

MA 1101 : Mathematics I

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September 13, 2019

Solution 1.(i) Let $P(n)$ be the statement

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1) \quad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{2}1(1+1)$. Thus, $P(1)$ is true.**Inductive step** We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= [1 + 2 + \cdots + k] + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) && \text{(From } P(k)) \\ &= \frac{1}{2}(k+2)(k+1) \\ &= \frac{1}{2}(k+1)((k+1)+1) \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □(ii) Let $P(n)$ be the statement

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1) \quad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{6}1(1+1)(2+1)$. Thus, $P(1)$ is true.**Inductive step** We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= [1^2 + 2^2 + \cdots + k^2] + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 && \text{(From } P(k)) \\ &= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6) \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □(iii) Let $P(n)$ be the statement

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n) \quad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{3}1(4-3)$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2 &= [1^2 + 3^2 + \cdots + (2k-1)^2] + (2k+1)^2 \\
&= \frac{1}{3}(4k^3 - k) + (2k+1)^2 && \text{(From } P(k)) \\
&= \frac{1}{3}(4k^3 - k + 12k^2 + 12k + 3) \\
&= \frac{1}{3}(4(k^3 + 3k^2 + 3k + 1) - k - 1) \\
&= \frac{1}{3}(4(k+1)^3 - (k+1))
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

(iv) Let $P(n)$ be the statement

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 \quad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly, $1 = \frac{1}{4}1(1+1)^2$. Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= [1^3 + 2^3 + \cdots + k^3] + (k+1)^3 \\
&= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 && \text{(From } P(k)) \\
&= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= \frac{1}{4}(k+1)^2((k+1)+1)^2
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

(v) Let $P(n)$ be the statement

$$\sum_{r=1}^n r(r+1) \cdots (r+9) = \frac{1}{11}n(n+1) \cdots (n+10) \quad \text{for all } n \in \mathbb{N}.$$

Base step We establish $P(1)$. Clearly,

$$1(1+1) \cdots (1+9) = \frac{1}{11}1(1+1) \cdots (1+9)(1+10)$$

Thus, $P(1)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
\sum_{r=1}^{k+1} r(r+1) \cdots (r+9) &= \left[\sum_{r=1}^k r(r+1) \cdots (r+9) \right] + (k+1)(k+2) \cdots (k+1+9) \\
&= \frac{1}{11}k(k+1) \cdots (k+10) + (k+1)(k+2) \cdots (k+1+9) && \text{(From } P(k)) \\
&= \frac{1}{11}(k+1) \cdots (k+10)(k+11) \\
&= \frac{1}{11}(k+1) \cdots ((k+1)+9)((k+1)+10)
\end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

Solution 2.

- (i) Let
- $P(n)$
- be the statement that for all
- $n \in \mathbb{N}$
- ,

$$3^n > n^2$$

Base step We establish $P(1)$ and $P(2)$. Clearly, $3^1 > 1^2$. Thus, $P(1)$ is true. Again, $3^2 = 9 > 8 = 2^2$. Thus, $P(2)$ is true.

Inductive step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$3^{k+1} = 3 \cdot 3^k > 3 \cdot k^2$$

We must show $3k^2 > (k+1)^2 \Leftrightarrow 3k^2 - (k+1)^2 > 0$.

$$3k^2 - (k+1)^2 = 2k^2 - 2k - 1 = k^2 + (k-1)^2 - 2$$

Clearly, for $k \geq 2$, $k^2 > 2$, so $k^2 + (k-1)^2 > 2$, and we are done.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

- (ii) Let
- $P(n)$
- be the statement that for all
- $n \in \mathbb{N}$
- and
- $x > -1$
- ,

$$(1+x)^n \geq 1+nx. \quad (\text{Bernoulli's Inequality})$$

Base Step We establish $P(1)$. Clearly, $(1+x)^1 \geq (1+1 \cdot x)$, thus $P(1)$ is true.

Inductive Step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k \cdot (1+x) \\ &\geq (1+kx) \cdot (1+x) && (x+1 > 0) \\ &= (1+x+kx+kx^2) \\ &\geq (1+(k+1)x) && (k > 0 \text{ and } x^2 \geq 0) \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

- (iii) Let
- $P(n)$
- be the statement that for all
- $n \geq 5$
- ,
- $n \in \mathbb{N}$
- ,

$$\binom{2n}{n} < 2^{2n-2}.$$

Base Step We establish $P(5)$. Now, $\binom{2n}{n} = 252$, while $2^{10-2} = 256$. Thus, $P(5)$ is true.

Inductive Step We assume $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned} \binom{2(k+1)}{k+1} &= \frac{(2k+2)!}{(k+1)!^2} \\ &= \frac{(2k+2)(2k+1)}{(k+1)^2} \binom{2n}{n} \\ &< 2 \cdot \frac{2k+1}{k+1} \cdot 2^{2k-2} \\ &< 2 \cdot \frac{2k+2}{k+1} \cdot 2^{2k-2} \\ &< 2^{2(k+1)-2} \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 5$, $n \in \mathbb{N}$. \square

Solution 3.

- (i) Let $P(n)$ be the statement that every $n \geq 2$, $n \in \mathbb{N}$ has a prime divisor. We prove this using the principle of strong mathematical induction.

Base Step We establish $P(2)$. Clearly, 2 is a prime divisor of itself, so $P(2)$ is true.

Inductive Step We assume that the statements $P(2), P(3), \dots, P(k-1)$ are all true. We will show that $P(k)$ is true.

If $k \geq 2$ is prime, then we are done, as k is a prime divisor of itself. Otherwise, if k is not prime, then $k = ab$ for some $1 < a, b < k$ and $a, b \in \mathbb{N}$. We see that $a \geq 2$, so by the induction hypothesis, a has a prime divisor $p \in \mathbb{N}$, i.e., $a = pc$ for some $c \in \mathbb{N}$. Thus, $k = (pc)b = p(cb)$, and $cb \in \mathbb{N}$, so p is a prime factor of k . This proves $P(k)$.

Hence, by the principle of strong induction, $P(n)$ is true for all $n \geq 2$, $n \in \mathbb{N}$. \square

- (ii) We define the Fibonacci sequence $(f_n)_{n \geq 0}$ as follows.

$$\begin{aligned} f_0 &:= 0 \\ f_1 &:= 1 \\ f_n &:= f_{n-1} + f_{n-2}, \quad \text{for all } n \geq 2 \end{aligned}$$

- (a) We wish to show that for all $n \in \mathbb{N}$,

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right] \quad (\text{Binet's formula})$$

We prove this using the principle of strong mathematical induction. Let $P(n)$ be the aforementioned statement, and let $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$. Note that φ and ψ both satisfy $x^2 = x + 1$.

$$\left(\frac{1 \pm \sqrt{5}}{2} \right)^2 = \frac{6 \pm 2\sqrt{5}}{4} = \frac{1 \pm \sqrt{5}}{2} + 1$$

Base Step We establish $P(1)$. Clearly, $f_1 = 1 = (\varphi - \psi)/\sqrt{5}$. Thus, $P(1)$ is true.

Inductive Step We assume that the statements $P(2), P(3), \dots, P(k)$ are all true. We will show that $P(k+1)$ is true.

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &= \frac{1}{\sqrt{5}}(\varphi^k - \psi^k) + \frac{1}{\sqrt{5}}(\varphi^{k-1} + \psi^{k-1}) \\ &= \frac{1}{\sqrt{5}}(\varphi^{k-1}(\varphi + 1) - \psi^{k-1}(\psi + 1)) \\ &= \frac{1}{\sqrt{5}}(\varphi^{k-1}(\varphi^2) - \psi^{k-1}(\psi^2)) \\ &= \frac{1}{\sqrt{5}}(\varphi^{k+1} - \psi^{k+1}) \end{aligned}$$

Hence, by the principle of strong induction, $P(n)$ is true for all $n \in \mathbb{N}$. \square

- (b) Let $P(n)$ be the statement that for all $n \in \mathbb{N}$,

$$f_1 + f_3 + \dots + f_{2n-1} = f_{2n}$$

Base Step We establish $P(1)$. Clearly, $f_1 = 1 = f_2$. Thus, $P(1)$ is true.

Inductive Step We assume that $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
 f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} &= [f_1 + f_3 + \cdots + f_{2k-1}] + f_{2k+1} \\
 &= f_{2k} + f_{2k+1} \\
 &= f_{2k+2} \\
 &= f_{2(k+1)}
 \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

(c) Let $P(n)$ be the statement that for all $n \in \mathbb{N}$,

$$f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$$

Base Step We establish $P(1)$. Clearly, $f_2 = 1 = 2 - 1 = f_3 - 1$. Thus, $P(1)$ is true.

Inductive Step We assume that $P(k)$ is true. We will show that $P(k+1)$ is true.

$$\begin{aligned}
 f_2 + f_4 + \cdots + f_{2k} + f_{2k+2} &= [f_2 + f_4 + \cdots + f_{2k}] + f_{2k+2} \\
 &= f_{2k+1} - 1 + f_{2k+2} \\
 &= f_{2k+3} - 1 \\
 &= f_{2(k+1)+1} - 1
 \end{aligned}$$

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □