

MA2201: ANALYSIS II

Integration

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Definition 3.1 (Partition). A partition P of an interval $[a, b]$ is a finite sequence of numbers

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The norm of a partition is defined as

$$\|P\| = \max |x_{j+1} - x_j|.$$

Definition 3.2 (Tagged partition). A tagged partition $\dot{P}(x_j, \xi_j)$ is a partition P together with a set of numbers ξ_j such that $\xi_j \in [x_j, x_{j+1}]$.

Definition 3.3 (Riemann sum). The Riemann sum of a function f on an interval $[a, b]$ with respect to a tagged partition \dot{P} is defined as

$$S(f, \dot{P}) = \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j).$$

Definition 3.4 (Riemann integral). A function f is called Riemann integrable on an interval $[a, b]$ if there is some $\ell \in \mathbb{R}$ where for every $\epsilon > 0$, there exists $\delta > 0$ such that all tagged partitions \dot{P} of $[a, b]$ with $\|\dot{P}\| < \delta$ satisfy

$$|S(f, \dot{P}) - \ell| < \epsilon.$$

The number ℓ is the value of the Riemann integral,

$$\int_a^b f = \ell.$$

Theorem 3.1. *If a function is Riemann integrable on an interval, then the value of the integral is unique.*

Proof. Let f be Riemann integrable on $[a, b]$, with integral values ℓ and ℓ' . Then, for every $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$,

$$|S(f, \dot{P}) - \ell| < \frac{\epsilon}{2}, \quad |S(f, \dot{P}) - \ell'| < \frac{\epsilon}{2}.$$

Note that such a partition \dot{P} always exists. Thus,

$$|\ell - \ell'| \leq |S(f, \dot{P}) - \ell| + |S(f, \dot{P}) - \ell'| < \epsilon$$

for all $\epsilon > 0$, which forces $\ell = \ell'$. □

Theorem 3.2. *If f is Riemann integrable on $[a, b]$, then f is bounded on that interval. Furthermore, if $M > 0$ is such that $|f(x)| \leq M$ for all $x \in [a, b]$, then*

$$-M(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Suppose not. Let the Riemann integral of f on $[a, b]$ be ℓ . For $\epsilon = 1$, we find $\delta > 0$ such that for all tagged partitions \dot{P} of $[a, b]$ with $\|\dot{P}\| < \delta$, we have $|S(f, \dot{P}) - \ell| < 1$. This means that

$$S(f, \dot{P}) < |\ell| + 1.$$

Let $Q = \{x_0, \dots, x_n\}$ be such a partition. The unboundedness of f means that we can find a subinterval $[x_k, x_{k+1}]$ where f is unbounded. Now, choose tags ξ_j creating the tagged partition \dot{Q} . We choose the tag $\xi_k \in [x_k, x_{k+1}]$ such that

$$|f(\xi_k)(x_{k+1} - x_k)| > |\ell| + 1 + \left| \sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j) \right|.$$

Now, observe that the triangle inequality demands

$$|S(f, \dot{Q})| \geq |f(\xi_k)(x_{k+1} - x_k)| - \left| \sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j) \right| > |\ell| + 1,$$

which is a contradiction. Thus, f must be bounded on $[a, b]$.

Next, for any tagged partition \dot{P} of $[a, b]$, we have

$$|S(f, \dot{P})| \leq \sum_{j=0}^{n-1} |f(\xi_j)|(x_{j+1} - x_j) \leq M(b-a).$$

Let the Riemann integral of f be ℓ . Thus, for all $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$,

$$||S(f, \dot{P})| - |\ell|| \leq |S(f, \dot{P}) - \ell| < \epsilon.$$

This gives

$$|\ell| < |S(f, \dot{P})| + \epsilon \leq M(b-a) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we may write

$$|\ell| \leq M(b-a). \quad \square$$

Theorem 3.3. *If f is Riemann integrable on $[a, b]$, and \dot{P}_n is any sequence of tagged partitions of $[a, b]$ such that $\|\dot{P}_n\| \rightarrow 0$, then*

$$\int_a^b f = \lim_{n \rightarrow \infty} S(f, \dot{P}_n).$$

Proof. Let $\epsilon > 0$. We find $\delta > 0$ such that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$, we have

$$|S(f, \dot{P}) - \int_a^b f| < \epsilon.$$

Now, since $\|\dot{P}_n\| \rightarrow 0$, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$, $\|\dot{P}_n\| < \delta$. Thus, for all $n \geq N$,

$$|S(f, \dot{P}_n) - \int_a^b f| < \epsilon.$$

In other words,

$$\int_a^b f = \lim_{n \rightarrow \infty} S(f, \dot{P}_n). \quad \square$$

Definition 3.5 (Refinement). A partition \tilde{P} is said to be a refinement of a partition P if $P \subset \tilde{P}$.

Definition 3.6 (Common refinement). Given two partitions P_1 and P_2 of an interval $[a, b]$, we say that \tilde{P} is their common refinement if $P_1 \cup P_2 \subset \tilde{P}$.

Definition 3.7 (Darboux sums). Given a partition P of $[a, b]$ and a bounded function f , define

$$m_j = \inf_{t \in [x_j, x_{j+1}]} f(t), \quad M_j = \sup_{t \in [x_j, x_{j+1}]} f(t).$$

The lower and upper Darboux sums are defined as

$$L(f, P) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j), \quad U(f, P) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j).$$

Lemma 3.4. *If P is a partition of an interval $[a, b]$, then*

$$L(f, P) \leq U(f, P).$$

Proof. This follows directly from the fact that the infimum is less than or equal to the supremum, i.e. $m_j \leq M_j$. \square

Theorem 3.5. *Let \tilde{P} be a refinement of a partition P of an interval $[a, b]$. Then,*

$$L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, P)$$

Proof. Suppose that

$$P = \{x_0, \dots, x_k, x_{k+1}, \dots, x_n\},$$

$$\tilde{P} = \{x_0, \dots, x_k, y, x_{k+1}, \dots, x_n\}.$$

Set

$$m_1 = \inf_{t \in [x_k, y]} f(t), \quad m_2 = \inf_{t \in [y, x_{k+1}]} f(t), \quad m = \inf_{t \in [x_k, x_{k+1}]} f(t).$$

Then, observe that

$$L(f, \tilde{P}) - L(f, P) = m_1(y - x_k) + m_2(x_{k+1} - y) - m(x_{k+1} - x_k).$$

Now, from the properties of the infimum, $m_1 \geq m$ and $m_2 \geq m$, so

$$L(f, \tilde{P}) - L(f, P) \geq m(y - x_k + x_{k+1} - y - x_{k+1} + x_k) = 0.$$

This procedure of adding one point can be repeated finitely many times to obtain the same conclusion for any refinement of P . The case for the upper sum is analogous. \square

Corollary 3.5.1. *For any two partitions P_1 and P_2 of $[a, b]$,*

$$L(f, P_1) \leq U(f, P_2).$$

Proof. Note that $P_1 \cup P_2$ is a common refinement of P_1 and P_2 , hence

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2). \quad \square$$

Corollary 3.5.2. *If $\{P_n\}$ is a sequence of refinements of a partition P_0 of $[a, b]$, then the following limits exist.*

$$L_{f, P_n} = \lim_{n \rightarrow \infty} L(f, P_n), \quad U_{f, P_n} = \lim_{n \rightarrow \infty} U(f, P_n).$$

Proof. This follows from the monotone convergence theorem, together with the fact that $U(f, P_0)$ and $L(f, P_0)$ are upper and lower bounds of the two respective sequences. \square

Corollary 3.5.3. *The following quantities exist, where the infimum and supremum is taken over all possible partitions P of $[a, b]$.*

$$L_f = \sup L(f, P), \quad U_f = \inf U(f, P).$$

Furthermore, for any partition P ,

$$L(f, P) \leq L_f \leq U_f \leq U(f, P).$$

Proof. First examine the set of all lower Darboux sums, $\{L(f, P)\}$. This set is non-empty, since any partition of $[a, b]$ gives a corresponding lower sum. Note that we have already demanded that f is bounded! This set is also bounded above, by any upper sum. Thus, the completeness of the reals guaranteed the existence of a supremum. The case for upper sums is analogous.

The outermost inequalities trivially follow from the definitions of the infimum and supremum. The middle inequality follows from the fact that if A and B are two subsets of \mathbb{R} such that $\alpha \in A, \beta \in B$ implies $\alpha \leq \beta$, then $\sup A \leq \inf B$. \square

Definition 3.8 (Darboux integrals). The lower and upper Darboux integrals of a function f are defined as

$$L_f = \sup L(f, P), \quad U_f = \inf U(f, P).$$

Here, the infimum and supremum is taken over all possible partitions P of $[a, b]$.

If $L_f = U_f$, then the common integral is simply called the Darboux integral,

$$\int_a^b f = L_f = U_f.$$

Such a function f is called Darboux integrable.

Theorem 3.6. A function f is Darboux integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Proof. First, assume that given $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

By the previous corollary,

$$U_f - L_f \leq U(f, P) - L(f, P) < \epsilon$$

for all $\epsilon > 0$, so $U_f = L_f$ giving Darboux integrability.

Now, suppose that f is Darboux integrable on $[a, b]$. This means that $U_f = L_f$. Using the definitions of supremum and infimum, for $\epsilon > 0$, there exists a partition P_1 such that $U(f, P_1) - U_f < \epsilon/2$ and a partition P_2 such that $L_f - L(f, P_2) < \epsilon/2$. Adding,

$$U(f, P_1) - L(f, P_2) < \epsilon.$$

Now, setting $P = P_1 \cup P_2$ as a common refinement of P_1 and P_2 , we have

$$U(f, P) - L(f, P) < U(f, P_1) - L(f, P_2) < \epsilon. \quad \square$$

Lemma 3.7. Let f be bounded on $[a, b]$, and let P' be any partition of that interval. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all partitions P with $\|P\| < \delta$,

$$U(f, P) - U(f, P \cup P') < \epsilon.$$

Proof. Using the boundedness of f , choose $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [a, b]$. Suppose that $P' = \{x_1, \dots, x_n\}$. Set $\delta = \epsilon/4nM$. Now, let P be any partition of $[a, b]$ with $\|P\| < \delta$. Write $P = \{y_i\}$, and $P \cup P' = \{z_i\}$ where all these sets are ordered. Now, note that if one of the subintervals $[z_j, z_{j+1}]$ does not contain a point x_i , then the term $M_j(y_{j+1} - y_j)$ cancels from $U(f, P) - U(f, P \cup P')$. Whenever there is an x_i in $[z_j, z_{j+1}]$, we have a term of the form

$$M_j(y_{j+1} - y_j) - M'(x_i - y_j) - M''(y_{j+1} - x_i),$$

where M', M'' are the supremums over the two pieces, each less than M . This means that this term is bounded by $4M\delta$. Since this can happen at most n times,

$$U(f, P) - U(f, P \cup P') < 4nM\delta = \epsilon. \quad \square$$

Theorem 3.8. *Riemann and Darboux integrability are equivalent and assign the same value to the integrals.*

Proof. First assume that f is Riemann integrable on $[a, b]$. By Theorem 3.2, f is bounded so the Darboux upper and lower sums are well defined. Given $\epsilon > 0$, we seek a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Now, Riemann integrability guarantees the existence of a $\delta > 0$ such that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$,

$$|S(f, \dot{P}) - \ell| < \frac{\epsilon}{3}$$

where ℓ is the value of the Riemann integral. Choose P with n subintervals. Now, let \dot{P}_ξ be tagged with ξ_j and \dot{P}_ζ be tagged with ζ_j . From the definitions of the infimum and supremum, we choose our tags such that

$$f(\xi_j) - m_j < \frac{\epsilon}{6(b-a)}, \quad M_j - f(\zeta_j) < \frac{\epsilon}{6(b-a)}.$$

This gives

$$M_j - m_j < f(\xi_j) - f(\zeta_j) + \frac{\epsilon}{3(b-a)}.$$

Thus,

$$\begin{aligned} U(f, P) - L(f, P) &< \sum_{j=0}^{n-1} \left(f(\xi_j) - f(\zeta_j) + \frac{\epsilon}{3(b-a)} \right) (x_{j+1} - x_j) \\ &< S(f, \dot{P}_\xi) - S(f, \dot{P}_\zeta) + \frac{\epsilon}{3(b-a)} \cdot (b-a) \\ &< |S(f, \dot{P}_\xi) - \ell| + |\ell - S(f, \dot{P}_\zeta)| + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

This proves that f is Darboux integrable on $[a, b]$, i.e. $U_f = L_f$. We now wish to show that $U_f = L_f = \ell$. Let $\epsilon > 0$. Using the properties of the infimum and supremum, we find partitions P_1, P_2 and \dot{P}_3 such that

$$L_f - L(f, P_1) < \frac{\epsilon}{6}, \quad U(f, P_2) - U_f < \frac{\epsilon}{6}, \quad |S(f, \dot{P}_3) - \ell| < \frac{\epsilon}{3}.$$

Setting $P = P_1 \cup P_2 \cup P_3$,

$$L_f - L(f, P) < \frac{\epsilon}{6}, \quad U(f, P) - U_f < \frac{\epsilon}{6}, \quad |S(f, \dot{P}) - \ell| < \frac{\epsilon}{3}.$$

Now,

$$L(f, P) \leq S(f, \dot{P}) \leq U(f, P) < L(f, P) + \frac{\epsilon}{3}.$$

This means that $S(f, \dot{P}) - L(f, P) < \epsilon/3$. Now,

$$|\ell - L_f| \leq |\ell - S(f, \dot{P})| + |S(f, \dot{P}) - L(f, P)| + |L(f, P) - L_f| < \epsilon.$$

This forces $U_f = L_f = \ell$.

Now assume that f is Darboux integrable on $[a, b]$. This means that $U_f = L_f$. For $\epsilon > 0$, choose a partition P' such that

$$U(f, P') - U_f < \frac{\epsilon}{2}.$$

Set $\delta_1 = \epsilon/8nM$, and use our previous lemma to conclude that for any partition P of $[a, b]$ with $\|P\| < \delta_1$,

$$U(f, P) < U(f, P \cup P') + \frac{\epsilon}{2} \leq U(f, P') + \frac{\epsilon}{2} < U_f + \epsilon.$$

Similarly, we can choose $\delta_2 > 0$ such that for all partitions P with $\|P\| < \delta_2$,

$$L(f, P) > L_f - \epsilon.$$

Setting $\delta = \min\{\delta_1, \delta_2\}$, we have

$$L_f - \epsilon < L(f, P) < S(f, \dot{P}) < U(f, P) < U_f + \epsilon.$$

Thus, for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$, we have

$$|S(f, \dot{P}) - U_f| < \epsilon. \quad \square$$

Theorem 3.9. *Any real continuous function on $[a, b]$ is Riemann integrable.*

Proof. Note that any continuous function on a compact interval is uniformly continuous. Thus, for $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [a, b]$, we have

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Now, construct a partition of $[a, b]$ which divides the interval into equal subintervals of length $(b-a)/n$, where n is chosen such that $\|P\| < \delta$. This immediately gives

$$U(f, P) - L(f, P) = \sum_{j=0}^{n-1} (M_j - m_j) \cdot \frac{1}{n}(b-a) \leq n \cdot \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} = \epsilon. \quad \square$$

Theorem 3.10. *Any bounded, monotone function on $[a, b]$ is Riemann integrable.*

Proof. Without loss of generality, suppose that f is monotonically increasing. Now, f on each interval attains its minimum and maximum at the endpoints, so

$$U(f, P) - L(f, P) = \sum_{j=0}^{n-1} (f(x_{j+1}) - f(x_j))(x_{j+1} - x_j).$$

Let $\epsilon > 0$. Choose P such that each subinterval has length $(b-a)/n$. Since f is bounded on $[a, b]$, we can choose n to be sufficiently large such that

$$U(f, P) - L(f, P) = (f(b) - f(a)) \frac{b-a}{n} < \epsilon. \quad \square$$

Theorem 3.11. *Let f be Riemann integrable on $[a, b]$, and let g be such that $g(x) = f(x)$ for all $x \in [a, b] \setminus \{c\}$, and $f(c) \neq g(c)$ for some $c \in [a, b]$. Then, g is also Riemann integrable on $[a, b]$.*

Proof. Let the integral of f on $[a, b]$ be ℓ . For $\epsilon > 0$, let $\delta > 0$ be such that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$, we have

$$|S(f, \dot{P}) - \ell| < \frac{\epsilon}{2}.$$

Now, note that

$$|S(f, \dot{P}) - S(g, \dot{P})| \leq 2\delta|f(c) - g(c)|.$$

This is because c can be a tag of at most 2 subintervals. Relabelling δ such that $2\delta|f(c) - g(c)| < \epsilon/2$, we see that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$,

$$|S(g, \dot{P}) - \ell| \leq |S(g, \dot{P}) - S(f, \dot{P})| + |S(f, \dot{P}) - \ell| < \epsilon. \quad \square$$

Corollary 3.11.1. *Let f be Riemann integrable on $[a, b]$, and let g be such that $g(x) = f(x)$ on all but finitely many points in $[a, b]$. Then, g is also Riemann integrable on $[a, b]$.*

Theorem 3.12. *Let f be a bounded function with a single point of discontinuity in $[a, b]$. Then, f is Riemann integrable on $[a, b]$.*

Proof. Let $c \in [a, b]$ be the point of discontinuity. From the boundedness of f , choose M such that $|f(x)| < M$ for all $x \in [a, b]$. Now, for $\epsilon > 0$, choose a partition of $[a, b]$ such that the subinterval containing c has length at most $\epsilon/2M$. The continuity and boundedness of f on the remaining subintervals means that it is Riemann integrable on them, so we can repartition them with P' such that $U(f, P') - L(f, P') < \epsilon/2$. Thus, the difference in the upper and lower sums for the total partition P is bounded as

$$U(f, P) - L(f, P) < \frac{\epsilon}{2} + (M_i - m_i)\frac{\epsilon}{2M} \leq \epsilon. \quad \square$$

Corollary 3.12.1. *Let f be a bounded function with finitely many points of discontinuity in $[a, b]$. Then, f is Riemann integrable on $[a, b]$.*

Corollary 3.12.2. *Let f be a bounded function whose points of discontinuities in $[a, b]$ have finitely many limit points. Then, f is Riemann integrable.*

Theorem 3.13. *The collection of all Riemann integrable functions on $[a, b]$ forms a vector space.*

Theorem 3.14. Let f and g be Riemann integrable on $[a, b]$, such that

$$f(x) \leq g(x)$$

for all $x \in [a, b]$. Then,

$$\int_a^b f \leq \int_a^b g.$$

Theorem 3.15. Let f be Riemann integrable on $[a, b]$, and let $c \in [a, b]$. Then,

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Definition 3.9. Let f be Riemann integrable on $[a, b]$. We define

$$\int_a^b f = - \int_b^a f.$$

Theorem 3.16. Let f be Riemann integrable on $[a, b]$, and let

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$. Also let $\varphi: [m, M] \rightarrow \mathbb{R}$ be continuous, and define $h = \varphi \circ f$ on $[a, b]$. Then, h is Riemann integrable on $[a, b]$.

Proof. Let $\epsilon > 0$. Note that φ must be uniformly continuous on $[a, b]$, which means that there exists $\delta > 0$ such that for all $x, y \in [m, M]$ with $|x - y| < \delta$, we have

$$|\varphi(x) - \varphi(y)| < \epsilon.$$

From the Riemann integrability of f on $[a, b]$, we find a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that $\|P\| < \delta$ and

$$U(f, P) - L(f, P) < \delta^2.$$

We define M_j and m_j in the usual way, with respect to f and P . Also define M_j^* and m_j^* with respect to $h = \varphi \circ f$ and P . Consider the sets

$$A = \{j \in \{0, \dots, n-1\} : M_j - m_j < \delta\},$$

$$B = \{j \in \{0, \dots, n-1\} : M_j - m_j \geq \delta\}.$$

For each $j \in A$, we use the uniform continuity of φ to get

$$M_j^* - m_j^* < \epsilon.$$

When $k \in B$, let $M' = \sup_{[a, b]} h = \sup_{[m, M]} \varphi$, so

$$M_k^* - m_k^* < 2M'.$$

Now,

$$\delta \sum_{k \in B} (x_{k+1} - x_k) \leq \sum_{k \in B} (M_k - m_k)(x_{k+1} - x_k) \leq \delta^2,$$

where the first inequality is because $k \in B$, and the second is because of the Riemann integrability of f . This gives

$$\sum_{k \in B} x_{k+1} - x_k < \delta.$$

Thus,

$$U(h, P) - U(h, P) = \sum_{j \in A} (M_j^* - m_j^*) \Delta x + \sum_{k \in B} (M_k^* - m_k^*) \Delta x \leq \epsilon(b - a) + 2M'\delta.$$

Choose $\delta < \epsilon$, whence

$$U(h, P) - L(h, P) \leq \epsilon(a - b + 2M'). \quad \square$$

Corollary 3.16.1. *Let f be Riemann integrable on $[a, b]$. Then, f^2 is also Riemann integrable on $[a, b]$.*

Corollary 3.16.2. *Let f and g be Riemann integrable on $[a, b]$. Then, fg is also Riemann integrable on $[a, b]$.*

Proof. Use

$$fg = \frac{1}{4} [(f + g)^2 - (f - g)^2]. \quad \square$$

Theorem 3.17 (Lebesgue-Vitali theorem). *A bounded function on a compact interval $[a, b]$ is Riemann integrable if and only if it is continuous almost everywhere, i.e. the set of its points of discontinuity has measure zero.*

Remark. A set $\mathcal{D} \subset \mathbb{R}$ has Lebesgue measure zero if for every $\epsilon > 0$, there exists countable collection of open intervals $\{\mathcal{I}_n\}$ such that

$$\mathcal{D} \subseteq \bigcup_{n=1}^{\infty} \mathcal{I}_n, \quad \sum_{n=1}^{\infty} \mu(\mathcal{I}_n) < \epsilon,$$

where $\mu(\mathcal{I}_n) = b_n - a_n$ is the length of each open interval $\mathcal{I}_n = (a_n, b_n)$.

Example. The Dirichlet function, defined as

$$f: [0, 1] \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

is not Riemann integrable since its set of discontinuities is the entire interval $[0, 1]$.

Lemma 3.18. *Any countable set has Lebesgue measure zero.*

Proof. Enclose each element of the set with an open interval of length $\epsilon/2^n$, for $n = 1, 2, \dots$ \square

Theorem 3.19. *Let f be Riemann integrable on $[a, b]$ and let $g: [a, b] \rightarrow \mathbb{R}$ be defined as*

$$g(x) = \int_a^x f(t) dt.$$

Then,

1. g is continuous on $[a, b]$.
2. If f is continuous at some $x_0 \in (a, b)$, then g is differentiable at x_0 and $g'(x_0) = f(x_0)$.

Proof. To show that g is continuous, first note that f is Riemann integrable so it must be bounded, i.e. we find $M > 0$ such that $|f(x)| < M$ on $[a, b]$. Now, for $x, y \in [a, b]$, $x > y$, note that

$$|g(x) - g(y)| = \left| \int_y^x f(t) dt \right| \leq M(x - y).$$

Therefore, for $\epsilon > 0$, set $\delta = \epsilon/M$ whereby for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have

$$|g(x) - g(y)| \leq M|x - y| < \epsilon.$$

This gives the continuity of g .

Now, suppose that f is continuous at some $x_0 \in (a, b)$. Thus, given $\epsilon > 0$, we can choose $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$. Choose $0 < h < \delta$, whence

$$g(x_0 + h) - g(x_0) = \int_{x_0}^{x_0+h} f(t) dt.$$

Rearranging, we can write

$$\frac{g(x_0 + h) - g(x_0)}{h} - f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} f(t) - f(x_0) dt.$$

Taking absolute values, we see that

$$\left| \frac{g(x_0 + h) - g(x_0)}{h} - f(x_0) \right| \leq \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)| dt \leq \frac{\epsilon h}{h} = \epsilon.$$

The case for $-\delta < h < 0$ is analogous. This gives

$$g'(x_0) = \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} = f(x_0). \quad \square$$

Theorem 3.20 (Fundamental Theorem of Calculus). *Let f be Riemann integrable on $[a, b]$. Suppose that g is continuous on $[a, b]$, differentiable on (a, b) , and satisfies $g'(x) = f(x)$ for all $x \in (a, b)$. Then,*

$$g(b) - g(a) = \int_a^b f(x) dx.$$

Proof. Since f is Riemann integrable on $[a, b]$, given $\epsilon > 0$, there exists a $\delta > 0$ such that for all tagged partitions \dot{P} with $\|\dot{P}\| < \delta$,

$$|S(f, \dot{P}) - \ell| < \epsilon,$$

where ℓ is the integral of f on $[a, b]$. Choose one such partition \dot{P} . Now, pick a subinterval $[x_j, x_{j+1}]$. Note that g is continuous on this closed interval, and differentiable on the open interval. Using the Mean Value Theorem, we choose $\xi_j \in (x_j, x_{j+1})$ such that

$$g(x_{j+1}) - g(x_j) = g'(\xi_j)(x_{j+1} - x_j) = f(\xi_j)(x_{j+1} - x_j).$$

Choosing such ξ_j for all $j = 0, \dots, n-1$, we have

$$\sum_{j=0}^{n-1} g(x_{j+1}) - g(x_j) = \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j) = S(f, \dot{P}_\xi),$$

where \dot{P}_ξ denotes the use of the tags ξ_j . Note that $\|\dot{P}\| = \|\dot{P}_\xi\| < \delta$. Also, the first sum telescopes to $g(b) - g(a)$. Thus,

$$|g(b) - g(a) - \ell| < \epsilon$$

for all $\epsilon > 0$, which gives the desired equality. \square

Theorem 3.21 (Integration by parts). *Let f and g be continuous on $[a, b]$ and differentiable on (a, b) . Also let f' and g' be Riemann integrable on $[a, b]$. Then,*

$$\int_a^b f(x)g'(x) dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x) dx.$$