DEFINITION 37. *A sequence $\{\alpha_n\}$ in a metric space (M, d) is said to* converge *to a point $p \in M$ (written $\alpha_n \to p$) if for every $r > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\alpha_n \in B_r(p)$.*

So, in particular if the set $S_{\{\alpha_n\}} := \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ has a unique limit point p, then the sequence $\{\alpha_n\}$ converges to the point p. Hence, we also write

$$\lim_{n\to\infty} \alpha_n = p. \tag{1.1}$$

It is customary to use the above notation whenever the sequence $\{\alpha_n\}$ converges to a point p. However, note that p may not always be a limit point the set $S_{\{\alpha_n\}}$. For example, if $\alpha_n = p$ for all $n \in \mathbb{N}$, then $S_{\{\alpha_n\}} = \{p\}$ and of course, the sequence α_n converges to p. Here, p is not a limit point of S (though p is indeed a closure point of S). Nevertheless, also in this case we denote the convergence of the sequence $\{\alpha_n\}$ to p by the notation (1.1).

If a sequence does not converge to any point, it is said to *diverge*. We call a sequence $\{\alpha_n\}$ *bounded* if the image of α is bounded. Note that in a metric space (M, d), a sequence $\{\alpha_n\}$ converges to a point p if and only if the sequence $\{d(\alpha_n, p)\}$ converges to 0 in $\mathbb{R}$.

THEOREM 24. *Let $\{\alpha_n\}$ be a sequence in a metric space (M, d).*
(a) If $\{\alpha_n\}$ converges to a and to b, then $a = b$.
(b) If $\{\alpha_n\}$ converges, then $\{\alpha_n\}$ is bounded.
(c) If ℓ is a limit point of $S \subseteq M$, then there is a sequence $\{s_n\}$ in S such that $\lim_{n\to\infty} s_n = \ell$.

PROOF. (*a*) Suppose, $a \neq b$. Then we have $r_0 := d(a, b) > 0$.

Note that for every $r > 0$, there exists N and $N' \in \mathbb{N}$ such that $n \geq \max\{N, N'\}$ implies that $\alpha_n \in B_r(a)$ and $\alpha_n \in B_r(b)$. Now, from the triangle inequality, we have

$$r_0 = d(a, b) \leq d(a, \alpha_n) + d(\alpha_n, b) \leq 2r.$$

Since r_0 is a positive constant, whereas r could be chosen to be as small as we wish, the above inequality leads to a contradiction. Hence, we conclude that $a = b$.

(*b*) Let the sequence $\{\alpha_n\}$ converge to p in the metric space (M, d). Then given $r_0 > 0$, there exists $N \in \mathbb{N}$ such that $\alpha_n \in B_{r_0}(p)$ for all $n > N$. Let $r_1 := \max\{r_0, d(\alpha_1, p), \ldots d(\alpha_N, p)\}$ Then we have $\alpha_n \in B_{r_1}(p)$ for all n. In other words, $\{\alpha_n\}$ is indeed bounded.

(*c*) If ℓ is a limit point of S, then for $n \in \mathbb{N}$, there exists a point $s_n \in S \smallsetminus \{\ell\}$ in the the open ball $B_{1/n}(\ell)$. From the Archimedean property of $\mathbb{R}$, it follows that given $r > 0$, there exists an $n \in \mathbb{N}$ with $r > 1/n$, which implies $s_n \in B_{1/n}(\ell) \subset B_r(\ell)$. Hence, $s_n \to \ell$. □

EXAMPLE 5. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in the Euclidean space $\mathbb{R}$ such that $\alpha_n \to a$ and $\beta_n \to b$. Let $c \in \mathbb{R}$.*
(a) Then $\alpha_n + \beta_n \to a + b$
(b) For $c \in \mathbb{R}$, we have $c\alpha_n \to ca$ and $c + \alpha_n \to c + a$.

(c) $\alpha_n\beta_n \to ab$.
(d) $\dfrac{1}{\alpha_n} \to \dfrac{1}{a}$ *if* $\alpha_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $a \neq 0$.

PROOF. (a) Since $\alpha_n \to a$ and $\beta_n \to b$, for every $r > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\alpha_n \in B_{r/2}(a)$ and for all $n \geq N_2$, we have $\beta_n \in B_{r/2}(b)$. Let $N = \max\{N_1, N_2\}$. Then the triangle inequality implies that for all $n \geq N$, we have

$$\begin{aligned} d(\alpha_n + \beta_n, a+b) &\leq d(\alpha_n + \beta_n, a + \beta_n) + d(a + \beta_n, a + b) \\ &= |\alpha_n - a| + |\beta_n - b| < \frac{r}{2} + \frac{r}{2} = r, \end{aligned}$$

where the second equality follows from the definition of the Euclidean metric on $\mathbb{R}$. Thus, we obtain that $\alpha_n + \beta_n \to a + b$.

(b) The claim is trivial for $c = 0$. So, let us assume that $c \neq 0$. Since $\alpha_n \to a$, for every $r > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\alpha_n \in B_{\max\{r, r/|c|\}}(a)$. Hence, for all $n \geq N$,

$$d(c\alpha_n, ca) = |c\alpha_n - ca| = |c| \cdot |\alpha_n - a| < |c| \cdot \frac{r}{|c|} = r.$$

and

$$d(c + \alpha_n, c + a) = |(c + \alpha_n) - (c + a)| = |\alpha_n - a| < r.$$

Thus, we obtain that $c\alpha_n \to ca$ and $c + \alpha_n \to c + a$.

(c) Since $\{\alpha_n\}$ is a convergent sequence, Theorem 24.(b) implies that $\{\alpha_n\}$ is bounded, i.e. there exists a point $p \in \mathbb{R}$ and some $s > 0$ such that $\alpha_n \in B_s(p)$ for all n. Let $M = |p| + s$. Since $\alpha_n \in B_s(p)$ for all n, it follows that $|\alpha_n| < M$ for all n.

Since $\alpha_n \to a$ and $\beta_n \to b$, for every $r > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\alpha_n \in B_{r/(M+|b|)}(a)$ and for all $n \geq N_2$, we have $\beta_n \in B_{r/(M+|b|)}(b)$. Let $N = \max\{N_1, N_2\}$. Then the triangle inequality implies that for all $n \geq N$, we have

$$\begin{aligned} d(\alpha_n\beta_n, ab) &\leq d(\alpha_n\beta_n, b\alpha_n) + d(b\alpha_n, ab) \\ &= |\alpha_n\beta_n - b\alpha_n| + |b\alpha_n - ab| \\ &= |\alpha_n| \cdot |\beta_n - b| + |b| \cdot |\alpha_n - a| \\ &< M \cdot \frac{r}{M + |b|} + |b| \cdot \frac{r}{M + |b|} \\ &= r. \end{aligned}$$

Hence, we conclude that $\alpha_n\beta_n \to ab$.

(d) Since $\alpha_n \to a \neq 0$, for every $r > 0$, there exists an $N \in \mathbb{N}$ such that $\alpha_n \in B_{\max\{|a|,\, r|a|^2\}/2}(a)$ for all $n \geq N$. That implies, for all $n \geq N$, we have

$$\begin{aligned} d(1/\alpha_n, 1/a) &= \left|\frac{1}{\alpha_n} - \frac{1}{a}\right| = \left|\frac{\alpha_n - a}{a\alpha_n}\right| \\ &< \frac{r|a|^2}{2|a\alpha_n|} = \frac{r|a|}{2|\alpha_n|} < r, \end{aligned}$$

where the last inequality follows from the fact $|\alpha_n| > |a|/2$, which holds because $\alpha_n \in B_{|a|/2}(a)$ for all $n \geq N$. Hence, we conclude that

$$\frac{1}{\alpha_n} \to \frac{1}{a}.$$

□

2. Subsequences

Definition 38 (Subsequence). *Given a sequence $\{\alpha_n\}$, consider an increasing sequence n_k of positive integers. Then the sequence $\{\alpha_{n_k}\}$ is called a subsequence of $\{\alpha_n\}$. If $\{\alpha_{n_k}\}$ converges, its limit is called a subsequential limit of $\{\alpha_n\}$.*

Theorem 25. (*a*) *Let (M, d) be a compact metric space and let $\{\alpha_n\}$ be a sequence in M. Then some subsequence of $\{\alpha_n\}$ converges to a point of M.*
(*b*) *Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.*

Proof. (*a*) The claim is trivial if $\{\alpha_n\}$ is a finite sequence. So, we may assume that the sequence $\{\alpha_n\}$ is infinite. Since every infinite subset of a compact set has a limit point, the claim follows from Theorem 24.(c).

(*b*) Again, the claim is trivial if we have a finite sequence in $\mathbb{R}^n$. So, we may assume that the sequence is infinite. Then the claim follows from the Bolzano-Weierstrass theorem and Theorem 24.(c). □

3. Sequentially compact spaces

Definition 39. *A subset S of a metric space (M, d) is called sequentially compact if every sequence of points in S has a convergent subsequence which converges to a point in S.*

From the Heine-Borel theorem, it follows that

Corollary 9. *A subset S of an Euclidean space is compact if and only if S is sequentially compact.*

The above corollary is in fact, true for arbitrary metric spaces, the proof of which you will see in a later course.

4. Cauchy sequences

THEOREM 26. *Let* (M,d) *be a metric space and let* $\{\alpha_n\}$ *be a convergent sequence in* M. *Then for every* $\varepsilon > 0$, *there exists an* $N \in \mathbb{N}$ *such that*

$$d(\alpha_m, \alpha_n) < \varepsilon$$

if $\min\{m,n\} \geq N$.

PROOF. Let $\alpha_n \to a$ in M. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $\alpha_m, \alpha_n \in B_{\varepsilon/2}(a)$. The triangle inequality implies that

$$d(\alpha_m, \alpha_n) \leq d(\alpha_m, a) + d(a, \alpha_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

DEFINITION 40 (CAUCHY SEQUENCE). *A sequence* $\{\alpha_n\}$ *in a metric space* (M,d) *is called a Cauchy sequence if for every* $\varepsilon > 0$, *there is an integer* N *such that*

$$d(\alpha_m, \alpha_n) < \varepsilon \tag{4.1}$$

if $\min\{m,n\} \geq N$.

We call Condition (4.1) the *Cauchy condition.*

THEOREM 27. *If a subsequence* $\{\alpha_{n_k}\}$ *of a Cauchy sequence* $\{\alpha_n\}$ *converges to a limit point* ℓ, *then the entire sequence* $\{\alpha_n\}$ *converges to* ℓ.

PROOF. Since $\alpha_{n_k} \to \ell$, for every $r > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n_k \geq N$, we have $\alpha_{n_k} \in B_{r/2}(\ell)$. Again, the Cauchy condition implies that there exists $N_2 \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $\min\{m,n\} \geq N$, we have $d(\alpha_m, \alpha_n) < r/2$. Let $N = \max\{N_1, N_2\}$ and let α_{n_r} be an element in the subsequence with $n_r \geq N$. Then the triangle inequality implies that for all $n \geq N$,

$$d(\alpha_n, \ell) \leq d(\alpha_n, \alpha_{n_r}) + d(\alpha_{n_r}, \ell) < \frac{r}{2} + \frac{r}{2} = r.$$

In other words, $\alpha_n \in B_r(\ell)$. Hence, we conclude that $\alpha_n \to \ell$ □

5. Cauchy completeness

DEFINITION 41 (CAUCHY COMPLETE METRIC SPACES). *A metric space* (M,d) *is called Cauchy complete if every Cauchy sequence in* M *converges in* M.

THEOREM 28. *Every compact subset of a metric space is Cauchy complete.*

PROOF. Let $\{\alpha_n\}$ be a Cauchy sequence in a compact subset K of a metric space (M,d). If the sequence $\{\alpha_n\}$ is finite, then the Cauchy condition (4.1) implies that the sequence $\{\alpha_n\}$ eventually stabilizes. In other words, $\{\alpha_n\}$ converges to a unique point in the image of α.

Next, we may assume that the image of α is infinite. Since every infinite subset of a compact space has a limit point, Theorem 24.(c) implies that there exists a subsequence of $\{\alpha_n\}$ which converges to a point $p \in K$. Now, from the Cauchy condition (4.1), it follows that the entire sequence $\{\alpha_n\}$ converges to p. $\square$

Theorem 29. *All Euclidean spaces are Cauchy complete.*

Proof. Let $\{\alpha_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Then for $\varepsilon_0 > 0$, there is an integer N such that $d(\alpha_n, \alpha_N) < \varepsilon_0$ if $n \geq N$. Let

$$r_0 := \max\{\varepsilon_0, d(\alpha_1, \alpha_N), \ldots d(\alpha_{N-1}, \alpha_N)\}.$$

Then we have $\alpha_n \in B_{r_0}(\alpha_N)$ for all n. In other words, $\{\alpha_n\}$ is a bounded sequence in $\mathbb{R}^k$. Hence Theorem 25.(b) implies that there exists a subsequence of $\{\alpha_n\}$ which converges to a point $p \in \mathbb{R}^k$. Now, from the Cauchy condition (4.1), it follows that the entire sequence $\{\alpha_n\}$ converges to p. $\square$

Corollary 10 (Cauchy criterion). *A sequence in an Euclidean space converges if and only if it is a Cauchy sequence.* $\square$

6. Construction of the real numbers

Let $\mathcal{C}$ denote the set of rational Cauchy sequences. Let $\{\alpha_n\}, \{\beta_n\} \in \mathbb{S}$. Then we say that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equivalent (written $\{\alpha_n\} \sim \{\beta_n\}$) if $|\alpha_n - \beta_n| \to 0$.

We define $\mathbb{R}$ as the set of the rational Cauchy sequences modulo the above equivalence relation, i.e.

$$\mathbb{R} := \mathcal{C}/\sim .$$

We embed $\mathbb{Q}$ in $\mathbb{R}$ as constant sequences and we define the addition and multiplication on $\mathbb{R}$ as termwise addition and multiplication of Cauchy sequences. For $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{R}$, the equality $\{\alpha_n\} = \{\beta_n\}$ holds if and only if the Cauchy sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are equivalent, i.e. if $|\alpha_n - \beta_n| \to 0$.

The decimal notation can be interpreted with Cauchy sequences in a natural way. For example, the notation $\pi = 3.1415...$ means that π is the equivalence class of the Cauchy sequence $\{3, 3.1, 3.14, 3.141, 3.1415, ...\}$. The equality $0.999... = 1$ means that the Cauchy sequences $\{0, 0.9, 0.99, 0.999, ...\}$ and $\{1, 1, 1, 1, ...\}$ are equivalent, i.e. their termwise difference converges to 0.

We define the order on $\mathbb{R}$ as follows: Given two Cauchy sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we write $\{\alpha_n\} > \{\beta_n\}$ if there exists and $N_1 \in \mathbb{N}$ such that $\alpha_n > \beta_n$ for all $n \geq N_1$ and if $|\alpha_n - \beta_n|$ does not converge to 0, i.e. if there exists some $\varepsilon > 0$ such that

$$|\alpha_{n_k} - \beta_{n_k}| > 3\varepsilon$$

for all $n_k \in \mathbb{N}$, where $\{\alpha_{n_k}\}$ and $\{\beta_{n_k}\}$ are some subsequences of $\{\alpha_n\}$ and $\{\beta_n\}$, respectively. Since both $\{\alpha_n\}$ and $\{\beta_n\}$ are Cauchy sequences, if follows that there exists an $N_2 \in \mathbb{N}$ such that

$$|\alpha_n - \alpha_{n_k}| < \varepsilon \text{ and } |\beta_n - \beta_{n_k}| < \varepsilon$$

for all $n, n_k \geq N_2$. Hence,

$$\begin{aligned} 3\varepsilon < |\alpha_{n_k} - \beta_{n_k}| &\leq |\alpha_{n_k} - \alpha_n| + |\alpha_n - \beta_{n_k}| \\ &\leq |\alpha_{n_k} - \alpha_n| + |\alpha_n - \beta_n| + |\beta_n - \beta_{n_k}| \end{aligned}$$

which implies that

$$\alpha_n - \beta_n = |\alpha_n - \beta_n| > \varepsilon$$

for all $n \geq N$, where $N := \max\{N_1, N_2\}$.

Thus, we conclude that for two rational Cauchy sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we have $\{\alpha_n\} > \{\beta_n\}$ if there exists some $\varepsilon > 0$ and some natural number N such that

$$\alpha_n - \beta_n > \varepsilon$$

for all $n \geq N$.

Note that, by the above From the above construction of $\mathbb{R}$, it follows easily that $\mathbb{R}$ is an ordered field. Hence, to prove Theorem 2, it suffices to show that $\mathbb{R}$ is *complete*:

Proof. Let S be a nonempty subset of $\mathbb{R}$ and let $u_1 \in \mathbb{Q}$ be an upper bound of S. Let $s \in S$ and let $\ell_1 \in \mathbb{Q}$ be such that $\ell_1 < s$. We define two rational Cauchy sequences $\{u_n\}$ and $\{\ell_n\}$ as follows:

$$\begin{aligned} u_{n+1} &= \begin{cases} \frac{u_n+\ell_n}{2} & \text{if } (u_n+\ell_n)/2 \text{ is an upper bound of } S \\ u_n & \text{otherwise.} \end{cases} \\ \ell_{n+1} &= \begin{cases} \frac{u_n+\ell_n}{2} & \text{if } (u_n+\ell_n)/2 \text{ is not an upper bound of } S \\ \ell_n & \text{otherwise.} \end{cases} \end{aligned}$$

By induction, it follows that u_n is an upper bound of S and ℓ_n is never an upper bound of S for all $n \in \mathbb{N}$.

Suppose, $\{u_n\} < \{s_n\}$ for some $\{s_n\} \in S$. Then for any sufficiently large n_0, we have $u_{n_0} < s_{n_0}$. In particular, u_{n_0}, i.e. the constant sequence $\{u_{n_0}\}$, is not an upper bound of S. Thus, we get a contradition! Hence, $\{u_n\}$ is an upper bound of S.

Since

$$u_n - \ell_n = \frac{u_1 - \ell_1}{2^{n-1}},$$

if follows that $\{u_n\} \sim \{\ell_n\}$. Let $\{a_n\} \in \mathbb{R}$ be an upper bound of S such that $\{a_n\} < \{u_n\}$. Since $\mathbb{R}$ was defined as the set of the equivalence classes of rational Cauchy sequences, we have $\{u_n\} = \{\ell_n\}$ in $\mathbb{R}$. It follows that $\{a_n\} < \{\ell_n\}$ in $\mathbb{R}$. That implies, there exists an $\varepsilon > 0$ and an $N_1 \in \mathbb{N}$ such that $\ell_n - a_n > \varepsilon$ for all $n \geq N_1$. However, since $\{\ell_n\}$ is a Cauchy sequence, there exists N_2 such that for all $m, n \geq N_2$, we have $|\ell_m - \ell_n| < \varepsilon/2$. Let $N := \max\{N_1, N_2\}$. Since $\{\ell_n\}$ is a nondecreasing sequence, it follows that $\ell_n - \ell_N = |\ell_n - \ell_N| < \varepsilon/2$ for all $n \geq N$. Therefore, we have

$$\ell_N - a_n \geq (\ell_N - \ell_n) + (\ell_n - a_n) > -\frac{\varepsilon}{2} + \varepsilon = \varepsilon/2$$

for all $n \geq N$. So, $\{a_n\}$ is less than the constant sequence $\{\ell_N\}$. But ℓ_N was not an upper bound of S. Hence, $\{a_n\}$ is also not an upper bound of S. Thus we get a contradiction! Hence, $\{u_n\}$ is the least upper bound of the set S. Therefore, we conclude that $\mathbb{R}$ is complete. $\square$

7. Diameter of a set

DEFINITION 42. *Diameter Let (M, d) be a metric space and let $S \subseteq M$ be nonempty and let*

$$D := \{d(s_1, s_2) \mid \text{ for all } s_1, s_2 \in S\}.$$

Then $\sup D$ *is called the diameter of S and denoted by* $\operatorname{diam} S$.

Let $\{\alpha_n\}$ be a sequence in a metric space (M, d) and for $N \in \mathbb{N}$, let

$$S_N := \{\alpha_n \mid n \geq N\}.$$

Then $\{\alpha_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \operatorname{diam} S_N = 0.$$

THEOREM 30. *Let (M, d) be a metric space.*
(a) Let $S \subseteq M$. Then

$$\operatorname{diam} \overline{S} = \operatorname{diam} S.$$

(b) If K_n is a sequence of nonempty compact sets in M such that $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{N\to\infty} \operatorname{diam} K_N = 0$, then there exists a point $p \in M$ such that

$$\bigcap_{n\in\mathbb{N}} K_n = \{p\}.$$

PROOF. (a) Since $S \subseteq \overline{S}$, it follows that $\operatorname{diam} S \leq \operatorname{diam} \overline{S}$. Suppose, $\operatorname{diam} S < \operatorname{diam} \overline{S}$ and let $r_0 := \operatorname{diam} \overline{S} - \operatorname{diam} S$. Let $c_1, c_2 \in \overline{S}$. Then given $r > 0$, there exists $s_1 s_2 \in S$ such that $s_1 \in B_r(c_1)$ and $s_2 \in B_r(c_2)$. Now, the triangle inequality implies that

$$\begin{aligned} d(c_1, c_2) &\leq d(c_1, s_2) + d(s_2, c_2) \\ &\leq d(c_1, s_1) + d(s_1, s_2) + d(s_2, c_2) \\ &\leq r + d(s_1, s_2) + r \end{aligned}$$

Hence, we have

$$\sup\{d(c_1, c_2) \mid c_1, c_2 \in \overline{S}\} \leq \sup\{d(s_1, s_2) \mid c_1, c_2 \in S\} + 2r.$$

That implies

$$r_0 := \operatorname{diam} \overline{S} - \operatorname{diam} S \leq 2r.$$

Since the above equality holds for all positive values of r, in particular, choosing $r \in (0, r_0/2)$, we obtain a contradiction! Hence, we conclude that $\operatorname{diam} S = \operatorname{diam} \overline{S}$.

(b) By Corollary 8, $K := \bigcap_{n \in \mathbb{N}} K_n$ is nonempty. Suppose, there are two distinct points $p_1, p_2 \in K$, Let $r_0 := d(p_1, p_2) > 0$. Since $\lim_{N \to \infty} \operatorname{diam} K_N = 0$, There exists an $m \in \mathbb{N}$ such that $\operatorname{diam} K_m < r_0$. However, since $K \subset K_m$, we have $\operatorname{diam} K \leq \operatorname{diam} K_m$. It follows that

$$r_0 = d(p_1, p_2) \leq \operatorname{diam} K \leq \operatorname{diam} K_m < r_0.$$

Thus, we obtain a contradiction. So, the nonempty set K does not contain any two distinct points. Therefore, there is a unique $p \in M$ such that $K = \{p\}$. $\square$

8. Monotone convergence theorem

DEFINITION 43. *Every nondecreasing sequence as well as every nonincreasing sequence* is called a monotone / monotonic sequence.*

THEOREM 31. *Every bounded monotonic sequence converges in* $\mathbb{R}$.

PROOF. Let $\{\alpha_n\}$ be a bounded monotonic sequence. If necessary, replacing α_n with $-\alpha_n$ for all $n \in \mathbb{N}$, we may assume that $\{\alpha_n\}$ is nondecreasing. Let $S = \{\alpha_n \mid n \in \mathbb{N}\}$. Since $\{\alpha_n\}$ is bounded, so is $S \subseteq \mathbb{R}$. Let $u = \sup S \in \mathbb{R}$. Then $\alpha_n \leq u$ for all $n \in \mathbb{N}$. Since u is the least upper bound of S, for every $r > 0$, there exists $N \in \mathbb{N}$ such that $u - r < \alpha_N \leq u$. Since $\{\alpha_n\}$ is nondecreasing, it follows that for all $n \geq N$, we have $u - r < \alpha_n \leq u$, i.e. $\alpha_n \in B_r(u)$ for all $n \geq N$. In other words, $\alpha_n \to u$. $\square$

9. Upper and lower limits

THEOREM 32. *Let (M, d) be a metric space and let $\{a_n\}$ be a sequence in M and let S be the set of all the subsequential limits† of $\{a_n\}$ in M. Then S is closed.*

PROOF. Let ℓ be a limit point of S. Then Theorem 24.(c) implies that there is a sequence $\{s_n\}$ in S which converges to ℓ, i.e. for every $r > 0$, there is an $N \in \mathbb{N}$ such that for all $m \geq N$, we have $s_m \in B_{r/2}(\ell)$. Let $m_0 \geq N$ be an integer. Since s_{m_0} is a subsequential limit of the sequence $\{a_n\}$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ and an $N' \in \mathbb{N}$

*See Definition 36.

†See Definition 38.

such that for all $n_k \geq N'$, we have $a_{n_k} \in B_{r/2}(s_{m_0})$. From the triangle inequality, it follows that for all $n_k \geq N'$,

$$d(\ell, a_{n_k}) \leq d(\ell, s_{m_0}) + d(s_{m_0}, a_{n_k}) < r/2 + r/2 = r.$$

In other words, for very $r > 0$, there is an element in the set $A := \{a_1, a_2, a_3, \ldots\}$ which lies in $B_r(\ell)$. Hence, ℓ is a limit point of the set A. So, theorem 24.(c) implies that there is sequence in A which converges to ℓ. Since every sequence in A is a subsequence of $\{a_n\}$ and since S is the set of subsequential limits of $\{a_n\}$, it follows that $\ell \in S$. Therefore, S is closed. □

Definition 44 (Divergence to $\pm\infty$). *Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences such that for every $M \in \mathbb{R}$, there exists and $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\alpha_n \geq M$ and $\beta_n \leq M$. Then we say that α_n (resp. β_n) diverges to ∞ (resp. to $-\infty$) and write*

$$\alpha_n \to \infty \quad \textit{and} \quad \beta_n \to -\infty.$$

Definition 45 (Upper and lower limits). *Let $\{\alpha_n\}$ be a real sequence and let S be the set of all the subsequential limits of $\{\alpha_n\}$. Also, if any subsequence of $\{\alpha_n\}$ diverges to ∞ or $-\infty$, we let ∞ or $-\infty \in S$, respectively. Then $s^* := \sup S$ and $s_* := \inf S$ exist in the extended real number system, which we call the upper and lower limits, respectively and denote them as follows*

$$\limsup_{n\to\infty} \alpha_n := s^* \quad \textit{and} \quad \liminf_{n\to\infty} \alpha_n := s_*.$$

Example 6. *Let's consider the upper and lower limits of the following sequences:*

(*i*) $\alpha_n = n$. *Then* $\limsup\limits_{n\to\infty} \alpha_n = \infty$ *and* $\liminf\limits_{n\to\infty} \alpha_n = \infty$.
(*ii*) $\alpha_n = (-1)^n$. *Then* $\limsup\limits_{n\to\infty} \alpha_n = 1$ *and* $\liminf\limits_{n\to\infty} \alpha_n = -1$.
(*iii*) $\alpha_n = 1/n$. *Then* $\limsup\limits_{n\to\infty} \alpha_n = 0$ *and* $\liminf\limits_{n\to\infty} \alpha_n = 0$.

In general, for a real sequence $\{\alpha_n\}$, we have $\lim\limits_{n\to\infty} \alpha_n = \ell$ if and only if

$$\limsup_{n\to\infty} \alpha_n = \liminf_{n\to\infty} \alpha_n = \ell.$$

Definition 46 (Oscillation of a sequence). *Let $\{\alpha_n\}$ be a sequence of real numbers. If $\limsup\limits_{n\to\infty} \alpha_n \neq \liminf\limits_{n\to\infty} \alpha_n$, we say that the series $\{\alpha_n\}$* oscillates. *The* oscillation $\omega(\{\alpha_n\})$ *of the sequence $\{\alpha_n\}$ is defined by*

$$\omega(\{\alpha_n\}) = \begin{cases} \limsup\limits_{n\to\infty} \alpha_n - \liminf\limits_{n\to\infty} \alpha_n & \textit{if both of the limits are finite} \\ \infty & \textit{otherwise.} \end{cases}$$

Theorem 33. *Let $\{\alpha_n\}$ be a sequence of real numbers. Let S be the set of all the subsequential limits of $\{\alpha_n\}$. Also, if any subsequence of*

$\{\alpha_n\}$ diverges to ∞ or $-\infty$, we include ∞ or $-\infty$ in S, respectively. Let s^* denote the least upper bound of S. Then
(a) $s^* \in S$.
(b) If $x > s^*$, then there exists and integer N such that for all $n \geq N$, we have $\alpha_n < x$.
(c) There is no other element in $\mathbb{R} \cup \{-\infty, \infty\}$ except s^* which satisfies both of the properties (a) and (b).

Proof. (*a*) There are only the following three possible cases:

Case 1 ($s^* \in \mathbb{R}$)
Then S is bounded above. So, we may choose an increasing subsequence of $\{\alpha_n\}$ which by the Monotone convergence theorem, converges to some real number $s \in S$. That implies, $S \cap \mathbb{R}$ is nonempty. Theorem 32 implies that $S \cap \mathbb{R}$ is closed. Since every nonempty closed subset of $\mathbb{R}$ which is bounded above, has a maximum (see Theorem 10), it follows that

$$s^* = \max S \cap \mathbb{R} = \max S \in S,$$

where the second inequality holds since $\infty \notin S \subseteq \mathbb{R} \cup \{-\infty, \infty\}$.

Case 2 ($s^* = \infty$)
Then S is not bounded above. That implies, the sequence $\{\alpha_n\}$ is not bounded above and hence, there is a subsequence $\{\alpha_{n_k}\}$ such that $\alpha_{n_k} \to \infty$. It follows that $\infty \in S$.

Case 3 ($s^* = -\infty$)
Since

$$\sup S = -\infty = \inf(\mathbb{R} \cup \{\infty\})$$

and since $-\infty$ does not belong to the later set, it follows that

$$S \cap (\mathbb{R} \cup \{\infty\}) = \emptyset.$$

Since there are no subsequential limits of $\{\alpha_n\}$ in $\mathbb{R} \cup \{\infty\}$, it follows that $\{\alpha_n\}$ diverges to $-\infty$ (otherwise, either the Monotone convergence theorem implies that $\{\alpha_n\}$ has a limit point in $\mathbb{R}$ which in turn, implies that $\sup S > -\infty$ or we have, $\{\alpha_n\}$ is unbounded above which implies $s^* = \sup S = \infty$). Hence, $-\infty \in S$.

(*b*) If the claim does not hold, then there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_k} \geq x$ for all $n_k \in \mathbb{N}$.

Since $\alpha_{n_k} \in \mathbb{R}$ for all $n_k \in \mathbb{N}$, it follows that $x \neq \infty$. If there is an unbounded subsequence of $\{\alpha_{n_k}\}$, then we have $\infty \in S$ and $\infty > x > s^* = \sup S$. Thus, we get a contradiction! So, $\{\alpha_{n_k}\}$ is a bounded sequence. Since every bounded sequence has a convergent subsequence and since $\alpha_{n_k} \geq x$ for all $n_k \in \mathbb{N}$, it follows that there exists a subsequential limit $\ell \in S$ of $\{\alpha_{n_k}\}$ such that $\ell \geq x > s^* = \sup S$. Thus, we obtain a contradiction!

(c) Suppose, there exists an $s' \in \mathbb{R} \cup \{-\infty, \infty\}$ different from s^* such that s' also satisfies both (a) and (b). Let $x \in \mathbb{R}$ be such that

$$\min\{s', s^*\} < x < \max\{s', s^*\}.$$

Since $\min\{s', s^*\}$ satisfies (b), there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\alpha_n < x$. It follows that $\max\{s', s^*\}$ is not a subsequential limit of the sequence $\{\alpha_n\}$, i.e. $\max\{s', s^*\} \notin S$. Thus, we get a contradiction! $\square$

Similarly as above, we can also prove the following theorem:

THEOREM 34. *Let $\{\alpha_n\}$ be a sequence of real numbers. Let S be the set of all the subsequential limits of $\{\alpha_n\}$. Also, if any subsequence of $\{\alpha_n\}$ diverges to ∞ or $-\infty$, we include ∞ or $-\infty$ in S, respectively. Let s_* denote the greatest lower bound of S. Then*
(a) $s_* \in S$.
(b) If $x < s_*$, then there exists and integer N such that for all $n \geq N$, we have $\alpha_n > x$.
(c) There is no other element in $\mathbb{R} \cup \{-\infty, \infty\}$ except s_* which satisfies both of the properties (a) and (b). $\square$

10. Series

DEFINITION 47 (PARTIAL SUMS). *Given a sequence $\{\alpha_n\}$ in $\mathbb{R}$, we define its k-th* partial sum *as*

$$\sum_{n=1}^{k} \alpha_n = \alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

DEFINITION 48 (CONVERGENT AND DIVERGENT SERIES). *Given a sequence $\{\alpha_n\}$, let $\{s_k\}$ denote the sequence of its partial sums. If $\{s_k\} \to s \in \mathbb{R}$, we say that the* series $\alpha_1 + \alpha_2 + \alpha_3 + \ldots$ *converges to s and denote it by*

$$\sum_{n=1}^{\infty} \alpha_n = s.$$

Otherwise, we say that the above series diverges. In particular, if the sequence $\{s_k\}$ diverges to ∞ or $-\infty$, we also write

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad or \quad \sum_{n=1}^{\infty} \alpha_n = -\infty$$

accordingly. We call α_m the m-th term of the series $\displaystyle\sum_{n=1}^{\infty} \alpha_n$.

The Cauchy criterion for the sequence $\{s_k\}$ of the partial sums of the sequence $\{\alpha_j\}$ implies that

THEOREM 35. *The series* $\sum_{j=1}^{\infty} \alpha_j$ *converges if and only if for every* $\varepsilon > 0$, *there exists an integer* N *such that for all integers* $n > m \geq N$, *we have*

$$\Big|\sum_{j=m+1}^{n} \alpha_j\Big| < \varepsilon.$$

□

In particular, taking $n = m + 1$ in the above theorem, we obtain the following corollary:

COROLLARY 11. *If the series* $\sum_{n=1}^{\infty} \alpha_n$ *converges, then* $\alpha_n \to 0$.

Note that, the condition $a_n \to 0$ is necessary but not sufficient in ensuring the convergence of the series $\sum_{n=1}^{\infty} \alpha_n$. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to infinity. In fact, from the Monotone convergence theorem, it follows that

THEOREM 36. *A series of nonnegative real numbers converges if and only if its partial sums form a bounded sequence.* □

THEOREM 37 (CONVERGENCE OF GEOMETRIC SERIES).
(*a*) *For* $a \in (-1, 1) \setminus \{0\}$, *we have*

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

(*b*) *If* $|a| \geq 1$, *then the series* $\sum_{n=0}^{\infty} a^n$ *diverges.*

PROOF. (*a*) It follows by induction that for all $a \notin \{0, 1\}$, the k-th partial sum of the sequence $\{a^n\}_{n=0}^{\infty}$ is given by

$$s_k = \sum_{n=0}^{k} a^n = \frac{1 - a^{k+1}}{1 - a}.$$

Let $\ell = 1/(1 - a)$. Then we have

$$|s_k - \ell| = |\ell| \cdot |a|^{k+1}.$$

Since $\lim_{n\to\infty} |a|^n = 0$ if and only if $|a| < 1$, it follows from the above equation that $\lim_{k\to\infty} s_k = \ell$ if and only if $|a| < 1$.

(b) Since $\lim_{n\to\infty} |a|^n = 0$ if and only if $|a| < 1$, from Corollary 11 it follows that if $|a| \geq 1$, then the series $\sum_{n=0}^{\infty} a^n$ diverges. $\square$

11. Tests for convergence

THEOREM 38 (COMPARISON TEST). *Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences.*
(a) *Let there be an $N \in \mathbb{N}$ such that $|\alpha_n| \leq \beta_n$ for all $n \geq N$. Then*

$$\textit{if } \sum_{n=1}^{\infty} \beta_n \textit{ converges, so does } \sum_{n=1}^{\infty} \alpha_n.$$

(b) *Let there be an $N \in \mathbb{N}$ such that $0 \leq \alpha_n \leq \beta_n$ for all $n \geq N$. Then*

$$\textit{if } \sum_{n=1}^{\infty} \alpha_n \textit{ diverges, so does } \sum_{n=1}^{\infty} \beta_n.$$

PROOF. (a) Since $\sum_{n=1}^{\infty} \beta_n$ converges, the Cauchy criterion implies that for every $\varepsilon > 0$, there exists an integer $N' \geq N$ such that for all integers $n > m \geq N'$, we have $\sum_{j=m+1}^{n} \beta_j < \varepsilon$. It follows that

$$\Big| \sum_{j=m+1}^{n} \alpha_j \Big| \leq \sum_{j=m+1}^{n} |\alpha_j| \leq \sum_{j=m+1}^{n} \beta_j < \varepsilon.$$

Hence, the claim follows from the Cauchy criterion.

(b) Suppose, $\sum_{n=1}^{\infty} \beta_n$ converges. Then from (a), it follows that $\sum_{n=1}^{\infty} \alpha_n$ converges, which leads to a contradiction. $\square$

THEOREM 39 (CAUCHY CONDENSATION TEST, 1821). *Let $\{\alpha_n\}$ be a nonincreasing sequence of nonnegative real numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n$ converges if and only if the series*

$$\sum_{m=0}^{\infty} 2^m \alpha_{2^m} = \alpha_1 + 2\alpha_2 + 4\alpha_4 + 8\alpha_8 + \cdots$$

converges.

PROOF. Since the terms of both the sequences $\{\alpha_n\}$ and $\{2^n \alpha_{2^n}\}$ are nonnegative real numbers, Theorem 36 implies that both of these two sequences converge if and only if both of their partial sums form bounded sequences.

Hence, it suffices to show that the partial sums of the sequence $\{\alpha_n\}$ remains bounded if and only if the partial sums of $\{2^n\alpha_{2^n}\}$ remains bounded. Let s_k and s'_k denote the k-th partial sums of the sequences $\{\alpha_n\}$ and $\{2^n\alpha_{2^n}\}$, respectively. Then for $k_1 \leq 2^m$, we have

$$\begin{aligned} s_{k_1} &\leq \alpha_1 + (\alpha_2 + \alpha_3) + \cdots + (\alpha_{2^m} + \cdots + \alpha_{2^{m+1}-1}) \\ &\leq \alpha_1 + 2\alpha_2 + \cdots + 2^m\alpha_{2^m} \\ &= s'_m, \end{aligned}$$

Again, for $k_2 \geq 2^m$,

$$\begin{aligned} s_{k_2} &\geq \alpha_1 + \alpha_2 + (\alpha_3 + \alpha_4) + \cdots + (\alpha_{2^{m-1}+1} + \cdots + \alpha_{2^m}) \\ &\geq \frac{1}{2}\alpha_1 + \alpha_2 + 2\alpha_4 + \cdots + 2^{m-1}\alpha_{2^m} \\ &= \frac{1}{2}s'_m. \end{aligned}$$

Hence, we conclude that for all $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq 2^m \leq k_2$, we have

$$s_{k_1} \leq s'_m \leq 2s_{k_2},$$

which implies that the partial sums s_k remains bounded if and only if the partial sums s'_m remains bounded. $\square$

Using the Cauchy condensation test, we obtain the following result.

THEOREM 40. *The series* $\displaystyle\sum_{j=1}^{\infty} \frac{1}{n^j}$ *converges if and only if* $j > 1$.

PROOF. If $j \leq 0$, then the terms of the series do not converge to zero. Hence, Corollary 11 implies the divergence of the series. If $j > 0$, then we use the Cauchy condensation test: The convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^j}$ is equivalent to the convergence of the series

$$\sum_{m=0}^{\infty} 2^m \frac{1}{2^{mj}} = \sum_{m=0}^{\infty} 2^{(1-j)m}.$$

Since the above geometric series converges if and only if $2^{1-j} < 1$, i.e. if and only if $1 < j$, the claim follows. $\square$

THEOREM 41 (ROOT TEST (CAUCHY, 1821)). *Let* $\{\alpha_n\}$ *be a real sequence.*

(a) *If* $\limsup_{n\to\infty} (|\alpha_n|)^{1/n} < 1$, *the series* $\displaystyle\sum_{n=1}^{\infty} \alpha_n$ *converges.*

(b) *If* $\limsup_{n\to\infty} (|\alpha_n|)^{1/n} > 1$, *the series* $\displaystyle\sum_{n=1}^{\infty} \alpha_n$ *diverges.*

(c) *If* $\limsup_{n\to\infty} (|\alpha_n|)^{1/n} = 1$, *the series could either converge of diverge.*

Proof. Let $a := \limsup_{n\to\infty} (|\alpha_n|)^{1/n}$.

(*a*) If $a < 1$, there exists $b \in \mathbb{R}$ such that $a < b < 1$. Since $a = \limsup_{n\to\infty} (|\alpha_n|)^{1/n} < b$, it follows that there exists an integer $N \in \mathbb{N}$ such that $(|\alpha_n|)^{1/n} < b$ for all $n \geq N$, i.e.

$$|\alpha_n| < b^n$$

for all $n \geq N$. Since $b \in (0,1)$, the geometric series $\sum_{n\in\mathbb{N}} b^n$ converges. Hence, by Comparison test, we conclude that $\sum_{n\in\mathbb{N}} \alpha_n$ converges.

(*b*) There exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $(|\alpha_{n_k}|)^{1/n_k} \to a$. If $a > 1$, there exists $N \in \mathbb{N}$ such that $(|\alpha_{n_k}|)^{1/n_k} \in B_{a-1}(a)$ for all $n_k \geq N$. In other words, $(|\alpha_{n_k}|)^{1/n_k} > 1$, i.e.

$$|\alpha_{n_k}| > 1.$$

for all $n_k \geq N$. Hence, the Cauchy criterion implies the divergence of the series.

(*c*) Let $\alpha_n = \frac{1}{n^j}$ and let a be defined as above. Then we have $a = 1$ for all $j \in \mathbb{Z}$. However, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^j}$ converges only for $j > 1$. □

Theorem 42 (Ratio test (d'Alembert, 1768)). *Let $\{\alpha_n\}$ be a real sequence such that there exists an $N \in \mathbb{N}$ with $\alpha_n \neq 0$ for all $n \geq N$.*

(*a*) *If* $\limsup_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| < 1$, *the series* $\displaystyle\sum_{n=1}^{\infty} \alpha_n$ *converges.*

(*b*) *If* $\limsup_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| > 1$, *the series* $\displaystyle\sum_{n=1}^{\infty} \alpha_n$ *diverges.*

(*c*) *If* $\limsup_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| = 1$, *the series could either converge of diverge.*

Proof. Let $a := \limsup_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right|$.

(*a*) If $a < 1$, there exists a $b \in \mathbb{R}$ such that $a < b < 1$. Since $a = \limsup_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| < b$, it follows that there exists an integer $N \in \mathbb{N}$ such that $\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < b$ for all $n \geq N$, i.e.

$$|\alpha_{n+1}| < b|\alpha_n| < b^2|\alpha_{n-1}| < \cdots < b^{n-N}|\alpha_N|$$

for all $n \geq N$. Since $b \in (0,1)$, the geometric series $\sum_{n\in\mathbb{N}} b^n$ converges. Hence, by Comparison test, we conclude that $\sum_{n\in\mathbb{N}} \alpha_n$ converges.

(b) There exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $\left|\frac{\alpha_{n_k+1}}{\alpha_{n_k}}\right| \to a$. If $a > 1$, there exists $N \in \mathbb{N}$ such that $\left|\frac{\alpha_{n_k+1}}{\alpha_{n_k}}\right| \in B_{a-1}(a)$ for all $n_k \geq N$. In other words, $\left|\frac{\alpha_{n_{k+1}}}{\alpha_{n_k}}\right| > 1$, i.e.

$$|\alpha_{n_{k+1}}| > |\alpha_{n_k}|$$

for all $n_k \geq N$. Hence, the Cauchy criterion implies the divergence of the series.

(c) Let $\alpha_n = \frac{1}{n^j}$ and let a be defined as above. Then we have $a = 1$ for all $j \in \mathbb{Z}$. However, the series $\sum_{n=1}^{\infty} \frac{1}{n^j}$ converges only for $j > 1$. □

12. Power series

DEFINITION 49. *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence. The series*

$$\sum_{n=0}^{\infty} \alpha_n x^n$$

is called a power series. The numbers α_n are called the coefficients of the series.

It depends on the choice of x whether the above power series converges or diverges.

THEOREM 43 (CAUCHY-HADAMARD THEOREM, 1821). *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence and let $a = \limsup_{n\to\infty} (|\alpha_n|)^{1/n}$. Let*

$$R := \begin{cases} 0 & \text{if } a = \infty \\ \infty & \text{if } a = 0 \\ 1/a & \text{otherwise.} \end{cases}$$

Then the power series $\sum_{n=0}^{\infty} \alpha_n x^n$ converges if $|x| < R$ and diverges if $|x| > R$. (R is called the radius of convergence of the series $\sum_{n=0}^{\infty} \alpha_n x^n$).

PROOF. Let $\beta_n = \alpha_n x^n$. From root test, we know that the series

$$\sum_{n=0}^{\infty} \beta_n$$

converges if $\limsup_{n\to\infty} (|\beta_n|)^{1/n} < 1$ and diverges if $\limsup_{n\to\infty} (|\beta_n|)^{1/n} > 1$. Since

$$\limsup_{n\to\infty} (|\beta_n|)^{1/n} = |x| \cdot \limsup_{n\to\infty} (|\alpha_n|)^{1/n} = |x| \cdot a,$$

the claim follows. □

13. Absolute and conditional convergence

DEFINITION 50. *Let* $\{\alpha_n\}$ *be a real sequence. we say that the sequence* $\sum_{n=1}^{\infty} \alpha_n$ converges absolutely *if the series* $\sum_{n=1}^{\infty} |\alpha_n|$ *converges.*

THEOREM 44. *If* $\sum_{n=1}^{\infty} \alpha_n$ *converges absolutely, then the series* $\sum_{n=1}^{\infty} \alpha_n$ *converges.*

PROOF. The claim follows immediately from the Cauchy criterion, since for all $m, n \in \mathbb{N}$, we have

$$\Big|\sum_{j=m}^{n} \alpha_j\Big| \leq \sum_{j=m}^{n} |\alpha_j|.$$

□

DEFINITION 51. *Let* $\{\alpha_n\}$ *be a real sequence. we say that the sequence* $\sum_{n=1}^{\infty} \alpha_n$ converges conditionally *if the series* $\sum_{n=1}^{\infty} \alpha_n$ *converges but* $\sum_{n=1}^{\infty} |\alpha_n|$ *diverges.*

For example, in the next section we shall show that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges conditionally.

14. Summation by parts

LEMMA 1 (ABEL'S LEMMA/ PARTIAL SUMMATION FORMULA). *Given two sequences* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$, *let* s_k *denote the* k*-th partial sum*

$$s_k := \sum_{j=0}^{k} \alpha_j.$$

Define $s_k := 0$ *if* $k \notin \mathbb{N}$. *Then for* $0 \leq m \leq n$, *we have*

$$\sum_{j=m}^{n} \alpha_j \beta_j = \sum_{j=m}^{n-1} s_j(\beta_j - \beta_{j+1}) + s_n\beta_n - s_{m-1}\beta_m.$$

PROOF.

$$\begin{aligned}\sum_{j=m}^{n} \alpha_j\beta_j &= \sum_{j=m}^{n}(s_j - s_{j-1})\beta_j \\ &= \sum_{j=m}^{n} s_j\beta_j - \sum_{j=m-1}^{n-1} s_j\beta_{j+1} \\ &= \sum_{j=m}^{n-1} s_j(\beta_j - \beta_{j+1}) + s_n\beta_n - s_{m-1}\beta_m.\end{aligned}$$

□

THEOREM 45. *Given two real sequences* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$, *let* s_k *denote the* k*-th partial sum* $s_k := \sum_{j=0}^{k} \alpha_j$ *and let* $s_k := 0$ *if* $k \notin \mathbb{N}$. *Suppose,*

(a) *The partial sums* $\{s_k\}$ *form a bounded sequence.*
(b) *The sequence* $\{\beta_n\}$ *is nonincreasing.*
(c) $\beta_n \to 0$.

Then $\sum_{j=0}^{\infty} \alpha_j\beta_j$ *converges.*

PROOF. Since $\{s_k\}$ is a bounded sequence, there exists $u \in \mathbb{R}_{>0}$ such that $|s_k| \le u$ for all k. Since $\beta_n \to 0$, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\beta_n \in B_{\varepsilon/(2u)}(0)$ for all $n \ge N$. In particular, that implies $\beta_N \le \varepsilon/(2u)$. For $n \ge m \ge N$, we have

$$\begin{aligned}\Big|\sum_{j=m}^{n} \alpha_j\beta_j\Big| &= \Big|\sum_{j=m}^{n-1} s_j(\beta_j - \beta_{j+1}) + s_n\beta_n - s_{m-1}\beta_m\Big| \\ &\le u \cdot \Big|\sum_{j=m}^{n-1}(\beta_j - \beta_{j+1}) + \beta_n + \beta_m\Big| \\ &= 2u\beta_m \le 2u\beta_N \le \varepsilon.\end{aligned}$$

Now, the convergence of the series $\sum_{j=0}^{\infty} \alpha_j\beta_j$ follows from the Cauchy criterion. □

COROLLARY 12. *Let* $\{a_n\}$ *be a nonincreasing sequence such that* $a_n \to 0$. *Let the radius of convergence of the power series* $\sum_{n=0}^{\infty} a_n x^n$ *be 1. Then* $\sum_{n=0}^{\infty} a_n x^n$ *converges at* $x = -1$.

PROOF. Follows immediately from the previous theorem if we let $\alpha_n = (-1)^n$ and $\beta_n = a_n$. □

COROLLARY 13. *The series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ *converges.*

THEOREM 46 (LEIBNIZ, 1690). *Let* $\{a_n\}$ *be a real sequence such that*

(a) *The sequence* $\{|a_n|\}$ *is nonincreasing.*
(b) $a_{2m-1} \geq 0$ *and* $a_{2m} \leq 0$ *for all* $m \in \mathbb{N}$.
(c) $a_n \to 0$.

Then $\sum_{n=0}^{\infty} a_n$ *converges.*

PROOF. Follows immediately from the previous theorem if we let $\alpha_n = (-1)^{n-1}$ and $\beta_n = |a_n|$. □

15. Rearrangements of series

Let the alternating harmonic series converge to $s \in \mathbb{R}$, i.e.

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

Since all the partial sums of the above series are greater than 1/2, we conclude that $s > 0$.* Multiplying both sides of the above equality by 2, we get

$$2s = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \cdots$$

Now, collecting the terms with the same denominator, we obtain

$$\begin{aligned} 2s &= (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\ &= s \end{aligned}$$

Since $s \neq 0$, the above equality leads to a contradiction! What might have gone wrong?

DEFINITION 52 (REARRANGEMENTS OF SERIES). *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a bijection. Then we say that the series*

$$\sum_{n=1}^{\infty} \alpha_{f(n)}$$

is a rearrangement *of the series*

$$\sum_{n=1}^{\infty} \alpha_n.$$

*You will see in a later course in Analysis that the alternating harmonic series is the evaluation of the Taylor series of $\log(1+x)$ (which is also called the Newton-Mercator series) at $x = 1$. So, the alternating harmonic series converges to $\log 2$.

THEOREM 47 (DIRICHLET, 1837). *Let* $\sum_{n=1}^{\infty} \alpha_n$ *be an absolutely convergent series of real numbers and let* $f : \mathbb{N} \to \mathbb{N}$ *be a bijection. Then* $\sum_{n=1}^{\infty} \alpha_{f(n)}$ *is also absolutely convergent and*

$$\sum_{n=1}^{\infty} \alpha_{f(n)} = \sum_{n=1}^{\infty} \alpha_n.$$

PROOF. Note that proving only the last equality suffices, since it implies the absolute convergence of the series, if we replace all the terms in both of the series by their absolute values.

Let s_k and s'_k denote the k-th partial sum of the sequences $\{\alpha_n\}$ and $\{\alpha_{f(n)}\}$, respectively. Since $\sum_{n=1}^{\infty} \alpha_n$ converges absolutely, the Cauchy criterion implies that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m' \geq m \geq N$, we have

$$\sum_{n=m+1}^{m'} |\alpha_n| < \varepsilon.$$

Since $f : \mathbb{N} \to \mathbb{N}$ is a bijection, there exists an integer $M \geq N$ such that $\{1, 2, \ldots, N\} \subseteq \{f(1), f(2), \ldots, f(M)\}$. Then for $k \geq M$, we have

$$|s'_k - s_k| \leq \sum_{n=N+1}^{k'} |\alpha_n| < \varepsilon,$$

where $k' := \max\{k, f(1), f(2), \ldots, f(M)\}$. Hence, the sequence of the partial sums $\{s'_k\}$ converges to the same limit as that of $\{s_k\}$. $\square$

Dirichlet's theorem only says that had the alternating series been absolutely convergent, every reordering of it would have the same sum. However, except from saying that all the series which could be rearranged to yield a different sum, are not absolutely convergent, the above theorem does not shed any light on precisely which series could be rearranged to yield which sums. The following theorem of Riemann provides in particular, a complete answer to this.

THEOREM 48 (RIEMANN SERIES THEOREM, 1853). *Let* $\sum_{n=1}^{\infty} \alpha_n$ *be a conditionally convergent series of real numbers. Then for each pair of* a *and* b *in the extended real number system with* $a \leq b$*, the given series has a rearrangement* $\sum_{n=1}^{\infty} \alpha_{f(n)}$ *with partial sums* s_k *such that*

$$\liminf_{k\to\infty} s_k = a \quad \textit{and} \quad \limsup_{k\to\infty} s_k = b.$$

Proof. Define the sequences $\{\alpha_n^+\}$ and $\{\alpha_n^-\}$ by

$$\alpha_n^+ = \frac{\alpha_n + |\alpha_n|}{2} \quad \text{and} \quad \alpha_n^- = \frac{\alpha_n - |\alpha_n|}{2}.$$

In other words,

$$\alpha_n^+ = \begin{cases} \alpha_n & \text{if } \alpha_n > 0 \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad \alpha_n^- = \begin{cases} \alpha_n & \text{if } \alpha_n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\sum_{n=1}^{\infty}(\alpha_n^+ - \alpha_n^-) = \sum_{n=1}^{\infty}|\alpha_n|$$

does not converge, at least one of the series $\sum_{n=1}^{\infty}\alpha_n^+$ and $\sum_{n=1}^{\infty}\alpha_n^-$ diverges.

If exactly one of these two series diverge, then the series

$$\sum_{n=1}^{\infty}(\alpha_n^+ + \alpha_n^-) = \sum_{n=1}^{\infty}\alpha_n$$

diverges. Thus, we get a contradiction! Hence, both $\sum_{n=1}^{\infty}\alpha_n^+$ and $\sum_{n=1}^{\infty}\alpha_n^-$ diverges. Let $p_1 < p_2 < p_3 \cdots$ be the sequence of indices such that each α_{p_i} is nonnegative, and let $q_1 < q_2 < q_3 < \cdots$ be the indices such that each α_{q_j} is negative. Then each natural number appears in exactly one of the sequences $\{p_i\}$ and $\{q_j\}$. The series $\sum_{i=1}^{\infty}\alpha_{p_i}$, $\sum_{j=1}^{\infty}\alpha_{q_j}$ differ from $\sum_{n=1}^{\infty}\alpha_n^+$, $\sum_{n=1}^{\infty}\alpha_n^-$ only by zero terms and hence, are also divergent.

Let k_1 and ℓ_1 be the smallest integers such that

$$S_{k_1} := \sum_{i=1}^{k_1}\alpha_{p_i} > b \quad \text{and} \quad S'_{\ell_1} := S_{k_1} + \sum_{j=1}^{\ell_1}\alpha_{q_j} < a.$$

For the integers $m > 1$, we define k_m and ℓ_m as the smallest integers such that

$$S_{k_m} := S'_{\ell_{m-1}} + \sum_{i=k_{m-1}+1}^{k_m}\alpha_{p_i} > b \quad \text{and} \quad S'_{\ell_m} := S_{k_m} + \sum_{j=\ell_{m-1}+1}^{\ell_m}\alpha_{q_j} < a.$$

It follows that for all $m \in \mathbb{N}$, we have

$$S_{k_m} - \alpha_{p_{k_m}} \le b < S_{k_m} \quad \text{and} \quad S'_{\ell_m} - \alpha_{q_{\ell_m}} \ge a > S'_{\ell_m}.$$

That implies

$$|S_{k_m} - b| \le \alpha_{p_{k_m}} \quad \text{and} \quad |S'_{\ell_m} - a| \le -\alpha_{q_{\ell_m}} = |\alpha_{q_{\ell_m}}|.$$

Since $\sum_{n=1}^{\infty} \alpha_n$ converges, the Cauchy criterion implies that $\alpha_n \to 0$. Hence, it follows from the last two inequalities that the subsequences S_{k_m} and S'_{ℓ_m} of the partial sums of the rearrangement of the given series converge (or diverge) to b and a, respectively.

Thus, we provided a rearrangement of the given series so that a and b are subsequential limits of the rearranged series. It only remains to show that no subsequential limit of this rearranged series is less than a or greater than b. Since $S'_{\ell_m} \to a$, for each $a' < a$, there exists $N \in \mathbb{N}$ such that for all $m \geq N$, we have $S'_{\ell_m} \in B_{(a-a')/2}(a)$. Since all the other partial sums of the rearranged series are larger than a, it follows that the ball $B_{(a-a')/2}(a')$ contains at most only finitely many partial sums of the rearranged series. Similarly, for each $b' > b$, there are at most only finitely many partial sums of the rearranged series in the ball $B_{(b'-b)/2}(b')$. Hence, no $a' < a$ or $b' > b$ could be a subsequential limit of the sequence of the partial sums of the rearranged series.

$\square$

For $j \in \mathbb{N}$, let k_j and ℓ_j be defined as above and let $k_0 := 0$ and $\ell_0 := 0$. Note that the rearrangement function $f : \mathbb{N} \to \mathbb{N}$ for the above rearrangement is given by

$$f(n) = \begin{cases} p_{n-\sum_{j=0}^{r} \ell_j} & \text{if } \displaystyle\sum_{j=0}^{r}(k_j + \ell_j) < n \leq k_{r+1} + \sum_{j=0}^{r}(k_j + \ell_j) \\ q_{n-\sum_{j=0}^{r+1} k_j} & \text{if } \displaystyle k_{r+1} + \sum_{j=0}^{r}(k_j + \ell_j) < n \leq \sum_{j=0}^{r+1}(k_j + \ell_j), \end{cases}$$

where r is a nonnegative integer.

Corollary 14. *Let $\sum_{n=1}^{\infty} \alpha_n$ be a conditionally convergent series. Then for each point p in the extended real number system, there exists a rearrangement $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\sum_{n=1}^{\infty} \alpha_{f(n)} = p.$$

16. A conjecture on divergent series

Conjecture (Erdős, 1976). *Let $\{\alpha_n\}$ be a sequence of positive integers. If the series $\sum_{i=1}^{\infty} \frac{1}{\alpha_n}$ diverges, then for all $m \in \mathbb{N}$, the sequence $\{\alpha_n\}$ contains an arithmetic progression of length m.*

Though the above conjecture remains still open, some special cases of it have been proven recently:

THEOREM 49 (Green-Tao, 2004). *The prime numbers contain arbitrarily long arithmetic progressions.*

EXAMPLES. $\{3,5,7\}$, $\{5,11,17,23,29\}$, $\{7,37,67,97,127,157\}$.

THEOREM 50 (Bloom-Sisask, 2020). *If the series* $\sum_{i=1}^{\infty} \frac{1}{\alpha_n}$ *diverges, then the sequence* $\{\alpha_n\}$ *contains an arithmetic progression of length* 3.

17. The number e

Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{17.1}$$

Since $n! > 2^{n-1}$, and since the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges to 2, it follows from comparison test that the series (17.1) converges to some number in the interval $(2,3]$. Denote that number by e. That is

$$e := \sum_{n=0}^{\infty} \frac{1}{n!}.$$

THEOREM 51 (BERNOULLI, 1683).

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

PROOF. For integers $k \geq 2$, let

$$s_k := \sum_{n=0}^{k} \frac{1}{n!} \quad \text{and} \quad p_k := \left(1 + \frac{1}{k}\right)^k.$$

Then it follows from the binomial theorem that

$$\begin{aligned}
p_k &= \sum_{r=0}^{k} \binom{k}{r} \frac{1}{k^r} \\
&= 1 + 1 + \sum_{r=2}^{k} \frac{k(k-1)\cdots(k-r+1)}{r!} \frac{1}{k^r} \\
&= 2 + \sum_{r=2}^{k} \frac{1}{r!} \prod_{j=1}^{r} \frac{k-j+1}{k} \\
&= 2 + \sum_{r=2}^{k} \frac{1}{r!} \prod_{j=1}^{r} \left(1 - \frac{j-1}{k}\right) \\
&< 2 + \sum_{r=2}^{k} \frac{1}{r!} \\
&= s_k.
\end{aligned}$$

That implies

$$\limsup_{k\to\infty} p_k \le e. \tag{17.2}$$

Let $m \in \mathbb{N}$. For $k \ge m$, we have

$$\begin{aligned} p_k &= 2 + \sum_{r=2}^{k} \frac{1}{r!} \prod_{j=1}^{r} \left(1 - \frac{j-1}{k}\right) \\ &\ge 2 + \sum_{r=2}^{m} \frac{1}{r!} \prod_{j=1}^{r} \left(1 - \frac{j-1}{k}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \liminf_{k\to\infty} p_k &\ge \liminf_{k\to\infty} \left(2 + \sum_{r=2}^{m} \frac{1}{r!} \prod_{j=1}^{r} \left(1 - \frac{j-1}{k}\right)\right) \\ &= 2 + \sum_{r=2}^{m} \frac{1}{r!} \\ &= s_m. \end{aligned}$$

Letting $m \to \infty$, we get

$$\liminf_{k\to\infty} p_k \ge e. \tag{17.3}$$

Now, from (17.2) and (17.3), we conclude that

$$\lim_{k\to\infty} p_k = e.$$

$\square$

CHAPTER 4

Continuity

1. Limit of a function

DEFINITION 53. *Let (M, d) and (M', d') be metric spaces. Let $S \subseteq M$ and let there be a function $f : S \to M'$. Let a be a limit point of S. For $b \in M'$, we write $f(x) \to b$ as $x \to a$ or*

$$\lim_{x \to a} f(x) = b$$

if the preimage[†] *of every open ball around b contains all those points in S which belong to an open ball around a.*

EXAMPLE 7. *Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$ be a function. Let a be a limit point of S. Then we write $\lim\limits_{x \to a} f(x) = b$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$B_\delta(a) \cap S \subseteq f^{-1}(B_\varepsilon(b)),$$

i.e. for all $x \in S$ with $|x - a| < \delta$, we have $|f(x) - b| < \varepsilon$.

THEOREM 52. *Let (M, d) and (M', d') be metric spaces. Let $S \subseteq M$ and let there be a function $f : S \to M'$. Let a be a limit point of S. Then*

$$\lim_{x \to a} f(x) = b$$

if and only if for every sequence $\{\alpha_n\}$ in S such that $\lim\limits_{n \to \infty} \alpha_n = a$, we have

$$\lim_{n \to \infty} f(\alpha_n) = b.$$

PROOF. Let $\lim\limits_{x \to a} f(x) = b$ and let $\{\alpha_n\}$ in S be a sequence converging to a. Then for every open ball $B_\varepsilon(b) \subseteq M'$, there exists an open ball $B_\delta(a) \subseteq M$ such that

$$B_\delta(a) \cap S \subseteq f^{-1}(B_\varepsilon(b)).$$

Since $\alpha_n \to a$, there exists an $N \in \mathbb{N}$ such that $\alpha_n \in B_\delta(a)$ for all $n \geq N$. It follows that for all $n \geq N$, we have $f(\alpha_n) \in B_\varepsilon(b)$. Hence, we conclude that $\lim\limits_{n \to \infty} f(\alpha_n) = b$.

[†]For a function $f : A \to B$, the *preimage* or the *inverse image* of any subset $E \subseteq B$ is defined as $f^{-1}(E) := \{a \in A \mid f(a) \in E\}$.

Now, suppose that $\lim_{x \to a} f(x) \neq b$ but

$$\lim_{n\to\infty} f(\alpha_n) = b$$

for every sequence $\{\alpha_n\}$ in S such that $\lim_{n\to\infty} \alpha_n = a$. Then there exists an open ball $B_\varepsilon(b) \subseteq M'$ such that

$$B_\delta(a) \cap S \nsubseteq f^{-1}(B_\varepsilon(b))$$

for all $\delta > 0$. Since a is a limit point of S, there exists an $\alpha_n \neq a$ in $B_{1/n}(a) \cap S$ for each $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \alpha_n = a$ but $f(\alpha_n) \notin B_\varepsilon(b)$ for all $n \in \mathbb{N}$. That implies,

$$\lim_{n\to\infty} f(\alpha_n) \neq b.$$

Thus, we get a contradition! $\square$

Since the limit of a convergent sequence is unique (see Theorem 24.(a)), from the above theorem, we conclude:

COROLLARY 15. *If a function f has a limit at a point a, then this limit is unique.* $\square$

DEFINITION 54. *Let (M, d) be a metric space. Let $S \subseteq M$ and let $f, g : S \to \mathbb{R}$ be two functions. Then we define*

$$(f + g)(x) := f(x) + g(x),$$

$$(f - g)(x) := f(x) - g(x),$$

$$fg(x) := f(x)g(x),$$

$$\textit{and} \quad (f/g)(x) := f(x)/g(x) \quad \textit{if } g(x) \neq 0.$$

THEOREM 53. *Let (M, d) be a metric space. Let $S \subseteq M$ and let $f, g : S \to \mathbb{R}$ be two functions. Let a be a limit point of S and let*

$$\lim_{x\to a} f(x) = b \quad \textit{and} \quad \lim_{x\to a} g(x) = c.$$

Then we have

(a) $\lim_{x\to a}(f + g)(x) = b + c$.
(b) $\lim_{x\to a}(fg)(x) = bc$.
(c) $\lim_{x\to a}(f/g)(x) = b/c$ *if* $c \neq 0$.

PROOF. Follows from Theorem 52 and the corresponding properties of sequences (see Example 5). $\square$

2. Continuous functions

DEFINITION 55. *Let (M,d) and (M',d') be metric spaces. Let $S \subseteq M$ and let there be a function $f : S \to M'$. The function f is said to be continuous* at $a \in S$ if for every open set $\mathcal{O}' \subseteq M'$ containing $f(a)$, there exists an open set $\mathcal{O} \subseteq S$ containing a such that*

$$\mathcal{O} \subseteq f^{-1}(\mathcal{O}').$$

If f is continuous at every point of S, then we say that f is continuous on S.

EXAMPLE 8. *Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$ be a function. Then the function f is continuous at $a \in S$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$B_\delta(a) \cap S \subseteq f^{-1}(B_\varepsilon(f(a))),$$

i.e. for all $x \in S$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$.

Note that, from the definition of continuous functions, it follows trivially that every function defined on a subset S of a metric space is continuous at the isolated points of S.

THEOREM 54. *Let (M,d) and (M',d') be metric spaces. Let $S \subseteq M$ and let there be a function $f : S \to M'$. Let $a \in S$ be a limit point of S. Then f is continuous at a if and only if*

$$\lim_{x \to a} f(x) = f(a).$$

PROOF. Follows immediately from Definition 53 and Definition 12. □

COROLLARY 16. *A function f is continuous on a set S if and only if f has a limit at every limit point in S.* □

COROLLARY 17. *Let (M,d) and (M',d') be metric spaces and let f be a continuous function from M to M'. Then for every sequence $\{\alpha_n\}$ in M which converges to a point $p \in M$, we have*

$$\lim_{n \to \infty} f(\alpha_n) = f(p).$$

□

THEOREM 55. *Let (M,d) and (M',d') be metric spaces and let*

$$f : M \to M'$$

be a function. Then f is continuous on M if and only if $f^{-1}(\mathcal{O}')$ is open in M for every open set $\mathcal{O}' \subseteq M'$.

---

*Note that, unlike when f has a limit at a (see Definition 53), first f needs to be defined at a in order to be continuous at a.

Proof. Let f be continuous on M and let $\mathcal{O}' \subseteq M'$ be an open set. It suffices to show that every point $a \in f^{-1}(\mathcal{O}')$ is an interior point of $f^{-1}(\mathcal{O}')$. Since $\mathcal{O}'$ is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq \mathcal{O}'$. Since f is continuous at a, there exists a $\delta > 0$ such that

$$B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a))) \subseteq f^{-1}(\mathcal{O}').$$

Hence, a is an interior point of $f^{-1}(\mathcal{O}')$. So, $f^{-1}(\mathcal{O}')$ is open.

Now, suppose that the preimage of every open set in M' is open in M. Let $a \in M$ and let $\mathcal{O}' \subseteq M'$ be an open set containing $f(a)$. Then $\mathcal{O} := f^{-1}(\mathcal{O}')$ is an open set in M containing a. Hence, f is continuous on M. □

Corollary 18. *Let (M,d) and (M',d') be metric spaces and let*

$$f : M \to M'$$

be a function. Then f is continuous on M if and only if $f^{-1}(\mathcal{C}')$ is closed in M for every closed set $\mathcal{C}' \subseteq M'$.

Proof. Follows immediately from the last theorem, since a set is closed if and only if its complement is open and since

$$f^{-1}(\mathcal{O}'^c) = \left(f^{-1}(\mathcal{O}')\right)^c$$

for all $\mathcal{O}' \subseteq M'$. □

Theorem 56. *Let (M,d) be a metric space. Let $S \subseteq M$ and let $f, g : S \to \mathbb{R}$ be two functions. Then both $f+g$ and fg are continuous on M. Moreover, if $g(x) \neq 0$ for all $x \in M$, then f/g is also continuous on M.*

Proof. Since every function on M is continuous at the isolated points of M, it suffices to show that the above functions are continuous at the limit points of M, which follows trivially from Theorem 54 and Theorem 53. □

Theorem 57. *Let $(M,d), (M',d')$ and (M'',d'') be metric spaces, let $f : M \to M'$, $g : f(M) \to M''$ and let $h : M \to M''$ be defined by*[*]

$$h(x) := g(f(x)).$$

Then if f is continuous at $a \in S$ and if g is continuous at $f(a) \in f(M)$, then h is continuous at a.

Proof. Let $\mathcal{O}'' \in M''$ be an open set containing $h(a) = g(f(a))$. Since g is continuous at $f(a)$, there exists an open set $\mathcal{O}' \subseteq M'$ containing $f(a)$ such that

$$\mathcal{O}' \subseteq g^{-1}(\mathcal{O}'').$$

[*]The function h, denoted by $g \circ f$ is called the *composition* of f and g.

Again, since f is continuous at a, there exists an open set $\mathcal{O} \subseteq M$ containing a such that

$$\mathcal{O} \subseteq f^{-1}(\mathcal{O}') \subseteq f^{-1}(g^{-1}(\mathcal{O}'')) = h^{-1}(\mathcal{O}'').$$

Hence, h is continuous at a. □

3. Discontinuities

Definition 56. *If a function f is not continuous at a point x in its domain of definition, we say that f is discontinuous at x or f has a discontinuity at x.*

Definition 57 (Right-hand and left-hand limits). *Let $a, b \in \mathbb{R}$ with $a < b$ and let (M, d) be a metric space. Let $f : (a, b) \to M$ be a function. For $x \in [a, b)$, we write*

$$f(x+) = \alpha$$

if $f(\alpha_n) \to \alpha$ as $n \to \infty$ for all sequences $\{\alpha_n\}$ in (x, b) such that $\alpha_n \to x$. Also, for $x \in (a, b]$, we write

$$f(x-) = \beta$$

if $f(\beta_n) \to \beta$ as $n \to \infty$ for all sequences $\{\beta_n\}$ in (a, x) such that $\beta_n \to x$.

Definition 58 (Discontinuities of first and second kinds). *Let $a, b \in \mathbb{R}$ with $a < b$ and let (M, d) be a metric space. Let $f : (a, b) \to M$ be a function. If f is discontinuous at $x \in (a, b)$ and if both $f(x+)$ and $f(x-)$ exist, then we say that f has a discontinuity of the first kind or a simple discontinuity at x. Otherwise, we say that f has a discontinuity of the second kind at x.*

Example 9 (Simple discontinuity). *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ -x & \text{if } 0 \leq x < 1 \\ x & \text{if } 1 \leq x \end{cases}$$

Then f has a simple discontinuity at 1.

Example 10 (Discontinuity of the second kind). *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

Then f has a discontinuity of the second kind at every point in $\mathbb{R}$ since neither $f(x+)$ nor $f(x-)$ exists.

Example 11 (Discontinuity of the second kind). *Let* $f : \mathbb{R} \to [-1, 1]$ *be defined by*

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then f *has a discontinuity of the second kind at* 0 *since none of the limits* $f(0+)$ *and* $f(0-)$ *exists.*

4. Continuity and connectedness

Definition 59 (Connected space). *A metric space is said to be* connected *if it is not the union of two disjoint nonempty open subsets.*

Theorem 58. *Continuous image of a connected set is connected.*

Proof. Let (M, d) and (M', d') be metric spaces, let M be connected and let $f : M \to M'$ be a continuous function. We require to show that $f(M)$ is connected. Suppose, $f(M)$ is not connected. Then there exists two nonempty open subsets $S, T \subset f(M)$ such that

$$S \cap T = \emptyset \quad \text{and} \quad S \cup T = f(M).$$

Then neither $f^{-1}(S)$ nor $f^{-1}(T)$ is empty (since they are preimages of nonempty subsets of the image of f) and both of them are open, since f is continuous. Let I denote the intersection of $f^{-1}(S)$ and $f^{-1}(T)$. Then $f(I) \subseteq S \cap T = \emptyset$. Hence, we conclude that $I = \emptyset$. So, $f^{-1}(S)$ and $f^{-1}(T)$ are disjoint nonempty open sets such that

$$M = f^{-1}(f(M)) = f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$$

which contradicts the connectedness of M. Thus, we obtain a contradiction! Hence, $f(M)$ is connected. □

Theorem 59. *Each nonempty interval* in* $\mathbb{R}$ *is connected.*

Proof. Let, $I \subset \mathbb{R}$ be an interval and suppose, I is not connected. Then there are two nonempty open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in the subspace topology of I such that

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \quad \text{and} \quad \mathcal{O}_1 \cup \mathcal{O}_1 = I.$$

It follows from Theorem 11 that there exists two open sets U_1 and U_2 in $\mathbb{R}$ such that $\mathcal{O}_1 = I \cap U_1$ and $\mathcal{O}_1 = I \cap U_2$. Let I^0 denote the interior of the interval I. Since $I \cap U_1 \neq \emptyset$, there exists a point $x \in (I \cap U_1)$. Since U_1 is open, U_1 contains a neighbourhood N of x. Since every point in an interval is a limit point of its interior, $x \in I$ is a limit point of the interior $I^0 \subseteq I$, and hence, $N \cap I^0 \neq \emptyset$. Since $N \cap I^0 \subseteq U_1 \cap I^0$, it follows that $A := U_1 \cap I^0 \neq \emptyset$. Similarly, we obtain that $B := U_2 \cap I^0 \neq \emptyset$. Since $A \subseteq U_1 \cap I = \mathcal{O}_1$, $B \subseteq U_2 \cap I = \mathcal{O}_2$ and since $\mathcal{O}_1 \cap \mathcal{O}_1 = \emptyset$,

$$A \cap B = \emptyset.$$

*Here, we also allow $\pm\infty$ to be the supremum or infimum of the intervals

Again since $I^0 \subseteq I \subseteq U_1 \cup U_2$, it follows that

$$A \cup B = I^0.$$

Since intersection of two open sets is open, both of the nonempty sets A and B are open in $\mathbb{R}$. Let $a \in A$ and $b \in B$. Without loss of generality, we may assume that $a < b$. Let

$$S := \{x \in I^0 \mid [a, x] \subseteq A\}.$$

If there exists $x \in S$ such that $b \leq x$, then $[a, b] \subseteq [a, x] \subseteq A$. That implies $b \in A \cap B = \emptyset$, which is absurd! So, we conclude that $b > x$ for for all $x \in S$. Since b is an upper bound of $S \subset \mathbb{R}$ and since $\mathbb{R}$ is complete, S has a supremum in $\mathbb{R}$. Let $s := \sup S$. Since $a \leq s \leq b$, and since I^0 is an interval containing a and b, it follows that $s \in I^0 = A \cup B$. So, either $s \in A$ or $s \in B$.

Case 1. $(s \in A)$
Since A open, there exists an $r > 0$ such that the open ball $B_r(s) = (s - r, s + r) \subseteq A$. Since $s = \sup S$, there exists an $s' \in I^0$ with $s - r < s' \leq s$ such that $[a, s'] \subset A$. That implies $[a, s + r/2] = [a, s'] \cup [s', s + r/2] \subseteq [a, s'] \cup (s - r, s + r) \subseteq A$. So, we have $s + r/2 \in S$, which is absurd, since $s = \sup S$.

Case 2. $(s \in B)$
Since B open, there exists an $r > 0$ such that the open ball $B_r(s) = (s - r, s + r) \subseteq B$. Since $s = \sup S$, there exists an $s' \in I^0$ with $s - r < s' \leq s$ such that $[a, s'] \subset A$. In particular, we have $s' \in [a, s'] \subset A$. However, we also have, $s' \in (s - r, s + r) \subseteq B$. Therefore, $s' \in A \cap B = \emptyset$, which is absurd!

Since both of the above two cases lead to contradiction, we conclude that I is connected. $\square$

THEOREM 60. *Each connected subset of* $\mathbb{R}$ *is an interval.*

PROOF. Let S be a connected subset of $\mathbb{R}$ and suppose, S is not an interval. Then there exists $t \in \mathbb{R} \setminus S$ and $s, s' \in S$ such that $s < t < s'$. Let

$$A = S \cap (-\infty, t) \quad \text{and} \quad B = S \cap (t, \infty).$$

Then A and B are nonempty open subsets in the subspace topology of S such that

$$A \cap B = \emptyset \quad \text{and} \quad A \cup B = S$$

which contradicts the connectedness of S! Hence, S is an interval. $\square$

THEOREM 61 (Intermediate value theorem, (Bolzano, 1817)). *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Let*

$\alpha := \min\{f(a), f(b)\}$ and $\beta := \max\{f(a), f(b)\}$. Then for every $y \in [\alpha, \beta]$, there exists an $x \in [a, b]$ such that

$$f(x) = y.$$

Proof. Since $[a, b]$ is connected, so is $f([a, b])$. Hence, $f([a, b])$ is an interval. since $\alpha, \beta \in f([a, b])$ and since $f([a, b])$ is an interval, it follows that $[\alpha, \beta] \subseteq f([a, b])$. Hence, for every $y \in [\alpha, \beta]$, there exists an $x \in [a, b]$ such that $f(x) = y$. □

The converse of the Intermediate value theorem is false. For example, the function $f : \mathbb{R} \to [-1, 1]$ defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

assumes every value in the interval $[-1, 1]$. However, f has a discontinuity of second kind at 0, since none of the limits $f(0+)$ and $f(0-)$ exists.

5. Continuity and compactness

Theorem 62. *Continuous image of a compact set is compact.*

Proof. Let (M, d) and (M', d') be metric spaces, let M be compact and let $f : M \to M'$ be a continuous function. We require to show that $f(M)$ is compact. Let $\{\mathcal{O}_n\}$ be an open cover of $f(M)$. Since f is continuous, each of the sets $f^{-1}(\mathcal{O}_n)$ is open. Hence $\{f^{-1}(\mathcal{O}_n)\}$ forms an open cover of M. Since M is compact, there are finitely many indices $\{n_1, n_2, \ldots, n_k\}$ such that

$$M \subseteq f^{-1}(\mathcal{O}_{n_1}) \cup \cdots \cup f^{-1}(\mathcal{O}_{n_k}).$$

Since for every $S \subseteq M'$, we have* $f(f^{-1}(S)) \subseteq S$, the above equality implies that

$$f(M) \subseteq \mathcal{O}_{n_1} \cup \cdots \cup \mathcal{O}_{n_k}.$$

Hence, $f(M)$ is compact. □

Definition 60 (Bounded function). *A function f from a set S to the Euclidean space $\mathbb{R}^n$ is called bounded if there is an $M \in \mathbb{R}_{>0}$ such that*

$$\|f(x)\| \leq M$$

for all $x \in S$, where $\|\|$ denotes the Euclidean norm†.

Theorem 63. *Every continuous function from a compact set to an Euclidean space is bounded.*

Proof. Follows immediately from the previous theorem and the Heine-Borel theorem (Theorem 22). □

*Note that, on the other hand, we have $S \subseteq f^{-1}(f(S)$ for all $S \subseteq M'$.

†The norm $\|x\|$ of $x \in \mathbb{R}^n$ denotes the distance from x to the origin in $\mathbb{R}^n$.

THEOREM 64. *Let (M, d) be a compact metric space and let $f : M \to \mathbb{R}$ be a continuous function. Then f attains both its minimum and maximum on M.*

PROOF. We require to show that the set $f(M)$ has a minimum and a maximum. Theorem 62 implies that $f(M)$ is compact. Hence, it follows from the Heine-Borel theorem (Theorem 22) that $f(M)$ is closed and bounded. Now, the claim follows from Theorem 10. □

THEOREM 65. *Let (M, d) and (M', d') be metric spaces, let M be compact and let $f : M \to M'$ be a continuous bijection. Then $f^{-1} : M' \to M$ defined by*

$$f^{-1}(f(x)) = x \quad \text{for all } x \in M$$

is also continuous.

PROOF. Theroem 55 implies, in order to show that f^{-1} is countinuous, It suffices to prove that $f(\mathcal{O})$ is open for every open set $\mathcal{O} \subseteq M$. Note that since $\mathcal{O} \subseteq M$ is open, $\mathcal{O}^c$ is a closed subset of the compact set M. Hence, by Theorem 14, $\mathcal{O}^c$ is compact. So, by Theorem 62, $f(\mathcal{O}^c)$ is a compact subset of M' and therefore, by Theorem 13, $f(\mathcal{O}^c)$ is closed. As f is a bijection, $f(\mathcal{O}^c) = f(\mathcal{O})^c$. Since $f(\mathcal{O})^c$ is closed, it follows that $f(\mathcal{O})$ is open. □

6. Uniform continuity

DEFINITION 61 (Uniformly continuous function). *Let (M, d) and (M', d') be metric spaces and let $f : M \to M'$ be a function. We say that f is uniformly continuous if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that we have*

$$d'(f(x), f(y)) < \varepsilon \text{ for all } x, y \in M \text{ with } d(x, y) < \delta. \tag{6.1}$$

EXAMPLE 12. *Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) := x^2$. Then*

$$|f(x) - f(y)| = |x + y| \cdot |x - y| < 2|x - y|.$$

Hence, f is uniformly continuous.

EXAMPLE 13. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \sin x$. Then*

$$\begin{aligned} |f(x) - f(y)| &= |\sin x - \sin y| = \left|2 \sin \frac{x - y}{2} \cos \frac{x + y}{2}\right| \\ &\leq 2\left|\sin \frac{x - y}{2}\right| \leq 2\left|\frac{x - y}{2}\right| \\ &= |x - y|. \end{aligned}$$

Hence, f is uniformly continuous.

Of course, a uniformly continuous function is continuous. But the converse does not hold in general: Though if a function g is continuous at a point $a \in M$, then given $\varepsilon' > 0$, there exists a $\delta' > 0$ such that $g(y) \in B_{\varepsilon'}(g(a))$ whenever $y \in B_{\delta'}(a)$, but this δ' depends on both ε'

and a. If g were uniformly continuous, then there would have been no dependency of δ' on a.

Note that since the derivative of x^2 is $2x < 2$ on the interval $(0, 1)$ and since the derivative of $\sin x$ is $\cos x \leq 1$, both of the inequalities relating $|f(x) - f(y)|$ with $|x - y|$ in Example 12 and Example 13, follows immediately from the mean value theorem. More generally, the mean value theorem implies that a differentiable function f on an interval $I \subseteq \mathbb{R}$ with bounded derivative is uniformly continuous, provided f is continuously extendable to the closure of I. However, once we know that f has a continousous extension to the closure of a bounded interval I, we do not require anymore to appeal to the mean value theorem to prove that f is uniformly continousous on I (see Theorem 68 and Theorem 69).

In a later course in Analysis, you will see that the notion of uniform continuity (see Theorem 68) is essential in showing that if f is a continuous function on a closed interval $[a, b]$, then f is Riemann integrable on $[a, b]$.

EXAMPLE 14. *The function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, defined by $f(x) := 1/x$ is continuous but not uniformly continuous.*

PROOF. The continuity of the function $f(x)$ follows from the continuity of x on $\mathbb{R}_{>0}$ via Theorem 54 and Theorem 53.(c).

Now suppose, f is uniformly continuous. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that (6.1) is satisfied. Let $x = \min\{\delta, 1\}$ and $y = x/(\varepsilon + 1)$. Then $|x - y| < \delta$ but $|f(x) - f(y)| = \varepsilon/x \geq \varepsilon$. Thus, we get a contradiction! Hence, f is not uniformly continuous. □

In contrast to the notion of continuity at a single point, uniform continuity is a property of a function on its entire domain of definition. In particular, it does not make much sense to ask whether a function is uniformly continuous at a single point.

THEOREM 66 (Sequential criterion for non-uniformity). *Let (M, d) and (M', d') be metric spaces and let $f : M \to M'$ be a function. If f is not uniformly continuous, then there are sequences $\{x_n\}$ and $\{y_n\}$ in M and some $\varepsilon > 0$, such that $d(x_n, y_n) \to 0$ but $d'(f(x_n), f(y_n)) \geq \varepsilon$ for all $n \in \mathbb{N}$.*

PROOF. If f is not uniformly continuous, then there exists some $\varepsilon > 0$ such that for every $\delta > 0$, there exists $x, y \in M$ with $d(x, y) < \delta$ and $d'(f(x), f(y)) \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$, there exists $x_n, y_n \in M$ with $d(x_n, y_n) < 1/n$ and $d'(f(x_n), f(y_n)) \geq \varepsilon$, which proves the claim. □

We show the usefulness of the above criterion by applying it to obtain an easier proof the non-uniformity of the function $f(x) = 1/x$ which we encountered in Example 14.

EXAMPLE 15. *The function* $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, *defined by* $f(x) := 1/x$ *is not uniformly continuous.*

PROOF. Define the sequences $\{x_n\}$ and $\{y_n\}$ by $x_n := 1/n$ and $y_n := 1/(n+1)$ for all $n \in \mathbb{N}$. Then $d(x_n, y_n) = 1/(n(n+1))$ which goes to 0 as $n \to \infty$. However, $d'(f(x_n), f(y_n)) = 1$ for all $n \in \mathbb{N}$. Hence, f is not uniformly continuous. □

THEOREM 67. *The image of a Cauchy sequence under a uniformly continousous function is a Cauchy sequence.*

PROOF. Let (M, d) and (M', d') be metric spaces, $f : M \to M'$ a uniformly continuous function and $\{\alpha_n\}$ a Cauchy sequence in M. Then we require to show that $\{f(\alpha_n)\}$ is a Cauchy sequence in M'. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that (6.1) is satisfied for the given ε. Since $\{\alpha_n\}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that $d(\alpha_m, \alpha_n) < \delta$ for all $m, n \geq N$. Hence, it follows that

$$d'(f(\alpha_m), f(\alpha_n)) < \varepsilon \quad \text{for all } m, n \geq N.$$

So, $\{f(\alpha_n)\}$ is a Cauchy sequence. □

THEOREM 68. *Continuous functions on compact metric spaces are uniformly continuous.*

PROOF. Let (M, d) and (M', d') be metric spaces, let M be compact and let $f : M \to M'$ be a continuous function. Suppose, f is not uniformly continuous. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in M and some $\varepsilon > 0$, such that $d(x_n, y_n) \to 0$ but

$$d'(f(x_n), f(y_n)) \geq \varepsilon \tag{6.2}$$

for all $n \in \mathbb{N}$. Since M is sequentially compact (see Corollary 9), there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to some point $\ell \in M$. Since $d(x_n, y_n) \to 0$, it follows from the triangle inequality that there is also a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which also converges to ℓ. Since f is continuous, Corollary 17 implies that

$$\lim_{j\to\infty} f(x_{n_j}) = f(\ell) = \lim_{j\to\infty} f(y_{n_j}).$$

That implies, both $d'(f(\ell), f(x_{n_j})) \to 0$ and $d'(f(\ell), f(y_{n_j})) \to 0$. Therefore, again by the triangle inequality, we conclude that

$$d'(f(x_{n_j}), f(y_{n_j})) \to 0,$$

which contradicts (6.2). Hence, f is uniformly continuous. □

THEOREM 69 (Criterion for uniformity: Extendability to the closure). *Let f be a continuous function from a bounded subset S of a Euclidean space to a Euclidean space. Then f is uniformly continuous if and only if for every point p in the set** $\overline{S} \setminus S$, *the limit* $\lim_{x\to p} f(x)$ *exists.*

*This is the set of limit points of S on its boundary ∂S (see Definition 31).

PROOF. First assuming that $\lim_{x\to p} f(x)$ exists for every point $p \in \overline{S} \setminus S$, we show that f is uniformly continuous.

Since S is bounded, there exists $x \in S$ and $r > 0$ such that $S \subseteq B_r(x)$. Hence, we have $\overline{S} \subseteq \overline{B_r(x)} \subset B_{r+1}(x)$. In particular, $\overline{S}$ is also bounded. Since $\overline{S}$ is a closed and bounded subset of a Euclidean space, the Heine-Borel theorem (Theorem 22) implies that $\overline{S}$ is compact. Define $\widetilde{f} : \overline{S} \to M$ by

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in S \\ \lim_{x\to p} f(x) & \text{otherwise.} \end{cases}$$

Since f is continuous, from Theorem 54, we obtain that $\widetilde{f}$ is also continuous on $\overline{S}$. Since $\overline{S}$ is compact, Theorem 68 implies that $\widetilde{f}$ is uniformly continuous. Since $\widetilde{f}(x) = f(x)$ for all $x \in S$, it follows that f is uniformly continuous.

Next, we assume that f is uniformly continuous and we show that $\lim_{x\to p} f(x)$ exists for each $p \in \overline{S} \setminus S$. Since such a point p is necessarily a limit point of S, by Theorem 24.(c), there exists a sequence $\{\alpha_n\}$ in S, which converges to p. So, Theorem 26 implies that $\{\alpha_n\}$ is a Cauchy sequence. Since f is uniformly continuous, it follows from Theorem 67 that $\{f(\alpha_n)\}$ is a Cauchy sequence. Since the image of f is contained in a Euclidean space, the Cauchy criterion again implies that $\{f(\alpha_n)\}$ converges. Let $\lim_{n\to\infty} f(\alpha_n) = \ell$. Theorem 52 implies that in order to prove the existence of $\lim_{x\to p} f(x)$ it suffices to prove that every sequence in S which converges to p is mapped by f to a sequence which converges to ℓ. Let $\{\beta_n\}$ be a sequence in S which converges to p. Then we require to show that $\lim_{n\to\infty} f(\beta_n) = \ell$. Define a sequence $\{a_n\}$ in S by

$$a_n = \begin{cases} \alpha_{(n+1)/2} & \text{if } n \text{ is odd} \\ \beta_{n/2} & \text{otherwise.} \end{cases}$$

Then $a_n \to p$. Hence, similarly as before, we obtain the convergence of the sequence $\{f(a_n)\}$. Since every subsequence of a convergent sequence converges to the same limit and since $\{\alpha_n\}$ and $\{\beta_n\}$ are subsequences of $\{a_n\}$, it follows that

$$\lim_{n\to\infty} f(\beta_n) = \lim_{n\to\infty} f(\alpha_n) = \ell.$$

Hence, $\lim_{x\to p} f(x)$ exists for each $p \in \overline{S} \setminus S$. □

7. Periodic functions

DEFINITION 62. *A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there exists $P > 0$ such that $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. The number P is called the period of f.*

EXAMPLE 16. *The functions* $\sin x$ *and* $\cos x$ *are* 2π*-periodic.*

THEOREM 70. *Every periodic continuous function* $f : \mathbb{R} \to \mathbb{R}$ *is uniformly continuous.*

PROOF. Let P be the period of f. Since f is continutous, by Theorem 68, f is uniformly continuous $[0, 2P]$, because the interval $[0, 2P]$ is compact by the Heine-Borel Theorem*. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z, w \in [0, 2P]$ with $|z - w| < \delta$, we have $|f(z) - f(w)| < \varepsilon$. Let $\delta' = \min\{\delta, P\}$ and let $x, y \in \mathbb{R}$ with $|x - y| < \delta'$. It suffices to show that $|f(x) - f(y)| < \varepsilon$. Let $m := \lfloor x/P \rfloor$ and $n := \lfloor y/P \rfloor$. Then there exists $x_0, y_0 \in [0, P)$ such that $x = mP + x_0$ and $y = nP + y_0$. Hence, $f(x) = f(x_0)$ and $f(y) = f(y_0)$. Without loss of generality, we may assume that $x \geq y$. That implies $\lfloor x/P \rfloor \geq \lfloor y/P \rfloor$, i.e. $m \geq n$. If $m = n$, then we have $x_0 \geq y_0$. If $m > n$, then if necessary, replacing m by $m - 1$ and x_0 by $x_0 + P$, we may still assume that $m \geq n$ and $x_0 \geq y_0$, however with $x_0, y_0 \in [0, 2P)$. Since f is P periodic, even after replacing x_0 with $x_0 + P$, we still have $f(x) = f(x_0)$. Since

$$|x - y| = |x_0 - y_0 + (m - n)P| < \min\{\delta, P\},$$

and since both $x_0 - y_0$ and the integer $m - n$ are nonnegative, it follows that $m - n = 0$. Therefore, we have $|x_0 - y_0| = |x - y| < \delta$. Since $x_0, y_0 \in [0, 2P]$, it follows that

$$|f(x) - f(y)| = |f(x_0) - f(y_0)| < \varepsilon.$$

□

*See Theorem 22.