

MA3101

Analysis III

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1 Euclidean spaces

1.1 \mathbb{R}^n as a vector space

We are familiar with the vector space \mathbb{R}^n , with the standard inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is defined as

$$\|\mathbf{x} - \mathbf{y}\|^2 = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = \sum_{k=1}^n (x_k - y_k)^2.$$

Exercise 1.1. What are all possible inner products on \mathbb{R}^n ?

Solution. Note that an inner product is a bilinear, symmetric map such that $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Thus, an product map on \mathbb{R}^n is completely and uniquely determined by the values $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = a_{ij}$. Let A be the $n \times n$ matrix with entries a_{ij} . Note that A is a real symmetric matrix with positive entries. Now,

$$\langle \mathbf{x}, \mathbf{e}_j \rangle = x_1 a_{1j} + \cdots + x_n a_{nj} = \mathbf{x}^\top \mathbf{a}_j,$$

where \mathbf{a}_j is the j^{th} column of A . Thus,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{a}_1 y_1 + \cdots + \mathbf{x}^\top \mathbf{a}_n y_n = \mathbf{x}^\top A \mathbf{y}.$$

Furthermore, any choice of real symmetric A with positive entries produces an inner product.

Theorem 1.1 (Cauchy-Schwarz). *Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have*

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

Proof. This is trivial when $\mathbf{w} = \mathbf{0}$. When $\mathbf{w} \neq \mathbf{0}$, set $\lambda = \langle \mathbf{v}, \mathbf{w} \rangle / \|\mathbf{w}\|^2$. Thus,

$$0 \leq \|\mathbf{v} - \lambda \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\lambda \langle \mathbf{v}, \mathbf{w} \rangle + \lambda^2 \|\mathbf{w}\|^2.$$

Simplifying,

$$0 \leq \|\mathbf{v}\|^2 - \frac{|\langle \mathbf{v}, \mathbf{w} \rangle|^2}{\|\mathbf{w}\|^2}.$$

This gives the desired result. Clearly, equality holds if and only if $\mathbf{v} = \lambda \mathbf{w}$. □

Theorem 1.2 (Triangle inequality). *Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have*

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

Proof. Write

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \leq \|\mathbf{v}\|^2 + 2|\langle \mathbf{v}, \mathbf{w} \rangle| + \|\mathbf{w}\|^2.$$

Applying Cauchy-Schwarz gives

$$\|\mathbf{v} + \mathbf{w}\|^2 \leq (\|\mathbf{v}\| + \|\mathbf{w}\|)^2.$$

Equality holds if and only if $\mathbf{v} = \lambda \mathbf{w}$ for $\lambda \geq 0$. □

1.2 \mathbb{R}^n as a metric space

Our previous observations allow us to define the standard metric on \mathbb{R}^n , seen as a point set.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Definition 1.1. For any $\delta > 0$, the set

$$B_\delta(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{y}) < \delta\}$$

is called the open ball centred at $\mathbf{x} \in \mathbb{R}^n$ with radius δ . This is also called the δ neighbourhood of \mathbf{x} .

Definition 1.2. A set U is open in \mathbb{R}^n if for every $\mathbf{x} \in U$, there exists an open ball $B_\delta(\mathbf{x}) \subset U$.

Remark. Every open ball in \mathbb{R}^n is open.

Remark. Both \emptyset and \mathbb{R}^n are open.

Definition 1.3. A set F is closed in \mathbb{R}^n if its complement $\mathbb{R}^n \setminus F$ is open in \mathbb{R}^n .

Remark. Both \emptyset and \mathbb{R}^n are closed.

Remark. Finite sets in \mathbb{R}^n are closed.

Theorem 1.3. *Unions and finite intersections of open sets are open.*

Corollary 1.3.1. *Intersections and finite unions of closed sets are closed.*

Definition 1.4. An interior point x of a set $S \subseteq \mathbb{R}^n$ is such that there is a neighbourhood of x contained within S .

Example. Every point in an open set is an interior point by definition. The interior of a set is the largest open set contained within it.

Definition 1.5. An exterior point x of a set $S \subseteq \mathbb{R}^n$ is an interior point of the complement $\mathbb{R}^n \setminus S$.

Definition 1.6. A boundary point of a set is neither an interior point, nor an exterior point.

Example. The boundary of the unit open ball $B_1(0) \subset \mathbb{R}^n$ is the sphere S^{n-1} .

Definition 1.7. A limit point x of a set $S \subseteq \mathbb{R}^n$ is such that every neighbourhood of x contains a point from S other than itself.

Definition 1.8. The closure of a set $S \subseteq \mathbb{R}^n$ is the union of S and its limit points.

Remark. The closure of a set is the smallest closed set containing it.

Lemma 1.4. Every open set in \mathbb{R}^n is a union of open balls.

Proof. Let $U \subseteq \mathbb{R}^n$ be open. Thus, for every $\mathbf{x} \in \mathbb{R}^n$, we can choose $\delta_{\mathbf{x}} > 0$ such that $B_{\delta_{\mathbf{x}}}(\mathbf{x}) \subset U$. The union of all such open balls is precisely the set U . \square

1.3 \mathbb{R}^n as a topological space

Definition 1.9. A topology on a set X is a collection τ of subsets of X such that

1. $\emptyset \in \tau$
2. $X \in \tau$
3. Arbitrary union of sets from τ belong to τ .
4. Finite intersections of sets from τ belong to τ .

Sets from τ are called open sets.

Example. The Euclidean metric induces the standard topology on \mathbb{R}^n .

Example. The discrete topology on a set X is one where every singleton set is open. This is the topology induced by the discrete metric,

$$d_{\text{discrete}}: X \times X \rightarrow \mathbb{R}, \quad (x, y) \mapsto \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Example. Let X be an infinite set. The collection of sets consisting of \emptyset along with all sets A such that $X \setminus A$ is finite is a topology on X . This is called the Zariski topology.

Example. Consider the set of real numbers, and let τ be the collection \emptyset, \mathbb{R} , and all intervals $(-x, +x)$ for $x > 0$. This constitutes a topology on \mathbb{R} , very different from the usual one.

This topology cannot be induced by a metric; it is not metrizable.

Consider the constant sequence of zeros. In this topology (\mathbb{R}, τ) , this sequence converges to *every* point in \mathbb{R} . Given any $\ell \in \mathbb{R}$, the open neighbourhoods of ℓ are precisely the sets \mathbb{R} and the open intervals $(-x, +x)$ for $x > |\ell|$. The tail of the constant sequence of zeros is contained within every such neighbourhood of ℓ , hence $0 \rightarrow \ell$. Indeed, the element zero belongs to every open set apart from \emptyset in this topology.

Definition 1.10. A topological space is called Hausdorff if for every distinct $x, y \in X$, there exist disjoint neighbourhoods of x and y .

Example. Every metric space is Hausdorff. Given distinct x, y in a metric space (X, d) , set $\delta = d(x, y)/3$ and consider the open balls $B_\delta(x)$ and $B_\delta(y)$.

Lemma 1.5. Every convergent sequence in a Hausdorff space has exactly one limit.

Proof. Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$, and suppose that it converges to distinct x_1 and x_2 . Construct disjoint neighbourhoods U_1 and U_2 around x_1 and x_2 . Now, convergence implies that both U_1 and U_2 contain the tail of $\{x_n\}$, which is impossible since they are disjoint and hence contain no elements in common. \square

Definition 1.11. Given a topological space (X, τ) and a subset $Y \subseteq X$, the collection of sets $U \cap Y$ where $U \in \tau$ is a topology τ_Y on Y . We call this collection the subspace topology on Y , induced by the topology on X .

1.4 Compact sets in \mathbb{R}^n

Definition 1.12. A set $K \subset X$ in a topological space is compact if every open cover of K has a finite sub-cover. That is, for every collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that K is contained in their union, there exists a finite sub-collection $U_{\alpha_1}, \dots, U_{\alpha_k}$ such that K is also contained in their union.

Example. All finite sets are compact.

Example. Given a convergent sequence of real numbers $x_n \rightarrow x$, the collection $\{x_n\}_{n \in \mathbb{N}} \cup \{x\}$ is compact.

Example. In \mathbb{R}^n , compact sets are precisely those sets which are closed and bounded. This is the Heine-Borel Theorem.

Theorem 1.6. The closed intervals $[a, b] \subset \mathbb{R}$ are compact.

Remark. This can be extended to show that any k -cell $[a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $[a, b]$, and suppose that $I_1 = [a, b]$ has no finite sub-cover. Then, at least one of the intervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$ must not have a finite sub-cover; pick one and call it I_2 . Similarly, one of the halves of I_2 must not have a finite

sub-cover; call it I_3 . In this process, we generate a sequence of closed intervals $I_1 \supset I_2 \supset \dots$, none of which have a finite sub-cover. The length of each interval is given by

$$|I_n| = 2^{-n+1}\|b - a\| \rightarrow 0.$$

Now, pick a sequence of points $\{x_n\}$ where each $x_n \in I_n$. Then, $\{x_n\}$ is a Cauchy sequence. To see this, given any $\epsilon > 0$, we can find sufficiently large n_0 such that $2^{-n_0+1}\|b - a\| < \epsilon$. Thus, $x_n \in I_n \subset I_{n_0}$ for all $n \geq n_0$, which means that for any $m, n \geq n_0$, we have $x_m, x_n \in I_{n_0}$ forcing¹

$$\|x_m - x_n\| \leq |I_{n_0}| = 2^{-n_0+1}\|b - a\| < \epsilon.$$

From the completeness of \mathbb{R} , this sequence must converge in \mathbb{R} , specifically in $[a, b]$. Thus, $x_n \rightarrow x$ for some $x \in [a, b]$. It can also be seen that the limit $x \in I_n$ for all $n \in \mathbb{N}$; if not, say $x \notin I_{n_0}$, then $x \in [a, b] \setminus I_{n_0}$ which is open, hence there is an open interval such that $(x - \delta, x + \delta) \cap I_{n_0} = \emptyset$. However, I_{n_0} contains all $x_{n \geq n_0}$, thus this δ -neighbourhood of x would miss out a tail of $\{x_n\}$.

Now, pick the open set $U \in \{U_\alpha\}$ which covers the point x . Thus, $x \in U$ so U contains some non-empty open interval $(x - \delta, x + \delta)$ around x . Choose n_0 such that $2^{-n_0+1}\|b - a\| < \delta$; this immediately gives $I_{n_0} \subseteq (x - \delta, x + \delta) \subset U$. This contradicts that fact that I_{n_0} has no finite sub-cover from $\{U_\alpha\}$, completing the proof. \square

Remark. The fact that Cauchy sequences in \mathbb{R}^n converge isn't immediately obvious; it is a consequence of the completeness of \mathbb{R}^n . Start by noting that \mathbb{R} has the Least Upper Bound property, from which the Monotone Convergence Theorem follows; every monotonic, bounded sequence of reals converges. It can also be shown that any sequence of reals with contain a monotone subsequence, from which it follows that every bounded sequence contains a convergent subsequence (Bolzano-Weierstrass). Finally, it can be shown that if a subsequence of a Cauchy sequence converges, then the entire sequence also converges to the same limit, giving us the desired result for \mathbb{R} . For sequence in \mathbb{R}^n , we may apply this coordinate-wise to obtain the result.

Lemma 1.7. *Compact sets in \mathbb{R}^n are closed and bounded.*

Proof. Consider a compact set $K \subset \mathbb{R}^n$. Let $x \in \mathbb{R}^n \setminus K$, and let $y \in K$. Since $x \neq y$, we choose open balls U_y around y and V_y around x such that $U_y \cap V_y = \emptyset$. Repeating this for all $y \in K$, we generate an open cover $\{U_y\}$ of K consisting of open balls. The compactness of K guarantees that this has a finite sub-cover, i.e. there is a finite set Y such that the collection $\{U_y\}_{y \in Y}$ covers K . As a result, the finite intersection of all V_y for $y \in Y$ is contained within $\mathbb{R}^n \setminus K$. Thus, x is in the exterior of K . Since x was chosen arbitrarily from $\mathbb{R}^n \setminus K$, we see that K is closed.

Now, consider the open cover $\{B_1(x)\}_{x \in K}$, and extract a finite sub-cover of unit open balls. The distance between any two points in K is at most the maximum distance between the centres of any two balls in our sub-cover, plus two. \square

Lemma 1.8. *The intersection of a closed set and a compact set is compact.*

¹If $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, note that $a \leq x_1 < x_2 \leq b$, so

$$|x_2 - x_1| = x_2 - x_1 \leq b - a.$$

Proof. Let $F \subseteq \mathbb{R}^n$ be closed and let $K \subseteq \mathbb{R}^n$ be compact. Suppose that the open cover $\{U_\alpha\}$ of $F \cap K$ has no finite sub-cover. Now the complement $U = F^c$ is open in \mathbb{R}^n , hence the collection $\{U_\alpha\} \cup \{U\}$ is an open cover of K , and hence must admit a finite sub-cover of K . In particular, this must be a finite sub-cover of $F \cap K$. However, we can remove the set U from this sub-cover since it shares no element with $F \cap K$; as a result, our sub-cover must be a finite sub-collection of sets U_α , contradicting our assumption. This shows that $F \cap K$ is compact. \square

Lemma 1.9 (Finite intersection property). *Let $\{K_\alpha\}$ be a collection of compact sets in \mathbb{R}^n which have the property that any finite intersection of them is non-empty. Then,*

$$\bigcap_{\alpha} K_{\alpha} \neq \emptyset.$$

Proof. Suppose to the contrary that the intersection of all K_α is empty. Fix an index β , and note that no element of K_β lies in every K_α . Set $J_\alpha = K_\alpha^c$, whence the collection $\{J_\alpha : \alpha \neq \beta\}$ is an open cover of K_β . This must admit a finite sub-cover $\{J_{\alpha_1}, \dots, J_{\alpha_k}\}$ of K_β . Thus, we must have

$$K_\beta^c \cup J_{\alpha_1} \cup \dots \cup J_{\alpha_k} = \mathbb{R}^n.$$

This immediately gives the contradiction

$$K_\beta \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_k} = \emptyset. \quad \square$$

Theorem 1.10 (Heine-Borel). *Compact sets in \mathbb{R}^n are precisely those that are closed and bounded.*

Proof. Given a compact set in \mathbb{R}^n , we have already shown that it must be closed and bounded. Next, if $F \subset \mathbb{R}^n$ is closed and bounded, it can be enclosed within a k -cell which we know is compact. Thus, F is the intersection of the closed set F and the compact k -cell, proving that F must be compact. \square

1.5 Continuous maps

Definition 1.13. A map $f: X \rightarrow Y$ is continuous if the pre-image of every open set from Y is open in X .

Lemma 1.11. *A map $f: X \rightarrow Y$ is continuous if the pre-image of every closed set from Y is closed in X .*

Theorem 1.12. *The projection maps $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $\mathbf{x} \mapsto x_i$ are continuous.*

Proof. Let $U \subseteq \mathbb{R}$ be open; we claim that $\pi_i^{-1}(U)$ is open. Pick $\mathbf{x} \in \pi_i^{-1}(U)$, and note that $\pi_i(\mathbf{x}) = x_i \in U$. Thus, there exists $\delta > 0$ such that $(x_i - \delta, x_i + \delta) \subset U$. Now examine $B_\delta(\mathbf{x})$; for any point \mathbf{y} within this open ball, we have $d(\mathbf{x}, \mathbf{y}) < \delta$ hence

$$|x_i - y_i|^2 \leq \sum_{k=1}^n (x_k - y_k)^2 = d(\mathbf{x}, \mathbf{y})^2 < \delta^2.$$

In other words, $\pi_i(\mathbf{y}) = y_i \in (x_i - \delta, x_i + \delta)$, hence $\pi_i B_\delta(\mathbf{x}) \subseteq (x_i - \delta, x_i + \delta) \subset U$. Thus, given arbitrary $\mathbf{x} \in \pi_i^{-1}(U)$, we have found an open ball $B_\delta(\mathbf{x}) \subset \pi_i^{-1}(U)$. \square

Lemma 1.13. *Finite sums, products, and compositions of continuous functions are continuous.*

Corollary 1.13.1. *A function $f: [a, b] \rightarrow \mathbb{R}^n$ is continuous if and only if the components, $\pi_i \circ f$, are continuous.*

Theorem 1.14. *All polynomial functions of the coordinates in \mathbb{R}^n are continuous.*

Example. The unit sphere $S^{n-1} \subset \mathbb{R}^n$ is closed. It is by definition the pre-image of the singleton closed set $\{1\}$ under the continuous map

$$\mathbf{x} \mapsto x_1^2 + \cdots + x_n^2.$$

Theorem 1.15. *The continuous image of a compact set is compact.*

Proof. Let $f: X \rightarrow Y$ be continuous, where Y is the image of the compact set X , and let $\{U_\alpha\}$ be an open cover of Y . Then, the collection $\{f^{-1}(U_\alpha)\}$ is an open cover of X . Using the compactness of X , extract a finite sub-cover $f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_k})$ of X . It follows that the collection $U_{\alpha_1}, \dots, U_{\alpha_k}$ is a finite sub-cover of Y . \square

1.6 Connectedness

Definition 1.14. Let X be a topological space. A separation of X is a pair U, V of non-empty disjoint open subsets such that $X = U \cup V$.

Definition 1.15. A connected topological space is one which cannot be separated.

Lemma 1.16. *A topological space X is connected if and only if the only sets which are both open and closed are \emptyset and X .*

Example. The intervals $(a, b) \subset \mathbb{R}$ are connected. To see this, suppose that U, V is a separation of (a, b) . Pick $x \in U, y \in V$, and without loss of generality let $x < y$. Define $S = [x, y] \cap U$, and set $c = \sup S$. It can be argued that $c \in (a, b)$, but $c \notin U, c \notin V$, using the properties of the supremum.

Theorem 1.17. *The continuous image of a connected set is connected.*

Proof. Let f be a continuous map on the connected set X , and let Y be the image of X . If U, V is a separation of Y , then it can be shown that $f^{-1}(U), f^{-1}(V)$ constitutes a separation of X , which is a contradiction. \square

Definition 1.16. A path γ joining two points $x, y \in X$ is a continuous map $\gamma: [a, b] \rightarrow X$ such that $\gamma(a) = x, \gamma(b) = y$.

Definition 1.17. A set in X is path connected if given any two distinct points in X , there exists a path joining them.

Lemma 1.18. *Every path connected set is connected.*

Proof. Let X be path connected, and suppose that U, V is a separation of X . Then, pick $x \in U, y \in V$, and choose a path $\gamma: [0, 1] \rightarrow X$ between x and y . The sets $f^{-1}(U)$ and $f^{-1}(V)$ separate the interval $[0, 1]$, which is a contradiction. \square

Example. All connected sets are not path connected. Consider the topologist's sine curve,

$$\left\{ \left(x, \sin \frac{1}{x} \right) : 0 < x \leq 1 \right\} \cup \{(0, 0)\}.$$

Definition 1.18. The ϵ neighbourhood of a set K in a metric space X is defined as

$$\bigcup_{a \in K} B_\epsilon(a) = \bigcup_{a \in K} \{x \in X : d(x, a) < \epsilon\}.$$

Exercise 1.2. Let $K \subseteq \mathbb{R}^n$ be compact, and define $f: \mathbb{R}^n \rightarrow \mathbb{R}$,

$$f(x) = \text{dist}(x, K) = \inf_{a \in K} d(x, a).$$

Show that f is continuous on \mathbb{R}^n , and $f^{-1}(\{0\}) = K$.

Exercise 1.3. If $K \subseteq \mathbb{R}^n$ is compact and $K \cap L = \emptyset$, then

$$\text{dist}(K, L) = \inf_{a \in K} \text{dist}(a, L) > 0.$$

Exercise 1.4. If $K \subseteq \mathbb{R}^n$ is compact and U is an open set containing K , then there exists $\epsilon > 0$ such that U contains the ϵ neighbourhood of K .

Is the compactness of K necessary?

1.7 Differentiability

Definition 1.19. Let $f: (a, b) \rightarrow \mathbb{R}^n$, and let $f_i = \pi_i \circ f$ be its components. Then, f is differentiable at $x \in (a, b)$ if the following limit exists.

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Definition 1.20. Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \rightarrow \mathbb{R}^m$. Then, f is differentiable at $x \in U$ if there exists a linear transformation $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - \lambda h\|}{\|h\|} = 0.$$