Exercises from a course* on

Representation Theory of Finite Groups

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The base field is $\mathbb{C}$.

Problem 1

Show that every irreducible representation of a finite abelian group is 1-dimensional.

Solution: Let G be a finite abelian group, and let (σ, V) be an irreducible representation of G. Then for each $g \in G$, the map $\sigma(g) \in \mathrm{GL}(V)$ is G-invariant; observe that for $g' \in G$,

$$\sigma(g)(\sigma(g')v) = (\sigma(g)\sigma(g'))(v) = \sigma(gg')(v) = \sigma(g'g)(v) = \sigma(g')(\sigma(g)(v)).$$

Thus, by Schur's Lemma, each $\sigma(g)$ must be a scalar map of the form $v \mapsto \lambda_g v$. As a result, every 1-dimensional subspace of V is G-stable. For V to be irreducible, it must be the case that $\dim(V) = 1$.

Problem 2

Let V, W be vector spaces on which G acts. Let $\mathrm{Hom}(V, W)$ denote the vector space of all linear maps from V to W. Define an action of G on $\mathrm{Hom}(V, W)$ as follows. Let $g \in G$ and $f \in \mathrm{Hom}(V, W)$. Then define $gf \in \mathrm{Hom}(V, W)$ by

$$(gf)(v) = gf(g^{-1}v).$$

(a) Show that this indeed defines an action of G on $\mathrm{Hom}(V, W)$.
(b) Suppose now that $W = \mathbb{C}$ (the action of G being trivial). Then $\mathrm{Hom}(V, W)$ is the dual space V^* of V. It is called the representation dual to V. Compute the character of V^* in terms of the character of V.

Solution:

(a) Note that $(1f)(v) = 1f(1v) = f(v)$ so $1f = f$. Next, for $g_1, g_2 \in G$, we have

$$\begin{aligned}(g_1(g_2 f))(v) &= g_1((g_2 f)(g_1^{-1}v)) = g_1(g_2(f(g_2^{-1}(g_1^{-1}v)))) \\ &= (g_1 g_2)\Big(f\Big((g_1 g_2)^{-1}v\Big)\Big) = ((g_1 g_2)f)(v).\end{aligned}$$

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(b) Let χ be the character of V, let $g \in G$, and let $\{v_i\}$ be a basis of V with respect to which the action of g is diagonal (this is permitted since the base field is $\mathbb{C}$). Then, each $gv_i = \lambda_i v_i$ for some constants $\{\lambda_i\}$. Observe that $\{v_i^*\}$ forms a basis of V^*, where each v_i^* is the map determined by $v_j \mapsto \delta_{ij} v_i$. Then,

$$(gv_i^*)(v_j) = v_i^*(g^{-1}v_j) = v_i^*(\lambda_i^{-1}v_j) = \lambda_i^{-1}v_i^*(v_j),$$

so $gv_i^* = \lambda_i^{-1}v_i^*$. Thus, the matrix of g in V^* is precisely $\sigma^*(g) = \sigma(g)^{-1}$, where $\sigma(g)$ is the matrix of g in V. Thus, $\operatorname{tr}(\sigma^*(g)) = \operatorname{tr}\left(\sigma(g)^{-1}\right) = \operatorname{tr}(\sigma(g^{-1}))$. Denoting the character of V as χ^*, we have

$$\chi^*(g) = \chi(g^{-1}) = \overline{\chi(g)}, \qquad \chi^* = \overline{\chi}.$$

Problem 3

We want to compute the determinant, say D, of the character table of a group G. Since its rows (characters) and columns (conjugacy classes) can be written in any order, D is only well defined upto a sign.

(a) Show that D is either real or purely imaginary.
(b) Compute $|D|^2$ using the orthogonality relations.
(c) Use (a) and (b) to determine D (upto a sign).

Solution: Let A be the matrix representing the character table, of order $k \times k$, and let $n_1, ..., n_k$ be the sizes of the corresponding conjugacy classes.

(a) Observe that if χ is an irreducible character, so is $\overline{\chi}$ via the previous exercise. Thus, for every row in A, its complex conjugate is also present. This means that the rows of $\overline{A}$ are just a permutation of the rows of A, whence $\det(A) = \pm\det\left(\overline{A}\right) = \pm\overline{\det(A)}$. As a result, $D \mp \overline{D} = 0$, from which D is either real $(-)$ or purely imaginary $(+)$.

(b) Let B be the diagonal matrix with $n_1, ..., n_k$ along the diagonal, i.e. $B_{ij} = \delta_{ij} n_i$. for each $1 \leq i, j \leq k$. Then, compute

$$\left[ABA^\dagger\right]_{ij} = \sum_{1 \leq l, m \leq k} A_{il} B_{lm} A^\dagger_{mj} = \sum_{1 \leq l, m \leq k} A_{il}\delta_{lm} n_l \overline{A_{jm}} = \sum_{1 \leq l \leq k} n_l A_{il} \overline{A_{jl}} = \delta_{ij}\ |G|.$$

The last step follows from the row orthogonality relation

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij},$$

and the fact that characters are class functions. Thus, $ABA^\dagger$ is the matrix $|G|\ \mathbb{I}_k$. Taking determinants, $D \det(B) \overline{D} = |G|^k$; but $\det(B) = \prod_i n_i$. Thus,

$$|D|^2 = \frac{|G|^k}{\prod_{1 \leq i \leq k} n_i}.$$

(c) We may write

$$D = i^\ell \sqrt{\frac{|G|^k}{\prod_{1 \leq i \leq k} n_i}}.$$

for some $\ell \in \{0, 1, 2, 3\}$.

Problem 4

Let G_1, G_2 be two finite groups and let χ_1, χ_2 be two irreducible characters of G_1, G_2 respectively. Let V_1 (resp. V_2) be a vector space on which G_1 (resp. G_2) acts with character χ_1 (resp. χ_2).

(a) Let $V = V_1 \otimes V_2$. Define an action of $G_1 \times G_2$ on V by setting

$$(g_1, g_2)(v_1 \otimes v_2) = g_1(v_1) \otimes g_2(v_2).$$

Let χ be the character of this representation. Show that

$$\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2).$$

(b) Show that χ is irreducible.

(c) Show that every irreducible character of $G_1 \times G_2$ is obtained in this way.

Solution:

(a) Fix $g_1 \in G_1$, $g_2 \in G_2$. Let $\{v_i^1\}$ be a basis of V_1, and let $\{v_j^2\}$ be a basis of V_2. Then, $\{v_i^1 \otimes v_j^2\}$ is a basis of V. Furthermore, let $\{v_i^1\}$, $\{v_j^2\}$ have been chosen so that the actions of g_1, g_2 are diagonal (this is permitted since we are working in $\mathrm{GL}(\mathbb{C})$). Thus, we may write $g_1 v_i^1 = \lambda_i^1 v_i^1$, $g_2 v_j^2 = \lambda_j^2 v_j^2$ for constants $\{\lambda_i^1\}$, $\{\lambda_j^2\}$. Thus, each

$$(g_1, g_2)\left(v_i^1 \otimes v_j^2\right) = \left(\lambda_i^1 v_i^1\right) \otimes \left(\lambda_j^2 v_j^2\right) = \lambda_i^1 \lambda_j^2 \left(v_i^1 \otimes v_j^2\right).$$

From this, we immediately have

$$\chi(g_1, g_2) = \sum_{i,j} \lambda_i^1 \lambda_j^2 = \left(\sum_i \lambda_i^1\right)\left(\sum_j \lambda_j^2\right) = \chi_1(g_1)\chi_2(g_2).$$

(b) Note that

$$\sum_{(g_1,g_2)\in G_1\times G_2} |\chi(g_1, g_2)|^2 = \sum_{\substack{g_1\in G_1 \\ g_2 \in G_2}} |\chi_1(g_1)|^2 \ |\chi_2(g_2)|^2 = \left(\sum_{g_1\in G_1} |\chi_1(g_1)|^2\right)\left(\sum_{g_2\in G_2} |\chi_2(g_2)|^2\right),$$

so

$$|G_1 \times G_2| \ \langle \chi, \chi \rangle = |G_1| \ \langle \chi_1, \chi_1 \rangle \cdot |G_2| \ \langle \chi_2, \chi_2 \rangle = |G_1| \cdot |G_2|$$

since χ_1, χ_2 are irreducible. Thus, $\langle \chi, \chi \rangle = 1$, whence χ is irreducible.

(c) Let $\{\chi_i^1\}$ be the m irreducible characters of G_1, and let $\{\chi_j^2\}$ be the n irreducible characters of G_2. Set $\chi = \chi_i^1 \chi_j^2$, $\chi' = \chi_{i'}^1 \chi_{j'}^2$ for some $1 \le i, i' \le m$, $1 \le j, j' \le n$. Then,

$$\begin{aligned}
\langle \chi, \chi' \rangle &= \frac{1}{|G_1 \times G_2|} \sum_{(g_1, g_2) \in G_1 \times G_2} \chi(g_1, g_2) \overline{\chi'(g_1, g_2)} \\
&= \frac{1}{|G_1|\,|G_2|} \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} \chi_i^1(g_1) \chi_j^2(g_2) \overline{\chi_{i'}^1(g_1) \chi_{j'}^2(g_2)} \\
&= \frac{1}{|G_1|\,|G_2|} \sum_{g_1 \in G_1} \chi_i^1(g_1) \overline{\chi_{i'}^1(g_1)} \sum_{g_2 \in G_2} \chi_j^1(g_2) \overline{\chi_{j'}^2(g_2)} \\
&= \langle \chi_i^1, \chi_{i'}^1 \rangle \langle \chi_j^2, \chi_{j'}^2 \rangle \\
&= \delta_{ii'} \delta_{jj'}.
\end{aligned}$$

Thus, all mn irreducible characters obtained in this manner are orthonormal, hence distinct. However, there are precisely mn irreducible characters of $G_1 \times G_2$, since this is the number of conjugacy classes (the conjugacy classes of $G_1 \times G_2$ look like $C_1 \times C_2$ where C_1 is a conjugacy class of G_1, C_2 is a conjugacy class of G_2). Thus, all irreducible characters of $G_1 \times G_2$ have been found.

Problem 5

(a) Let N be a normal subgroup of G. Show that every irreducible representation of G/N gives rise to an irreducible representation of G.

(b) Let A_4 be the alternating group on 4 elements. It contains a normal subgroup K of order 4, the Klein group. Using (a), this gives three 1-dimensional representations of A_4. Show that there exists exactly one more irreducible representation. Write down the character table of A_4 (first show that there are exactly 4 conjugacy classes in A_4).

Solution:

(a) Let χ be an irreducible character of G/N, and let G/N act on a vector space V such that its character is χ. We define an action of G on V as follows: for $g \in G$ and $v \in V$, let

$$gv = (gN)(v).$$

This is indeed an action of G on V; note that $1v = (N)(v) = v$, and for $g_1, g_2 \in G$, we must have

$$g_1(g_2(v)) = (g_1 N)((g_2 N)(v)) = ((g_1 N)(g_2 N))(v) = (g_1 g_2 N)(v) = (g_1 g_2)(v).$$

Thus, the corresponding character χ' of G is given by

$$\chi'(g) = \chi(gN).$$

Furthermore, observe that

$$
\begin{aligned}
\langle \chi', \chi' \rangle &= \frac{1}{|G|} \sum_{g \in G} |\chi'(g)|^2 \\
&= \frac{1}{|G/N|\ |N|} \sum_{g'N \in G/N} \sum_{g \in g'N} |\chi(gN)|^2 \\
&= \frac{1}{|G/N|\ |N|} \sum_{g'N \in G/N} \sum_{g \in g'N} |\chi(g'N)|^2 \\
&= \frac{1}{|G/N|\ |N|} \sum_{g'N \in G/N} |N| \cdot |\chi(g'N)|^2 \\
&= \langle \chi, \chi \rangle \\
&= 1.
\end{aligned}
$$

Thus, χ' is an irreducible character of G.

(b) Note that the (non-trivial) cycle types in A_4 are 2^2 and 3. The Klein group may be represented as

$$K = \{e, (12)(34), (13)(24), (14)(23)\} := \{e, a, b, c\}.$$

Note that K is abelian, and $ab = c$, $bc = a$, $ca = b$. Now,

$$A_4/K = \{K, (123)K, (132)K\}.$$

Indeed,

$$
\begin{aligned}
K &= \{e, (12)(34), (13)(24), (14)(23)\}, \\
(123)K &= \{(123), (134), (243), (142)\}, \\
(132)K &= \{(132), (234), (124), (143)\}.
\end{aligned}
$$

Thus, $A_4/K \cong C_3 = \{e, (123), (132)\}$. This is also abelian, and admits the following three irreducible representations ($\omega = e^{2\pi i/3}$).

$$
\begin{aligned}
&\sigma_0 : A_4/K \to \mathbb{C}, \quad && K \mapsto 1, \quad && (123)K \mapsto 1, \quad && (132)K \mapsto 1, \\
&\sigma_1 : A_4/K \to \mathbb{C}, \quad && K \mapsto 1, \quad && (123)K \mapsto \omega, \quad && (132)K \mapsto \omega^2, \\
&\sigma_2 : A_4/K \to \mathbb{C}, \quad && K \mapsto 1, \quad && (123)K \mapsto \omega^2, \quad && (132)K \mapsto \omega.
\end{aligned}
$$

Using (a), these give rise to three irreducible representations of A_4.

Problem 6

You are only given the character table of a group. Can you decide whether it is simple or not?

Solution: A group is not simple if and only if there exists non-trivial $g \in G$ and a non-trivial irreducible character χ of G such that $\chi(g) = \chi(1)$. The latter condition is satisfied if and only if there is a (non-trivial) row in the character table of G where the first value (corresponding to $\chi(1)$) is equal to some other value in that row.

To prove this, first let $g \in G$, $g \neq 1$ such that $\chi(g) = \chi(1)$ for some non-trivial irreducible character χ. Now, $\chi(g)$ is the sum of $d = \chi(1)$ roots of unity (the eigenvalues of $\sigma(g)$). By the triangle inequality, this can be equal to $d = \chi(1)$ only when all these eigenvalues are 1, hence $\sigma(g) = \mathrm{id}$. With this, consider $N = \{g \in G : \chi(g) = \chi(1)\}$. Then, N is a normal subgroup of G; note that $N = \ker(\sigma)$, where $\sigma : G \to \mathrm{GL}(V)$ is a group homomorphism. Furthermore, $N \neq \{1\}$ by the existence of $g \neq 1$ such that

$\chi(g) = \chi(1)$, and $N \neq G$ since σ is a non-trivial representation (χ is a non-trivial character). Thus, G is not simple.

Conversely, suppose that G is not simple. Find a normal subgroup N of G where $N \neq \{1\}, G$. Then, G/N admits some non-trivial irreducible character χ (if not, that would imply that G/N has only one conjugacy class, forcing $G/N = \{1\}$). Use the previous exercise to define an irreducible, non-trivial character χ' of G where $\chi'(g) = \chi(gN)$. Then, we can pick $g \in N, g \neq 1$ and have $\chi'(g) = \chi(gN) = \chi(N) = \chi'(1)$.

Problem 7

Let χ be an irreducible character of a group G. Show that its complex conjugate $\overline{\chi}$ is also an irreducible character.

Solution: $\overline{\chi}$ is the dual character of χ, via Problem 2(b).

Problem 8

Let g be an element of order 2. Show that $\chi(g)$ is always an integer for any character χ.

Solution: Note that $\sigma(g)^2 = \sigma(g^2) = \sigma(1) = \mathrm{id}_V$, so the minimal polynomial of $\sigma(g)$ is a factor of $x^2 - 1$. As a result, the eigenvalues of $\sigma(g)$ are in $\{\pm 1\}$. Thus, $\chi(g)$ being the sum of eigenvalues of $\sigma(g)$ must be an integer.

Problem 9

(a) Let C be a conjugacy class and let $C^{-1} = \{g^{-1} : g \in C\}$. Show that C^{-1} is also a conjugacy class.

(b) Show that if $C = C^{-1}$, then $\chi(C)$ is real for any character χ.

Solution:

(a) Note that any two elements from C^{-1} can be written as g_1^{-1}, g_2^{-1} for conjugates $g_1, g_2 \in C$. Thus, we can find $h \in G$ such that $g_1 = hg_2h^{-1}$, hence $g_1^{-1} = hg_2^{-1}h^{-1}$. This shows that g_1^{-1}, g_2^{-1} are conjugate. Furthermore, fixing $g \in G$, we have

$$C = \{hgh^{-1} : h \in G\},$$

so

$$C^{-1} = \left\{\left(hgh^{-1}\right)^{-1} : h \in G\right\} = \{hgh^{-1} : h \in G\}$$

is the set of all conjugates of g^{-1}.

(b) Pick $g \in C$, whence $g^{-1} \in C^{-1} = C$, so $g^{-1} = hgh^{-1}$ for some $h \in G$. Thus, $\overline{\chi(g)} = \chi(g^{-1}) = \chi(hgh^{-1}) = \chi(g)$. This means that $\chi(g)$ must be real.

Problem 10

We want to construct the character table of A_5.

Problem 11

Consider the action of S_n on $\mathbb{C}^n$, where S_n acts by permuting the coordinates. We want to show that this representation is the direct sum of the trivial representation and another irreducible representation (here $n \geq 2$).

(a) For $\sigma \in S_n$, let $\chi(\sigma)$ denote the number of fixed points of σ. Show that χ is the character of the permutation representation.
(b) Let $X = \{1, ..., n\}$. Consider the action of S_n on $X \times X$. Observe that the number of fixed points of σ in $X \times X$ is $\chi(\sigma)^2$. Evaluate

$$\sum_{\sigma \in S_n} \chi(\sigma)^2$$

using Burnside's Lemma.
(c) Deduce that $\langle \chi, \chi \rangle = 2$. Conclude.

Solution:

(a) Let $\{e_i\}$ be the standard basis of $\mathbb{C}^n$. Then, σ acts on $\mathbb{C}^n$ via $\sigma e_i = e_{\sigma(i)}$; this is equal to e_i precisely when $\sigma(i) = i$, i.e. i is a fixed point of σ. Thus, the trace of the representation of σ is precisely the number of such fixed points of σ, which is $\chi(\sigma)$.

(b) Note that $\sigma(i, j) = (\sigma(i), \sigma(j))$; this is equal to (i, j) precisely when $\sigma(i) = i$ and $\sigma(j) = j$, i.e. i, j are both fixed points of σ. The number of such tuples (i, j) is precisely $\chi(\sigma)^2$.

Now, Burnside's Lemma tells us that

$$|(X \times X)/S_n| = \frac{1}{|S_n|} \sum_{\sigma \in S_n} |(X \times X)^\sigma| = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \chi(\sigma)^2.$$

We claim that the number of orbits $|(X \times X)/S_n| = 2$, the orbits being the diagonal $\Delta = \{(i, i) : i \in X\}$ and the complement $\Delta' = (X \times X) \setminus \Delta$. Indeed, $\sigma(1, 1) = (\sigma(1), \sigma(1)) \in \Delta$ for all σ, and $(1i)(1, 1) = (i, i)$ for any $i \in X$, so Δ is indeed the orbit of $(1, 1)$. Next, given any $i, j \in X$ with $i \neq j$, it is always possible to find a permutation σ such that $\sigma(1, 2) = (i, j)$.

With this,

$$\sum_{\sigma \in S_n} \chi(\sigma)^2 = 2\ |S_n| = 2n!.$$

(c) The previous equation immediately gives $\langle \chi, \chi \rangle = 2 = 1^2 + 1^2$. But $\mathrm{span}_{\mathbb{C}}\{e_1 + ... + e_n\}$ is a trivial sub-representation of $\mathbb{C}^n$; subtracting it from χ leaves us with an irreducible representation of S_n.

Problem 12

Let X be a finite set on which a finite group G acts. Let V be the vector space which has the elements of X as a basis. Note that G acts on V by permuting its basis. Let χ (rep. $\mathbf{1}_G$) denote that character of V (resp. the trivial character). Show that

$$\sum_{g \in G} \chi(g) = |G| \cdot m,$$

where m is the number of orbits of G in X. Using this, show that the number of times $\mathbf{1}_G$ occurs in V is the same as the number of orbits of G in X. In particular, if G acts transitively on X, then $\mathbf{1}_G$ occurs exactly once in χ.

Solution: By the same argument as before, $\chi(g)$ is precisely the number of fixed points of $g \in G$ when acting on X. Thus, Burnside's Lemma immediately gives

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) = |X/G| = m.$$

However, this is also precisely $\langle \chi, \mathbf{1}_G \rangle$, hence the number of times $\mathbf{1}_G$ occurs in χ is the number of orbits of G in X, namely m.

Problem 13

Suppose now that G acts doubly transitively on X, i.e. given elements x, y and z, w in X such that $x \neq y$ and $z \neq w$, there exists $g \in G$ such that

$$g(x) = z, \qquad g(y) = w.$$

Note that the action of G is transitive (why?). Thus, we can write

$$\chi = \mathbf{1}_G + \theta$$

where θ is a character in which $\mathbf{1}_G$ does not appear. Show that θ is irreducible.

Solution: The fact that G acts transitively on X can be checked by setting $z = y$, $w = x$ for $x, y \in X$, $x \neq y$. With this, the previous exercise gives us the representation θ which does not contain $\mathbf{1}_G$. Consider the action of G on $X \times X$; like before, the number of fixed points of $g \in G$ is $\chi(g)^2$, and the two orbits of G in $X \times X$ are the diagonal $\Delta = \{(x, x) : x \in X\}$ and $\Delta' = (X \times X) \setminus \Delta$ (via the doubly transitive property). Thus, Burnside's Lemma gives

$$\langle \chi, \chi \rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)^2 = \frac{1}{|G|}\sum_{g\in G}|(X \times X)^g| = 2 = 1^2 + 1^2.$$

Thus, χ is the sum of two irreducible characters each of degree 1; since $\mathbf{1}_G$ is one of them, θ must be the other.

Problem 14

Let N be a normal subgroup of G and let χ be a character of N. If $\tilde{\chi}$ is the character of G induced from N, show that $\tilde{\chi}(g) = 0$ if $g \neq N$.

Solution: Recall that

$$\tilde{\chi}(g) = \frac{1}{|N|}\sum_{\substack{h\in G\\ hgh^{-1}\in N}}\chi(hgh^{-1}).$$

It is enough to show that $hgh^{-1} \notin N$ for $g \notin N$, $h \in G$. Indeed, if $hgh^{-1} = g' \in N$, then $g = h^{-1}g'h \notin N$, contradicting the normality of N in G.

Problem 15

For each irreducible representation of S_3, find the character of the representation obtained by inducing it to S_4. Decompose the induced characters into irreducibles.

Problem 16

Induce the sign representation of S_4 to S_5 and decompose it into irreducibles using the character table of S_5.

Problem 17

Let D_n be the group of symmetries of a regular n-gon. Note that $|D_n| = 2n$ and D_n contains a cyclic subgroup C_n of order n, consisting of rotations in D_n.

(a) Let χ be a character of C_n. Note that since C_n is abelian, χ is necessarily 1-dimensional. Suppose that $\chi^2 \neq 1$. Show that $\chi' := \operatorname{Ind}_{C_n}^{D_n} \chi$ is irreducible.
(b) Using (a), compute the character table of D_n.

Solution: Use the presentation $D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\tau\sigma)^2 = 1 \rangle$, with $C_n = \{1, \sigma, ..., \sigma^{n-1}\}$. Observe that $\tau\sigma = \sigma^{n-1}\tau$, hence every element of D_n is either of the form σ^k or $\sigma^k\tau$.

(a) Representing $D_n / C_n = \{C_n, \tau C_n\}$, we have

$$\chi'(g) = \sum_{\substack{r \in \{1,\tau\} \\ rgr^{-1} \in C_n}} \chi(rgr^{-1}).$$

Since $[D_n : C_n] = 2$, C_n is a normal subgroup of D_n. Thus, $\chi'(g) = 0$ for all $g \notin C_n$ by Problem 14. Now for $g \in C_n$, write $g = \sigma^k$ for some $0 \leq k < n$, so $\tau g \tau^{-1} = \tau\sigma^k\tau = \sigma^{k(n-1)} = \sigma^{-k} \in C_n$. Thus,

$$\chi'(\sigma^k) = \chi(\sigma^k) + \chi(\sigma^{-k}) = \chi(\sigma^k) + \overline{\chi(\sigma^k)}.$$

Thus,

$$\langle \chi', \chi' \rangle = \frac{1}{2n} \sum_{g \in D_n} |\chi'(g)|^2 = \frac{1}{2n} \sum_{g \in C_n} \left(\chi(g) + \overline{\chi(g)}\right)^2$$

Now, observe that since χ is 1-dimensional, $\chi(g)$ must be a root of unity. Specifically, χ is completely determined by the value of $\chi(\sigma) = \xi$, where $\xi = e^{2\pi i \ell / n}$; every other $\chi(\sigma^k) = \xi^k$. Thus,

$$\langle \chi', \chi' \rangle = \frac{1}{2n} \sum_{k=0}^{n-1} \left(\xi^k + \xi^{-k}\right)^2 = \frac{1}{2n} \sum_{k=0}^{n-1} [\xi^{2k} + \xi^{-2k} + 2] = \frac{1}{n} \left(\sum_{k=0}^{n-1} \xi^{2k} \right) + 1.$$

But $\xi^2 = \chi(\sigma)^2 \neq 1$, hence

$$\sum_{k=0}^{n-1} \xi^{2k} = \frac{\xi^{2n} - 1}{\xi^2 - 1} = 0,$$

so $\langle \chi', \chi' \rangle = 1$ whence χ' is irreducible.