MA5121: Nonparametric Statistics

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Problem 1 We use the Wilcoxon rank-sum test. This yields $Z = -0.11399$, with a p-value of 0.926 for the two-sided test. Thus, we fail to reject the null hypothesis: that the medians of the underlying distributions of the scores in the morning and afternoon are identical. Similarly, the p-values for the alternative hypotheses that the median in the afternoon is greater than or less than the median in the morning are 0.552 and 0.463, hence we fail to reject the null hypothesis again.

We may also use a two-sample Kolmogorov-Smirnov test. This yields $D = 0.2$, $D^+ = 0.2$, and $D^- = 0.2$, with p-values of 0.9719, 0.6101, 0.6101 for the two-sided, one-sided (greater), and one-sided (lesser) tests. Again, we fail to reject the null hypothesis that the morning and afternoon scores have the same underlying distribution.

Problem 2 We say that a test for $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ is unbiased of level α when the power function β obeys $\beta \leq \alpha$ on Θ_0 and $\beta \geq \alpha$ on Θ_1.

In the setup of the sign test, let $H_0 : \theta = \theta_0$ and $H_1 : \theta < \theta_0$. Recall that the test statistic is

$$S^- = \sum_{i=1}^{n} \chi_{(-\infty,\theta_0]}(X_i) \sim \text{Binomial}(n, p_\theta), \qquad p_\theta = F(\theta_0).$$

We reject H_0 if S^- is large. Now, the power function $\beta(\theta) = P_\theta(S^- \in \omega)$ is of the form

$$\beta(\theta) = q\binom{n}{k} p_\theta^k (1-p_\theta)^{n-k} + \sum_{j=k+1}^{n} \binom{n}{j} p_\theta^j (1-p_\theta)^{n-j}.$$

This is because we reject H_0 if $S^- > k$, or with probability q if $S^- = k$ where k, q are chosen such that $\beta(\theta_0) = \alpha$; using $p_{\theta_0} = 1/2$,

$$2^n \alpha = q\binom{n}{k} + \sum_{j=k+1}^{n} \binom{n}{j}.$$

Now, $p_\theta \geq 1/2$ when $\theta < \theta_0$. Thus, it is enough to show that the function

$$f(x) = q\binom{n}{k} x^k (1-x)^{n-k} + \sum_{j=k+1}^{n} \binom{n}{j} x^j (1-x)^{n-j}.$$

is increasing on $[0, 1]$ in order to establish $\beta(\theta) \geq \alpha$ when $\theta < \theta_0$.

Check that

$$\begin{aligned}
f'(x) &= qk\binom{n}{k} x^{k-1}(1-x)^{n-k} - q(n-k)\binom{n}{k} x^k (1-x)^{n-k-1} \\
&\quad + \sum_{j=k+1}^{n} j\binom{n}{j} x^{j-1}(1-x)^{n-j} - \sum_{j=k+1}^{n-1} (n-j)\binom{n}{j} x^j (1-x)^{n-j-1} \\
&= qn\binom{n-1}{k-1} x^{k-1}(1-x)^{n-k} - qn\binom{n-1}{k} x^k (1-x)^{n-k-1} \\
&\quad + \sum_{j=k+1}^{n} n\binom{n-1}{j-1} x^{j-1}(1-x)^{n-j} - \sum_{j=k+1}^{n-1} n\binom{n-1}{j} x^j (1-x)^{n-j-1}.
\end{aligned}$$

Cancelling terms,

$$f'(x)/n = q\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k} + (1-q)\binom{n-1}{k}x^k(1-x)^{n-k-1} > 0.$$

Thus, we have $f'(x) > 0$ as desired, whence the one-sided sign test H_0 versus H_1 is unbiased.

Note that the same can be said of the one-sided sign test H_0 versus $H_2 : \theta > \theta_0$.

For the two-sided sign test H_0 versus $H_3 : \theta \neq \theta_0$, we reject H_0 if $S^- < k$, or $S^- > n-k$, or with probability q if either $S^- = k$ or $S^- = n-k$. This gives a power function of the form

$$\begin{aligned}
\beta(\theta) &= \sum_{j=0}^{k-1}\binom{n}{j}p_\theta^j(1-p_\theta)^{n-j} + q\binom{n}{k}p_\theta^k(1-p_\theta)^{n-k} \\
&\quad + q\binom{n}{n-k}p_\theta^{n-k}(1-p_\theta)^k + \sum_{j=n-k+1}^{n}\binom{n}{j}p_\theta^j(1-p_\theta)^{n-j} \\
&= \sum_{j=0}^{k-1}\binom{n}{j}\left[p_\theta^j(1-p_\theta)^{n-j} + p_\theta^{n-j}(1-p_\theta)^j\right] + q\binom{n}{k}\left[p_\theta^k(1-p_\theta)^{n-k} + p_\theta^{n-k}(1-p_\theta)^k\right].
\end{aligned}$$

Note that k, q are chosen such that $\beta(\theta_0) = \alpha$, i.e.

$$2^n\alpha = 2\sum_{j=0}^{k-1}\binom{n}{j} + 2q\binom{n}{k}.$$

We claim that $\beta(\theta) \geq \alpha$ when $\theta \neq \theta_0$. Thus, it is enough to show that the function

$$g(x) = \sum_{j=0}^{k-1}\binom{n}{j}\left[x^j(1-x)^{n-j} + x^{n-j}(1-x)^j\right] + q\binom{n}{k}\left[x^k(1-x)^{n-k} + x^{n-k}(1-x)^k\right]$$

is minimum at $x = 1/2$. For this, it is enough to show that g is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$. But $g(x) = h(x) + h(1-x) = g(1-x)$, where

$$h(x) = \sum_{j=0}^{k-1}\binom{n}{j}x^j(1-x)^{n-j} + q\binom{n}{k}x^k(1-x)^{n-k}.$$

Thus, it is enough to show that g is decreasing on $[0, 1/2]$. Calculate

$$\begin{aligned}
h'(x) &= \sum_{j=1}^{k-1} j\binom{n}{j}x^{j-1}(1-x)^{n-j} - \sum_{j=0}^{k-1}(n-j)\binom{n}{j}x^j(1-x)^{n-j-1} \\
&\quad + qk\binom{n}{k}x^{k-1}(1-x)^{n-k} - q(n-k)\binom{n}{k}x^k(1-x)^{n-k-1} \\
&= \sum_{j=1}^{k-1} n\binom{n-1}{j-1}x^{j-1}(1-x)^{n-j} - \sum_{j=0}^{k-1} n\binom{n-1}{j}x^j(1-x)^{n-j-1} \\
&\quad + qn\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k} - qn\binom{n-1}{k}x^k(1-x)^{n-k-1} \\
&= -(1-q)n\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k} - qn\binom{n-1}{k}x^k(1-x)^{n-k-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
g'(x) &= h'(x) - h'(1-x) \\
&= -(1-q)n\binom{n-1}{k-1}\left[x^{k-1}(1-x)^{n-k} - x^{n-k}(1-x)^{k-1}\right] \\
&\quad - qn\binom{n-1}{k}\left[x^{k}(1-x)^{n-k-1} - x^{n-k-1}(1-x)^{k}\right] \\
&= -(1-q)n\binom{n-1}{k-1}x^{n-1}\left[(1/x-1)^{n-k} - (1/x-1)^{k-1}\right] \\
&\quad - qn\binom{n-1}{k}x^{n-1}\left[(1/x-1)^{n-k-1} - (1/x-1)^{k}\right].
\end{aligned}$$

Finally, note that when $x \in (0, 1/2)$, we have $(1/x - 1) \in (1, \infty)$. Since $k < n/2$, we have the exponents $n - k > k - 1$, and $n - k - 1 > k$. All together, we have $g'(x) < 0$ when $x \in (0, 1/2)$, as desired.

Thus, the two-sided sign test H_0 versus H_3 as formulated above is unbiased.

Problem 3 Recall that a test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ is consistent if the power functions $\beta_n \to 1$ pointwise on Θ_1 as the number of observations $n \to \infty$. We state and prove the following result.

Proposition. Let g be a function such that $g = \theta_0$ on Θ_0, and $g \neq \theta_0$ on Θ_1. Suppose that T_n is a test statistic based on n observations such that $E_\theta(T_n) \to g(\theta)$ and $V_\theta(T_n) \to 0$ as $n \to \infty$ for all $\theta \in \Theta$. Then, the test of size α which rejects H_0 when $|T - \theta_0| > c_\alpha$ is consistent.

Proof: Using Markov's inequality, note that for any $\delta > 0$, we have

$$P_\theta(|T_n - g(\theta)| \geq \delta) \leq \frac{E_\theta((T_n - g(\theta))^2)}{\delta^2} = \frac{V_\theta(T_n) + (E_\theta(T_n) - g(\theta))^2}{\delta^2} \to 0$$

as $n \to \infty$. Thus, $P_\theta(|T_n - g(\theta)| < \delta) \to 1$. Now, when $\theta \in \Theta_1$, we have $|g(\theta) - \theta_0| = c > 0$. ??.

With this, consider the test statistic $T = S^-/n$. Observe that $E(T_n) = F(\theta_0)$, which is 1/2 if and only if $\theta = \theta_0$ (with the assumption that F is continuous and increasing around the median θ). Furthermore, $V(T_n) = F(\theta_0)(1 - F(\theta_0))/n \to 0$ as $n \to \infty$. Thus, the test $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$ where we reject H_0 when $|S^-/n - 1/2| > c_\alpha$ is consistent.

Problem 4 Given that the joint density of X_1, X_2, X_3 is

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = 162\, x_1^2 x_2^2 x_3^2\ \chi_{0<x_1<x_2<x_3}(x_1, x_2, x_3).$$

Consider the probability density function

$$f(x) = 3x^2\ \chi_{(0,1)}(x).$$

Then, given an i.i.d. sample $Y_1, Y_2, Y_3 \sim f$, observe that f_{X_1,X_2,X_3} is precisely the joint density of the order statistics $Y_{(1)}, Y_{(2)}, Y_{(3)}$. Thus, (X_1, X_2, X_3) and $(Y_{(1)}, Y_{(2)}, Y_{(3)})$ are identically distributed. Furthermore, we must have $Y_1 + Y_2 + Y_3 = Y_{(1)} + Y_{(2)} + Y_{(3)}$. This means that

$$\operatorname{var}(X_1 + X_2 + X_3) = \operatorname{var}(Y_1 + Y_2 + Y_3) = 9\operatorname{var}(Y_1).$$

Now,

$$E(Y_1) = \int_0^1 x \cdot 3x^2\, dx = \frac{3}{4}, \qquad E(Y_1^2) = \int_0^1 x^2 \cdot 3x^2\, dx = \frac{3}{5},$$

hence

$$\operatorname{var}(Y_1) = E(Y_1^2) - E(Y_1)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

Thus,

$$\operatorname{var}(X_1 + X_2 + X_3) = \frac{27}{80}.$$

Problem 5 Given a probability density function f with positive support, $\theta > 0$, and $Y \sim f$, $X \sim f(x/\theta)/\theta$.

(a) Clearly, for $a \geq 0$,

$$F_{X/\theta}(a) = P(X/\theta \leq a) = P(X \leq a\theta) = F_X(a\theta),$$

hence

$$F_{X/\theta}(a) = \int_0^{a\theta} f_X(x)\,dx = \int_0^{a\theta} \frac{1}{\theta} f\left(\frac{x}{\theta}\right)\,dx = \int_0^a f(x)\,dx = F_Y(a).$$

Thus, X/θ and Y are identically distributed.

(b) Let $\theta > 1$. We claim that Y is stochastically smaller than X, i.e. that $F_Y \geq F_X$, with strict inequality at at least one point. Indeed, for $x \geq 0$

$$F_X(x) = \int_0^x \frac{1}{\theta} f\left(\frac{x}{\theta}\right)\,dx = \int_0^{x/\theta} f(x)\,dx \leq \int_0^x f(x)\,dx = F_Y(x),$$

because $x/\theta < x$. Furthermore, the inequality must be strict at at least one $x > 0$; if not, then we would have

$$\int_{x/\theta}^x f(x)\,dx = 0$$

for all $x \geq 0$. This would force $f = 0$ almost everywhere on the intervals (θ^n, θ^{n+1}) for all $n \in \mathbb{Z}$. The (countable) union of these intervals is $(0, \infty)$ save for countably many points, whence $f = 0$ almost everywhere, a contradiction.

Problem 6 Note that‘ we have

$$\int (x - \mu)\,dF(x) = 0.$$

Thus,

$$E(X_{(n)} - \mu) = \int (x - \mu)(nF(x)^{n-1} - 1)\,dF(x).$$

Using Cauchy-Schwarz,

$$\begin{aligned}
[E(X_{(n)} - \mu)]^2 &\leq \int (x - \mu)^2\,dF(x) \int (nF(x)^{n-1} - 1)^2\,dF(x) \\
&= \sigma^2 \int n^2 F(x)^{2n-2} - 2nF(x)^{n-1} + 1\,dF(x) \\
&= \sigma^2 \left[\frac{n^2}{2n-1} - 1\right] \\
&= \sigma^2 \frac{(n-1)^2}{2n-1}.
\end{aligned}$$

This immediately gives

$$E(X_{(n)}) \leq \mu + \sigma \frac{n-1}{\sqrt{2n-1}}.$$

Problem 7 We use the empirical cumulative distribution function of pH values, as shown in Figure 1 to estimate

$$P(\text{pH} > 4.75) \approx 1 - \hat{F}_n(4.75) = 0.269.$$

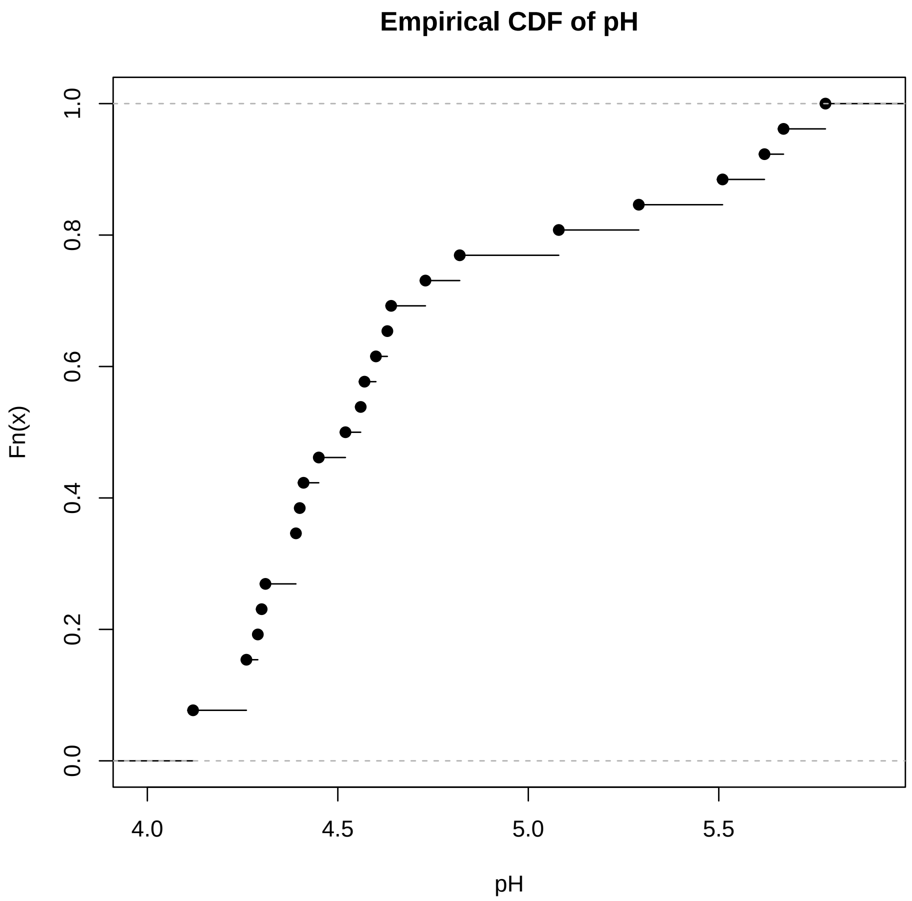
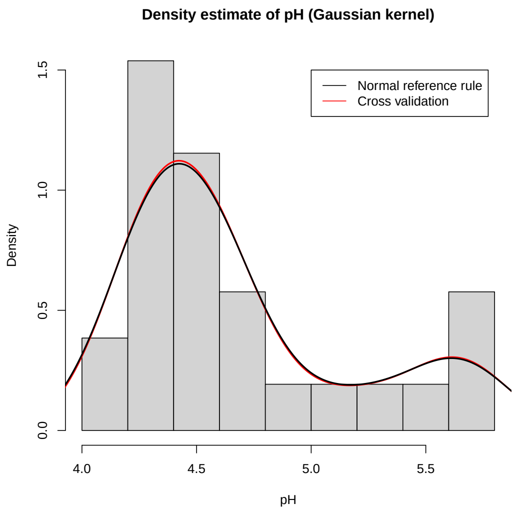


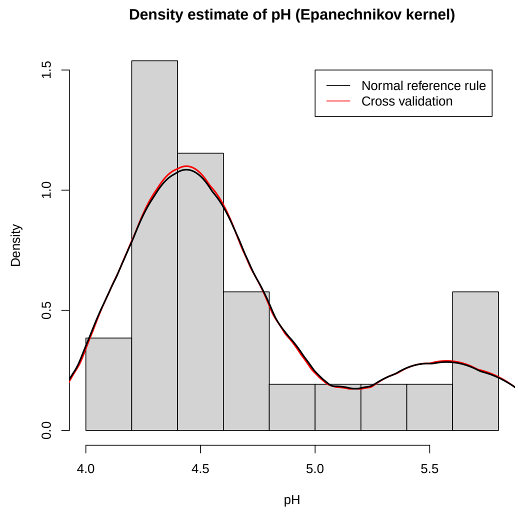
Figure 1: Empirical CDF of pH.

Kernel density estimates of pH, along with a simple histogram, have been shown in Figure 2. We have used the Gaussian, Epanechnikov, rectangular, and triangular kernels.

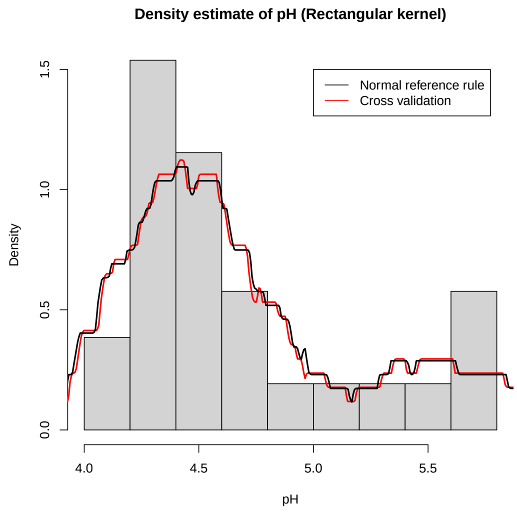
The kernel density estimates are practically identical, save for some excess roughness when using a rectangular kernel.



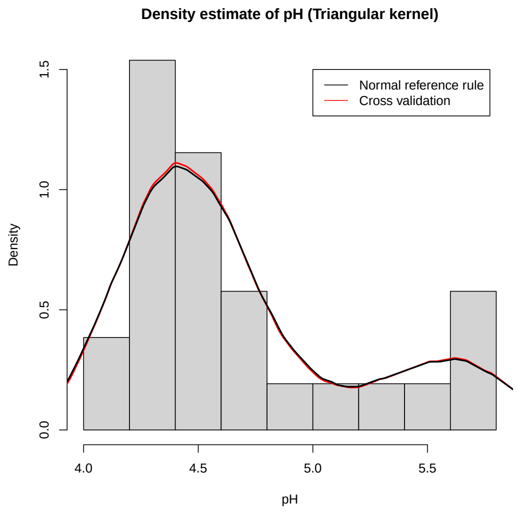
(a) Gaussian kernel



(b) Epanechnikov kernel



(c) Rectangular kernel



(d) Triangular kernel

Figure 2: Kernel density estimates of pH.