

# MA 1101 : Mathematics I

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**Solution 1.**

Let  $X, Y, Z \neq \emptyset$ , let  $f : X \rightarrow Y$  and let  $g : Y \rightarrow Z$ . We have

$$\begin{aligned} g \circ f : X &\rightarrow Z, \\ x &\mapsto g(f(x)) \end{aligned}$$

- (i) If  $f$  and  $g$  are injective, for arbitrary  $x_1, x_2 \in X$ ,

$$\begin{aligned} (g \circ f)(x_1) &= (g \circ f)(x_2) \\ \Rightarrow g(f(x_1)) &= g(f(x_2)) \\ \Rightarrow f(x_1) &= f(x_2) && \text{(Injectivity of } g) \\ \Rightarrow x_1 &= x_2 && \text{(Injectivity of } f) \end{aligned}$$

Hence,  $g \circ f$  is injective.  $\square$

- (ii) If  $g$  is surjective, it follows that for all  $z_i \in Z$ , there exists  $y_i \in Y$  such that  $g(y_i) = z_i$ . If  $f$  is also surjective, it follows that for all these  $y_i$ , there exists  $x_i \in X$  such that  $f(x_i) = y_i$ . Hence, for all  $z_i \in Z$ , there exists  $x_i \in X$  such that  $(g \circ f)(x_i) = g(f(x_i)) = z_i$ . Therefore,  $g \circ f$  is surjective.  $\square$

- (iii) If  $f$  and  $g$  are bijective,  $g \circ f$  must be injective from (i) and surjective from (ii). Therefore,  $g \circ f$  is bijective.  $\square$

- (iv) If  $g \circ f$  is surjective, it follows that for all  $z_i \in Z$ , there exists  $x_i \in X$  such that  $g(f(x_i)) = z_i$ . Since  $f$  is a function, for all these  $x_i$ , there must exist  $y_i \in Y$  such that  $f(x_i) = y_i$ . Hence, for all  $z_i \in Z$ , there exists  $y_i \in Y$  such that  $g(y_i) = z_i$ . Therefore,  $g$  is surjective.  $\square$

Consider

$$\begin{aligned} f : \{0, 1, 2\} &\rightarrow \{0, 1\}, \\ x &\mapsto 0. \\ g : \{0, 1\} &\rightarrow \{0\}, \\ x &\mapsto 0. \end{aligned}$$

Clearly, we have  $g \circ f : \{0, 1, 2\} \rightarrow \{0\}, x \mapsto 0$  is surjective, yet  $f$  is not surjective since there is no  $x \in \{0, 1, 2\}$  such that  $f(x) = 1$ .

- (v) Let  $x_1, x_2 \in X$  such that  $x_1 \neq x_2$ . We have two cases :  $f(x_1) = f(x_2)$  or  $f(x_1) \neq f(x_2)$ . If  $f(x_1) = f(x_2) = y \in Y$ , we must have  $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$ . This contradicts the injectivity of  $g \circ f$ . Hence, we must have  $f(x_1) \neq f(x_2)$ . Therefore,  $f$  is injective.  $\square$

Consider

$$\begin{aligned} f : \{0\} &\rightarrow \{0, 1\}, \\ x &\mapsto 0. \\ g : \{0, 1\} &\rightarrow \{0\}, \\ x &\mapsto 0. \end{aligned}$$

Clearly, we have  $g \circ f : \{0\} \rightarrow \{0\}, x \mapsto 0$  is injective, yet  $g$  is not injective since  $g(0) = g(1) = 0$ .

- (vi) We have  $g \circ f$  is injective and  $f$  is surjective. Let  $y_1, y_2 \in Y$  such that  $g(y_1) = g(y_2)$ . The surjectivity of  $f$  implies that there exist  $x_1, x_2 \in X$  such that  $f(x_1) = y_1$  and  $f(x_2) = y_2$ . Hence, we have  $g(f(x_1)) = g(f(x_2)) \Leftrightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$ . The injectivity of  $g \circ f$  implies  $x_1 = x_2$ , from which we have  $y_1 = y_2$ . Therefore,  $g$  is injective.  $\square$

**Solution 2.**

Let  $W, X, Y, Z \neq \emptyset$ , and let  $f : W \rightarrow X$ ,  $g : X \rightarrow Y$  and  $h : Y \rightarrow Z$ . We will show that

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Clearly, we have  $h \circ g : X \rightarrow Z$ , hence  $(h \circ g) \circ f : W \rightarrow Z$ . Also,  $g \circ f : W \rightarrow Y$ , hence  $h \circ (g \circ f) : W \rightarrow Z$ . Thus, the domains and codomains of both these functions are equal.

Let  $w \in W$ ,  $x = f(w) \in X$ ,  $y = g(x) \in Y$ ,  $z = h(y) \in Z$ . Thus,  $(h \circ g)(x) = h(g(x)) = h(y) = z$ , so  $((h \circ g) \circ f)(w) = (h \circ g)(f(w)) = (h \circ g)(x) = z$ .

Again,  $(g \circ f)(w) = g(f(w)) = g(x) = y$ , so  $(h \circ (g \circ f))(w) = h((g \circ f)(w)) = h(y) = z$ .

Hence, for all  $w \in W$ ,  $((h \circ g) \circ f)(w) = (h \circ (g \circ f))(w) \in Z$ . Therefore, these two functions are equal.  $\square$

**Solution 3.**

(i) We examine

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}, \\ x &\mapsto x^2 + x. \end{aligned}$$

Clearly,  $f$  is not injective, since  $f(0) = f(-1) = 0$ .

Note that for all  $x \in \mathbb{R}$ ,

$$f(x) = x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4}$$

Hence, for all  $y < -1/4$ , e.g.,  $y = -1$ , there is no  $x \in \mathbb{R}$  such that  $f(x) = y$ .

Therefore,  $f$  is neither injective, nor surjective.  $\square$

(ii) We examine

$$\begin{aligned} f : \mathbb{N} &\rightarrow \mathbb{N}, \\ n &\mapsto \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

Clearly,  $f$  is not injective, since  $f(1) = f(2) = 1$ .

Note that for all  $k \in \mathbb{N}$ ,  $f(2k-1) = k$ . Also,  $2k-1 \in \mathbb{N}$ .

Therefore,  $f$  is not injective, but is surjective.  $\square$

(iii) We examine

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}, \\ x &\mapsto x + \lfloor x \rfloor. \end{aligned}$$

Let  $x_1, x_2 \in \mathbb{R}$ . Thus,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow x_1 + \lfloor x_1 \rfloor &= x_2 + \lfloor x_2 \rfloor \\ \Rightarrow x_1 - x_2 &= -\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \end{aligned}$$

It follows that  $k = x_1 - x_2 \in \mathbb{Z}$ , so

$$\begin{aligned} \lfloor x_1 \rfloor &= \lfloor k + x_2 \rfloor \\ &= k + \lfloor x_2 \rfloor \\ &= x_1 - x_2 + \lfloor x_2 \rfloor \\ x_1 - x_2 &= \lfloor x_1 \rfloor - \lfloor x_2 \rfloor \end{aligned}$$

Hence, we have  $x_1 = x_2$ . Therefore,  $f$  is injective.

For  $f(x) = 2k+1 \in \mathbb{Z} \subset \mathbb{R}$ ,  $k \in \mathbb{Z}$ , we must have  $x + \lfloor x \rfloor = 2k+1$ , so  $x \in \mathbb{Z}$ . Thus,  $f(x) = 2x = 2k+1 \Rightarrow x = k + \frac{1}{2} \notin \mathbb{Z}$ , a contradiction. Hence, there is no  $x \in \mathbb{R}$  such that  $f(x) = 2k+1$ ,  $k \in \mathbb{Z}$ .

Therefore,  $f$  is injective, but not surjective.  $\square$

(iv) We examine

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}, \\ x &\mapsto x - \lfloor x \rfloor. \end{aligned}$$

Clearly,  $f$  is not injective, since  $f(0) = f(1) = 0$ .

Note that  $\lfloor x \rfloor$  is the *greatest* integer less than or equal to  $x$ . Let  $x - \lfloor x \rfloor = \delta$ , where  $\delta \in \mathbb{R}$ . We must have  $\lfloor x \rfloor \leq x$ , so  $\delta \geq 0$ . If  $\delta \geq 1$ , we would have  $x - (1 + \lfloor x \rfloor) = \delta - 1 \geq 0 \Rightarrow x \geq 1 + \lfloor x \rfloor$ , a contradiction. Hence,  $\delta < 1$ , and  $f(x) < 1$  for all  $x \in \mathbb{R}$ , i.e., there is no  $x \in \mathbb{R}$  such that  $f(x) = 2$ .

Therefore,  $f$  is neither injective, nor surjective.  $\square$

(v) We examine

$$\begin{aligned} f : \mathbb{R} \setminus \{1\} &\rightarrow \mathbb{R}, \\ x &\mapsto \frac{x+1}{x-1}. \end{aligned}$$

Let  $x_1, x_2 \in \mathbb{R} \setminus \{1\}$ . Thus,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow \frac{x_1+1}{x_1-1} &= \frac{x_2+1}{x_2-1} \\ \Rightarrow (x_1+1)(x_2-1) &= (x_1-1)(x_2+1) & (x \neq 1) \\ \Rightarrow x_1x_2 - x_1 + x_2 - 1 &= x_1x_2 + x_1 - x_2 - 1 \\ \Rightarrow x_1 &= x_2 \end{aligned}$$

Hence, we have  $x_1 = x_2$ . Therefore,  $f$  is injective.

Note that for  $f(x) = 1 \in \mathbb{R}$ , we require  $x+1 = x-1$ , a contradiction. Hence, there is no  $x \in \mathbb{R} \setminus \{1\}$  such that  $f(x) = 1$ .

Therefore,  $f$  is injective, but not surjective.  $\square$

(vi) We examine

$$\begin{aligned} f : (-1, 1) &\rightarrow \mathbb{R}, \\ x &\mapsto \frac{x}{1-|x|}. \end{aligned}$$

Let  $x_1, x_2 \in (-1, 1)$ . Thus,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow \frac{x_1}{1-|x_1|} &= \frac{x_2}{1-|x_2|} \\ \Rightarrow x_1(1-|x_2|) &= x_2(1-|x_1|) & (|x| \neq 1) \\ \Rightarrow x_1 - x_2 &= x_1|x_2| - x_2|x_1| \end{aligned}$$

If either  $x_1$  or  $x_2$ , we are forced to have  $x_1 = x_2 = 0$ .

Note that  $x_1$  and  $x_2$  cannot have opposite signs, since  $1 - |x| > 0$  for all  $x \in (-1, +1)$ .

We are left with  $x_1$  and  $x_2$  sharing the same sign. Thus, we have  $x_1/|x_1| = x_2/|x_2| = \pm 1$ , so  $x_1|x_2| = x_2|x_1|$ , and  $x_1 = x_2$ .

In all cases, we have  $x_1 = x_2$ . Therefore,  $f$  is injective.

We will now show that  $f$  is surjective. Let  $y = f(x) \in \mathbb{R}$ .

For  $y = 0$ , we have  $x = 0$ .

For  $y > 0$ , we have  $x > 0$ , so

$$y = \frac{x}{1-x} \Rightarrow x = \frac{y}{1+y} < 1 \quad (1+y > y > 0)$$

Clearly, for every  $y > 0$ , there exists  $x \in (0, 1)$  such that  $f(x) = y$ .

For  $y < 0$ , we have  $x < 0$ , so

$$y = \frac{x}{1+x} \Rightarrow x = \frac{y}{1-y} > -1 \quad (0 > y > y-1)$$

Again, for every  $y < 0$ , there exists  $x \in (-1, 0)$  such that  $f(x) = y$ .

Therefore,  $f$  is both injective and surjective. □