MA3104

# Linear Algebra II

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## Contents



## 1 Linear operators on a vector space

### 1.1 Preliminaries

We discuss finite dimensional vector spaces $V$ over some field $\mathbb{F}$, along with linear operators $T\colon V \to V$. We also assume that $V$ has the inner product $\langle \cdot, \cdot \rangle$.

**Theorem 1.1.** *Let $\mathscr{L}(V)$ be the set of all linear operators on the vector space $V$. Then, $\mathscr{L}(V)$ is a linear algebra over the field $\mathbb{F}$.*

### 1.2 Ideals in a ring

**Definition 1.1.** Let $(R, +, \cdot)$ be a ring, where $(R, +)$ is its additive subgroup. A set $I \subseteq R$ is a left ideal of $R$ if $(I, +)$ is a subgroup of $(R, +)$, and $rx \in I$ for every $r \in R$, $x \in I$.

*Example.* Let $\mathbb{Z}$ be the ring of integers. For some $n \in \mathbb{N}$, the set $n\mathbb{Z}$ is an ideal. In fact, these are the only ideals (along with $\{0\}$).

**Definition 1.2.** The principal left ideal generated by $x \in R$ is the set

$$I_x = Rx = \{rx : r \in R\}.$$

*Example.* In the ring of integers $\mathbb{Z}$, every ideal is a principal ideal. This follows directly from the fact that $(\mathbb{Z}, +)$ is a cyclic group, thus any subgroup is cyclic and thus generated by a single element.

Let $I \subseteq \mathbb{Z}$ be an ideal. If $I = \{0\}$, we are done. Otherwise, let $n$ be the smallest positive integer in $I$ (note that if $a \in I$, then $-a \in I$ which means that $I$ must contain positive integers). This immediately gives $I \supseteq n\mathbb{Z}$. Now for any $m \in I$, use Euclid's Division Lemma to write $m = nq + r$, where $q \in \mathbb{Z}$, $0 \leq r < n$. Since $I$ is an ideal, $nq \in I$ hence $m - nq = r \in I$. The minimality of $n$ in $I$ forces $r = 0$, hence $m = nq$ and $I \subseteq n\mathbb{Z}$. This proves $I = n\mathbb{Z}$.

**Theorem 1.2.** *Let $\mathbb{F}$ be a field and let $\mathbb{F}[x]$ denote the ring of polynomials with coefficients from $\mathbb{F}$. Then, every ideal in $\mathbb{F}[x]$ is a principal ideal.*

*Remark.* This is analogous to the theorem which states that every subgroup of a cyclic group is cyclic. Both lead to a precise definition of the greatest common divisor.

**Corollary 1.2.1.** *Let $I$ be a non-trivial ideal in $\mathbb{F}[x]$. Then, there exists a unique monic polynomial $p \in \mathbb{F}[x]$ (leading coefficient $1$) such that $I$ is precisely the principal ideal generated by $p$.*

### 1.3 Eigenvalues and eigenvectors

**Definition 1.3.** Let $T \in \mathscr{L}(V)$ and $c \in \mathbb{F}$. We say that $c$ is an eigenvalue or characteristic value of $T$ if $T\boldsymbol{v} = c\boldsymbol{v}$ for some non-zero $\boldsymbol{v} \in V$. The vector $\boldsymbol{v}$ is called an eigenvector of $T$.

**Theorem 1.3.** *Let $T \in \mathscr{L}(V)$ and $c \in \mathbb{F}$. The following are equivalent.*

1. *$c$ is an eigenvalue of $T$.*
2. *$T - cI$ is singular.*
3. $\det(T - cI) = 0$.

**Definition 1.4.** The polynomial $\det(T - xI)$ is called the characteristic polynomial of $T$.

**Definition 1.5.** Two linear operators $S, T \in \mathscr{L}(V)$ are similar if there exists an invertible operator $X \in \mathscr{L}(V)$ such that $S = X^{-1}TX$.

*Remark.* Similarity is an equivalence relation on $\mathscr{L}(V)$, thus partitioning it into similarity classes.

**Lemma 1.4.** *Similar linear operators have the same characteristic polynomial.*

*Proof.* Let $S, T$ be similar with $S = X^{-1}TX$. Then,

$$\begin{aligned}\det(S - xI) &= \det(X^{-1}TX - xX^{-1}X)\\ &= \det(X^{-1})\det(T - xI)\det(X)\\ &= \det(T - xI).\end{aligned}$$

□

**Definition 1.6.** A linear operator $T \in \mathscr{L}(V)$ is diagonalizable if there is a basis of $V$ consisting of eigenvectors of $T$.

*Remark.* The matrix of $T$ with respect to such a basis is diagonal.

### 1.4 Annihilating polynomials

**Definition 1.7.** An polynomial $p$ such that $p(T) = 0$ for a given linear operator $T \in \mathscr{L}(V)$ is called an annihilating polynomial of $T$.

**Lemma 1.5.** *Every linear operator $T \in \mathscr{L}(V)$, where $V$ is finite dimensional, has a non-trivial annihilating polynomial.*

*Proof.* Note that the operators $I, T, T^2, \ldots, T^{n^2} \in \mathscr{L}(V)$, of which there are $n^2 + 1$, are linearly dependent, since $\dim \mathscr{L}(V) = n^2$. □

**Lemma 1.6.** *The annihilating polynomials of $T$ form an ideal in $\mathbb{F}[x]$.*

**Definition 1.8.** The minimal polynomial of $T$ is the unique monic generator of the annihilating polynomials of $T$.

*Remark.* The minimal polynomial of $T$ divides all its annihilating polynomials.

**Theorem 1.7.** *The minimal polynomial and characteristic polynomial of $T$ share the same roots, except for multiplicities.*

*Proof.* Let $p$ be the minimal polynomial of $T$ and let $f$ be its characteristic polynomial.

First, let $c \in \mathbb{F}$ be a root of the minimal polynomial, i.e. $p(c) = 0$. The Division Algorithm guarantees

$$p(x) = (x - c)\, q(x)$$

for some monic polynomial $q$. By the minimality of the degree of $p$, we have $q(T) \neq 0$, hence there exists non-zero $\boldsymbol{v} \in V$ such that $\boldsymbol{w} = q(T)\,\boldsymbol{v} \neq \boldsymbol{0}$. Thus, $p(T)\,\boldsymbol{v} = \boldsymbol{0}$ gives

$$(T - cI)\,q(T)\,\boldsymbol{v} = \boldsymbol{0}, \qquad T\boldsymbol{w} = c\boldsymbol{w},$$

which shows that $c$ is an eigenvalue, i.e. a root of the characteristic polynomial $f$.

Next, suppose that $c$ is a root of the characteristic polynomial, i.e. $f(c) = 0$. Thus, $c$ is an eigenvalue of $T$, hence there exists non-zero $\boldsymbol{v} \in V$ such that $T\boldsymbol{v} = c\boldsymbol{v}$. This gives $p(T)\,\boldsymbol{v} = p(c)\,\boldsymbol{v}$, but $p(T) = 0$ identically, forcing $p(c) = 0$. $\square$

**Theorem 1.8** (Cayley-Hamilton)**.** *The characteristic polynomial of $T$ annihilates $T$.*

**Corollary 1.8.1.** *The minimal polynomial of $T$ divides its characteristic polynomial.*

**Corollary 1.8.2.** *The minimal polynomial of $T$ in a finite-dimensional vector space $V$ is at most* $\dim V$*.*