

MA3201

# Topology

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## 1 Introduction

### 1.1 Topological spaces

**Definition 1.1.** A topology on some set  $X$  is a family  $\tau$  of subsets of  $X$ , satisfying the following.

1.  $\emptyset, X \in \tau$ .
2. All unions of elements from  $\tau$  are in  $\tau$ .
3. All finite intersections of elements from  $\tau$  are in  $\tau$ .

The sets from  $\tau$  are declared to be open sets in the topological space  $(X, \tau)$ .

*Example.* Any set  $X$  admits the indiscrete topology  $\tau_{id} = \{\emptyset, X\}$ , as well as the discrete topology  $\tau_d = \mathcal{P}(X)$ . Both of these are trivial examples.

*Example.* Let  $X$  be a set. The cofinite topology on  $X$  is the collection of complements of finite sets, along with the empty set. Note that when  $X$  is finite, this is simply the discrete topology.

**Definition 1.2.** Let  $\tau, \tau'$  be two topologies on the set  $X$ . We say that  $\tau$  is finer than  $\tau'$  if  $\tau$  has more open sets than  $\tau'$ . In such a case, we also say that  $\tau'$  is coarser than  $\tau$ .

## 1.2 Topological bases

**Definition 1.3.** Let  $(X, \tau)$  be a topological space. We say that  $\beta \subseteq \tau$  is a base of the topology  $\tau$  such that every open set  $U \in \tau$  is expressible as a union of elements from  $\beta$ .

**Definition 1.4.** Let  $X$  be a set, and let  $\beta$  be a collection of subsets of  $X$  satisfying the following.

1. For every  $x \in X$ , there exists  $x \in B \in \beta$ .
2. For every  $x \in X$  such that  $x \in B_1 \cap B_2$ ,  $B_1, B_2 \in \beta$ , there exists  $B \in \beta$  such that  $x \in B \subseteq B_1 \cap B_2$ .

Then,  $\beta$  generates a topology on  $X$ , namely the collection of all unions of elements of  $\beta$ .

**Lemma 1.1.** Let  $\tau$  be a topology on  $X$ , and let  $\beta \subseteq \tau$  be a collection of open sets. Then,  $\beta$  is a basis of  $\tau$ , or generates  $\tau$ , if for every  $x \in U \in \tau$ , there exists  $B \in \beta$  such that  $x \in B \subseteq U$ .

*Example.* The collection of all open balls in  $\mathbb{R}^n$  form a basis of the usual topology.

**Lemma 1.2.** Let  $X$  be equipped with the topologies  $\tau$  and  $\tau'$ , and let  $\beta$  and  $\beta'$  be the respective bases of these topologies. Then,  $\tau$  is finer than  $\tau'$  if and only if given  $x \in B' \in \beta'$ , there exists  $x \in B \in \beta$  such that  $B \subseteq B'$ .

*Example.* The collections of open balls in  $\mathbb{R}^n$  generate the same topology as the collection of all open rectangles in  $\mathbb{R}^n$ .

*Example.* Consider the topologies on  $\mathbb{R}$  generated by the following bases.

1.  $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ .
2.  $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ .
3.  $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$  where  $K = \{1/n : n \in \mathbb{Z}\}$ .

We call the topology generated by  $\beta_2$  the lower limit topology, denoted  $\mathbb{R}_\ell$ . The topology generated by  $\beta_3$  is denoted  $\mathbb{R}_K$ . Both of these are strictly finer than the standard topology.

**Definition 1.5.** A sub-basis for some topology on  $X$  is a collection  $\rho$  of subsets of  $X$  whose union is the whole of  $X$ . The topology generated by  $\rho$  is defined to be the topology generated by the collection of all finite intersections of elements of  $\rho$ .

### 1.3 Product topology

**Definition 1.6.** Let  $(X_1, \tau_1), (X_2, \tau_2)$  be topological spaces. Then  $\tau_1 \times \tau_2$  generates the product topology on  $X_1 \times X_2$ .

*Example.* The product topology on  $\mathbb{R} \times \mathbb{R}$ , where  $\mathbb{R}$  is equipped with the standard topology, coincides with the standard topology on  $\mathbb{R}^2$ .

**Lemma 1.3.** If  $\beta_1, \beta_2$  are bases of the topologies  $\tau_1, \tau_2$ , then  $\beta_1 \times \beta_2$  and  $\tau_1 \times \tau_2$  generate the same product topology.

*Proof.* Given  $(x_1, x_2) \in U$  where  $U \subseteq X_1 \times X_2$  is open in the product topology, recall that  $U$  can be written as a union of the basic open sets  $U_{1i} \times U_{2i}$ , where  $U_{1i} \in \tau_1$  and  $U_{2i} \in \tau_2$ . Suppose that  $(x_1, x_2) \in U_1 \times U_2$ . Thus, we can choose  $B_1 \in \beta_1, B_2 \in \beta_2$  such that  $x_1 \in B_1 \subseteq U_1$  and  $x_2 \in B_2 \subseteq U_2$ . Thus,  $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$ .  $\square$

**Definition 1.7.** The projection maps are defined as  $\pi_i: X_1 \times \cdots \times X_k \rightarrow X_i, (x_1, \dots, x_k) \mapsto x_i$ .

**Lemma 1.4.** The collection of elements of the form  $\pi_1^{-1}(U_1)$  or  $\pi_2^{-1}(U_2)$ , where  $U_1 \in \tau_1$  and  $U_2 \in \tau_2$ , forms a sub-basis of the product topology on  $X_1 \times X_2$ .

*Proof.* Note that  $\pi_1^{-1}(X_1) = X_1 \times X_2$ . Now it is easy to see that finite intersections of elements of the form  $U_1 \times X_2$  or  $X_1 \times U_2$  where  $U_1, U_2$  are open, are all of the form  $U_1 \times U_2$  which is precisely a basis of the product topology.  $\square$

**Corollary 1.4.1.** We can restrict ourselves to the sub-basis of elements of the form  $\pi_1^{-1}(B_1)$  or  $\pi_2^{-1}(B_2)$ , where  $B_1 \in \beta_1, B_2 \in \beta_2$  for some bases  $\beta_1, \beta_2$  of  $\tau_1, \tau_2$ .

### 1.4 Subspace topology

**Definition 1.8.** Let  $(X, \tau)$  be a topological space, and let  $Y \subset X$ . Then the collection  $U \cap Y$  for all  $U \in \tau$  comprises the subspace topology  $\tau_Y$  on  $Y$  induced by the topology  $\tau$  on  $X$ .

**Lemma 1.5.** *If  $\beta$  is a basis for the topology on  $X$ , and  $Y \subset X$ , then the collection  $B \cap Y$  for all  $B \in \beta$  generates the subspace topology on  $Y$ .*

**Lemma 1.6.** *An open set of  $Y$  is open in  $X$  if  $Y$  is open in  $X$ .*

*Proof.* Let  $U \subset Y$  be open in  $Y$ , then  $U = V \cap Y$  for some open set  $V$  in  $X$ . If additionally  $Y$  is open in  $X$ , this immediately shows that  $U$  is open in  $X$ .  $\square$

**Theorem 1.7.** *Let  $(X, \tau_X)$ ,  $(Y, \tau_Y)$  be topological spaces, and let  $A \subseteq X$ ,  $B \subseteq Y$ . Then, there are two ways of assigning a natural topology on  $A \times B$ .*

1. *Take the product topology on  $X \times Y$ , and consider the subspace topology induced by it on  $A \times B$ .*
2. *Take the subspace topologies on  $A$  induced by  $\tau_X$ ,  $B$  induced by  $\tau_Y$ , and consider the product topology generated by them on  $A \times B$ .*

*These two methods generate the same topology on  $A \times B$ .*

*Proof.* Open sets in 1 look like  $(U \times V) \cap (A \times B)$ , where  $U \in \tau_X$ ,  $V \in \tau_Y$ . Open sets in 2 look like  $(U' \cap A) \times (V' \cap B)$ , where  $U' \in \tau_X$ ,  $V' \in \tau_Y$ , which can be rewritten as  $(U' \times V') \cap (A \times B)$ . It is easy to see that these describe precisely the same sets.  $\square$

## 1.5 Order topology

**Definition 1.9.** Let  $X$  be a set with a simple order  $<$ . Then the collection of sets of the form  $(a, b)$ ,  $[a_0, b)$ ,  $(a, b_0]$  where  $a_0$  is the minimal element of  $X$ ,  $b_0$  is the maximal element of  $X$ , generate the order topology on  $X$ .

*Example.* The order topology on  $\mathbb{N}$  is precisely the discrete topology.

**Definition 1.10.** Let  $X_1, X_2$  be simply ordered sets. The dictionary order on  $X_1 \times X_2$  is defined as follows:  $(x_1, x_2) < (y_1, y_2)$  if  $x_1 < y_1$ , or if  $x_1 = y_1$  and  $x_2 < y_2$ .

*Example.* Consider  $X = \{1, 2\} \times \mathbb{N}$ , where both  $\{1, 2\}$  and  $\mathbb{N}$  are endowed with the discrete topology. Note that the product topology on  $X$  is the discrete topology.

Now consider the dictionary order on  $X$ . Here,  $(1, 1)$  is the smallest element, so we can list the elements of  $X$  in ascending order. Note that every  $(1, m) < (2, n)$ , for all  $m, n \in \mathbb{N}$ . Now, note that all singletons  $\{(1, m)\}$  are open in the order topology on  $X$ . The same is true for the singletons  $\{(1, n)\}$  for all  $n > 1$ . However, the singleton  $\{(2, 1)\}$  is *not* open in the order topology.

*Example.* Consider  $\mathbb{R}$  with the usual topology, and  $X = [0, 1) \cup \{2\}$ . Then,  $\{2\}$  is open in the subspace topology on  $X$ , but it is not open in the order topology on  $X$ .

**Lemma 1.8.** *The open rays of the form  $(a, +\infty)$  and  $(-\infty, a)$  in  $X$  form a sub-basis of the order topology on  $X$ .*

*Proof.* Note that  $(a, b) = (-\infty, b) \cap (a, +\infty)$ ,  $[a_0, b) = (-\infty, b)$ , and  $(a, b_0] = (a, +\infty)$ .  $\square$

**Definition 1.11.** Let  $X$  be a simply ordered set, and  $Y \subseteq X$ . Then, we say that  $Y$  is convex in  $X$  if given  $a, b \in Y$  such that  $a < b$ , the interval  $(a, b) = \{x \in X : a < x < b\} \subseteq Y$ .

**Theorem 1.9.** *Let  $Y$  be convex in  $X$ . Then, the subspace topology and the order topology on  $Y$  induced from the order topology on  $X$  coincide.*

## 1.6 Closed sets

**Definition 1.12.** Let  $(X, \tau)$  be a topological space. A set  $F \subseteq X$  is said to be closed in  $X$  if  $F^c = X \setminus F \in \tau$ .

*Example.* The sets  $\emptyset, X$  are closed in every topological space  $(X, \tau)$ .

*Example.* In a set equipped with the discrete topology, every set is both open and closed.

**Lemma 1.10.** *Arbitrary intersections, and finite unions of closed sets are closed.*

**Theorem 1.11.** *Let  $(X, \tau)$  be a topological space, and let  $Y \subset X$  be equipped with the subspace topology. Then, a set  $F \subseteq Y$  is closed in  $Y$  if and only if  $F = Y \cap G$ , where  $G$  is closed in  $X$ .*

*Proof.* Let  $F \subset Y$ . Now,  $F$  is closed in  $Y$ ,  $Y \setminus F = Y \cap F^c$  is open in  $Y$ ,  $Y \cap F^c = Y \cap U$  where  $U$  is open in  $X$ ,  $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$  where  $U^c$  is closed. The steps are reversible.  $\square$

**Lemma 1.12.** *A closed set of  $Y$  is closed in  $X$  if  $Y$  is closed in  $X$ .*

## 1.7 Interiors and closures

**Definition 1.13.** Let  $A \subseteq X$  where  $(X, \tau)$  is a topological space.

1. The interior of  $A$  is defined as the union of all open sets contained in  $A$ . This is denoted by  $A^\circ$ .
2. The closure of  $A$  is defined as the intersection of all closed sets containing  $A$ . This is denoted by  $\overline{A}$ .

*Remark.* The interior of a set is open, and the closure of a set is closed.

**Lemma 1.13.** Let  $Y \subset X$  be topological spaces, and let  $A \subseteq Y$ . Also let  $\overline{A}_X, \overline{A}_Y$  denote the closures of  $A$  in  $X, Y$  respectively. Then,  $\overline{A}_Y = \overline{A}_X \cap Y$ .

**Theorem 1.14.** Let  $A \subset X$ . Then,

1.  $x \in \overline{A}$  if and only if every open set containing  $x$  has non-empty intersection with  $A$ .
2.  $x \in \overline{A}$  if and only if every basic open set containing  $x$  has non-empty intersection with  $A$ , given that the topology on  $X$  is generated by those basic open sets.

**Definition 1.14.** Let  $A \subseteq X$  where  $(X, \tau)$  is a topological space. We say that  $x \in X$  is a limit point of  $X$  if for every open set  $U$  containing  $x$ , the deleted neighbourhood  $U \setminus \{x\}$  has non-empty intersection with  $A$ . The set of limit points of  $A$  is denoted by  $A'$ .

*Example.* Let  $X$  be a set endowed with the discrete topology. Then, given any set  $A \subseteq X$ , we have  $A' = \emptyset$ .

**Lemma 1.15.** A closed set contains all its limit points.

*Proof.* Let  $F \subseteq X$  be closed in  $X$ , and let  $x \in F'$ . Then given any open set containing  $x$ , we have  $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$ , hence  $x \in \overline{F} = F$ .  $\square$

**Lemma 1.16.** Let  $A \subseteq X$  where  $(X, \tau)$  is a topological space. Then,  $\overline{A} = A \cup A'$ .

*Proof.* It is clear that  $\overline{A} \supseteq A \cup A'$ . Now pick  $x \in \overline{A}$ . If  $x \notin A$ , then we know that given any open neighbourhood  $U$  of  $x$ , we have non-empty  $U \cap A$ . Furthermore, this intersection can never contain  $x$ , hence  $x \in A'$ . This proves that  $\overline{A} \subseteq A \cup A'$ .  $\square$

## 1.8 Convergence of sequences

**Definition 1.15.** Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of points from  $(X, \tau)$ , and let  $x \in X$ . We say that this sequence converges to  $x$ , denoted  $x_n \rightarrow x$ , if every open neighbourhood of  $x$  contains the tail of this sequence. In other words, given  $U \in \tau$  such that  $x \in U$ , there must exist  $N \in \mathbb{N}$  such that  $x_n \in U$  for all  $n \geq N$ .

*Example.* Let  $X = \{a, b, c\}$ , and  $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ . Then, the constant sequence of  $b$ 's converges to all three points  $a, b, c$ .

*Example.* Let  $X = \mathbb{R}$ , and  $\tau$  be the collection of all intervals  $(-a, a)$  together with  $\emptyset, \mathbb{R}$ . Then, the constant sequence of 0's converges to every point in  $\mathbb{R}$ .

**Definition 1.16.** Let  $(X, \tau)$  be a topological space. We say that this topological space is Hausdorff if given any two distinct points  $x, y \in X$ , there exist open sets  $U, V \in \tau$  such that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ .

*Example.* The real numbers under the standard topology is Hausdorff.

**Theorem 1.17.** Let  $(X, \tau)$  be a Hausdorff topological space, and let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of points in  $X$ . Then, this sequence can converge to at most one point in  $X$ .

*Proof.* Suppose that  $\{x_n\}_{n=1}^{\infty}$  converges to distinct points  $x, y \in X$ . Then there exist disjoint open neighbourhoods  $U, V$  such that  $x \in U$ ,  $y \in V$ . Convergence means that both  $U$  and  $V$  contain a tail of the sequence, which is a contradiction.  $\square$

**Lemma 1.18.** The singleton sets in a Hausdorff space are closed.

*Proof.* Let  $x \in X$  where  $(X, \tau)$  is Hausdorff. Pick  $y \neq x$ , whence there exist  $U_y, V_y \in \tau$ , such that  $x \in U_y$ ,  $y \in V_y$ , and  $U_y \cap V_y = \emptyset$ . In particular,  $\{x\} \cap V_y = \emptyset$ . We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open.  $\square$

**Theorem 1.19.** The topology induced by a metric is Hausdorff.

*Proof.* Given a metric space  $X$  and distinct points  $x, y \in X$ , we set  $r = |x - y|$ ,  $U = B(x, r/3)$ ,  $V = B(y, r/3)$ .  $\square$

## 2 Continuous maps

**Definition 2.1.** Let  $f: X \rightarrow Y$  be a function between the topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ . We say that  $f$  is continuous if for every  $U \in \tau_Y$ , we have  $f^{-1}(U) \in \tau_X$ . In other words, the pre-image of every open set in  $Y$  must be open in  $X$ .

**Lemma 2.1.** A function  $f: X \rightarrow Y$  is continuous if and only if given a base  $\beta$  of  $Y$ , we have  $f^{-1}(U) \in \tau_X$  for every  $U \in \beta$ .

*Example.* The identity function  $\text{id}: \mathbb{R}_\ell \rightarrow \mathbb{R}$  is continuous, while the identity function  $\text{id}: \mathbb{R} \rightarrow \mathbb{R}_\ell$  is not. This is because the topology on  $\mathbb{R}_\ell$  is strictly finer than that on  $\mathbb{R}$ .

**Lemma 2.2.** A function  $f: X \rightarrow Y$  is continuous if and only if for every closed set  $F \subseteq Y$ , we have  $f^{-1}(F)$  closed in  $X$ .

**Lemma 2.3.** A function  $f: X \rightarrow Y$  is continuous if and only if given any  $x \in X$  and an open set  $V \subseteq Y$  such that  $f(x) \in V$ , there exists an open set  $U \subseteq X$  such that  $x \in U$ ,  $f(U) \subseteq V$ .

**Theorem 2.4.** The composition of continuous functions is continuous.

### 2.1 Restricting and enlarging the domain

**Lemma 2.5.** Let  $f: X \rightarrow Y$  be continuous, and let  $A \subset X$ . Then the restriction of  $f$  to  $A$  is continuous.

**Theorem 2.6.** Let  $f: X \rightarrow Y$ , and let  $X$  be the union of the collection of open sets  $\{A_\lambda\}_{\lambda \in \Lambda}$ . If the restrictions of  $f$  to each  $A_\lambda$  are continuous, then  $f$  is continuous.

*Proof.* Pick  $x \in X$ , hence  $x \in A_\lambda$  for some  $\lambda \in \Lambda$ . Now if  $f(x) \in V \subset Y$ , where  $V$  is open in  $Y$ , then the continuity of the restriction of  $f$  to  $A_\lambda$  gives us an open set  $U \subseteq A_\lambda$  such that  $f(U) \subseteq V$ . Finally since  $A_\lambda$  is open in  $X$ , so is  $U$ .  $\square$

**Definition 2.2.** Let  $X$  be the union of the collection of open sets  $\{A_\lambda\}_{\lambda \in \Lambda}$ . We say that this collection is a locally finite cover of  $X$  if given  $x \in X$ , there exists a neighbourhood  $U$  of  $x$  such that  $U \cap A_\lambda$  is non-empty for only finitely many  $\lambda \in \Lambda$ .



**Theorem 2.7.** *Let  $f: X \rightarrow Y$ , and let  $\{F_\lambda\}_{\lambda \in \Lambda}$  be a locally finite collection of closed sets covering  $X$ . If the restrictions of  $f$  to each  $F_\lambda$  are continuous, then  $f$  is continuous.*

**Corollary 2.7.1** (Pasting lemma). *Let  $X = A \cup B$ , with  $A, B$  closed in  $X$ . Let  $f: A \rightarrow Y$ ,  $g: B \rightarrow Y$  be continuous, with  $f(x) = g(x)$  on  $A \cap B$ . Then the function  $h: X \rightarrow Y$ , defined by  $x \mapsto f(x)$  on  $A$  and  $x \mapsto g(x)$  on  $B$ , is continuous.*

**Definition 2.3.** A path is a continuous function  $\gamma: [0, 1] \rightarrow X$ .

**Lemma 2.8.** *Two paths  $\gamma_1, \gamma_2$  can be concatenated when  $\gamma_1(1) = \gamma_2(0)$ .*

## 2.2 Projection maps

**Theorem 2.9.** *The projection maps  $\pi_i: X_1 \times \cdots \times X_k \rightarrow X_i$  are continuous, when the domain is equipped with the product topology. Furthermore, the product topology is the coarsest topology on the domain for which the projection maps are continuous.*

**Lemma 2.10.** *Let  $f: A \rightarrow X_1 \times \cdots \times X_k$ , where the co-domain is equipped with the product topology. Then,  $f$  is continuous if and only if the component functions  $f_i = \pi_i \circ f$  are continuous.*

*Proof.* Note that if  $f$  is continuous, the compositions  $\pi_i \circ f$  are immediately continuous. Conversely suppose that each  $f_i$  is continuous, and write

$$f(t) = (f_1(t), \dots, f_k(t)).$$

The sets  $U_1 \times \cdots \times U_k$ , where each  $U_i \subseteq X_i$  is open, form a basis of the co-domain. Furthermore, their pre-images under  $f$  are  $f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k)$ , which are open in  $A$ . This shows that  $f$  is continuous.  $\square$

## 2.3 Homeomorphisms

**Definition 2.4.** Let  $f: X \rightarrow Y$  be a function between the topological spaces  $(X, \tau_X)$  and  $(Y, \tau_Y)$ . We say that  $f$  is a homeomorphism if  $f$  is continuous,  $f$  is bijective, and  $f^{-1}$  is continuous. We also say that  $X$  and  $Y$  are homeomorphic when such a homeomorphism between them exists.

*Example.* The interval  $(-1, 1)$  is homeomorphic to  $\mathbb{R}$ ; for instance, the map  $x \mapsto \tan(2x/\pi)$  on  $(-1, 1)$  is a homeomorphism. A simpler construction is the map  $x \mapsto x/(1 - x^2)$ .