MA3101

Analysis III

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Contents

1 Euclidean spaces

We are familiar with the vector space $\mathbb{R}^n$, with the standard inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is define as

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle = \sum_{k=1}^{n} (x_i - y_i)^2.$$

Exercise 1.1. What are all possible inner products on $\mathbb{R}^n$?

Theorem 1.1 (Cauchy-Schwarz)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|.$$

Theorem 1.2 (Triangle inequality)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

This allows us to define the standard metric on $\mathbb{R}^n$, seen as a point set.

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Definition 1.1. For any $\epsilon > 0$, the set

$$B_\epsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < \epsilon\}$$

is called the open ball centred at $\boldsymbol{x} \in \mathbb{R}^n$ with radius ϵ. This is also called the ϵ neighbourhood of $\boldsymbol{x}$.

Definition 1.2. A set U is open in $\mathbb{R}^n$ if for every $\boldsymbol{x} \in U$, there exists an open ball $B_\epsilon(\boldsymbol{x}) \subset U$.

Remark. Every open ball in $\mathbb{R}^n$ is open.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are open.

Definition 1.3. A set F is closed in $\mathbb{R}^n$ if its complement $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are closed.