MA3201

Topology

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Contents

1 Introduction

1.1 Topological spaces

Definition 1.1. A topology on some set X is a family τ of subsets of X, satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from τ are in τ.
3. All finite intersections of elements from τ are in τ.

The sets from τ are declared to be open sets in the topological space (X, τ).

Example. Any set X admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

Example. Let X be a set. The cofinite topology on X is the collection of complements of finite sets, along with the empty set. Note that when X is finite, this is simply the discrete topology.

Definition 1.2. Let τ, τ' be two topologies on the set X. We say that τ is finer than τ' if τ has more open sets than τ'. In such a case, we also say that τ' is coarser than τ.

1.2 Topological bases

Definition 1.3. Let (X, τ) be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology τ such that every open set $U \in \tau$ is expressible as a union of elements from β.

Definition 1.4. Let X be a set, and let β be a collection of subsets of X satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then, β generates a topology on X, namely the collection of all unions of elements of β.

Lemma 1.1. *Let τ be a topology on X, and let $\beta \subseteq \tau$ be a collection of open sets. Then, β is a basis of τ, or generates τ, if for every $x \in U \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U$.*

Example. The collection of all open balls in $\mathbb{R}^n$ form a basis of the usual topology.

Lemma 1.2. *Let X be equipped with the topologies τ and τ', and let β and β' be the respective bases of these topologies. Then, τ is finer than τ' if and only if given $x \in B' \in \beta'$, there exists $x \in B \in \beta$ such that $B \subseteq B'$.*

Example. The collections of open balls in $\mathbb{R}^n$ generate the same topology as the collection of all open rectangles in $\mathbb{R}^n$.

Example. Consider the topologies on $\mathbb{R}$ generated by the following bases.

1. $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.
2. $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.
3. $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$ where $K = \{1/n : n \in \mathbb{Z}\}$.

We call the topology generated by β_2 the lower limit topology, denoted $\mathbb{R}_\ell$. The topology generated by β_3 is denoted $\mathbb{R}_K$. Both of these are strictly finer than the standard topology.

Definition 1.5. A sub-basis for some topology on X is a collection ρ of subsets of X whose union is the whole of X. The topology generated by ρ is defined to be the topology generated by the collection of all finite intersections of elements of ρ.

1.3 Product topology

Definition 1.6. Let (X_1, τ_1), (X_2, τ_2) be topological spaces. Then $\tau_1 \times \tau_2$ generates the product topology on $X_1 \times X_2$.

Example. The product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard topology, coincides with the standard topology on $\mathbb{R}^2$.

Lemma 1.3. *If β_1, β_2 are bases of the topologies τ_1, τ_2, then $\beta_1 \times \beta_2$ and $\tau_1 \times \tau_2$ generate the same product topology.*

Proof. Given $(x_1, x_2) \in U$ where $U \subseteq X_1 \times X_2$ is open in the product topology, recall that U can be written as a union of the basic open sets $U_{1i} \times U_{2i}$, where $U_{1i} \in \tau_1$ and $U_{2i} \in \tau_2$. Suppose that $(x_1, x_2) \in U_1 \times U_2$. Thus, we can choose $B_1 \in \beta_1$, $B_2 \in \beta_2$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$. Thus, $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$. □

Definition 1.7. The projection maps are defined as $\pi_i \colon X_1 \times \cdots X_k \to X_i$, $(x_1, \ldots, x_k) \mapsto x_i$.

Lemma 1.4. *The collection of elements of the form $\pi_1^{-1}(U_1)$ or $\pi_2^{-1}(U_2)$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$, forms a sub-basis of the product topology on $X_1 \times X_2$.*

Proof. Note that $\pi_1^{-1}(X_1) = X_1 \times X_2$. Now it is easy to see that finite intersections of elements of the form $U_1 \times X_2$ or $X_1 \times U_2$ where U_1, U_2 are open, are all of the form $U_1 \times U_2$ which is precisely a basis of the product topology. □

Corollary 1.4.1. *We can restrict ourselves to the sub-basis of elements of the form $\pi_1^{-1}(B_1)$ or $\pi_2^{-1}(B_2)$, where $B_1 \in \beta_1$, $B_2 \in \beta_2$ for some bases β_1, β_2 of τ_1, τ_2.*

1.4 Subspace topology

Definition 1.8. Let (X, τ) be a topological space, and let $Y \subset X$. Then the collection $U \cap Y$ for all $U \in \tau$ comprises the subspace topology τ_Y on Y induced by the topology τ on X.

Lemma 1.5. *If β is a basis for the topology on X, and $Y \subset X$, then the collection $B \cap Y$ for all $B \in \beta$ generates the subspace topology on Y.*

Lemma 1.6. *An open set of Y is open in X if Y is open in X.*

Proof. Let $U \subset Y$ be open in Y, then $U = V \cap Y$ for some open set V in X. If additionally Y is open in X, this immediately shows that U is open in X. $\square$

Theorem 1.7. *Let (X, τ_X), (Y, τ_Y) be topological spaces, and let $A \subseteq X$, $B \subseteq Y$. Then, there are two ways of assigning a natural topology on $A \times B$.*

1. *Take the product topology on $X \times Y$, and consider the subspace topology induced by it on $A \times B$.*
2. *Take the subspace topologies on A induced by τ_X, B induced by τ_Y, and consider the product topology generated by them on $A \times B$.*

These two methods generate the same topology on $A \times B$.

Proof. Open sets in 1 look like $(U \times V) \cap (A \times B)$, where $U \in \tau_X$, $V \in \tau_Y$). Open sets in 2 look like $(U' \cap A) \times (V' \cap B)$, where $U' \in \tau_X$, $V' \in \tau_Y$, which can be rewritten as $(U' \times V') \cap (A \times B)$. It is easy to see that these describe precisely the same sets. $\square$

1.5 Order topology

Definition 1.9. Let X be a set with a simple order $<$. Then the collection of sets of the form (a, b), $[a_0, b)$, $(a, b_0]$ where a_0 is the minimal element of X, b_0 is the maximal element of X, generate the order topology on X.

Example. The order topology on $\mathbb{N}$ is precisely the discrete topology.

Definition 1.10. Let X_1, X_2 be simply ordered sets. The dictionary order on $X_1 \times X_2$ is defined as follows: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$.

Example. Consider $X = \{1, 2\} \times \mathbb{N}$, where both $\{1, 2\}$ and $\mathbb{N}$ are endowed with the discrete topology. Note that the product topology on X is the discrete topology.

Now consider the dictionary order on X. Here, $(1, 1)$ is the smallest element, so we can list the elements of X in ascending order. Note that every $(1, m) < (2, n)$, for all $m, n \in \mathbb{N}$. Now, note that all singletons $\{(1, m)\}$ are open in the order topology on X. The same is true for the singletons $\{(1, n)\}$ for all $n > 1$. However, the singleton $\{(2, 1)\}$ is *not* open in the order topology.

Example. Consider $\mathbb{R}$ with the usual topology, and $X = [0,1) \cup \{2\}$. Then, $\{2\}$ is open in the subspace topology on X, but it is not open in the order topology on X.

Lemma 1.8. *The open rays of the form $(a, +\infty)$ and $(-\infty, a)$ in X form a sub-basis of the order topology on X.*

Proof. Note that $(a,b) = (-\infty, b) \cap (a, +\infty)$, $[a_0, b) = (-\infty, b)$, and $(a, b_0] = (a, +\infty)$. ☐

Definition 1.11. Let X be a simply ordered set, and $Y \subseteq X$. Then, we say that Y is convex in X if given $a, b \in Y$ such that $a < b$, the interval $(a,b) = \{x \in X : a < x < b\} \subseteq Y$.

Theorem 1.9. *Let Y be convex in X. Then, the subspace topology and the order topology on Y induced from the order topology on X coincide.*

1.6 Closed sets

Definition 1.12. Let (X, τ) be a topological space. A set $F \subseteq X$ is said to be closed in X if $F^c = X \setminus F \in \tau$.

Example. The sets $\emptyset, X$ are closed in every topological space (X, τ).

Example. In a set equipped with the discrete topology, every set is both open and closed.

Lemma 1.10. *Arbitrary intersections, and finite unions of closed sets are closed.*

Theorem 1.11. *Let (X, τ) be a topological space, and let $Y \subset X$ be equipped with the subspace topology. Then, a set $F \subseteq Y$ is closed in Y if and only if $F = Y \cap G$, where G is closed in X.*

Proof. Let $F \subset Y$. Now, F is closed in Y, $Y \setminus F = Y \cap F^c$ is open in Y, $Y \cap F^c = Y \cap U$ where U is open in X, $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$ where U^c is closed. The steps are reversible. ☐

Lemma 1.12. *A closed set of Y is closed in X if Y is closed in X.*

1.7 Interiors and closures

Definition 1.13. Let $A \subseteq X$ where (X, τ) is a topological space.

1. The interior of A is defined as the union of all open sets contained in A. This is denoted by A°.
2. The closure of A is defined as the intersection of all closed sets containing A. This is denoted by $\overline{A}$.

Remark. The interior of a set is open, and the closure of a set is closed.

Lemma 1.13. *Let $Y \subset X$ be topological spaces, and let $A \subseteq Y$. Also let $\overline{A}_X$, $\overline{A}_Y$ denote the closures of A in X, Y respectively. Then, $\overline{A}_Y = \overline{A}_X \cap Y$.*

Theorem 1.14. *Let $A \subset X$. Then,*

1. *$x \in \overline{A}$ if and only if every open set containing x has non-empty intersection with A.*
2. *$x \in \overline{A}$ if and only if every basic open set containing x has non-empty intersection with A, given that the topology on X is generated by those basic open sets.*

Definition 1.14. Let $A \subseteq X$ where (X, τ) is a topological space. We say that $x \in X$ is a limit point of X if for every open set U containing x, the deleted neighbourhood $U \setminus \{x\}$ has non-empty intersection with A. The set of limit points of A is denoted by A'.

Example. Let X be a set endowed with the discrete topology. Then, given any set $A \subseteq X$, we have $A' = \emptyset$.

Lemma 1.15. *A closed set contains all its limit points.*

Proof. Let $F \subseteq X$ be closed in X, and let $x \in F'$. Then given any open set containing x, we have $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$, hence $x \in \overline{F} = F$. □

Lemma 1.16. *Let $A \subseteq X$ where (X, τ) is a topological space. Then, $\overline{A} = A \cup A'$.*

Proof. It is clear that $\overline{A} \supseteq A \cup A'$. Now pick $x \in \overline{A}$. If $x \notin A$, then we know that given any open neighbourhood U of x, we have non-empty $U \cap A$. Furthermore, this intersection can never contain x, hence $x \in A'$. This proves that $\overline{A} \subseteq A \cup A'$. □

1.8 Convergence of sequences

Definition 1.15. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points from (X, τ), and let $x \in X$. We say that this sequence converges to x, denoted $x_n \to x$, if every open neighbourhood of x contains the tail of this sequence. In other words, given $U \in \tau$ such that $x \in U$, there must exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then, the constant sequence of b's converges to all three points a, b, c.

Example. Let $X = \mathbb{R}$, and τ be the collection of all intervals $(-a, a)$ together with $\emptyset, \mathbb{R}$. Then, the constant sequence of 0's converges to every point in $\mathbb{R}$.

Definition 1.16. Let (X, τ) be a topological space. We say that this topological space is Hausdorff if given any two distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Example. The real numbers under the standard topology is Hausdorff.

Theorem 1.17. *Let (X, τ) be a Hausdorff topological space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X. Then, this sequence can converge to at most one point in X.*

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to distinct points $x, y \in X$. Then there exist disjoint open neighbourhoods U, V such that $x \in U$, $y \in V$. Convergence means that both U and V contain a tail of the sequence, which is a contradiction. □

Lemma 1.18. *The singleton sets in a Hausdorff space are closed.*

Proof. Let $x \in X$ where (X, τ) is Hausdorff. Pick $y \neq x$, whence there exist $U_y, V_y \in \tau$, such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. In particular, $\{x\} \cap V_y = \emptyset$. We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open. □

Theorem 1.19. *The topology induced by a metric is Hausdorff.*

Proof. Given a metric space X and distinct points $x, y \in X$, we set $r = |x - y|$, $U = B(x, r/3)$, $V = B(y, r/3)$. □

2 Continuous maps

Definition 2.1. Let $f\colon X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in Y must be open in X.

Lemma 2.1. *A function $f\colon X \to Y$ is continuous if and only if given a base β of Y, we have $f^{-1}(U) \in \tau_X$ for every $U \in \beta$.*

Example. The identity function $\mathrm{id}\colon \mathbb{R}_\ell \to \mathbb{R}$ is continuous, while the identity function $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}_\ell$ is not. This is because the topology on $\mathbb{R}_\ell$ is strictly finer than that on $\mathbb{R}$.

Lemma 2.2. *A function $f\colon X \to Y$ is continuous if and only if for every closed set $F \subseteq Y$, we have $f^{-1}(F)$ closed in X.*

Lemma 2.3. *A function $f\colon X \to Y$ is continuous if and only if given any $x \in X$ and an open set $V \subseteq Y$ such that $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$, $f(U) \subseteq V$.*

Theorem 2.4. *The composition of continuous functions is continuous.*

2.1 Restricting and enlarging the domain

Lemma 2.5. *Let $f\colon X \to Y$ be continuous, and let $A \subset X$. Then the restriction of f to A is continuous.*

Theorem 2.6. *Let $f\colon X \to Y$, and let X be the union of the collection of open sets $\{A_\lambda\}_{\lambda\in\Lambda}$. If the restrictions of f to each A_λ are continuous, then f is continuous.*

Proof. Pick $x \in X$, hence $x \in A_\lambda$ for some $\lambda \in \Lambda$. Now if $f(x) \in V \subset Y$, where V is open in Y, then the continuity of the restriction of f to A_λ gives us an open set $U \subseteq A_\lambda$ such that $f(U) \subseteq V$. Finally since A_λ is open in X, so is U. $\square$

Definition 2.2. Let X be the union of the collection of open sets $\{A_\lambda\}_{\lambda\in\Lambda}$. We say that this collection is a locally finite cover of X if given $x \in X$, there exists a neighbourhood U of x such that $U \cap A_\lambda$ is non-empty for only finitely many $\lambda \in \Lambda$.

Theorem 2.7. *Let $f\colon X \to Y$, and let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a locally finite collection of closed sets covering X. If the restrictions of f to each F_λ are continuous, then f is continuous.*

Corollary 2.7.1 (Pasting lemma)**.** *Let $X = A \cup B$, with A, B closed in X. Let $f\colon A \to Y$, $g\colon B \to Y$ be continuous, with $f(x) = g(x)$ on $A \cap B$. Then the function $h\colon X \to Y$, defined by $x \mapsto f(x)$ on A and $x \mapsto g(x)$ on B, is continuous.*

Definition 2.3. A path is a continuous function $\gamma\colon [0,1] \to X$.

Lemma 2.8. *Two paths γ_1, γ_2 can be concatenated when $\gamma_1(1) = \gamma_2(0)$.*

2.2 Projection maps

Theorem 2.9. *The projection maps $\pi_i\colon X_1 \times \cdots \times X_k \to X_i$ are continuous, when the domain is equipped with the product topology. Furthermore, the product topology is the coarsest topology on the domain for which the projection maps are continuous.*

Lemma 2.10. *Let $f\colon A \to X_1 \times \cdots \times X_k$, where the co-domain is equipped with the product topology. Then, f is continuous if and only if the component functions $f_i = \pi_i \circ f$ are continuous.*

Proof. Note that if f is continuous, the compositions $\pi_i \circ f$ are immediately continuous. Conversely suppose that each f_i is continuous, and write

$$f(t) = (f_1(t), \ldots, f_k(t)).$$

The sets $U_1 \times \cdots \times U_k$, where each $U_i \subseteq X_i$ is open, form a basis of the co-domain. Furthermore, their pre-images under f are $f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k)$, which are open in A. This shows that f is continuous. $\square$

Definition 2.4. Let J be an arbitrary index set. A J-tuple of elements in a set X is a function $x\colon J \to X$, formally denoted $(x_\alpha)_{\alpha\in J}$. If $\{X_\alpha\}_{\alpha\in J}$ is a family of sets, their Cartesian product is defined as

$$\prod_{\alpha\in J} X_\alpha = \{x\colon J \to \bigcup_{\alpha\in J} X_\alpha \colon x_\alpha \in X_\alpha\}.$$

Remark. The fact that we can choose an element from each set in an uncountable collection relies on the Axiom of Choice.

Definition 2.5. Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The topology generated by $\prod_{\alpha\in J} U_\alpha$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the box topology on $\prod_{\alpha\in J} X_\alpha$.

Definition 2.6. Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The topology generated by the sub-basis $\pi_\alpha^{-1}(U_\alpha)$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the product topology on $\prod_{\alpha\in J} X_\alpha$.

Remark. The basic open sets are of the form $\pi_{\alpha\in J} U_\alpha$, where all but finitely many $U_\alpha = X_\alpha$. Thus, this is a coarser topology than the box topology.

Lemma 2.11. *Let $\prod_{\alpha\in J} X_\alpha$ be equipped with the box or product topology. Then, $\overline{\prod A_\alpha} = \prod \overline{A}_\alpha$, where each $A_\alpha \in X_\alpha$.*

Lemma 2.12. *Let $f\colon A \to \prod_{\alpha\in J} X_\alpha$, where the co-domain is equipped with the product topology. Then, f is continuous if and only if the component functions $f_\alpha = \pi_\alpha \circ f$ are continuous.*

Remark. This fails when $\prod_{\alpha\in J}$ is equipped with the box topology. Consider $f\colon \mathbb{R} \to \prod_{n=1}^{\infty} \mathbb{R}$, $x \mapsto (x, x, \dots)$. Then, the product $\prod_{n=1}^{\infty}(-1/n, 1/n)$ is open in the box topology, but its pre-image under f is $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is not open in $\mathbb{R}$.

2.3 Homeomorphisms

Definition 2.7. Let $f\colon X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is a homeomorphism if f is continuous, f is bijective, and f^{-1} is continuous. We also say that X and Y are homeomorphic when such a homeomorphism between them exists.

Example. The interval $(-1, 1)$ is homeomorphic to $\mathbb{R}$; for instance, the map $x \mapsto \tan(\pi x/2)$ on $(-1, 1)$ is a homeomorphism. A simpler construction is the map $x \mapsto x/(1 - x^2)$.