MA2201: Analysis II

Sequences of functions

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Definition 1.1 (Sequences of functions)**.** Let the functions $f_n \colon X \to Y$ be defined for all $n \in \mathbb{N}$ and let the sequences $\{f_n(x)\}$ converge for all $x \in X$. Define the function $f \colon X \to Y$ as

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in X$. We call f the limit of $\{f_n\}$, or say that $\{f_n\}$ converges to f pointwise on X.

Example. Consider the functions $f_n \colon [0,1] \to \mathbb{R}$, $x \mapsto x^n$. It can be shown that $x^n \to 0$ when $x \in [0,1)$ and $x^n \to 1$ when $x = 1$. Thus, $f = \lim_{n\to\infty} f_n$ is well defined.

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1 \end{cases}.$$

Note that while each f_n is continuous in this example, the limit f is not.

Example. Consider the functions $f_n \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x/n$. We see that $f_n \to 0$. Note that 0 here denotes the zero function.

Definition 1.2 (Series of functions)**.** Let the functions $f_n \colon X \to Y$ be defined for all $n \in \mathbb{N}$ and let the series $\sum f_n(x)$ converge for all $x \in X$. Define the function $f \colon X \to Y$ as

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in X$. We call f the sum of the series $\sum f_n$.

Example. Consider the functions $f_n\colon (0,1) \to \mathbb{R}$, $x \mapsto x^n$. Note that the sum

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

does indeed converge for all $x \in (0,1)$. Thus, the sum $f = \sum f_n$ is well defined.

$$f(x) = \frac{x}{1-x}.$$

1 Uniform convergence

Definition 1.3 (Uniform convergence)**.** Let the functions $f_n\colon X \to Y$ be defined for all $n \in \mathbb{N}$. We say that the sequence $\{f_n\}$ converges uniformly on X to f if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in X$, we have

$$|f_n(x) - f(x)| < \epsilon.$$

Remark. Note that for convergence $f_n \to f$, we need only find N depending on ϵ and x. Uniform convergence requires N depending on ϵ which ensures the inequality for *all* $x \in X$.

Lemma 1.1. *The sequence of functions $\{f_n\}$ does not converge uniformly on X to its pointwise limit f if there exists some $\epsilon_0 > 0$, some subsequence $\{f_{n_k}\}$ and some sequence $\{x_k\}$ in X such that for all $k \in \mathbb{N}$,*

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

Example. The sequence of functions $\{f_n\}$ where $f_n\colon [0,1] \to \mathbb{R}$, $x \mapsto x^n$ does not converge uniformly on $[0,1]$. We have already described $f = \lim_{n\to\infty} f_n$. Set $\epsilon_0 = 1/2$, $x_k = (1/2)^{1/k}$ and $n_k = k$. Thus,

$$|f_{n_k}(x_k) - f(x_k)| = \frac{1}{2} \geq \epsilon_0.$$

Note that $x_k \to 1$, which is the point of discontinuity of f.

Example. The sequence of functions $\{f_n\}$ where $f_n\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x/n$ does not converge uniformly on $\mathbb{R}$. Recall that $f_n \to 0$, but when $\epsilon_0 = 1$, $n_k = x_k = k$, we have

$$|f_{n_k}(x_k) - f(x_k)| = 1 \geq \epsilon_0.$$

Theorem 1.2 (Cauchy criterion for uniform convergence)**.** *The sequence of functions $\{f_n\}$ converges uniformly on X if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and $x \in X$, we have*

$$|f_n(x) - f_m(x)| < \epsilon.$$

Proof. First suppose that $\{f_n\}$ converges uniformly on X, and $f_n \to f$. This means that given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all $n \geq N$, $x \in X$. Thus, for all $m, n \geq N$ and $x \in X$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

Now suppose that the Cauchy criterion holds. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and $x \in X$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Recall that the Cauchy criterion for real sequences guarantees that the sequence $\{f_n(x)\}$ converges, thus the function $f = \lim_{n\to\infty} f_n$ is well defined. To show that the convergence of $f_n \to f$ is uniform, fix n and let $m \to \infty$, so $f_m(x) \to f(x)$. Thus for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| < \epsilon,$$

as desired. □

Theorem 1.3. *Let $f_n \colon X \to Y$ and let $f_n \to f$. Set*

$$M_n = \sup |f_n(x) - f(x)|.$$

Then, $\{f_n\}$ converges uniformly on X to f if and only if $M_n \to 0$.

Proof. Suppose that $f_n \to f$ uniformly on X. Let $\epsilon > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

This means that for all $n \geq N$,

$$M_n = \sup |f_n(x_n) - f(x_n)| \leq \frac{\epsilon}{2} < \epsilon.$$

Also note that all $M_n \geq 0$, since they are the supremums of non-negative quantities. This means that $M_n \to 0$, as desired.

Now suppose that $M_n \to 0$. This means that for arbitrary $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|M_n| = \sup |f_n(x) - f(x)| < \epsilon.$$

Now, from the properties of the supremum, we see that for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| \leq \sup |f_n(x) - f(x)| < \epsilon.$$

This proves that $f_n \to f$ uniformly. □

Example. Consider $f_n \colon [0, 1/2] \to \mathbb{R}$, $x \mapsto x^n$. We see that $f_n \to 0$, and that

$$M_n = \sup |f_n(x) - f(x)| = \frac{1}{2^n} \to 0.$$

Thus, $\{f_n\}$ converges uniformly on $[0, 1/2]$ to 0.

Theorem 1.4 (Weierstrass M-test)**.** *Let $f_n\colon X \to Y$ and suppose that for all $n \in \mathbb{N}$ and $x \in X$,*

$$|f_n(x)| \leq M_n.$$

Then the series $\sum f_n$ converges uniformly on X if $\sum M_n$ converges.

Proof. Let $\epsilon > 0$. Since $\sum M_n$ converges, we can use the Cauchy criterion for the convergence of real series to choose $N \in \mathbb{N}$ such that for all $m \geq n \geq N$,

$$\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} M_k \leq \epsilon$$

for all $x \in X$. Note that the left hand side is simply $|s_m(x) - s_{n-1}(x)|$ where $s_k(x)$ is the k^{th} partial sum of the series $\sum f_n(x)$. Thus, the Cauchy criterion gives the uniform convergence of $\{s_n\}$, hence the uniform convergence of the series $\sum f_n$. $\square$

Remark. The converse is not true. Simply setting $f_n = 0$, we observe that the series $\sum f_n$ converges uniformly on $\mathbb{R}$ to 0. On the other hand, $|f_n(x)| \leq 1$ for all $x \in \mathbb{R}$, and the series $\sum 1$ diverges to ∞.