Notes from a course* on

Representation Theory of Finite Groups

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1. Linear representations of groups

Definition 1.1 (Linear representation): Let G be a finite group, and let V be a vector space. A linear representation (σ, V) of G is a homomorphism

$$\sigma : G \to \mathrm{GL}(V).$$

Example 1.1.1: The *trivial* representation of G is defined by $g \mapsto \mathrm{id}_V$.

Example 1.1.2: Consider a vector space V of dimension $\mathrm{ord}(G)$, and pick a basis $\{e_h\}_{h \in G}$. The *regular* representation $\tau : G \to \mathrm{GL}(V)$ of G is defined as follows: $\tau(g)$ sends each of the basis vectors $e_h \mapsto e_{gh}$.

The following propositions show that it is possible to define group representations in terms of a special class of group actions of G on the vector space V.

Proposition 1.2: Let G be a finite group, and let V be a vector space. Let $\rho : G \times V \to V$ be a group action of G on V, such that each for each G, the map $v \mapsto \rho(g, v)$ is linear. Then, (σ, V) is a linear representation, where

$$\sigma : G \to \mathrm{GL}(V), \qquad g \mapsto (v \mapsto \rho(g, v)).$$

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Proposition 1.3: Let (σ, V) be a linear representation. Then, the map

$$\rho : G \times V \to V, \qquad (g, v) \mapsto \sigma(g)(v)$$

is a group action of G on V, where for each $g \in G$, the map $v \mapsto \rho(g, v)$ is linear.

In this discussion, we will always work with finite groups, as well as finite dimensional vector spaces over a base field K. Typically, we will consider $K = \mathbb{C}$.

We will often abbreviate (σ, V) with V, and $\sigma(g)$ with g when the presence of σ is clear from context.

Definition 1.4: The dimension of a representation (σ, V) is $\dim(V)$.

Example 1.4.1: The only one dimensional representation of S_3 in C^* is the sign homomorphism. To see this, consider an arbitrary homomorphism $\sigma : S_3 \to C^*$. Note that $\ker(\sigma)$ must be a normal subgroup of S_3, hence must be one of $\{e\}, A_3, S_3$. The third option yields the trivial representation $\sigma = \mathrm{id}_{C^*}$, and the first option gives the contradiction $S_3 \cong \mathrm{im}(\sigma) \subset C^*$ (the right side is abelian while the left is not). This leaves $\ker(\sigma) = A_3$, i.e. $\sigma(g) = 1$ for all even permutations $g \in S_3$. The remaining elements of S_3 (the odd permutations) must be sent to -1, since for any odd permutation $h \in S_3$, the permutation h^2 is even, so $\sigma(h)^2 = \sigma(h^2) = 1$. The result is precisely the sign homomorphism

$$\sigma : S_3 \to C^*, \qquad g \mapsto \begin{cases} +1 \text{ if } g \in A_3 \\ -1 \text{ if } g \notin A_3. \end{cases}$$

Example 1.4.2: Construct an equilateral triangle in $\mathbb{C}^2$ centered at the origin, and consider the natural action of S_3 on it (permuting its vertices v_1, v_2, v_3). This induces a two dimensional representation $\sigma : S_3 \to \mathrm{GL}(\mathbb{C}^2)$. Note that $\{v_1, v_2\}$ forms a basis of $\mathbb{C}^2$; the third vertex can be obtained via $v_3 = -v_1 - v_2$. With this, we can calculate the image of (v_1, v_2) under the action of each $g \in S_3$, and hence the matrices of $\sigma(g)$ in the given basis as follows.

g	$(\sigma(g)(v_1), \sigma(g)(v_2))$	**Matrix of** $\sigma(g)$
e	(v_1, v_2)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(12)	(v_2, v_1)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(23)	$(v_1, v_3) = (v_1, -v_1 - v_2)$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$
(31)	$(v_3, v_2) = (-v_1 - v_2, v_2)$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(123)	$(v_2, v_3) = (v_2, -v_1 - v_2)$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

(321)	$(v_3, v_1) = (-v_1 - v_2, v_1)$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

Consider the setting $K = \mathbb{C}$. The fact that G is a finite group means that each element $g \in G$ has finite order, hence satisfies $g^m = 1$ for some $m \mid \operatorname{ord}(G)$. This means that $\sigma(g)^m = \operatorname{id}_V$, whence $x^m - 1$ is an annihilating polynomial for $\sigma(g)$. A consequence of this is that the minimal polynomial of $\sigma(g)$ is a factor of $x^m - 1$; but the latter splits into distinct linear factors. Furthermore, all eigenvalues of $\sigma(g)$ are roots of its minimal polynomial. This yields the following result.

Proposition 1.5: Suppose that $K = \mathbb{C}$. Let (σ, V) be a representation of G, and let $g \in G$. Then, $\sigma(g)$ is diagonalizable, and its eigenvalues are roots of unity.

2. Subrepresentations

Definition 2.1 (Stable subspace): Let (σ, V) be a representation of G, and let $W \subseteq V$ be a subspace of V. We say that W is a stable subspace of V if it is invariant under the action of G, i.e. $gw \in W$ for all $g \in G$, $w \in W$.

Example 2.1.1: Let S_3 act on $\mathbb{C}^3$ by permuting the basis vectors $\{e_1, e_2, e_3\}$. Then, the subspace $\operatorname{span}\{e_1 + e_2 + e_3\}$ is stable.

Definition 2.2 (Subrepresentation): Let W be a stable subspace of V. We say that (σ, W) is a subrepresentation of (σ, V).

Theorem 2.3: Suppose that $\operatorname{char}(K) \nmid \operatorname{ord}(G)$. Let W be a stable subspace of V. Then, there exists a stable subspace W' of V such that $V = W \oplus W'$.

When working with the field $K = \mathbb{C}$, Theorem 2.3 admits a simpler form by invoking the orthocomplement of $W \subseteq V$, with respect to a suitable Hermitian form on V. We say that a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is G-invariant if for all $g \in G$, $v, v' \in V$, we have $\langle gv, gv' \rangle = \langle v, v' \rangle$.

Theorem 2.4: Suppose that $K = \mathbb{C}$. If W is a stable subspace of V, then $W^\perp$ is a stable subspace of V, with $V = W \oplus W^\perp$.

Remark: The subspace $W^\perp$ is defined with respect to a non-degenerate G-invariant Hermitian form.

Proof: For all $g \in G$, $w \in W$, $w' \in W^\perp$, observe that $g^{-1}w \in W$, so

$$\langle gw', w \rangle = \langle gw', gg^{-1}w \rangle = \langle w', g^{-1}w \rangle = 0,$$

whence $gw' \in W^\perp$. $\square$

Example 2.4.1: Continuing Example 2.1.1, the subspace $\operatorname{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$ is also stable under the action of S_3. This gives a two dimensional subrepresentation of S_3. In fact, it is easy to check that the matrices describing this representation in the basis $\{2e_1 - e_2 - e_3, 2e_2 - e_3 - e_1\}$ are precisely the same as those in Example 1.4.2, making these two representations identical in some sense.

Remark: Given any Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to C$, we can obtain a G-invariant Hermitian form on V defined by

$$(u, v) \mapsto \sum_{g \in G} \langle gu, gv \rangle.$$

Returning to Theorem 2.3, observe that if π_W is a projection onto the subspace W, then we may write $V = W \oplus \ker(\pi_W)$. With this in mind, we will construct the required subspace W' as the kernel of a suitable projection map π_W. For this, we demand that π_W be G-invariant.

Definition 2.5: A linear map $f : V \to V'$ is called G-invariant if it is compatible with the G-action, i.e. for all $g \in G$, $v \in V$, we have $f(gv) = gf(v)$.

Note that the above definition implicitly deals with *two* representations (σ, V) and (σ', V) of G. The indicated property really looks like $\sigma'(g)(f(v)) = f(\sigma(g)(v))$ when written in full.

Lemma 2.6: Let $f : V \to V'$ be G-invariant. Then,

1. $\ker(f)$ is a stable subspace of V.
2. $\operatorname{im}(f)$ is a stable subspace of V'.

Given any linear map $f : V \to V'$, we can construct a G-invariant linear map via

$$\tilde{f} : V \to V', \qquad v \mapsto \sum_{g \in G} g f(g^{-1} v).$$

With this, we are ready to furnish a proof of our theorem.

Proof of Theorem 2.3: Let $\pi : V \to W$ be any projection onto W. Observe that

$$\pi_W : V \to W, \qquad v \mapsto \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} g\pi(g^{-1} v)$$

is a G-invariant projection onto W. Setting $W' = \ker(\pi_W)$ completes the proof. □

Remark: Note how the assumption that $\operatorname{char}(\mathrm{K}) \nmid \operatorname{ord}(G)$ is crucial for defining the projection π_W.

3. Irreducible representations

Definition 3.1 (Irreducible representations): We way that a representation is irreducible if it admits no proper non-trivial subrepresentations.

In other words, a representation (σ, V) is irreducible *if and only if* the only G-invariant subspaces of V are $\{0\}, V$.

Example 3.1.1: All one dimensional representations are irreducible.

Theorem 3.2 (Maschke's Theorem): Suppose that $\operatorname{char}(K) \nmid \operatorname{ord}(G)$. Then, every representation of G over the field K can be written as a direct sum of irreducible representations of G.

Proof: Follows immediately from Theorem 2.3. □

Example 3.2.1: Combining Examples 2.1.1 and 2.4.1, we have the decomposition

$$\mathbb{C}^3 = \operatorname{span}\{e_1 + e_2 + e_3\} \oplus \operatorname{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$$

into irreducible subrepresentations of S_3.

When we say that two representations (σ, V) and (σ', V') are isomorphic, denoted $V \cong V'$, we mean that there exists a G-invariant linear bijection $f : V \to V'$. The following result offers a very powerful characterization of G-invariant maps between irreducible representations.

Theorem 3.3 (Schur's Lemma): Let V, V' be two irreducible representations of G, and let $f : V \to V'$ be a G-invariant linear map.

1. If $V \ncong V'$, then $f = 0$.
2. If $V = V'$ and K is algebraically closed, then f is a scalar map, i.e. $f = \lambda \ \mathrm{id}_V$ for some $\lambda \in K$.

Proof:

1. Suppose that $f \neq 0$. It suffices to show that f is an isomorphism; to do so, we make extensive use of Lemma 2.6.

   First, $\ker(f) \subseteq V$ is stable, hence must be one of $\{0\}, V$ by the irreducibility of V. The assumption $f \neq 0$ forces $\ker(f) = \{0\}$, whence f is injective.

   Next, $\operatorname{im}(f) \subseteq V'$ is stable, hence must be one of $\{0\}, V'$ by the irreducibility of V'. Again, $f \neq 0$ forces $\operatorname{im}(f) = V'$, whence f is surjective.

2. We have a G-invariant linear bijection $f : V \to V$; suppose that $f \neq 0$. Let λ be an eigenvalue of f, and observe that the map $(f - \lambda \ \mathrm{id}_V)$ is also G-invariant; indeed, for all $g \in G$, $v \in V$,

   $$(f - \lambda)(gv) = f(gv) - \lambda gv = gf(v) - g\lambda v = g(f - \lambda)(v).$$

   Since λ is an eigenvalue of f, we have $\ker(f - \lambda) \neq \{0\}$. Since $\ker(f - \lambda) \subseteq V$ is stable and V is irreducible, we must have $\ker(f - \lambda) = V$, whence $f - \lambda \, \mathrm{id}_V = 0$.

□

Remark: Note how the existence of the scalar $\lambda \in K$ is guaranteed by the fact that K is algebraically closed.

Corollary 3.3.1: All $\mathbb{C}$-linear irreducible representations of finite abelian groups are one dimensional.

Proof: Let (σ, V) be an irreducible representation of a finite abelian group G. Check that for each $g \in G$, the linear map $\sigma(g) : V \to V$ is G-invariant, since it commutes with all $\sigma(h)$ for $h \in G$. From Schur's Lemma (Theorem 3.3), each $\sigma(g)$ is a scalar map. As a result, every one dimensional subspace of V is stable. The result now follows from the irreducibility of V. □

4. Characters

Definition 4.1 (Character): The character χ_V of a representation (σ, V) of G is the function

$$\chi_V : G \to K, \qquad g \mapsto \operatorname{tr}(\sigma(g)).$$

Example 4.1.1: $\chi_V(1) = \dim(V)$.

Observe that $\chi_V(g)$ is precisely the sum of eigenvalues of $\sigma(g)$. The eigenvalues of $\chi_V(g^{-1})$ are simply reciprocals of those of $\chi_V(g)$; in the setting $K = \mathbb{C}$, the following result is immediate from Proposition 1.5.

Proposition 4.2: Suppose that $K = \mathbb{C}$. Then, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

The fact that the trace is invariant under conjugation, i.e. $\operatorname{tr}(tst^{-1}) = \operatorname{tr}(s)$, yields the following result.

Lemma 4.3: χ_V is a class function, i.e. χ_V is constant on conjugacy classes of G.

Lemma 4.4: Isomorphic representations have the same character.

Proof: Let $f : V \to V'$ be an isomorphism of representations (σ, V) and (σ', V') of G. Then for each $g \in G$, we have $f \circ \sigma(g) = \sigma'(g) \circ f$, hence $\sigma(g) = f^{-1} \circ \sigma'(g) \circ f$. Taking the trace of both sides and using the cyclic property gives $\operatorname{tr}(\sigma(g)) = \operatorname{tr}(\sigma'(g))$ as desired. □

The space K^G of all maps $G \to K$ forms a vector space over K, with dimension $\operatorname{ord}(G)$. In the setting $K = \mathbb{C}$, we may define the following inner product.

$$\langle \cdot, \cdot \rangle : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}, \qquad (\varphi, \psi) \mapsto \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

Remark: For characters χ, χ', Proposition 4.2 gives

$$\langle \chi, \chi' \rangle = \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} \chi(g)\chi'(g^{-1}).$$

Theorem 4.5 (Orthogonality of characters): Suppose that $K = \mathbb{C}$. Let (σ, V), (σ', V') be two irreducible representations of G.

1. If $V \ncong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 0$.
2. If $V \cong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 1$.

Proof: Let $\{v_1, ..., v_n\}$ be a basis of V, and let $\{v_1', ..., v_m'\}$ be a basis of V'. Given any linear map $f : V \to V'$, we will denote $\tilde{f} = \sum_{g \in G} \sigma'(g) \circ f \circ \sigma(g)^{-1}$; recall that $\tilde{f}$ is G-invariant.

1. Observe that Schur's Lemma (Theorem 3.3) forces all such $\tilde{f} = 0$. In particular, consider the maps e_{ij} defined for each $1 \leq i \leq n$, $1 \leq j \leq m$ as
$$e_{ij} : V \to V', \qquad \sum_i \alpha_i v_i \mapsto \alpha_i v'_j.$$
These maps $\{e_{ij}\}$ form a basis of $\mathcal{L}(V, V')$. Check that the matrix entries obey
$$\left[a \circ e_{ij} \circ b\right]_{k\ell} = [a]_{ki}[b]_{j\ell},$$
so using $\tilde{e}_{ij} = 0$ gives the relations
$$\left[\tilde{e}_{ij}\right]_{kl} = \sum_{g\in G} \left[\sigma'(g) \circ e_{ij} \circ \sigma(g)^{-1}\right]_{kl} = \sum_{g\in G} [\sigma'(g)]_{ki}\left[\sigma(g)^{-1}\right]_{j\ell} = 0$$
for all $1 \leq i, k \leq n$, $1 \leq j, \ell \leq m$. These hold in particular for $i = k$, $j = \ell$; summing over $1 \leq i \leq n$, $1 \leq j \leq m$, we have
$$\begin{aligned} 0 = \sum_{ij} \sum_{g\in G} [\sigma'(g)]_{ii}\left[\sigma(g)^{-1}\right]_{jj} &= \sum_{g\in G} \left(\left(\sum_i [\sigma(g)]_{ii}\right)\left(\sum_j \left[\sigma(g)^{-1}\right]_{jj}\right)\right) \\ &= \sum_{g\in G} \chi_V(g)\chi_{V'}(g^{-1}) \\ &= \operatorname{ord}(G)\langle \chi_V, \chi_{V'}\rangle. \end{aligned}$$
2. Schur's Lemma (Theorem 3.3) forces all such $\tilde{f} = \lambda_f \operatorname{id}_V$ for scalars $\lambda_f \in \mathbb{C}$. To extract λ_f, take the trace of both sides to obtain
$$n\lambda_f = \dim(V)\lambda_f = \sum_{g\in G} \operatorname{tr}\left(\sigma'(g) \circ f \circ \sigma(g)^{-1}\right) = \operatorname{ord}(G)\operatorname{tr}(f).$$
With this, each $\tilde{e}_{ij} = \lambda_{ij}\delta_{ij} \operatorname{id}_V$, where $\lambda_{ij} = \operatorname{ord}(G)/n$. Thus, we obtain the relations
$$\sum_{g\in G} [\sigma'(g)]_{ki}\left[\sigma(g)^{-1}\right]_{j\ell} = \frac{1}{n}\operatorname{ord}(G)\delta_{ij}\delta_{kl}$$
for all $1 \leq i, j, k, \ell \leq n$. Following a similar process as before,
$$\begin{aligned} \operatorname{ord}(G)\langle \chi_V, \chi_{V'}\rangle &= \sum_{g\in G} \left(\left(\sum_i [\sigma(g)]_{ii}\right)\left(\sum_j \left[\sigma(g)^{-1}\right]_{jj}\right)\right) \\ &= \sum_{ij} \sum_{g\in G} [\sigma'(g)]_{ii}\left[\sigma(g)^{-1}\right]_{jj} \\ &= \sum_{ij} \frac{1}{n}\operatorname{ord}(G)\delta_{ij} \\ &= \operatorname{ord}(G) \end{aligned}$$

This completes the proof. □

Corollary 4.5.1: The number of irreducible representations of G (up to isomorphism) is at most the number of conjugacy classes of G.

Example 4.5.2: We have now established that the trivial representation, the one dimensional representation from Example 1.4.1, and the two dimensional representation from Example 1.4.2 are the only irreducible representations of S_3. Note that S_3 has three conjugacy classes: $\{e\}$, $\{(12), (23), (31)\}$, and $\{(123), (321)\}$. With this, we can construct the *character table* for S_3, with each row containing the characters of the group elements with respect to the given representation.

S_3	e	(12)	(23)	(31)	(123)	(321)
Trivial	1	1	1	1	1	1
Sign	1	−1	−1	−1	1	1
Standard	2	0	0	0	−1	−1

Observe that the rows of this table are orthogonal; indeed, so are the columns!