

MA3101

# Analysis III

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## 1 Euclidean spaces

### 1.1 $\mathbb{R}^n$ as a vector space

We are familiar with the vector space  $\mathbb{R}^n$ , with the standard inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is defined as

$$\|\mathbf{x} - \mathbf{y}\|^2 = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = \sum_{k=1}^n (x_k - y_k)^2.$$

**Exercise 1.1.** What are all possible inner products on  $\mathbb{R}^n$ ?

*Solution.* Note that an inner product is a bilinear, symmetric map such that  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ , and  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  if and only if  $\mathbf{x} = \mathbf{0}$ . Thus, an product map on  $\mathbb{R}^n$  is completely and uniquely determined by the values  $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = a_{ij}$ . Let  $A$  be the  $n \times n$  matrix with entries  $a_{ij}$ . Note that  $A$  is a real symmetric matrix with positive entries. Now,

$$\langle \mathbf{x}, \mathbf{e}_j \rangle = x_1 a_{1j} + \cdots + x_n a_{nj} = \mathbf{x}^\top \mathbf{a}_j,$$

where  $\mathbf{a}_j$  is the  $j^{\text{th}}$  column of  $A$ . Thus,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{a}_1 y_1 + \cdots + \mathbf{x}^\top \mathbf{a}_n y_n = \mathbf{x}^\top A \mathbf{y}.$$

Furthermore, any choice of real symmetric  $A$  with positive entries produces an inner product.

**Theorem 1.1** (Cauchy-Schwarz). *Given two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ , we have*

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

*Proof.* This is trivial when  $\mathbf{w} = \mathbf{0}$ . When  $\mathbf{w} \neq \mathbf{0}$ , set  $\lambda = \langle \mathbf{v}, \mathbf{w} \rangle / \|\mathbf{w}\|^2$ . Thus,

$$0 \leq \|\mathbf{v} - \lambda \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\lambda \langle \mathbf{v}, \mathbf{w} \rangle + \lambda^2 \|\mathbf{w}\|^2.$$

Simplifying,

$$0 \leq \|\mathbf{v}\|^2 - \frac{|\langle \mathbf{v}, \mathbf{w} \rangle|^2}{\|\mathbf{w}\|^2}.$$

This gives the desired result. Clearly, equality holds if and only if  $\mathbf{v} = \lambda \mathbf{w}$ .  $\square$

**Theorem 1.2** (Triangle inequality). *Given two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ , we have*

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

*Proof.* Write

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \leq \|\mathbf{v}\|^2 + 2|\langle \mathbf{v}, \mathbf{w} \rangle| + \|\mathbf{w}\|^2.$$

Applying Cauchy-Schwarz gives

$$\|\mathbf{v} + \mathbf{w}\|^2 \leq (\|\mathbf{v}\| + \|\mathbf{w}\|)^2.$$

Equality holds if and only if  $\mathbf{v} = \lambda \mathbf{w}$  for  $\lambda \geq 0$ .  $\square$

## 1.2 $\mathbb{R}^n$ as a metric space

Our previous observations allow us to define the standard metric on  $\mathbb{R}^n$ , seen as a point set.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

**Definition 1.1.** For any  $\delta > 0$ , the set

$$B_\delta(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{y}) < \delta\}$$

is called the open ball centred at  $\mathbf{x} \in \mathbb{R}^n$  with radius  $\delta$ . This is also called the  $\delta$  neighbourhood of  $\mathbf{x}$ .

**Definition 1.2.** A set  $U$  is open in  $\mathbb{R}^n$  if for every  $\mathbf{x} \in U$ , there exists an open ball  $B_\delta(\mathbf{x}) \subset U$ .

*Remark.* Every open ball in  $\mathbb{R}^n$  is open.

*Remark.* Both  $\emptyset$  and  $\mathbb{R}^n$  are open.

**Definition 1.3.** A set  $F$  is closed in  $\mathbb{R}^n$  if its complement  $\mathbb{R}^n \setminus F$  is open in  $\mathbb{R}^n$ .

*Remark.* Both  $\emptyset$  and  $\mathbb{R}^n$  are closed.

*Remark.* Finite sets in  $\mathbb{R}^n$  are closed.

**Theorem 1.3.** *Unions and finite intersections of open sets are open.*

**Corollary 1.3.1.** *Intersections and finite unions of closed sets are closed.*

**Definition 1.4.** An interior point  $x$  of a set  $S \subseteq \mathbb{R}^n$  is such that there is a neighbourhood of  $x$  contained within  $S$ .

*Example.* Every point in an open set is an interior point by definition. The interior of a set is the largest open set contained within it.

**Definition 1.5.** An exterior point  $x$  of a set  $S \subseteq \mathbb{R}^n$  is an interior point of the complement  $\mathbb{R}^n \setminus S$ .

**Definition 1.6.** A boundary point of a set is neither an interior point, nor an exterior point.

*Example.* The boundary of the unit open ball  $B_1(0) \subset \mathbb{R}^n$  is the sphere  $S^{n-1}$ .

**Definition 1.7.** A limit point  $x$  of a set  $S \subseteq \mathbb{R}^n$  is such that every neighbourhood of  $x$  contains a point from  $S$  other than itself.

**Definition 1.8.** The closure of a set  $S \subseteq \mathbb{R}^n$  is the union of  $S$  and its limit points.

*Remark.* The closure of a set is the smallest closed set containing it.

**Lemma 1.4.** *Every open set in  $\mathbb{R}^n$  is a union of open balls.*

*Proof.* Let  $U \subseteq \mathbb{R}^n$  be open. Thus, for every  $\mathbf{x} \in \mathbb{R}^n$ , we can choose  $\delta_{\mathbf{x}} > 0$  such that  $B_{\delta_{\mathbf{x}}}(\mathbf{x}) \subset U$ . The union of all such open balls is precisely the set  $U$ .  $\square$

### 1.3 $\mathbb{R}^n$ as a topological space

**Definition 1.9.** A topology on a set  $X$  is a collection  $\tau$  of subsets of  $X$  such that

1.  $\emptyset \in \tau$
2.  $X \in \tau$
3. Arbitrary union of sets from  $\tau$  belong to  $\tau$ .
4. Finite intersections of sets from  $\tau$  belong to  $\tau$ .

Sets from  $\tau$  are called open sets.

*Example.* The discrete topology on a set  $X$  is one where every singleton set is open. This is the topology induced by the discrete metric,

$$d_{\text{discrete}}: X \times X \rightarrow \mathbb{R}, \quad (x, y) \mapsto \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

**Theorem 1.5.** The Euclidean metric  $d$  induces the standard topology  $\tau_d$  on  $\mathbb{R}^n$ .

**Definition 1.10.** Given a topological space  $(X, \tau)$  and a subset  $Y \subseteq X$ , the collection of sets  $U \cap Y$  where  $U \in \tau$  is a topology  $\tau_Y$  on  $Y$ . We call this collection the subspace topology on  $Y$ , induced by the topology on  $X$ .

**Definition 1.11.** A set  $K \subset X$  in a topological space is compact if every open cover of  $K$  has a finite sub-cover. That is, for every collection of open sets  $\{U_\alpha\}_{\alpha \in A}$  such that  $K$  is contained in their union, there exists a finite sub-collection  $U_{\alpha_1}, \dots, U_{\alpha_k}$  such that  $K$  is also contained in their union.

*Example.* All finite sets are compact.

*Example.* Given a convergent sequence of real numbers  $x_n \rightarrow x$ , the collection  $\{x_n\}_{n \in \mathbb{N}} \cup \{x\}$  is compact.

*Example.* In  $\mathbb{R}^n$ , compact sets are precisely those sets which are closed and bounded. This is the Heine-Borel Theorem.

**Theorem 1.6.** The closed intervals  $[a, b] \subset \mathbb{R}$  are compact.

*Remark.* This can be extended to show that any  $k$ -cell  $[a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$  is compact.

*Proof.* Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $[a, b]$ , and suppose that  $I_1 = [a, b]$  has no finite sub-cover. Then, at least one of the intervals  $[a, (a+b)/2]$  and  $[(a+b)/2, b]$  must not have a finite sub-cover; pick one and call it  $I_2$ . Similarly, one of the halves of  $I_2$  must not have a finite sub-cover; call it  $I_3$ . In this process, we generate a sequence of closed intervals  $I_1 \supset I_2 \supset \dots$ , none of which have a finite sub-cover. The length of each interval is given by

$$|I_n| = 2^{-n+1} \|b - a\| \rightarrow 0.$$

Now, pick a sequence of points  $\{x_n\}$  where each  $x_n \in I_n$ . Then,  $\{x_n\}$  is a Cauchy sequence. To see this, given any  $\epsilon > 0$ , we can find sufficiently large  $n_0$  such that  $2^{-n_0+1} \|b - a\| < \epsilon$ . Thus,  $x_n \in I_n \subset I_{n_0}$  for all  $n \geq n_0$ , which means that for any  $m, n \geq n_0$ , we have  $x_m, x_n \in I_{n_0}$  forcing<sup>1</sup>

$$\|x_m - x_n\| \leq |I_{n_0}| = 2^{-n_0+1} \|b - a\| < \epsilon.$$

From the completeness of  $\mathbb{R}$ , this sequence must converge in  $\mathbb{R}$ , specifically in  $[a, b]$ . Thus,  $x_n \rightarrow x$  for some  $x \in [a, b]$ . It can also be seen that the limit  $x \in I_n$  for all  $n \in \mathbb{N}$ ; if not, say  $x \notin I_{n_0}$ , then  $x \in [a, b] \setminus I_{n_0}$  which is open, hence there is an open interval such that  $(x - \delta, x + \delta) \cap I_{n_0} = \emptyset$ . However,  $I_{n_0}$  contains all  $x_{n \geq n_0}$ , thus this  $\delta$ -neighbourhood of  $x$  would miss out a tail of  $\{x_n\}$ .

Now, pick the open set  $U \in \{U_\alpha\}$  which covers the point  $x$ . Thus,  $x \in U$  so  $U$  contains some non-empty open interval  $(x - \delta, x + \delta)$  around  $x$ . Choose  $n_0$  such that  $2^{-n_0+1} \|b - a\| < \delta$ ; this immediately gives  $I_{n_0} \subseteq (x - \delta, x + \delta) \subset U$ . This contradicts that fact that  $I_{n_0}$  has no finite sub-cover from  $\{U_\alpha\}$ , completing the proof.  $\square$

*Remark.* The fact that Cauchy sequences in  $\mathbb{R}^n$  converge isn't immediately obvious; it is a consequence of the completeness of  $\mathbb{R}^n$ . Start by noting that  $\mathbb{R}$  has the Least Upper Bound property, from which the Monotone Convergence Theorem follows; every monotonic, bounded sequence of reals converges. It can also be shown that any sequence of reals with contain a monotone subsequence, from which it follows that every bounded sequence contains a convergent subsequence (Bolzano-Weierstrass). Finally, it can be shown that if a subsequence of a Cauchy sequence converges, then the entire sequence also converges to the same limit, giving us the desired result for  $\mathbb{R}$ . For sequence in  $\mathbb{R}^n$ , we may apply this coordinate-wise to obtain the result.

**Lemma 1.7.** *Compact sets in  $\mathbb{R}^n$  are closed and bounded.*

*Proof.* Consider a compact set  $K \subset \mathbb{R}^n$ . Let  $x \in \mathbb{R}^n \setminus K$ , and let  $y \in K$ . Since  $x \neq y$ , we choose open balls  $U_y$  around  $y$  and  $V_y$  around  $x$  such that  $U_y \cap V_y = \emptyset$ . Repeating this for all  $y \in K$ , we generate an open cover  $\{U_y\}$  of  $K$  consisting of open balls. The compactness of  $K$  guarantees that this has a finite sub-cover, i.e. there is a finite set  $Y$  such that the collection  $\{U_y\}_{y \in Y}$  covers  $K$ . As a result, the finite intersection of all  $V_y$  for  $y \in Y$  is contained within  $\mathbb{R}^n \setminus K$ . Thus,  $x$  is in the exterior of  $K$ . Since  $x$  was chosen arbitrarily from  $\mathbb{R}^n \setminus K$ , we see that  $K$  is closed.

Now, consider the open cover  $\{B_1(x)\}_{x \in K}$ , and extract a finite sub-cover of unit open balls. The distance between any two points in  $K$  is at most the maximum distance between the centres of any two balls in our sub-cover, plus two.  $\square$

<sup>1</sup>If  $x_1, x_2 \in [a, b]$  with  $x_1 < x_2$ , note that  $a \leq x_1 < x_2 \leq b$ , so

$$|x_2 - x_1| = x_2 - x_1 \leq b - a.$$

**Lemma 1.8.** *The intersection of a closed set and a compact set is compact.*

*Proof.* Let  $F \subseteq \mathbb{R}^n$  be closed and let  $K \subseteq \mathbb{R}^n$  be compact. Suppose that the open cover  $\{U_\alpha\}$  of  $F \cap K$  has no finite sub-cover. Now the complement  $U = F^c$  is open in  $\mathbb{R}^n$ , hence the collection  $\{U_\alpha\} \cup \{U\}$  is an open cover of  $K$ , and hence must admit a finite sub-cover of  $K$ . In particular, this must be a finite sub-cover of  $F \cap K$ . However, we can remove the set  $U$  from this sub-cover since it shares no element with  $F \cap K$ ; as a result, our sub-cover must be a finite sub-collection of sets  $U_\alpha$ , contradicting our assumption. This shows that  $F \cap K$  is compact.  $\square$

**Lemma 1.9** (Finite intersection property). *Let  $\{K_\alpha\}$  be a collection of compact sets in  $\mathbb{R}^n$  which have the property that any finite intersection of them is non-empty. Then,*

$$\bigcap_{\alpha} K_{\alpha} \neq \emptyset.$$

*Proof.* Suppose to the contrary that the intersection of all  $K_\alpha$  is empty. Fix an index  $\beta$ , and note that no element of  $K_\beta$  lies in every  $K_\alpha$ . Set  $J_\alpha = K_\alpha^c$ , whence the collection  $\{J_\alpha : \alpha \neq \beta\}$  is an open cover of  $K_\beta$ . This must admit a finite sub-cover  $\{J_{\alpha_1}, \dots, J_{\alpha_k}\}$  of  $K_\beta$ . Thus, we must have

$$K_\beta^c \cup J_{\alpha_1} \cup \dots \cup J_{\alpha_k} = \mathbb{R}^n.$$

This immediately gives the contradiction

$$K_\beta \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_k} = \emptyset. \quad \square$$

**Theorem 1.10** (Heine-Borel). *Compact sets in  $\mathbb{R}^n$  are precisely those that are closed and bounded.*

*Proof.* Given a compact set in  $\mathbb{R}^n$ , we have already shown that it must be closed and bounded. Next, if  $F \subset \mathbb{R}^n$  is closed and bounded, it can be enclosed within a  $k$ -cell which we know is compact. Thus,  $F$  is the intersection of the closed set  $F$  and the compact  $k$ -cell, proving that  $F$  must be compact.  $\square$

## 1.4 Continuous maps

**Definition 1.12.** A map  $f: X \rightarrow Y$  is continuous if the pre-image of every open set from  $Y$  is open in  $X$ .

**Lemma 1.11.** *A map  $f: X \rightarrow Y$  is continuous if the pre-image of every closed set from  $Y$  is closed in  $X$ .*

**Theorem 1.12.** *The projection maps  $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{x} \mapsto x_i$  are continuous.*

*Proof.* Let  $U \subseteq \mathbb{R}$  be open; we claim that  $\pi_i^{-1}(U)$  is open. Pick  $\mathbf{x} \in \pi_i^{-1}(U)$ , and note that  $\pi_i(\mathbf{x}) = x_i \in U$ . Thus, there exists  $\delta > 0$  such that  $(x_i - \delta, x_i + \delta) \subset U$ . Now examine  $B_\delta(\mathbf{x})$ ; for any point  $\mathbf{y}$  within this open ball, we have  $d(\mathbf{x}, \mathbf{y}) < \delta$  hence

$$|x_i - y_i|^2 \leq \sum_{k=1}^n (x_k - y_k)^2 = d(\mathbf{x}, \mathbf{y})^2 < \delta^2.$$

In other words,  $\pi_i(\mathbf{y}) = y_i \in (x_i - \delta, x_i + \delta)$ , hence  $\pi_i B_\delta(\mathbf{x}) \subseteq (x_i - \delta, x_i + \delta) \subset U$ . Thus, given arbitrary  $\mathbf{x} \in \pi_i^{-1}(U)$ , we have found an open ball  $B_\delta(\mathbf{x}) \subset \pi_i^{-1}(U)$ .  $\square$

**Lemma 1.13.** *Finite sums, products, and compositions of continuous functions are continuous.*

**Theorem 1.14.** *All polynomial functions of the coordinates in  $\mathbb{R}^n$  are continuous.*

*Example.* The unit sphere  $S^{n-1} \subset \mathbb{R}^n$  is closed. It is by definition the pre-image of the singleton closed set  $\{1\}$  under the continuous map

$$\mathbf{x} \mapsto x_1^2 + \cdots + x_n^2.$$