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Solutions to exercises from Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 1

The Real and Complex Number Systems

Exercise 1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution. Use the fact that the field of rationals is closed under additiona and multiplication, as well as the existence of the additive inverse $-r$ and the multiplicative inverse $1/r$. If $r + x$ and rx were rational, then both

$$(-r) + r + x = x, \qquad (1/r)rx = x$$

must also be rational. These are contradictions.

Exercise 2. Prove that there is no rational number whose square is 12.

Solution. Suppose that $x \in \mathbb{Q}$, $x^2 = 12$, and $x = p/q$ where $q \neq 0$ and p and q are coprime integers. This would imply that

$$p^2 = 12q^2 = 3(2q)^2,$$

so 3 divides p^2, hence 3 divides p. Write $p = 3m$ for some integer m, giving

$$3(2q)^2 = p^2 = (3m)^2 = 9m^2, \qquad (2q)^2 = 3m^2.$$

This means that 3 divides $(2q)^2$, hence 3 divides $2q$, hence 3 divides q. This contradicts the fact that p and q are coprime, which means that there is no rational number whose square is 12.

Exercise 3. Prove that the axioms of multiplication in a field imply the following statements.

(a) If $x \neq 0$ and $xy = xz$, then $y = z$.

(b) If $x \neq 0$ and $xy = x$, then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Solution. The axioms of multiplication guarantee the existence of an element $1/x$ such that $x(1/x) = 1$. Left multiply on both sides of $xy = xz$, use associativity and $1w = w$ for all w in the field to get

$$(1/x)xy = (1/x)xz, \qquad y = z.$$

This proves (a). Setting $z = 1$ proves (b), and setting $z = 1/x$ proves (c). Using $x(1/x) = 1$, replace x with $1/x$ in (c) to give

$$(1/x)(1/(1/x)) = 1,$$

then left multiply with x yielding

$$x(1/x)(1/(1/x)) = x, \qquad 1/(1/x) = x.$$

Exercise 4. Let E be a non-empty subset of an ordered set; suppose that α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution. By definition, $\alpha \leq x$ for all $x \in E$ and $x \leq \beta$ for all $\in E$. Since E is non-empty, simply select some $x \in E$, whence $\alpha \leq x \leq \beta$. Thus, we either have $\alpha = x = \beta$, $\alpha = x < \beta$, $\alpha < x = \beta$, or $\alpha < x < \beta$. In the last case, transitivity gives $\alpha < \beta$. Hence, $\alpha \leq \beta$.

Exercise 5. Let A be a non-empty subset of the real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution. Fix $\alpha = -\sup(-A)$. We claim that $\alpha = \inf A$, i.e. $\beta \leq \alpha \leq x$ for all lower bounds β of A and for all $x \in A$.

First, note that $-\alpha = \sup(-A)$, which means that $-\alpha \geq x$ for all $x \in -A$, whence $\alpha \leq -x$ for all $-x \in A$. However, for each $x \in A$, we have $-x \in -A$ so $\alpha \leq x$ for all $x \in A$.

Now, let β be a lower bound of A. This means that $\beta \leq x$ for all $x \in A$, so $-\beta \geq -x$ for all $x \in A$. Again, $-x \in -A$ for all $x \in A$, so $-\beta \geq x$ for all $x \in -A$. This means that β is an upper bound of $-A$, which means $-\beta \geq \sup(-A) = -\alpha$. Thus, $\beta \leq \alpha$.

This proves that $\inf A = -\sup(-A)$.

Exercise 6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r).$$

Hence is makes sense to define $b^x = \sup B(x)$ for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution.

(a) Write r with the common denominator $s = nq$, so $r = mq/s = pn/s$. Now, note that

$$\left((b^m)^{1/n}\right)^s = (b^m)^q = b^{mq}, \qquad \left((b^p)^{1/q}\right)^s = (b^p)^n = b^{np},$$

but $mq = np = rs$. Setting $b^{rs} = x$, use Theorem 1.21 to conclude that there is a unique y such that $y^s = x = b^{rs}$. However, we have just verified two such y, hence

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

(b) Set $r = m/n$, $s = p/q$ with $n > 0$, $q > 0$. Then,

$$b^{r+s} = b^{(mq+np)/nq} = (b^{mq+np})^{1/nq} = (b^{mq}b^{np})^{1/nq}.$$

The corollary of Theorem 1.21 lets us distribute the integer root over the product, giving

$$b^{r+s} = b^{mq/nq}b^{np/nq} = b^{m/n}b^{p/q} = b^r b^s.$$

(c) First, we show that $b^n - 1 \geq n(b-1)$ for all positive integers n. This is trivially true for $n = 1$. For $n > 1$, write $b = 1 + a$ where $a > 0$. Hence the Binomial Theorem gives

$$b^n = (1+a)^n = 1 + na + \frac{1}{2}n(n-1)a^2 + \cdots + a^n > 1 + na,$$

hence

$$b^n - 1 > na = n(b-1).$$

Note that this inequality becomes strict for $n > 1$. Replacing b with $b^{1/n} > 1$, we have $b - 1 > n(b^{1/n} - 1)$ for all positive integers n.

Now, given some $t > 1$, we can choose a positive integer $n > (b-1)/(t-1)$, which implies $n(t-1) > b - 1 > n(b^{1/n} - 1)$, hence $t > b^{1/n}$.

Now, note that for all $x \in B(r)$, $x = b^t$ for some rational t. First, note that for all rational $t \leq r$, we have $b^t \leq b^r$. This is because if we write t and r with a common positive integer denominator, $t = m/q$, $r = n/q$, then $m \leq n$ so $(b^{1/q})^m \leq (b^{1/q})^n$. Thus, b^r is an upper bound for $B(r)$.

Next, we show that b^r is the least upper bound to $B(r)$. Suppose that $\alpha = \sup B(r)$, and $b^t \leq \alpha < b^r$ for all $t \leq r$. Using the previously proven inequality, find a large enough integer n such that $b^{1/n} < b^r/\alpha$. Thus, $\alpha < b^{r-1/n}$, and $r - 1/n < r$ so $b^{r-1/n} \in B(r)$, which contradicts the fact that α is the supremum of $B(r)$. Hence, b^r is the least upper bound of $B(r)$, so

$$b^r = \sup B(r).$$

(d) We have been given

$$b^x = \sup B(x), \qquad b^y = \sup B^y, \qquad b^{x+y} = \sup B(x+y)$$

by definition for real x and y. Choose some rational $t \leq x + y$, so $b^t \in B(x+y)$. By choosing a rational r such that $t - y < r < x$ and setting $s = t - r$, we have $t = r + s$ and $r < x$, $s < y$. Thus, $b^r \in B(x)$ and $b^s \in B(y)$, so every element $b^t \in B(x+y)$ can be written as $b^{r+s} = b^r b^s$, which is the product of an element each from $B(x)$ and $B(y)$. Conversely, given elements $b^r \in B(x)$ and $b^s \in B(y)$, we have $r \leq x$ and $s \leq y$ so $t = r + s \leq x + y$, hence $b^{r+s} = b^t \in B(x+y)$. Thus, we have

$$B(x+y) = \{wz : w \in B(x), z \in B(y)\}.$$

Thus, for any element $wz \in B(x+y)$, $w \in B(x)$, $z \in B(y)$, we have $w \leq \sup B(x) = b^x$ and $z \leq \sup B(y) = b^y$, so $wz \leq b^x b^y$. This means that $b^x b^y$ is an upper bound of $B(x+y)$.

Now suppose that $\alpha = \sup B(x+y)$ such that $wz \leq \alpha < b^x b^y$ for all $wz \in B(x+y)$, where $w \in B(x)$ and $z \in B(y)$. Then, $\alpha/b^x < b^y$, so choose β such that $\alpha/b^x < \beta < b^y$. In other words, $\alpha/\beta < b^x$ and $\beta < b^y$, so we can choose rational $r < x$, $s < y$ such that $\alpha/\beta \leq b^r \in B(x)$ and $\beta \leq b^s \in B(y)$. Note that $r \neq x$ and $s \neq y$. Thus, the product $(\alpha/\beta)\beta = \alpha \leq b^r b^s \in B(x+y)$. However, recall that we chose α such that $b^r b^s \leq \alpha$ for all $b^r \in B(x)$, $b^s \in B(y)$, so we must have $\alpha = b^r b^s$ for our choice of r and s. Now, we can choose rational r' and s' such that $r < r' < x$ and $s < s' < y$, hence $b^r < b^{r'} \in B(x)$ and $b^s < b^{s'} \in B(y)$. This gives $\alpha = b^r b^s < b^{r'} b^{s'} \in B(x+y)$, which contradicts the fact that α is an upper bound. Thus, $b^x b^y$ must be the least upper bound of $B(x+y)$, so

$$b^{x+y} = b^x b^y.$$

Exercise 7. Fix $b > 1$, $y > 0$, and show the following.

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) Hence, $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for sufficiently large n.

(e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Solution.

(a) See Exercise 1 (c).

(b) See Exercise 1 (c).

(c) See Exercise 1 (c).

(d) Set $t = yb^{-w} > 1$, and using the previous inequality, choose sufficiently large n such that $b^{1/n} < t = yb^{-w}$. Thus,

$$b^{w+1/n} < y.$$

(e) Set $t = (1/y)b^w > 1$, and using the inequality in (c), choose sufficiently large n such that $b^{1/n} < t = (1/y)b^w$. Thus,

$$y < b^{w-1/n}.$$

(f) Exactly one of the following must be true; $b^x < y$, $b^x = y$, $b^x > y$. If $b^x < y$, then $x \in A$ by definition. Using (d), we can find sufficiently large n such that

$$b^{x+1/n} < y,$$

hence $x < x + 1/n \in A$, contradicting the fact that x is an upper bound of A. If $b^x > y$, then using (e), we can find sufficiently large n such that

$$y < b^{x-1/n},$$

which means that $x - 1/n$ is also an upper bound of A, contradicting the fact that x is the lowest upper bound of A. This leaves us with $b^x = y$.

(g) Suppose that $x \neq x'$, and without loss of generality $x < x'$. Set $x' - x = h > 0$, and note that $b^{x'} = b^{x+h} = b^x b^h$. Now, $b > 1$ and $h > 0$, so $b^h > 1$. Thus, $b^{x'} > b^x$, which means that $b^{x'} \neq b^x$ for $x' \neq x$. Thus, if $b^x = y$, then x is unique.

Exercise 8. Prove that no order can be defined in the complex field that turns it into an ordered field.

Solution. In an ordered field, if $x > 0$, then we must have $-x < 0$, and vice versa by Proposition 1.18. The same proposition gives that if $x \neq 0$, then $x^2 > 0$. This forces $i^2 = -1 > 0$. Applying the same proposition again, this forces $(-1)^2 = 1 > 0$, which is a contradiction because we cannot have both $-1 > 0$ and $1 > 0$.

Exercise 9. Suppose $z = a+bi$, $w = c+di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b = d$. Prove that this turns the set of all complex numbers into an ordered set. Does this ordered set have the least-upper-bound property?

Solution. First, we show that for arbitrary $z = a+bi$ and $w = c+di$, exactly one of the following is true: $z < w$, $z = w$, $z > w$. To do this, note that the real numbers are ordered, so either $a < c$, $a = c$, or $a > c$. In the case $a < c$, we have $z < w$ and since $a \neq c$, $z \neq w$. Also, this excludes $w < z$. In the case $a > c$, the roles of z and w are interchanged, so $z > w$. In the case $a = c$, we note that either $b < d$, $b = d$, or $b > d$; when $b < d$, $z < w$ and when $b > d$, $z > w$. Finally, when $a = c$ and $b = d$, we have $z = w$.

Next, we show that transitivity holds, i.e. if $z < w$ and $w < x$, then $z < x$. Write $z = a+bi$, $w = c+di$ and $x = e+fi$. Note that the conditions $z < w$ and $w < x$ imply $a \leq c$ and $c \leq e$. This has to be further split into four cases.

Case 1 If $a < c$ and $c < e$, then $a < e$ so $z < x$.

Case 2 If $a = c$ and $c < e$, then $a < e$ again so $z < x$.

Case 3 If $a < c$ and $c = e$, then $a < e$ again so $z < x$.

Case 4 If $a = c$ and $c = e$, then we must have had $b < d$ and $d < f$, so $a = e$ and $b < f$ gives $z < x$.

No, this ordered set does not have the least upper bound property. Consider the set of complex numbers $S = \{a + bi : 0 < a < 1, b = 0\}$. If $w = c + di$ is to be an upper bound of S, i.e. $z \leq w$ for all $z \in S$, then either $z = w$ for some $z \in S$ or $z < w$ for all $z \in S$. The former implies that $w = a+0i$ for some $0 < a < 1$, in which case we have $w = a+0i < (a+1)/2+0i \in S$, a contradiction. The latter implies that $a \leq c$ for all $0 < a < 1$, which forces $1 \leq c$. If $w = c+di$ is the least upper bound of S with $1 < c$, then note that $(1 + c)/2 + di < c + di = w$ is smaller upper bound of S. Otherwise, if $w = 1 + di$ is the least upper bound of S, then $1 + (d - 1)i < 1 + di = w$ is a smaller upper bound. This means that the set S have no least upper bound.

Exercise 10. Suppose $z = a + bi$, $w = u + vi$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception) has two complex square roots.

Solution. Write

$$z^2 = (a + bi)^2 = a^2 - b^2 + 2abi, \qquad \overline{z}^2 = (a - bi)^2 = a^2 - b^2 - 2abi.$$

Now,

$$a^2 - b^2 = \frac{1}{2}(|w| + u) - \frac{1}{2}(|w| - v) = u,$$

and

$$\begin{aligned} 2ab &= 2\left(\frac{|w| + u}{2}\right)^{1/2}\left(\frac{|w| - u}{2}\right)^{1/2} \\ &= 2\left(\frac{(|w| + u)(|w| - u)}{4}\right)^{1/2} \\ &= 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} \\ &= 2\left(\left(\frac{v}{2}\right)^2\right)^{1/2}. \end{aligned}$$

Recall that $(x^2)^{1/2} = x$ if $x \geq 0$ and $(x^2)^{1/2} = -x$ if $x \leq 0$. Thus, when $v \geq 0$, we have $2ab = v$ and when $v \leq 0$, we have $2ab = -v$. This means that $w = u + 2abi = z^2$ when $v \geq 0$ and $w = u - 2abi = (\overline{z})^2$ when $v \leq 0$.

Note that when $w = 0$, it has only one square root, namely 0. Otherwise, every non-zero complex number $w = u + iv$ has two square roots, either $z, -z$ or $\overline{z}, -\overline{z}$ depending on the sign of v.

Exercise 11. If z is a complex number, prove that there exists an $r \geq 0$, a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Solution. Write $z = a + bi$, and if $z \neq 0$ define

$$r = \sqrt{a^2 + b^2}, \qquad w = z/r = \frac{a}{\sqrt{a^2+b^2}} + \frac{bi}{\sqrt{a^2+b^2}}.$$

If $z = 0$, simply take $r = 0$ and $w = 1$. Thus, $z = rw$.

When $z \neq 0$, this choice is unique, since $z = rw$ forces $|z| = |rw| = |r||w| = r$, hence $r = |z| = \sqrt{a^2 + b^2}$ and $w = z/r$. Otherwise for $z = 0$, we can choose any w (say $w = \pm 1$) as long as $r = 0$.

Exercise 12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Solution. We prove this by induction. The case $n = 1$ is trivially true. For $n = 2$, see Theorem 1.33. If this holds for some $n \geq 1$, then use the $n = 2$ case on $z_1 + \cdots + z_n$ and z_{n+1}, then the induction hypothesis to get

$$|z_1 + \cdots + z_n + z_{n+1}| \leq |z_1 + \cdots + z_n| + |z_{n+1}| \leq |z_1| + \cdots + |z_n| + |z_{n+1}|.$$

This proves the desired statement by induction.

Exercise 13. If x and y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Solution. Use the triangle inequality to write

$$|x| = |x - y + y| \leq |x - y| + |y|, \qquad |y| = |y - x + x| \leq |x - y| + |x|.$$

Thus, if $|x| > |y|$, then $||x| - |y|| = |x| - |y| \leq |x - y|$ by the first inequality. If $|x| < |y|$, then $||x| - |y|| = |y| - |x| \leq |x - y|$ by the second inequality. If $|x| = |y|$, then $||x| - |y|| = 0$, so the inequality holds trivially.

Exercise 14. If z is a complex number such that $|z| = 1$, that is, such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Solution. Write $z = a+bi$, so $a^2+b^2 = 1$. Now, $|1+z|^2 = (a+1)^2+b^2$, and $|1-z|^2 = (a-1)^2+b^2$. Adding,

$$|1 + z|^2 + |1 - z|^2 = 2(a^2 + b^2 + 1) + 2a - 2a = 4.$$

Exercise 15. Under what conditions does equality hold in the Schwarz inequality?

Solution. In Theorem 1.35, recall that

$$A = \sum |a_i|^2, \qquad B = \sum |b_i|^2, \quad C = \sum a_i \overline{b_i},$$

and the desired inequality was $AB \geq C^2$. If $B = 0$, then all $b_i = 0$ so equality holds. Otherwise, we concluded that with $B > 0$,

$$\sum |Ba_i - Cb_i|^2 = B(AB - |C|^2) \geq 0.$$

Here, equality means $AB = |C|^2$, so every $|Ba_i - Cb_i| = 0$, hence $a_i = (C/B)b_i$ for all i.

Exercise 16. Suppose $k \geq 3$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$, $|\boldsymbol{x} - \boldsymbol{y}| = d > 0$ and $r > 0$. Prove the following.

(a) If $2r > d$, then there are infinitely many $\boldsymbol{z} \in \mathbb{R}^k$ such that

$$|\boldsymbol{z} - \boldsymbol{x}| = |\boldsymbol{z} - \boldsymbol{y}| = r.$$

(b) If $2r = d$, there is exactly one such $\boldsymbol{z}$.

(c) If $2r < d$, there is no such $\boldsymbol{z}$.

Solution. Note that by translating all the variables $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{y}$, $\boldsymbol{y}' = \boldsymbol{0}$, our system of equations looks identical, with $|\boldsymbol{x}' - \boldsymbol{y}'| = d$ and the solutions are related by $\boldsymbol{z}' = \boldsymbol{z} - \boldsymbol{y}$. Thus, we may instead consider the system $|\boldsymbol{x}| = d$,

$$|\boldsymbol{z} - \boldsymbol{x}| = |\boldsymbol{z}| = r.$$

Consider an arbitrary solution $\boldsymbol{z}$ and write $\boldsymbol{v} = \boldsymbol{z} - \frac{1}{2}\boldsymbol{x}$. Now,

$$|\boldsymbol{z}|^2 = (\frac{1}{2}\boldsymbol{x} + \boldsymbol{v}) \cdot (\frac{1}{2}\boldsymbol{x} + \boldsymbol{v}) = \frac{1}{4}|\boldsymbol{x}|^2 + |\boldsymbol{v}|^2 + \boldsymbol{x} \cdot \boldsymbol{v}.$$

Also,

$$|\boldsymbol{z} - \boldsymbol{x}|^2 = (-\frac{1}{2}\boldsymbol{x} + \boldsymbol{v}) \cdot (-\frac{1}{2}\boldsymbol{x} + \boldsymbol{v}) = \frac{1}{4}|\boldsymbol{x}|^2 + |\boldsymbol{v}|^2 - \boldsymbol{x} \cdot \boldsymbol{v}.$$

Adding the above equations gives

$$|\boldsymbol{z}|^2 + |\boldsymbol{z} - \boldsymbol{x}|^2 = \frac{1}{2}|\boldsymbol{x}|^2 + 2|\boldsymbol{v}|^2, \qquad |\boldsymbol{v}|^2 = r^2 - \frac{d^2}{4}.$$

Subtracting the two equations gives $\boldsymbol{v} \cdot \boldsymbol{x} = 0$.

These conditions on $\boldsymbol{v}$ are necessary and sufficient to generate solutions $\boldsymbol{z} = \frac{1}{2}\boldsymbol{x} + \boldsymbol{v}$.

(a) Pick a unit vector $\hat{\boldsymbol{v}}$ perpendicular to $\boldsymbol{x}$, i.e. $\hat{\boldsymbol{v}} \cdot \boldsymbol{x} = 0$. Note that the components satisfy

$$v_1 x_1 + \cdots + v_k x_k = 0.$$

Since $d > 0$, we have $\boldsymbol{x} \neq \boldsymbol{0}$, so without loss of generality let $x_1 \neq 0$. Then we have

$$v_1 = -\frac{1}{x_1}(v_2 x_2 + \cdots + v_k x_k).$$

Therefore, we may choose the components $v_2, \ldots, v_k$ arbitrarily. For example, fix $v_2 = 1$, vary $v_3 = 0, 1, 2, \ldots$ and vary the remaining components arbitrarily, then normalize. All of the generated unit vectors are distinct, because the ratio of components v_2 and v_3 is different in each case. Thus, we have generated infinitely many unit vectors $\hat{\boldsymbol{v}}$ this way.

Now define the real number $v \geq 0$, $v^2 = r^2 - d^2/4$. Then, all the vectors $\boldsymbol{z} = \frac{1}{2}\boldsymbol{x} + \boldsymbol{v}$ are solutions, where $\boldsymbol{v} = v\hat{\boldsymbol{v}}$.

(b) We have $|\boldsymbol{x}| = d = 2r$, which means

$$|\boldsymbol{v}|^2 = r^2 - \frac{1}{4}(2r)^2 = 0,$$

forcing $|\boldsymbol{v}| = 0$, $\boldsymbol{v} = \mathbf{0}$. Thus, there is only one solution, namely $\boldsymbol{z} = \frac{1}{2}\boldsymbol{x}$.

(c) When $2r < d$

$$|\boldsymbol{v}|^2 = r^2 - \frac{d^2}{4} < 0,$$

which is impossible. Thus, there are no solutions $\boldsymbol{z}$ of this system.

Note that when $k = 2$, we can only generate 2 unit vectors $\hat{\boldsymbol{v}}$ such that $\hat{\boldsymbol{v}} \cdot \boldsymbol{x} = 0$. This is because we want

$$v_1 x_1 + v_2 x_2 = 0, \qquad v_1 = -\frac{v_2 x_2}{x_1}, \qquad v_1^2 = 1 - v_2^2.$$

Thus, there are only two solutions, when $2r > d$. When $k = 1$, the condition $vx = 0$ with $x \neq 0$ forces $v = 0$, yet we require $v^2 = r^2 - d^2/4 > 0$ when $2r > d$, so there are no solutions.

The remaining parts (b) and (c) remain identical for $k = 1, 2$.

Exercise 17. Prove that

$$|\boldsymbol{x} + \boldsymbol{y}|^2 + |\boldsymbol{x} - \boldsymbol{y}|^2 = 2|\boldsymbol{x}|^2 + 2|\boldsymbol{y}|^2$$

if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Solution. Calculate

$$|\boldsymbol{x} + \boldsymbol{y}|^2 = (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) = |\boldsymbol{x}|^2 + |\boldsymbol{y}|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y},$$

$$|\boldsymbol{x} - \boldsymbol{y}|^2 = (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) = |\boldsymbol{x}|^2 + |\boldsymbol{y}|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y}.$$

Adding the two gives the desired equation.

If we interpret $\boldsymbol{x}$ and $\boldsymbol{y}$ to be two adjacent legs of a parallelogram, then $\boldsymbol{x} + \boldsymbol{y}$ and $\boldsymbol{x} - \boldsymbol{y}$ represent its diagonals. Thus, the sum of squares of the diagonals of a parallelogram is equal to twice the sum of squares of two adjacent sides.

Exercise 18. If $k \geq 2$ and $\boldsymbol{x} \in \mathbb{R}^k$, prove that there exists $\boldsymbol{y} \in \mathbb{R}^k$ such that $\boldsymbol{y} \neq \mathbf{0}$ but $\boldsymbol{x} \cdot \boldsymbol{y} = 0$. Is this also true if $k = 1$?

Solution. If $\boldsymbol{x} = \mathbf{0}$, then any non-zero vector in $\boldsymbol{y} \in \mathbb{R}^k$ satisfies $\boldsymbol{x} \cdot \boldsymbol{y} = 0$. Otherwise, $\boldsymbol{x} = (x_1, x_2, \ldots, x_k) \neq \mathbf{0}$ so without loss of generality let the component $x_1 \neq 0$. Set

$$\boldsymbol{y} = (-x_2, x_1, 0, \ldots, 0) \in \mathbb{R}^k,$$

so

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1(-x_2) + x_2(x_1) + 0 + \cdots + 0 = 0.$$

This is clearly not possible in $\mathbb{R}$ unless $x = 0$, because the product of any two non-zero real numbers is also non-zero.

Exercise 19. Suppose $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$. Find $\boldsymbol{c} \in \mathbb{R}^k$ such that

$$|\boldsymbol{x} - \boldsymbol{a}| = 2|\boldsymbol{x} - \boldsymbol{b}|$$

if and only if $|\boldsymbol{x} - \boldsymbol{c}| = r$.

Solution. Write $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{a}$, $\boldsymbol{b}' = \boldsymbol{b} - \boldsymbol{a}$, $\boldsymbol{c}' = \boldsymbol{c} - \boldsymbol{a}$, so we want to find $\boldsymbol{c}'$ such that

$$|\boldsymbol{x}'| = 2|\boldsymbol{x}' - \boldsymbol{b}'|$$

if and only if $|\boldsymbol{x}' - \boldsymbol{c}'| = r$.

Write $\boldsymbol{x}' = \frac{4}{3}\boldsymbol{b}' + \boldsymbol{r}$. Then

$$|\boldsymbol{x}'|^2 = \frac{16}{9}|\boldsymbol{b}'|^2 + |\boldsymbol{r}|^2 + \frac{8}{3}\boldsymbol{b}' \cdot \boldsymbol{r},$$

and

$$|\boldsymbol{x}' - \boldsymbol{b}'|^2 = |\frac{1}{3}\boldsymbol{b}' + \boldsymbol{r}|^2 = \frac{1}{9}|\boldsymbol{b}'|^2 + |\boldsymbol{r}|^2 + \frac{2}{3}\boldsymbol{b}' \cdot \boldsymbol{r}.$$

Using $|\boldsymbol{x}'|^2 = 4|\boldsymbol{x}' - \boldsymbol{b}'|^2$, we have

$$\frac{12}{9}|\boldsymbol{b}'|^2 = 3|\boldsymbol{r}|^2, \qquad |\boldsymbol{r}| = \frac{2}{3}|\boldsymbol{b}'|.$$

Thus, $|\boldsymbol{x}' - \frac{4}{3}\boldsymbol{b}'| = \frac{2}{3}|\boldsymbol{b}'|$, which is both necessary and sufficient. This means that $\boldsymbol{c}' = \frac{4}{3}\boldsymbol{b}'$ and $r = \frac{2}{3}|\boldsymbol{b}'|$. Translating everything back by $\boldsymbol{a}$, we have

$$\boldsymbol{c} = \frac{4}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{a}, \qquad r = \frac{2}{3}|\boldsymbol{b} - \boldsymbol{a}|.$$

Exercise 20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

Solution. We define a cut as any set $\alpha \subset \mathbb{Q}$ with the following properties.

(I) α is not empty, $\alpha \neq \mathbb{Q}$.

(II) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.

Property (III) used to state that if $p \in \alpha$, then $p < r$ for some $r \in \alpha$, which meant that α had no maximal element. Property (II) implies that if $p \in \alpha$ and $q \notin \alpha$, then $p < q$ (take the contrapositive, and note that $p \neq q$). It also implies that if $r \notin \alpha$ and $r < s$, then $s \notin \alpha$ ($s \in \alpha$ would have forced $r \in \alpha$).

Call the set of all these cuts $\mathbb{R}'$. Like before, the order $\alpha < \beta$ is defined to mean $\alpha \subset \beta$, for $\alpha, \beta \in \mathbb{R}'$. Again, $\mathbb{R}'$ has the least upper bound property.

To see this, let A be any non-empty subset of $\mathbb{R}'$ bounded above by $\beta \in \mathbb{R}'$, and let γ be the union of all $\alpha \in A$. Thus, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. To verify that γ is indeed a cut, note that A is non-empty so there is at least one element $\alpha_0 \in A$ which is non-empty, so $\alpha_0 \subset \gamma$ with γ non-empty. Also, $\gamma \subset \beta$ since β being an upper bound means that $\alpha < \beta$ for all $\alpha \in A$, which in turn means $\alpha \subset \beta$ for all $\alpha \in A$, hence $\gamma = \cup_{\alpha \in A}\alpha \subset \beta$. This verifies property (I). To verify property (II), pick $p \in \gamma$, and suppose that $p \in \alpha_1$ for some $\alpha \in A$. If $q \in \mathbb{Q}$ with $q < p$, this gives $q \in \alpha_1$, hence $q \in \gamma$. Thus, γ is indeed a cut, i.e. $\gamma \in \mathbb{R}'$.

Now, we claim that $\gamma = \sup A$. Clearly, for any $\alpha \in A$, we have $\alpha \subset \gamma$ by definition to $\alpha \leq \gamma$ for all $\alpha \in A$, meaning γ is an upper bound of A. Now suppose that $\delta \in \mathbb{R}'$, and $\delta < \gamma$. This means that δ is a proper subset of γ, so there is some $p \in \gamma$ such that $\notin \delta$. However, we must have $p \in \alpha_1$ for some $\alpha_1 \in A$, so α cannot be a proper subset of δ, meaning that δ is not an upper bound of A. Thus, γ is the least upper bound of A.

Like before, for $\alpha, \beta \in \mathbb{R}'$, define addition $\alpha + \beta$ as the set of sums $r + s$ with $r \in \alpha$, $s \in \beta$. We must now verify the axioms of addition.

(A1) We demand closure, which is easily seen because $\alpha + \beta$ is a non-empty proper subset of $\mathbb{Q}$, and if $p \in \alpha + \beta$, then we must be able to write $p = r + s$ for some $r \in \alpha$, $s \in \beta$. Now if $q \in \mathbb{Q}$ and $q < p$, then $q - s < p - s = r$, so $q - s \in \alpha$, hence $q = (q - s) + s \in \alpha + \beta$.

(A2) We demand commutativity, which follows trivially. $\alpha + \beta = \beta + \alpha$, both being the set of $r + s = s + r$ with $r \in \alpha$, $s \in \beta$.

(A3) We demand associativity, which follows again from the associativity of the rational numbers. Note that if $\alpha, \beta, \gamma \in \mathbb{R}'$, with $r \in \alpha$, $s \in \beta$, $t \in \gamma$, then $r + (s + t) = (r + s) + t$.

(A4) Here, select $0' = \{r \in \mathbb{Q} : r \leq 0\}$. To show that for any $\alpha \in \mathbb{R}'$, $0' + \alpha = \alpha$, note that $0' + \alpha$ is the set of all rational numbers $r + s$ with $r \leq 0$ and $s \in \alpha$, so $r + s \leq s \in \alpha$ hence $0' + \alpha \subseteq \alpha$. Now, if $s \in \alpha$, then $0 + s \in 0' + \alpha$ since $0 \in 0'$ and $s \in \alpha$, so $\alpha \subseteq 0' + \alpha$. This proves $0' + \alpha = \alpha$.

(A5) We demand the existence of an additive inverse $-\alpha$ for every α, such that $\alpha + (-\alpha) = 0'$. This fails with the choice $\alpha = 0^* = \{r \in \mathbb{Q} : r < 0\}$. Note that if $0^* + (-0^*) = 0'$, we require $r + s \leq 0$ for all $r \in 0^*$, $s \in -0^*$. There must also be some $r_0 \in 0^*$, $s_0 \in -0^*$ such that $r_0 + s_0 = 0$. Since $r_0 \in 0^*$, $r_0 < 0$, so $s_0 = -r_0 > 0$. Now, note that $-s_0/2 < 0$ so $-s_0/2 \in 0^*$, but the sum $(-s_0/2) + s_0 = s_0/2 > 0$, which is a contradiction.

In addition, note that 0^* does not serve as a zero element, since $0^* + 0' = 0'$, not 0^*. Furthermore, there is no choice of a zero element, say α_0, which makes (A1-4) hold as well as (A5), since our choice of the zero element $0'$ is forced (we have already shown that $0' + \alpha_0 = \alpha_0$, not $0'$ if $\alpha_0 \neq 0'$; the field axioms imply that the zero element once found is unique).

Chapter 2

Basic Topology

Exercise 1. Prove that the empty set is a subset of every set.

Solution. Suppose that there exists a set A such that $\emptyset \not\subseteq A$. Note that $\emptyset \neq A$, so $\emptyset \subset A$ is a proper subset. Thus, there must be some element $x \in \emptyset$, $x \notin A$ which is absurd since the empty set $\emptyset$ contains no elements.

Exercise 2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable.

Solution. Let A_n be the set of all algebraic numbers which are the roots of a polynomial with integral coefficients of degree n. Each of these algebraic numbers can be mapped to a tuple $(a_0, \ldots, a_n) \in \mathbb{Z}^n$, denoting the coefficients of the polynomial it is a root of. This particular polynomial can have at most n distinct roots, hence any such tuple can have at most n algebraic numbers in its pre-image. Thus, we can map each $x \in A_n$ injectively to a tuple $(a_0, \ldots, a_n, k) \in \mathbb{Z}^n \times \mathbb{Z}$, where k is an index. Theorem 2.13 guarantees that $\mathbb{Z}^n \times \mathbb{Z}$ is countable, and Theorem 2.8 guarantees that the range of this map, which is an infinite subset of $\mathbb{Z}^n \times \mathbb{Z}$, is also countably infinite. Thus, A_n is countably infinite. Finally, Theorem 2.12 guarantees that the countable union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is countably infinite. Noting that A is the set of all algebraic numbers gives the desired result.

Exercise 3. Prove that there exist real numbers which are not algebraic.

Solution. Let $S = \mathbb{R} \setminus A$ be the set of all real numbers which are not algebraic. Note that the set of algebraic numbers A is countable. If S were empty, or even countable, then the union $\mathbb{R} = S \cup A$ would also be countable, which is a contradiction. Thus, there are an uncountably infinite number of real numbers which are not algebraic.

Exercise 4. Is the set of all irrational real numbers countable?

Solution. We use the same argument as before, noting that if $\mathbb{R} \setminus \mathbb{Q}$ were countable, then the union $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ would also be countable, which is a contradiction.

Exercise 5. Construct a bounded set of real numbers with exactly three limit points.

Solution. For $x \in \mathbb{R}$, define the set

$$S_x = \{x + 1/n : n \in \mathbb{N}\}.$$

Then, $S = S_0 \cup S_1 \cup S_2$ has exactly three limit points, namely $0, 1, 2$.

Exercise 6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

Solution. We must show that E' contains all its limit points. Let x' be a limit point of E', which means that given $r > 0$, we can find $y \in E'$ such that $0 < d(x, y) < r$. Let h be the positive real number such that $d(x, y) = r - h$. Now, $y \in E'$ is a limit point of E, which means that we can find $z \in E$ such that $0 < d(y, z) < h$. Thus, $d(x, z) \leq d(x, y) + d(y, z) < r$. Also, we can choose $z \neq x$ since there are infinitely many $z \in E$ to choose from for each neighbourhood of y. Thus, x is a limit point of E, so $x \in E'$.

Suppose that x is a limit point of E. This means that every deleted neighbourhood of x contains some point $y \in E$, hence $y \in E \cup E' = \overline{E}$, which means that x is a limit point of $\overline{E}$. Next, suppose that x is a limit point of $\overline{E}$, which means that for any $r > 0$, we can find some point $y \in E \cup E'$ such that $0 < d(x, y) < r$. Let $h > 0$ such that $d(x, y) = r - h$. If $y \in E'$, then y is a limit point of E which means that we can find $z \in E$ such that $z \neq x$ and $0 < d(y, z) < r - h$. Thus, $0 < d(x, z) \leq d(x, y) + d(y, z) < r$. If $y \in E$, then simply set $z = y$. In either case, we have shown that x is a limit point of E. Thus, E and $\overline{E}$ have precisely the same limit points.

Note that E and E' need not have the same limit points. If $E = \{1/n : n \in \mathbb{N}\}$, then $E' = \{0\}$ and $E'' = \emptyset$.

Exercise 7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \cup_{i=1}^n A_i$, prove that $\overline{B}_n = \cup_{i=1}^n \overline{A}_i$ for $n = 1, 2, 3, \ldots$.

(b) If $B = \cup_{i=1}^\infty A_i$, prove that $\overline{B} \supset \cup_{i=1}^\infty \overline{A}_i$.

Solution.

(a) Note that each $\overline{A}_i$ is closed, hence the finite union $S = \cup_{i=1}^n \overline{A}_i$ is closed. We wish to show that $\overline{B}_n = S$. First, each $A_i \subseteq \overline{A}_i$ so $B_n = \cup_{i=1}^n A_i \subseteq \cup_{i=1}^n \overline{A}_i \subseteq S$. Theorem 2.27 (c) guarantees that $\overline{B}_n \subseteq S$. Next, each $A_i \subseteq B_n \subseteq \overline{B}_n$ which is closed, so Theorem 2.27 (c) guarantees that $\overline{A}_i \subseteq \overline{B}_n$. Thus, the union $S \subseteq \overline{B}_n$. Together, this proves that $\overline{B}_n = S$.

(b) Note that each $A_i \subseteq B \subseteq \overline{B}$ and $\overline{B}$ is closed, so Theorem 2.27 (c) guarantees that $\overline{A}_i \subseteq \overline{B}$. Thus, the union $\cup_{i=1}^\infty \overline{A}_i \subseteq \overline{B}$.

Exercise 8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution. In $\mathbb{R}^2$, every interior point of a set E is a limit point of E. This is because given every $x \in E^\circ$, we can find a neighbourhood $N_r(x) \subseteq E$. Therefore, all the points $x + (h, 0) \in E$ where $0 < h < r$. For any $\epsilon > 0$, we set $x' = x + (\min\{\epsilon, r\}, 0)$ so $x' \in N_r(x) \subseteq E$ and $0 < d(x, x') < \epsilon$. When E is open, $E = E^\circ$ hence every point of E is a limit point of E.

This is false for closed subsets in $\mathbb{R}^2$. Consider the finite set $\{(0, 0)\}$, which has no limit points.

Exercise 9. Let E° denote the set of all interior points of a set E.

(a) Prove that E° is always open.

(b) Prove that E is open if and only if $E^\circ = E$.

(c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

(e) Do E and $\overline{E}$ always have the same interiors?

(f) Do E and E° always have the same closures?

Solution.

(a) We wish to show that every point of E° is an interior point of E°. Suppose that $x \in E^\circ$, which means that for some $r > 0$, there is a neighbourhood $N_r(x) \subseteq E$. Now, $N_r(x)$ is open, so for any $y \in N_r(x)$, we there is a neighbourhood $N_s(y) \subseteq N_r(x) \subseteq E$. Thus, y is an interior point of E, i.e. $y \in E^\circ$. This gives $N_r(x) \subseteq E^\circ$, hence x is an interior point of E°.

(b) First suppose that E is open. This means that every point of E is an interior point of E, so $E \subseteq E^\circ$. However, every $x \in E^\circ$ has a neighbourhood $N_r(x) \subseteq E$, so $x \in E$, thus $E^\circ \subseteq E$. This gives $E^\circ = E$.

Next, suppose that $E^\circ = E$. Part (a) directly gives E is open.

(c) Pick $g \in G$, hence $g \in E$. Since E is open, $E = E^\circ$, so $g \in E^\circ$. Thus, $G \subseteq E^\circ$.

(d) First, pick $x \in (E^\circ)^c$. This means that x is not an interior point of E, so every neighbourhood of x will contain some point $y \notin E$. If $x \in E^c$, then $x \in \overline{E^c}$ as required. Otherwise, $x \in E$, so $x \neq y$ for each chosen neighbourhood, which means that x is a limit point of E^c. Thus, $x \in (E^c)'$, so $x \in \overline{E^c}$. This gives $(E^\circ)^c \subseteq \overline{E^c}$.

Next, pick $x \in \overline{E^c}$. If $x \in E^c$, then $x \notin E$ so $x \notin E^\circ \subseteq E$. Otherwise, x must be a limit point of E^c, so every neighbourhood of x contains a point $y \in E^c$. In other words, no neighbourhood of x is wholly contained within E, so x cannot be an interior point of E, i.e. $xin(E^\circ)^c$. This gives $\overline{E^c} \subseteq (E^\circ)^c$.

(e) Consider $E = (0,1)\cup(1,2)$, with $\overline{E} = [0,2]$. Note that $E^\circ = (0,1)\cup(1,2)$ but $(\overline{E})^\circ = (1,2)$.

(f) Consider $E = \{1\}$, with $E^\circ = \emptyset$. Note that $\overline{E} = \{1\}$ but $\overline{E^\circ} = \emptyset$.

Exercise 10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1, & \text{if } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which subsets closed? Which are compact?

Solution. Clearly, $d(p,q) = 1 > 0$ if $p \neq q$ and $d(p,p) = 0$ for $p, q \in X$. Also, if $p = q$, $d(p,q) = 0 = d(p,q)$, otherwise if $p \neq q$, then $q \neq p$, so $d(p,q) = 1 = d(q,p)$. Finally, $0 \leq d(p,q) \leq 1$, so for any $p,q,r \in X$,

$$d(p,q) + d(q,r) = \begin{cases} 2 > d(p,r), & \text{if } p \neq q, q \neq r, \\ 1 = d(p,r), & \text{if } p \neq q, q = r \text{ or } p = q, q \neq r, \\ 0 = d(p,r), & \text{if } p = q = r, \end{cases}$$

This proves that X is a metric space under this distance function.

All subsets of this metric space are open. To see this, note that every singleton set $\{x\}$ for $x \in X$ is open, because $\{x\} = N_{1/2}(x)$. Thus, any subset $E \subseteq X$ is the union $\cup_{x\in E}\{x\}$, and is hence open. The empty set is vacuously open.

All subsets of this metric space are closed. This is because their complements are open by the previous remark.

All finite sets are compact, since for any open cover $\cup\{\mathcal{O}_n\}$ of a finite set, we can choose an open set covering each element x_n of the finite set, which means that their union makes a finite sub-cover.

Any infinite set $E \subseteq X$ cannot be compact, since the open cover $\cup_{x\in E}\{N_{1/2}(x)\} = E$ has no finite sub-cover. This is because each $x \in E$ is covered by precisely one neighbourhood, $N_{1/2}(x)$.

Exercise 11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2-y^2| \\ d_4(x,y) &= |x-2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution. The first distance function fails the triangle inequality. Note that

$$d_1(1,0) + d_1(0,-1) = 2 < 4 = d(1,-1).$$

The second distance function is a metric. Note that $d_2(x,y) = \sqrt{|x-y|} > 0$ when $x \neq y$ and $d_2(x,x) = 0$. Furthermore, let $x, y, z \in \mathbb{R}$. We also claim that

$$d_2(x,y) + d_2(y,z) \geq d_2(x,z).$$

Set $a = |x-y|$, $b = |y-z|$, and $c = |x-z| \leq |x-y| + |y-z| = a+b$ by the triangle inequality. The desired inequality is equivalent to showing

$$\sqrt{a} + \sqrt{b} \geq \sqrt{c}, \qquad a + b + 2\sqrt{ab} \geq c,$$

the last of which is clearly true, since $2\sqrt{ab} \geq 0$.

The third distance function is not a metric, since $1 \neq -1$ yet $d_3(1,-1) = |1-1| = 0$.

The fourth distance function is not a metric, since $d_4(1,1) = |1-2| = 1 \neq 0$.

The fifth distance function is a metric. Note that $d_5(x,y) \neq 0$ when $x \neq y$, and $d_5(x,x) = 0$. Furthermore, for $x, y, z \in \mathbb{R}$, we claim that

$$d_2(x,y) + d_2(y,z) \geq d_2(x,z).$$

Set $a = |x-y|$, $b = |y-z|$, and $c = |x-z| \leq |x-y| + |y-z| = a+b$ by the triangle inequality. The desired inequality is equivalent to showing

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c}.$$

Since $d(x,y) \geq 0$, this is equivalent to

$$\begin{gathered} a(1+b)(1+c) + b(1+a)(1+c) \geq c(1+a)(1+b), \\ a + ab + ac + abc + b + ab + bc + abc \geq c + ac + bc + abc, \\ a + b + 2ab + abc \geq c, \end{gathered}$$

which is true since $2ab + abc \geq 0$.

Exercise 12. Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition.

Solution. Let $\{\mathscr{O}_\alpha\}$ be an open cover of K. This means that

$$K \subseteq \bigcup_\alpha \mathscr{O}_\alpha,$$

so there is some α_0 such that $0 \in \mathscr{O}_{\alpha_0}$. Since $\mathscr{O}_{\alpha_0}$ is an open set, there exists $r > 0$ such that $N_r(0) \subseteq \mathscr{O}_{\alpha_0}$, therefore $x \in \mathscr{O}_{\alpha_0}$ whenever $0 \leq x < r$. This means that $\mathscr{O}_{\alpha_0}$ covers all elements $1/n \in K$ where $n > 1/r$. This leaves finitely many elements $1/n > r$, since there are only finitely many $n = 1, 2, 3, \ldots$ such that $n < 1/r$. For all such $n < 1/r$, pick α_n such that $1/n \in \mathscr{O}_{\alpha_n}$. As a result,

$$K \subseteq \mathscr{O}_{\alpha_0} \cup \mathscr{O}_{\alpha_1} \cup \cdots \cup \mathscr{O}_{\alpha_n}$$

hence we have constructed a finite sub-cover of K, proving that K is compact.

Exercise 13. Construct a compact set of real numbers whose limit points form a countable set.

Solution. Write

$$S = \left\{ \frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N} \right\}.$$

It can be shown that the set of limit points of S is

$$S' = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Thus, $\overline{S} = S \cup S'$ has countably many limit points. Furthermore, $\overline{S}$ is bounded and closed, hence compact.

Exercise 14. Give an example of an open cover of the segment $(0, 1)$ which has no finite sub-cover.

Solution. For every $n \in \mathbb{N}$, define

$$\mathscr{O}_n = (1/n, 1).$$

We claim that $(0, 1) = \cup_{n=1}^{\infty} \mathscr{O}_n$. Note that for any $x \in (0, 1)$, we can choose $n \in \mathbb{N}$ such that $nx > 1$, hence $x \in (1/n, 1) = \mathscr{O}_n$. If $\{\mathscr{O}_n\}_{n \in S}$ were a finite sub-cover, set $N = \max S$ and note that $1/2N < 1/N$, hence $1/2N \notin \mathscr{O}_n$ for all $n \in S$, which is a contradiction.

Exercise 15. Show that Theorem 2.36 and its Corollary become false if the word "compact" is replaced by "closed" or "bounded".

Solution. Consider the collection of closed sets in $\mathbb{R}$,

$$\mathscr{C}_n = [n, \infty), \qquad \text{for all } n \in \mathbb{N}.$$

The intersection of any finite number of them is non-empty and $\mathscr{C}_n \supset \mathscr{C}_{n+1}$, but the intersection of all of them must be empty, since for any $x \in R$, there exists $n \in N$ such that $x < n$, hence $x \notin \mathscr{C}_n$.

Consider the collection of bounded sets in $\mathbb{R}$,

$$\mathscr{B}_n = (0, 1/n), \qquad \text{for all } n \in \mathbb{N}.$$

The intersection of any finite number of them is non-empty and $\mathscr{B}_n \supset \mathscr{B}_{n+1}$, but the intersection of all of them must be empty, since for any $x > 0$, there exists $n \in N$ such that $nx > 1$, so $x > 1/n$, hence $x \notin \mathscr{B}_n$.

Exercise 16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Solution. It is clear that E is bounded, because $E \subset N_3(0)$. If $p \in E$, then $p^2 < 3$, hence $p^2 < 9$ or $|p - 0| < 3$, so $p \in N_3(0)$.

In order to show that E is closed, we'll show that it contains all its limit points. If $x \notin E$ is a limit point of E, then $x^2 \leq 2$ or $x^2 \geq 3$. Since x is rational, we have $x \neq 2$, $x \neq 3$. If $x^2 < 2$, set $r = \sqrt{2} - |x| > 0$. We claim that $N_r(x) \cap E = \emptyset$. This is because for $y \in N_r(x)$, we have $|y - x| < r$ hence $|y| \leq |x| + |y - x| < |x| + r \leq \sqrt{2}$, so $y^2 < 2$. Similarly, if $x^2 > 3$, set $r = |x| - 3 > 0$, whence $N_r(x) \cap E = \emptyset$. This is because for $y \in N_r(x)$, $|y - x| < r$ hence $|x| \leq |x - y| + |y|$, or $|y| \geq |x| - |x - y| > |x| - r \geq \sqrt{3}$, so $y^2 > 3$. In either case, we have found a neighbourhood of x not containing any point of E, which contradicts the fact that x is a limit point of E. Therefore, E contains all its limit points and is closed in $\mathbb{Q}$.

In order to show that E is not compact, consider the open cover $\{\mathscr{O}_n\}_{n\in\mathbb{N}}$, where

$$\mathscr{O}_n = \mathbb{Q} \cap (-\infty, \sqrt{3 - 1/n}).$$

Each $\mathscr{O}$ is open relative to $\mathbb{Q}$ because $\mathbb{Q} \subset \mathbb{R}$, and $(-\infty, \sqrt{3-1/n})$ is open in $\mathbb{R}$. $\mathscr{O}_1$ already covers all $p < 0$. For any $p \in E$, $p > 0$, note that $p^2 < 3$, so there exists some $n \in N$ such that $n(3 - p^2) > 1$, whence $p^2 < 3 - 1/n$ or $p < \sqrt{3 - 1/n}$. This means that p is covered by $\mathscr{O}_n$. On the other hand, if this cover contains a finite sub-cover $\{\mathscr{O}_n\}_{n\in J}$, let $m = \max J$. Since $\mathscr{O}_n \subset \mathscr{O}_{n+1}$, we see that the union $\cup_{n\in J}\mathscr{O}_n = \mathscr{O}_m$. Now, $\sqrt{3 - 1/m} < \sqrt{3}$, so there exists a rational number p between these real numbers such that $3 - 1/m < p^2 < 3$, which means that $p \notin \mathscr{O}_m$ but $p \in E$, which is a contradiction.

In order to show that E is open in $\mathbb{Q}$, note that $E = \mathbb{Q} \cap ((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3}))$.

Exercise 17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Solution. The set E is uncountable, which can be shown by applying Cantor's diagonalization argument with the symbols 4 and 7 in lieu of 0 and 1. The set E is not dense in $[0, 1]$, since the neighbourhood $(-0.1, +0.1) \cap [0, 1] = [0, 0.1)$ contains no elements of E.

In order to show that the bounded set E is compact, it is sufficient to show that it is closed and then apply the Heine-Borel Theorem. Suppose that the number $a = 0.a_1a_2 \ldots a_n \cdots \in [0, 1] \setminus E$, where all $a_{k<n}$ being either 4 or 7 and a_n being neither 4 nor 7. This means that for any $b = 0.b_1b_2 \cdots \in E$, we have $a \neq b$, and the first digit which differs from a, say $b_m \neq a_m$ must be such that $m \leq n$ (clearly the nth digit of a differs from that of b). Thus,

$$a - b = \sum_{k=m}^{\infty} a_k 10^{-k}, \qquad |a - b| \geq |a_m - b_m|10^{-m} - \sum_{k=m+1}^{\infty} |a_k - b_k|10^{-k}.$$

Now, clearly $|a_m - b_m| \geq 1$. Also, each $|a_k - b_k| \leq 7$; this is because b_k is limited to either 4 or 7, and each of these digits is at most 7 away from $0 \leq a_k < 10$. Thus, the infinite sum is bounded above by

$$\sum_{k=m+1}^{\infty} 7 \cdot 10^{-k} = \frac{7}{9}10^{-m}, \qquad |a - b| \geq \frac{2}{9}10^{-m} \geq \frac{2}{9}10^{-n}.$$

Note that $|a - b|$ is bounded below by the same quantity regardless of our choice of $b \in E$, hence the corresponding neighbourhood of a contains no points from E. Thus, a cannot be a limit point of E. In other words, E contains all its limit points, and is hence closed. This further shows that $E \subset [0, 1]$ is compact.

In order to show that E is perfect, it is sufficient to show that every point in E is a limit point. Again, given $b = 0.b_1b_2 \cdots \in E$, toggling the digit b_n between 4 and 7 creating the number $b' \in E$ gives a difference of $3 \cdot 10^{-n}$. Since $10^{-n} \to 0$, given any $\epsilon > 0$ we can find sufficiently large n such that $3 \cdot 10^{-n} < \epsilon$, hence $|b - b'| < \epsilon$. Thus, every point $b \in E$ is a limit point of E.

Exercise 18. Is there a non-empty perfect set in $\mathbb{R}^1$ which contains no rational number?

Exercise 19.

(a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly with $>$ in place of $<$. Prove that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable.

Solution.

(a) Note that since A and B are closed, we have $A = \overline{A}$ and $B = \overline{B}$. This immediately gives $A \cap \overline{B} = B \cap \overline{A} = \emptyset$.

(b) Suppose that $A \cap \overline{B} \neq \emptyset$, i.e. there is a limit point x of B which is contained in A. Thus, there is an open ball $B_r(x) \subseteq A$, and since x is a limit point of B there are an infinite number of point $b_i \in B \cap B_r(x) \subseteq A$, which contradicts the assumption that $A \cap B = \emptyset$. A symmetric argument shows that $B \cap \overline{A} =$.

(c) Note that A and B are open sets; A is an open ball, and B is the complement of the closed ball described by $d(p,q) \leq \delta$. In addition, A and B are disjoint, since we cannot have both $\delta < d(p,q) < \delta$. Thus, A and B are separated by the previous exercise.

(d) Choose distinct $p, q \in X$, set $\delta = d(p,q) > 0$. Now, let $0 \leq \delta' \leq \delta$, and examine all points x such that $d(p,x) = \delta$. If no such x existed, then the open sets described by $d(p,x) < \delta'$ and $d(p,x) > \delta'$ would cover X, thus separating it. In this manner, we choose $x_{\delta'}$ for every $\delta' \in [0,1]$. Note that all such $x_{\delta'}$ are distinct by construction, thus we have a bijection between $\{x_{\delta'}\}$ and the uncountable interval $[0,1]$, hence $\{x_{\delta'}\} \subseteq X$ is uncountable.

Exercise 20. Are closures and interiors of connected sets always connected?

Solution. The closures of connected sets are always connected. Suppose that A is connected but $\overline{A}$ is not, i.e. we can choose non-empty disjoint open sets X, Y such that $\overline{A} = X \cup Y$. Consider $X_0 = X \cap A$ and $Y_0 = Y \cap A$. Suppose that $A \subseteq X$; this would imply that all the points in the non-empty set Y are limit points of A. However, since $A \subseteq X$, these are also limit points of X, which contradicts the fact that X is open. Similarly, we cannot have $A \subseteq Y$. Therefore, X_0 and Y_0 are both non-empty. Furthermore, $A = X_0 \cup Y_0$, because $X_0 \cup Y_0 = (X \cap A) \cup (Y \cap A) = (X \cup Y) \cap A = \overline{A} \cap A = A$. Thus, X_0 and Y_0 separate A, which is a contradiction.

Consider the sets A and B in $\mathbb{R}^2$ described by $d(x,(0,0)) \leq 1$ and $d(x,(2,0)) \leq 1$. These are closed discs with the common point $(1,0)$. This means that the union $A \cup B$ is connected. However, their interiors are open discs which are disjoint, and hence their union is disconnected.

Exercise 21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$, and define

$$p(t) = (1-t)\boldsymbol{a} + t\boldsymbol{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = p^{-1}(A)$, $B_0 = p^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.

(b) Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

Solution.

(a) Suppose that $A_0 \cap \overline{B_0} \neq \emptyset$, i.e. let $x \in A_0$ be a limit point of B_0. Thus, for any $r > 0$, we can find $y \in B_0$ such that $d(x,y) < r$. Note that $p(x) = (1-x)\boldsymbol{a} + x\boldsymbol{b} \in A$, and $p(y) = (1-y)\boldsymbol{a} + y\boldsymbol{b} \in B$ by construction, and

$$d(p(x), p(y)) = |(y-x)\boldsymbol{a} + (x-y)\boldsymbol{b}| = |x-y||\boldsymbol{a} - \boldsymbol{b}|.$$

Since $|\boldsymbol{a}-\boldsymbol{b}|$ is fixed and positive ($A \cap B = \emptyset$), for any $\epsilon > 0$, we can choose sufficiently small r such that $|x-y| < r$ gives $d(p(x), p(y)) < \epsilon$. Thus, we have shown that $p(x)$ is a limit point of B, with every ϵ neighbourhood of $p(x) \in A$ containing the points $p(y) \in B$. This contradicts the fact that A and B are separated, so $A \cap \overline{B} = \emptyset$. A symmetric argument shows that we must have ${}_0 \cap \overline{A_0} = \emptyset$, which proves that A_0 and B_0 are separated.

(b) Suppose that $p(t) \in A \cup B$ for all $t \in [0,1]$. Set $X = [0,1] \cap A_0$, $Y = [0,1] \cap B_0$. Note that $A_0 \cup B_0 = p^{-1}(A) \cup p^{-1}(B) = [0,1]$, since for every $t \in [0,1]$, either $t \in A$ or $t \in B$. Thus, $X \cup Y = [0,1]$, with $X \cap Y = \emptyset$. This constitutes a disconnection of the connected set $[0,1]$, which is a contradiction. Thus, there must exist some point $t_0 \in [0,1]$ such that $p(t_0) \notin A \cup B$. Furthermore, $p(0) = \boldsymbol{a} \in A$ and $p(1) = \boldsymbol{b} \in B$, so we must have $t_0 \in (0,1)$.

(c) A convex set E is such that if $\boldsymbol{a}, \boldsymbol{b} \in E$, then the line segment joining $\boldsymbol{a}$ and $\boldsymbol{b}$ is contained in E, i.e. $p([0,1]) \subseteq E$. Thus, if a convex set E were separated as $E = A \cup B$, then we could use the previous exercise to find t_0 such that $p(t_0) \notin E$, which is a contradiction.

Exercise 22. A metric space is called separable if it contains a countable dense set. Show that $\mathbb{R}^k$ is separable.

Solution. We claim that $\mathbb{Q}^k$ is a countable dense set in $\mathbb{R}^k$. This follows from the fact that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. Given any point $\boldsymbol{x} \in \mathbb{R}^k$ and $\epsilon > 0$, we can always find points $y_i \in \mathbb{Q}$ such that $|x_i - y_i| < \epsilon/\sqrt{k}$ for each component $x_i \in \mathbb{R}$ of $\boldsymbol{x}$. Thus, we have found $\boldsymbol{y} \in \mathbb{Q}^k$ where

$$d(\boldsymbol{x}, \boldsymbol{y})^2 = \sum_{i=1}^{k} |x_i - y_i|^2 < k \cdot \frac{\epsilon^2}{k} = \epsilon^2.$$

Exercise 23. A collection $\{V_\alpha\}$ of open sets of X is called a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a sub-collection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base.

Solution. Let $\{x_i\}_{i\in\mathbb{N}}$ be a countable dense subset of the separable metric space X. Define $V_{ij} = B_{q_j}(x_i)$ for every positive rational q_j. Note that $\{V_{ij}\}$ is countable, since the indexing set $\mathbb{N} \times \mathbb{N}$ is countable. We claim that $\{V_{ij}\}$ is a base for X. Indeed, for every $x \in X$ and every open set $G \subset X$ with $x \in G$, we can find a point $x_i \in G$ such that $d(x, x_i) < r/2$ for any r; choose this r such that the open ball $B_r(x) \subset G$ (this can be done since G is open). Now choose a positive rational q_j such that $d(x, x_i) < q_j < r/2$. This immediately gives $x \in V_{ij} = B_{q_j}(x_i) \subset B_r(x) \subset G$.

Exercise 24. Let X be a metric space in which every infinite set has a limit point. Prove that X is separable.

Solution. Fix $\delta > 0$ and $x_i \in X$. Construct the sequence of points $x_1, x_2, \ldots$ such that every new point $x_{j+i} \in X$ is chosen to satisfy $d(x_{j+1}, x_i) \geq \delta$ for all previous $i = 1, \ldots, j$. We claim that this process terminates, i.e. X can be covered by finitely many many neighbourhoods of radius δ. If this process did not terminate, then the infinite set $\{x_i\}$ would not have any limit point; any point $x \in X$ cannot have more than one point of $\{x_i\}$ in the neighbourhood $B_{\delta/2}(x)$ since every point x_i is separated from the others by at least δ. Thus, we have found a finite set X_δ of open balls $B_\delta(x_i)$ which cover X. Consider the collection of the centres x_{in} of all such covers $X_{1/n}$. This is a countable set which is dense in X. This is because given any $x \in X$ and $\epsilon > 0$, we can find k such that $0 < 1/k < \epsilon$, and furthermore X is covered by $X_{1/k}$ so there is an open ball $B_{1/k}(x_{in})$ containing the point x.

Exercise 25. Prove that every compact metric space K has a countable base, and that K is therefore separable.

Solution. Given $n \in N$, consider the set of open balls $B_{1/n}(x)$ for all $x \in K$. This is an open cover of K, and hence must contain a finite sub-cover $X_{1/n}$ of K. Repeating this for all $n \in N$, we see that the collection of the centres x_{in} of all such $X_{1/n}$ is a countable dense subset of X, by the previous exercise. This proves that X is separable. The union of all $X_{1/n}$, i.e. the collection of all open balls $B_{1/n}(x_{in})$, is a countable base of X.

Exercise 26. Let X be a metric space in which every infinite set set has a limit point. Show that X is compact.

Solution. By Exercise 24, X is separable, and by Exercise 23, X has a countable base $\{V_i\}$. Suppose that $\{X_\alpha\}$ is an open cover of X. Since for every $x \in X$ we can choose V_i such that $x \in V_i \subset X_\alpha$, we can manufacture a countable sub-cover $\{X_{\alpha_i}\} \equiv \{X_j\}$. Now suppose that no *finite* collection of such X_j covers X. This means that for every finite union $X_1 \cup \cdots \cup X_n$, the complement Y_n is non-empty. However, the intersection of all such Y_n is empty, since the infinite union $X_1 \cup \ldots$ covers X. Thus, choose $y_n \in Y_n$ for each n, note that the infinite set $E = \{y_n\}$ must have a limit point $y \in X$. Since $\{X_i\}$ covers X, we have $y \in X_n$ for some n, and since X_n is open, we can find an open ball $B_r(y) \subset X_n$. Thus, $B_r(y) \cap Y_n = \emptyset$, and since $Y_k \supset Y_{k+1}$ we have $B_r(y) \cap Y_{k \geq n} = \emptyset$. Thus, we have found a neighbourhood of y not containing any points in $Y_{k \geq n}$, specifically not containing any y_n. This contradicts the fact that y is a limit point of E.

Exercise 27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighbourhood of p contains uncountably many points of E.

Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable.

Solution. Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, and let W be the union of those V_n such that $E \cap V_n$ is at most countable. Then, $E \cap W = \bigcup_n (E \cap V_n)$ is at most countable since it is countable union of at most countable sets. We claim that $P = W^c$.

First, let $x \in W^c$. Since $\{V_n\}$ is a countable base, given any open ball $B_r(x)$, we can find $x \in V_k \subset B_r(x)$. Now, $x \notin W$ hence $x \notin V_n$ where $E \cap V_n$ is countable. Hence, $x \in V_k$ where $E \cap V_k$ is uncountable. Thus, the neighbourhood $B_r(x)$ contains uncountably many points of E, so x is a condensation point of E, i.e. $x \in P$. This gives $W^c \subseteq P$.

Next, let $x \in W$. This means that $x \in V_n$ where $E \cap V_n$ is countable. Thus, any neighbourhood of x contained within V_n (recall that the countable base in a separable metric space can consist of open balls) can contain at most countably many points of E, so x is not a condensation point of E. Thus, $x \notin P$, i.e. $W \subseteq P^c$. Together, we have $W^c = P$.

To show that P is perfect, we must show that it is closed and that every point of P is a limit point. Note that the complement $P^c = W$ is open, being a union of open sets V_n, immediately showing that P is closed. Now, let $x \in P$, and let $B_r(x)$ be a neighbourhood of x, with $B_r(x) \cap E$ being uncountable. Suppose that there is no point $y \in B_r(x) \cap P$, $y \neq x$. This means that all the points $y \in B_r(x) \setminus \{x\}$ are contained in $P^c = W$, hence all such y belong to some V_n where $E \cap V_n$ is countable. Thus, $B_r(x) \setminus \{x\} \subset W$. Since $E \cap W$ contains at most countably many points, we have $E \cap (B_r(x) \setminus \{x\})$ containing at most countably many points, hence the addition of the single point x gives $E \cap B_r(x)$ containing at most countably many points. This contradicts the fact that x is a condensation point of E. Thus, every point in P must be a limit point of P, proving that P is perfect.

Exercise 28. Prove that every closed set in a metric space is the union of a (possibly empty) perfect set and a set which is at most countable. As a corollary, every countable closed set in $\mathbb{R}^k$ has isolated points.

Solution. Given any closed set E, note that its set of condensation points P is perfect, and the set $P^c \cap E$ is at most countable, with their union $P \cup (P^c \cap E) = E$. This is because $P \subseteq \overline{E} = E$, since every condensation point is also a limit point.

If E were countable, then P must be empty (there are no uncountably many points of E to choose from). Thus, the set $P^c \cap E = E$ is countable and hence not perfect. This means that at least one of its points is not a limit point, and is hence an isolated point.

Exercise 29. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.

Solution. Note that $\mathbb{R}$ is separable and hence has a countable base $\{V_i\}$ of open intervals (a_i, b_i). Thus, given any open set $O \subseteq \mathbb{R}$, for each $x \in O$ choose V_i such that $x \in V_j \subset O$. The countable union of all such V_j gives precisely O. This is because any $x \in O$ belongs to some V_j, and any x in the union of V_j belongs to some V_j which is contained in O.

Now we refine our collection of V_j based on the following rules. If $V_j \subset V_k$, we eliminate V_j. If V_j and V_k overlap such that $a_j < a_k < b_j < b_k$, then replace V_j and V_k with $V_j \cup V_k$. This new collection of open intervals is thus by construction disjoint as required.

Exercise 30. Imitate the proof of Theorem 2.43 to obtain the following result.

> If $\mathbb{R}^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a non-empty interior.
>
> *Equivalent statement:* If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \dots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

Solution. First we show that the statements are equivalent. If the first statement is true, and we have been given dense open subsets $G_n \subseteq \mathbb{R}^k$, set $F_n = G_n^c$ for all n. All F_n must have empty interiors since every G_n is dense in $\mathbb{R}^k$. Thus, their union cannot be the entirety of R^k, hence the union of all G_n must be non-empty.

If the second statement is true, and we have been given closed subsets $F_n \subseteq \mathbb{R}^n$ whose union is $\mathbb{R}^k$, set $G_n = F_n^c$. Note that the intersection of all such G_n is empty, which means that at least one such G_k is not dense in $\mathbb{R}^k$. The corresponding F_k thus has a non-empty interior.

We prove the second statement. Let G_n be dense open sets of $\mathbb{R}^k$. Pick any open non-empty set $O \subseteq \mathbb{R}^k$, and note that since G_1 is dense, the set $O \cap G_1$ is non-empty and open (note that $\overline{G_1} = \mathbb{R}^k$, so G_1^c is a collection of limit points of G_1, which is thus closed). Thus, pick $x_1 \in O \cap G_1$ and construct the open ball $B_{r_1}(x_1)$ contained within it. This is another non-empty open set, hence $O_1 = B_{r_1}(x_1) \cap G_2$ is non-empty and open. Pick a point x_2 in this set, and construct the open ball $B_{r_2}(x_2) \subset O_1 \subseteq B_{r_1}(x_1)$. Continuing in this manner, we have generated nested sequence of open balls such that $B_{r_k}(x_k) \supseteq B_{r_{k+1}}(x_{k+1})$. Now, the closure of each of these balls is compact by the Heine-Borel Theorem, hence the intersection of all these compact balls must be non-empty by Theorem 2.36. Any point x in this intersection is a common element of all the compact balls, but every one of these compact balls is contained within its parent open set $O_n = B_{r_n}(x_n) \cap G_{n+1}$. Hence, $x \in O \cap G_n$ for all G_n, thus the set $G = \bigcap_1^\infty G_n$ intersects with every open set O in at least one point, i.e. every non-empty neighbourhood O contains some point of G, which means that G is dense in $\mathbb{R}^k$.