MA3103

Introduction to Graph Theory and Combinatorics

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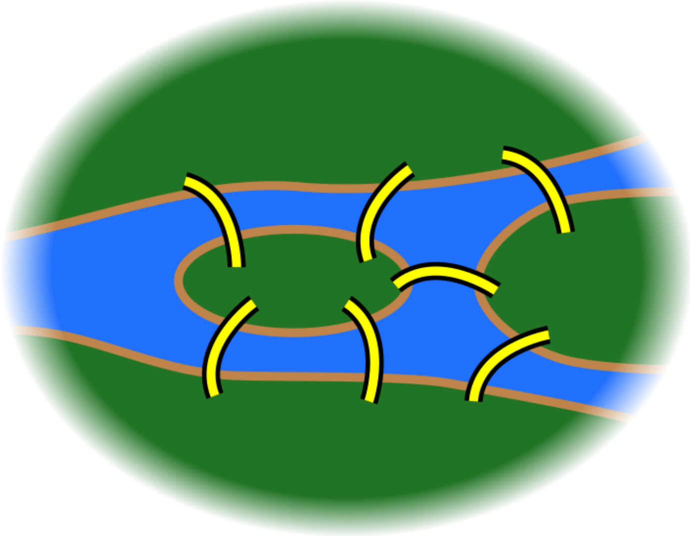
Contents

1 Introduction

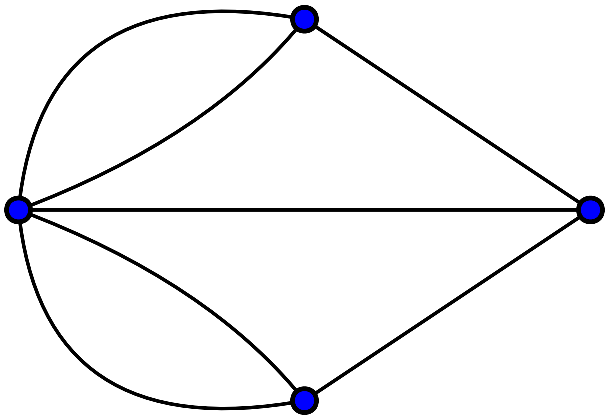
1.1 The Seven Bridges of Königsberg

The diagram below depicts a region in the city of Königsberg, Prussia. There are two islands, connected with the mainland and to each other via seven bridges. The Seven Bridges Problem is posed as follows: is it possible to walk through the entire city, visiting each one of the four landmasses by crossing each of the bridges exactly once?



Leonhard Euler showed that this is impossible; no such walk exists. The techniques he developed in doing so laid the foundations of *graph theory*.

The first thing to note is that the exact shape of the walk/trail is immaterial; all that matters is the sequence of landmasses visited and bridges crossed. Thus, each landmass can be compacted to a single point or *vertex*, and each bridge a line or *edge* connecting two such points. The resulting figure is a graph. Note that the orientations or placements of the points and lines are irrelevant, as long as the connections are undisturbed.



Now, examine a landmass which is on the trail but is neither our starting point, nor our ending point. In order to reach this landmass, we must enter via a bridge; but we cannot stay in the landmass, so we must leave via another a bridge. Thus, for each time we pass through this landmass, we can cross off two bridges joined to it. Once we are done, no bridge may remain unused; this means that we must have started with an even number of bridges joined to this landmass.

However, all four vertices in our graph connect to an odd number of edges. Since we require at least two vertices to act as intermediate points on our path, the desired walk is impossible.

1.2 Basic definitions

Definition 1.1. A graph $G(V, E)$ is an ordered pair of the set of vertices V and the set of edges E.

Definition 1.2. A simple graph is undirected, unweighted, and contains no self-loops or multiple edges joining vertices.

Definition 1.3. For a simple undirected graph, the set of edges E consists of two-element subsets of the set of vertices V.

Remark. For a directed, unweighted graph, the set of edges E consists of ordered pairs of elements from the set of vertices V.

Definition 1.4. A vertex is incident to an edge if that edge joins that vertex.

Definition 1.5. Two vertices are adjacent if there exists an edge connecting them. Two edges are adjacent if they connect to a common vertex.

Definition 1.6. The neighbours of a vertex consist of all vertices adjacent to it. The neighbours of an edge consist of all edges adjacent to it.

The number of neighbours of a vertex is called the degree of that vertex.

Definition 1.7. A complete graph is such that every pair of vertices is connected by an edge. The complete (simple) graph of n vertices is denoted by K_n.

1.3 Some principles

Lemma 1.1 (Pigeonhole Principle). *If $n+1$ objects are placed in n boxes, then we can fin a box containing at least 2 objects.*

Proof. If every box contains at most 1 objects, then the total number of objects falls short. □

Theorem 1.2. *There are no simple graphs where the degrees of all vertices are distinct.*

Proof. Let $G(V,E)$ be a simple graph with n vertices. The degrees of each of these vertices must be an integer among $0, 1, \ldots, n-1$. We now consider two cases.

Case I: There is a vertex of degree 0. Thus, this vertex is adjacent to no other vertex, which means that no vertex can have the full degree $n-1$. This means that the remaining vertices have degrees among $1, 2, \ldots, n-2$, i.e. $n-2$ choices of degree for $n-1$ vertices.

Case II: There is no vertex of degree 0. Thus, the vertices have degrees among $1, 2, \ldots, n-1$, i.e. $n-1$ choices of degree for n vertices.

In either case, the Pigeonhole Principle forces at least two vertices to share the same degree. □

Lemma 1.3 (Strong Pigeonhole Principle)**.** *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$N = q_1 + \cdots + q_n - n + 1$$

objects are placed in n boxes, then we can find a box i containing at least q_i objects.

Proof. If every box i contains at most $q_i - 1$ objects, then the total number of objects falls short.

$$N \leq (q_1 - 1) + \cdots + (q_n - 1) = q_1 + \cdots + q_n - n = N - 1$$

□

Theorem 1.4. *The sum of the degrees of all vertices in a simple graph is twice the number of its edges.*

Proof. Let $G((V, E)$ be a simple graph. Define the incidence function $I \colon E \times V \to \{0, 1\}$, such that $I(e, v) = 1$ if e and v are incident, 0 otherwise. We perform the double counting,

$$\sum_{v \in V} \sum_{e \in E} I(e, v) = \sum_{e \in E} \sum_{v \in V} I(e, v).$$

Now, the number of edges incident to a vertex is simply its degree, so $\sum_{e \in E} I(e, v) = d(v)$. Also, every edge is incident to exactly two vertices, so $\sum_{v \in V} I(e, v) = 2$. Thus, we have

$$\sum_{v \in V} d(v) = 2|E|.$$

□

Lemma 1.5 (Inclusion-Exclusion Principle)**.** *For finite sets $A_1, A_2, \ldots, A_n$, the number of elements in their union is given by*

$$\sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

Theorem 1.6. *There are $2^{\binom{n}{2}}$ simple graphs with n vertices.*

Exercise 1.1. How many simple graphs are there with n vertices and m edges?

Theorem 1.7. *Let $n, k \in \mathbb{N}$ such that $n > 3$ and $n/2 < k < n$. Let there be n points on a plane such that no three points are collinear. If every point is connected to at least k other points by segments, then there must be at least three segments forming a triangle.*

Proof. Consider a graph $G(V, E)$ with n vertices, such that every vertex has degree at least k. Pick an edge, say $\{x, y\}$, and let A be the neighbours of x apart from y, B be the neighbours of y apart from x. Note that A, B have at least $k-1$ elements each. Suppose that $A \cap B = \emptyset$, i.e. the edge $\{x, y\}$ doe not form a triangle. Thus,

$$|A \cup B| = |A| + |B| - |A \cap B| \geq 2(k-1).$$

However, $|A \cup B| \leq n-2$, hence $n \geq 2k$, or $k \leq n/2$. This is a contradiction. □

Remark. We have shown that *every* segment is part of a triangle. The number of segments here is

$$|E| \geq nk > \frac{n^2}{4}.$$

Exercise 1.2. Is the condition $|E| > n^2/4$ sufficient to ensure the existence of a triangle?

Lemma 1.8 (Cauchy-Schwarz)**.** *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive reals. Then,*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Equality holds if and only if every $a_i = \lambda b_i$ for some fixed real λ.

Theorem 1.9 (Mantel)**.** *In a simple graph with n vertices, the condition $|E| > n^2/4$ is sufficient to ensure the existence of a triangle.*

Proof. Let G be a simple graph with n vertices which is triangle-free. Thus, for any edge $\{x, y\} \in E$, the neighbour sets A and B of x and y intersect at no vertex. Thus, we can write

$$d(x) + d(y) = |A \cup B| \leq n.$$

Sum this over all possible edges. On the right, we have $n|E|$. On the left, we have the sum

$$\sum_{x \in V} d(x)^2 \geq \frac{1}{n} \left(\sum_{x \in V} d(x) \right)^2 \geq \frac{1}{n} \cdot 4|E|^2$$

This gives

$$\frac{4|E|^2}{n} \leq n|E|, \qquad |E| \leq \frac{n^2}{4}.$$ □

Example. Consider a circle, with 21 points on its circumference. It follows that among the angles subtended by these points at the center, at most 110 are greater than $2\pi/3$.

Note that there are $\binom{21}{2} = 210$ angles. Furthermore, given any 3 points on the circle (forming a triangle), all three angles subtended by them cannot be greater than $2\pi/3$. Construct a graph with these $n = 21$ points as vertices, such that two vertices are connected by an edge if and only if the angle subtended by them is greater than $2\pi/3$. Now, note that $n^2/4 = 110.25$, thus if there are more than 110 edges, there must exist a triangle of vertices in which all three angles are greater than $2\pi/3$ – a contradiction!

1.4 Bipartite graphs

Definition 1.8. A graph $G(V, E)$ is called bipartite if the vertex set V can be partitioned into 2 parts V_1, V_2 such that every edge in E joins a vertex of V_1 to a vertex of V_2. In other words, there exists a 2- colouring of the vertices such that no edge connects two vertices of the same colour.

Remark. The sum of the degree of the vertices in one part is exactly equal to the number of edges, which in turn is equal to the sum of the degrees of the vertices in the other part.

Definition 1.9. A complete bipartite graph is such that each vertex in one part is connected to every vertex in the other part. Such a graph is denoted by $K_{m,n}$, where the parts have m and n vertices respectively.

Remark. The total number of edges must be the product of the numbers of vertices in each part.

Definition 1.10. A set of vertices (or edges) in a graph is called independent if no two elements in that set are adjacent.

Lemma 1.10. *A bipartite graph is triangle free.*

Corollary 1.10.1. *If we choose even n, we can achieve a triangle free graph with $n^2/4$ edges, namely $K_{n/2,n/2}$. Similarly, if n is odd but $\lfloor n^2/4 \rfloor$ factors into natural numbers $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, then $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ achieves the upper bound again.*

Lemma 1.11. *An r-partite graph is K_{r+1} free.*

Exercise 1.3. Consider a complete r-partite graph on n vertices. What is the maximum number of edges possible?

Solution. Consider $K_{n_1,\dots,n_r}$, where $n = n_1 + \cdots + n_r$. The number of edges is

$$|E| = \sum_{i<j} n_i n_j.$$

Cauchy-Schwarz gives

$$n^2 = \sum_i n_i^2 + 2\sum_{i<j} n_i n_j \geq \frac{n^2}{r} + 2|E|.$$

Thus,

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Equality is achieved when $n_1 = \cdots = n_r$.

Definition 1.11. The complete r-partite graph $K_{n_1,\dots,n_r}$ on n vertices, such that $|n_i - n_j| \leq 1$ for all i, j is called Turan's graph, $T_{n,r}$.

Theorem 1.12 (Turan's Theorem)**.** *The number of edges in a K_{r+1} free graph on n vertices is at most*

$$|E(T_{n,r})| = \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Proof. Fix r; we prove the theorem by induction on n. The base case $n = 2$ has already been shown. Suppose that this holds for all K_{r+1} free graphs with less than n vertices. Note that whenever $n \leq r$, the claim is obvious, since

$$|E| \leq \frac{n(n-1)}{2} \leq \frac{n^2}{2}\left(1 - \frac{1}{r}\right).$$

Otherwise, we have $n \geq n + 1$. Let G have the maximum number of edges such that it is K_{r+1} free. We argue that G must contain K_r; if not, there is still scope for adding edges. Call the vertices in this subset A, and the remaining vertices B. Clearly, $|A| = r$ and $|B| = n - r$. Set e_A equal to the number of edges within A, e_B the number of edges within B, and e_{AB} the number of edges between A and B. We must have $|E| = e_A + e_B + e_B$. Now, A has the structure of K_r, so $e_A = r(r-1)/2$. Since the structure of B is K_{r+1} free, we can apply the induction hypothesis on it, giving $e_B \leq (n-r)^2(r-1)/2r$. Finally, no vertex in B can be connected to every vertex in A, so we have $e_{AB} \leq (r-1)(n-r)$. Adding everything together,

$$\begin{aligned} |E| &\leq \frac{r(r-1)}{2} + \frac{(n-r)^2(r-1)}{2r} + (r-1)(n-r) \\ &= \frac{1}{2}\left[r^2 + (n-r)^2 + 2r(n-r)\right]\frac{r-1}{r} \\ &= \frac{n^2}{2}\left(1 - \frac{1}{r}\right). \end{aligned}$$

Note that for equality to hold, we require every vertex in B to be connected to $r - 1$ vertices in A. This means that B is the Turan's graph $T_{n-r,r}$, so G is the Turan's graph $T_{n,r}$. □

Example. We give a second proof of Mantel's Theorem. Let G be a triangle free graph on n vertices. Let A be an independent set of G, and let B be the set of remaining vertices. Furthermore, let A be a *largest* independent set of G. Now note that in a triangle free graph, the neighbouring set of any vertex must be an independent set. This gives an upper bound of $|A|$ on the degree of any vertex. Also note that given an arbitrary edge, one of its endpoints must lie in B (both endpoints cannot lie in A since it is an independent set). This forces

$$|E| \leq \sum_{x \in B} d(x) \leq |A||B| \leq \frac{1}{4}(|A| + |B|)^2 = \frac{n^2}{4}.$$

Now, note that for equality in the first case, we require no edges in B, i.e. B must be independent. This forces G to be bipartite. For the second equality, we need every vertex in B to have the full degree $|A|$, so G is a complete bipartite graph. For the third equality, we demand $|A| = |B|$, so $G = T_{n,2}$.

1.5 Subgraphs

Definition 1.12. Let $G(V, E)$ be a graph. We say that $G'(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We write $G' \subseteq G$.

Definition 1.13. The subgraph G' induced by a set of vertices $V' \subseteq V$ is such that G' contains all the edges of G that connect vertices from V'. We write $G' = G[V']$.

Definition 1.14. The subgraph G' spans its parent G if the vertex sets $V' = V$.

Definition 1.15. A k-clique of G is an induced subgraph on k vertices which is complete.

Definition 1.16. The Ramsey number $R(s, t)$ is the least positive integer for which every complete graph on that many vertices, with its edges coloured in red and blue, must contain either a red s-clique or a blue t-clique.

Example. We can see that $R(s, t) = R(t, s)$, $R(1, r) = 1$, $R(2, r) = r$.

Example. We can show that $R(3, 3) = 6$. Indeed, every colouring of K_6 must contain at least *two* monochromatic triangles.

Example. Consider any graph G on 6 vertices. Construct a new graph G' on 6 vertices where two vertices are joined by a red edge if there exists a corresponding edge in G, and blue if not. Thus, G' is a 2-coloured complete graph and hence contains a monochromatic triangle. This means that there are 3 vertices in G where either all of them are connected to each other, or none of them are.

Lemma 1.13. *The number $R(s,t)$ is the smallest positive integer such that any graph on $R(s,t)$ vertices contains either an independent set of size s, or a t-clique.*

Theorem 1.14 (Ramsey Theorem)**.** *The Ramsey number $R(s,t)$ is always finite.*

Proof. We show this for all $s,t \geq 3$ by induction on $s+t$. Note that when $s+t=6$, we know that $R(1,5)$, $R(2,4)$, $R(3,3)$ are all finite. Furthermore, $R(s,1) = R(1,t) = 1$. Suppose that $R(s-1,t)$ and $R(s,t-1)$ are both finite; we claim that $R(s,t) < R(s-1,t)+R(s,t-1)$. Without loss of generality, let $s \geq t$. Consider a complete graph K_n on $R(s-1,t)+R(s,t-1) = n$ vertices. Choose a vertex v, and let V_R be the set of its neighbours connected by red edges, V_B be the neighbours connected by blue edges. Clearly, $n = |V_R|+|V_B|+1$, so either $|V_R| \geq R(s-1,t)$ or $|V_B| \geq R(s,t-1)$. In the first case, the consider the subgraph induced by V_R; either it contains a blue t-clique, or a red $s-1$ clique which means that $V_R \cup \{v\}$ contains a s-clique. In either case, we are done. The case with V_B is analogous. □

Remark. This upper bound can be sharpened to $R(s-1,t)+R(s,t-1)-1$ when both $R(s-1,t)$, $R(s,t-1)$ are even.

Example. Consider $R(4,3)$. We have $R(3,3) = 6$ and $R(4,2) = 4$, hence $R(4,3) \leq 6+4-1 = 9$. We show this is a different way. Note that $R(4,3) = R(3,4)$. In the manner of the previous proof, look at the case $|V_R| \geq |V_B|$. Since $|V_B|+|V_R|+1 = 9$, we have $|V_R| \geq 4$. If V_R contains one red edge, then we have found a red 3-clique. Otherwise, V_R contains a blue 4-clique, so we are done.

Definition 1.17. The Ramsey number $R(n_1,\dots,n_r)$ is the least positive integer for which every complete graph on that many vertices, with the edges coloured in r different colours, must contain some n_i-clique with colour i.

Theorem 1.15. *The Ramsey number $R(n_1,\dots,n_r)$ is always finite.*

Proof. Apply induction on the number of colours r. Note that we have already proved the theorem for $r=2$. Suppose that the statement holds for $r-1$ colours. We claim that

$$R(n_1,\dots,n_r) \leq R(n_1,\dots,n_{r-2},R(n_{r-1},n_r)).$$

Consider a complete graph K_n on $R(n_1,\dots,n_{r-2},R(n_{r-1},r)) = n$ vertices, with the edges coloured in $1,\dots,r$. Thus, the induction hypothesis gurantees that this graph must contain at least one n_i-clique in colour i for $1 \leq i \leq r-2$, or an $R(n_{r-1},n_r)$-clique in colour $r-1$ and r. However, the latter case means that the clique contains either an n_{r-1}-clique in colour $r-1$, or an n_r-clique in colour r. □

Theorem 1.16.

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

Proof. Perform induction on $s+t$. This is true whenever $s+t \leq 5$. Suppose that this holds for all $s+t-1$. Now,

$$R(s,t) \leq R(s,t-1) + R(s-1,t) \leq \binom{s+t-3}{s} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}.$$

□

Lemma 1.17.

$$\binom{2k}{k} \leq 2^{2k}.$$

Corollary 1.17.1.

$$R(s,s) \leq \binom{2s-2}{s-1} \leq 2^{2s-2} < 4^s.$$

Theorem 1.18 (Erdős)**.** *For all $s > 3$, $R(s,s) > \lfloor 2^{s/2} \rfloor$.*

Proof. We show that there exists a way of colouring K_n, $n = \lfloor 2^{s/2} \rfloor$ in red and blue such that there is no monochromatic s-clique.

Let A_R denote the event in which an induced subgraph $K_n[R]$ on $R \subseteq V$ vertices is monochromatic. For any edge, let the probability of it being coloured red or blue be the same, i.e. 1/2. Thus, K_s is red with probability

$$\left(\frac{1}{2}\right)^{\binom{s}{2}}.$$

When $|R| = s$, $P(A_R)$ is twice the above probability. We will show that the probability of the existence of a monochromatic s-clique is strictly less than 1. To see this, apply the Inclusion-Exclusion principle, whence the required probability is

$$\sum_{|R|=s} 2\left(\frac{1}{2}\right)^{\binom{s}{s}} = \binom{n}{s} 2^{1-\binom{s}{2}} \leq \frac{n^s}{s!} \cdot 2^{1+s/2} \cdot 2^{-s^2/2}.$$

Putting the value of n, we see that this is

$$\frac{2^{s^2/2}}{s!} \cdot 2^{1+s/2} \cdot 2^{-s^2/2} = \frac{1}{s!} 2^{1+s/2}.$$

However, it is easy to see that $s! > 2^s$ for all $s \geq 4$, which gives the result. □

Theorem 1.19. *For every $n \geq 1$, there is a lower bound p_0 such that for every prime $p \geq p_0$, the following congruence has a solution.*

$$x^n + y^n \equiv z^n \pmod{p}.$$

Theorem 1.20 (Schur's Theorem)**.** *For any positive integer r, there exists a positive integer $S(r)$ such that for every partition of the integers $\{1, 2, \ldots, S(r)\}$ into r parts, there exists one part which contains integers x, y, z where $x + y = z$.*

Remark. This can be rephrased in the following manner. For any r colouring of the integers $1, 2, \ldots, S(r)$, one can pick integers x, y, z all of the same colour such that $x + y = z$.

Remark. The integers x, y, z are not necessarily distinct!

Proof. We show this for $r \geq 2$. Let $n = R(3, 3, \ldots, 3)$ where there are r colours; we claim that this choice of n satisfies the desired property, i.e. $S(r) \leq n$.

Let $C\colon \{1, 2, \ldots, n\} \to \{1, \ldots, r\}$ be an arbitrary colouring of the integers. Construct the graph K_n, and colour its edges using the following map.

$$\chi\colon E(K_n) \to \{1, \ldots, r\}, \qquad \{v_1, v_2\} \mapsto C(|i - j|).$$

We immediately deduce the existence of a monochromatic triangle, say v_i, v_j, v_j with $i < j < k$. Set $x = j - i$, $y = k - j$, $z = k - i$. Then, $x + y = z$ and $C(x) = C(y) = C(z)$. $\square$

1.6 Degree sequences

Definition 1.18. Let G be a graph on n vertices, labelled $1, \ldots, n$. Then, we call the sequence $d(1), \ldots, d(n)$ the degree sequence of the graph.

Remark. Recall that given a degree sequence, the sum of the numbers is always twice the number of edges, i.e. the sum is always even. Also, we know that at least two vertices have the same degree.

Theorem 1.21. *Let d_i be a graphic sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then, there is a simple graph with the vertex set $\{x_1, \ldots, x_n\}$ such that $d(x_i) = d_i$ and the neighbour set*

$$N(x_1) = \{x_2, x_3, \ldots, x_{d_1+1}\}.$$

Proof. Let G be one of the graphs with the degree sequence d_i, $d(x_i) = d_i$. Furthermore, choose G such that the following number is maximised.

$$r_G = |N(x_1) \cap \{x_2, \ldots, x_{d_1+1}\}|.$$

If $r = d_1$, we are done. Otherwise, suppose that $r_G < d_1$, in which case one of the vertices x_s, $2 \leq s \leq d_1 + 1$ which is not adjacent to x_1. This also means that there is some vertex x_t, $t > d_1 + 1$ adjacent to x_1. Note that $1 \leq d(x_t) \leq d(x_s)$, so x_s is connected to at least one vertex $x_k \neq x_1$; we can also choose $x_k \neq x_t$, and x_k not connected to x_t. This is simply because $d_s \geq d_t$: every neighbour of x_s cannot be connected to x_t as well. Now, we simply remove the edges $\{x_1, x_t\}$, $\{x_s, x_k\}$, and add the edges $\{x_1, x_s\}$, $\{x_t, x_k\}$ to obtain the graph G'. In doing so, we have not preserved the degrees of every vertex, but observe that $r_{G'} > r_G$, contradicting the maximality of r_G. $\square$

Corollary 1.21.1 (Havel-Hakimi)**.** *A sequence d_i with $d_1 \geq \cdots \geq d_n$ is graphic if and only if the sequence $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphic.*

Proof. Simply delete the highest degree vertex in the first case to reach the second, and vice versa. $\square$

1.7 Independent sets and vertex covers

Definition 1.19. An independent set in a graph is maximal if it is not a subset of any other independent set.

Definition 1.20. A largest maximal independent set in a graph is called a maximum independent set. We denote its cardinality as $\alpha(\cdot)$.

Example. Given a path P_n, we have $\alpha(P_n) = \lceil n/2 \rceil$. Similarly, given a cycle C_n, we have $\alpha(C_n) = \lfloor n/2 \rfloor$.

Definition 1.21. A vertex covers an edge and vice versa if they are incident.

Definition 1.22. A vertex cover of a graph is a set of vertices which cover all its edges.

Definition 1.23. A minimal vertex cover of a graph is one which has no vertex cover as a subset.

Definition 1.24. A smallest vertex cover of a graph is called a minimum vertex cover. We denote its cardinality as $\beta(\cdot)$.

Lemma 1.22. *The complement of a vertex cover of a graph is independent, and vice versa.*

Proof. Consider a vertex cover K, and pick two vertices x, y in its complement. If x is adjacent to y, then $\{x, y\}$ is not covered by K.

Consider an independent set U, and pick an arbitrary edge $\{x, y\}$. Both x and y cannot be in U, hence the complement of U covers this edge. $\square$

Corollary 1.22.1. *Given any graph G on n vertices, we have $\alpha(G) + \beta(G) = n$. The complement of any maximum independent set is a minimum vertex cover, and vice versa.*

Example. Consider a river crossing problem, involving a boat with k extra seats, and n objects on one side. Let G represent the conflict graph between these objects. Then, the number of extra seats must satisfy

$$\beta(G) \leq k \leq \beta(G) + 1.$$

To see this, we must take a vertex cover of G with us on the first step so that we leave an independent set behind. On the other hand, it is enough to keep the minimum vertex cover permanently on the boat, and ferry the remaining objects (which are independent) one by one.

1.8 Dominating vertex sets

Definition 1.25. A set of vertices in a graph is called a dominating set if every vertex in the graph is either part of this set, or a neighbour of some vertex in this set.

Definition 1.26. A smallest dominating set of a graph is called a minimum dominating set. We denote its cardinality as $\gamma(\cdot)$.

Lemma 1.23. *In a connected graph, every vertex cover is also a dominating set.*

Lemma 1.24. *Every maximal independent set is also a dominating set. This immediately gives*

$$\alpha(G) \geq \gamma(G).$$

Corollary 1.24.1. *Every connected graph has at least two disjoint dominating sets. This gives*

$$\gamma(G) \leq \frac{n}{2}.$$

Definition 1.27. Any dominating set which is independent is also maximally independent.

1.9 Matching edges

Definition 1.28. A set of independent edges is called a matching set.

Definition 1.29. We denote the cardinality of the maximum matching set as $\alpha'(\cdot)$.

Definition 1.30. A perfect matching covers all the vertices in the graph.

Definition 1.31. A complete matching from independent sets A to B is one which covers all vertices in the smaller set A.

Example. Let $\mathscr{A} = \{A_1, \ldots, A_n\}$ be a family of subsets of X. The problem of finding n distinct elements such that each $x_i \in A_i$ can be reframed as a graph theoretical one. Construct a bipartite graph with the sets A_i on one side and the elements x_j on the other, and connect A_i with x_j if $x_j \in A_i$. We seek a complete matching from $\mathscr{A}$ to X.

Example. Observe that for a complete matching from A to B to exist, we must have the following: for any subset X of A, we need $|X| \leq |N(X)|$ where $N(X)$ is the neighbouring set of X. Indeed, this is sufficient.

Theorem 1.25 (Hall's Marriage theorem)**.** *A bipartite graph G with vertex sets V_1 and V_2 contains a complete matching from V_1 to V_2 if and only if given any subset $X \subseteq V_1$, we have $|X| \leq |N(X)|$.*

Proof. It is clear that this condition is necessary, by employing the Pigeonhole Principle. To show that this is sufficient, we use induction on $|V_1| = m$. This is trivial for $m = 1$; suppose that this holds for all $1, \ldots, m-1$. Consider the following cases.

Case I: All groups of k members from V_1, with $1 \leq k < m$, are connected to at least $k+1$ members from V_2. Note that every vertex from V_1 has degree 2. Fix one arbitrary edge. The remaining graph satisfies the induction hypothesis, hence there exists a complete matching.

Case II: There are some groups of k members from V_1, with $1 \leq k < m$, which are connected to exactly k members from V_2. Fix such k, label this group X, and note that X can be completely matched with its neighbouring set by the induction hypothesis. Now, we claim that the remaining sets, namely $V_1 \setminus X$ and $V_2 \setminus N(X)$, satisfy the induction hypothesis. Suppose that some set $Y \subseteq V_1 \setminus X$ has too few neighbours, i.e. $|Y| > |N(Y)|$. Then, examine the union $X \cup Y$, which has size $k + |Y|$ but has strictly less than $k + |N(Y)|$ neighbours. This is a contradiction, hence the remaining elements also admit a complete matching. □

1.10 Walks

Definition 1.32. A walk on a graph is a sequence of alternating vertices and edges, with two adjacent elements in the sequence being incident in the graph. The number of edges involved is called the length of the walk.

Definition 1.33. A walk with distinct edges is called a trail.

Definition 1.34. A walk with distinct vertices is called a path.

Remark. We can immediately see that every path is also a trail.

Definition 1.35. A path that starts and ends on the same vertex (closed) is called a cycle.

Definition 1.36. A closed trail is called a circuit.

Definition 1.37. A circuit which contains all the edges in a graph is called an Eulerian circuit. A graph which admits such an Eulerian circuit is called an Eulerian graph.

Definition 1.38. A cycle which contains all the vertices in a graph is called a Hamiltonian cycle. A graph which admits such a Hamiltonian cycle is called a Hamiltonian graph.

Lemma 1.26. *In an Eulerian graph, every vertex has even degree.*

Theorem 1.27. *The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.*

Proof. Consider a graph which is the union of edge disjoint cycles. It is clear that a vertex which is part of k different cycles must have degree $2k$.

Conversely, consider a graph in which every vertex has even degree. Let $x_0, \ldots, x_l$ be a path of maximal length l in G. Since $d(x_0) \geq 2$, there must exist another vertex $y \neq x_1$ connected to x_0. Now, if y is not one of $x_2, \ldots, x_l$, then we can extend out path by starting from y, contradicting the maximality of the path. Thus, $y = x_i$ for some $2 \leq i \leq l$, hence we have found a cycle $x_0, \ldots, x_i$. Remove these edges from the graph, and note that every vertex in the new graph still has even degree. By repeating this procedure, we will exhaust every edge. □

Definition 1.39. A connected graph is one in which given any pair of vertices x, y, there exists a path between them.

Definition 1.40. In a connected graph, the distance between two vertices is the length of the shortest path joining them. We denote this as $d(\cdot, \cdot)$.

Definition 1.41. A maximal connected subgraph is called a component of the graph.

Definition 1.42. A vertex whose deletion would increase the number of components in the graph is called a cut vertex.

Definition 1.43. An edge whose deletion would increase the number of components in the graph is called a bridge.

Definition 1.44. The maximum distance between a given vertex and the other vertices in a graph is called its eccentricity. We denote this as $\epsilon(\cdot)$.

Definition 1.45. The minimum eccentricity in a graph is called its radius. The maximum eccentricity is called its diameter.

A central vertex is one whose eccentricity is equal to the radius.

Definition 1.46. A graph in which every vertex has the same degree k is called a k-regular graph.

Example. A cube Q_k is a k-regular graph.

Example. The Petersen graph is a 3-regular graph.

Theorem 1.28 (Ore's theorem)**.** *Let G be a simple graph of order $n \geq 3$ such that given any non-adjacent vertices x, y, we have $d(x) + d(y) \geq n$. Then, G is Hamiltonian.*

Proof. Let G be a non-Hamiltonian graph; we claim that there exists a pair of non-adjacent vertices x, y, $d(x) + d(y) < n$. Suppose that G has maximal edges (addition of any edge makes it Hamiltonian). Pick two non-adjacent vertices x, y: by construction, $G + \{x, y\}$ contains a Hamiltonian cycle, hence G contains a Hamiltonian path whose endpoints are x and y, say $x = v_1, \ldots, v_{n-1}, v_n = y$. Now, we cannot have both $v_1 \sim v_{i+1}$ and $v_i \sim v_n$ where $2 \leq i \leq n-2$ – if so, we can see that $v_1, v_2, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1}, v_1$ is a Hamiltonian cycle. Thus, if x is connected to $v_{i_1}, \ldots, v_{i_k}$, then y cannot be connected to $v_{i_1-1}, \ldots, v_{i_k-1}$, hence $d(y) \leq n - 1 - d(x)$. $\square$

Lemma 1.29. *A graph on n vertices with at least $\binom{n-1}{2}+1$ edges is connected.*

Proof. Let G be an arbitrary graph on n vertices with $\binom{n-1}{2}+1$ vertices, and suppose that it is disconnected. Partition the vertices of G into two groups of size n_1 and n_2, such that the corresponding subgraphs are disconnected. Then, the total number of edges in G is at most

$$\binom{n_1}{2}+\binom{n_2}{2}.$$

However, with the constraint $n_1+n_2=n$, this quantity is at most $\binom{n-1}{2}$, a contradiction. □

Lemma 1.30. *A graph on n vertices with at least $\binom{n-1}{2}+2$ edges is Hamiltonian.*

1.11 Labelled trees

Definition 1.47. Two graphs are the same if they have the same vertex and edge sets. Two graphs are same up to relabelling if there is a way of relabelling the vertices of one to obtain the other graph.

Theorem 1.31 (Cayley)**.** *There are n^{n-2} distinct labelled trees on n vertices.*

Proof. Given a tree on n vertices, we shall construct a sequence $a_1,\dots,a_{n-2}$ — called a Prüfer code — of $n-2$ elements (not necessarily distinct) from $\{1,2,\dots,n\}$. In the first step, pick a leaf with the smallest label, and call it b_1. Set a_1 to the adjacent vertex of b_1. Now, delete b_1 from the tree. The Prf̈er sequence of our original tree is a_1, followed by the sequence corresponding to the new tree.

Given a Prüfer sequence, we can construct a tree. Let b_1 be the smallest number not appearing in the sequence $a_1,\dots,a_{n-2}$, and attach a_1 to it. Delete a_1 and b_1 from consideration (a_1 from the sequence, b_1 from the vertex list), and repeat the process. Finally, connect the two remaining vertices.

It can be shown that there is a bijection between distinct labelled trees and all possible Prüfer sequences, which immediately completes the proof. □

Corollary 1.31.1. *There are n^{n-2} spanning trees of K_n.*

1.12 Graph isomorphisms

Definition 1.48. An isomorphism from a graph G to a graph H is a bijective mapping f from the vertex set of G to that of H, such that it preserves the adjacency function. This means that

$$\{x,y\}\in E(G) \iff \{f(x),f(y)\}\in E(H).$$

Definition 1.49. A graph automorphism is an isomorphism from a graph to itself.

Example. The complete graph K_n has $n!$ automorphisms, since any permutation of its vertices preserves its structure.

Lemma 1.32. *The automorphisms of a graph form a group, under composition.*

Example. The automorphism group of any graph is isomorphic to that of its complement graph.

Theorem 1.33 (Frucht)**.** *For every finite group, there exists a finite graph such that their automorphism groups are isomorphic.*

Definition 1.50. A vertex transitive graph is one in which all vertices are similar. In other words, given any two arbitrary vertices, there exists an isomorphism of the graph which sends one to the other.

Example. Every vertex transitive graph is regular. The converse is not true.

Definition 1.51. A proper colouring of a graph is an assignment of colours to its vertices such that adjacent vertices have distinct colours.

Remark. This is equivalent to realizing the graph as a k-partite graph.

1.13 Graph colourings

Definition 1.52. The chromatic number of a graph is the minimum number of colours needed to construct a proper colouring. We denote it as $\chi(\cdot)$.

Example. For any complete graph, $\chi(K_n) = n$.

Example. For cycles, $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$.

Example. For any k-partite graph G, we clearly have $\chi(G) \leq k$.

Lemma 1.34. *The greedy algorithm produces a colouring where at most $d+1$ colours are used, where d is the highest degree of the vertices in the graph. As a result, $\chi(G) \leq d+1$.*

Theorem 1.35 (Szekeres-Wilf)**.** *Let $\delta(G)$ denote the minimum vertex degree of G. Then, $\chi(G) \leq 1 + \max(\delta(G'))$, where the maximum is taken over all induced subgraphs G' of G.*

Proof. Let $\chi(G) = k$, and let H be a minimal induced subgraph, in the sense that $\chi(H) = k$ and the chromatic number of any of its subgraphs drops, i.e. $H - v$ is always $k-1$ colourable. This immediately shows that for any $v \in H$, $d(v) \geq k-1$; note that if v had fewer neighbours, then they could not exhaust the $k-1$ colours of $H - v$, leaving one free for v and thus making H $k-1$ colourable. In other words, $\max(\delta(G')) \geq \delta(H) \geq k-1$, completing the proof. ☐

Lemma 1.36. *Let the vertices of a graph be labelled in decreasing order of their degrees. Then, applying the greedy algorithm will give*

$$\chi(G) \leq \max(\min(d_i + 1, i)).$$

Lemma 1.37. *Given any vertex in G, we can see that*

$$\chi(G) - 1 \leq \chi(G - v) \leq \chi(G).$$

In other words, removing a vertex from G causes its chromatic number to drop by one of 0 or 1.

Definition 1.53. A k-chromatic graph G is called critically k-chromatic if the deletion of any of its vertices causes its chromatic number to decrease (by one).

Example. All complete graphs K_n are critically n-chromatic.

Lemma 1.38. *Any critically k-chromatic graph is connected.*

Proof. If a critically k-chromatic graph G has at least two components, then deleting a vertex v from either component gives a $k-1$ colouring of $G - v$. In other words, the other component was always $k-1$ colourable, giving $\chi(G) \leq k-1$, a contradiction. ☐

Lemma 1.39. *Any critically k-chromatic graph has no cut vertex.*

Proof. If a critically k-chromatic graph G has a cut vertex v, then $G - v$ partitions into components G_i. Let $G_i' = G_i + v$; note that each of these is $k-1$ colourable. Thus, it is possible to colour each of these induced subgraphs with $k-1$ colours, and this can be done keeping a common colour for v without any conflicts. This shows that G is $k-1$ colourable, a contradiction. ☐

Lemma 1.40. *If G is critically k-chromatic, then $\delta(G) \geq k - 1$.*

Proof. Suppose that the minimum degree vertex v has $d(v) < k - 1$. Note that $G - v$ is $k - 1$ colourable; the neighbours of v cannot exhaust all of these colours. This gives a $k - 1$ colouring of G, a contradiction. □

Definition 1.54. Let G be a simple graph, and let $p_G(k)$ be the number of ways in which G can be properly coloured with k colours. Then, p_G is called the chromatic function/polynomial of G.

Example. For a connected graph K_n, we have

$$p_{K_n}(k) = k \cdot (k-1) \cdots (k-n+1) = \frac{k!}{(k-n)!}.$$

Example. For a path P_n, we have

$$p_{P_n}(k) = k \cdot (k-1)^{n-1}.$$

Theorem 1.41. *Consider a simple graph G, and perform an edge contraction to get G/e, i.e. pick an edge $e = \{x, y\}$ and collapse it into a single vertex v. Then,*

$$p_G = p_{G-e} - p_{G/e}.$$

Proof. When colouring $G - e$, the vertices x, y either get the same colour, or different colours. We must discard the former case; each of these cases directly corresponds to a colouring of G/e. □

Example. Given a cycle C_n, we have

$$p_{C_n} = p_{P_n} - p_{C_{n-1}}.$$

Theorem 1.42. *We can write*

$$p_G(k) = \sum_r f(r)\, \mathcal{K}_{(r)},$$

where $f(r)$ is the number of ways of partitioning the vertices of G into r independent sets, and $\mathcal{K}_{(r)}$ is the number of ways of colouring the parts of such a partition.

Theorem 1.43 (Brook)**.** *If G is a connected graph that is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

1.14 Planar graphs

Definition 1.55. A planar graph is one which can be drawn on a plane without any edge crossings.

Example. The graphs Q_3, K_4, $K_{2,n}$ are all planar. The graphs K_5, $K_{3,3}$ are not.

Example. Any tree is planar.

Definition 1.56. A planar graph partitions the plane into a number of arc-wise connected open sets, called faces.

Theorem 1.44 (Euler)**.** *A connected planar graph with n vertices, m edges, and f faces obeys*

$$n - m + f = 2.$$

Remark. If G is planar but not connected, say with k components, then

$$n - m + f = k + 1.$$

Definition 1.57. The girth of a graph G is the length of the smallest cycle present in it.

Remark. We set the girth of an acyclic graph to be ∞.

Theorem 1.45. *A simple planar graph on n vertices with girth at least $g \geq 3$ has at most* $\max(g(n-2)/(g-2), n-1)$ *edges.*

Proof. First suppose that G is acyclic. Note that a tree has exactly $n-1$ edges, thus a forest will have at most that many edges.

Otherwise for cyclic graphs G, we apply induction on n. The base case $n = 3$, $g = 3$ is easily checked. Now suppose that the theorem holds for any simple planar graph on less than n vertices. Without loss of generality, let G be a connected simple planar graph on n vertices.

Case I: G has a bridge, say $\{x, y\}$. In other words, removing this edge gives two disjoint subgraphs G_1 and G_2. Now, the girths of each of these must be at least g. Then,

$$m \leq m_1 + m_2 + 1 \leq \max\left(\frac{g_1(n_1-2)}{g_1-2}, n_1 - 1\right) + \max\left(\frac{g_2(n_2-2)}{g_2-2}, n_2 - 1\right) + 1.$$

The desired inequality follows by taking cases (both have cycles, exactly one has a cycle).

Case II: G does not have a bridge. Then, every edge is part of some cycle, hence every edge separates two distinct faces. Let G have f faces, and let f_i be the number of faces incident to exactly i edges. Then,

$$f = \sum_i f_i, \qquad 2m = \sum_i i f_i = \sum_{i \geq g} i f_i \geq g f = g(2 - n + m).$$

Rearranging, $m \leq g(n-2)/(g-2)$ as desired. □

Corollary 1.45.1. *For any planar graph, $\delta(G) \leq 5$.*

Proof. This is trivial for $n \geq 3$. Otherwise,

$$\delta(G) \cdot n \leq \sum_i d_i = 2m \leq 2 \cdot 3(n-2) = 6n - 12.$$

Thus, $\delta(G) \leq 5$. □