The Stone-Weierstrass Theorem

Approximating continuous functions by smooth functions

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Approximation in metric spaces

Approximation

The object α approximates β, to some degree of accuracy ϵ.

$$\updownarrow$$

$$d(\alpha, \beta) < \epsilon$$

$$\updownarrow$$

α lies within a narrow region centred at β.

Approximation

The sequence $\{\alpha_n\}$ converges to β.

$$\updownarrow$$

Given any ϵ, there exists n_0 such that for all $n \geq n_0$,

$$d(\alpha_n, \beta) < \epsilon$$

$$\updownarrow$$

Every neighbourhood of β contains some tail of $\{\alpha_n\}$.

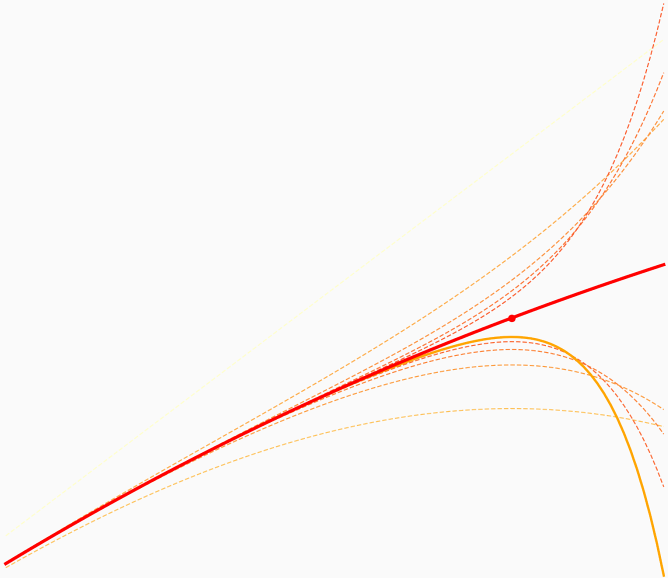
Approximation

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \to \frac{\pi}{4}$$

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \to \arctan x$$

This series converges *uniformly* on $[-1, +1]$.

Approximating $\arctan x$



Integrating a uniformly convergent series

A power series converges uniformly on its interval of convergence.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2k}}{(2k)!}$$
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2k+1}}{(2k+1)!}$$

$$\int_a^b \cos x \, dx = \sin b - \sin a.$$

Integrating a uniformly convergent series

If $f_n \to f$ uniformly on $[a, b]$, then given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x \in [a, b]$,

$$|f_n(x) - f(x)| < \epsilon.$$

$$-\epsilon(b-a) < \int_a^b f_n(x) - f(x)\, dx < \epsilon(b-a)$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$$

Metric spaces of functions

For every pair of functions f, g on E, define

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in E} |f(x) - g(x)|.$$

- $d(f, g) \geq 0$, and $d(f, g) = 0$ if and only if $f = g$.
- $d(f, g) = d(g, f)$.
- $d(f, h) \leq d(f, g) + d(g, h)$.

We require $d(f, g)$ to be finite, so all functions under consideration must be bounded.

Metric spaces of functions

The function g approximates f, to some degree of accuracy ϵ.

$$\updownarrow$$

$$d(g, f) < \epsilon$$

$$\updownarrow$$

The curve g lies within a narrow strip centred at f.

$$f - \epsilon \ \leq \ g \ \leq \ f + \epsilon$$

The sequence $\{f_n\}$ converges to f.

$$\updownarrow$$

Given any ϵ, there exists n_0 such that for all $n \geq n_0$,

$$d(f_n, f) < \epsilon$$

$$\updownarrow$$

The sequence $f_n \to f$ *uniformly*.

Given a real valued function f on a domain X, can we find a sequence of 'nice' functions (typically polynomials) which converge uniformly to f?

Uniform Limit Theorem

Let $\{f_n\}$ be a sequence of continuous, real valued functions on X. If $f_n \to f$ uniformly on X, then f is continuous.

Fix $x_0 \in X$. Then for sufficiently high n, there is a δ neighbourhood of x_0 on which

$$\begin{aligned}|f(x) - f(x_0)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< 3\epsilon.\end{aligned}$$

$\square$

Given any real valued, continuous function on a compact interval $[a, b]$, there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[a, b]$.

The Bernstein polynomials can be used to prove this constructively.

The Bernstein polynomials

Bernstein polynomials

The Bernstein polynomials are defined as

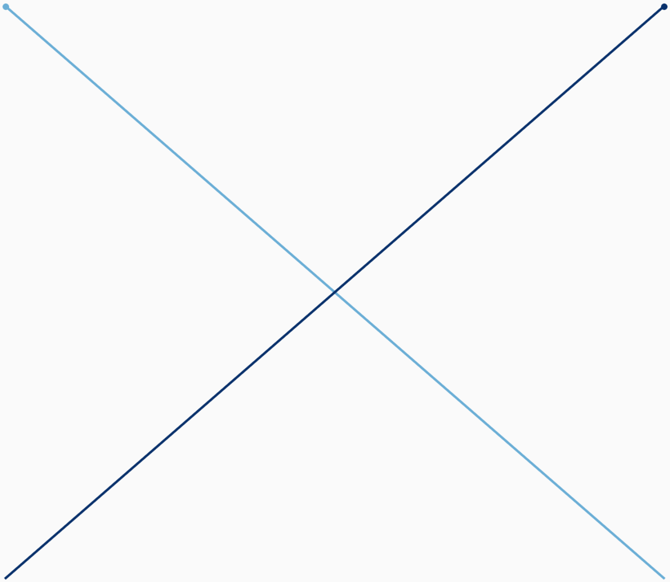
$$B_n^k(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

Note that B_n^k peaks at $x = k/n$.

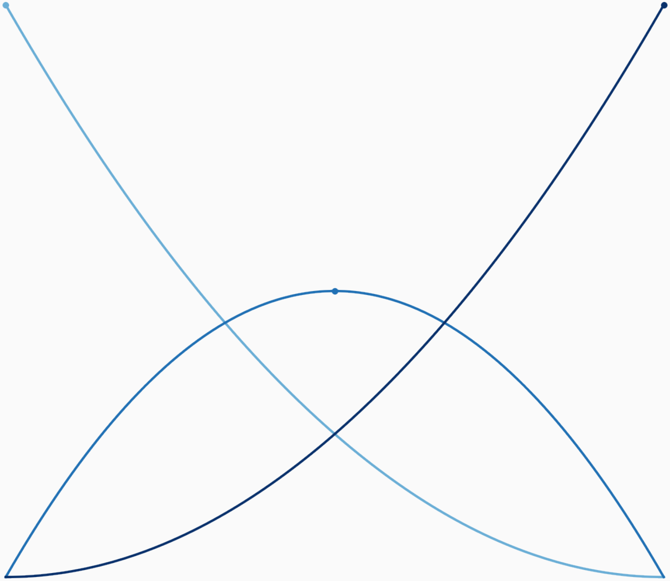
The Bernstein expansion of a function on $[0, 1]$ is defined as

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

Bernstein polynomials



Bernstein polynomials



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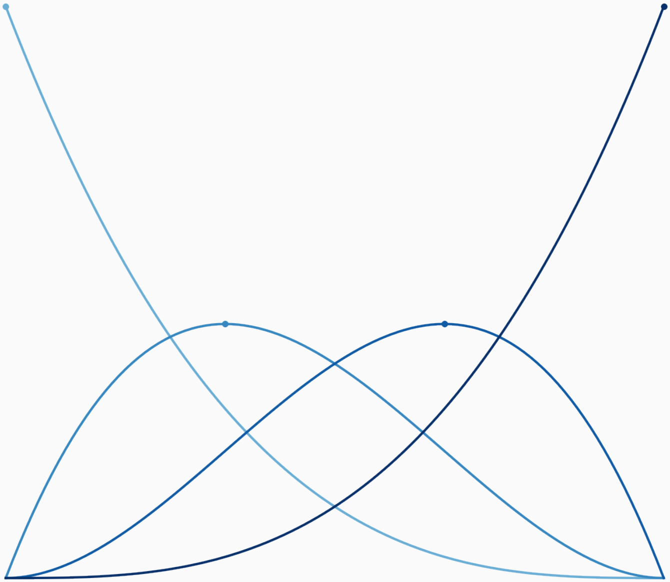

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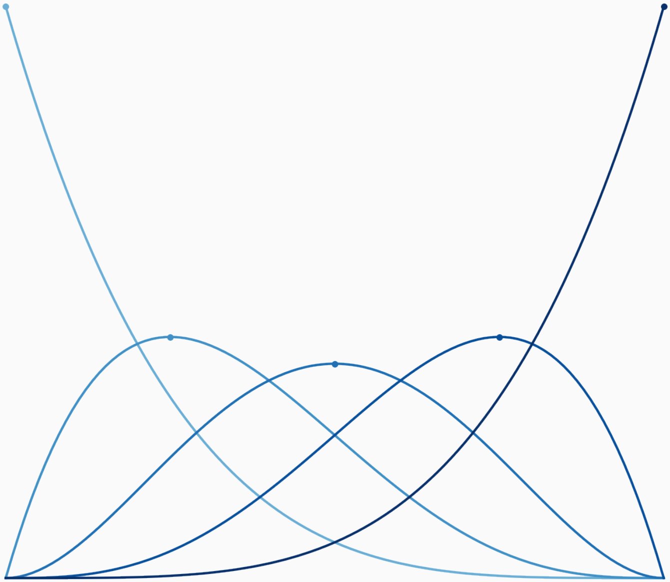

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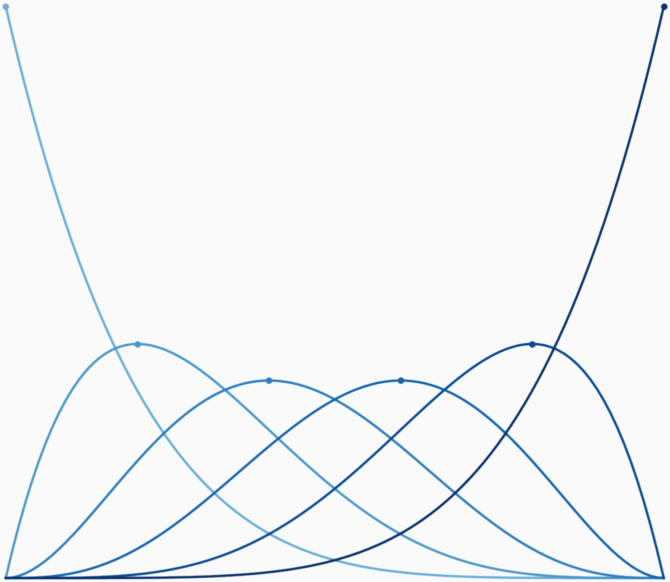

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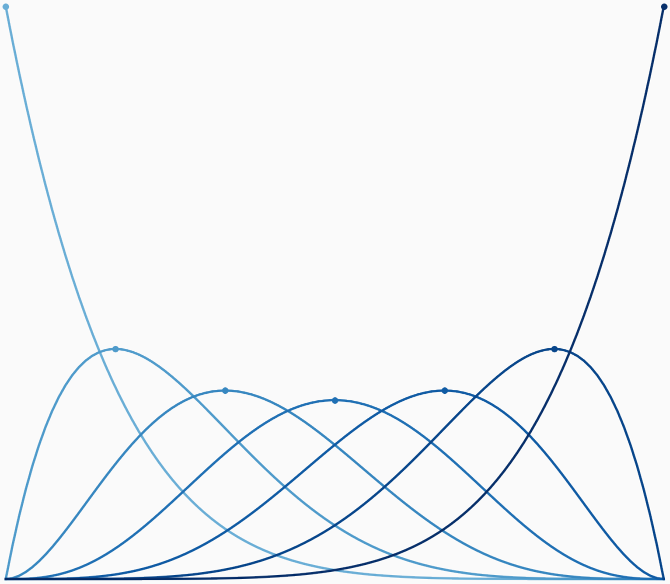

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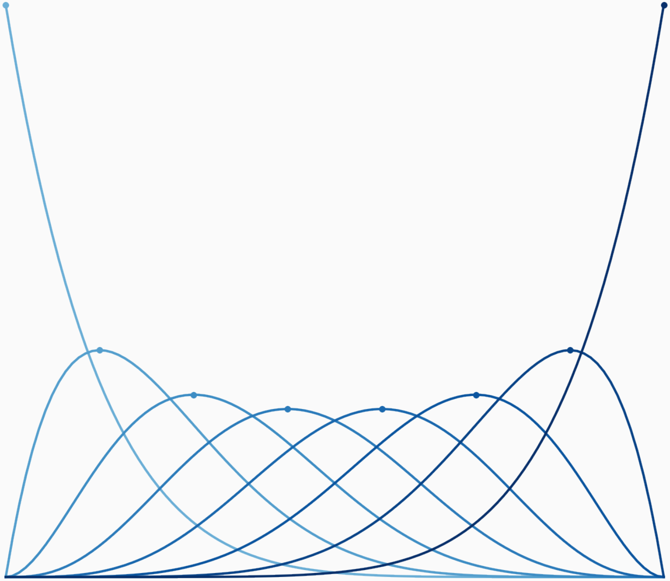
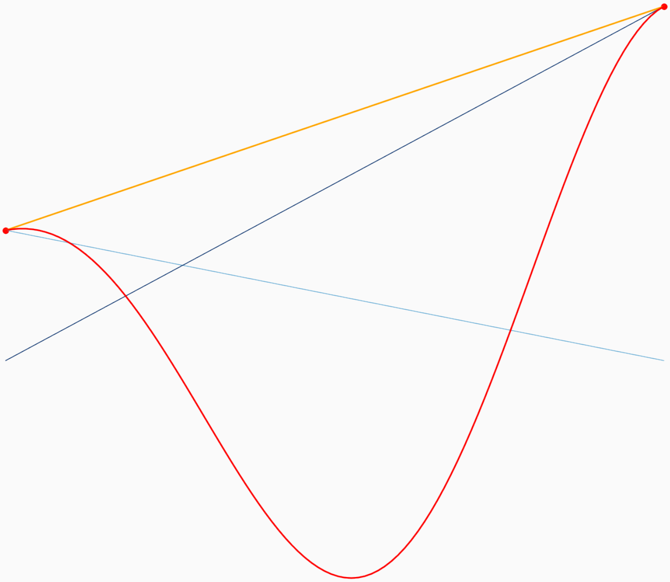


Bernstein expansion of $e^x \cos(2\pi x)$



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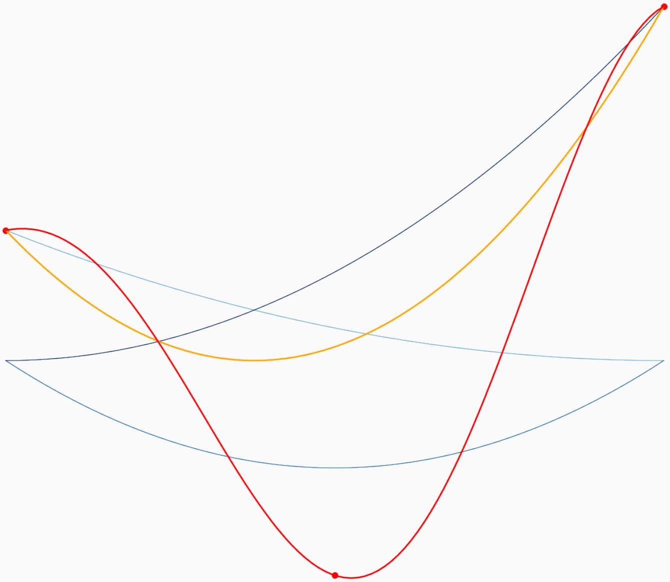

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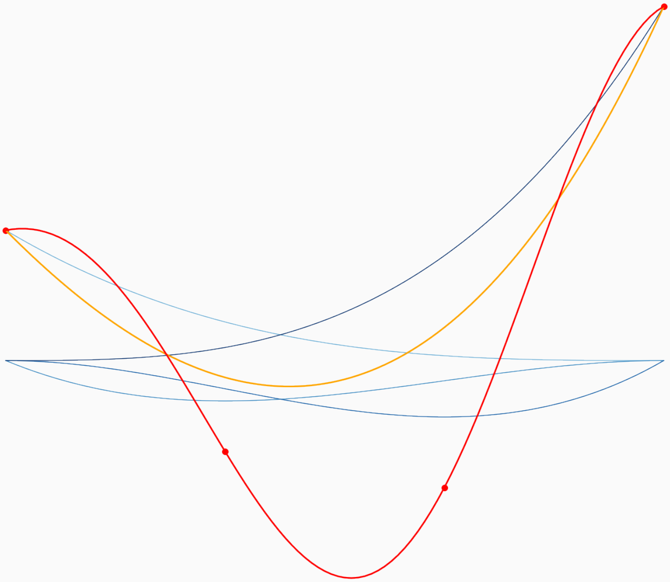

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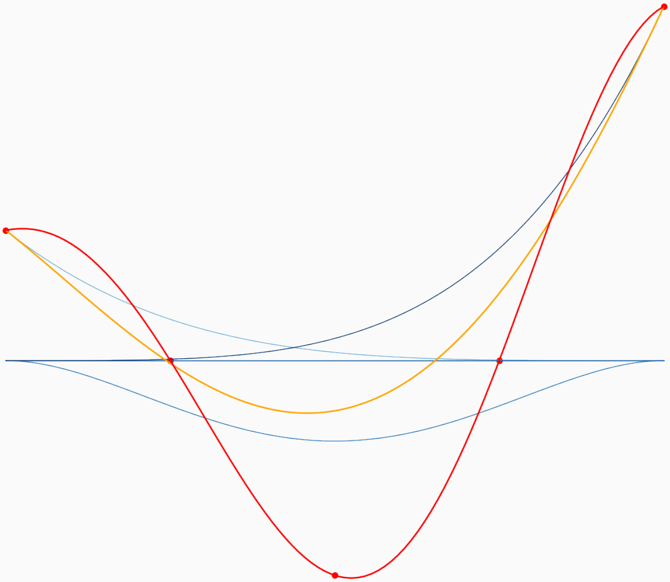

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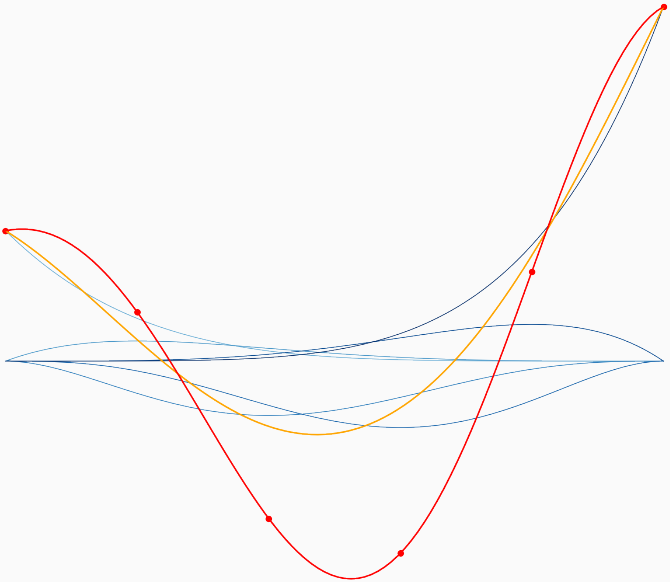

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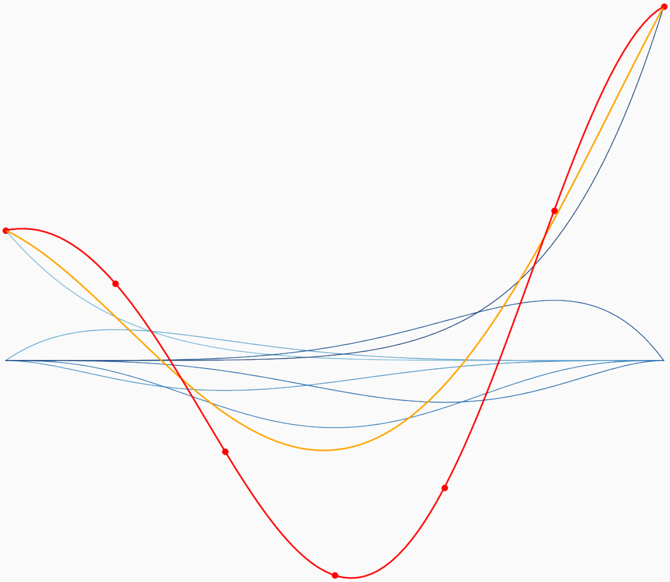

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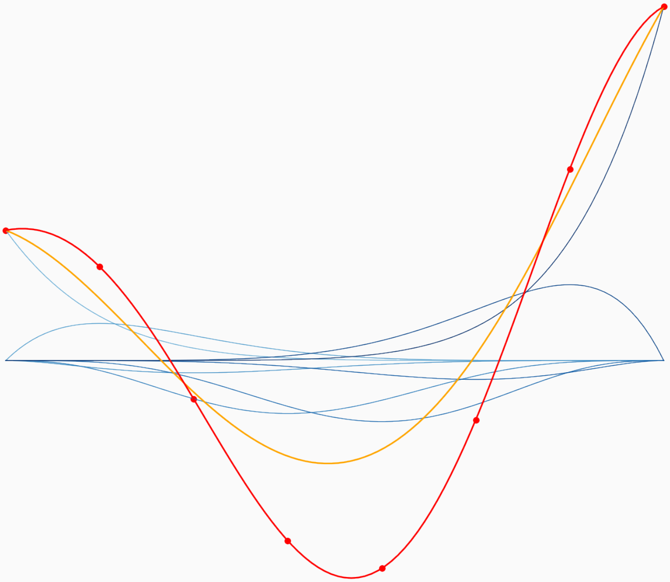

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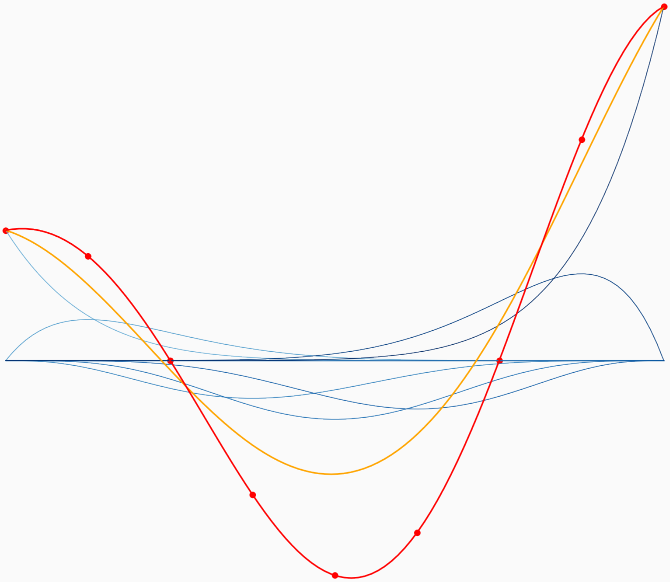

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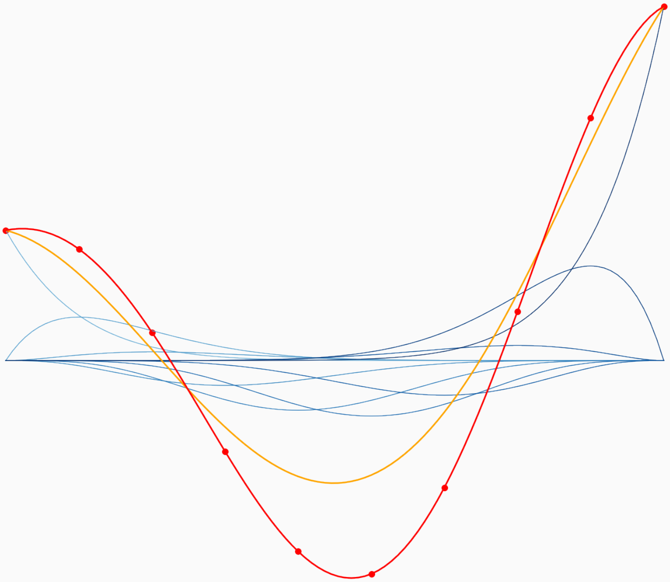

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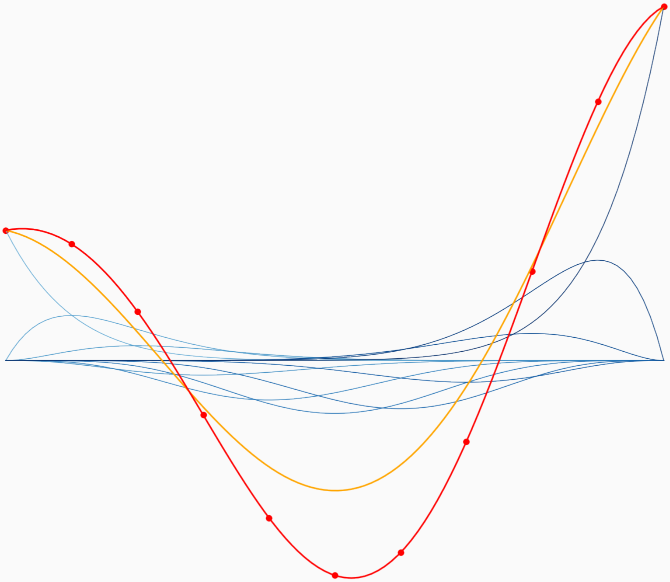

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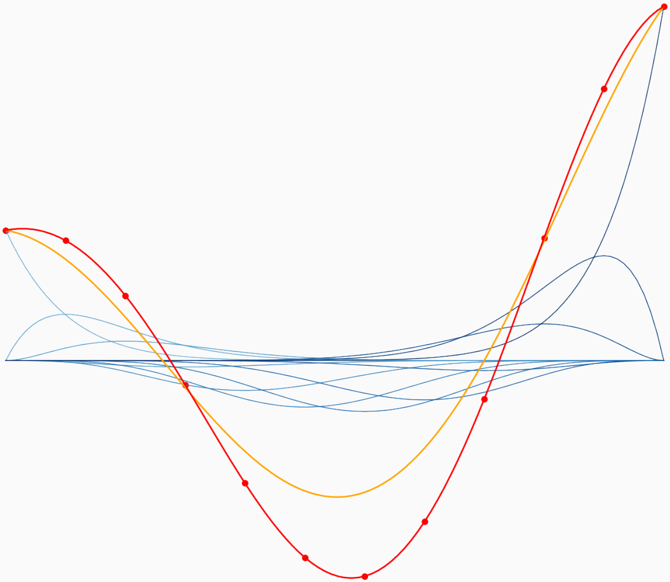

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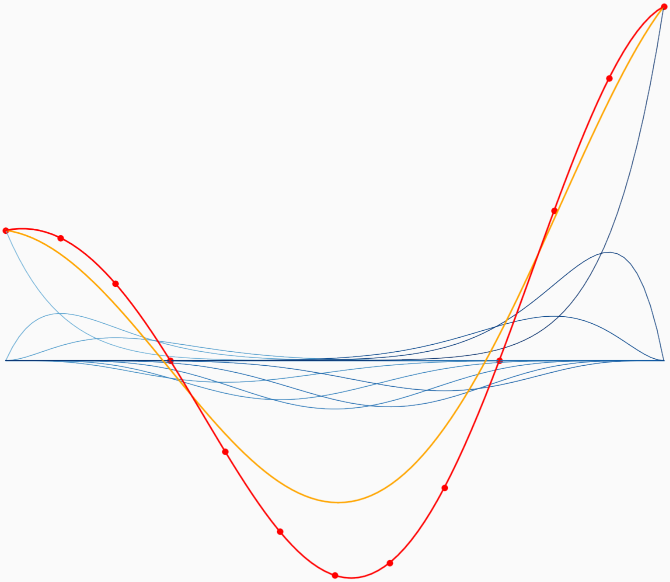

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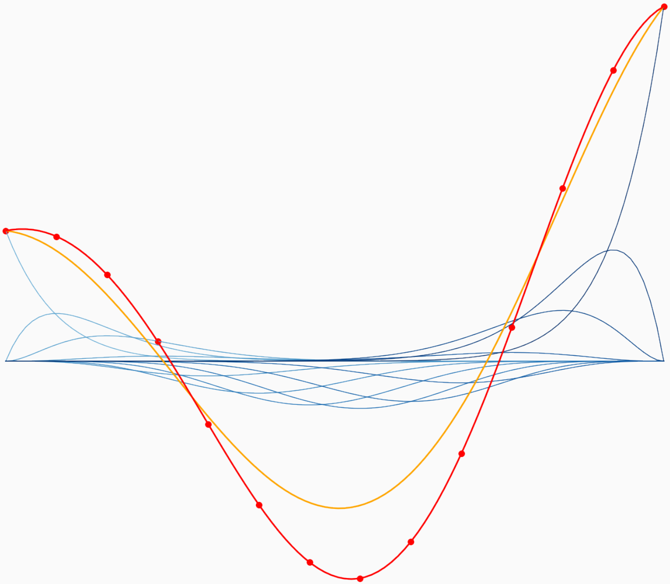

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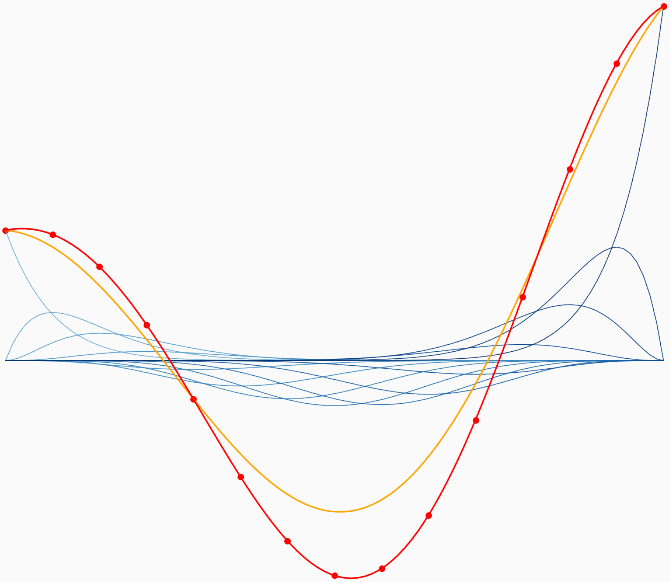

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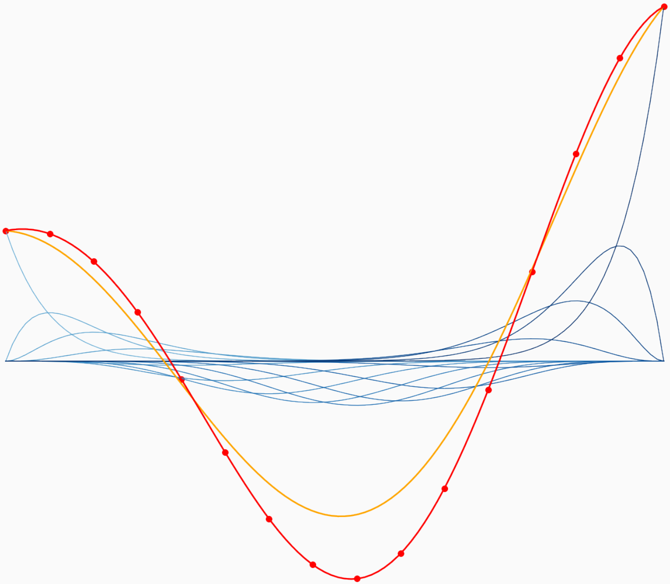

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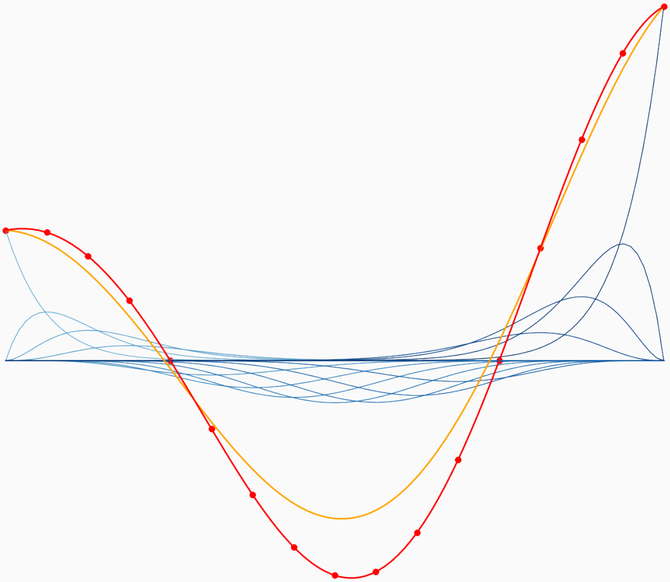

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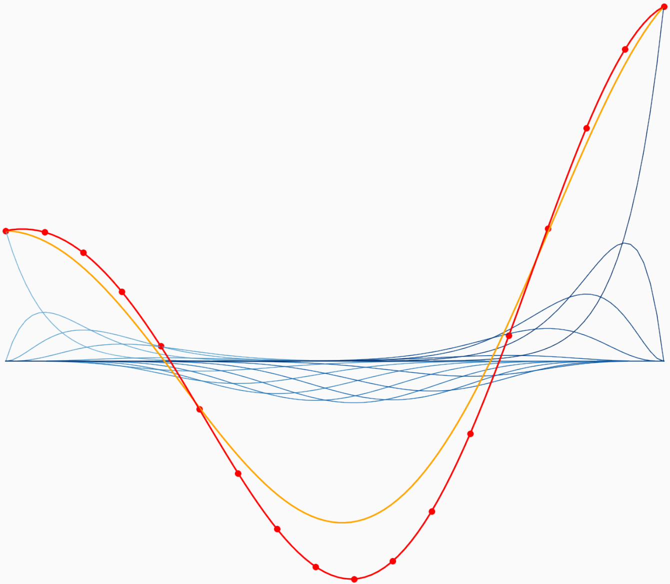

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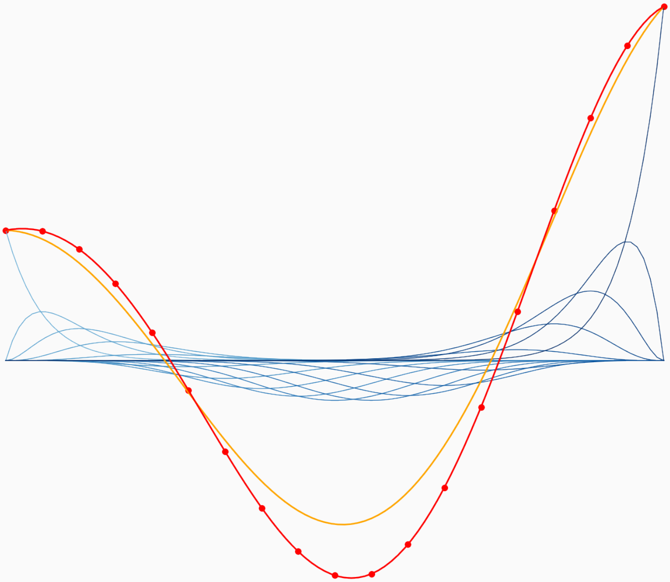

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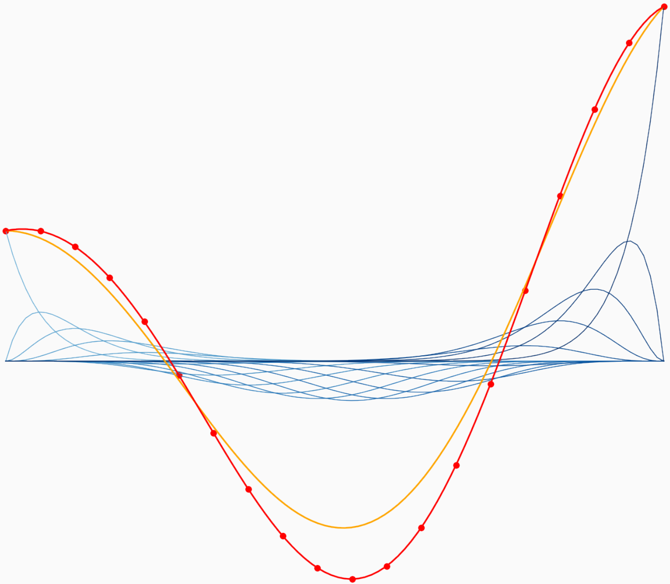

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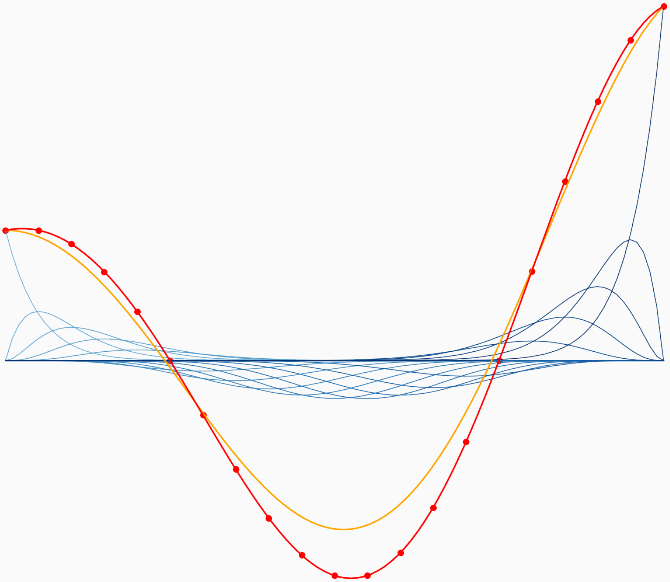

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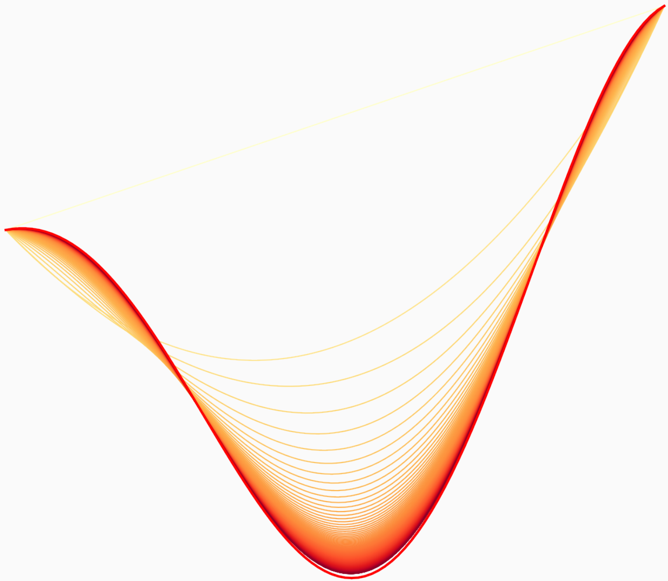


Generalizing polynomials

Algebras of functions

A collection of real valued functions $\mathscr{A}$ on a set E is called an algebra if

- $f \in \mathscr{A}, g \in \mathscr{A} \implies f + g \in \mathscr{A}$
- $f \in \mathscr{A}, g \in \mathscr{A} \implies fg \in \mathscr{A}$
- $f \in \mathscr{A}, c \in \mathbb{R} \implies cf \in \mathscr{A}$

If $f \in \mathscr{A}$ and p is a polynomial, then $p \circ f \in \mathscr{A}$.

Interpolation

An algebra $\mathscr{A}$ vanishes at no point of E if given $x \in E$, there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$.

An algebra separates points of E if given distinct $x_1, x_2 \in E$, there exists $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

Let the algebra $\mathscr{A}$ vanish at no point of E and separate points of E. Given distinct $x_1, x_2 \in E$ and $c_1, c_2 \in \mathbb{R}$, there exists $f \in \mathscr{A}$ such that $f(x_1) = c_1, f(x_2) = c_2$.

Let $f_1, f_2 \in \mathscr{A}$ such that $f_1(x_1) \neq 0$ and $f_2(x_2) \neq 0$, and let $g \in \mathscr{A}$ such that $g(x_1) \neq g(x_2)$. Define the functions

$$h_1 = \frac{g - g(x_2)}{g(x_1) - g(x_2)} \frac{f_1}{f_1(x_1)}, \qquad h_2 = \frac{g - g(x_1)}{g(x_2) - g(x_1)} \frac{f_2}{f_2(x_2)}.$$

Note that $h_i(x_j) = \delta_{ij}$. Finally, set

$$f = c_1 h_1 + x_2 h_2. \qquad \square$$

Note that this can be extended to arbitrarily many points, in the manner of Lagrange interpolation.

Interpolation and continuity

Let f, g be continuous, real valued functions on X such that $f(x_0) = g(x_0)$ for some $x_0 \in X$. Then, g approximates f to an arbitrary degree of accuracy ϵ on some neighbourhood of x_0.

Note that $h = g - f$ is continuous, hence the pre-image of the open interval $(-\epsilon, +\epsilon)$ is some open set $U \subseteq X$ containing x_0. Thus, on some neighbourhood $N_\delta(x_0) \subseteq U$, we have

$$-\epsilon < g - f < \epsilon$$

$$f - \epsilon < g < f + \epsilon \qquad \square$$

Closure

The set of uniform limits of functions from an algebra is called its uniform closure.

A uniformly closed algebra contains all uniform limits of its functions.

The uniform closure $\mathscr{B}$ of an algebra $\mathscr{A}$ of bounded functions is a uniformly closed algebra.

Closure

Let $\mathscr{A}$ be an algebra of real valued, bounded functions on X, and let $\mathscr{B}$ be its uniform closure. If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.

Let $\epsilon > 0$, let M be such that $|f| < M$. Pick a polynomial p such that for all $|x| < M$,

$$|p(x) - |x|| < \epsilon.$$

Then, for all $x \in X$, we have $|f(x)| < M$ so

$$|p(f(x)) - |f(x)|| < \epsilon.$$

Finally, note that $p \circ f \in \mathscr{B}$. $\square$

Closure

If $f, g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.

Note that

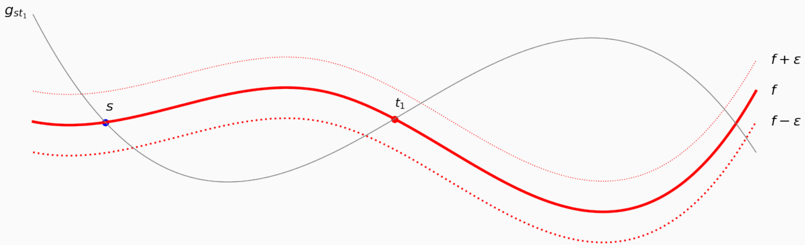
$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|,$$
$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|. \qquad \square$$

This gives us a way of ‘stitching’ functions from $\mathscr{B}$ together.

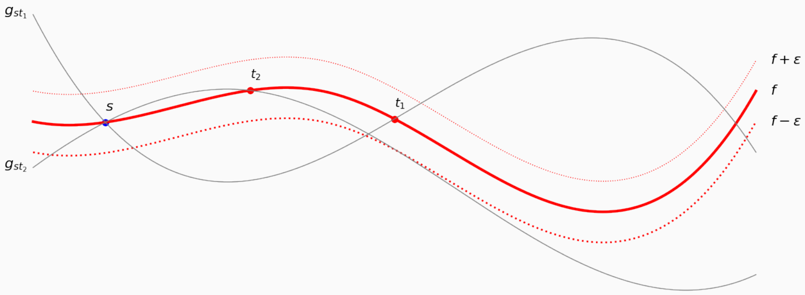
The Stone-Weierstrass Theorem

Let K be a compact metric space, and let $\mathscr{A}$ be an algebra of real continuous functions on K which separates points of K and vanishes at no point of K. The uniform closure of $\mathscr{A}$ consists of all real valued, continuous functions on K.

In other words, given any real valued continuous function f on K, there exists a sequence of functions $\{f_n\}$ from $\mathscr{A}$ such that $f_n \to f$ uniformly on K.



$$f - \epsilon < g_{st} \qquad \text{for all } x \in U_{st}$$



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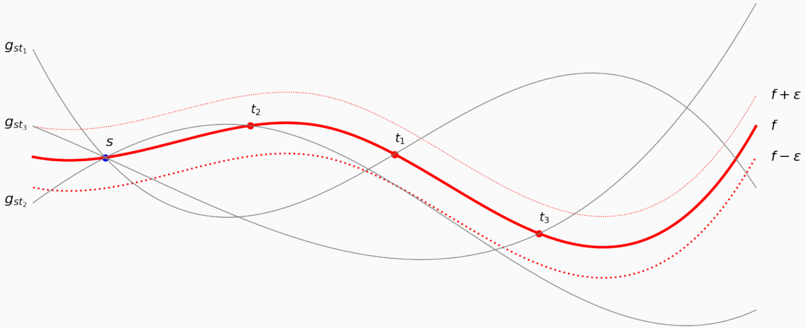

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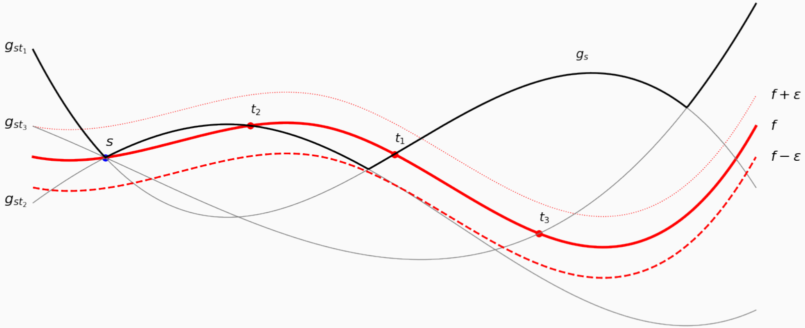
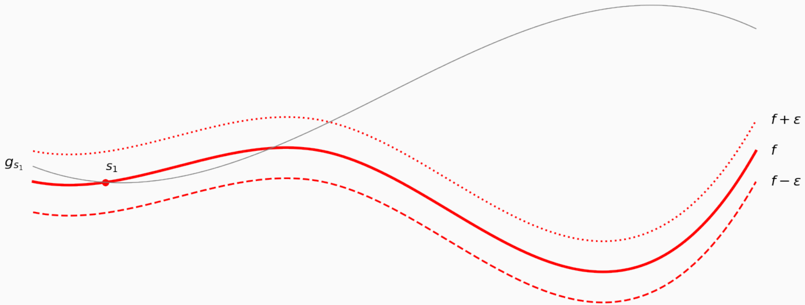
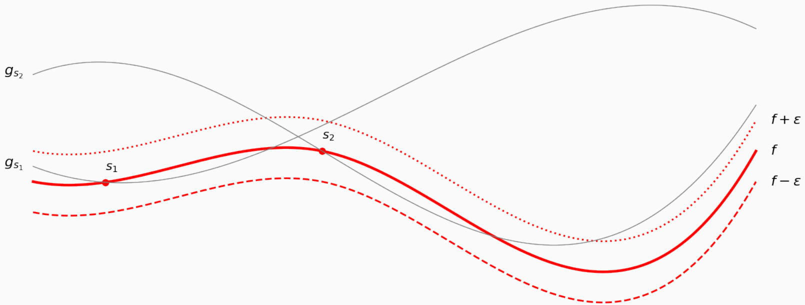


$$f - \epsilon < g_s \qquad \text{for all } x \in K$$



$$f - \epsilon < g_s < f + \epsilon \qquad \text{for all } x \in U_s$$



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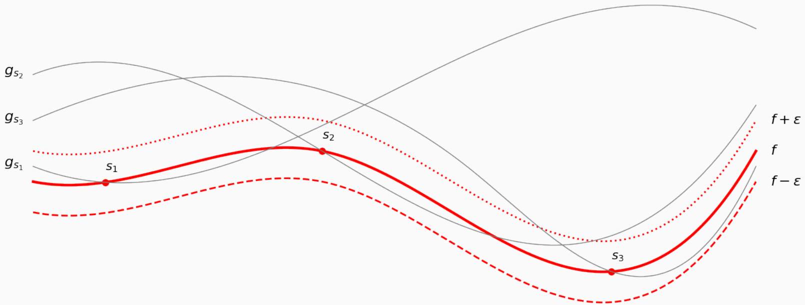

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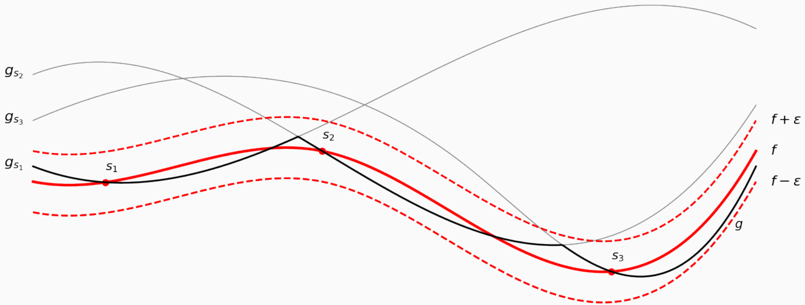

$$f - \epsilon < g < f + \epsilon \qquad \text{for all } x \in K \qquad \square$$

Polynomials

Let $\mathscr{P}$ be the set of polynomials on $\mathbb{R}^n$, for some $n \geq 1$. Then, $\mathscr{P}$ is an algebra of real continuous functions which separates points on $\mathbb{R}^n$ and vanishes at no point of $\mathbb{R}^n$.

In a sense, $\mathscr{P}$ is the algebra generated by the projection maps, $\pi_i(x_1, \ldots, x_n) = x_i$.

Any real continuous function on $\mathbb{R}^n$ can be uniformly approximated on a *compact subset* of $\mathbb{R}^n$.

Fourier-like functions

Let $\mathscr{F}$ be the set of functions of the form

$$f\colon S^1 \to \mathbb{R}, \qquad e^{ix} \mapsto a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx).$$

Here, we identify S^1 with the unit circle in $\mathbb{C}$; S^1 is compact. Also, $\mathscr{F}$ is an algebra of real continuous functions which separates points on S^1 and vanishes at no point of S^1.

Any real continuous function on S^1 can be uniformly approximated by functions from $\mathscr{F}$.