Presentation Problems - Linear Algebra (MA2102)

- There are **six** problems in each set and **sixteen** sets.
- Each set is assigned to five students. It is expected that you know how to solve **all six** problems in your set.
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Problem Set 1 It is assigned to the following students: *Priyangshu Datta, Aakash Ghosh, Ayussh Anubhav Patel, Choudhari Bhalchandra Sitaram, Deepak Verma*.

1. Let $V = \{(x, y) \mid x, y \in \mathbb{C}\}$. Under the standard addition and scalar multiplication for ordered pairs of complex numbers, is V a vector space over $\mathbb{C}$? Over $\mathbb{R}$? If so, find the dimension of V.

2. Let $A \in M_n(\mathbb{R})$ such that $AA^t = I$. Characterize geometrically what A looks like. If A has exactly one invariant one dimensional subspace, then what can you say about n?

3. Let $\mathbf{v} \in V$ be a non-zero vector and $T : V \to V$ be a linear map. Show that Span($\{\mathbf{v}\}$) is invariant under T if and only if $\mathbf{v}$ is an eigenvector for T.

4. Compute the determinant of the matrix
$$\begin{pmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{pmatrix}.$$

5. Let $P : V \to V$ be a linear map transformation such that $P^2 = P$. Describe the largest subspace which is pointwise invariant under P.

6. Let A be an $n \times n$ real matrix. The *exponential* of A is defined as
$$e^A := I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$
provided each entry of the matrix converges. Assuming e^A makes sense and P is an invertible matrix show that Pe^AP^{-1} is also an exponential matrix.

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Problem Set 2 It is assigned to the following students: *Sounak Sinha Biswas, Zuiner Jamir, Ramanarayanan K, Soham Mukherjee, Arikta Saha*.

1. Show that $S = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \,|\, a, b, c, d \in \mathbb{Q}\}$ is a field. [Hint: You may use the fact that if $a_1 + b_1\sqrt{d} = a_2 + b_2\sqrt{d}$ for rational numbers a_i, b_i and d a square-free integer, then $a_1 = a_2, b_1 = b_2$.]

2. Let $A \in M_n(\mathbb{R})$ and consider the map $T_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $T_A(X) = AX - XA$, called the *commutator* of A and X. Show that T_A is linear and zero is an eigenvalue of T_A.

3. Prove or disprove: *The set* $\{p(x) \in \mathcal{P}(\mathbb{R}) \,|\, p(-x) = -p(x)\}$ *is a subspace of* $\mathcal{P}(\mathbb{R})$.

4. Suppose U is a subspace of V such that V/U is finite dimensional. Show that V is isomorphic to $U \times V/U$.

5. Let $T : V \to W$ be a linear transformation. If $\{T(\mathbf{v}_1), \cdots, T(\mathbf{v}_k)\}$ is linearly independent, then show that $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is linearly independent. Is the converse true?

6. Let $A, B \in M_n(\mathbb{C})$. Show that $r(AB) \leq \min\{r(A), r(B)\}$.

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Problem Set 3 It is assigned to the following students: *Anusha Biswas, Samiron Chakraborti, Sourin Chatterjee, T Deepak Rajan, G Sushanth*.

1. Show that $\mathbb{R}$ is a vector space over $\mathbb{Q}$ and find its dimension.

2. Let $A \in M_n(\mathbb{R})$ such that $A^k = 0$ for some positive integer k. Such a matrix is called *nilpotent*. Show that if A is nilpotent, then $I + \lambda A$ is invertible for any $\lambda \in \mathbb{R}$.

3. Let $W_1 \subset W_2$ be subspaces of V. Show that W_2/W_1 is a subspace of V/W_1.

4. Consider the linear map
$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R}),\ T(A) = A^t.$$
Check whether T is diagonalizable; if so, find a suitable basis in which the corresponding matrix of the linear operator is a diagonal matrix.

5. Let T be a linear transformation on a vector space V of dimension n. If for some vector $\mathbf{v}$, the vectors $\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})$ are linearly independent, then show that every eigenvalue of T has only one corresponding eigenvector up to a scalar multiplication.

6. Consider a linear transformation $T : V \to V$ on a finite dimensional vector space V. It is given that for every $\mathbf{v} \in V$ there exists a positive integer k, possibly depending on $\mathbf{v}$, such that $T^k\mathbf{v} = \mathbf{0}$. Show that there exists an integer N such that $T^N \equiv 0$, i.e., T^N is the zero linear transformation. Would this still be true if we drop the asumption of finite dimensionality?

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Problem Set 4 It is assigned to the following students: *Rudraksh Agarwal, Koushik Barai, Juee Dhar, Varun Kushwaha, Arkadeep Mitra*.

1. Let $A \in \mathrm{SO}_3(\mathbb{R})$. Show that A always has 1 as an eigenvalue. Can A have exactly two real eigenvalues?
2. Find a basis for $M_n^0(\mathbb{R})$, the vector space of $n \times n$ matrices with trace zero. Explain why the set you chose is indeed a basis.
3. Prove that V is infinite dimensional if and only if there is a sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of vectors in V such that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent for every positive integer m.
4. Let $T : V \to V$ be a linear transformation. We say a subspace W is *invariant* if $T(\mathbf{w}) \in W$ for any $\mathbf{w} \in W$. Let W be an invariant subspace of an invertible transformation T. Show that W is also invariant under T^{-1}.
5. For any matrix $A \in M_n(\mathbb{R})$, let $\mathrm{adj}(A)$ denote its adjoint. Prove or disprove: *The rank of A is* $n-1$ *if and only if the rank of* $\mathrm{adj}(A)$ *is* 1.
6. Let $A \in M_n(\mathbb{C})$. Show that $A^n = 0$ if $\mathrm{tr}(A^k) = 0$, for $k = 1, 2, \ldots, n$.

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Problem Set 5 It is assigned to the following students: *Sohom Gupta, Nimmana Uday Kiran, Rudra Mukhopadhyay, Swaraj Pradhan, Harsh Talwar*.

1. Let $A \in M_n(\mathbb{C})$. Show that $A^2 = A$ if and only if $r(A) + r(A - I) = n$.

2. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent in V. Determine the values of k for which the vectors $\mathbf{v}_2 - \mathbf{v}_1, k\mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_3$ linearly independent.

3. Consider $M_2(\mathbb{R})$, the vector space of all 2×2 real matrices. Let E_{ij} be the 2×2 matrix with $(i, j)^{\text{th}}$ entry 1 and other entries 0. Let
$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
and define $\mathcal{A}(B) = AB$ for any $B \in M_2(\mathbb{R})$. Show that $\mathcal{A}$ is a linear transformation on $M_2(\mathbb{R})$ and find the matrix of $\mathcal{A}$ under the basis $E_{ij}, i, j = 1, 2$.

4. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ be a subspace of $\mathbb{R}^3$. Show that the quotient space $\mathbb{R}^3/W$ is isomorphic to $\mathbb{R}$.

5. Let $A, B \in M_n(\mathbb{C})$. If $AB = 0$, then show that for any positive integer k,
$$tr(A + B)^k = tr(A^k) + tr(B^k).$$

6. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $P_m(\mathbb{R})$ such that $p_j(2) = 0$ for each j. Show that $\{p_0, p_1, \ldots, p_m\}$ is not linearly independent in $P_m(\mathbb{R})$.

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Problem Set 6 It is assigned to the following students: *Souvik Das, Karthikeyan Ganesan, Subham Murmu, Sougata Sarkar, Balwinder Singh.*

1. Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\operatorname{range} T = \operatorname{null} T$.

2. Compute the dimension of $S = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \mid a, b, c, d \in \mathbb{Q}\}$ as a vector space over $\mathbb{Q}$.

3. Let $T : V \to V$ be an invertible linear transformation. Compare the eigenvalues of T and T^{-1}.

4. If the matrix
$$A = \begin{pmatrix} 0 & 0 & 1 \\ x & 1 & y \\ 1 & 0 & 0 \end{pmatrix}$$
has three linearly independent eigenvectors, then show that $x + y = 0$.

5. Let T be a linear transformation on $\mathbb{R}^3$ whose matrix (relative to the usual basis for $\mathbb{R}^3$) is both symmetric and orthogonal. Prove that T is either plus or minus the identity, or a rotation by 180° about some axis in $\mathbb{R}^3$, or a reflection about some two-dimensional subspace of $\mathbb{R}^3$.

6. Let $\alpha_1, \dots, \alpha_n$ be distinct real numbers. Show that the n exponential functions $e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_n x}$ are linearly independent (over the real numbers) in $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

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Problem Set 7 It is assigned to the following students: *Akash Chandra Behera, Madhav B Nair, Sirswa Kuldeep Shree Ram, Satvik Saha, Soham Sanyashiv.*

1. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

   and then find an invertible matrix P such that $P^{-1}AP$ is diagonal.

2. If a square $n \times n$ real matrix A is such that AA^t is diagonal with each diagonal entry non-zero, then show that the rows of A are orthogonal. Is it true that the columns are orthogonal?

3. Let V and W be vector spaces over the same field F. Show that the set $\mathcal{L}(V, W)$, consisting of linear maps from V to W, is a vector space. If V and W are finite dimensional, then find the dimension of $\mathcal{L}(V, W)$.

4. Show that a matrix A is of rank 1 if and only if $A = \mathbf{x}\mathbf{y}^t$ for some non-zero column vectors $\mathbf{x}$ and $\mathbf{y}$.

5. Let $a \in \mathbb{R}$ and consider the set $S_a = \{1, (x-a), (x-a)^2, \cdots, (x-a)^n\}$. Show that S_a is a basis for $P_n(\mathbb{R})$, the space of polynomials of degree at most n.

6. Find a basis of the quotient space $M_n(\mathbb{R})/\mathrm{Sym}_n(\mathbb{R})$, where $\mathrm{Sym}_n(\mathbb{R})$ is the subspace of symmetric matrices.

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Problem Set 8 It is assigned to the following students: *Aditya Dwarkesh, Nishant Gaurav, Temjeninla Jamir, Sanmoy Saha, Abhishek Yadav.*

1. Construct a field of size 4.

2. Let A be an $n \times n$ matrix, all of whose main diagonal entries are 0 and elsewhere 1. Show that A is invertible and compute A^{-1}.

3. Let $T : V \to V$ be a projection operator, i.e., $T^2 = T$. If V is finite dimensional, then show that $\mathrm{tr}(T)$ is the dimension of the subspace being projected onto.

4. Let $A \in M_n(\mathbb{R})$ and A^t be its transpose. Show that A^tA and A^t have the same range.

5. Let T be a linear transformation on a vector space V of dimension n. If $T^{n-1}(\mathbf{v}) \neq \mathbf{0}$ but $T^n(\mathbf{v}) = \mathbf{0}$, for some $\mathbf{v} \in V$, then show that $\{\mathbf{v}, T(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is linearly independent, and thus form a basis of V. What are the eigenvalues of T?

6. Let $C(\mathbb{R})$ denote the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that it is a vector space.

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Problem Set 9 It is assigned to the following students: *Subhadip Basak, Alok Ranjan Kerketta, Amarjeet Kumar, Shivam Kumar, D Nivedaa*.

1. Let $C(\mathbb{R})$ denote the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $\sin x$ and $\cos x$ are not linearly dependent. What about $\sin^2 x$ and $\cos^2 x$?

2. Let A be the $n \times n$ matrix whose entries a_{ij} are given by
$$a_{ij} = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$
Prove that the eigenvalues of A are symmetric with respect to the origin.

3. Suppose that P and Q are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$, and $I - (P+Q)$ is invertible. Show that P and Q have the same rank.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that the matrix A of T, in the usual basis of $\mathbb{R}^3$, is an orthogonal matrix. Suppose that A has determinant 1 and $A^3 = I$. Show that there exists an orthogonal matrix B such that
$$BAB^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$
[Hint: You may assume that A admits an eigenvector with eigenvalue 1.]

5. Given vector spaces V and W over a field F, define a vector space structure on $V \times W$.

6. Let $M_n^0(\mathbb{R})$ denote the subspace of matrices with trace zero. Show that the quotient space $M_n(\mathbb{R})/M_n^0(\mathbb{R})$ is one dimensional.

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Problem Set 10 It is assigned to the following students: *Akhil Bhogal, Ananya Biswas, Saikat Ghosh, Hadi Muhammad M, Krishna Das Saren*.

1. If A is a matrix such that $A^3 = 2I$, then show that $B = A^2 - 2A + 2I$ is invertible.
2. Let A be a square matrix. If $r(A) = r(A^2)$, then show that the system of equations $A\mathbf{x} = \mathbf{0}$ and $A^2\mathbf{x} = \mathbf{0}$ have the same solution space.
3. Prove that the union of three subspaces of V, a vector space over $\mathbb{R}$, is a subspace of V if and only if one of the subspaces contains the other two.
4. Find $n+1$ vectors in $\mathbb{C}^n$ that are linearly independent over $\mathbb{R}$. [It is assumed that you have verified that $\mathbb{C}^n$ is a vector space over $\mathbb{R}$.]
5. Let $A \in M_n(\mathbb{R})$ be expressible as BB^t for some $B \in M_n(\mathbb{R})$. Show that the real eigenvalues of A are non-negative.
6. Let A be an $n \times n$ matrix. Show that

$$\frac{d}{dt}\bigg|_{t=0} \det(I + tA) = \mathrm{tr}(A).$$

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Problem Set 11 It is assigned to the following students: *Anchit Bhair, Patel Meetkumar Bharatbhai, Mamani Bhumij, Soumya Kanti Saha, Aayush Srivastav.*

1. Find a basis for the quotient space $\mathbb{R}^4/W$, where
$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1 = 2x_2, 3x_3 = 4x_4\}.$$

2. Prove or disprove: *The set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is a basis for* V *if and only if the set* $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_n + \mathbf{v}_1\}$ *is a basis.*

3. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be non-zero column vectors orthogonal to each other, i.e., $\mathbf{v}^t\mathbf{u} = 0$. Find all eigenvalues of $A = \mathbf{u}\mathbf{v}^t$ and corresponding eigenvectors. Is A similar to a diagonal matrix?

4. Classify all matrices $A \in M_2(\mathbb{R})$ such that $A^2 = I$.

5. Let $A \in M_n(\mathbb{R})$. Show that A has rank n if and only if its adjoint adj(A) has rank n.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number p such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove or disprove: *The set of periodic functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *is a subspace of* $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

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Problem Set 12 It is assigned to the following students: *Rishik Bhoumick, Sayan Dutta, Sourasish Karmakar, Nitish Kumar Mishra, Mayukh Roy.*

1. If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent in a vector space V and $\mathbf{v}_1, \cdots, \mathbf{v}_n, \mathbf{w}$ are linearly dependent, then show that $\mathbf{w}$ can be expressed uniquely as a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_n$.

2. Let $A \in M_n(\mathbb{C})$ satisfy $AA^t = I$ and $\det A \neq 1$. Compute $\det(A + I)$. Explain how this fits in with your idea of a reflection matrix A in $\mathbb{R}^2$.

3. Let S be the subspace of $M_n(\mathbb{R})$ generated by all matrices of the form $AB - BA$ with $A, B \in M_n(\mathbb{R})$. Prove that $\dim S = n^2 - 1$.

4. Let A be the tridiagonal matrix given by
$$A = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$
with $b_j \neq 0$. Show that A has n distinct eigenvalues.

5. For each non-negative integer n, consider the subset of polynomials
$$W = \{p(x) \mid p(0) = 0, p'(0) = 0, \ldots, p^{(n)}(0) = 0\}.$$
Show that W is a subspace of $P(\mathbb{R})$.

6. If A, B, C, D are $n \times n$ matrices such that $ABCD = I$, then show that
$$ABCD = DABC = CDAB = BCDA = I.$$

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Problem Set 13 It is assigned to the following students: *Chaudhari Mukesh Kishor, Amitesh Mishra, Manoj Kumar Pandiri, Bipradeep Saha, Mounadeep Saha*.

1. If $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are eigenvectors associated to distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ of a linear transformation, then show that $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent.
2. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a *linear isometry* if $\|T(\mathbf{v})\| = \|\mathbf{v}\|$. Show that the matrix (with respect to the usual basis) of a linear isometry of $\mathbb{R}^2$ is an orthogonal matrix.
3. Given a square matrix $A \in M_n(\mathbb{R})$, show that $V = \{X \in M_n(\mathbb{R}) \,|\, AX = XA\}$, the set of the matrices commuting with A, is a vector space. Is V closed under matrix multiplication? Is V a field?
4. Prove that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has a one-dimensional invariant subspace, and a two-dimensional invariant subspace.
5. Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$. Show that $AB - I_m$ is invertible if and only if $BA - I_n$ is invertible.
6. Show that $W_1 + W_2$ is a direct sum if and only if $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$.

Presentation Problems - Linear Algebra (MA2102)

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- Each set is assigned to five students. It is expected that you know how to solve **all six** problems in your set.
- There will be five presentations for each set, with one student presenting one question. The problem that you will present will be decided (by me) on the day of the presentation; date and time will be announced soon.
- Since the problems are declared ahead of time, nothing short of a crystal clear presentation from each one of you is expected.
- Each presentation should not last more than 10 **minutes** and carries 10 **points**.
- You are also expected to submit a write-up of the problem you had presented. This carries 5 **points**.

Problem Set 14 It is assigned to the following students: *Firoz Alam, Ankit Das, Gayan Tushar Sanjay, Sahil Saha, Pratik Saren*.

1. Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.
2. Let $A \in M_3(\mathbb{R})$ such that $AA^t = I$ and $\det A = 1$. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the columns of A, then show that $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.
3. Compute the dimension of the space $V = \{X \in M_n(\mathbb{R}) \mid AX = XA\}$, the set of the matrices commuting with A, where
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$
4. Given $A \in M_n(\mathbb{R})$, we define a linear transformation on $M_n(\mathbb{R})$ by $\mathcal{L}(X) = AX$ for $X \in M_n(\mathbb{R})$. Show that $\mathcal{L}$ and A have the same set of eigenvalues. How are the characteristic polynomials of $\mathcal{L}$ and A related?
5. Let A and B denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that A and B have a common eigenvector in $\mathbb{R}^n$.
6. If $A \in M_2(\mathbb{R})$ satisfies $A^2 + I = 0$, then show that A is similar to the matrix
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

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Problem Set 15 It is assigned to the following students: *Adeeb Mohammed Kunjali, Gourish Majumdar, V. Hema Padmaja, Jyotiska Panda, Satbhav Voleti.*

1. Show that a vector space over $\mathbb{R}$ cannot be the union of a finite number of proper subspaces.

2. Consider $P_n(\mathbb{R})$ and define
$$\mathcal{A} : P_n(\mathbb{R}) \to P_n(\mathbb{R}),\ \mathcal{A}(p(x)) = xp'(x) - p(x).$$
Show that $\mathcal{A}$ is a linear transformation. Moreover, find $\operatorname{Ker}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$.

3. Find a basis and compute the dimension of the space of all polynomials in A over $\mathbb{R}$, where
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},\ \omega = \frac{-1+\sqrt{3}i}{2}.$$
For instance, $A^3 - 5A^2 + 2I$ is one such polynomial.

4. Let A be a symmetric $n \times n$ matrix over $\mathbb{R}$ of rank $n-1$. Prove there is a $k \in \{1, 2, \ldots, n\}$ such that the matrix resulting from deletion of the k^{th} row and k^{th} column from A has rank $n-1$.

5. Let L be the line in $\mathbb{R}^2$ passing through $(1,1)$ and $(0,0)$. Show that $\mathbb{R}^2/L$ is isomorphic to the x-axis.

6. Let λ_1 and λ_2 be two different eigenvalues of a matrix A and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively. Show that $\mathbf{u}_1 + \mathbf{u}_2$ is not an eigenvector of A.

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Problem Set 16 It is assigned to the following students: *Naravane Adwait Bipinchandra, Siddharth Gupta, Samrik Ray, Nagpal Avni*. [This set is assigned to four students only.]

1. Let W_1, W_2 be subspaces of a vector space V. Determine when $W_1 \cup W_2$ is a subspace of V.
2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, where $n > 1$. Prove that there is a 2-dimensional subspace $V \subset \mathbb{R}^n$ such that $T(V) \subseteq V$.
3. Let A be a $n \times n$ real matrix. Show that if $A^2 + I = 0$, then n must be even. Does this remain true for complex matrices?
4. Let $\mathcal{D}$ be the differential operator on $\mathcal{P}_n(\mathbb{R})$ defined as follows
$$\mathcal{D}(a_0 + a_1x + \cdots + a_nx^n) := a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}.$$
Show that $\mathcal{D} : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ is a linear transformation and find the eigenvalues of $\mathcal{D}$ and $I + \mathcal{D}$.
5. Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in V. If U, W are subspaces of V such that $\mathbf{v}_1 + U = \mathbf{v}_2 + W$, then show that $U = W$.
6. Let A and B be $n \times n$ matrices over a field F and $\operatorname{Ker} A$ and $\operatorname{Ker} B$ be the null spaces of A and B with dimensions l and m, respectively. Show that the null space of AB has dimension at least $\max\{l, m\}$.