Term presentation
Problem 5

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MA2102: Linear Algebra I

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Problem statement

Let $a \in \mathbb{R}$. Consider the set

$$S_a^n = \left\{1, (x-a), (x-a)^2, \ldots, (x-a)^n\right\}.$$

Show that S_a^n is a basis for $P_n(\mathbb{R})$, the space of polynomials of degree at most n.

Preliminaries

Any polynomial in $P_n(\mathbb{R})$ is of the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

In other words, the set $S_0^n = \{1, x, \ldots, x^n\}$ is a basis of $P_n(\mathbb{R})$. This gives $\dim P_n(\mathbb{R}) = n + 1$.

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The binomial theorem gives

$$(x-a)^n = x^n - nax^{n-1} + \binom{n}{2}a^2x^{n-2} + \cdots + (-1)^n a^n.$$

Specifically, the coefficient of x^n in $(x-a)^n$ is 1.

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$$(x-a)^n - x^n \in P_{n-1}(\mathbb{R}) \subset P_n(\mathbb{R}).$$

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Proof by induction: Base case

For $n = 0$, the claim is trivial. We have $S_a^0 = S_0^0 = \{1\}$, which is a linearly independent set.

For $n = 1$, consider the linear combination of elements from $S_a^1 = \{1, (x - a)\}$

$$c_0 + c_1(x - a) \;=\; \mathbf{0},$$

for arbitrary $c_0, c_1 \in \mathbb{R}$.

Successively set $x = a$ and $x = 0$.
Thus, $c_0 = 0$ and $c_0 - c_1 a = 0$, whence $c_0 = c_1 = 0$.

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Proof by induction: Induction step

Suppose that for $n = k$, the set $S_a^k = \{1, (x-a), \dots, (x-a)^k\}$ is linearly independent.

Consider the linear combination of elements from S_a^{k+1},

$$c_0 + c_1(x-a) + \cdots + c_k(x-a)^k + c_{k+1}(x-a)^{k+1} = \mathbf{0}.$$

Subtract and add $c_{k+1}x^{k+1}$.

$$\Big[c_0 + c_1(x-a) + \dots \\ + c_k(x-a)^k + c_{k+1}\left((x-a)^{k+1} - x^{k+1}\right)\Big] + c_{k+1}x^{k+1} = \mathbf{0}.$$

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Proof by induction: Induction step

The bracketed portion is a polynomial of degree at most k, so we write it in the form

$$p_k(x) \;=\; a_0 + a_1 x + \cdots + a_k x^k \in P_k(\mathbb{R}).$$

Replacing this in the previous equation,

$$a_0 + a_1 x + \cdots + a_k x^k + c_{k+1} x^{k+1} \;=\; \mathbf{0}.$$

The linear independence of $S_0^{k+1} = \{1, x, \ldots, x^{k+1}\}$ gives $a_0 = a_1 = \cdots = a_k = c_{k+1} = 0$.

Substituting this back into the original linear combination,

$$c_0 + c_1(x-a) + \cdots + c_k(x-a)^k \;=\; \mathbf{0}.$$

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Proof by induction: Conclusion

Thus, by the principle of mathematical induction, the set S_a^n is linearly independent for all integers $n \geq 0$.

Specifically, the set S_a^n is a linearly independent set of size $n+1$, in the space $P_n(\mathbb{R})$ which has dimension $n+1$.

Hence, S_a^n is a basis of $P_n(\mathbb{R})$.

Appendix

To show that $\{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n(\mathbb{R})$, it suffices to prove its linear independence. Consider the linear combination

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \ = \ \mathbf{0}.$$

We choose $n + 1$ distinct reals x, which we exhibit as $n + 1$ roots of the polynomial on the left. However, the degree of this polynomial is at most n.

We conclude that the polynomial on the left is the zero polynomial, so $c_0 = c_1 = \cdots = c_n$.

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