Term presentation
Problem 3

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MA2102: Linear Algebra I

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Let V and W be vector spaces over the same field F. Show that the set $\mathcal{L}(V, W)$, consisting of linear maps from V to W, is a vector space. If V and W are finite dimensional, then find the dimension of $\mathcal{L}(V, W)$.

Preliminaries

A vector space V over a field F is a set equipped with a binary operation $+\colon V \times V \to F$ called addition, and an operation $\cdot\colon F \times V \to V$ called scalar multiplication, such that

1. $\boldsymbol{u} + \boldsymbol{v} \in V$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$.
2. $\lambda \boldsymbol{u} \in V$, for all $\boldsymbol{u} \in V$, $\lambda \in F$.
3. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$.
4. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$.
5. There exists $\mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$.
6. For all $\boldsymbol{v} \in V$, there exists $\boldsymbol{u} \in V$ such that $\boldsymbol{v} + \boldsymbol{u} = \mathbf{0}$. We denote $\boldsymbol{u} = -\boldsymbol{v}$.
7. $\lambda(\boldsymbol{u} + \boldsymbol{v}) = \lambda \boldsymbol{u} + \lambda \boldsymbol{v}$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\lambda \in F$.
8. $(\lambda\mu)\boldsymbol{v} = \lambda(\mu \boldsymbol{v})$ for all $\boldsymbol{v} \in V$, $\lambda, \mu \in F$.
9. $(\lambda + \mu)\boldsymbol{v} = \lambda \boldsymbol{v} + \mu \boldsymbol{v}$, for all $\boldsymbol{v} \in V$, $\lambda, \mu \in F$.
10. There exists $1 \in F$ such that $1\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$.

A basis of a vector space V over a field F is a set of linearly independent vectors in V such that any element of V can be written as a finite linear combination of them.

The dimension of a vector space V is equal to number of elements in a basis of V. This is well defined by the Replacement Theorem, which guarantees that any two bases will have the same size.

A linear map between the vector spaces V and W is a map $T\colon V \to W$ such that for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\lambda \in F$,

$$\begin{aligned} T(\boldsymbol{u} + \boldsymbol{v}) &= T(\boldsymbol{u}) + T(\boldsymbol{v}), \\ T(\lambda \boldsymbol{v}) &= \lambda T(\boldsymbol{v}). \end{aligned}$$

$\mathcal{L}(V, W)$ as a vector space

Let $T, T_1, T_2 \colon V \to W$ be linear maps and let $\lambda \in F$. We define addition and scalar multiplication on $\mathcal{L}(V, W)$ as follows.

$$\begin{aligned} (T_1 + T_2)(\boldsymbol{v}) &= T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) && \text{for all } \boldsymbol{v} \in V, \\ (\lambda T)(\boldsymbol{v}) &= \lambda T(\boldsymbol{v}) && \text{for all } \boldsymbol{v} \in V. \end{aligned}$$

$\mathcal{L}(V, W)$ as a vector space: Closure

$T_1 + T_2$ and λT are both linear maps in $\mathcal{L}(V, W)$.

$$\begin{aligned}(T_1 + T_2)(\boldsymbol{u} + \mu\boldsymbol{v}) &= T_1(\boldsymbol{u} + \mu\boldsymbol{v}) + T_2(\boldsymbol{u} + \mu\boldsymbol{v}) \\ &= T_1(\boldsymbol{u}) + \mu T_1(\boldsymbol{v}) + T_2(\boldsymbol{u}) + \mu T_2(\boldsymbol{v}) \\ &= (T_1 + T_2)(\boldsymbol{u}) + \mu(T_1 + T_2)(\boldsymbol{v}).\end{aligned}$$

$$\begin{aligned}(\lambda T)(\boldsymbol{u} + \mu\boldsymbol{v}) &= \lambda T(\boldsymbol{u} + \mu\boldsymbol{v}) \\ &= \lambda T(\boldsymbol{u}) + \lambda\mu T(\boldsymbol{v})) \\ &= (\lambda T)(\boldsymbol{u}) + \mu(\lambda T)(\boldsymbol{v}).\end{aligned}$$

$\mathcal{L}(V, W)$ as a vector space: Commutativity and Associativity of addition

For all $\boldsymbol{v} \in V$, note that the commutativity of addition in W gives

$$(T_1 + T_2)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) = T_2(\boldsymbol{v}) + T_1(\boldsymbol{v}) = (T_2 + T_1)(\boldsymbol{v}).$$

The associativity of addition in W gives

$$((T_1 + T_2) + T_3)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + (T_2 + T_3)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) + T_3(\boldsymbol{v}),$$
$$(T_1 + (T_2 + T_3))(\boldsymbol{v}) = (T_1 + T_2)(\boldsymbol{v}) + T_3(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) + T_3(\boldsymbol{v}).$$

Thus, $T_1 + T_2 = T_2 + T_1$ and $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$.

$\mathcal{L}(V, W)$ as a vector space: Existence of an additive identity and inverses

Define the linear map $\mathbf{0}_{\mathcal{L}} \colon V \to W$, $\mathbf{v} \mapsto \mathbf{0}_W$. For any $T \in \mathcal{L}(V, W)$, for all $\mathbf{v} \in V$.

$$(\mathbf{0}_{\mathcal{L}} + T)(\mathbf{v}) = \mathbf{0}_{\mathcal{L}}(\mathbf{v}) + T(\mathbf{v}) = \mathbf{0}_W + T(\mathbf{v}) = T(\mathbf{v}).$$

Define $T' \colon V \to W$, $\mathbf{v} \mapsto -T(\mathbf{v})$. Then,

$$(T + T')(\mathbf{v}) = T(\mathbf{v}) + T'(\mathbf{v}) = T(\mathbf{v}) - T(\mathbf{v}) = \mathbf{0}_W = \mathbf{0}_{\mathcal{L}}(\mathbf{v}).$$

Thus, $\mathbf{0}_{\mathcal{L}} + T = T$ and $T + T' = \mathbf{0}_{\mathcal{L}}$.

$\mathcal{L}(V, W)$ as a vector space: Distributivity of scaling

For $\lambda, \mu \in F$, for all $\boldsymbol{v} \in V$,

$$\begin{aligned}(\lambda(T_1 + T_2))(\boldsymbol{v}) &= \lambda(T_1 + T_2)(\boldsymbol{v}) \\ &= \lambda(T_1(\boldsymbol{v}) + T_2(\boldsymbol{v})) \\ &= \lambda T_1(\boldsymbol{v}) + \lambda T_2(\boldsymbol{v}) \\ &= (\lambda T_1)(\boldsymbol{v}) + (\lambda T_2)(\boldsymbol{v}) \\ &= (\lambda T_1 + \lambda T_2)(\boldsymbol{v}).\end{aligned}$$

$$\begin{aligned}((\lambda + \mu)T)(\boldsymbol{v}) &= (\lambda + \mu)T(\boldsymbol{v}) \\ &= \lambda T(\boldsymbol{v}) + \mu T(\boldsymbol{v}) \\ &= (\lambda T)(\boldsymbol{v}) + (\mu T)(\boldsymbol{v}) \\ &= (\lambda T + \mu T)(\boldsymbol{v}).\end{aligned}$$

Thus, $\lambda(T_1 + T_2) = \lambda T_1 + \lambda T_2$ and $(\lambda + \mu)T = \lambda T + \mu T$.

$\mathcal{L}(V, W)$ as a vector space: Scaling

For $\lambda, \mu \in F$, for all $\mathbf{v} \in V$,

$$\begin{aligned} ((\lambda\mu)T)(\mathbf{v}) &= (\lambda\mu)T(\mathbf{v}) \\ &= \lambda(\mu T(\mathbf{v})) \\ &= \lambda(\mu T)(\mathbf{v}) \\ &= (\lambda(\mu T))(\mathbf{v}). \end{aligned}$$

Thus, $(\lambda\mu)T = \lambda(\mu T)$.

Pick the scalar $1 \in F$ which satisfies $1\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in W$. Then

$$(1T)(\mathbf{v}) = 1(T(\mathbf{v})) = T(\mathbf{v}),$$

so $1T = T$.

Thus, we have verified that $\mathcal{L}(V, W)$ is a vector space, with the given structure of addition and scaling.

Let $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V and let $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be a basis of W.

Define the linear maps

$$T_{ij} \colon V \to W, \qquad \mathbf{v}_k \mapsto \delta_{ik}\mathbf{w}_j,$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We claim that the set of all such T_{ij} comprises a basis of $\mathcal{L}(V, W)$.

Note that

$$T_{ij}(\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) = \lambda_i\mathbf{w}_j.$$

Dimension of $\mathcal{L}(V, W)$ when V and W are finite dimensional

Let $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V and let $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be a basis of W.

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$\text{span}\{T_{ij}\} = \mathcal{L}(V, W)$

Suppose $T\colon V \to W$ is a linear map in $\mathcal{L}(V, W)$. For each of the basis vectors $\boldsymbol{v}_i \in \beta$, there exist unique scalars a_{ij} such that

$$T(\boldsymbol{v}_i) \;=\; a_{i1}\boldsymbol{w}_1 + a_{i2}\boldsymbol{w}_2 + \cdots + a_{im}\boldsymbol{w}_m.$$

We see that

$$T \;=\; \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij}.$$

To prove this, pick any $\boldsymbol{v} \in V$ and write $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \cdots + \lambda_n\boldsymbol{v}_n$. Then,

$$T(\boldsymbol{v}) = \sum_{i=1}^{n} \lambda_i T(\boldsymbol{v}_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i a_{ij}\boldsymbol{w}_j = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}T_{ij}(\boldsymbol{v}).$$

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$\{T_{ij}\}$ is linearly independent

Consider the linear combination

$$\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}T_{ij} = \mathbf{0}.$$

By successively evaluating this map on $\mathbf{v}_k$ for $k = 1, \ldots, n$, we see that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}T_{ij}(\mathbf{v}_k) = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\delta_{ik}\mathbf{w}_j = \sum_{j=1}^{m} c_{kj}\mathbf{w}_j = \mathbf{0}.$$

The linear independence of $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ forces $c_{kj} = 0$.

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The linear independence of $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ forces $c_{kj} = 0$.

Thus, the set of all T_{ij} is a linearly independent set which spans $\mathcal{L}(V, W)$. Hence, this comprises a basis of $\mathcal{L}(V, W)$.

This basis contains mn elements. Thus,

$$\dim \mathcal{L}(V, W) = mn,$$

where $n = \dim V$ and $m = \dim W$ are finite.