

MA3201

Topology

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1 Introduction

1.1 Topological spaces

Definition 1.1. A topology on some set X is a family τ of subsets of X , satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from τ are in τ .
3. All finite intersections of elements from τ are in τ .

The sets from τ are declared to be open sets in the topological space (X, τ) .

Example. Any set X admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

Example. Let X be a set. The cofinite topology on X is the collection of complements of finite sets, along with the empty set. Note that when X is finite, this is simply the discrete topology.

Definition 1.2. Let τ, τ' be two topologies on the set X . We say that τ is finer than τ' if τ has more open sets than τ' . In such a case, we also say that τ' is coarser than τ .

1.2 Topological bases

Definition 1.3. Let (X, τ) be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology τ such that every open set $U \in \tau$ is expressible as a union of elements from β .

Definition 1.4. Let X be a set, and let β be a collection of subsets of X satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then, β generates a topology on X , namely the collection of all unions of elements of β .

Lemma 1.1. Let τ be a topology on X , and let $\beta \subseteq \tau$ be a collection of open sets. Then, β is a basis of τ , or generates τ , if for every $x \in U \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U$.

Example. The collection of all open balls in \mathbb{R}^n form a basis of the usual topology.

Lemma 1.2. Let X be equipped with the topologies τ and τ' , and let β and β' be the respective bases of these topologies. Then, τ is finer than τ' if and only if given $x \in B' \in \beta'$, there exists $x \in B \in \beta$ such that $B \subseteq B'$.

Example. The collections of open balls in \mathbb{R}^n generate the same topology as the collection of all open rectangles in \mathbb{R}^n .

Example. Consider the topologies on \mathbb{R} generated by the following bases.

1. $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.
2. $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.
3. $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$ where $K = \{1/n : n \in \mathbb{Z}\}$.

We call the topology generated by β_2 the lower limit topology, denoted \mathbb{R}_ℓ . The topology generated by β_3 is denoted \mathbb{R}_K . Both of these are strictly finer than the standard topology.

Definition 1.5. A sub-basis for some topology on X is a collection ρ of subsets of X whose union is the whole of X . The topology generated by ρ is defined to be the topology generated by the collection of all finite intersections of elements of ρ .

1.3 Product topology

Definition 1.6. Let $(X_1, \tau_1), (X_2, \tau_2)$ be topological spaces. Then $\tau_1 \times \tau_2$ generates the product topology on $X_1 \times X_2$.

Example. The product topology on $\mathbb{R} \times \mathbb{R}$, where \mathbb{R} is equipped with the standard topology, coincides with the standard topology on \mathbb{R}^2 .

Lemma 1.3. If β_1, β_2 are bases of the topologies τ_1, τ_2 , then $\beta_1 \times \beta_2$ and $\tau_1 \times \tau_2$ generate the same product topology.

Proof. Given $(x_1, x_2) \in U$ where $U \subseteq X_1 \times X_2$ is open in the product topology, recall that U can be written as a union of the basic open sets $U_{1i} \times U_{2i}$, where $U_{1i} \in \tau_1$ and $U_{2i} \in \tau_2$. Suppose that $(x_1, x_2) \in U_1 \times U_2$. Thus, we can choose $B_1 \in \beta_1, B_2 \in \beta_2$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$. Thus, $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$. \square

Definition 1.7. The projection maps are defined as $\pi_i: X_1 \times \cdots \times X_k \rightarrow X_i, (x_1, \dots, x_k) \mapsto x_i$.

Lemma 1.4. The collection of elements of the form $\pi_1^{-1}(U_1)$ or $\pi_2^{-1}(U_2)$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$, forms a sub-basis of the product topology on $X_1 \times X_2$.

Proof. Note that $\pi_1^{-1}(X_1) = X_1 \times X_2$. Now it is easy to see that finite intersections of elements of the form $U_1 \times X_2$ or $X_1 \times U_2$ where U_1, U_2 are open, are all of the form $U_1 \times U_2$ which is precisely a basis of the product topology. \square

Corollary 1.4.1. We can restrict ourselves to the sub-basis of elements of the form $\pi_1^{-1}(B_1)$ or $\pi_2^{-1}(B_2)$, where $B_1 \in \beta_1, B_2 \in \beta_2$ for some bases β_1, β_2 of τ_1, τ_2 .

1.4 Subspace topology

Definition 1.8. Let (X, τ) be a topological space, and let $Y \subset X$. Then the collection $U \cap Y$ for all $U \in \tau$ comprises the subspace topology τ_Y on Y induced by the topology τ on X .

Lemma 1.5. *If β is a basis for the topology on X , and $Y \subset X$, then the collection $B \cap Y$ for all $B \in \beta$ generates the subspace topology on Y .*

Lemma 1.6. *An open set of Y is open in X if Y is open in X .*

Proof. Let $U \subset Y$ be open in Y , then $U = V \cap Y$ for some open set V in X . If additionally Y is open in X , this immediately shows that U is open in X . \square

Theorem 1.7. *Let (X, τ_X) , (Y, τ_Y) be topological spaces, and let $A \subseteq X$, $B \subseteq Y$. Then, there are two ways of assigning a natural topology on $A \times B$.*

1. *Take the product topology on $X \times Y$, and consider the subspace topology induced by it on $A \times B$.*
2. *Take the subspace topologies on A induced by τ_X , B induced by τ_Y , and consider the product topology generated by them on $A \times B$.*

These two methods generate the same topology on $A \times B$.

Proof. Open sets in 1 look like $(U \times V) \cap (A \times B)$, where $U \in \tau_X$, $V \in \tau_Y$. Open sets in 2 look like $(U' \cap A) \times (V' \cap B)$, where $U' \in \tau_X$, $V' \in \tau_Y$, which can be rewritten as $(U' \times V') \cap (A \times B)$. It is easy to see that these describe precisely the same sets. \square

1.5 Order topology

Definition 1.9. Let X be a set with a simple order $<$. Then the collection of sets of the form (a, b) , $[a_0, b)$, $(a, b_0]$ where a_0 is the minimal element of X , b_0 is the maximal element of X , generate the order topology on X .

Example. The order topology on \mathbb{N} is precisely the discrete topology.

Definition 1.10. Let X_1, X_2 be simply ordered sets. The dictionary order on $X_1 \times X_2$ is defined as follows: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$.

Example. Consider $X = \{1, 2\} \times \mathbb{N}$, where both $\{1, 2\}$ and \mathbb{N} are endowed with the discrete topology. Note that the product topology on X is the discrete topology.

Now consider the dictionary order on X . Here, $(1, 1)$ is the smallest element, so we can list the elements of X in ascending order. Note that every $(1, m) < (2, n)$, for all $m, n \in \mathbb{N}$. Now, note that all singletons $\{(1, m)\}$ are open in the order topology on X . The same is true for the singletons $\{(1, n)\}$ for all $n > 1$. However, the singleton $\{(2, 1)\}$ is *not* open in the order topology.

Example. Consider \mathbb{R} with the usual topology, and $X = [0, 1) \cup \{2\}$. Then, $\{2\}$ is open in the subspace topology on X , but it is not open in the order topology on X .

Lemma 1.8. *The open rays of the form $(a, +\infty)$ and $(-\infty, a)$ in X form a sub-basis of the order topology on X .*

Proof. Note that $(a, b) = (-\infty, b) \cap (a, +\infty)$, $[a_0, b) = (-\infty, b)$, and $(a, b_0] = (a, +\infty)$. \square

Definition 1.11. Let X be a simply ordered set, and $Y \subseteq X$. Then, we say that Y is convex in X if given $a, b \in Y$ such that $a < b$, the interval $(a, b) = \{x \in X : a < x < b\} \subseteq Y$.

Theorem 1.9. *Let Y be convex in X . Then, the subspace topology and the order topology on Y induced from the order topology on X coincide.*

1.6 Closed sets

Definition 1.12. Let (X, τ) be a topological space. A set $F \subseteq X$ is said to be closed in X if $F^c = X \setminus F \in \tau$.

Example. The sets \emptyset, X are closed in every topological space (X, τ) .

Example. In a set equipped with the discrete topology, every set is both open and closed.

Lemma 1.10. *Arbitrary intersections, and finite unions of closed sets are closed.*

Theorem 1.11. *Let (X, τ) be a topological space, and let $Y \subset X$ be equipped with the subspace topology. Then, a set $F \subseteq Y$ is closed in Y if and only if $F = Y \cap G$, where G is closed in X .*

Proof. Let $F \subset Y$. Now, F is closed in Y , $Y \setminus F = Y \cap F^c$ is open in Y , $Y \cap F^c = Y \cap U$ where U is open in X , $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$ where U^c is closed. The steps are reversible. \square

Lemma 1.12. *A closed set of Y is closed in X if Y is closed in X .*

1.7 Interiors and closures

Definition 1.13. Let $A \subseteq X$ where (X, τ) is a topological space.

1. The interior of A is defined as the union of all open sets contained in A . This is denoted by A° .
2. The closure of A is defined as the intersection of all closed sets containing A . This is denoted by \overline{A} .

Remark. The interior of a set is open, and the closure of a set is closed.

Lemma 1.13. Let $Y \subset X$ be topological spaces, and let $A \subseteq Y$. Also let $\overline{A}_X, \overline{A}_Y$ denote the closures of A in X, Y respectively. Then, $\overline{A}_Y = \overline{A}_X \cap Y$.

Theorem 1.14. Let $A \subset X$. Then,

1. $x \in \overline{A}$ if and only if every open set containing x has non-empty intersection with A .
2. $x \in \overline{A}$ if and only if every basic open set containing x has non-empty intersection with A , given that the topology on X is generated by those basic open sets.

Definition 1.14. Let $A \subseteq X$ where (X, τ) is a topological space. We say that $x \in X$ is a limit point of X if for every open set U containing x , the deleted neighbourhood $U \setminus \{x\}$ has non-empty intersection with A . The set of limit points of A is denoted by A' .

Example. Let X be a set endowed with the discrete topology. Then, given any set $A \subseteq X$, we have $A' = \emptyset$.

Lemma 1.15. A closed set contains all its limit points.

Proof. Let $F \subseteq X$ be closed in X , and let $x \in F'$. Then given any open set containing x , we have $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$, hence $x \in \overline{F} = F$. \square

Lemma 1.16. Let $A \subseteq X$ where (X, τ) is a topological space. Then, $\overline{A} = A \cup A'$.

Proof. It is clear that $\overline{A} \supseteq A \cup A'$. Now pick $x \in \overline{A}$. If $x \notin A$, then we know that given any open neighbourhood U of x , we have non-empty $U \cap A$. Furthermore, this intersection can never contain x , hence $x \in A'$. This proves that $\overline{A} \subseteq A \cup A'$. \square

1.8 Convergence of sequences

Definition 1.15. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points from (X, τ) , and let $x \in X$. We say that this sequence converges to x , denoted $x_n \rightarrow x$, if every open neighbourhood of x contains the tail of this sequence. In other words, given $U \in \tau$ such that $x \in U$, there must exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then, the constant sequence of b 's converges to all three points a, b, c .

Example. Let $X = \mathbb{R}$, and τ be the collection of all intervals $(-a, a)$ together with \emptyset, \mathbb{R} . Then, the constant sequence of 0's converges to every point in \mathbb{R} .

Definition 1.16. Let (X, τ) be a topological space. We say that this topological space is Hausdorff if given any two distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Example. The real numbers under the standard topology is Hausdorff.

Theorem 1.17. Let (X, τ) be a Hausdorff topological space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X . Then, this sequence can converge to at most one point in X .

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to distinct points $x, y \in X$. Then there exist disjoint open neighbourhoods U, V such that $x \in U$, $y \in V$. Convergence means that both U and V contain a tail of the sequence, which is a contradiction. \square

Lemma 1.18. The singleton sets in a Hausdorff space are closed.

Proof. Let $x \in X$ where (X, τ) is Hausdorff. Pick $y \neq x$, whence there exist $U_y, V_y \in \tau$, such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. In particular, $\{x\} \cap V_y = \emptyset$. We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open. \square

Theorem 1.19. The topology induced by a metric is Hausdorff.

Proof. Given a metric space X and distinct points $x, y \in X$, we set $r = |x - y|$, $U = B(x, r/3)$, $V = B(y, r/3)$. \square

2 Continuous maps

Definition 2.1. Let $f: X \rightarrow Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y) . We say that f is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in Y must be open in X .

Lemma 2.1. A function $f: X \rightarrow Y$ is continuous if and only if given a base β of Y , we have $f^{-1}(U) \in \tau_X$ for every $U \in \beta$.

Example. The identity function $\text{id}: \mathbb{R}_\ell \rightarrow \mathbb{R}$ is continuous, while the identity function $\text{id}: \mathbb{R} \rightarrow \mathbb{R}_\ell$ is not. This is because the topology on \mathbb{R}_ℓ is strictly finer than that on \mathbb{R} .

Definition 2.2. Let $f: X \rightarrow Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y) . We say that f is a homeomorphism if f is continuous, f is invertible, and f^{-1} is continuous. We also say that X and Y are homeomorphic when such a homeomorphism between them exists.