MA2201: Analysis II

Differentiation

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The origins of differential calculus lie in our attempts to approximate various functions using linear ones. Suppose that we have been given a curve described by the function f, and we want to *locally* approximate the function around a point x using a straight line. In other words, for a small shift h, we want to write

$$f(x+h) \approx f(x) + kh.$$

Here, k is the slope of the straight line. In order to obtain k, we can rearrange the above to get

$$k \approx \frac{f(x+h) - f(x)}{h}.$$

As we pick smaller and smaller neighbourhoods of x, we want our approximation to get better and better. Thus, if such an approximation is possible, then the value of k must stabilize. This means that the limit

$$k = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

must exist. Note that this immediately forces the continuity of f, since

$$\lim_{h\to 0} f(x+h) - f(x) = \lim_{h\to 0} h \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 0k = 0,$$

whereby $\lim_{x\to a} f(x) = f(a)$. Splitting the limit is justified because the individual limits exist. If such a limit k exists, we call it the derivative of f at x, denoted $f'(x)$. We are now able to write

$$f(x+h) \approx f(x) + f'(x)h.$$

Definition 2.1 (Derivative)**.** The derivative of a function $f\colon [a,b] \to \mathbb{R}$ at a point $x \in [a,b]$ is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

if such a limit exists. Note that we only demand a one-sided limit if x is an endpoint. If the derivative of f exists at every point in $[a,b]$, we say that f is differentiable on $[a,b]$.

Example. Consider the map $x \mapsto x^n$, where $n \in \mathbb{N}$. Using the binomial theorem, we can write

$$(x+h)^n = x^n + nx^{n-1}h + \cdots + h^n,$$

which means that

$$\frac{d}{dx}x^n = \lim_{h \to 0} \frac{1}{h}\left[(x+h)^n - x^n\right] = \lim_{h \to 0}\left[nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1}\right] = nx^{n-1}.$$

Note that the process of differentiation we described can be generalised to multivariable functions. The idea is to locally approximate a function with an affine function.

Theorem 2.1. *If $f\colon (a,b) \to \mathbb{R}$ is differentiable on (a,b), then it is also continuous on (a,b).*

Theorem 2.2. *Let $f\colon I \to \mathbb{R}$ be a continuous function. Then,*

1. *f maps compact sets to compact sets.*
2. *f maps connected sets to connected sets.*

Corollary 2.2.1. *A continuous function $f\colon I \to \mathbb{R}$ maps intervals to intervals.*

Corollary 2.2.2. *A continuous function $f\colon [a,b] \to \mathbb{R}$ attains its minimum and maximum on $[a,b]$.*

Definition 2.2. Given $f\colon (a,b) \to \mathbb{R}$, a point $c \in (a,b)$ is said to be a point of local maximum if there exists a neighbourhood I_c of c such that

$$f(c) > f(x),$$

for all $x \in I_c \setminus \{c\}$. There is an analogous definition for a local minimum.

Theorem 2.3. *If $f\colon (a,b) \to \mathbb{R}$ is differentiable and $c \in (a,b)$ is a point of local minimum or maximum, then $f'(c) = 0$.*

Remark. The converse is not true. Note that the derivative of $x \mapsto x^3$ vanishes at $x = 0$, but that is not a local minimum or maximum.

Proof. Let c be a local minimum or maximum of f, but suppose that $f'(c) \neq 0$. Define the function

$$g\colon (a,b) \to \mathbb{R}, \qquad g(x) = \begin{cases} (f(x) - f(c))/(x-c), & \text{if } x \neq c \\ f'(c), & \text{if } x = c \end{cases}$$

We note that g is continuous. Also, $f'(c) = g(c) \neq 0$. If $g(c) > 0$, there exists a neighbourhood $I_\delta = (c - \delta, c + \delta)$ such that for all $x \in I_\delta$, $g(x) > 0$, from the continuity of g. This means that on I_c,

$$\frac{f(x) - f(c)}{x - c} > 0,$$

which gives $f(x) > f(c)$ on $(c, c + \delta)$ and $f(x) < f(c)$ on $(c - \delta, c)$. This means that c cannot be a local minimum, nor a local maximum. There is an analogous case assuming $g(c) < 0$, which leads to the same contradiction. Thus, we must have $f'(c) = g(c) = 0$. □

Theorem 2.4. *If $f\colon (a, b) \to \mathbb{R}$ is twice differentiable, and $c \in (a, b)$ is such that $f'(c) = 0$ and $f''(c) < 0$, then c is a point of local maximum. If $f'(c) = 0$ and $f''(c) > 0$, then c is a point of local minimum.*

Theorem 2.5 (Rolle's Theorem)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b), with $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Set $f(a) = f(b) = \kappa$. From the continuity of f, note that the image of the closed interval $[a, b]$ is another closed interval $[\alpha, \beta]$. This means that $\alpha \leq \kappa \leq \beta$. Note that if $\alpha = \beta = \kappa$, then the function f is identically equal to the constant κ, hence $f'(x) = 0$ everywhere on $[a, b]$. By the continuity of f, it must attain its maximum and minimum on $[a, b]$. If $\beta > \kappa$, then the maximum is al least β and is hence not attained at the endpoints, which means that the point of maximum lies in (a, b). If $\alpha < \kappa$, then the same argument shows that f attains a minimum in (a, b). Thus, in either case, we have found $c \in (a, b)$ which is either a maximum or minimum of f, i.e. $f'(c) = 0$. □

Theorem 2.6 (Mean Value Theorem)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)\,(b - a).$$

Proof. Apply Rolle's Theorem on the function defined as

$$g\colon [a, b] \to \mathbb{R}, \qquad g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

Note that g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b) = 0$. □

Theorem 2.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable, and $f'(x) > 0$ for all $x \in \mathbb{R}$. Then, f is strictly increasing on $\mathbb{R}$.*

Proof. Let $x_2 > x_1$. By the mean value theorem, we pick $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0.$$

□

Remark. The converse is not true. The map $x \mapsto x^3$ is strictly increasing, but its derivative vanishes at 0.

Theorem 2.8 (Chain rule)**.** *Let f and g be differentiable on $\mathbb{R}$. Then, $f \circ g$ is also differentiable, with*
$$(f \circ g)' = (f' \circ g) \cdot g'.$$

Proof. Fix $a \in \mathbb{R}$. Define the functions

$$\varphi\colon \mathbb{R} \to \mathbb{R}, \qquad \varphi(x) = \begin{cases} (g(x) - g(a))/(x-a) & \text{if } x \neq a \\ g'(a), & \text{if } x = a \end{cases},$$

$$\psi\colon \mathbb{R} \to \mathbb{R}, \qquad \psi(y) = \begin{cases} (f(y) - f(b))/(y-b) & \text{if } y \neq b \\ f'(b), & \text{if } y = b \end{cases}.$$

Note that φ and ψ are continuous. Also, when $x \neq a$, we have

$$g(x) - g(a) = \varphi(x)(x-a).$$

Set $b = g(a)$, and write

$$f(g(x)) - f(g(a)) = \psi(g(x))(g(x) - g(a)) = \psi(g(x))\,\varphi(x)(x-a).$$

Setting $h = f \circ g$, we have
$$\frac{h(x) - h(a)}{x-a} = \psi(g(x))\varphi(x).$$

Taking limits $x \to a$, we use the continuity of φ, ψ and g to conclude that the derivative of h is indeed defined at a, and

$$h'(a) = \psi(g(a))\,\varphi(a) = f'(g(a))\,g'(a).$$ □

Definition 2.3 (Intermediate Value Property)**.** Let $f\colon (a,b) \to \mathbb{R}$ be such that for all $c, d \in (a,b)$ such that $f(c) < f(d)$ and $\lambda \in (f(c), f(d))$, there exists $x_0 \in (a,b)$ such that $f(x_0) = \lambda$. Then, we say that f has the intermediate value property.

Theorem 2.9 (Intermediate Value Theorem)**.** *All continuous functions $f\colon (a,b) \to \mathbb{R}$ have the intermediate value property.*

Theorem 2.10. *Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. Then, f' satisfies the intermediate value property.*

Proof. Let $c, d \in (a,b)$ and let $\lambda \in \mathbb{R}$ such that $\lambda \in (f'(c), f'(d))$. We wish to find $x_0 \in (a,b)$ such that $f'(x_0) = \lambda$. Define

$$g\colon (a,b) \to \mathbb{R}, \qquad g(x) = f(x) - \lambda x.$$

Note that $g'(x) = f'(x) - \lambda$, so $g'(c) < 0$ and $g'(d) > 0$. Thus, g is decreasing near c and increasing near d, so we can find $t_1, t_2 \in (c,d)$ such that $g(t_1) < g(c)$ and $g(t_2) < g(d)$. This means that g has no local minimum at c nor d. From the continuity of g, there exists $x_0 \in [c,d]$ such that $g(x_0) = \inf_{[c,d]} g(x)$. We have already shown that x_0 is neither c, nor d, so $x_0 \in (c,d)$. Hence, $g'(x_0) = 0$, which gives $f'(x_0) = \lambda$. □

Lemma 2.11. *If $f\colon (a,b) \to (c,d)$ is surjective, continuous and strictly increasing, then f is invertible with a continuous inverse.*

Theorem 2.12 (Inverse function theorem)**.** *Let $f\colon (a,b) \to (c,d)$ be surjective and differentiable, with $f'(x) \neq 0$ everywhere. Then, f is invertible, with a differentiable inverse whose derivative is given by*

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Proof. Given $f'(x) \neq 0$ on (a,b). Then intermediate value property gives either $f'(x) > 0$ for all $x \in (a,b)$, or $f'(x) < 0$. Without loss of generality, assume the former. This means that f is strictly increasing on (a,b), continuous, and surjective. Our lemma gives the existence of a continuous inverse f^{-1}.

Let $y \in (c,d)$, and let $x = f^{-1}(y)$. From the continuity of f^{-1}, we can always write $f^{-1}(y+\kappa) = x + h$. Thus,

$$\lim_{\kappa\to 0} \frac{f^{-1}(y+\kappa) - f^{-1}(y)}{\kappa} = \lim_{\kappa\to 0} \frac{x+h-x}{\kappa} = \lim_{\kappa\to 0} \frac{h}{\kappa}.$$

Note that $h \to 0$ as $\kappa \to 0$. Thus, this limit can be written as

$$(f^{-1})'(y) = \lim_{h\to 0} \frac{h}{f(x+h) - f(x)} = \frac{1}{f'(x)}.$$

□

Corollary 2.12.1. *Let f be continuously differentiable on $\mathbb{R}$, with $f'(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Then, there exists some neighbourhood of x_0 on which f is invertible, with a continuously differentiable inverse.*

Theorem 2.13. *Let $f_n \to f$ pointwise and $\{f_n'\}$ converge uniformly on some interval (a,b). Then, $f_n \to f$ uniformly.*

Proof. We show that $\{f_n\}$ is uniformly Cauchy on E. Note that for some fixed t, we can write

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| + |f_n(t) - f_m(t)|.$$

Using the Mean Value Theorem, the first term can be bounded as

$$|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| = (f_n' - f_m')(x_0)|x - t|,$$

where x_0 is between x and t. From the pointwise convergence of $f_n \to f$, we have

$$|f_n(t) - f_m(t)| < \frac{\epsilon}{2}$$

for all $n, m \geq N_t$. The uniform convergence of $\{f_n'\}$ means that we can find N_0 such that

$$|f_n'(x_0) - f_m'(x_0)| < \frac{\epsilon}{2(b-a)}$$

for all $n, m > N_0$. Thus, for all $x \in [a,b]$, and $n, m \geq N_t + N_0$, we have

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2(b-a)} \cdot (b-a) + \frac{\epsilon}{2} = \epsilon.$$

This means that $\{f_n\}$ is uniformly Cauchy on $[a,b]$, which gives the uniform convergence of $\{f_n\}$. □

Remark. We only needed to use the pointwise convergence of $\{f_n\}$ at one point t. By using pointwise convergence everywhere, we can allow for unbounded intervals, or the entirety of $\mathbb{R}$.

Theorem 2.14. *Let $\{f_n\}$ be a sequence of differentiable functions on some bounded interval (a,b) such that $f_n \to f$ pointwise and $\{f_n'\}$ converges uniformly on every $[\alpha,\beta] \subset (a,b)$. Then, f is differentiable and $f_n' \to f'$.*

Remark. We allow a, b to be $\pm\infty$.

Proof. Let $x_0 \in (a,b)$. We wish to show that the following limit exists.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Define $\varphi\colon (a,b) \setminus \{x_0\} \to \mathbb{R}$,

$$\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0}.$$

Also define the functions $\varphi_n\colon (a,b) \to \mathbb{R}$,

$$\varphi_n(x) = \begin{cases} (f_n(x) - f_n(x_0))/(x - x_0) & \text{if } x \neq x_0, \\ f_n'(x_0) & \text{if } x = x_0. \end{cases}$$

Note that φ_n are continuous, from the continuity of each f_n. When $x \neq x_0$, we see that $\varphi_n(x) \to \varphi(x)$. For $x = x_0$, we know that f_n' converges hence $\varphi_n(x_0)$ also converges. This gives us pointwise convergence.

We want to show that $\{\varphi_n\}$ converges uniformly. Fix $[\alpha,\beta] \subset (a,b)$ such that $x_0 \in (\alpha,\beta)$. For $x \neq x_0$, we have

$$|\varphi_n(x) - \varphi_m(x)| = \left| \frac{(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))}{x - x_0} \right|.$$

Using the Mean Value Theorem on $g = f_n - f_m$, we choose c between x and x_0 such that $(x - x_0)g'(c) = g(x) - g(x_0)$. Thus,

$$|\varphi_n(x) - \varphi_m(x)| = |f_n'(c) - f_m'(c)| < \epsilon$$

for all $m, n \geq N$ for some N, given by the uniform convergence of $\{f_n'\}$. This shows that $\{\varphi_n\}$ also converges uniformly on $[\alpha,\beta]$. Note that when $x = x_0$, $|f_n'(x_0) - f_m'(x_0)|$ is similarly bounded.

Now that $\{\varphi_n\}$ converges uniformly, we know that the limit function is continuous. Since it converges pointwise to φ on $x \neq x_0$ and to $\lim_{n\to\infty} f_n'(x_0)$ when $x = x_0$, continuity gives the existence of the desired limit and

$$\lim_{n\to\infty} f_n'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

which gives the differentiability of f. Also note that $f_n' \to f'$. □