MA3101 : Introduction to Graph Theory and Combinatorics

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Exercise 1 Prove that the number of edge disjoint Hamiltonian cycles in K_{2n+1} for $n \geq 3$ is n.

Solution It is clear that there are at most n edge disjoint Hamiltonian cycles in K_{2n+1}. To see this, pick an arbitrary vertex x and note that there are $2n$ edges emanating from it. Each Hamiltonian cycle contains x, and hence exhausts two of these edges (one entering, one exiting). This means that there are at most n edge disjoint Hamiltonian cycles; if there were more, that would necessitate more than $2n$ distinct edges coming out of x.

Next, we show that there are at least n edge disjoint Hamiltonian cycles.

Note that if $2n + 1 = p$ is prime, then the n edge disjoint Hamiltonian cycles are precisely the sequences

$$x_0, x_d, x_{2d}, \dots, x_{(p-1)d}$$

where the indices are taken modulo p, and $1 \leq d \leq n$. This is easily seen from the fact that every non-identity element of the cyclic group $\mathbb{Z}_p$ generates it[1]; thus, the arithmetic progression dk modulo p cycles through every element of $\mathbb{Z}_p$. Any two such cycles are edge disjoint, since the (absolute) difference modulo p between the indices of any two consecutive elements in such a cycle is unique to that cycle[2]. Note that $n + 1 \leq d \leq 2n + 1$ generate exactly the same cycles but in reverse, since $-d \equiv (2n + 1) - d \pmod{p}$.

Exercise 2 Find the number of distinct Hamiltonian cycles in K_n containing a particular edge.

Solution Note that any permutation of the vertices of K_n gives a Hamiltonian cycle, since there is an edge available between any two vertices. Thus, there are $(n - 1)!/2$ Hamiltonian cycles in total (we divide by $2n$ to account for the 'dihedral symmetry', i.e. given a permutation of vertices, we can cycle the vertices in n ways, or reverse the direction of each of these, without getting a new Hamiltonian cycle[3]).

Let $n \geq 3$. Fix the edge $\{x_1, x_2\}$, and declare x_1, x_2 to be the start of our Hamiltonian cycle. There are $n - 2$ choices for the next vertex, $n - 3$ for the fourth, and so on. Thus, the remaining vertices can be permuted in $(n - 2)!$ ways giving that many cycles. Note that any two cycles obtained in this way are distinct – neither sequence of vertices can be rotated nor reflected onto the other. Also, these sequences form a complete list of desired cycles – every Hamiltonian cycle containing $\{x_1, x_2\}$ can be laid out as a sequence of vertices, with x_1 leading and x_2 in the second place. Thus, there are precisely $(n - 2)!$ Hamiltonian cycles containing a particular edge.

Exercise 3 Show that if n is odd, it is not possible for a knight to visit all the squares of an $n \times n$ chessboard exactly once by knight's moves and return to its starting point.

[1] Every non-identity element generates a non-trivial subgroup of $\mathbb{Z}_p$ whose order must divide p; this can only happen if the order is precisely p, i.e the element generated the entire group $\mathbb{Z}_p$.

[2] Suppose that the edge $\{x_i, x_j\}$ appears in two such cycles, generated by d_1 and d_2. Then, we know that $j - i \equiv \pm d_1 \pmod{p}$ and $j - i \equiv \pm d_2 \pmod{p}$ by construction. Thus, the quantity $|j - i| \pmod{p}$ must be exactly one of d_1 or $p - d_1$, and exactly one of d_2 or $p - d_2$. If this happens to be d_1, then $p - d_2 > n \geq d_1$, so $d_1 = d_2$. Again if this happens to be $p - d_1$, then $p - d_1 > n \geq d_2$, so $p - d_1 = p - d_2$ or $d_1 = d_2$. This shows that this edge belongs to only one of these cycles.

[3] Formally, we are looking at the dihedral group D_{2n} acting on the set of all permuted strings of vertices, and counting the number of orbits. Another way to derive this is to use the Orbit Stabilizer theorem; let X be the set of all Hamiltonian cycles of K_n, and let S_n act on it in the natural way, permuting the order of the vertices in a given cycle . To make this operation well-defined, we need to decide upon some 'canonical' representation of a Hamiltonian cycle; label the vertices $1, \dots, n$ beforehand, have the representation of each cycle start with 1, and have the lower-labelled neighbour of 1 follow. For instance, the cycle $3, 1, 4, 2$ is canonically written as $1, 3, 2, 4$; now, the permutation of these labels is well-defined. Note that any Hamiltonian cycle x can be mapped to another simply by permuting the order of the vertices, hence the orbit of x is the entirety of X. Cycling the order of the vertices in a Hamiltonian cycle, or reversing their order, gives back the same cycle; in addition, these are the only operations which do so. Thus, the stabilizer of a cycle x is the dihedral group D_{2n}. Thus, $|S_n| = |X| \cdot |D_{2n}|$, or $|X| = n!/2n = (n - 1)!/2$

Solution We claim that for an odd chessboard, there exists no *Knight's Tour*, i.e. a Hamiltonian cycle in the chessboard graph, whose vertices are the squares of the chessboard, two of them connected if and only if they are a knight's move away.

It is clear that on an $n \times n$ chessboard where n is odd, there are $(n^2-1)/2$ and $(n^2+1)/2$ squares of each colour, say black and white without loss of generality (put white on the bottom right and count). This induces a natural colouring of the vertices of our graph. Now, a knight on a chessboard always alternates colours when it moves on a chessboard (something very familiar to chess players!); a knight on a white square can only reach black squares, and vice versa. This immediately shows that our chessboard graph is bipartite, with black vertices in one part and white vertices in the other. As a result, the chessboard graph contains no odd cycles[4], hence no Hamiltonian cycle of length n^2 (which is odd).

Exercise 4 Let G be a Hamiltonian graph and let S be any set of k vertices in G. Prove that the graph $G - S$ has at most k components.

Solution Let

$$x_1, x_2, \ldots, x_{n-1}, x_n \,(, x_1)$$

be a Hamiltonian cycle in G where $x_1, \ldots, x_n$ are its vertices. This means that there are edges $\{x_i, x_{i+1}\}$ for $1 \leq i < n$ and the edge $\{x_n, x_1\}$. Now, suppose that $S = \{x_{i_1}, \ldots, x_{i_k}\}$, where the indices i_j are in ascending order. Then for some index $1 \leq j < k$ note that the path

$$x_{i_j+1}, x_{i_j+2}, \ldots, x_{i_{j+1}-1}$$

is contained in $G - S$ (we have simply sampled from the original Hamiltonian cycle; note that none of the edges in this path have been removed). In addition, the path

$$x_{i_k+1}, \ldots, x_n, x_1, \ldots, x_{i_1-1}$$

is also in $G - S$. Thus, we have found at most k paths (some of these paths may be empty, if two indices i_j are consecutive) in $G - S$, such that every vertex from $G - S$ is part of one of these paths. This proves that there can be at most k components in $G - S$.

Exercise 5 Prove that the complement of a d regular graph of order $2d + 2$ for $d \geq 1$ is Hamiltonian.

Solution Suppose that G is a d regular graph on $n = 2d + 2$ vertices. Thus, each vertex is connected to d other vertices. This means that in the complement graph G', that vertex is connected to precisely the other $(n - 1) - d = d + 1$ vertices. The sum of degrees of any two vertices in G' is thus $2(d + 1) = n$. Thus, G' is Hamiltonian by Ore's Theorem.

Exercise 6 Show that if G is a simple graph with minimum vertex degree k, then there exists a path of length k. Moreover, if $k \geq 2$, then there exists a cycle of length at least $k + 1$.

Solution Let G have n vertices; clearly, $n \geq k + 1$ since each vertex has at least k neighbours. Now, pick an arbitrary vertex and call it x_1. This has at least k neighbours; pick one of them and call it x_2. Now, x_2 has at least k neighbours; at least $k - 2$ of them which are neither x_1 nor x_2. Pick one of them and call it x_3. In this way, at each stage where we have a path $x_1, \ldots, x_j$ of length $j - 1$, where $1 < j \leq k$, the last vertex x_j has at least k neighbours, out of which at most $j - 1$ of them $(x_1, \ldots, x_{j-1})$ of them may have been exhausted. This leaves at least $k - (j - 1) \geq 1$ new vertices to choose from; pick one and call it x_{j+1}, thus growing the path. We can keep doing thus until we have obtained a path $x_1, \ldots, x_k, x_{k+1}$ of length k, as desired.

Now, suppose that $k \geq 2$. Obtain the path $x_1, \ldots, x_{k+1}$ as before, and continue applying our algorithm. Note that our algorithm must terminate, since there are finitely many vertices in G. There are precisely two ways in which this can happen.

[4]Suppose that $x_1, \ldots, x_{2k+1}$ is an odd cycle in a bipartite graph, with parts A and B. Then, each edge joins a vertex from A to one of B. Without loss of generality, let $x_1 \in A$, hence $x_2 \in B$, $x_3 \in A$, $\ldots$, $x_{2k+1} \in A$ going forwards. This forces the edge $\{x_1, x_{2k+1}\}$, contradicting the independence of the set of vertices A.

Case I: We have run out of vertices, i.e. we have found a path $x_1, \dots, x_n$. Then, x_1 has at least k neighbours; call (some of) them $x_{i_1}, \dots, x_{i_k}$ where the indices i_j are in ascending order, i.e. $2 \leq i_1 < i_2 < \cdots < i_k$. This makes it clear that $i_k \geq k+1$, hence we have found a cycle $x_1, x_2, \dots, x_{i_k}\,(, x_1)$ of length $i_k \geq k+1$.

Case II: We have run out of neighbours, i.e. we have found a path $x_1, \dots, x_m$, $m \geq k+1$, where x_m has no neighbours apart from $x_1, \dots, x_{m-1}$. Like before, call (some of) these neighbours $x_{i_1}, \dots, x_{i_k}$ where the indices are in descending order, i.e. $m-1 \geq i_1 > i_2 > \cdots > i_k$. This makes it clear that $i_k \leq m-k$, hence we have found a cycle $x_{i_k}, x_{i_k+1}, \dots, x_m\,(, x_{i_k})$ of length $m - i_k + 1 \geq k+1$.

Lemma 1. *Every tree contains at least one leaf.*

Proof. If the degree of every vertex in a tree on n vertices is at least 2, then the previous exercise guarantees the existence of a cycle of length at least 3, violating the acylicity of the tree. □

Lemma 2. *A tree on n vertices has exactly $n-1$ edges.*

Proof. We use induction on n. This is clearly true for $n = 1, 2$. Suppose that this holds for all trees with at most n vertices, and let T be a tree on $n+1$ vertices. Then, choose a leaf x of T, and remove it to obtain T'. Note that T' is still a tree, since we have removed only one edge. T' is also still connected since any path between two vertices $u, v \in V(T)$, $u, v \neq x$ is also a valid path in T'; no such path could have included x since there is only one edge coming out of x. Thus, T' has $n-2$ edges, hence T must have $n-1$ edges. □

Lemma 3. *Every connected graph has a spanning tree.*

Proof. Let G be a connected graph on n vertices. Then, G can have finitely many cycles; there are finitely many choices for the k vertices in a k-cycle, and there are finitely many cycle lengths (from 3 to n). Now, suppose that G contains a cycle $x_1, \dots, x_k\,(, x_1)$. Then, we can remove the edge $\{x_1, x_k\}$ and see that the new graph G' is still connected. In this way, we can refine our graph, removing cycles at each step; this process must terminate since the number of cycles drops at each step. At the end, we are left with an acyclic, connected graph on all n vertices, i.e. a spanning tree. □

Exercise 7 Let G be a graph with n vertices, m edges, and k components. Show that

$$n - k \leq m \leq \frac{1}{2}(n-k)(n-k+1).$$

Solution Let the components of G have vertex sets $V_1, \dots, V_k$, and let each component $G[V_i]$ contain n_i vertices, m_i edges. First, we claim that $n_i - 1 \leq m_i$; this is because each connected component with n_i vertices contains a spanning tree with $n_i - 1$ edges, hence the component has at least $n_i - 1$ edges. Summing this over all k components gives

$$m = \sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} n_i - 1 = n - k.$$

Note that the first equality is justified since there are no edges between components.

Now, we claim that $m_i \leq \binom{n_i}{2} = n_i(n_i - 1)/2$; this is because we can have one edge between any two vertices in a connected component, and no more. Thus,

$$m = \sum_{i=1}^{k} m_i \leq \frac{1}{2} \sum_{i=1}^{k} n_i^2 - n_i.$$

We examine the expression

$$x^2 - x + (N-x)^2 - (N-x) = 2x^2 - 2Nx + N^2 - N$$

where $N \geq 3$ is an integer. It is clear that this quadratic represents a parabola open upwards, with its vertex at $2N/4 = N/2$. Thus, this expression is decreasing on $[1, N/2]$ and increasing on $[N/2, N-1]$

(this can also be verified by differentiation), which means that it attains its maximum (on the interval $[1, N-1]$) at either $x = 1, N-1$. Indeed, both substitutions give

$$x^2 - x + (N-x)^2 - (N-x) \leq (N-1)^2 - (N-1).$$

Returning to our original sum, apply the above repeatedly, peeling off terms one at a time to get

$$\begin{aligned}
\sum_{i=1}^{k} n_i^2 - n_i &= n_1^2 - n_1 + \sum_{i=2}^{k} n_i^2 - n_i \\
&\leq (n_1 + n_2 - 1)^2 - (n_1 + n_2 - 1) + \sum_{i=3}^{k} n_i^2 - n_i \\
&\vdots \\
&\leq (n_1 + n_2 + \cdots + n_k - k + 1)^2 - (n_1 + n_2 + \cdots + n_k - k + 1) \\
&= (n - k + 1)[(n - k + 1) - 1] \\
&= (n - k + 1)(n - k).
\end{aligned}$$

Thus,

$$m \leq \frac{1}{2} \sum_{i=1}^{k} n_i^2 - n_i \leq \frac{1}{2}(n-k)(n-k+1).$$

Exercise 8 Prove that $\operatorname{girth}(G) \leq 2\operatorname{diam}(G) + 1$. Moreover show that if $\operatorname{girth}(G) = 2\operatorname{diam}(G) + 1$, then the length of each cycle in that graph is $2\operatorname{diam}(G) + 1$.

Solution If G contains no cycle, we declare this result to be vacuously true.

Otherwise, let $g = \operatorname{girth}(G)$, $d = \operatorname{diam}(G)$, and suppose that $x_1, \ldots, x_k\,(, x_1)$ is a smallest cycle of length g. If $g \geq 2d+2$, that means that the vertices x_1 and x_{d+2} have distance at least $d+1$. To see this, note that the path $x_1, \ldots, x_{d+2}$ has length $d+1$; if there is another, shorter path, it must be of the form $x_1, y_2, \ldots, y_s, x_{d+2}$ where each y_i may or may not be one of the x_i. By joining these paths (travel from x_1 to x_{d+2} along one, and back to x_1 along the other), we obtain at least one cycle of length strictly less than $(d+1) + (d+1) = 2d+2$, i.e. less than the girth which is a contradiction. As a result, x_1 and x_{d+2} have distance at least $d+1$, which is greater than the diameter, a contradiction. This forces $g \leq 2d+1$.

Exercise 9 Prove that for any graph G, $\operatorname{diam}(G) \leq 2 \cdot \mathrm{r}(G)$.

Solution Denote the radius and diameter of G by r and d respectively. This means that there exists a vertex v with eccentricity r. Now, let x, y be two arbitrary vertices in G. Then, the distances $d(x, v) \leq r$ and $d(y, v) \leq r$. Thus, we have a sequence of vertices from x to v to y, which can be trimmed down (by removing redundant loops/repetitions) to a path of length at most $2r$. Thus, every vertex x has eccentricity at most $2r$, which means that the diameter $d \leq 2r$.

This argument works perfectly when G has only one component: otherwise, we use the convention that the distance between disconnected vertices is infinite.

Exercise 10 Let $S = \{1, 2, 3, 4, 5, 6\}$ and $G = (V, E)$ be a graph such that V is the collection of all 2 element subsets of S. Two vertices $u, v \in V$ are adjacent in G if and only if u and v are disjoint sets.

(a) Find the value of $|E|$.

(b) Prove that every pair of non-adjacent vertices in G has exactly 3 common neighbours.

(c) Find the diameter of G.

(d) Find the number of triangles in G.

Solution Note that the number of vertices in G is

$$|V| = \binom{6}{2} = 15.$$

(a) Pick an arbitrary vertex $v = \{s_1, s_2\}$, and let the remaining elements of S be s_3, s_4, s_5, s_6. Then, v is adjacent to a vertex $\{s_i, s_j\}$ if and only if none of s_i, s_j is s_1, s_2, i.e. $3 \le i, j \le 6$. This shows that v is adjacent to $\binom{4}{2} = 6$ other vertices, hence every vertex in G has degree 6. Thus,

$$|E| = \frac{1}{2}\sum_{v \in V} d(v) = \frac{1}{2} \cdot 15 \cdot 6 = 45.$$

(b) Let $u = \{s_1, s_2\}$ and $v = \{s_2, s_3\}$ be non-adjacent vertices in G; clearly, any two such vertices must be of this form since they must share exactly one element. This leaves s_4, s_5, s_6 from S, which means that the common neighbours of u, v are those comprised of these numbers. This gives $\binom{3}{2} = 3$ common neighbours.

(c) Let $u = \{s_1, s_2\}$ and $v = \{s_2, s_3\}$ be non-adjacent vertices. Then, $u \sim \{s_4, s_5\} \sim v$, hence the maximum distance between any two vertices is 2. This shows that the diameter of G is precisely 2.

(d) Let $\{u, v\} \in E$, hence $u \sim v$ so we must have the form $u = \{s_1, s_2\}$ and $v = \{s_3, s_4\}$. If x is a common neighbour of u and v, then it is disjoint from both u and v, which only leaves one choice $x = \{s_5, s_6\}$. Thus, every edge contributes exactly one triangle, and every triangle is comprised of three edges. The total number of triangles in G is thus $|E|/3 = 15$.

Exercise 11 Prove or disprove that an n-cube Q_n is bipartite for $n \ge 2$.

Solution This is indeed true. Let the vertices of Q_n be labelled by the binary strings $a_1 a_2 \ldots a_n$ where each $a_i \in \{0, 1\}$; two vertices are adjacent if and only if their binary strings differ in exactly one place. Now, let A be the set of vertices with an even number of 1's in their binary string, and B be the set of vertices with an odd number of 1's in their binary string. We claim that A is an independent set; this is clear because any two vertices in A have an even number of 1's in their binary string, and hence differ in some even number of places (if a binary string has k 1's, then changing one bit can only change the number of 1's by one, i.e. flip the parity of k). Similarly, B is an independent set, since any two vertices in B have an odd number of 1's in their binary string, and again differ in an even number of places. The sets A and B exhaust all vertices in Q_n, hence Q_n is bipartite.

Exercise 12 Prove that all 3-regular Hamiltonian graph on 10 vertices have girth less than 5.

Solution Suppose note, i.e. let G be a 3-regular Hamiltonian graph on 10 vertices with girth at least 5. Let $x_1, x_2, \ldots, x_{10}\,(, x_1)$ be a Hamiltonian cycle: thus, each x_i is adjacent to x_{i-1} and x_{i+1}, along with another vertex (we are using cyclic indices, so $x_{11} \equiv x_1$, etc). For instance, take x_1: if $x_1 \sim x_i$ for any $i = 3, 4, 8, 9$, then we have found a cycle of length at most 4 (namely, $x_1, x_2, x_3\,(, x_1)$, or $x_1, x_2, x_3, x_4\,(, x_1)$, or $x_1, x_{10}, x_9, x_8\,(, x_1)$, or $x_1, x_{10}, x_9\,(, x_1)$ respectively). Thus, we must have $x_1 \sim$ one of x_5, x_6, x_7. Similarly, each $x_i \sim$ one of $x_{i+4}, x_{i+5}, x_{i+6}$.

Case I $x_1 \sim x_5$: Then, $x_6 \sim$ one of x_{10}, x_1, x_2; discard x_1 since every vertex has degree 3, discard x_{10} since $x_1, x_5, x_6, x_{10}\,(, x_1)$ is a 4-cycle, discard x_2 since $x_1, x_5, x_6, x_2\,(, x_1)$ is a 4-cycle.

Case II $x_1 \sim x_7$: Again, $x_6 \sim$ one of x_{10}, x_1, x_2; discard x_1 since every vertex has degree 3, discard x_{10} since $x_1, x_7, x_6, x_{10}\,(, x_1)$ is a 4-cycle, discard x_2 since $x_1, x_7, x_6, x_2\,(, x_1)$ is a 4-cycle.

Case III $x_1 \sim x_6$: Then, $x_7 \sim$ one of x_1, x_2, x_3; discard x_1 since every vertex has degree 3, discard x_2 since $x_1, x_6, x_7, x_2\,(, x_1)$ is a 4-cycle. This forces $x_7 \sim x_3$. Now, $x_2 \sim$ one of x_6, x_7, x_8; we have already used x_6, x_7 so $x_2 \sim x_8$. However, $x_2, x_8, x_7, x_3\,(, x_2)$ is a 4-cycle.

In all cases, we have arrived at a contradiction. Thus, the girth of such a graph must be less than 5.

Exercise 13 Find the number of spanning trees of $K_{2,n}$.

Solution Let $A = \{a_1, a_2\}$ be one part, and let $B = \{b_1, b_2, \dots, b_n\}$ be the other; A and B are independent sets. Consider a spanning tree of this graph. It is clear by inspection that $K_{2,1}$ has one spanning tree (the path a_1, b_1, a_2). Otherwise when $n \geq 2$, we know that a_1 and a_2 must be connected by some path, and the only possibility is $a_1 \sim b_i \sim a_2$ for some i. No other b_j can have degree 2: that would imply that $b_j \sim a_1, a_2$, giving a cycle $a_1, b_i, a_2, b_j\ (, a_1)$. Thus, each of the remaining $n-1$ vertices must be connected to exactly one of a_1 or a_2 (we cannot leave them isolated of course). The result is indeed a spanning tree: every b_i is connected to at least one of a_1, a_2, and these are connected to each other. Additionally, there are no cycles since only a_1, a_2, b_i can have degree greater than 1, and they do not form a triangle. Thus, the number of spanning trees of $K_{2,n}$ is precisely $n \cdot 2^{n-1}$.

Exercise 14 Show that if ℓ is the label of a vertex with degree n, then ℓ occurs $n-1$ times in the Prüfer code.

Solution Suppose that ℓ has neighbours $\ell_1, \dots, \ell_n$ in the original tree. Our algorithm terminates when there is only one edge remaining. This immediately shows that if $n = 1$, then ℓ never appears in the Prüfer code; for it to appear, it would have to be the parent of a (soon to be deleted) leaf, but if its only neighbour ℓ_1 becomes a leaf, then $\{\ell, \ell_1\}$ is the only edge remaining in the tree (there can be no other connected vertices, both ℓ, ℓ_1 are leaves), hence the algorithm terminates. Otherwise, assume $n \geq 2$. Suppose that ℓ survives in the algorithm; thus, the edge $\{\ell, \ell_1\}$ (without loss of generality) survives, so the $n-1$ vertices $\ell_2, \dots, \ell_n$ become deleted. Their parent ℓ thus appears in the code $n-1$ times before termination. Otherwise, suppose that ℓ does not survive. During its deletion, ℓ must have been a leaf, i.e. $n-1$ of its neighbours were removed first, again contributing ℓ to the Prüfer code $n-1$ times. The final neighbour cannot also be deleted, since that would leave ℓ isolated, hence ℓ does not appear any more times. In either case, ℓ appears precisely $n-1$ times in the Prüfer code.

The converse is clearly true, i.e. if a label ℓ appears $n-1$ times in a Prüfer code, then that label ℓ has degree n in the corresponding tree. Note that its appearance $n-1$ times indicates that it had at least $n-1$ neighbours connected to it before being deleted during the construction of the code. At each of these stages, ℓ was still connected to the tree (otherwise, it couldn't be listed as the parent of the lowest labelled leaf) and hence had one more neighbour, raising its original degree to at least n. Finally, its degree cannot have been more than n; after the last ($n-1$ th) appearance of the label ℓ in the code, there are two possible configurations of the tree at that stage of the algorithm. Either ℓ is a leaf, and hence contributes that single extra neighbour; or it was connected to two or more labels, say $\ell_1, \dots, \ell_k$. Note that the algorithm must terminate with at most one of the edges $\{\ell, \ell_i\}$ surviving at the end, which means that the remaining neighbours must be deleted at some point. Indeed, if none of the edges $\{\ell, \ell_i\}$ survive, that means that $k-1 \geq 1$ of the ℓ_i must have be deleted first (i.e. they must become leaves of ℓ), before the deletion of the final ℓ or ℓ_i; this contradicts the fact that ℓ never appears again. Again, if $\{\ell, \ell_i\}$ survives (thus terminating the algorithm), the remaining $k-1 \geq 1$ of the ℓ_i must be deleted, with the same contradiction. This proves that ℓ appears $n-1$ times in the Prüfer code if and only if it has degree n in the corresponding tree.

If each label i appears $d_i - 1$ times in the Prüfer code, the total number of elements in the code is

$$\sum_{i=1}^{n} d_i - 1 = 2|E| - n = 2(n-1) - n = n-2,$$

as expected.

Exercise 15 Show that the number of different labelled trees on n vertices such that the vertex i has degree d_i is

$$\frac{(n-2)!}{\prod (d_i - 1)!}.$$

Solution Note that for a tree,

$$\sum_{i=1}^{n} d_i = 2|E| = 2n-2, \qquad \sum_{i=1}^{n} d_i - 1 = 2|E| - n = n-2.$$

Thus, the given number is a multinomial coefficient,

$$\binom{n-2}{d_1-1, d_2-1, \ldots, d_n-1}.$$

Recall that a vertex of degree d_i appears $d_i - 1$ times in a Prüfer code and vice versa. Also, every possible Prüfer code (sequence of $n-2$ labels) corresponds to exactly one tree. Thus, we need only count the number of sequences of length $n-2$ in which the label i appears $d_i - 1$ times. This is precisely the given multinomial coefficient. To see this, note that we need to choose $d_i - 1$ out of $n-2$ places for each label i to appear in the Prüfer code. Another way is to note that the $n-2$ labels can be arranged in $(n-2)!$ ways; each label i appears $d_i - 1$ times and hence we have over-counted by a factor of $(d_i - 1)!$ for each (note that the $(d_i - 1)!$ ways in which the identical labels i can be permuted are redundant). This means that we divide $(n-2)!$ by each $(d_i - 1)!$, giving the desired formula.