MA3101 : Introduction to Graph Theory and Combinatorics

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Exercise 1 Prove that the automorphism group of a graph G is equal to the automorphism group if its complement $\overline{G}$.

Solution Let $g \in \operatorname{aut}(G)$ be an automorphism, i.e. a permutation of the vertices of G; we claim that $g \in \operatorname{aut}(\overline{G})$. To see this, pick $x, y \in V$ (here, V is the set of vertices, common to G and $\overline{G}$). There are two possible cases:

Case I: $x \sim y$ in $\overline{G}$. This means that $x \not\sim y$ in G, hence $g(x) \not\sim g(y)$ in G. Thus, $g(x) \sim g(y)$ in $\overline{G}$.
Case II: $x \not\sim y$ in $\overline{G}$. This means that $x \sim y$ in G, hence $g(x) \sim g(y)$ in G. Thus, $g(x) \not\sim g(y)$ in $\overline{G}$.

In either case, g preserves the adjacency relationships between vertices in $\overline{G}$. This shows that $\operatorname{aut}(\overline{G}) \subseteq \operatorname{aut}(G)$. Exactly the same argument can be repeated, interchanging the roles of G and $\overline{G}$; or simply note that $\overline{\overline{G}} = G$ to conclude that $\operatorname{aut}(\overline{\overline{G}}) \subseteq \operatorname{aut}(\overline{G}) \subseteq \operatorname{aut} G$ gives $\operatorname{aut}(G) = \operatorname{aut}(\overline{G})$.

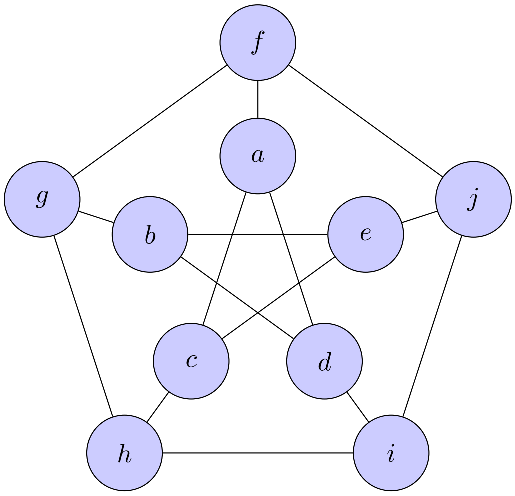
Exercise 2 If x and y are vertices of X and $g \in \operatorname{aut}(X)$, prove that $d(x, y) = d(g(x), g(y))$.

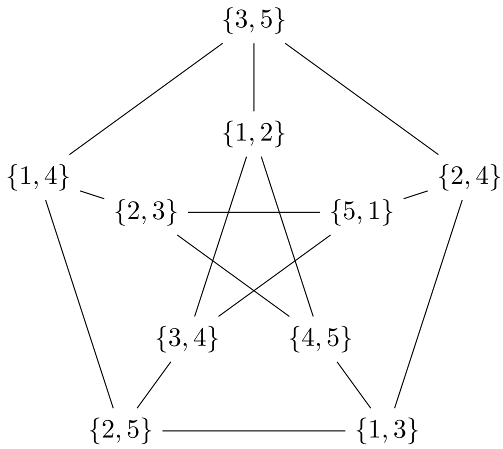
Solution First suppose that x and y are not connected, i.e. there is no path joining them. If $g(x)$ and $g(y)$ were connected, that would give a path $g(x) \sim v_1 \sim \cdots \sim v_k \sim g(y)$; now, applying the automorphism g^{-1} gives a path $x \sim g^{-1}(v_1) \sim \cdots \sim g^{-1}(v_k) \sim y$, a contradiction. Thus, $d(x, y) = \infty$ implies that $d(g(x), g(y)) = \infty$. Conversely, if $g(x)$ and $g(y)$ are not connected but x and y are via some path, applying g to that path yields a path between $g(x)$ and $g(y)$ exactly as before. Thus, $d(g(x), g(y)) = \infty$ also implies that $d(x, y) = \infty$.

Let $d(x, y) = k + 1 < \infty$ (we have $k \geq 0$). This means that there exists a path $x \sim v_1 \sim \cdots \sim v_k \sim y$. Applying g yields a path $g(x) \sim g(v_1) \sim \cdots \sim g(v_k) \sim g(y)$, hence $d(g(x), g(y)) \leq d(x, y)$. Now if there was a shorter path $g(x) \sim u_1 \sim \cdots \sim u_l \sim g(y)$, $l < k$ applying g^{-1} would give the shorter path $x \sim g^{-1}(u_1) \sim \cdots \sim g^{-1}(u_l) \sim y$ between x and y, contradicting the minimality of $d(x, y) = k + 1$. Thus, $d(x, y) = d(g(x), g(y))$.

Exercise 3 Count the number of automorphisms of the Petersen graph.

Solution The Petersen graph G can be described as follows: let the vertices be the two element subsets of $S = \{1, 2, 3, 4, 5\}$ (of which there are 10), and let two vertices be connected if and only if their intersection is empty.





Let σ be a permutation of S. This defined a corresponding permutation of the vertices, sending each vertex $\{x, y\} \to \{\sigma(x), \sigma(y)\}$. We claim that this is an automorphism of the graph. To see this, pick two vertices $\{x, y\}$, $\{p, q\}$. If they form an edge in G, that means that x, y, p, q are all distinct elements, hence so are $\sigma(x), \sigma(y), \sigma(p), \sigma(q)$ meaning that $\{\sigma(x), \sigma(y)\}$, $\{\sigma(p), \sigma(q)\}$ is also an edge. Otherwise, one of these elements is repeated, hence applying σ keeps them repeated so this maps non-edges to non-edges. Thus, we have found as many automorphisms of G as there are permutations σ, i.e. $5! = 120$ of them.

We now show that these are all the automorphisms of G. Note that the vertex $a \equiv \{1, 2\}$ can be mapped to any other vertex by choosing suitable σ, thus the orbit of a comprises of all 10 vertices. The orbit stabilizer theorem thus guarantees that

$$|\operatorname{aut}(G)| = |H| \cdot 10,$$

where H is the stabilizer of a. Now consider the action of H on the graph, specifically on the vertex $c \equiv \{3, 4\}$. The permutation (45) sends $c \to f$, and (35) sends $c \to d$. There are no other places to send c, since we must preserve the edge $\{a, c\}$. Thus, the orbit of c consists of the vertices c, d, f, hence

$$|H| = |K| \cdot 3$$

where K is the stabilizer of c under the action of H. Thus, the action of K on the graph fixes both a and c. Consider where K can send the vertex d. The permutation (34) sends $d \to f$. There are no other places to send d, since we must preserve the edge $\{a, d\}$. Thus, the orbit of d consists of the vertices d, f, hence

$$|K| = |N| \cdot 2$$

where N is the stabilizer of d under the action of K. Thus, the action of N on the graph fixes a, c, d, which also means that f must be fixed. Consider the action of N on the graph, and examine the vertex h. The permutation (12) sends $h \to e$, and there are no other places where h can be sent. Thus, the orbit of h consists of the vertices h, e, hence

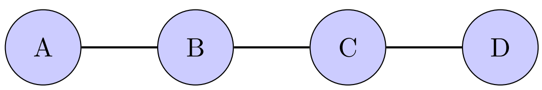
$$|N| = |O| \cdot 2,$$

where O is the stabilizer of h under the action of N. Now, the action of O on the graph fixes a, c, d, h. We claim that this also fixes all other elements, i.e. O is the trivial group. Indeed, examining the neighbours of c show that a, h are fixed so e must also be fixed. The vertex i is the only common neighbour of d and h, and hence must be fixed. Examining the neighbours of h show that c, i are fixed so g must also be fixed. Similarly examining the neighbours of i show that d, h are fixed so j must also be fixed. This means that the final vertex f is also fixed. Thus, $|O| = 1$, hence

$$|\operatorname{aut}(G)| = 10 \cdot 3 \cdot 2 \cdot 2 = 120.$$

Exercise 4 Find the automorphism group of the following graphs.

(a)



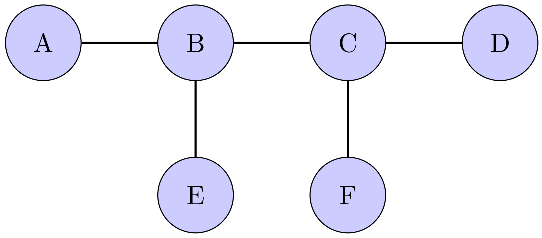
Solution We first show that an automorphism must preserve the degree of a vertex. Indeed, given a vertex $x \in V(G)$ whose neighbours are $x_1, \ldots, x_k$ (k being the degree of x), if g is an automorphism of the graph G then $g(x_1), \ldots, g(x_k)$ are all neighbours of $g(x)$; furthermore, the remaining vertices $y_1, \ldots, y_l$ from $V(G)$ (vertices other than x, $x_1, \ldots, x_k$) are also mapped such that $g(y_1), \ldots, g(y_l)$ are not neighbours of $g(x)$ by construction of g.

Secondly, an automorphism preserves edges. These two restrictions allow us to identify the automorphisms of the given graphs.

Here, B must be mapped to either B or C. If $B \to B$, then $\{A, B\} \to \{X, B\}$ forcing $X = A$, hence $A \to A$. This also forces $C \to C$, and $D \to D$. Similarly if $B \to C$, then $\{A, B\} \to \{X, C\}$ forcing $X = D$, hence $A \to D$. This also forces $C \to B$, and $D \to A$.

Thus, $\mathrm{aut}(G) \cong C_2$, the group with two elements.

(b)



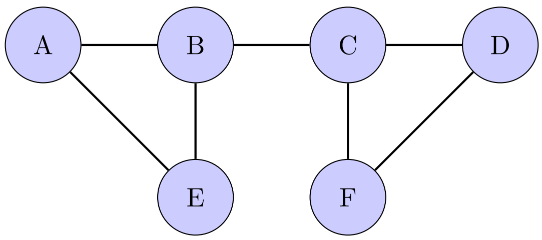
Solution Note that if $B \to B$, we can have $(A, E) \to (A, E)$ or $(A, E) \to (E, A)$, either one independently of $(D, F) \to (D, F)$ or $(D, F) \to (F, D)$. (Here, $(X, Y) \to (P, Q)$ is shorthand for $X \to P$ and $Y \to Q$). By letting e denote the identity permutation, l denote the swap (AB), r denote the swap (DF), and c denote the permutation $(BC)(AD)(EF)$, we can see that $\mathrm{aut}(G)$ consists of the following elements:

$$e, r, l, rl, c, rc, lc, rlc.$$

In other words,

$$\mathrm{aut}(G) = \{e, r\} \times \{e, l\} \times \{e, c\} \cong C_2 \times C_2 \times C_2.$$

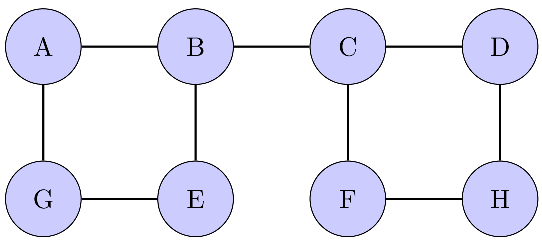
(c)



Solution The arguments in the previous solution remain unchanged, with

$$\mathrm{aut}(G) = \{e, r\} \times \{e, l\} \times \{e, c\} \cong C_2 \times C_2 \times C_2.$$

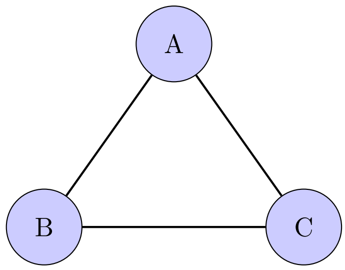
(d)



Solution If $B \to B$, then A can be mapped to either A, E and E gets the remaining spot. Similarly, C can be mapped to either C, H and H gets the remaining spot. The places of G and H are fixed. There is an analogous case when $B \leftrightarrow C$. Thus, define the permutations $l = (AE)$, $r = (DF)$, $c = (BC)(AD)(EF)(GH)$. Then,

$$\mathrm{aut}(G) = \{e, r\} \times \{e, l\} \times \{e, c\} \cong C_2 \times C_2 \times C_2.$$

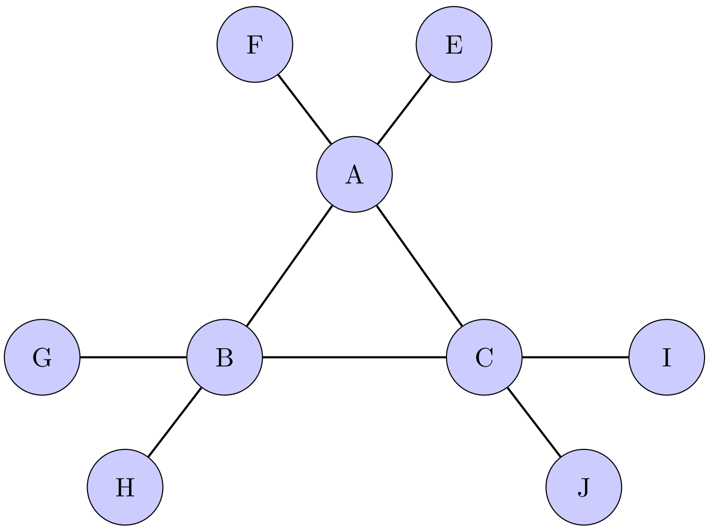
(e)



Solution It is clear that any permutation of A, B, C gives a graph automorphism, since this is the connected graph K_3. Thus,

$$\mathrm{aut}(G) \cong S_3.$$

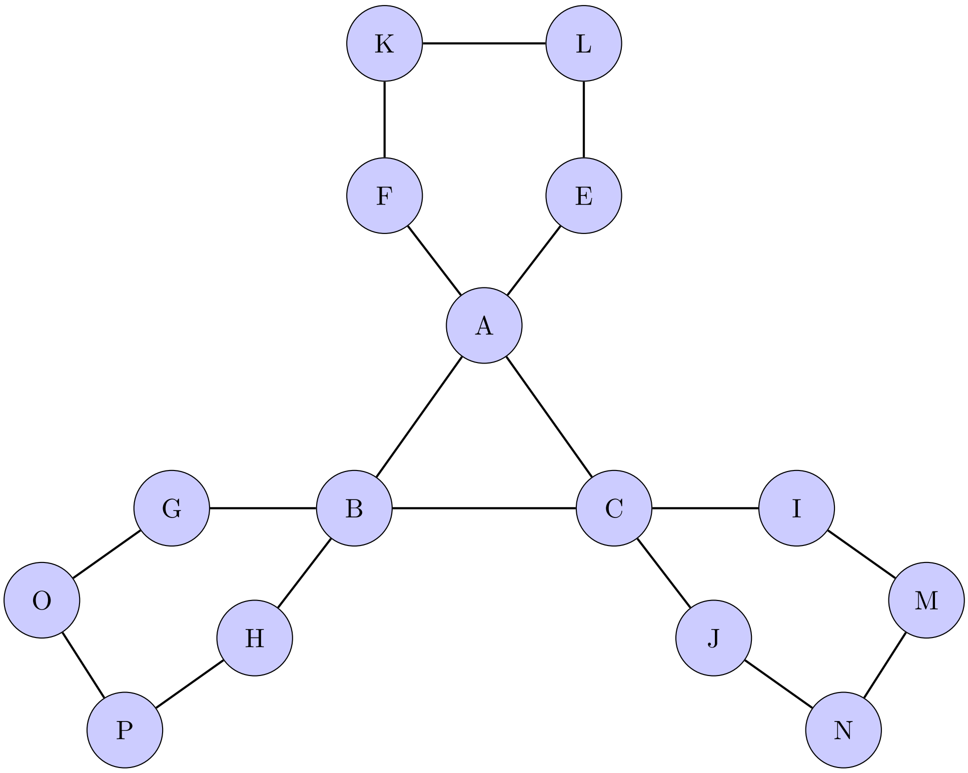
(f)



Solution Like before, A, B, C can be permuted in any way. After this, E, F can occupy the two positions next to where A lands in 2 ways, and the same goes for G, H next to B, I, J next to C. Thus,

$$\mathrm{aut}(G) \cong S_3 \times C_2 \times C_2 \times C_2.$$

(g)

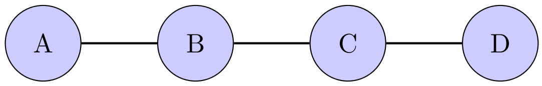


Solution We repeat exactly the same arguments as before: A, B, C permute freely, the elements F, E attach to A in two ways, and the remaining lobe K, L is forced by the positioning of F, E. Thus we have

$$\mathrm{aut}(G) \cong S_3 \times C_2 \times C_2 \times C_2.$$

Exercise 5 Find the orbits of all the vertices of the following graphs.

(a)

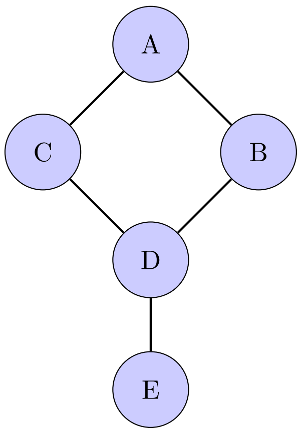


Solution Recall that we have computed the automorphism group of this graph. The orbits are

$$\{B, C\}, \qquad \{A, D\}.$$

In other words, the orbit of B and the orbit of C is $\{B, C\}$; the orbit of A and the orbit of D is $\{A, D\}$.

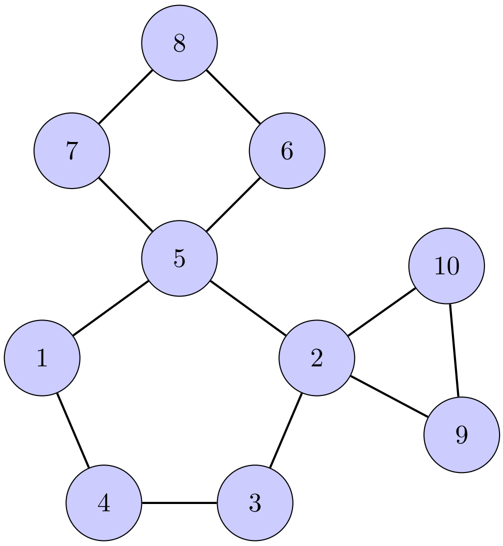
(b)



Solution Note that D can only be mapped to D, A can only be mapped to A, and E only to E. The permutation (BC) does give an automorphism. Thus, the orbits are

$$\{A\}, \qquad \{B, C\}, \qquad \{E\}, \qquad \{E\}.$$

(c)



Solution Note that if 5 is mapped to 2, then 6,7 must be mapped to 9, 10 leaving no room for mapping 8. Thus, 5 must remain fixed, 2 must remain fixed. 6 and 7 must be mapped amongst themselves; if say 6 is mapped to 1, then 8 must be mapped to 4, 7 to 3 which breaks the edge $7 \sim 5$. Thus, 8 must remain fixed. This in turn fixes 1, 3, 4. Finally, 9 and 10 can be mapped amongst themselves. The orbits are

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6, 7\}, \{8\}, \{9, 10\}.$$

Exercise 6 Prove that the n-cube is vertex transitive for any $n \in \mathbb{N}$.

Solution Label the each vertex of the n-cube with binary strings of length n, i.e. let each vertex be denoted by $x \equiv x_1 x_2 \dots x_n$ where each $x_i \in \{0, 1\}$. Two vertices are adjacent if and only if they differ in exactly one place, i.e. one bit.

Denote the bit complements $a' = 1 - a$ ($0' = 1$ and $1' = 0$). Define the bit flip operations $f_i \colon V \to V$, where $x \mapsto x_1 x_2 \dots x_{i-1} x_i' x_{i+1} \dots x_n$. In other words, f_i flips the ith bit of each vertex. It is clear that this is a bijection: no two vertices can be mapped to the same vertex, and every vertex has a pre-image. Indeed, $f_i^2(x) = x$ because $a'' = a$ for any bit a; this proves both injectivity[1] and surjectivity[2]. Furthermore, we claim that each f_i is an automorphism of the n-cube. To see this, pick two vertices x and y, and suppose that they differ in k bits. Specifically, if $x_i = y_i$, then $x_i' = y_i'$; if $x_i \neq y_i$, then $x_i' \neq y_i'$. This shows that $f_i(x)$ and $f_i(y)$ still differ in k bits. Thus, f_i preserves the edges of the n-cube.

Now, pick an arbitrary vertex x. Suppose that the binary string of x has 1's in precisely the indices $i_1, i_2, \dots, i_k$. Then, it is clear that $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}(x) = 0$ (the vertex whose binary string only consists of 0's). Note that this composition of automorphisms is itself an automorphism. Thus, every vertex x contains the common vertex 0 in its orbit, proving that the n-cube is vertex transitive.

Exercise 7 If G is a vertex transitive simple connected graph of order ≥ 3, then prove that G does not contain a cut-vertex.

[1] If $f_i(x) = f_i(y)$, then $f_i^2(x) = f_i^2(y)$ gives $x = y$.
[2] The pre-image of x is $f_i(x)$.

Solution Note that such a graph is regular (an automorphism preserves the degrees of each vertex) with minimum degree 2. The latter follows because if every vertex had degree 1, then given an edge $\{x, y\}$, there can be no path from x to a third vertex z: neither x nor y can contribute another edge.

First, we claim that an automorphism of G cannot map a non cut-vertex to a cut-vertex. Let x not be a cut-vertex, and suppose that $g \in \mathrm{aut}(G)$ is such that $g(x)$ is a cut vertex. This means that there exist vertices y', z' such that every path between them passes through $g(x)$. Set $y = g^{-1}(y')$, $z = g^{-1}(z')$; since x is not a cut vertex, there exists a path $y, v_1, \ldots, v_k, z$ not passing through x. The image, $y', g(v_1), \ldots, g(v_k), z'$ is still a path, and none of the $g(v_i) = g(x)$ since that would imply $v_i = x$. This contradicts the fact that $g(x)$ is a cut vertex.

Next, we claim that every connected graph has a non cut-vertex. The lemmas proved in Assignment 3 show that every connected graph G has a spanning tree, and every tree contains a leaf – pick such a spanning tree T and a leaf x. Now pick two vertices y, z from G; since T is a spanning tree, there exists a path between y and z within the tree T i.e. only using edges from T. Such a path cannot include the vertex x: note that x is not an endpoint of the path, but x has degree 1 so it cannot be an intermediate vertex in the path either. Thus, the removal of x still keeps T, hence G connected, which means that x is not a cut-vertex.

The above immediately show that a vertex transitive simple connected graph cannot contain a cut-vertex: simply pick a non cut-vertex and note that it can be mapped to any other vertex via an automorphism. Thus, none of these vertices can be a cut-vertex either.

Exercise 8 Prove that the group action of the automorphism group of a vertex transitive graph on the vertex set of the graph can have only one orbit.

Solution Let G be a vertex transitive graph and let $x \in V(G)$. Then, for any $y \in V(G)$, there exists an automorphism $g \in \mathrm{aut}(G)$ such that $gx = y$. In other words, y is in the orbit of x for all $y \in V(G)$, so the orbit of x is all of $V(G)$. This is true for any $x \in V(G)$, hence there is only one orbit.

Exercise 9 Prove that

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

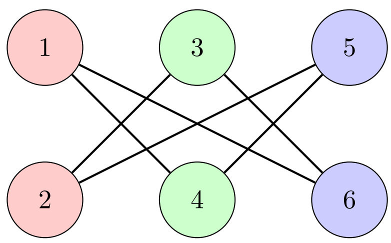
Solution Let the vertices of G be coloured (in the proper manner) in $k = \chi(G)$ colours, and let $V_1, \ldots, V_k$ be sets of vertices where V_i contains vertices of the ith colour. This is a partition of the vertices of G. Furthermore, each V_i is an independent set by construction: no two vertices of the same colour are adjacent. Thus, $\alpha(G) \geq |V_i|$ for each i, giving

$$k \cdot \alpha(G) \geq \sum_{i=1}^{k} |V_i| = |V(G)|$$

as desired.

Exercise 10 Construct a graph G that is neither a clique nor an odd cycle but has a vertex ordering relative to which the greedy colouring uses $\Delta(G) + 1$ colours.

Solution Let G be the following graph, with vertices labelled 1,. . . , 6 and colours Red, Green, Blue in order.



It is easily checked that this is indeed the colouring produced by the greedy algorithm; after colouring 1, 2 in red, 3, 4 must be given a new colour and 5, 6 yet another new colour. Thus, we have used $3 = \Delta(G) + 1$ colours. However, it is clear that $\chi(G) = 2$ since it is bipartite.

Exercise 11 Prove that a graph G is 2^k-colourable if and only if G is the union of k bipartite graphs.

Solution First suppose that G is the union of k bipartite graphs, $G_1, \dots, G_k$. Let the two parts of each G_i be A_i and B_i. Without loss of generality, let $V(G) = V(G_i)$ — do this by adding any remaining vertices not in G_i to the part A_i, and note that this keeps each G_i bipartite (we aren't adding any new edges). Now, given a vertex x in G and an index i, x is present in exactly one of A_i or B_i. Define $x_i = 1$ if $x \in A_i$, and $x_i = 0$ if $x \in B_i$. In this manner, colour each vertex $x \in G$ with the binary string $x_1 x_2 \dots x_k$. There are at most 2^k possible colours. We now show that this is indeed a proper colouring of G. To do this, suppose that two vertices x and y have the same colour, i.e. their binary strings match. For each index i, we have $x_i = y_i$; if this is 1, then $x, y \in A_i$ and if this is 0, then $x, y \in B_i$. In either case, x, y belong to the same part in G_i, hence G_i does not contribute the edge $\{x, y\}$. However, every edge in G must come from some G_i, since it is the union of all G_i; this proves that G does not contain the edge $\{x, y\}$, hence our colouring is proper.

Next, suppose that G is 2^k-colourable; like before, label these colours with binary strings of length k and colour each vertex with them. For each index $1 \leq i \leq k$, create the sets A_i and B_i. Now given a vertex x in G, look at the ith place in the binary string, i.e. the bit x_i, and define $x \in A_i$, $x \notin B_i$ if $x_i = 1$, otherwise $x \notin A_i$, $x \in B_i$ if $x_i = 0$. Finally, construct each G_i by starting with G, then removing all edges within A_i (that is, removing all edges with both endpoints in A_i) and removing all edges within B_i. Note that each G_i is clearly bipartite by construction. We now claim that their union gives all of G, i.e. every edge in G can be found in some G_i. Indeed, pick an edge $\{x, y\}$ from G; note that by our colouring scheme, x and y must have different colours, hence their binary strings differ in some index i, with $x_i \neq y_i$. Thus, $x_i \in A_i$ and $y_i \in B_i$ (without loss of generality), so G_i contains the edge $\{x, y\}$ (we have not removed this edge when constructing G_i).

Exercise 12 Prove that every graph G has a vertex ordering relative to which the greedy colouring uses $\chi(G)$ colours.

Solution Let G be coloured using $k = \chi(G)$ colours, and let $V_1, \dots, V_k$ be the sets of vertices of G such that V_i contains vertices of the ith colour. Then, each V_i is an independent set. Now, label the vertices of each V_i in any order, as $V_i = \{v_{i1}, v_{i2}, \dots, v_{it}\}$. Finally, label the vertices of G in order $[V_1, V_2, \dots, V_k]$, i.e. $v_{11}, v_{12}, \dots, v_{1t_1}, v_{21}, v_{22}, \dots, v_{kt_k}$. Here, the vertices appear sorted in ascending order of their colour.

Now, apply the greedy algorithm. For the first t_1 vertices taken from V_1, all can be assigned the lowest in index colour 1, since there are no conflicts (V_1 is independent). Next introduce the vertices from V_2; here, each vertex can be coloured with either 1, 2. There is no need to introduce the colour 3 since no vertex from V_2 is connected to another from V_2, which means that they are only connected to vertices from V_1 coloured in 1. Similarly, at the stage when vertices of V_{j+1} are introduced, the previous vertices all being given colours $1, \dots, j$, note that each new vertex can be given a colour from $1, \dots, j, j+1$ since they are only connected to vertices from vertices from $V_1, \dots, V_j$ coloured in $1, \dots, j$. This shows that the greedy algorithm will terminate by using only $k = \chi(G)$ colours.

Exercise 13 Prove that $\chi(G) = \omega(G)$ when $\overline{G}$ is bipartite, where $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices (called a clique) in G.

Solution Recall that if any subgraph od G has chromatic number k, then $\chi(G) \geq k$ (otherwise would imply the $< k$ colorability of the subgraph). Let X be a largest clique in G of $\omega(G)$ vertices; we have $\chi(G) \geq \omega(G)$.

Exercise 14 Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \leq 1 + m\sqrt{m-1}$.

Solution First, let G be k-chromatic, and let $V_1, \dots, V_k$ be the vertex sets where V_i contains vertices of the ith colour. Examine any pair V_i, V_j, with $i \neq j$. Suppose that there is no edge between them (there is no edge with one endpoint in V_i, the other in V_j). Then, recolouring V_j to be the same colour as V_i gives us a proper $k-1$ colouring of G, contradicting the minimality of k. Thus, there must be at least one edge associated with each pair i, j, of which there are $\binom{k}{2}$.

Now, let G be the union of m complete graphs of order m. Then, G has at most $m \cdot \binom{m}{2}$ edges. Set $\chi(G) = k$, hence

$$m \cdot \binom{m}{2} \geq |E(G)| \geq \binom{k}{2}, \qquad m^2(m-1) \geq k(k-1) \geq (k-1)^2.$$

Taking a square root, $k - 1 \leq m\sqrt{m-1}$ or $k \leq 1 + m\sqrt{m-1}$ as desired.

Lemma 1. *The chromatic polynomial of a graph with two components is the product of the chromatic polynomials of those components.*

Proof. Let G_1, G_2 be the two components of G. Then, there are no edges between them. Given some k, there are $P_{G_1}(k)$ ways of colouring G_1, and for each of these there are $P_{G_2}(k)$ ways of colouring G_2. This gives a total of $P_{G_1}(k)P_{G_2}(k)$ colourings of G. Furthermore, every colouring of G can be expressed in this way, i.e. every colouring of G gives a colouring of G_1 and a colouring of G_2, hence we have accounted for all possible k-colourings of G. □

Exercise 15 Prove that if T is a tree with n vertices, then $P_T(k) = k(k-1)^{n-1}$.

Solution We use induction on n. This is trivial for $n = 2$, since the tree on 2 vertices clearly shows $P_T(k) = k(k-1)$. Now, pick a tree T with $n > 2$ vertices, and suppose that this statement holds for all trees with less than n vertices. Let x be a leaf of T, and let $e = \{x, y\}$ be the only edge of x. Then, $T' = T/e$ is a tree on $n-1$ vertices. Furthermore, $T - e$ gives the same tree T' along with an extra isolated vertex x. Using the relation $P_G = P_{G-e} - P_{G/e}$ together with Lemma 1 (the chromatic polynomial of a single isolated vertex is just k), we can write

$$P_T(k) = k \cdot P_{T'}(k) - P_{T'}(k) = (k-1)P_{T'}(k) = (k-1) \cdot k(k-1)^{n-2} = k(k-1)^{n-1}.$$

Exercise 16 Show that the chromatic polynomial $P_G(k)$ has degree $|V(G)|$, with integer coefficients alternating in sign and beginning $1, -e(G), \ldots,$. Here, $e(G)$ is the number of edges in G.

Solution This is easily verified for all graphs G with at most 2 vertices. We proceed by induction on n; suppose that this holds for all graphs will fewer than $n > 2$ vertices. Pick a graph G on n vertices. If $e(G) = 0$, i.e. G only contains isolated vertices, it is clear that $P_G(k) = k^n$ which is of the given form. Otherwise, we perform induction on $e(G)$; suppose that this statement holds for all G with n vertices, and fewer than $e(G)$ edges. Pick an edge e from G, and note that $G - e$ has n vertices, $e(G) - 1$ edges while G/e has $n-1$ vertices. Thus, write their chromatic polynomials as

$$\begin{aligned} P_{G-e}(k) &= k^n - (e(G)-1)k^{n-1} + a_{n-2}k^{n-2} + a_{n-3}k^{n-3} - \cdots + (-1)^n a_0, \\ P_{G/e}(k) &= k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} + \cdots + (-1)^{n-1} b_0. \end{aligned}$$

Here, all $a_i, b_i \geq 0$. Using $P_G = P_{G-e} - P_{G/e}$ gives us

$$P_G(k) = k^n - e(G)k^{n-1} + (a_{n-2} + b_{n-2})k^{n-2} - (a_{n-3} + b_{n-3})k^{n-3} + \cdots + (-1)^n(a_0 + b_0),$$

which is of the desired form.

Exercise 17 Prove that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial.

Solution Denote this expression as $p(k)$. Note that

$$p(k) = k^2(k^2 - 4k + 3) = k^2(k-1)(k-3).$$

If p was the chromatic polynomial of some graph G, this would imply that G has $p(3) = 0$ number of 3-colourings, but $p(2) = -4$ number of 2-colourings, which is absurd.

Exercise 18 Prove that

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1).$$

Solution We show this by induction. This is clearly true for $n = 3$, since there are $k \cdot (k-1) \cdot (k-2)$ ways of colouring $C_3 = K_3$ with k colours, and this is just $k^3 - 3k^2 + 2k = (k-1)^3 - (k-1)$.

Now, suppose that this holds for cycles of length less than $n > 3$. Pick an edge e from C_n, and note that $C_n - e = P_n$, $C_n/e = C_{n-1}$. Thus, our reduction formula gives

$$P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k) = k(k-1)^{n-1} - \left[(k-1)^{n-1} + (-1)^{n-1}(k-1)\right].$$

Simplifying, this gives

$$P_{C_n}(k) = (k-1)^{n-1}(k-1) - (-1)^{n-1}(k-1) = (k-1)^n + (-1)^n(k-1).$$

Lemma 2. *The chromatic polynomial of any graph has integer coefficients.*

Proof. Use the induction method of Exercise 16; if all the coefficients of P_{G-e} and $P_{G/e}$ are integers, then so are the coefficients of P_G. □

Lemma 3. *The chromatic polynomial of any graph can only have integer roots.*

Proof. This follows directly from the rational root theorem. Suppose that

$$P_G(k) = k^t \left[k^{n-t} - mk^{n-t-1} + \cdots + a_1 k + a_0\right],$$

where $a_0 \neq 0$ (if P_G has no such coefficient, then $P_G(k) = k^n$ and we are done). Then, the roots of the bracketed polynomial must be rational numbers p/q, $\gcd(p,q) = 1$ such that $p|a_0$ and $q|1$, i.e. the roots must be integers. This is also easily seen from the fact that if

$$k^{n-t} - mk^{n-t-1} + \cdots + a_1 k + a_0 = 0,$$

all the powers of k are divisible by k, hence a_0 must also be divisible by k. □

Exercise 19 Prove that the chromatic polynomial of an n-vertex graph has no real root larger than $n - 1$.

Solution Note that $\chi(G) \leq n$ for any graph, hence its chromatic polynomial $P_G(k)$ is a strictly positive integer for all integers $k \geq n$ (any $\chi(G)$ colouring gives a valid $k \geq n$ colouring). Thus, P_G has no integer roots $k > n - 1$; but P_G can only have integer roots by the above lemma, so P_G has no real roots $x > n - 1$.

Lemma 4. *The chromatic polynomial of any graph has no negative roots.*

Proof. Since the coefficients of P_G have alternating signs, $P_G(k)$ for negative k is strictly positive when the degree n is even, and strictly negative when the degree n is odd. Hence, there can be no negative roots. □

Lemma 5. *The roots of a chromatic polynomial must among the integers* $0, 1, \ldots, \chi(G) - 1$.

Exercise 20 Prove that the last non-zero term in the chromatic polynomial of G is the term whose exponent is the number of components of G.

Solution We will show that when G is connected, the highest power of k dividing $P_G(k)$ is exactly k. This in turn will show that when G has r components, the highest power of k dividing $P_G(k)$ will be k^r (Lemma 1 shows that the chromatic polynomials of the components multiply).

Let G be a connected graph on n vertices. Then it is clear that $P_G(0) = 0$ since there is no way of colouring G with zero colours, thus $k|P_G(k)$. Another way to see this is to note that for any $k \geq \chi(G)$ and given some proper k-colouring of G, cyclically permuting the k colours yields k distinct proper colourings. Furthermore, none of these can be obtained by cyclically permuting the colours of some other k-colouring not present here. Thus, the $P_G(k)$ many k-colourings can be partitioned into groups of k, hence $k|P_G(k)$.

Note that the highest power of k dividing the chromatic polynomial of any connected graphs on 2 vertices is just k, as desired. Suppose that this result holds for all connected graphs on fewer than $n > 2$ vertices, and let G be connected with n vertices. If G has $n - 1$ edges, it is a tree so we know that $P_G(k) = k(k-1)^{n-1}$. Further suppose that the result holds for all G on fewer than n vertices, fewer than $e(G) > n-1$ edges. Since $e(G) > n-1$, it contains a cycle. Then, we can find an edge e whose removal keeps G connected (there are edges in G apart from those in its spanning tree, choose one of these). Now, $G-e$ is connected with fewer than $e(G)$ edges, and G/e is connected with fewer than n vertices. Thus, write

$$\begin{aligned} P_{G-e}(k) &= k^n - (e(G)-1)k^{n-1} + a_{n-2}k^{n-2} + a_{n-3}k^{n-3} - \cdots - (-1)^n a_1 k, \\ P_{G/e}(k) &= k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} + \cdots - (-1)^{n-1} b_1 k. \end{aligned}$$

Here, all $a_i, b_i \geq 0$ and $a_1, b_1 \neq 0$ since k^2 does not divide either of these polynomials by the induction hypothesis. Subtracting,

$$P_G(k) = k^n - e(G)k^{n-1} + (a_{n-2} + b_{n-2})k^{n-2} - (a_{n-3} + b_{n-3})k^{n-3} + \cdots - (-1)^n (a_1 + b_1)k.$$

Again, $a_1 + b_1 \neq 0$ hence k^2 does not divide $P_G(k)$. This proves the result by induction.