MA3201

Topology

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Contents

1 Introduction

1.1 Topological spaces

Definition 1.1. A topology on some set X is a family τ of subsets of X, satisfying the following.

1. $\emptyset, X \in \tau$.
2. All unions of elements from τ are in τ.
3. All finite intersections of elements from τ are in τ.

The sets from τ are declared to be open sets in the topological space (X, τ).

Example. Any set X admits the indiscrete topology $\tau_{id} = \{\emptyset, X\}$, as well as the discrete topology $\tau_d = \mathcal{P}(X)$. Both of these are trivial examples.

Example. Let X be a set. The cofinite topology on X is the collection of complements of finite sets, along with the empty set. Note that when X is finite, this is simply the discrete topology.

Definition 1.2. Let τ, τ' be two topologies on the set X. We say that τ is finer than τ' if τ has more open sets than τ'. In such a case, we also say that τ' is coarser than τ.

1.2 Topological bases

Definition 1.3. Let (X, τ) be a topological space. We say that $\beta \subseteq \tau$ is a base of the topology τ such that every open set $U \in \tau$ is expressible as a union of elements from β.

Definition 1.4. Let X be a set, and let β be a collection of subsets of X satisfying the following.

1. For every $x \in X$, there exists $x \in B \in \beta$.
2. For every $x \in X$ such that $x \in B_1 \cap B_2$, $B_1, B_2 \in \beta$, there exists $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then, β generates a topology on X, namely the collection of all unions of elements of β.

Lemma 1.1. *Let τ be a topology on X, and let $\beta \subseteq \tau$ be a collection of open sets. Then, β is a basis of τ, or generates τ, if for every $x \in U \in \tau$, there exists $B \in \beta$ such that $x \in B \subseteq U$.*

Example. The collection of all open balls in $\mathbb{R}^n$ form a basis of the usual topology.

Lemma 1.2. *Let X be equipped with the topologies τ and τ', and let β and β' be the respective bases of these topologies. Then, τ is finer than τ' if and only if given $x \in B' \in \beta'$, there exists $x \in B \in \beta$ such that $B \subseteq B'$.*

Example. The collections of open balls in $\mathbb{R}^n$ generate the same topology as the collection of all open rectangles in $\mathbb{R}^n$.

Example. Consider the topologies on $\mathbb{R}$ generated by the following bases.

1. $\beta_1 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.
2. $\beta_2 = \{[a, b) : a, b \in \mathbb{R}, a < b\}$.
3. $\beta_3 = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K\}$ where $K = \{1/n : n \in \mathbb{Z}\}$.

We call the topology generated by β_2 the lower limit topology, denoted $\mathbb{R}_\ell$. The topology generated by β_3 is denoted $\mathbb{R}_K$. Both of these are strictly finer than the standard topology.

Definition 1.5. A sub-basis for some topology on X is a collection ρ of subsets of X whose union is the whole of X. The topology generated by ρ is defined to be the topology generated by the collection of all finite intersections of elements of ρ.

1.3 Product topology

Definition 1.6. Let (X_1, τ_1), (X_2, τ_2) be topological spaces. Then $\tau_1 \times \tau_2$ generates the product topology on $X_1 \times X_2$.

Example. The product topology on $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard topology, coincides with the standard topology on $\mathbb{R}^2$.

Lemma 1.3. *If β_1, β_2 are bases of the topologies τ_1, τ_2, then $\beta_1 \times \beta_2$ and $\tau_1 \times \tau_2$ generate the same product topology.*

Proof. Given $(x_1, x_2) \in U$ where $U \subseteq X_1 \times X_2$ is open in the product topology, recall that U can be written as a union of the basic open sets $U_{1i} \times U_{2i}$, where $U_{1i} \in \tau_1$ and $U_{2i} \in \tau_2$. Suppose that $(x_1, x_2) \in U_1 \times U_2$. Thus, we can choose $B_1 \in \beta_1$, $B_2 \in \beta_2$ such that $x_1 \in B_1 \subseteq U_1$ and $x_2 \in B_2 \subseteq U_2$. Thus, $(x_1, x_2) \in B_1 \times B_2 \subseteq U_1 \times U_2 \subseteq U$. $\square$

Definition 1.7. The projection maps are defined as $\pi_i \colon X_1 \times \cdots X_k \to X_i$, $(x_1, \ldots, x_k) \mapsto x_i$.

Lemma 1.4. *The collection of elements of the form $\pi_1^{-1}(U_1)$ or $\pi_2^{-1}(U_2)$, where $U_1 \in \tau_1$ and $U_2 \in \tau_2$, forms a sub-basis of the product topology on $X_1 \times X_2$.*

Proof. Note that $\pi_1^{-1}(X_1) = X_1 \times X_2$. Now it is easy to see that finite intersections of elements of the form $U_1 \times X_2$ or $X_1 \times U_2$ where U_1, U_2 are open, are all of the form $U_1 \times U_2$ which is precisely a basis of the product topology. □

Corollary 1.4.1. *We can restrict ourselves to the sub-basis of elements of the form $\pi_1^{-1}(B_1)$ or $\pi_2^{-1}(B_2)$, where $B_1 \in \beta_1$, $B_2 \in \beta_2$ for some bases β_1, β_2 of τ_1, τ_2.*

1.4 Subspace topology

Definition 1.8. Let (X, τ) be a topological space, and let $Y \subset X$. Then the collection $U \cap Y$ for all $U \in \tau$ comprises the subspace topology τ_Y on Y induced by the topology τ on X.

Lemma 1.5. *If β is a basis for the topology on X, and $Y \subset X$, then the collection $B \cap Y$ for all $B \in \beta$ generates the subspace topology on Y.*

Lemma 1.6. *An open set of Y is open in X if Y is open in X.*

Proof. Let $U \subset Y$ be open in Y, then $U = V \cap Y$ for some open set V in X. If additionally Y is open in X, this immediately shows that U is open in X. □

Theorem 1.7. *Let (X, τ_X), (Y, τ_Y) be topological spaces, and let $A \subseteq X$, $B \subseteq Y$. Then, there are two ways of assigning a natural topology on $A \times B$.*

1. *Take the product topology on $X \times Y$, and consider the subspace topology induced by it on $A \times B$.*
2. *Take the subspace topologies on A induced by τ_X, B induced by τ_Y, and consider the product topology generated by them on $A \times B$.*

These two methods generate the same topology on $A \times B$.

Proof. Open sets in 1 look like $(U \times V) \cap (A \times B)$, where $U \in \tau_X$, $V \in \tau_Y$). Open sets in 2 look like $(U' \cap A) \times (V' \cap B)$, where $U' \in \tau_X$, $V' \in \tau_Y$, which can be rewritten as $(U' \times V') \cap (A \times B)$. It is easy to see that these describe precisely the same sets. □

1.5 Order topology

Definition 1.9. Let X be a set with a simple order $<$. Then the collection of sets of the form (a, b), $[a_0, b)$, $(a, b_0]$ where a_0 is the minimal element of X, b_0 is the maximal element of X, generate the order topology on X.

Example. The order topology on $\mathbb{N}$ is precisely the discrete topology.

Definition 1.10. Let X_1, X_2 be simply ordered sets. The dictionary order on $X_1 \times X_2$ is defined as follows: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$.

Example. Consider $X = \{1, 2\} \times \mathbb{N}$, where both $\{1, 2\}$ and $\mathbb{N}$ are endowed with the discrete topology. Note that the product topology on X is the discrete topology.

Now consider the dictionary order on X. Here, $(1, 1)$ is the smallest element, so we can list the elements of X in ascending order. Note that every $(1, m) < (2, n)$, for all $m, n \in \mathbb{N}$. Now, note that all singletons $\{(1, m)\}$ are open in the order topology on X. The same is true for the singletons $\{(1, n)\}$ for all $n > 1$. However, the singleton $\{(2, 1)\}$ is *not* open in the order topology.

Example. Consider $\mathbb{R}$ with the usual topology, and $X = [0, 1) \cup \{2\}$. Then, $\{2\}$ is open in the subspace topology on X, but it is not open in the order topology on X.

Lemma 1.8. *The open rays of the form $(a, +\infty)$ and $(-\infty, a)$ in X form a sub-basis of the order topology on X.*

Proof. Note that $(a, b) = (-\infty, b) \cap (a, +\infty)$, $[a_0, b) = (-\infty, b)$, and $(a, b_0] = (a, +\infty)$. □

Definition 1.11. Let X be a simply ordered set, and $Y \subseteq X$. Then, we say that Y is convex in X if given $a, b \in Y$ such that $a < b$, the interval $(a, b) = \{x \in X : a < x < b\} \subseteq Y$.

Theorem 1.9. *Let Y be convex in X. Then, the subspace topology and the order topology on Y induced from the order topology on X coincide.*

1.6 Closed sets

Definition 1.12. Let (X, τ) be a topological space. A set $F \subseteq X$ is said to be closed in X if $F^c = X \setminus F \in \tau$.

Example. The sets $\emptyset, X$ are closed in every topological space (X, τ).

Example. In a set equipped with the discrete topology, every set is both open and closed.

Lemma 1.10. *Arbitrary intersections, and finite unions of closed sets are closed.*

Theorem 1.11. *Let (X, τ) be a topological space, and let $Y \subset X$ be equipped with the subspace topology. Then, a set $F \subseteq Y$ is closed in Y if and only if $F = Y \cap G$, where G is closed in X.*

Proof. Let $F \subset Y$. Now, F is closed in Y, $Y \setminus F = Y \cap F^c$ is open in Y, $Y \cap F^c = Y \cap U$ where U is open in X, $F = Y \cap (Y \cap F^c)^c = Y \cap (Y \cap U)^c = Y \cap U^c$ where U^c is closed. The steps are reversible. □

Lemma 1.12. *A closed set of Y is closed in X if Y is closed in X.*

1.7 Interiors and closures

Definition 1.13. Let $A \subseteq X$ where (X, τ) is a topological space.

1. The interior of A is defined as the union of all open sets contained in A. This is denoted by A°.
2. The closure of A is defined as the intersection of all closed sets containing A. This is denoted by $\overline{A}$.

Remark. The interior of a set is open, and the closure of a set is closed.

Lemma 1.13. *Let $Y \subset X$ be topological spaces, and let $A \subseteq Y$. Also let $\overline{A}_X$, $\overline{A}_Y$ denote the closures of A in X, Y respectively. Then, $\overline{A}_Y = \overline{A}_X \cap Y$.*

Theorem 1.14. *Let $A \subset X$. Then,*

1. *$x \in \overline{A}$ if and only if every open set containing x has non-empty intersection with A.*
2. *$x \in \overline{A}$ if and only if every basic open set containing x has non-empty intersection with A, given that the topology on X is generated by those basic open sets.*

Definition 1.14. Let $A \subseteq X$ where (X, τ) is a topological space. We say that $x \in X$ is a limit point of X if for every open set U containing x, the deleted neighbourhood $U \setminus \{x\}$ has non-empty intersection with A. The set of limit points of A is denoted by A'.

Example. Let X be a set endowed with the discrete topology. Then, given any set $A \subseteq X$, we have $A' = \emptyset$.

Lemma 1.15. *A closed set contains all its limit points.*

Proof. Let $F \subseteq X$ be closed in X, and let $x \in F'$. Then given any open set containing x, we have $U \cap F \supseteq (U \setminus \{x\}) \cap F \neq \emptyset$, hence $x \in \overline{F} = F$. □

Lemma 1.16. *Let $A \subseteq X$ where (X, τ) is a topological space. Then, $\overline{A} = A \cup A'$.*

Proof. It is clear that $\overline{A} \supseteq A \cup A'$. Now pick $x \in \overline{A}$. If $x \notin A$, then we know that given any open neighbourhood U of x, we have non-empty $U \cap A$. Furthermore, this intersection can never contain x, hence $x \in A'$. This proves that $\overline{A} \subseteq A \cup A'$. □

1.8 Convergence of sequences

Definition 1.15. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points from (X, τ), and let $x \in X$. We say that this sequence converges to x, denoted $x_n \to x$, if every open neighbourhood of x contains the tail of this sequence. In other words, given $U \in \tau$ such that $x \in U$, there must exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example. Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then, the constant sequence of b's converges to all three points a, b, c.

Example. Let $X = \mathbb{R}$, and τ be the collection of all intervals $(-a, a)$ together with $\emptyset, \mathbb{R}$. Then, the constant sequence of 0's converges to every point in $\mathbb{R}$.

Definition 1.16. Let (X, τ) be a topological space. We say that this topological space is Hausdorff if given any two distinct points $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Example. The real numbers under the standard topology is Hausdorff.

Theorem 1.17. *Let (X, τ) be a Hausdorff topological space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in X. Then, this sequence can converge to at most one point in X.*

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to distinct points $x, y \in X$. Then there exist disjoint open neighbourhoods U, V such that $x \in U$, $y \in V$. Convergence means that both U and V contain a tail of the sequence, which is a contradiction. □

Lemma 1.18. *The singleton sets in a Hausdorff space are closed.*

Proof. Let $x \in X$ where (X, τ) is Hausdorff. Pick $y \neq x$, whence there exist $U_y, V_y \in \tau$, such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. In particular, $\{x\} \cap V_y = \emptyset$. We now have

$$X \setminus \{x\} = \bigcup_{y \neq x} V_y,$$

which is open. □

Theorem 1.19. *The topology induced by a metric is Hausdorff.*

Proof. Given a metric space X and distinct points $x, y \in X$, we set $r = |x - y|$, $U = B(x, r/3)$, $V = B(y, r/3)$. □

2 Continuous maps

Definition 2.1. Let $f : X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is continuous if for every $U \in \tau_Y$, we have $f^{-1}(U) \in \tau_X$. In other words, the pre-image of every open set in Y must be open in X.

Lemma 2.1. *A function $f : X \to Y$ is continuous if and only if given a base β of Y, we have $f^{-1}(U) \in \tau_X$ for every $U \in \beta$.*

Example. The identity function $\mathrm{id} : \mathbb{R}_\ell \to \mathbb{R}$ is continuous, while the identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}_\ell$ is not. This is because the topology on $\mathbb{R}_\ell$ is strictly finer than that on $\mathbb{R}$.

Lemma 2.2. *A function $f : X \to Y$ is continuous if and only if for every closed set $F \subseteq Y$, we have $f^{-1}(F)$ closed in X.*

Lemma 2.3. *A function $f : X \to Y$ is continuous if and only if given any $x \in X$ and an open set $V \subseteq Y$ such that $f(x) \in V$, there exists an open set $U \subseteq X$ such that $x \in U$, $f(U) \subseteq V$.*

Theorem 2.4. *The composition of continuous functions is continuous.*

2.1 Restricting and enlarging the domain

Lemma 2.5. *Let $f\colon X \to Y$ be continuous, and let $A \subset X$. Then the restriction of f to A is continuous.*

Theorem 2.6. *Let $f\colon X \to Y$, and let X be the union of the collection of open sets $\{A_\lambda\}_{\lambda\in\Lambda}$. If the restrictions of f to each A_λ are continuous, then f is continuous.*

Proof. Pick $x \in X$, hence $x \in A_\lambda$ for some $\lambda \in \Lambda$. Now if $f(x) \in V \subset Y$, where V is open in Y, then the continuity of the restriction of f to A_λ gives us an open set $U \subseteq A_\lambda$ such that $f(U) \subseteq V$. Finally since A_λ is open in X, so is U. □

Definition 2.2. Let X be the union of the collection of open sets $\{A_\lambda\}_{\lambda\in\Lambda}$. We say that this collection is a locally finite cover of X if given $x \in X$, there exists a neighbourhood U of x such that $U \cap A_\lambda$ is non-empty for only finitely many $\lambda \in \Lambda$.

Theorem 2.7. *Let $f\colon X \to Y$, and let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a locally finite collection of closed sets covering X. If the restrictions of f to each F_λ are continuous, then f is continuous.*

Corollary 2.7.1 (Pasting lemma)**.** *Let $X = A \cup B$, with A, B closed in X. Let $f\colon A \to Y$, $g\colon B \to Y$ be continuous, with $f(x) = g(x)$ on $A \cap B$. Then the function $h\colon X \to Y$, defined by $x \mapsto f(x)$ on A and $x \mapsto g(x)$ on B, is continuous.*

Definition 2.3. A path is a continuous function $\gamma\colon [0,1] \to X$.

Lemma 2.8. *Two paths γ_1, γ_2 can be concatenated when $\gamma_1(1) = \gamma_2(0)$.*

2.2 Homeomorphisms

Definition 2.4. Let $f\colon X \to Y$ be a function between the topological spaces (X, τ_X) and (Y, τ_Y). We say that f is a homeomorphism if f is continuous, f is bijective, and f^{-1} is continuous. We also say that X and Y are homeomorphic when such a homeomorphism between them exists.

Example. The interval $(-1,1)$ is homeomorphic to $\mathbb{R}$; for instance, the map $x \mapsto \tan(\pi x/2)$ on $(-1,1)$ is a homeomorphism. A simpler construction is the map $x \mapsto x/(1-x^2)$.

2.3 Projection maps

Theorem 2.9. *The projection maps $\pi_i \colon X_1 \times \cdots \times X_k \to X_i$ are continuous, when the domain is equipped with the product topology. Furthermore, the product topology is the coarsest topology on the domain for which the projection maps are continuous.*

Lemma 2.10. *Let $f \colon A \to X_1 \times \cdots \times X_k$, where the co-domain is equipped with the product topology. Then, f is continuous if and only if the component functions $f_i = \pi_i \circ f$ are continuous.*

Proof. Note that if f is continuous, the compositions $\pi_i \circ f$ are immediately continuous. Conversely suppose that each f_i is continuous, and write

$$f(t) = (f_1(t), \dots, f_k(t)).$$

The sets $U_1 \times \cdots \times U_k$, where each $U_i \subseteq X_i$ is open, form a basis of the co-domain. Furthermore, their pre-images under f are $f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k)$, which are open in A. This shows that f is continuous. $\square$

Definition 2.5. Let J be an arbitrary index set. A J-tuple of elements in a set X is a function $x \colon J \to X$, formally denoted $(x_\alpha)_{\alpha\in J}$. If $\{X_\alpha\}_{\alpha\in J}$ is a family of sets, their Cartesian product is defined as

$$\prod_{\alpha\in J} X_\alpha = \{x \colon J \to \bigcup_{\alpha\in J} X_\alpha \colon x_\alpha \in X_\alpha\}.$$

Remark. The fact that we can choose an element from each set in an uncountable collection relies on the Axiom of Choice.

Definition 2.6. Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The topology generated by $\prod_{\alpha\in J} U_\alpha$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the box topology on $\prod_{\alpha\in J} X_\alpha$.

Definition 2.7. Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces. The topology generated by the sub-basis $\pi_\alpha^{-1}(U_\alpha)$, where each $U_\alpha \subseteq X_\alpha$ is open, is called the product topology on $\prod_{\alpha\in J} X_\alpha$.

Remark. The basic open sets are of the form $\pi_{\alpha\in J} U_\alpha$, where all but finitely many $U_\alpha = X_\alpha$. Thus, this is a coarser topology than the box topology.

Lemma 2.11. *Let $\prod_{\alpha\in J} X_\alpha$ be equipped with the box or product topology. Then, $\overline{\prod A_\alpha} = \prod \overline{A}_\alpha$, where each $A_\alpha \in X_\alpha$.*

Lemma 2.12. *Let $f\colon A \to \prod_{\alpha \in J} X_\alpha$, where the co-domain is equipped with the product topology. Then, f is continuous if and only if the component functions $f_\alpha = \pi_\alpha \circ f$ are continuous.*

Remark. This fails when $\prod_{\alpha \in J}$ is equipped with the box topology. Consider $f\colon \mathbb{R} \to \prod_{n=1}^{\infty} \mathbb{R}$, $x \mapsto (x, x, \dots)$. Then, the product $\prod_{n=1}^{\infty}(-1/n, 1/n)$ is open in the box topology, but its pre-image under f is $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is not open in $\mathbb{R}$.

3 Metric spaces

Definition 3.1. A metric space (X, d) is a set equipped with a metric $d\colon X \times X \to \mathbb{R}$, such that

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 3.2. An open ball in a metric spaces is the set of points

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Lemma 3.1. *The collection of open balls in a metric space generates its standard topology.*

Example. Consider a set X, equipped with the metric

$$d\colon X \times X \to \mathbb{R}, \qquad (x, y) \mapsto \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then, this metric induces the discrete topology on X.

3.1 Metrizable spaces

Definition 3.3. A topological space (X, τ) is called metrizable if there exists a metric $d\colon X \times X \to \mathbb{R}$ which induces the topology τ on X.

Definition 3.4. Let $A \subseteq X$. The diameter of A is defined to be

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

If $\operatorname{diam}(A)$ is finite, we say that A is bounded.

Example. The metric

$$(x, y) \mapsto \frac{|x - y|}{1 + |x - y|}$$

generates the standard topology on $\mathbb{R}$. Note that $\mathbb{R}$ is unbounded in the standard metric, but bounded in this one.

Definition 3.5. Let (X, d) be a metric space. Then the standard bounded metric corresponding to d is defined as

$$\bar{d}\colon X \times X \to \mathbb{R}, \qquad (x, y) \mapsto \min\{d(x, y), 1\}.$$

Lemma 3.2. *Both d and $\bar{d}$ generate the same topology.*

Theorem 3.3. *The product topology on $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \dots$ is metrizable, using the metric*

$$D(x, y) = \sup_n \left\{ \frac{1}{n} \bar{d}(x, y) \right\}.$$

Lemma 3.4 (Sequence lemma)**.** *Let $A \subseteq X$, let $x \in X$, and let the sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in A$ converge with $x_n \to x$. Then, $x \in \overline{A}$.*

Remark. The converse holds if X is metrizable.

Example. Consider $X = \mathbb{R}^\omega$ equipped with the box topology. Choose $A = \{(x_1, x_2, \dots) : x_i > 0\}$. Then, $0 = (0, 0, \dots) \in \overline{A}$; this is clear from the fact that any open set around 0 contains the basic open set $\prod_i (a_i, b_i)$ with $a_i < 0 < b_i$. However, there is no sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in A$, such that $x_n \to 0$. Note that if this were the case, then each $x_n = (x_{n1}, x_{n2}, \dots)$. Now, $B = \prod_i (-x_{ii}, x_{ii})$ contains none of the points x_n, since the nth coordinate of B eliminates the point n.

Corollary 3.4.1. $\mathbb{R}^\omega$ *equipped with the box topology is not metrizable.*

4 Compactness

Definition 4.1. Let X be a topological space. We say that X is compact if every open cover of X has a finite subcover.

Lemma 4.1. *Let $Y \subseteq X$. Then, Y is compact if and only if every open cover of Y by open sets in X has a finite subcover.*

4.1 Compact subspaces

Lemma 4.2. *All compact sets in a metric space are bounded.*

Proof. Let $K \subseteq X$ be compact. Then, K admits an open cover of open balls $B(0,n)$ from which we can extract a finite subcover; however, this can be reduced to just one open ball $B(0,N)$ for some N. Thus $K \subset B(0,N)$ is bounded. □

Lemma 4.3. *A closed subset of a compact space is compact.*

Proof. Let K be compact, and $F \subseteq K$ be closed. Consider an open cover $\{U_\alpha\}_{\alpha \in J}$ of F. By adding $K \setminus F$ to this collection, we have an open cover of K, from which we can extract a finite subcover $U_{i_1}, U_{i_2}, \ldots, U_{i_k}, K \setminus F$. By discarding the latter, we have found a finite subcover of F. □

Lemma 4.4. *In a Hausdorff space, every compact set is closed.*

Proof. Let X be Hausdorff, and $K \subseteq X$ be compact. Fix $x_0 \in X \setminus K$, and note that given any $y \in K$, there exist open neighbourhoods U_y, V_y such that $x_0 \in U_y$, $y \in V_y$, $U_y \cap V_y = \emptyset$. Thus, the collection of all such $\{V_y\}_{y \in K}$ is an open cover of K, from which we can extract a finite subcover $V_{y_1}, \ldots, V_{y_k}$. Corresponding to this, $x_0 \in U_{y_1} \cap \cdots \cap U_{y_k} \subseteq X \setminus K$. Thus, x_0 lies in the interior of $X \setminus K$. This shows that $X \setminus K$ is open, hence K is closed. □

Theorem 4.5. *The image of a compact space under a continuous map is compact.*

Lemma 4.6. *Let $f \colon X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We need only show that f is a closed map; now every closed set $F \subseteq X$ is compact because X is compact, hence $f(K) \subseteq Y$ is compact. Since Y is Hausdorff, the compact set $f(K)$ is closed. □

4.2 Products of compact spaces

Lemma 4.7 (Tube lemma)**.** *Let X, Y be topological spaces, and let Y be compact. Let $x_0 \in X$, and let $\{x_0\} \times Y \subset N \subseteq X \times Y$ where N is open. Then, there exists an open set $W \subseteq X$ such that $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.*

Proof. Note that $\{x_0\} \times Y$ is compact, being homeomorphic to Y. Thus, it can be covered with basic open sets $U_1 \times V_1, \ldots, U_k \times V_k$ such that each $U_i \times V_i \subset N$. Simply set $W = U_1 \cap \cdots \cap U_k$. □

Theorem 4.8. *Let X, Y be compact topological spaces. Then, $X \times Y$ is compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of $X \times Y$. Pick $x \in X$, whence $\{x\} \times Y$ is compact and admits a finite subcover $U_{xi_1}, \dots, U_{xi_k}$. Denote their union by U_x; the tube lemma guarantees an open set $W_x \subseteq X$ such that $\{x\} \times Y \subseteq W \times Y \subseteq U_x$. Now, the collection $\{W_x\}_{x\in X}$ is an open cover of X, hence admits a finite subcover $W_{x_1}, \dots, W_{x_n}$. This also means that $W_{x_1} \times Y, \dots, W_{x_n} \times Y$ is a finite cover of Y. However, each $W_{x_i} \times Y \subseteq U_{x_i}$ can be covered by finitely many U_α, which means that we have a finite subcover of $X \times Y$. □

4.3 Euclidean spaces

Lemma 4.9. *Let X be a simply ordered set with the least upper bound property. Then, the intervals $[a, b]$ are compact.*

Theorem 4.10 (Heine-Borel)**.** *Compact sets of $\mathbb{R}^n$ are precisely those which are closed and bounded.*

4.4 Limit point compactness

Definition 4.2. Let X be a topological space. We say that X is limit point compact if every infinite subset of X has a limit point.

Lemma 4.11. *A compact space is limit point compact.*

Proof. Let X be compact, and let $A \subseteq X$ have no limit points. Then, $A = A \cup A' = \overline{A}$ is closed in X, hence compact. Now given any $a \in A$, we know that a is not a limit point of A, hence we can choose an open neighbourhood U_a such that $U_a \cap A = \{a\}$. The collection $\{U_a\}_{a\in A}$ is now an open cover of A, and hence admits a finite subcover $U_{a_1}, \dots, U_{a_k}$. Let U denote their union, whence $A = A \cap U = \{a_1, \dots, a_k\}$ is finite. □

Example. Let $X = \mathbb{N} \times \{0, 1\}$, where $\mathbb{N}$ has the discrete topology, and $\{0, 1\}$ has the indiscrete topology. Then, every subset of X has a limit point; indeed, given any $\{(n, b)\}$, we have a limit point $(n, 1 - b)$. However, X is clearly not compact, since the open cover of sets $\{n\} \times \{0, 1\}$ does not admit any finite subcover.

Theorem 4.12. *Let X be a metrizable space. Then, X is limit point compact if and only if it is compact.*

5 Connectedness

Definition 5.1. Let X be a topological space, and let $U, V \subseteq X$ be open, non-empty, disjoint, with $U \cup V = X$. We say that U, V form a separation of X.

Definition 5.2. A topological space X is said to be connected if it admits no separation.

Lemma 5.1. *A topological space X is connected if and only if the only subsets that are both open and closed in it are $\emptyset, X$.*

Lemma 5.2. *Let X be a topological space, and let $Y \subseteq X$ be a subspace. Then, a separation of Y is a pair of open sets $A, B \subseteq X$ such that $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$.*

Lemma 5.3. *Let C, D form a separation of X, and let $Y \subseteq X$ be a connected subspace. Then, either $Y \subseteq C$, $Y \subseteq D$.*

Lemma 5.4. *The union of a collection of connected spaces with a common point is connected.*

Proof. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of connected spaces, with the common point x_0, and let X be their union. Suppose that U, V is a separation of X; then each of the connected X_α must be contained in one of U, V. However, since all X_α share the common point x_0, they must all lie in the same half, say U, forcing $V = \emptyset$, a contradiction. □

Lemma 5.5. *Let $A \subseteq X$ be connected, and let $A \subseteq B \subseteq \overline{A}$. Then, B is connected.*

Theorem 5.6. *The image of a connected space under a continuous maps is connected.*

Theorem 5.7. *A finite Cartesian product of connected spaces is connected.*

Proof. Let X, Y be connected spaces. Fix $(a, b) \in X \times Y$. Now, $X \times \{b\}$ is connected, being homeomorphic to X. Furthermore, each $\{x\} \times Y$ is connected, for each $x \in Y$. Now, the set $T_x = \{x\} \times Y \cup X \times \{b\}$ is connected, being the union of connected spaces with the common point (x, b). Finally, the union of all such T_x is connected, being the union of connected spaces with the common point (a, b). This union is just $X \times Y$, which is thus connected. □

Example. The countable product $\mathbb{R}^\omega$ with the box topology is disconnected. Consider

$$A = \text{set of all bounded sequences}, \qquad B = \text{set of all unbounded sequences}.$$

Now, $A \cap B = \emptyset$, $A \cup B = \mathbb{R}^\omega$, $A, B \neq \emptyset$. It can also be shown that A, B are open.

Example. The countable product $\mathbb{R}^\omega$ with the product topology is connected. To show this, define

$$\tilde{R}^n = \{(x_1, \dots, x_n, 0, 0, \dots) : x_i \in \mathbb{R}\}.$$

Then, set $X = \bigcup_{n=1}^{\infty} \tilde{R}^n$, and note that since each $\tilde{R}^n \cong \mathbb{R}^n$ is connected with all of them sharing the common point $(0, 0, \dots)$, X must be connected. We now show that $\overline{X} = \mathbb{R}^\omega$. Indeed, given $x = (x_1, x_2, \dots) \in \mathbb{R}^\omega$, an open neighbourhood of x looks like $U = U_1 \times U_2 \times \dots$, where all but finitely many $I_i = \mathbb{R}$. In other words, there exists sufficiently large $N in \mathbb{N}$ such that for all $n \geq N$, $U_n = \mathbb{R}$. Thus, the point $(x_1, x_2, \dots, x_n, 0, 0, \dots) \in U \cap \tilde{R}^N$.

Lemma 5.8. *The closed intervals $[a, b] \subset \mathbb{R}$ are connected.*

5.1 Path connectedness

Definition 5.3. A topological space X is said to be path connected if there exists a path joining any two points in X. In other words, given $a, b \in X$, there always exists a continuous map $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = a$, $\gamma(1) = b$.

Lemma 5.9. *All path connected spaces are connected.*

Proof. Note that if $X = U \cup V$ is a separation of the path connected space X, then $[0, 1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a separation of the connected interval $[0, 1]$, a contradiction. □

Lemma 5.10. *The image of a path connected space under a continuous map is path connected.*

Example. The unit sphere S^{n-1} is path connected. Note that the map

$$f\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}, \qquad x \mapsto x/\|x\|$$

is continuous and surjective. Thus, it maps the path connected set $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}, which must be path connected.

Example. The set $\overline{S}$, called the topologist's sine curve, is connected but not path connected.

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

Note that S is the continuous image of the connected interval $(0, 1]$, hence connected. This further shows that $\overline{S}$ is connected. Now,

$$\overline{S} = S \cup \{(0, y) : -1 \leq y \leq 1\}.$$

However, $\overline{S}$ is not path connected, since there exists no path joining $(0, 0)$ and $(1/\pi, 0)$. Indeed, given any path $\gamma\colon [0, 1] \to \overline{S}$ starting at $(0, 0)$, it cannot escape $\{0\} \times [-1, 1]$. To see this, write $\gamma = (\gamma_1, \gamma_2)$, $\gamma_2(0) = 0$. By continuity of γ_2, we can choose $\delta > 0$ such that $|\gamma_2(t)| < 1/2$ for all $0 \leq t \leq \delta$. Suppose that $\gamma_1(t^*) = \tau > 0$ for some $0 \leq t \leq \delta$. By the intermediate value theorem, γ_1 takes all the values between 0 and τ in the interval $[0, t^*]$. Choose N such that $2/\pi(2N + 1) < \tau$. Again, there must exist some $0 < t_0 < t^*$ such that $\gamma_1(t_0) = 2/\pi(2n + 1)$. Now, $\gamma_2(t_0) = \sin(1/\gamma_1(t_0)) = 1 > 1/2$, a contradiction. This means that $\gamma_1(t) = 0$ for all $t \in [0, \delta]$.

6 Quotient topology

Definition 6.1. Let X be a topological space, and let $\sim$ be an equivalence relation on X. Then $X/\!\sim$ denotes the set of all equivalence classes with respect to $\sim$. Its elements are of the form $[x] = \{y \in X : x \sim y\}$, for $x \in X$. Define the map

$$\pi\colon X \to X/\!\sim, \qquad x \mapsto [x].$$

The quotient topology on $X/\sim$ is the finest topology such that π is continuous. In other words, $U \subseteq X/\!\sim$ is open if $\pi^{-1}(U)$ is open in X.

Lemma 6.1. *Let $f\colon X \to Y$ be a continuous surjection, with X compact and Y Hausdorff. Define an equivalence relation $\sim$ on X such that $x \sim x' \Leftrightarrow f(x) = f(x')$. Then, $g\colon X/\!\sim \to Y$, $[x] \mapsto f(x)$ is a homeomorphism.*

Example. Consider the interval $[0, 1]$, with the equivalence relation $\sim$ which identifies $0 \sim 1$, and leaves all other points undisturbed. Then, the quotient space $[0, 1]/\!\sim$ is homeomorphic to the circle S^1.

Note that the quotient map on $[0, 1]$ is not open, since the image of the open set $[0, 1/2)$ is not open in $[0, 1]/\!\sim$.

Example. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$, and define an equivalence relation on X which identifies points on the same line through the origin together. Then, the resulting quotient space is called the real projective space, denoted $\mathbb{R}\mathrm{P}^n$.

Example. Let S^n denote the n-sphere in $\mathbb{R}^{n+1}$, and define an equivalence relation on S^n which identifies antipodal points. Then, the resulting quotient space is also $\mathbb{R}\mathrm{P}^n$. The quotient map here is an open map.

Lemma 6.2. *Let $f\colon X \to Y$ be an open, continuous, surjective map. Define an equivalence relation $\sim$ on X such that $x \sim x' \Leftrightarrow f(x) = f(x')$. Then, $g\colon X/\sim \to Y$, $[x] \mapsto f(x)$ is a homeomorphism.*

Example. By defining f as a composition of maps $\mathbb{R}^n \setminus \{0\} \to S^n \to S^n/\sim$, it can be shown that $\mathbb{R}\mathrm{P}^n$ is compact.

6.1 One-point compactification

Definition 6.2. Let X be a compact topological space, and let $A \subset X$ be closed. The one-point compactification of $X \setminus A$ is defined by

$$Y = (X \setminus A) \cup \{\infty\},$$

with the topology

$$\tau_Y = \{U \subseteq X \setminus A \text{ is open}\} \cup \{Y \setminus C \text{ where } C \text{ is compact in } X \setminus A\}.$$

Lemma 6.3. *If X is compact, Hausdorff, then so is the one-point compactification Y of $X \setminus A$.*

Lemma 6.4. *Let X be a compact, Hausdorff space and $A \subseteq X$ be a closed set. Define $\sim$ on X by identifying $x \sim x'$ whenever $x, x' \in A$ and leaving the remaining points undisturbed. Then, $X/\sim$ is homeomorphic to the one-point compactification $Y = X \setminus A \cup \{\infty\}$.*

7 Countability and separation axioms

7.1 First countability

Definition 7.1. Let X be a topological space. A countable basis at a point $x \in X$ is a countable collection β of neighbourhoods of x such that for any neighbourhood U of x, there is a basis element $B \in \beta$ such that $x \in B \subseteq U$.

Definition 7.2. A topological space X in which every element $x \in X$ admits a countable basis is called a first countable space.

Example. All metrizable spaces are first countable. Given an element x, the collection of all open balls centred at x with rational radii forms a countable basis.

Example. The space $\mathbb{R}_\ell$ is a non-metrizable space which is first countable.

Lemma 7.1. *The sequence lemma holds for first countable spaces, i.e. if X is first countable, $A \subseteq X$, and $x \in \overline{A}$, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in A$, such that $x_n \to x$.*

Example. The space $\mathbb{R}^\omega$ with the box topology is not first countable.

7.2 Second countability

Definition 7.3. A topological space X which admits a countable basis is called a second countable space.

Example. The Euclidean spaces $\mathbb{R}^n$ are second countable. The collection of all open balls with rational radii, centred at rational points, forms a countable basis.

Example. The space $\mathbb{R}^\omega$ with the product topology is second countable.

Lemma 7.2. *If a topological space X is second countable, then any discrete subspace $A \subseteq X$ must be countable.*

Proof. Let $\beta = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis of X. For each $a \in A$, note that $\{a\}$ is open in subspace topology A, hence there exists a basis element $B_a \in \beta$ such that $B_a \cap A = \{a\}$. Furthermore, this assignment $A \to \beta$, $a \mapsto B_a$ is injective, hence A must be countable. □

Example. The space $\mathbb{R}^\omega$ with the uniform topology is metrizable hence first countable, but not second countable. This topology is induced by the metric

$$\rho(x, y) = \sup_{i\in\mathbb{N}} \bar{d}(x_i, y_i) = \sup_{i\in\mathbb{N}} \min\{|x_i - y_i|, 1\}.$$

Consider the subspace $A \subset \mathbb{R}^\omega$, consisting of all binary sequences. This is clearly an uncountable set. However, for any two distinct members $x, y \in A$, we have $d(x, y) = 1$. This precisely describes the discrete topology on A. The contrapositive of the above lemma now shows that $\mathbb{R}^\omega$ with the uniform topology cannot be second countable.

Lemma 7.3. *Let X be a second countable space. Then, every open cover of X admits a countable subcover.*

Remark. A topological space in which every open cover admits a countable subcover is called a Lindelöf space.

Example. The space $\mathbb{R}_\ell$ is non-metrizable, first countable, and Lindelöf, but not second countable. To see the latter, let β be a basis of $\mathbb{R}_\ell$. Note that every $x \in [x, x+1) \subset \mathbb{R}_\ell$, hence there must exist $B_x \in \beta$, $x \in B_x \subseteq [x, x+1)$. Now, the assignment $x \mapsto B_x$ is injective, hence β must be uncountable.

Lemma 7.4. *Let X be a second countable space. Then, there exists a countable subset which is dense in X.*

Remark. A topological space in which there exists a dense countable subset is called a separable space.

Proof. Let $\{B_n\}_{n\in X}$ be a countable basis of X. Pick one element $x_n \in B_n$ for each $n \in \mathbb{N}$, whence the set $\{x_n\}_{n\in\mathbb{N}}$ is countable and dense in X. $\square$

Lemma 7.5. *Subspaces, countable products of first/second countable spaces are first/second countable.*

Example. The space $\mathbb{R}_\ell^2$, called the Sorgenfrey plane, is not Lindelöf, even though $\mathbb{R}_\ell$ is. This can be shown by considering the line $L = \{(x, -x) : x \in \mathbb{R}\}$. Note that $L \subset \mathbb{R}_\ell^2$ is closed, hence $\mathbb{R}_\ell^2 \setminus L$ is open. Start with this, and add the sets $[x, x+1) \times [x, x+1)$ to our collection. This is an open cover of $\mathbb{R}_\ell^2$ which admits no countable subcover.

7.3 Separation axioms

Definition 7.4. A topological space X in which any two distinct points $x, y \in X$ admit open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$, is called a Hausdorff space.

Definition 7.5. A topological space X in which any point $x \in X$ and a closed set $F \subseteq X$ (not containing x) can be separated is called a regular space.

Definition 7.6. A topological space X in which any point two disjoint closed sets $F, F' \subseteq X$ can be separated is called a normal space.

Lemma 7.6. *Consider topological spaces in which singleton sets are closed. Then, all such normal spaces are regular, and all such regular spaces are Hausdorff.*