

Notes from a course* on

Representation Theory of Finite Groups

Satvik Saha

Department of Mathematics and Statistics,
Indian Institute of Science Education and Research, Kolkata.

Table of Contents

1. Linear representations of groups	1
2. Subrepresentations	3
3. Irreducible representations	5
4. Characters	6
4.1. Orthogonality of characters	6
4.2. The character table for S_3	8
4.3. The character table for S_4	9
4.4. The character of the regular representation	10
4.5. The tensor product of representations	11
4.6. The character table for S_5	13

1. Linear representations of groups

Definition 1.1 (Linear representation): Let G be a finite group, and let V be a vector space. A linear representation (σ, V) of G is a homomorphism

$$\sigma : G \rightarrow \mathrm{GL}(V).$$

Example 1.1.1: The *trivial* representation of G is defined by $g \mapsto \mathrm{id}_V$.

Example 1.1.2: Consider a vector space V of dimension $\mathrm{ord}(G)$, and pick a basis $\{e_h\}_{h \in G}$. The *regular* representation $\tau : G \rightarrow \mathrm{GL}(V)$ of G is defined as follows: $\tau(g)$ sends each of the basis vectors $e_h \mapsto e_{gh}$.

The following propositions show that it is possible to define group representations in terms of a special class of group actions of G on the vector space V .

*MA4204, instructed by Dr. Swarnendu Datta.

Proposition 1.2: Let G be a finite group, and let V be a vector space. Let $\rho : G \times V \rightarrow V$ be a group action of G on V , such that each for each G , the map $v \mapsto \rho(g, v)$ is linear. Then, (σ, V) is a linear representation, where

$$\sigma : G \rightarrow \text{GL}(V), \quad g \mapsto (v \mapsto \rho(g, v)).$$

Proposition 1.3: Let (σ, V) be a linear representation. Then, the map

$$\rho : G \times V \rightarrow V, \quad (g, v) \mapsto \sigma(g)(v)$$

is a group action of G on V , where for each $g \in G$, the map $v \mapsto \rho(g, v)$ is linear.

In this discussion, we will always work with finite groups, as well as finite dimensional vector spaces over a base field K . Typically, we will consider $K = \mathbb{C}$.

We will often abbreviate (σ, V) with V , and $\sigma(g)$ with g when the presence of σ is clear from context.

Definition 1.4: The dimension of a representation (σ, V) is $\dim(V)$.

Example 1.4.1: The only one dimensional representation of S_3 in \mathbb{C}^\times is the sign homomorphism. To see this, consider an arbitrary homomorphism $\sigma : S_3 \rightarrow \mathbb{C}^\times$. Note that $\ker(\sigma)$ must be a normal subgroup of S_3 , hence must be one of $\{e\}, A_3, S_3$. The third option yields the trivial representation $\sigma = \text{id}_{\mathbb{C}^\times}$, and the first option gives the contradiction $S_3 \cong \text{im}(\sigma) \subset \mathbb{C}^\times$ (the right side is abelian while the left is not). This leaves $\ker(\sigma) = A_3$, i.e. $\sigma(g) = 1$ for all even permutations $g \in S_3$. The remaining elements of S_3 (the odd permutations) must be sent to -1 , since for any odd permutation $h \in S_3$, the permutation h^2 is even, so $\sigma(h)^2 = \sigma(h^2) = 1$. The result is precisely the sign homomorphism

$$\sigma : S_3 \rightarrow \mathbb{C}^\times, \quad g \mapsto \begin{cases} +1 & \text{if } g \in A_3 \\ -1 & \text{if } g \notin A_3. \end{cases}$$

Example 1.4.2: Construct an equilateral triangle in \mathbb{C}^2 centered at the origin, and consider the natural action of S_3 on it (permuting its vertices v_1, v_2, v_3). This induces a two dimensional representation $\sigma : S_3 \rightarrow \text{GL}(\mathbb{C}^2)$. Note that $\{v_1, v_2\}$ forms a basis of \mathbb{C}^2 ; the third vertex can be obtained via $v_3 = -v_1 - v_2$. With this, we can calculate the image of (v_1, v_2) under the action of each $g \in S_3$, and hence the matrices of $\sigma(g)$ in the given basis as follows.

g	$(\sigma(g)(v_1), \sigma(g)(v_2))$	Matrix of $\sigma(g)$
e	(v_1, v_2)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(12)	(v_2, v_1)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(23)	$(v_1, v_3) = (v_1, -v_1 - v_2)$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

(31)	$(v_3, v_2) = (-v_1 - v_2, v_2)$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(123)	$(v_2, v_3) = (v_2, -v_1 - v_2)$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
(321)	$(v_3, v_1) = (-v_1 - v_2, v_1)$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

Consider the setting $K = \mathbb{C}$. The fact that G is a finite group means that each element $g \in G$ has finite order, hence satisfies $g^m = 1$ for some $m \mid \text{ord}(G)$. This means that $\sigma(g)^m = \text{id}_V$, whence $x^m - 1$ is an annihilating polynomial for $\sigma(g)$. A consequence of this is that the minimal polynomial of $\sigma(g)$ is a factor of $x^m - 1$; but the latter splits into distinct linear factors. Furthermore, all eigenvalues of $\sigma(g)$ are roots of its minimal polynomial. This yields the following result.

Proposition 1.5: Suppose that $K = \mathbb{C}$. Let (σ, V) be a representation of G , and let $g \in G$. Then, $\sigma(g)$ is diagonalizable, and its eigenvalues are roots of unity.

2. Subrepresentations

Definition 2.1 (Stable subspace): Let (σ, V) be a representation of G , and let $W \subseteq V$ be a subspace of V . We say that W is a stable subspace of V if it is invariant under the action of G , i.e. $gw \in W$ for all $g \in G, w \in W$.

Example 2.1.1: Let S_3 act on \mathbb{C}^3 by permuting the basis vectors $\{e_1, e_2, e_3\}$. Then, the subspace $\text{span}\{e_1 + e_2 + e_3\}$ is stable.

Definition 2.2 (Subrepresentation): Let W be a stable subspace of V . We say that (σ, W) is a subrepresentation of (σ, V) .

Theorem 2.3: Suppose that $\text{char}(K) \nmid \text{ord}(G)$. Let W be a stable subspace of V . Then, there exists a stable subspace W' of V such that $V = W \oplus W'$.

When working with the field $K = \mathbb{C}$, [Theorem 2.3](#) admits a simpler form by invoking the orthocomplement of $W \subseteq V$, with respect to a suitable Hermitian form on V . We say that a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ is G -invariant if for all $g \in G, v, v' \in V$, we have $\langle gv, gv' \rangle = \langle v, v' \rangle$.

Theorem 2.4: Suppose that $K = \mathbb{C}$. If W is a stable subspace of V , then W^\perp is a stable subspace of V , with $V = W \oplus W^\perp$.

Remark: The subspace W^\perp is defined with respect to a non-degenerate G -invariant Hermitian form.

Proof: For all $g \in G$, $w \in W$, $w' \in W^\perp$, observe that $g^{-1}w \in W$, so

$$\langle gw', w \rangle = \langle gw', gg^{-1}w \rangle = \langle w', g^{-1}w \rangle = 0,$$

whence $gw' \in W^\perp$. □

Example 2.4.1: Continuing [Example 2.1.1](#), the subspace $\text{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$ is also stable under the action of S_3 . This gives a two dimensional subrepresentation of S_3 . In fact, it is easy to check that the matrices describing this representation in the basis $\{2e_1 - e_2 - e_3, 2e_2 - e_3 - e_1\}$ are precisely the same as those in [Example 1.4.2](#), making these two representations identical in some sense.

Remark: Given any Hermitian form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$, we can obtain a G -invariant Hermitian form on V defined by

$$(u, v) \mapsto \sum_{g \in G} \langle gu, gv \rangle.$$

Returning to [Theorem 2.3](#), observe that if π_W is a projection onto the subspace W , then we may write $V = W \oplus \ker(\pi_W)$. With this in mind, we will construct the required subspace W' as the kernel of a suitable projection map π_W . For this, we demand that π_W be G -invariant.

Definition 2.5: A linear map $f : V \rightarrow V'$ is called G -invariant if it is compatible with the G -action, i.e. for all $g \in G$, $v \in V$, we have $f(gv) = gf(v)$.

Note that the above definition implicitly deals with *two* representations (σ, V) and (σ', V') of G . The indicated property really looks like $\sigma'(g)(f(v)) = f(\sigma(g)(v))$ when written in full.

Lemma 2.6: Let $f : V \rightarrow V'$ be G -invariant. Then,

1. $\ker(f)$ is a stable subspace of V .
2. $\text{im}(f)$ is a stable subspace of V' .

Given any linear map $f : V \rightarrow V'$, we can construct a G -invariant linear map via

$$\tilde{f} : V \rightarrow V', \quad v \mapsto \sum_{g \in G} gf(g^{-1}v).$$

With this, we are ready to furnish a proof of our theorem.

Proof of Theorem 2.3: Let $\pi : V \rightarrow W$ be any projection onto W . Observe that

$$\pi_W : V \rightarrow W, \quad v \mapsto \frac{1}{\text{ord}(G)} \sum_{g \in G} g\pi(g^{-1}v)$$

is a G -invariant projection onto W . Setting $W' = \ker(\pi_W)$ completes the proof. □

Remark: Note how the assumption that $\text{char}(K) \nmid \text{ord}(G)$ is crucial for defining the projection π_W .

3. Irreducible representations

Definition 3.1 (Irreducible representations): We say that a representation is irreducible if it admits no proper non-trivial subrepresentations.

In other words, a representation (σ, V) is irreducible *if and only if* the only G -invariant subspaces of V are $\{0\}, V$.

Example 3.1.1: All one dimensional representations are irreducible.

Theorem 3.2 (Maschke's Theorem): Suppose that $\text{char}(K) \nmid \text{ord}(G)$. Then, every representation of G over the field K can be written as a direct sum of irreducible representations of G .

Proof: Follows immediately from [Theorem 2.3](#). □

Example 3.2.1: Combining [Examples 2.1.1](#) and [2.4.1](#), we have the decomposition

$$\mathbb{C}^3 = \text{span}\{e_1 + e_2 + e_3\} \oplus \text{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$$

into irreducible subrepresentations of S_3 .

When we say that two representations (σ, V) and (σ', V') are isomorphic, denoted $V \cong V'$, we mean that there exists a G -invariant linear bijection $f : V \rightarrow V'$. The following result offers a very powerful characterization of G -invariant maps between irreducible representations.

Theorem 3.3 (Schur's Lemma): Let V, V' be two irreducible representations of G , and let $f : V \rightarrow V'$ be a G -invariant linear map.

1. If $V \not\cong V'$, then $f = 0$.
2. If $V = V'$ and K is algebraically closed, then f is a scalar map, i.e. $f = \lambda \text{id}_V$ for some $\lambda \in K$.

Proof:

1. Suppose that $f \neq 0$. It suffices to show that f is an isomorphism; to do so, we make extensive use of [Lemma 2.6](#).

First, $\ker(f) \subseteq V$ is stable, hence must be one of $\{0\}, V$ by the irreducibility of V . The assumption $f \neq 0$ forces $\ker(f) = \{0\}$, whence f is injective.

Next, $\text{im}(f) \subseteq V'$ is stable, hence must be one of $\{0\}, V'$ by the irreducibility of V' . Again, $f \neq 0$ forces $\text{im}(f) = V'$, whence f is surjective.

2. We have a G -invariant linear bijection $f : V \rightarrow V$; suppose that $f \neq 0$. Let λ be an eigenvalue of f , and observe that the map $(f - \lambda \text{id}_V)$ is also G -invariant; indeed, for all $g \in G, v \in V$,

$$(f - \lambda)(gv) = f(gv) - \lambda gv = gf(v) - g\lambda v = g(f - \lambda)(v).$$

Since λ is an eigenvalue of f , we have $\ker(f - \lambda) \neq \{0\}$. Since $\ker(f - \lambda) \subseteq V$ is stable and V is irreducible, we must have $\ker(f - \lambda) = V$, whence $f - \lambda \text{id}_V = 0$.

□

Remark: Note how the existence of the scalar $\lambda \in K$ is guaranteed by the fact that K is algebraically closed.

Corollary 3.3.1: All \mathbb{C} -linear irreducible representations of finite abelian groups are one dimensional.

Proof: Let (σ, V) be an irreducible representation of a finite abelian group G . Check that for each $g \in G$, the linear map $\sigma(g) : V \rightarrow V$ is G -invariant, since it commutes with all $\sigma(h)$ for $h \in G$. From Schur's Lemma ([Theorem 3.3](#)), each $\sigma(g)$ is a scalar map. As a result, every one dimensional subspace of V is stable. The result now follows from the irreducibility of V . \square

4. Characters

Definition 4.1 (Character): The character χ_V of a representation (σ, V) of G is the function

$$\chi_V : G \rightarrow K, \quad g \mapsto \text{tr}(\sigma(g)).$$

Example 4.1.1: $\chi_V(1) = \dim(V)$.

Observe that $\chi_V(g)$ is precisely the sum of eigenvalues of $\sigma(g)$. The eigenvalues of $\chi_V(g^{-1})$ are simply reciprocals of those of $\chi_V(g)$; in the setting $K = \mathbb{C}$, the following result is immediate from [Proposition 1.5](#).

Proposition 4.2: Suppose that $K = \mathbb{C}$. Then, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

The fact that the trace is invariant under conjugation, i.e. $\text{tr}(tst^{-1}) = \text{tr}(s)$, yields the following result.

Lemma 4.3: χ_V is a class function, i.e. χ_V is constant on conjugacy classes of G .

Lemma 4.4: Isomorphic representations have the same character.

Proof: Let $f : V \rightarrow V'$ be an isomorphism of representations (σ, V) and (σ', V') of G . Then for each $g \in G$, we have $f \circ \sigma(g) = \sigma'(g) \circ f$, hence $\sigma(g) = f^{-1} \circ \sigma'(g) \circ f$. Taking the trace of both sides and using the cyclic property gives $\text{tr}(\sigma(g)) = \text{tr}(\sigma'(g))$ as desired. \square

4.1. Orthogonality of characters

The space K^G of all maps $G \rightarrow K$ forms a vector space over K , with dimension $\text{ord}(G)$. In the setting $K = \mathbb{C}$, we may define the following inner product.

$$\langle \cdot, \cdot \rangle : \mathbb{C}^G \times \mathbb{C}^G \rightarrow \mathbb{C}, \quad (\varphi, \psi) \mapsto \frac{1}{\text{ord}(G)} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

Remark: For characters χ, χ' , [Proposition 4.2](#) gives

$$\langle \chi, \chi' \rangle = \frac{1}{\text{ord}(G)} \sum_{g \in G} \chi(g) \chi'(g^{-1}).$$

Theorem 4.5 (Orthogonality of characters): Suppose that $K = \mathbb{C}$. Let (σ, V) , (σ', V') be two irreducible representations of G .

1. If $V \not\cong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 0$.
2. If $V \cong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 1$.

Proof: Let $\{v_1, \dots, v_n\}$ be a basis of V , and let $\{v'_1, \dots, v'_m\}$ be a basis of V' . Given any linear map $f : V \rightarrow V'$, we will denote $\tilde{f} = \sum_{g \in G} \sigma'(g) \circ f \circ \sigma(g)^{-1}$; recall that \tilde{f} is G -invariant.

1. Observe that Schur's Lemma ([Theorem 3.3](#)) forces all such $\tilde{f} = 0$. In particular, consider the maps e_{ij} defined for each $1 \leq i \leq n, 1 \leq j \leq m$ as

$$e_{ij} : V \rightarrow V', \quad \sum_i \alpha_i v_i \mapsto \alpha_i v'_j.$$

These maps $\{e_{ij}\}$ form a basis of $\mathcal{L}(V, V')$. Check that the matrix entries obey

$$[a \circ e_{ij} \circ b]_{k\ell} = [a]_{ki} [b]_{j\ell},$$

so using $\tilde{e}_{ij} = 0$ gives the relations

$$[\tilde{e}_{ij}]_{kl} = \sum_{g \in G} [\sigma'(g) \circ e_{ij} \circ \sigma(g)^{-1}]_{kl} = \sum_{g \in G} [\sigma'(g)]_{ki} [\sigma(g)^{-1}]_{j\ell} = 0$$

for all $1 \leq i, k \leq n, 1 \leq j, \ell \leq m$. These hold in particular for $i = k, j = \ell$; summing over $1 \leq i \leq n, 1 \leq j \leq m$, we have

$$\begin{aligned} 0 &= \sum_{ij} \sum_{g \in G} [\sigma'(g)]_{ii} [\sigma(g)^{-1}]_{jj} = \sum_{g \in G} \left(\left(\sum_i [\sigma'(g)]_{ii} \right) \left(\sum_j [\sigma(g)^{-1}]_{jj} \right) \right) \\ &= \sum_{g \in G} \chi_V(g) \chi_{V'}(g^{-1}) \\ &= \text{ord}(G) \langle \chi_V, \chi_{V'} \rangle. \end{aligned}$$

2. Schur's Lemma ([Theorem 3.3](#)) forces all such $\tilde{f} = \lambda_f \text{id}_V$ for scalars $\lambda_f \in \mathbb{C}$. To extract λ_f , take the trace of both sides to obtain

$$n \lambda_f = \dim(V) \lambda_f = \sum_{g \in G} \text{tr}(\sigma'(g) \circ f \circ \sigma(g)^{-1}) = \text{ord}(G) \text{tr}(f).$$

With this, each $\tilde{e}_{ij} = \lambda_{ij} \delta_{ij} \text{id}_V$, where $\lambda_{ij} = \text{ord}(G)/n$. Thus, we obtain the relations

$$\sum_{g \in G} [\sigma'(g)]_{ki} [\sigma(g)^{-1}]_{j\ell} = \frac{1}{n} \text{ord}(G) \delta_{ij} \delta_{kl}$$

for all $1 \leq i, j, k, \ell \leq n$. Following a similar process as before,

$$\begin{aligned}
\text{ord}(G) \langle \chi_V, \chi_{V'} \rangle &= \sum_{g \in G} \left(\left(\sum_i [\sigma'(g)]_{ii} \right) \left(\sum_j [\sigma(g)^{-1}]_{jj} \right) \right) \\
&= \sum_{ij} \sum_{g \in G} [\sigma'(g)]_{ii} [\sigma(g)^{-1}]_{jj} \\
&= \sum_{ij} \frac{1}{n} \text{ord}(G) \delta_{ij} \\
&= \text{ord}(G)
\end{aligned}$$

This completes the proof. \square

With this, the characters of irreducible representations form an orthonormal subset of class functions on G . To check whether a representation V is irreducible or not, it is enough to verify that $\langle \chi_V, \chi_V \rangle = 1$.

Corollary 4.5.1: The number of irreducible representations of G (up to isomorphism) is at most the number of conjugacy classes of G .

Given any representation V of G , we can use Maschke's Theorem ([Theorem 3.2](#)) to decompose it as a direct sum of (non-isomorphic) irreducible representations V_1, \dots, V_k , with multiplicities m_1, \dots, m_k . By representing the elements of G as matrices in block diagonal form, we can derive the following result.

Lemma 4.6: Let V_1, \dots, V_k be irreducible representations of G , and let

$$V \cong m_1 V_1 \oplus \dots \oplus m_k V_k.$$

Then,

$$\chi_V = m_1 \chi_{V_1} + \dots + m_k \chi_{V_k}.$$

The multiplicities can be recovered as $m_i = \langle \chi_V, \chi_{V_i} \rangle$.

This immediately tells us that $\chi_V = \chi_{V'}$ if and only if $V \cong V'$. Furthermore, we have the relation

$$\langle \chi_V, \chi_V \rangle = \sum_i m_i^2.$$

4.2. The character table for S_3

We have now established that the trivial representation, the one dimensional representation from [Example 1.4.1](#), and the two dimensional representation from [Example 1.4.2](#) are the only irreducible representations of S_3 . Note that S_3 has three conjugacy classes: $\{e\}$, $\{(12), (23), (31)\}$, and $\{(123), (321)\}$. With this, we can construct the *character table* for S_3 , with each row containing the characters of the group elements with respect to the given representation.

S_3	e	(12)	(23)	(31)	(123)	(321)
Trivial	1	1	1	1	1	1

Sign	1	-1	-1	-1	1	1
Standard	2	0	0	0	-1	-1

Observe that the rows of this table are orthogonal; indeed, so are the columns!

4.3. The character table for S_4

Let S_4 act on \mathbb{C}^4 by permuting the basis vectors $\{e_1, e_2, e_3, e_4\}$, and let (σ, V) denote the induced (natural) representation. Note that each matrix $\sigma(g)$ is a permutation, hence its trace $\chi_V(g)$ is precisely the number of elements of $\{1, 2, 3, 4\}$ fixed by the action of g . With this, we can compute χ_V for each conjugacy class (identified by its cycle type) as follows.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
Trivial	1	1	1	1	1
Sign	1	-1	1	1	-1
V	4	2	0	1	0

Compute

$$\langle \chi_V, \chi_V \rangle = \frac{1}{24}(4^2 + 6 \cdot 2^2 + 8 \cdot 1^2) = 2,$$

whence V is not irreducible. Indeed, we know that $W_1 = \text{span}\{e_1 + e_2 + e_3 + e_4\}$ is a trivial subrepresentation of V of dimension 1. Furthermore, $2 = 1^2 + 1^2$ is the only way of writing 2 as a sum of squares of integers, so V must decompose into precisely two irreducible subrepresentations with both multiplicities 1. This means that $V \cong W_1 \oplus W_3$ for some irreducible representation W_3 of dimension 3. Using $\chi_V = \chi_{W_1} + \chi_{W_3}$, we can compute the character χ_{W_3} and obtain the following.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
W_3	3	1	-1	0	-1

Next, we move on to a different representation of S_4 : consider all subsets of size 2 of $\{1, 2, 3, 4\}$ (of which there are 6), and consider the action on this collection induced by the permutations on the set $\{1, 2, 3, 4\}$.

Remark: If we wish to define a transitive action of G on a set X (and thereby examine the vector space $\text{span}\{e_x\}_{x \in X}$ with the action of G defined via $ge_x = e_{gx}$), we may invoke the Orbit-Stabilizer Theorem, along with the fact that there is only one orbit (all of X) to demand that $\text{ord}(X) \mid \text{ord}(G)$.

Let (τ, V') denote the induced representation. Again, $\chi_{V'}(g)$ is the number of 2-subsets fixed by the action of g . For instance, an element $(ab) \in S_4$ will only fix 2-subsets $\{a, b\}, \{c, d\}$, while an element $(abc) \in S_4$ fixes no 2-subset. With this, we have the following.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
V'	6	2	2	0	0

Compute $\langle \chi_{V'}, \chi_{V'} \rangle = 3 = 1^2 + 1^2 + 1^2$. Again, we may compute $\langle \chi_{V'}, \chi_{W_1} \rangle = 1$ and $\langle \chi_{V'}, \chi_{W_3} \rangle = 1$, which tells us that $V' \cong W_1 \oplus W_3 \oplus W_2$ for some irreducible representation W_2 of dimension 2. Using $\chi_{V'} = \chi_{W_1} + \chi_{W_3} + \chi_{W_2}$, we can compute the character χ_{W_2} .

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
W_2	2	0	2	-1	0

We now have 4 irreducible characters of S_4 ; indeed, we may combine W_3 with the sign representation to get another irreducible representation W'_3 , completing the character table of S_4 .

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
Trivial	1	1	1	1	1
Sign	1	-1	1	1	-1
W_2	2	0	2	-1	0
W_3	3	1	-1	0	-1
W'_3	3	-1	-1	0	1

The last trick uses the following proposition.

Proposition 4.7: Let (σ, V) and $(\tau, \mathbb{C}^\times)$ be representations of G . Then, $(\tau\sigma, V)$ is also a representation of G , where $(\tau\sigma)(g) = \tau(g)\sigma(g)$. Furthermore, $\chi_{\tau\sigma} = \chi_\tau\chi_\sigma$.

Remark: The above proposition is a special case of [Proposition 4.12](#).

4.4. The character of the regular representation

We focus our attention once again to the regular representation, as defined in [Example 1.1.2](#). Note that when G acts on itself by left multiplication, only the identity element 1 fixes all $\text{ord}(G)$ elements of G , while the remaining elements have no fixed points at all. With this, we have the following proposition.

Proposition 4.8: Let (τ, V_G) be the regular representation of G , and let χ_τ denote its character. Then,

$$\chi_\tau(g) = \begin{cases} \text{ord}(G) & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\chi_\tau = d_1\chi_1 + \dots + d_k\chi_k$$

where χ_1, \dots, χ_k are the irreducible characters of G , and each $d_i = \chi_i(1)$ is the dimension of the corresponding irreducible representation.

By simply evaluating $\chi_\tau(1)$, we have the following result.

Corollary 4.8.1: Let χ_1, \dots, χ_k be the irreducible characters of G , and let each $d_i = \chi_i(1)$. Then,

$$\sum_i d_i^2 = \text{ord}(G).$$

Proposition 4.9: Let $f : G \rightarrow \mathbb{C}$ be a class function, and let (σ, V) be a representation of G . Define

$$f_\sigma : V \rightarrow V, \quad v \mapsto \sum_{g \in G} f(g) \sigma(g)(v).$$

Then, f_σ is G -invariant. Furthermore, if (σ, V) is irreducible, then Schur's Lemma ([Theorem 3.3](#)) gives $f_\sigma = \lambda \text{id}_V$, where $\lambda = \text{ord}(G) \langle f, \overline{\chi_\sigma} \rangle / \dim(V)$.

With this construction, we can improve upon [Corollary 4.5.1](#) and precisely count the number of irreducible representations of a group G (up to isomorphism). However, we still do not have any simple way of calculating these representations explicitly.

Theorem 4.10: The number of irreducible representations of G (up to isomorphism) is precisely the number of conjugacy classes of G .

Proof: Let \mathcal{C} be the space of class functions on G , with dimension equal to the number of conjugacy classes of G . Let \mathcal{X} be the subspace of \mathcal{C} spanned by the irreducible characters $\{\chi_i\}$ of G . We claim that $\mathcal{X} = \mathcal{C}$. It is enough to show that the orthocomplement of \mathcal{X} in \mathcal{C} is trivial. For this, pick $f \in \mathcal{C}$ such that all $\langle f, \overline{\chi_i} \rangle = 0$. Let (τ, V_G) be the regular representation of G , and use [Proposition 4.8](#) to write

$$V_G \cong d_1 V_1 \oplus \dots \oplus d_k V_k$$

where $\{(\sigma_i, V_i)\}$ are the irreducible representations corresponding to the characters $\{\chi_i\}$. Using [Proposition 4.9](#), each $f_{\sigma_i} = 0$, hence $f_\tau = 0$. Evaluating f_τ at the element $e_1 \in V_G$, we have

$$\sum_{g \in G} f(g) \sigma(g)(e_1) = \sum_{g \in G} f(g) e_g = 0.$$

Since $\{e_g\}_{g \in G}$ forms a basis of V_G , we must have $f = 0$. □

Remark: The irreducible characters of G span the space of all class functions on G .

4.5. The tensor product of representations

The construction used in [Proposition 4.7](#) generalizes nicely to tensor products of representations, as follows.

Definition 4.11: Let V, V' be two representations of G . Then, the tensor product $V \otimes V'$ is a representation of G induced by the action defined by

$$g(v \otimes v') = (gv) \otimes (gv').$$

Recall that if $\{v_i\}$ is a basis of V and $\{v'_j\}$ is a basis of V' , then $\{v_i \otimes v'_j\}$ forms a basis of $V \otimes V'$. Using this, the next proposition follows.

Proposition 4.12: Let V, V' be two representations of G . Then, $\chi_{V \otimes V'} = \chi_V \chi_{V'}$.

Let's focus on the tensor product $V \otimes V$ and examine the involution defined by

$$\iota : V \otimes V \rightarrow V \otimes V, \quad v \otimes v' \mapsto v' \otimes v.$$

This map has two eigenspaces, corresponding to the eigenvalues 1 and -1 , which we define as $\text{Sym}^2(V)$ and $\text{Alt}^2(V)$ respectively. Furthermore, it is clear that these eigenspaces are stable under the action of G , since the action of G commutes with ι . This gives us the following decomposition of $V \otimes V$.

Proposition 4.13: Let V be a representation of G . Then,

$$V \otimes V \cong \text{Sym}^2(V) \oplus \text{Alt}^2(V).$$

Observe that $\text{Sym}^2(V)$ is spanned by elements of the form $v \otimes v' + v' \otimes v$, while $\text{Alt}^2(V)$ is spanned by elements of the form $v \otimes v' - v' \otimes v$. This, together with $\dim(V \otimes V) = n^2$, tells us that

$$\dim(\text{Sym}^2(V)) = \binom{n}{2} + n, \quad \dim(\text{Alt}^2(V)) = \binom{n}{2}.$$

To compute the characters of these representations, first note that

$$\chi_V^2 = \chi_{\text{Sym}^2(V)} + \chi_{\text{Alt}^2(V)}.$$

Fix $g \in G$ and choose a basis $\{v_i\}$ of V such that the action of g is diagonalized, i.e. $gv_i = \lambda_i v_i$; recall that this is always possible via [Proposition 1.5](#) when we are working over the field $K = \mathbb{C}$. We need only check the action of g on the basis elements of $\text{Sym}^2(V)$ and $\text{Alt}^2(V)$. To this end, compute

$$g(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i \lambda_j (v_i \otimes v_j + v_j \otimes v_i),$$

whence

$$\chi_{\text{Sym}^2(V)}(g) = \sum_{i \leq j} \lambda_i \lambda_j = \sum_{i < j} \lambda_i \lambda_j + \sum_i \lambda_i^2 = \frac{1}{2} \left(\sum_i \lambda_i \right)^2 + \frac{1}{2} \sum_i \lambda_i^2.$$

However, $\sum_i \lambda_i$ and $\sum_i \lambda_i^2$ are precisely $\chi_V(g)$ and $\chi_V(g^2)$. Thus, we have the following result.

Proposition 4.14:

$$\chi_{\text{Sym}^2(V)}(g) = \frac{1}{2} (\chi_V(g)^2 + \chi_V(g^2)), \quad \chi_{\text{Alt}^2(V)}(g) = \frac{1}{2} (\chi_V(g)^2 - \chi_V(g^2)).$$

4.6. The character table for S_5

We proceed much in the same manner as in the computation for S_4 ; the character of the natural representation V of S_5 (induced by the action of S_5 on 5 objects) can be computed by counting the number of fixed points of the corresponding conjugacy class.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
	$\times 1$	$\times 10$	$\times 15$	$\times 20$	$\times 20$	$\times 30$	$\times 24$
Trivial	1	1	1	1	1	1	1
Sign	1	-1	1	1	-1	-1	1
V	5	3	1	2	0	1	0

Since the inner product of the first and last rows is non-zero, V contains a copy of the trivial representation, which we subtract from its character to obtain the representation W_4 . This happens to be irreducible (verified by checking $\langle \chi_W, \chi_W \rangle = 1$), and we obtain another irreducible representation W'_4 by taking the tensor product with the sign representation.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
W_4	4	2	0	1	-1	0	-1
W'_4	4	-2	0	1	1	0	-1

We now compute the characters of $\text{Sym}^2(W_4)$ and $\text{Alt}^2(W_4)$ using [Proposition 4.14](#). To do so, we compute $\chi_{W_4}(g)^2$ and $\chi_{W_4}(g^2)$ for each conjugacy class; the latter can be computed by examining what happens to each cycle type after squaring.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
$\chi_{W_4}^2$	16	4	0	1	1	0	1
$\chi_{W_4}(g^2)$	4	4	4	1	1	0	-1
$\text{Alt}^2(W_4)$	6	0	-2	0	0	0	1
$\text{Sym}^2(W_4)$	10	4	2	1	1	0	0

Note that $\text{Alt}^2(W_4)$ is irreducible, but $\text{Sym}^2(W_4)$ is not. Indeed, the latter clearly has a positive inner product with the trivial representation, hence contains a copy of it which we take away giving us W_9 .

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
W_9	9	3	1	0	0	-1	-1

Looking back, we have found five irreducible representations of S_5 , and hence are left to discover two more. The sum of squares of their dimensions is 70; [Corollary 4.8.1](#) tells us that the sum of squares of the dimensions of the remaining two must be $\text{ord}(S_5) - 70 = 50$. Now, $50 = 1^2 + 7^2 = 5^2 + 5^2$, but the first decomposition is not possible since the trivial and sign representations are the only ones of dimension 1. Thus, both remaining representations are of dimension 5.

Observe that $\langle \chi_{W_9}, \chi_{W_9} \rangle = 2 = 1^2 + 1^2$, hence W_9 must be composed of two irreducible representations. The only way to write 9 as the sum of two dimensions of irreducible representations (which we know are 1, 1, 4, 4, 5, 5, 6) is 4 + 5. Indeed, $\langle \chi_{W_9}, \chi_{W_4} \rangle = 1$, so we take W_4 away leaving us with an irreducible representation W_5 . Taking the tensor product with the sign representation yields our final irreducible representation $W_{5'}$. Thus, the complete character table of S_5 is as follows.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
	$\times 1$	$\times 10$	$\times 15$	$\times 20$	$\times 20$	$\times 30$	$\times 24$
Trivial	1	1	1	1	1	1	1
Sign	1	-1	1	1	-1	-1	1
W_4	4	2	0	1	-1	0	-1
W_4'	4	-2	0	1	1	0	-1
W_5	5	1	1	-1	1	-1	0
W_5'	5	-1	1	-1	-1	1	0
$\text{Alt}^2(W_4)$	6	0	-2	0	0	0	1

Remark: Observe that the columns of this character table are pairwise orthogonal! Indeed, we already know from [Proposition 4.8](#) that the first column of any character table must be orthogonal to the rest. Furthermore, this character table strongly suggests a relation of the form

$$\sum_i |\chi_i(g)|^2 = \frac{\text{ord}(G)}{\text{ord}(\text{Cl}(g))},$$

where χ_i, \dots, χ_k are the irreducible characters of G , and $\text{Cl}(g)$ denotes the conjugacy class of $g \in G$.