Indian Institute of Science Education and Research, Kolkata

Department of Mathematics and Statistics

Summer Project Report, May-July 2022

The hyperbolic plane as the universal cover of the double torus

Submitted by

Satvik Saha

On

14 July, 2022

Under the supervision of

Dr. Somnath Basu

Department of Mathematics and Statistics
IISER Kolkata

Declaration

I declare here that the report included in this project entitled "The hyperbolic plane as the universal cover of the double torus" is the summer internship carried out by me in the Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, India from May 17 to July 14, 2020 under the supervision of Dr. Somnath Basu.

In keeping with general practice of reporting scientific observations, due acknowledgements have been made wherever the work described is based on the findings of other investigators.

July 14, 2022 Satvik Saha
IISER Kolkata

Certificate

It is certified that the summer research work included in the project report entitled "The hyperbolic plane as the universal cover of the double torus" has been carried out by Mr. Satvik Saha under my supervision and guidance. The content of this project report has not been submitted elsewhere for the award of any academic and professional degree.

July 14, 2022	Dr. Somnath Basu
IISER Kolkata	**Project Supervisor**

Contents

Acknowledgements

I would like to thank Dr. Somnath Basu for his guidance and support. I also like to thank my friends and teammates, who made working with them this summer an njoyable experience.

I sincerely thank IISER Kolkata for giving me the opportunity to carry out my summer internship.

July 14, 2022 Satvik Saha
IISER Kolkata

The hyperbolic plane as the universal cover of the double torus

Abstract

We study the double torus, a compact orientable manifold Σ_2 of genus 2, and its universal cover. In particular, we examine the upper half plane $\mathbb{H}^2$ and the Poincaré disc $\mathbb{D}^2$ models of the hyperbolic plane. We use this to compute the fundamental group of Σ_2 and discuss its relationship with the group of deck transformations on $\mathbb{D}^2$. We further note that this covering space induces a metric of constant negative Gaussian curvature on Σ_2.

1 Introduction

The double torus is our primary object of study. It can be described as a topological manifold, obtained from the connected sum of two tori. In other words, take two tori, remove a small open disc from both, then glue them together along their new boundaries. The result is a compact manifold of genus 2, which we refer to as Σ_2. The process of taking connected sums of tori can be repeated, with n contributing tori yielding a compact manifold Σ_g of genus g.

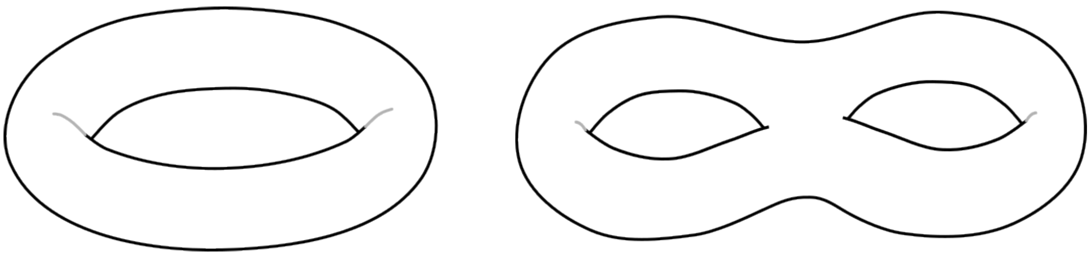

Figure 1: The torus and the double torus, representing the structures Σ_1 and Σ_2.

The manifold Σ_2 has many possible embeddings in the real space $\mathbb{R}^3$, and each of these induces a differential structure, hence a notion of curvature on the surface. In Figure. 1, we see that this representation of Σ_2 has regions of positive curvature where the surface bulges outwards, and regions of negative curvature where it caves inwards.

Theorem 1.1. *If $\Sigma \subseteq \mathbb{R}^3$ is a compact surface, then there exists a point $p \in \Sigma$ where the Gaussian curvature $K(p) > 0$.*

Proof. Consider the map

$$f : \Sigma \to \mathbb{R}, \qquad v \mapsto \|v\|^2.$$

From the compactness of Σ, this must attain a maximum and minimum at some points $p_+, p_- \in \Sigma$. At such points, we have $Df_p \equiv 0$, but

$$Df_p\colon T_p\Sigma \to T_{f(p)}\mathbb{R}, \qquad v \mapsto 2\langle p, v\rangle.$$

Thus, we must have $p_\pm \perp T_{p_\pm}\Sigma$. Now, pick a regular curve γ in Σ based at p_+. Then $f \circ \gamma$ must attain its maximum at 0, hence

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = 0, \qquad \frac{d^2}{dt^2} f(\gamma(t))\Big|_{t=0} \leq 0.$$

This means that

$$2\langle \gamma(0), \dot{\gamma}(0)\rangle = 0, \qquad 2[\|\dot{\gamma}(0)\|^2 + \langle \gamma(0), \ddot{\gamma}(0)\rangle] \leq 0.$$

By construction, $\|\dot{\gamma}\| = 1$, $\gamma(0) = p_+$, and $\|\ddot{\gamma}(0)\|$ is the curvature κ. The latter inequality gives $\kappa \geq 1/\|p_+\|$. Since this is true for any such γ, this must also hold for the principal curvatures k_1, k_2, hence the Gaussian curvature $K(p_+) = k_1 k_2 \geq 1/\|p_+\|^2$. $\square$

However, it is not necessary that all possible structures on Σ_2 can be faithfully represented in $\mathbb{R}^3$. Here, we show that it is possible to assign a constant negative curvature structure to Σ_2, by making use of its universal cover.

In the rest of this section, we run through three models of the torus Σ_1, which will be analogous to our final construction of Σ_2. We also list a few basic results regarding fundamental groups and covering spaces. Next, we give a very brief description of the hyperbolic plane, specifically the upper half plane and Poincaré disc models. Following that, we take a close look at the relationship between a discontinuous group acting on a space and its fundamental domain. There, we exhibit a classical theorem of Poincaré which determines when certain polygons with identified sides can be expressed as a fundamental polygon of a group action, and how to reconstruct that group using the identifications on the sides. Finally, we use these tools to construct our model of Σ_2 by starting with such a polygon in the Poincaré disc $\mathbb{D}^2$, and showing that it is indeed the fundamental polygon of a group of isometries of $\mathbb{D}^2$. Thus, Σ_2 can be expressed as a quotient of its universal cover $\mathbb{D}^2$, giving it the negative curvature structure we seek.

1.1 The torus

We use the torus Σ_1 as a motivating example. One manifestation of this manifold involves its embedding as the surface $T_{a,b}$ in $\mathbb{R}^3$, using the charts

$$\sigma_+\colon (0, 2\pi) \times (0, 2\pi) \to T_{a,b}, \qquad (u, v) \mapsto (+(a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$$

$$\sigma_-\colon (0, 2\pi) \times (0, 2\pi) \to T_{a,b}, \qquad (u, v) \mapsto (-(a - b\cos u)\cos v, (a - b\cos u)\sin v, b\sin u)$$

It can be shown that the First and Second Fundamental Forms of $T_{a,b}$ at some point $p = \sigma_+(u, v)$ are given by

$$\mathrm{I}_p^{\sigma_+} = b^2\, du^2 + (a + b\cos u)^2\, dv^2, \qquad \mathrm{II}_p^{\sigma_+} = b\, du^2 + (a + b\cos u)\cos u\, dv^2.$$

Thus, the principal curvatures at p are given by

$$k_1 = \frac{1}{b}, \qquad k_2 = \frac{\cos u}{a + b\cos u}.$$

Their product gives the Gaussian curvature,

$$K = \frac{\cos u}{b(a + b\cos u)}.$$

This is positive precisely when $u \in (0, \pi/2) \cup (3\pi/2, 2\pi)$, and negative when $u \in (\pi/2, 3\pi/2)$.

Another construction of Σ_1 involves starting with a unit square $I^2 = [0,1] \times [0,1]$, then identifying the sides via the relation described by $(x, 0) \sim (x, 1)$, $(0, y) \sim (1, y)$, along with their symmetric counterparts. The resultant space $I^2/\sim$ is homeomorphic to $T_{a,b}$, but the differential structure it inherits from $I^2 \subset \mathbb{R}^2$ is very different. Indeed, this manifestation of Σ_1 is completely flat, with constant Gaussian curvature of zero everywhere.

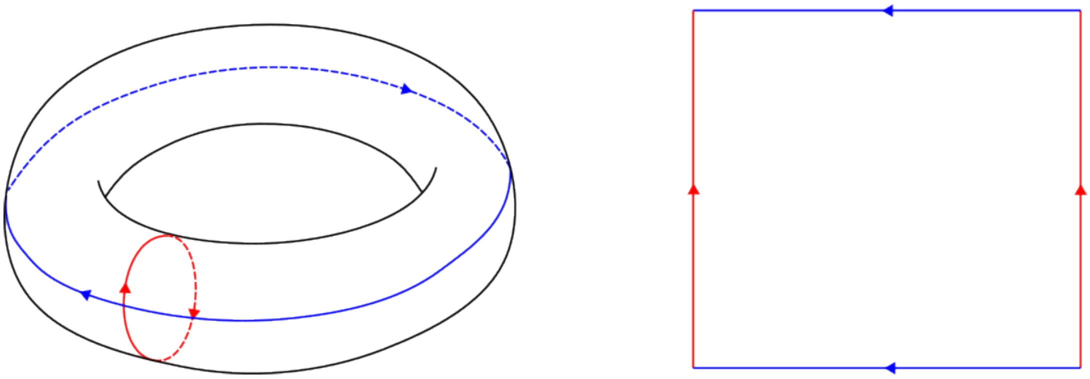

Figure 2: Cutting the torus along the red and blue lines yields the unit square with the marked identifications at the sides.

A third construction of Σ_1 starts with the plane $\mathbb{R}^2$, with the group $\mathbb{Z}^2$ acting on it via translations. In other words, $(m, n) \in \mathbb{Z}^2$ sends the point (x, y) to $(x + m, y + n)$. Identifying orbits together, the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ is again homeomorphic to $T_{a,b}$. This formulation is almost identical to the one with $I^2/\sim$. We will later see that $\mathbb{R}^2$ acts as a universal cover of Σ_1, with the group of deck transformations $\mathbb{Z}^2$. The unit square I^2 is a fundamental polygon of this action of $\mathbb{Z}^2$ on $\mathbb{R}^2$.

1.2 Fundamental groups

The following definitions and results have been paraphrased from *Hatcher* [1].

Given a space X and a point $x_0 \in X$, a path from x_0 to x_1 is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$. If $\gamma(1) = \gamma(0) = x_0$, we call this a loop with basepoint x_0. Two paths γ, η where $\gamma(1) = \eta(0)$ can be concatenated forming a new path $\gamma * \eta$, as follows.

$$\gamma * \eta\colon [0,1] \to X, \qquad t \mapsto \begin{cases} \gamma(2t), & \text{if } 0 \le t < 1/2, \\ \eta(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Furthermore, a curve γ can be reversed, forming

$$\overline{\gamma}\colon [0,1] \to X, \qquad t \mapsto \gamma(1-t).$$

Two paths γ, η with the same endpoints are said to be homotopic if there exists a continuous map $f\colon [0,1]\times[0,1] \to X$ such that $f(t,0) = \gamma(t)$, $f(t,1) = f_1(t) = \eta(t)$ for all $t \in [0,1]$, and $f(0,s) = \gamma(0) = \eta(0)$, $f(1,s) = \gamma(1) = \eta(1)$ for all $s \in [0,1]$. In other words, f is a homotopy between γ and η, relative to the endpoints x_0, x_1.

Lemma 1.2. *Homotopy describes an equivalence relation between paths with fixed endpoints.*

We denote $[\gamma]$ to be the equivalence class of a path γ under homotopy, i.e. the set of all paths homotopic to γ. This is also called the homotopy class of γ.

Consider the set $\Omega(X, x_0)$ of all loops in X based at $x_0 \in X$, and quotient this via homotopy equivalence. The resulting space, denoted $\pi_1(X, x_0)$, forms a group, with the group operation and inverse described as follows: given $[\gamma], [\eta] \in \pi_1(X, x_0)$, define

$$[\gamma]\cdot[\eta] = [\gamma * \eta], \qquad [\gamma]^{-1} = [\overline{\gamma}].$$

Lemma 1.3. *The space $\pi_1(X, x_0)$ comprised of homotopy classes of loops based at x_0 is a group.*

Suppose that $x_0, x_1 \in X$ are joined via a path $\alpha\colon [0,1] \to X$, with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Then, each curve $\gamma \in \Omega(X, x_0)$ can be associated with the curve $(\overline{\alpha} * \gamma) * \alpha \in \Omega(X, x_1)$. Note that the latter is *not* the same curve as $\overline{\alpha} * (\gamma * \alpha)$, but is homotopic to it via reparametrization. Thus, the homotopy class denoted $[\overline{\alpha} * \gamma * \alpha]$ is unambiguous. Note that if $\gamma_1, \gamma_2 \in [\gamma]$, then $(\overline{\alpha} * \gamma_1) * \alpha$ and $(\overline{\alpha} * \gamma_2) * \alpha$ are homotopic. This means that the association of homotopy classes $[\gamma]$ with $[\overline{\alpha} * \gamma * \alpha]$ is well-defined.

Lemma 1.4. *Let α be a path from $x_0 \in X$ to $x_1 \in X$. The map*

$$\Phi\colon \pi_1(X, x_0) \to \pi_1(X, x_1), \qquad [\gamma] \mapsto [\overline{\alpha} * \gamma * \alpha],$$

is a group isomorphism.

As a result, if X is path connected, we may simply speak of *the* fundamental group $\pi_1(X)$ of the space X.

We say that the space X is simply connected if X is path connected, and $\pi_1(X)$ is the trivial group.

Example. Given any two loops γ, η in $\mathbb{R}^n$ with basepoint x_0, the linear interpolation $(t,s) \mapsto (1-s)\gamma(t) + s\eta(t)$ is a homotopy. Thus, $\mathbb{R}^n$ is simply connected, with $\pi_1(\mathbb{R}^n) = 0$.

Lemma 1.5. *A space X is simply connected if and only if there exists a unique homotopy class of paths between any two points in X.*

One basic but important calculation is the fundamental group of the circle S^1.

Theorem 1.6. *The fundamental group of the unit circle S^1 is isomorphic to the infinite cyclic group $\mathbb{Z}$, via the isomorphism*

$$\Phi\colon \mathbb{Z} \to \pi_1(S^1), \qquad n \mapsto [\omega_n], \qquad \omega_n(t) = e^{2\pi i n t}.$$

If $f\colon X \to Y$ is a continuous map such that $f(x_0) = y_0$, it induces a map sending loops $\gamma \in \Omega(X, x_0)$ to loops $f \circ \gamma \in \Omega(Y, y_0)$. Again, if $\gamma_1, \gamma_2 \in [\gamma]$, then $f \circ \gamma_1$ and $f \circ \gamma_2$ are homotopic, making the map sending $[\gamma]$ to $[f \circ \gamma]$ well-defined.

Lemma 1.7. *Let $f\colon (X, x_0) \to (Y, y_0)$ be continuous. The induced map*

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \qquad [\gamma] \mapsto [f \circ \gamma]$$

is a group homomorphism.

Note that if $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ are continuous, then we have $(g \circ f)_* = g_* \circ f_*$.

Corollary 1.7.1. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a homeomorphism. The induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism.*

If $A \subseteq X$, then a retraction from X onto A is a map $f\colon X \to A$ such that f fixes A. Furthermore, a deformation retraction is a continuous map $f\colon X \times [0, 1] \to X$ such that $f(\cdot, 0) = f_0$ is the identity map on X, $f_1(\cdot, 0) = f_1$ is a retraction from X onto A, and each $f(\cdot, t) = f_t$ fixes A. In other words, it is a homotopy between the identity map on A and the retraction f_1, relative to A.

Lemma 1.8. *If X retracts onto $A \subseteq X$, then the group homomorphism $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ for $x_0 \in A$, induced by the inclusion $i\colon A \hookrightarrow X$ is injective. Furthermore, if A is a deformation retract of X, then i_* is a group isomorphism.*

Example. Any convex set K in a metric space deformation retracts to a point $x_0 \in K$, via the homotopy $(x, t) \mapsto x + (x_0 - x)t$. Thus, we trivially have $\pi_1(K) = 0$. This immediately shows that all $\pi_1(\mathbb{R}^n) = 0$.

1.3 Covering spaces

The following definitions and results have been paraphrased from *Hatcher* [1].

Consider a topological space X. A covering space of X consists of a space $\tilde{X}$ and a continuous map $p\colon \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X where each $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$. Furthermore, the restriction of p to each of these open subsets is a homeomorphism onto U_α.

Example. The unit circle S^1 has a covering space $\mathbb{R}$, with

$$p\colon \mathbb{R} \to S^1, \qquad t \mapsto e^{2\pi it}.$$

Note that the pre-image of a sufficiently small open neighbourhood of some point $e^{2\pi it} \in S^1$ consists of infinitely many shifted copies of the same open set in $\mathbb{R}$.

Let $f\colon Y \to X$ be a continuous map, and suppose that X has a covering space $p\colon \tilde{X} \to X$. Then, a map $\tilde{f}\colon Y \to \tilde{X}$ satisfying $p \circ \tilde{f} = f$ is called a lift of f.

Lemma 1.9 (Homotopy lifting)**.** *Let $p\colon \tilde{X} \to X$ be a covering space. Suppose that $f\colon Y \times [0,1] \to X$ is a homotopy with $f(\cdot, 0) = f_0$, and let $\tilde{f}_0$ be a lift of f_0. Then, there exists a unique homotopy $\tilde{f}\colon Y \times [0,1] \to \tilde{X}$ of $\tilde{f}_0$ that lifts f.*

Note that when $Y = \{x_0\}$ is a single point, this says given a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x_0$ and given $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique path $\tilde{\gamma}$ lifting γ with $\tilde{\gamma}(0) = \tilde{x}_0$.

Lemma 1.10. *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. The induced map*

$$p_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0), \qquad [\gamma] \mapsto [p \circ \gamma]$$

is an injective group homomorphism. Furthermore, the image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

We can also lift certain types of general maps.

Lemma 1.11 (Lifting criterion)**.** *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, X_0)$ be a covering space and let $f\colon (Y, y_0) \to (X, x_0)$ be continuous. Furthermore, let Y be path connected at locally path-connected. Then, there exists a lift $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if $f_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Lemma 1.12 (Unique lifting)**.** *Let $p\colon \tilde{X} \to X$ be a covering space and let $f\colon Y \to X$ be continuous, with two lifts $\tilde{f}_1, \tilde{f}_2\colon Y \to \tilde{X}$. If Y is connected and $\tilde{f}_1, \tilde{f}_2$ agree on at least one point of Y, then they must agree on all of Y.*

Consider two covering space $p_1\colon \tilde{X}_1 \to X$, $p_2\colon \tilde{X}_2 \to X$ of X. An isomorphism between them consists of homeomorphism $f\colon \tilde{X}_1 \to \tilde{X}_2$, such that $p_1 = p_2 \circ f$.

Theorem 1.13. *A simply connected covering space of a path connected, locally path connected space X is a covering space of every other path connected covering space of X.*

Such a simply connected covering space of X is called a universal cover.

If $p\colon \tilde{X} \to X$ is a covering space, the isomorphisms $\tilde{X} \to \tilde{X}$ form a group $G(\tilde{X})$ under composition, called the group of deck transformations of $\tilde{X}$. We say that a covering space is normal if given any pair of lifts $\tilde{x}_1, \tilde{x}_2$ of $x \in X$, there exists a deck transformation sending $\tilde{x}_1$ to $\tilde{x}_2$.

Theorem 1.14. *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path connected covering space of the path connected, locally path connected space X, and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$. Then,*

1. *This covering space is normal if and only if H is a normal subgroup.*
2. *The group of deck transformations $G(\tilde{X})$ is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

Note that if $\tilde{X}$ is a normal covering, then $G(\tilde{X}) \cong \pi_1(X, x_0)/H$. Furthermore, if $\tilde{X}$ is a universal covering, then $G(\tilde{X}) \cong \pi_1(X)$.

Theorem 1.15. *Let G be a group acting on a space Y, such that each $y \in Y$ has a neighbourhood U with all possible $g(U)$ for $g \in G$ disjoint. Then,*

1. *The quotient map*
$$p\colon Y \to Y/G, \qquad y \mapsto Gy$$
is a normal covering space.
2. *G is the group of deck transformations of this covering space, if Y is path connected.*
3. *$G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$ if Y is path connected and locally path connected.*

2 The hyperbolic plane

The upper half plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \colon y > 0\}$ (which can also be thought of as part of the complex plane), when equipped with the metric

$$\frac{dx^2 + dy^2}{y^2}$$

forms a metric space with constant negative curvature. Here, geodesics consist of parts of semicircular arcs whose centres lie on the $y = 0$ line, or vertical lines.

Another convenient model of the hyperbolic plane, the Poincaré disc model, can be obtained from the upper half plane via the conformal transformation

$$z \mapsto i\left(\frac{z - i}{z + i}\right).$$

This maps $\mathbb{H}^2$ precisely onto the unit disc $\mathbb{D}^2 = \{z \in \mathbb{C} \colon |z| < 1\}$. The metric now looks like

$$\frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \equiv \frac{4\, dz\, d\bar{z}}{(1 - |z|^2)^2}.$$

Geodesics in this model are also semicircular arcs whose centres lie on the unit circle, or diameters of $\mathbb{D}^2$.

Since both of these First Fundamental Forms arise from a conformal parametrization (the coefficients $E = G$, $F = 0$), angles in the hyperbolic plane (in either model) are precisely the same as the usual Euclidean angles.

3 Fundamental domains

We return to group actions on spaces; specifically, we consider discontinuous groups acting on a space X, which is either $\mathbb{R}^2$ or $\mathbb{D}^2$. Here, a group G acts discontinuously on X if there exists a non-empty open set $V \subseteq X$ such that no two distinct points $x, x' \in V$ satisfy $x' = gx$ for some $g \in G$, i.e. no two points are *equivalent* under G. We may further restrict our attention to groups G which act as *isometries* of the space X.

We say that a non-empty connected open set $D \subseteq X$ is a fundamental domain for G if no two distinct points in D are equivalent under G, and every point in X is equivalent under G to some point in the closure $\overline{D}$.

We are mostly interested in the case where D is a polygon, i.e. the boundary ∂D is a countable (preferably finite) union of *sides* s_i, with only finitely many sides meeting any compact set. Each side is a non-degenerate (contains at least 2 points) connected closed subset of a line. Furthermore, if $s_i \cap s_j \neq \emptyset$ for some $i \neq j$, then this intersection $s_i \cap s_j$ must be a single point z called a *vertex*. Here, z is an endpoint of both s_i and s_j. Finally, if a side s_i has a finite endpoint z, then must be exactly one other side s_j which also has the endpoint z. Such a domain is called a *fundamental polygon* for G.

Example. Recall the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ via translations, with $(m, n) \in \mathbb{Z}^2$ sending $(x, y) \in \mathbb{R}^2$ to $(x + m, y + n)$. Then, the open unit square $(0, 1) \times (0, 1)$ is a fundamental domain for $\mathbb{Z}^2$. Its closure, the unit square $I^2 = [0, 1] \times [0, 1]$, is a fundamental polygon with 4 sides, four vertices.

Lemma 3.1. *Every discontinuous group G of isometries of $\mathbb{R}^2$ or $\mathbb{D}^2$ has a fundamental polygon.*

Proof. Let D be the set of points in $\mathbb{R}^2$ or $\mathbb{D}$ whose distance from the origin 0 is less than their distance from any other point $g(0)$, where $g \in G$. This must be a domain; if $x' = gx$ for some $x, x' \in D$ and $g \in G$, without loss of generality assume that the distance $\rho(0, x) \leq \rho(0, x')$. From $x' \in D$, we must have $\rho(0, x') < \rho(x', g(0)) = \rho(g(x), g(0)) = \rho(x, 0)$ which is a contradiction. Such a construction is called a Dirichlet domain, and the remaining properties can also be demonstrated. $\square$

We are interested in the converse: when is a given polygon D a fundamental polygon of a discontinuous group G? This is answered by Poincaré's classical theorem for fundamental polygons. First, we require some additional information on which sides of the polygon are equivalent. This is done via an *identification*, where each side s is assigned another side s' and an isometry $A(s, s')$ such that

(a) $A(s, s')$ maps s onto s'.

(b) $(s')' = s$ and $A(s', s) = (A(s, s'))^{-1}$.

(c) If $s = s'$, then $A(s, s')$ is the identity on s.

(d) For each side s, there is a neighbourhood V of s such that $A(s, s')(V \cap D) \cap D = \emptyset$.

These isometries $A(s, s')$ are called generators, and they generate the group G. Note that if $A(s, s')$ has order 2, it must be a reflection in the side $s = s'$. This gives us a *reflection relation* $A^2 = 1$ for this generator. We will soon introduce another set of relations, called the *cycle relations* deal with angle sum properties at the vertices.

From D, we construct an *identified polygon* D^* by identifying the sides of D. Define the surjection $\pi\colon \overline{D} \to D^*$ where $\pi(x) = \pi(x')$ if there is a generator A such that $x' = Ax$. For $x, x' \in D^*$, we now define the metric

$$\rho^*(x, x') = \inf \sum_{i=1}^{n} \rho(z_i, z_i')$$

where $\pi(z_1) = x$, each $\pi(z_i') = \pi(z_{i+1})$, and $\pi(z_n') = x'$. Note that this immediately gives

$$\rho^*(\pi(x), \pi(x')) \leq \rho(x, x'),$$

making $\pi\colon \overline{D} \to D^*$ a continuous map. We say that D is *complete* if

(e) $\pi^{-1}(x)$ is finite for each $x \in D^*$, which in turn guarantees that ρ^* is indeed a metric.

(f) D^* is complete in the metric ρ^*.

We now return to cycle relations. Let z_i be a vertex of D, and s_1 be one of the two sides with endpoint z_1. This must be identified with the corresponding side s_1' via the generator $A_1 = A(s_1, s_1')$. Set $z_2 = A_1(z_1)$, and let $s_2 \neq s_1'$ be the (unique) other side with endpoint z_2. Repeat the above process to obtain a sequence of pairs of sides $(s_1, s_1'), (s_2, s_2'), \ldots,$ a sequence of generators $A_1, A_2, \ldots,$ and a sequence of vertices $z_1, z_2, \ldots$. Since all these vertices are related to z_1 via a product of generators, they all map to the same point in D^*. Condition (e) now guarantees that this sequence of vertices must be periodic; since each vertex admits only two sides adjoining it, the sequence of pairs of sides, along with the sequence of generators must also be periodic. Set m to be the least positive integer such that all three of these sequences have period m. We call $(z_1, z_2, \ldots, z_m)$ a cycle of vertices. By setting

$$B = A_m \circ A_{m-1} \circ \cdots \circ A_1,$$

called the *cycle transformation*, we have $Bz_1 = z_1$. Now, note that the images

$$D, A_1^{-1}(D), A_1^{-1} \circ A_2^{-1}(D), \ldots, A_1^{-1} \circ \cdots \circ A_m^{-1}(D) \tag{$\star$}$$

form a star shape around the vertex z_1 (which may not go all the way around). Let $\alpha(z_i)$ be the angle between sides s_{i-1}' and s_i at the vertex z_i, measured from inside the polygon. To ensure that the angle sum in the mentioned star shape is 2π, or a submultiple so that 'folding' around z_1 along these images of the sides leaves no 'gap' or 'overlap', we impose the following *cycle condition.*

(g) For each cycle $(z_1, z_2, \ldots, z_m)$, there exists an integer ν such that

$$\nu \sum_{i=1}^{m} \alpha(z_i) = 2\pi.$$

Note that the last transformation in $(\star)$ is just B^{-1}. By continuing the process $(\star)$ ν times, we rotate around z_1 by a full turn and must return to the original copy of D, thus guaranteeing that B is orientation preserving and that $B^\nu = 1$. In other words, $m\nu$ copies of D 'fit' around the vertex z_1, placed there by these products of generators. Each possible cycle gives us a relation of this form; these are called the *cycle relations.*

A polygon D satisfying all of these conditions (a-g) is called a *Poincaré polygon.*

Theorem 3.2 (Poincaré)**.** *Let D be a Poincaré polygon, and let G be the group generated by the identifying generators. Then, G is discontinuous, D is a fundamental polygon for G, and the cycle and reflection relations form a complete set of relations for G.*

Proof. See *Maskit* [2]. □

The copies $g(D)$ for $g \in G$ are all mutually disjoint, and the copies $g(\overline{D})$ fill up the entirety of the space X, overlapping only at the sides. This immediately shows that X is a covering space of $X/G \cong D^*$. When X is $\mathbb{R}^2$ or $\mathbb{D}^2$, this is a universal cover since these spaces are simply connected.

4 Constructing Σ_2 from a fundamental polygon

Consider the regular octagon D in $\mathbb{D}^2$ with each interior angle $2\pi/8$, illustrated below. Such a shape can be constructed by first creating a $(\pi/2, \pi/8, \pi/8)$ right triangle with one of the $\pi/8$ angles at the origin, then reflecting about the side through the origin to obtain 16 copies. Again, such a triangle can be shown to exist, since in the hyberbolic plane, right triangles satisfy the rules

$$\cos A = \cosh a \sin B, \qquad \tan A = \frac{\cot B}{\cosh c}.$$

where A, B are the angles apart from the right angle, and a, c are the absolute lengths of the sides opposite $A, \pi/2$. Thus, it is enough to draw two line segments starting at the origin with the appropriate lengths a, c (obtained from plugging in $A = B = \pi/8$) at an angle $\pi/8$ from each other, then draw the (unique) line joining the two endpoints.

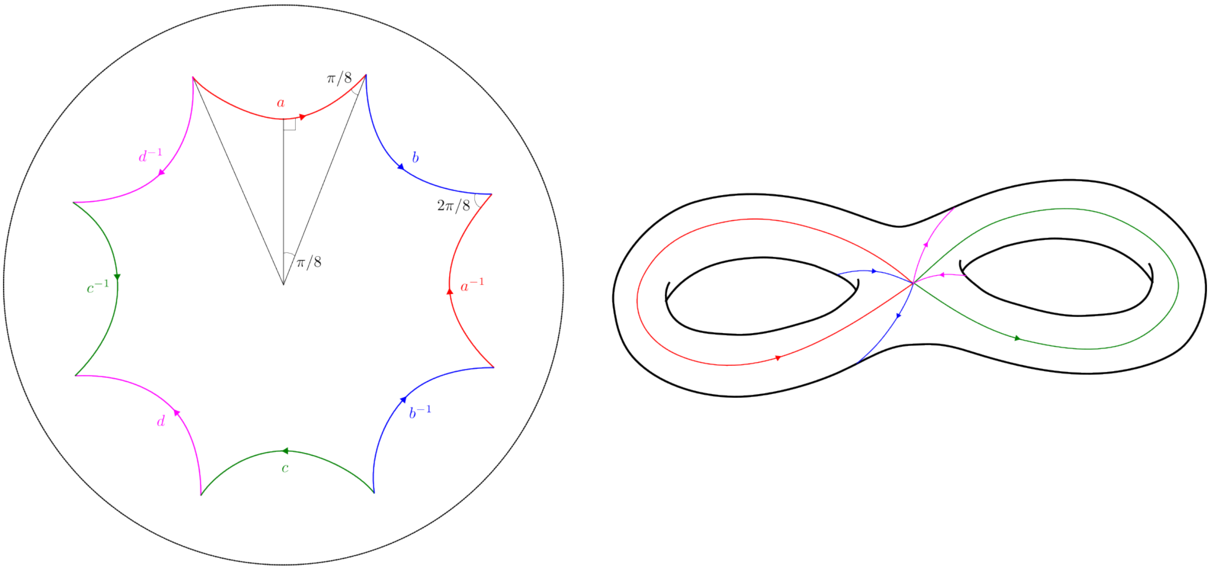

Figure 3: The identified octagonal polygon for the double torus in the Poincaré disc, and the corresponding cuts in the double torus.

It is clear that with the given identification of the sides of D, the corresponding identified polygon D^* is homeomorphic to Σ_2. This can also be shown by starting with a model of Σ_2, then introducing cuts to flatten it out. Since D has finitely many sides, the conditions (a-g) are easily checked. Crucially, the cycle conditions work out because we are in the hyperbolic plane, where eight regular octagons can fit around a point. Poincaré's theorem now guarantees that $\mathbb{D}^2$ is indeed a universal cover of $\Sigma_2 \cong D^* \cong \mathbb{D}^2/G$, where G is the group generated by the cycle relations (there are no reflection relations here). Theorem. 1.15 also guarantees that the quotient map $p\colon \mathbb{D}^2 \to \Sigma_2$ is a normal covering space, that G is the group of deck transformations of this covering space, and that $G \cong \pi_1(\Sigma_2)$.

The space Σ_2 naturally inherits a differential structure from its universal cover $\mathbb{D}^2$; given $x \in \Sigma_2$, look at its pre-image in the fundamental octagon $\overline{D}$ and use the metric there. Note that there is no ambiguity, since the First Fundamental Form in $\mathbb{D}^2$ is identical for points the same distance from the origin, and $\pi^{-1}(x)$ consists of such radially symmetric points. Since the Poincaré disc has a constant negative curvature of -1, so does this model of Σ_2.

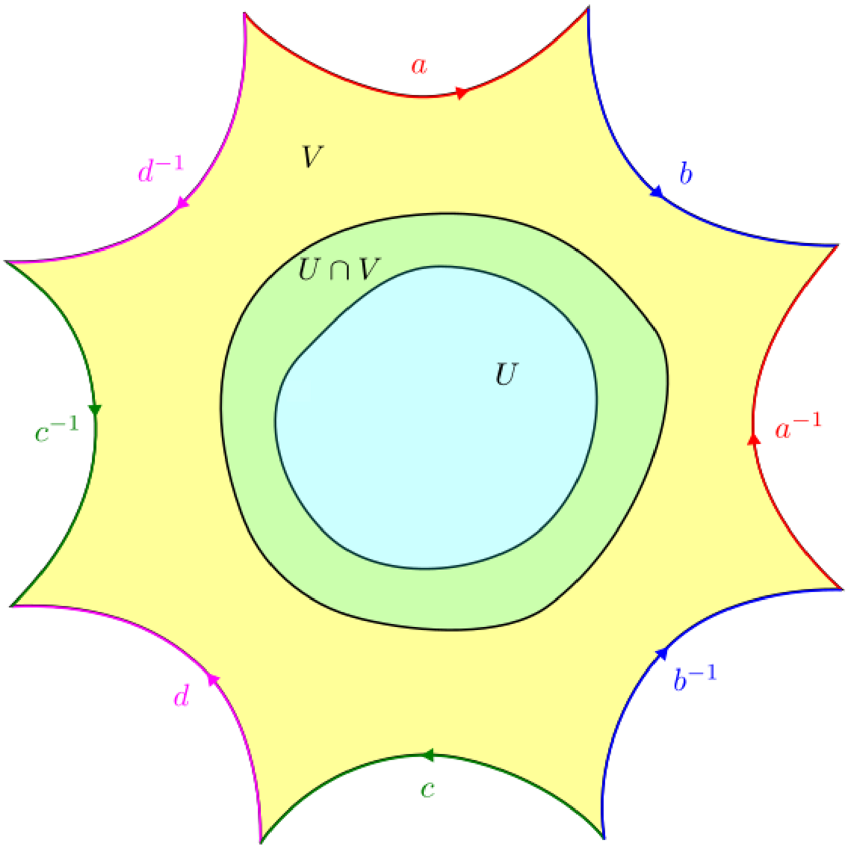

Figure 4: Decomposing the double torus into two sets U (cyan) and V (yellow), with their intersection $U \cap V$ (green). A loop travelling clockwise once in the green region can be pushed outwards to the boundary of the octagon.

We use Van Kampen's theorem to compute the fundamental group of Σ_2, by considering the two indicated open sets U, V in D^*. Note that U is contractible to a point hence its fundamental group is trivial. The set V deformation retracts to the boundary, which when properly identified consists of four loops sharing a single basepoint, i.e. the wedge product of four copies of S^1. Thus, the fundamental group of V is the free group on four generators a, b, c, d. Now, $U \cap V$ is an annulus which deformation retracts to a circle, hence has fundamental group $\mathbb{Z}$. The generator of $\pi_1(U \cap V)$, a single loop around the annulus in $U \cap V$, gets retracted to the boundary when viewed as a loop in V, hence corresponds to the homotopy class $[aba^{-1}b^{-1}cdc^{-1}d^{-1}] \in \pi_1(V)$. This loop must also correspond to the identity element in $\pi_1(U) = 0$, hence we have the relation $aba^{-1}b^{-1}cdc^{-1}d^{-1} = e$. Thus, we have

$$\pi_1(\Sigma_2) = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle.$$

This is isomorphic to the group G, and corresponds directly to its generators and cycle relations.

5 Conclusion

We have seen that it is possible to equip the topological manifold Σ_2 with a differential structure giving it constant negative curvature -1. This is consistent with the Gauus-Bonnet formula

$$\int_{\Sigma_g} K\, dA = 2\pi\chi(\Sigma_g) = 4\pi(1-g).$$

Note that plugging in $g = 2$ and $K = -1$ identically tells us that the total surface area of this structure is 4π. Now, the area of each of the hyperbolic $(\pi/2, \pi/8, \pi/8)$ triangles in the Poincaré disc, used in Figure. 3, is equal to the angle defect

$$\pi - \left(\frac{\pi}{2} + \frac{\pi}{8} + \frac{\pi}{8}\right) = \frac{\pi}{4}.$$

Since 16 of them combine to form our fundamental polygon, it has a total area of 4π, as expected.

We also note that the surfaces Σ_g for genus $g \geq 3$ can also be constructed in a similar manner, by identifying the sides of a $4g$-gon with interior angles $2\pi/4g$. The identification follows the same pattern as with the torus and the double torus. If we denote $[ab] = aba^{-1}b^{-1}$, then the sequence of identifications on the sides of the $4g$-gon read in order will be of the form $w = [a_1b_1][a_2b_2]\dots[a_gb_g]$. The fundamental group $\pi_1(\Sigma_g)$ can also be deduced as before, turning out to be the free group on the generators $a_1, b_1, \dots, a_g, b_g$ with the word w as its only relation.

References

[1] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[2] Bernard Maskit. On Poincaré's Theorem for Fundamental Polygons. *Advances in Mathematics*, 7:219–230, 1971.