MA 1101 : Mathematics I

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Solution 1.
Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be convex. We claim that

$$\max\{f(a), f(b)\} \geq f(x), \text{ for all } x \in (a, b).$$

To prove this, let $x \in (a, b)$ be given. We set $M = \max\{f(a), f(b)\}$ and $\lambda = (b - x)/(b - a)$. Clearly, $\lambda > 0$ and $1 - \lambda = (x - a)/(b - a) > 0$, thus $\lambda \in [0, 1]$.
By the convexity of f, we have

$$\begin{aligned} f(\lambda a + (1 - \lambda)b) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ f(x) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ &\leq \lambda M + (1 - \lambda)M \\ &= M \end{aligned}$$

This proves the desired statement. □

Solution 2.
Let $a < b$ and let $f : (a, b) \to \mathbb{R}$ be differentiable. We claim that f is convex if and only if

$$f(y) - f(x) \geq f'(x)(y - x), \text{ for all } x, y \in (a, b) \qquad (\star)$$

To prove this, we first assume that f is convex. We will show that $(\star)$ holds.
If $x = y$, the result follows trivially. Let $y > x$. We choose $\alpha \in (a, b)$ such that $y > x > \alpha > b$. Using the Rising Slope Theorem, we have

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(x) - f(\alpha)}{x - \alpha}.$$

Taking the limit as $\alpha \to x$, we have

$$\frac{f(y) - f(x)}{y - x} \geq f'(x).$$

Again, if $x > y$, we choose $\beta \in (a, b)$ such that $a > \beta > x > y$. Using the Rising Slope Theorem, we have

$$\frac{f(\beta) - f(x)}{\beta - x} \geq \frac{f(x) - f(y)}{x - y}.$$

Taking the limit as $\beta \to x$, we have

$$f'(x) \geq \frac{f(x) - f(y)}{x - y}.$$

In either case,

$$f(y) - f(x) \geq f'(x)(y - x), \text{ for all } x, y \in (a, b).$$

We now assume that $(\star)$ holds. We will show that f is convex.
Let $x, y, z \in (a, b)$, such that $x > y > z$. By Cauchy's Mean Value Theorem, there exist α, β such that $\alpha \in (x, y), \beta \in (y, z)$ and

$$f(x) - f(y) = f'(\alpha)(x - y)$$
$$f(y) - f(z) = f'(\beta)(y - z)$$

Using $(\star)$, we have

$$f(x) - f(y) = f'(\alpha)(x - y) \geq f'(y)(x - y)$$
$$f(z) - f(y) = f'(\beta)(z - y) \geq f'(y)(z - y)$$

Rearranging,

$$\frac{f(x)-f(y)}{x-y} \geq f'(y)$$

$$\frac{f(z)-f(y)}{z-y} \leq f'(y)$$

This is the same as

$$\frac{f(x)-f(y)}{x-y} \geq \frac{f(y)-f(z)}{y-z}.$$

Therefore, using the Rising Slope Theorem, f is convex.
This proves the desired result. $\square$

Solution 3.
Let $n \in \mathbb{N}$, let $a_i, \lambda_i > 0$ for all $i = 1, \dots, n$, and let $p \geq 1$. We claim that

$$\frac{\sum \lambda_i a_i^p}{\sum \lambda_i} \geq \left(\frac{\sum \lambda_i a_i}{\sum \lambda_i} \right)^p.$$

To prove this, we define $f\colon (0,\infty) \to \mathbb{R}$ by $f(x) := x^p$ for all $x \in (0,\infty)$. Note that $f''(x) = p(p-1)x^p \geq 0$ for all $x \in (0,\infty)$. Hence, f is convex.
Using Jensen's Inequality on a_i, with weights $\lambda_i / \sum \lambda_i$, we have

$$f\left(\frac{\sum \lambda_i a_i}{\sum \lambda_i} \right) \leq \frac{\sum \lambda_i f(a_i)}{\sum \lambda_i},$$

from which the desired statement follows directly. $\square$

Solution 4.

(i) Let $a > 0$. We claim that for all $x \geq y > 0$,

$$\frac{a^x - 1}{x} \geq \frac{a^y - 1}{y}.$$

To prove this, note that when $x = y$, the inequality follows trivially. Assume $x > y$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := a^x$ for all $x \in \mathbb{R}$. Note that $f''(x) = a^x(\log a)^2 \geq 0$ for all $x \in \mathbb{R}$. Hence, f is convex.

We set $\lambda = y/x$. Note that $\lambda > 0$ and $1 - \lambda = (x-y)/x > 0$. Thus, $\lambda \in [0,1]$.

By the convexity of f, we have

$$\begin{aligned} f(\lambda x + (1-\lambda)0) &\leq \lambda f(x) + (1-\lambda) f(0) \\ f(y) &\leq \frac{y}{x} f(x) + \frac{(x-y)}{x} f(0) \\ x a^y &\leq y a^x + (x-y) a^0 \\ x a^y - x &\leq y a^x - y \\ \frac{a^y - y}{y} &\leq \frac{a^x - 1}{x} \end{aligned}$$

This proves the desired statement. $\square$

(ii) We claim that for all $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

This result is trivial. $\square$