

# Statistical Depth Functions

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1. Depth Functions for Functional Data
2. Outlier detection for Functional Data
3. Local Depth Functions

# Depth Functions

A *depth function* quantifies how *central* a point  $\mathbf{x} \in \mathcal{X}$  is with respect to a distribution  $F$ .

This induces a *center-outwards* ordering on the space  $\mathcal{X}$ .

# Depth Functions in $\mathbb{R}^d$

We want  $D: \mathbb{R}^d \times \mathcal{F} \rightarrow \mathbb{R}$  to be bounded, non-negative, continuous, and satisfy the following properties.

P1. **Affine invariance:**  $D(A\mathbf{x} + b, F_{A\mathbf{x}+b}) = D(\mathbf{x}, F_X)$ .

P2. **Maximality at centre:**  $D(\theta, F_X) = \sup_{\mathbf{x} \in \mathbb{R}^d} D(\mathbf{x}, F)$ .

P3. **Monotonicity along rays:**  $D(\mathbf{x}, F) \leq D(\theta + \alpha(\mathbf{x} - \theta), F)$ .

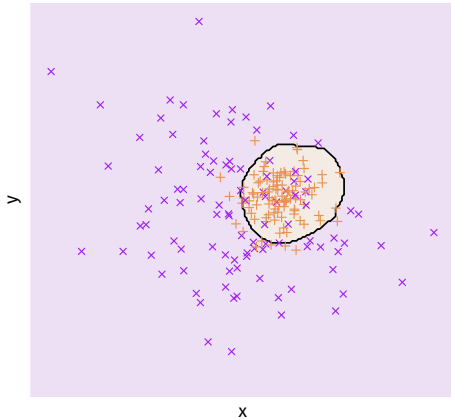
P4. **Vanish at infinity:**  $D(\mathbf{x}, F) \rightarrow 0$  as  $\|\mathbf{x}\| \rightarrow \infty$ .

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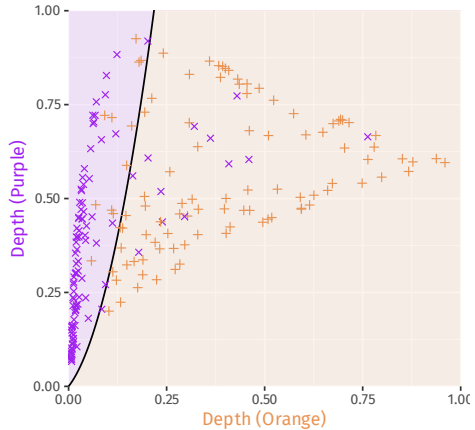
Zuo, Y., & Serfling, R. (2000) General notions of statistical depth function

# The DD classifier

Training Data



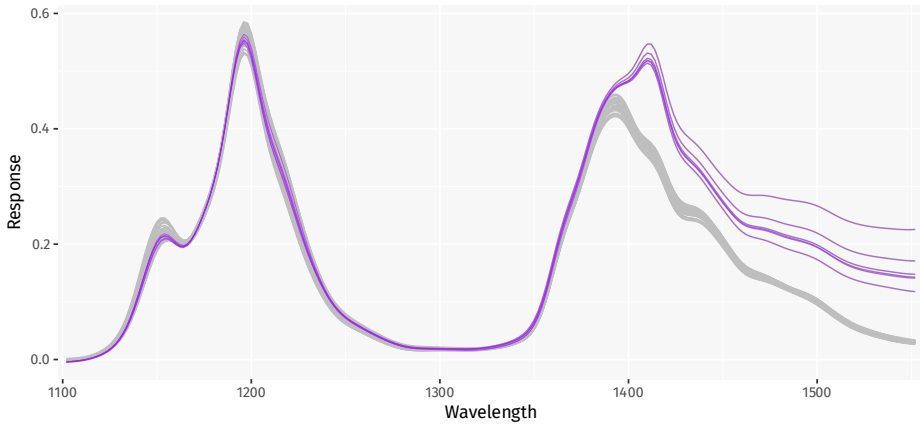
Test Data (DD Plot)



# Depth Functions for Functional Data

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NIR spectra of gasoline samples



# Depth Functions in Banach spaces $\mathcal{X}$

Let  $\mathcal{X}$  be a class of functions of the form  $\mathbf{x}: [0, 1] \rightarrow \mathbb{R}^d$ , equipped with a norm  $\|\cdot\|$ . We typically choose  $L^2[0, 1]$  or  $\mathcal{C}[0, 1]$ .

We want to generalize the Zuo-Serfling properties (P1-4) in this setting, for depth functions  $D: \mathcal{X} \times \mathcal{F} \rightarrow \mathbb{R}$ .

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Gijbels, I., & Nagy, S. (2017) On a General Definition of Depth for Functional Data



# Statistical Depth Functions

## └ Depth Functions for Functional Data

### └ Depth Functions in Banach spaces $\mathcal{X}$

Depth Functions in Banach spaces  $\mathcal{X}$

Let  $\mathcal{X}$  be a class of functions of the form  $x: [0, 1] \rightarrow \mathbb{R}^d$ , equipped with a norm  $\|\cdot\|$ . We typically choose  $L^2[0, 1]$  or  $C[0, 1]$ .

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Gijbels, I., & Nagys, S. (2017) On a General Definition of Depth for Functional Data

Properties  $P3$  (Monotonicity along rays) and  $P4$  (Vanish at infinity) carry over naturally.

# Non-degeneracy

P0. **Non-degeneracy:**  $\inf_{\mathbf{x} \in \mathcal{X}} D(\mathbf{x}, F) < \sup_{\mathbf{x} \in \mathcal{X}} D(\mathbf{x}, F)$ .

The naïve generalization of the halfspace/Tukey depth

$$D_H(\mathbf{x}, F) = \inf_{\mathbf{v} \in \mathcal{X}^*} P_{X \sim F}(\mathbf{v}^*(X) \leq \mathbf{v}^*(\mathbf{x})),$$

is degenerate for a wide class of distributions  $\mathcal{F}$ . For instance,  $\mathcal{X} = \mathcal{C}[0, 1]$ , Gaussian processes with positive definite covariance kernels.

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Chakraborty, A., & Chaudhuri, P. (2014) On data depth in infinite dimensional spaces

# Statistical Depth Functions

## └ Depth Functions for Functional Data

### └ Non-degeneracy

Non-degeneracy

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The naïve generalization of the halfspace/Tukey depth

$$D_H(x, F) = \inf_{v \in \mathbb{R}^n} P_{X \sim F}\{v^T(X) \leq v^T(x)\}.$$

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Chakraborty, A. & Chaudhuri, P (2014) On data depth in infinite dimensional spaces

This also applies to the functional analogue of the projection depth.

The functional analogue of the spatial depth

$$D_{Sp}(\mathbf{x}, F) = 1 - \left\| \mathbb{E}_{X \sim F} \left[ \frac{\mathbf{x} - X}{\|\mathbf{x} - X\|_2} \right] \right\|_2,$$

does not suffer from degeneracy.

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Chakraborty, A., & Chaudhuri, P. (2014) The spatial distribution in infinite dimensional spaces and related quantiles and depths

P1S. **Scalar-affine invariance:** For  $a, b \in \mathbb{R}$  with  $a$  non-zero and  $\mathbf{x} \in \mathcal{X}$ ,

$$D(a\mathbf{x} + b, F_{a\mathbf{x}+b}) = D(\mathbf{x}, F_{\mathbf{x}}).$$

P1F. **Function-affine invariance:** For  $a, b, \mathbf{x} \in \mathcal{X}$ , with  $a\mathbf{x} \in \mathcal{X}$ ,

$$D(a\mathbf{x} + b, F_{a\mathbf{x}+b}) = D(\mathbf{x}, F_{\mathbf{x}}).$$

# Maximality at center

We say that  $F_X$  is symmetric about  $\theta \in \mathcal{X}$  if for all  $\varphi \in \mathcal{X}^*$ , we have  $\varphi(X)$  symmetric about  $\varphi(\theta)$ .

- P2C. **Maximality at center of central symmetry:** For  $F \in \mathcal{X}$  centrally symmetric about  $\theta \in \mathcal{X}$ ,  $D(\theta, F) = \sup_{x \in \mathcal{X}} D(x, F)$ .
- P2H. **Maximality at center of halfspace symmetry:** For  $F \in \mathcal{X}$  halfspace symmetric about  $\theta \in \mathcal{X}$ ,  $D(\theta, F) = \sup_{x \in \mathcal{X}} D(x, F)$ .

# The Integrated and Infimal Depths

$$D_{FM}(\mathbf{x}, F_X) = \int_{[0,1]} D(\mathbf{x}(t), F_{X(t)}) w(t) dt.$$

$$D_{Inf}(\mathbf{x}, F_X) = \inf_{t \in [0,1]} D(\mathbf{x}(t), F_{X(t)}).$$

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Fraiman, R., & Muniz, G. (2001) Trimmed means for functional data  
Mosler, K. (2013) Depth Statistics

# The $J$ -th order Integrated and Infimal Depths

$$D_{FM}^J(\mathbf{x}, F_X) = \int_{[0,1]^J} D((\mathbf{x}(t_1), \dots, \mathbf{x}(t_j))^{\top}, F_{(\mathbf{x}(t_1), \dots, \mathbf{x}(t_j))^{\top}}) w(\mathbf{t}) d\mathbf{t}.$$

$$D_{Inf}^J(\mathbf{x}, F_X) = \inf_{\mathbf{t} \in [0,1]^J} D((\mathbf{x}(t_1), \dots, \mathbf{x}(t_j))^{\top}, F_{(\mathbf{x}(t_1), \dots, \mathbf{x}(t_j))^{\top}}).$$

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Nagy, S., Gijbels, I., & and Hlubinka, D. (2017) Depth-Based Recognition of Shape Outlying Functions



# Statistical Depth Functions

## └ Depth Functions for Functional Data

### └ The $J$ -th order Integrated and Infimal Depths

$$D_{int}^J(\mathbf{x}, F_X) = \int_{[0, \varphi]} D(\{\mathbf{x}(t_1), \dots, \mathbf{x}(t_j)\}^T, F_{\{\mathbf{x}(t_1), \dots, \mathbf{x}(t_j)\}^T}) w(t) dt.$$

$$D_{inf}^J(\mathbf{x}, F_X) = \inf_{t \in [0, \varphi]} D(\{\mathbf{x}(t_1), \dots, \mathbf{x}(t_j)\}^T, F_{\{\mathbf{x}(t_1), \dots, \mathbf{x}(t_j)\}^T}).$$

These  $J$ -th order depths carry information about the derivatives of the curves, of orders  $0, \dots, J - 1$ .

# The Band Depth

$$D_B^j(\mathbf{x}, F) = \sum_{j=2}^J P_{\mathbf{X}_j \sim F}(\mathbf{x} \in \text{conv}(\mathbf{X}_1, \dots, \mathbf{X}_j)).$$

This is the proportion of  $j$ -tuples of curves, for  $2 \leq j \leq J$ , which *completely* envelope  $\mathbf{x}$ .

The band depth becomes degenerate for  $\mathcal{X} = \mathcal{C}[0, 1]$ , Feller processes  $\mathbf{X}$  (e.g. Brownian motion) with  $P(\mathbf{X}_0 = 0) = 1$  and each  $\mathbf{X}_t$  for  $t > 1$  non-atomic and symmetric about 0.

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López Pintado, S., & Romo, J. (2009) On the concept of depth for functional data

# The Modified Band Depth

Define the *enveloping time*

$$\text{ET}(\mathbf{x}; \mathbf{x}_1, \dots, \mathbf{x}_j) = m_1(\{t \in [0, 1]: \mathbf{x}(t) \in \text{conv}(\mathbf{x}_1(t), \dots, \mathbf{x}_j(t))\})$$

The modified band depth is defined as

$$D_{\text{MBD}}(\mathbf{x}, F) = \sum_{j=2}^J \mathbb{E}_{\mathbf{x}_j \sim F} [\text{ET}(\mathbf{x}; \mathbf{X}_1, \dots, \mathbf{X}_j)] .$$

# The Half-Region Depth

We say that  $\mathbf{y}$  is in the hypograph (resp. epigraph) of  $\mathbf{x}$ , denoted  $\mathbf{y} \in H_{\mathbf{x}}$  (resp.  $E_{\mathbf{x}}$ ), if  $\mathbf{y}(t) \leq \mathbf{x}(t)$  (resp.  $\geq$ ) for all  $t \in [0, 1]$ .

The half-region depth is defined as

$$D_{HR}(\mathbf{x}, F) = \min \{P_F(H_{\mathbf{x}}), P_F(E_{\mathbf{x}})\}.$$

This suffers from the same degeneracy problems as the band depth.

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López Pintado, S., & Romo, J. (2011) A half-region depth for functional data

# The Modified Half-Region Depth

Define the Modified Hypograph (MHI) and Epigraph (MEI) Indices as

$$\text{MHI}_F(\mathbf{x}) = \mathbb{E}_{X \in F}[m_1\{t \in [0, 1]: \mathbf{x}(t) \geq X(t)\}],$$

$$\text{MEI}_F(\mathbf{x}) = \mathbb{E}_{X \in F}[m_1\{t \in [0, 1]: \mathbf{x}(t) \leq X(t)\}].$$

The modified half-region depth is defined as

$$D_{\text{MHR}}(\mathbf{x}, F) = \min \{ \text{MHI}_F(\mathbf{x}), \text{MEI}_F(\mathbf{x}) \}.$$

## Partially Observed Functional Data

Suppose that  $X \sim F_X$  is not observed on the entire interval  $[0, 1]$ , but rather on some random compact subinterval  $O \sim Q$  (independent of  $X$ ).

Given a dataset  $\mathcal{D} = \{(X_i, O_i)\}_{i=1}^n$  where  $(X_i, O_i) \stackrel{\text{iid}}{\sim} F_X \times Q$ , we keep track of the indices observed at time  $t \in [0, 1]$  as  $\mathcal{J}(t) = \{j: t \in O_j\}$ , as well as their number  $q(t) = |\mathcal{J}(t)|$ .

# The Partially Observed Integrated Functional Depth (POIFD)

We may define a depth function in this setting via

$$D_{POIFD}((\mathbf{x}, o), F_X \times Q) = \int_o D(\mathbf{x}(t), F_{X(t)}) w_o(t) dt,$$

where  $w_o(t) = q(t) / \int_0 q(t) dt$ .

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Elías, A., Jiménez, R., & Shang, H. L. (2023) Depth-based reconstruction method for incomplete functional data

# The functional reconstruction problem

Given  $(X, O)$ , can we estimate  $X$  on  $M = [0, 1] \setminus O$ ?

We may search for a reconstruction operator  $\mathcal{R}: L^2(O) \rightarrow L^2(M)$  that minimizes the mean integrated prediction squared error loss  $\mathbb{E}[\|X_M - \mathcal{R}(X_O)\|^2]$ . In this setup, the best predictor is the conditional expectation  $\mathbb{E}[X_M \mid X_O]$ .

We may also search for a continuous linear reconstruction operator  $\mathcal{A}$ , by estimating terms of the Karhunen-Loève expansion of  $X$ .



# The functional reconstruction problem

Another approach is to take a convex linear combination of curves from a suitable curve envelope with indices  $\mathcal{J}$ .

The enveloping curves  $\mathcal{J}$  may be chosen so that  $(X, O)$  is as deep as possible inside the curve envelope.

Additionally, we want  $\mathcal{J}$  to envelope  $(X, O)$  for as long as possible (in the sense of the enveloping time **ET**), and contain as many near curves (in an appropriately modified norm  $\|\cdot\|'$ ) to  $(X, O)$  as possible.

# Outlier detection for Functional Data

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## A naïve outlier detection scheme

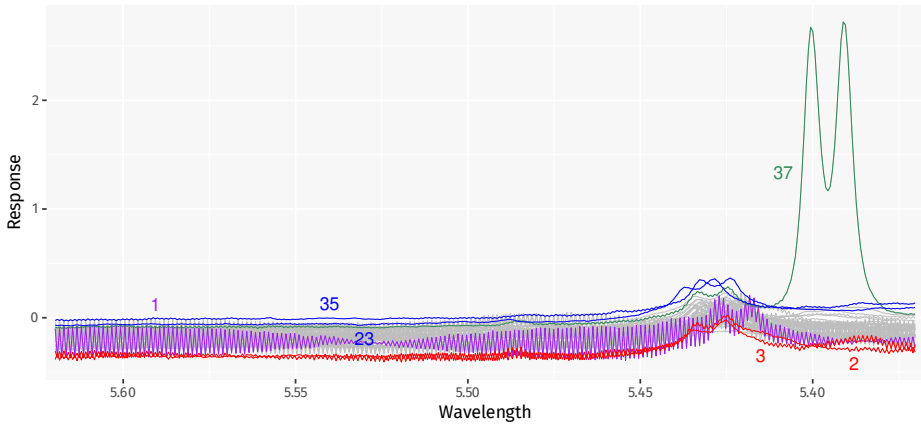
Given data  $\mathcal{D} = \{\mathbf{x}_i\}_{i=1}^n$ , we may extract ranks  $r_i = R(\mathbf{x}_i, \hat{F}_n)$ .

For instance, we may choose

$$R(\mathbf{x}, \hat{F}_n) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(D(\mathbf{x}_i, \hat{F}_n) \leq D(\mathbf{x}, \hat{F}_n)).$$

Declare those  $\mathbf{x}_i$  with unusually high ranks  $r_i$  as outliers, say greater than a cutoff  $Q_3 + 1.5 \text{ IQR}$ .

## NMR spectra of wine samples



# Functional outliers

A curve  $x: [0, 1] \rightarrow \mathbb{R}$  may exhibit outlying behaviour within a body of curves in many ways.

- **Isolated outlier:** Significant deviation over a short interval.
- **Persistent outlier:** Deviation over a large/entire interval.
  - Shape
  - Shift
  - Amplitude

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Hubert, M., Rousseeuw, P. J., & Segaert, P. (2015) Multivariate functional outlier detection

## Statistical Depth Functions

## └ Outlier detection for Functional Data

## └ Functional outliers

Functional outliers

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Hubert, M., Rousseeuw, P. J., & Seggent, P. (2015) Multivariate functional outlier detection

For a shape outlier, the slices  $\mathbf{x}(t)$  may all seem inconspicuous in the marginals  $F_{X(t)}$ .

# Shape outliers and derivatives

One way of incorporating shape information of a curve  $\mathbf{x}$  is to bundle it with its derivatives  $\mathbf{x}^{(j)}$ .

$$\int_{[0,1]} D((\mathbf{x}^{(0)}(t), \dots, \mathbf{x}^{(j)}(t))^{\top}, F_{(\mathbf{x}^{(0)}(t), \dots, \mathbf{x}^{(j)}(t))^{\top}}) w(t) dt.$$

## Statistical Depth Functions

└ Outlier detection for Functional Data

└ Shape outliers and derivatives

One way of incorporating shape information of a curve  $\mathbf{x}$  is to bundle it with its derivatives  $\mathbf{x}^{(j)}$ .

$$\int_{[0,1]} \mathcal{D}((\mathbf{x}^{(0)}(t), \dots, \mathbf{x}^{(J)}(t))^T, F_{\mathcal{D}}(\mathbf{x}^{(0)}(t), \dots, \mathbf{x}^{(J)}(t))^T) w(t) dt.$$

This suffers from errors in approximating derivatives, and the assumption of differentiability in the first place.



# Shape outliers and the $J$ -th order Integrated depth

We say that a curve  $\mathbf{x}$  is a  $J$ -th order outlier with respect to  $F_X$  if there exists  $\mathbf{t} \in [0, 1]^J$  such that the vector  $(\mathbf{x}(t_1), \dots, \mathbf{x}(t_J))^T$  is outlying with respect to  $F_{(\mathbf{x}(t_1), \dots, \mathbf{x}(t_J))^T}$ .

$$D_{FM}^J(\mathbf{x}, F_X) = \int_{[0,1]^J} D((\mathbf{x}(t_1), \dots, \mathbf{x}(t_J))^T, F_{(\mathbf{x}(t_1), \dots, \mathbf{x}(t_J))^T}) w(\mathbf{t}) d\mathbf{t}.$$

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Nagy, S., Gijbels, I., & and Hlubinka, D. (2017) Depth-Based Recognition of Shape Outlying Functions

## Statistical Depth Functions

## └ Outlier detection for Functional Data

└ Shape outliers and the  $J$ -th order Integrated depth

We say that a curve  $x$  is a  $J$ -th order outlier with respect to  $F_X$  if there exists  $\mathbf{t} \in [0, 1]^J$  such that the vector  $(x(t_1), \dots, x(t_J))^T$  is outlying with respect to  $F_{(X(t_1), \dots, X(t_J))^T}$ .

$$D_{J, \text{sh}}^{\text{out}}(x, F_X) = \int_{[0, 1]^J} D((x(t_1), \dots, x(t_J))^T, F_{(X(t_1), \dots, X(t_J))^T}) w(\mathbf{t}) d\mathbf{t}.$$

Naqvi, S., Gijbels, I. & Mizubata, D. (2017) Depth-Based Recognition of Shape Outlying Functions

- This process looks at points of the form  $(x(t), x(t+h), \dots)$ , thus encoding information about the derivatives.
- One may choose the weight function  $w(\cdot)$  to put emphasis on the diagonal.

# The Centrality-Stability scheme

Consider an outlyingness function  $O(\mathbf{x}(t))$  which measures the outlyingness of  $\mathbf{x}(t)$  with respect to  $F_{X(t)}$ . For instance, we may choose

$$O(\mathbf{x}(t)) = \frac{\mathbf{x}(t) - \text{med}(X(t))}{\text{MAD}(X(t))}.$$

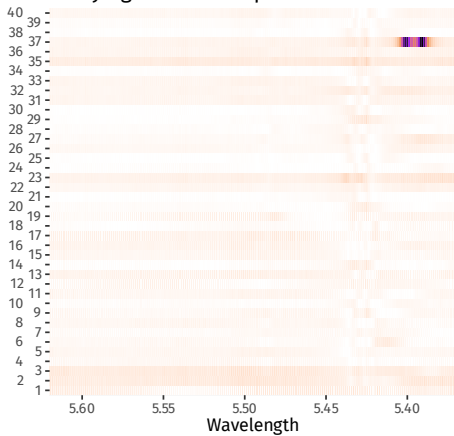
Then, we may define a depth function

$$D(\mathbf{x}, F_X) = \int_{[0,1]} (1 + |O(\mathbf{x}(t))|)^{-1} dt.$$

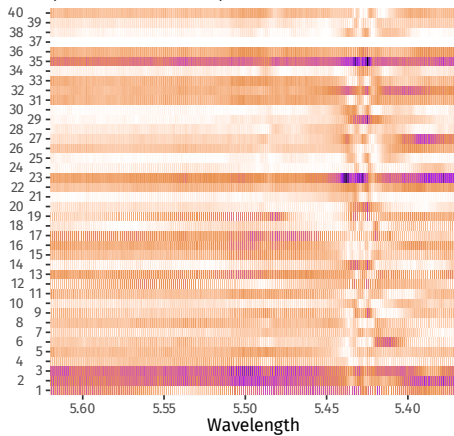
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Hubert, M., Rousseeuw, P. J., & Segaert, P. (2015) Multivariate functional outlier detection

Outlyingness heatmap



(Zeroed curve #37)



# The Centrality-Stability scheme

Define

$$\widetilde{MO}(\mathbf{x}, F_X) = \int_{[0,1]} |O(\mathbf{x}(t))| dt$$

Then, Cauchy-Schwarz gives

$$D(\mathbf{x}, F_X) \cdot (1 + \widetilde{MO}(\mathbf{x}, F_X)) \geq 1,$$

with equality when  $O(\mathbf{x}(\cdot))$  remains constant over time.

# The Centrality-Stability scheme

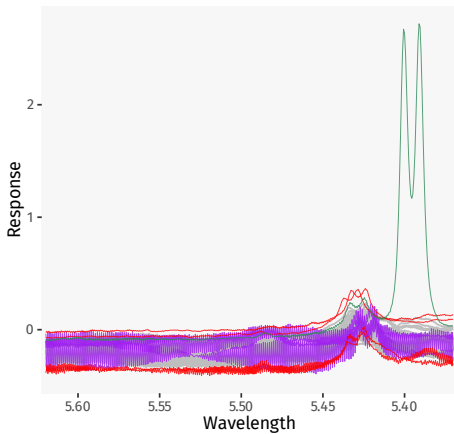
Any sudden deviation in outlyingness will be detected by the *stability deviation*

$$\Delta S = (1 + \widetilde{MO}(\mathbf{x}, F)) - \frac{1}{D(\mathbf{x}, F)}.$$

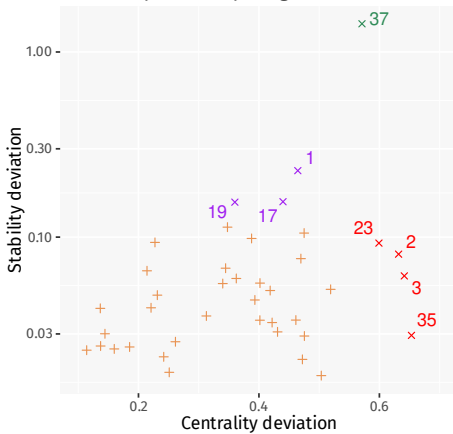
The *centrality deviation* is measured as

$$\Delta C = 1 - D(\mathbf{x}, F).$$

Data



Centrality-Stability diagram



# The MO-VO scheme

We may measure the variability in outlyingness over time more simply via

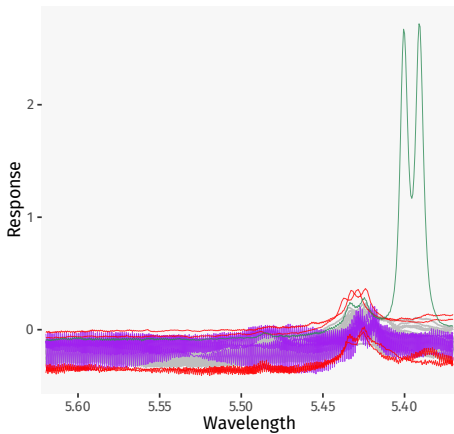
$$\begin{aligned}\mathbf{MO}(\mathbf{x}, F) &= \int_{[0,1]} O(\mathbf{x}(t)) \, dt, \\ \mathbf{VO}(\mathbf{x}, F) &= \int_{[0,1]} \|O(\mathbf{x}(t)) - \mathbf{MO}(\mathbf{x}, F)\|^2 \, dt.\end{aligned}$$

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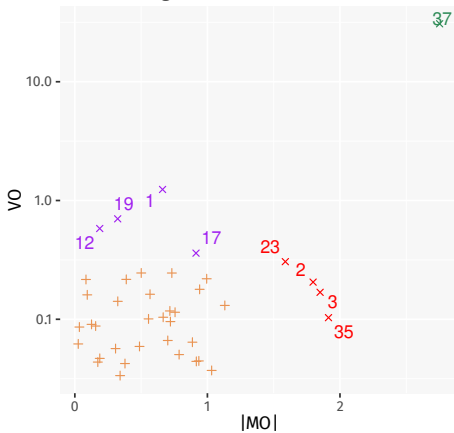
Dai, W., & Genton, M. G. (2018) An outlyingness matrix for multivariate functional data classification



Data



MO-VO diagram



# The Outliergram

Given a dataset of curves  $\mathcal{D} = \{\mathbf{x}_i\}_{i=1}^n$ , the distances

$$d_i = a_0 + a_1 \text{MEI}(\mathbf{x}_i) + a_2 n^2 \text{MEI}(\mathbf{x}_i)^2 - \text{MBD}(\mathbf{x}_i),$$

where  $a_0 = a_2 = -2/n(n+1)$ ,  $a_1 = 2(n+1)/(n-1)$ , are indicative of shape outlyingness.

Thus, one may declare  $\mathbf{x}_i$  as an outlier if  $d_i$  exceeds a cutoff such as  $Q_3 + 1.5 \text{IQR}$ .

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Arribas-Gil, A., & Romo, J. (2014). Shape outlier detection and visualization for functional data: the outliergram

# Statistical Depth Functions

## └ Outlier detection for Functional Data

### └ The Outliergram

The numbers  $d_i$  are always positive!

#### The Outliergram

Given a dataset of curves  $\mathcal{Y} = \{x_i\}_{i=1}^n$ , the distances

$$d_i = a_2 + a_1 \text{MEI}(x_i) + a_2 n^2 \text{MEI}(x_i)^2 - \text{MBD}(x_i),$$

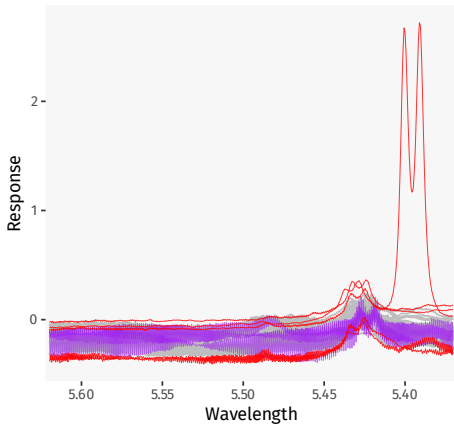
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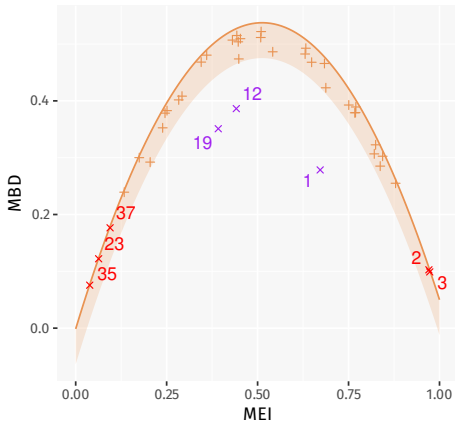
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Arratia-Gil, A., & Rønn, J. (2014). Shape outlier detection and visualization for functional data: the outliergram

Data



Outliergram



# Local Depth Functions

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# Elliptical distributions

We say that a distribution  $F$  is elliptical if it admits a density of the form

$$f_{\mathbf{x}}(\mathbf{x}) = c|\Sigma|^{-1/2}h\left((\mathbf{x} - \boldsymbol{\mu})^{\top}\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

for some strictly decreasing function  $h$ . Write  $F \in \text{Ell}(h; \boldsymbol{\mu}, \Sigma)$ .

An affine invariant depth function continuous in  $\mathbf{x}$  uniquely determines  $F$  within  $\text{Ell}(h; \cdot, \cdot)$ . The depth and density contours coincide.

## Statistical Depth Functions

## └ Local Depth Functions

## └ Elliptical distributions

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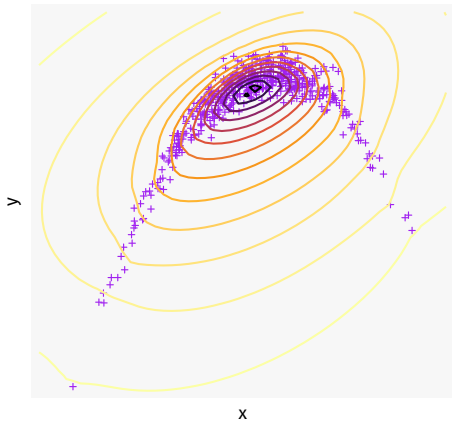
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for some strictly decreasing function  $h$ . Write  $F \in \text{Ell}(\mathbf{h}; \boldsymbol{\mu}, \Sigma)$ .

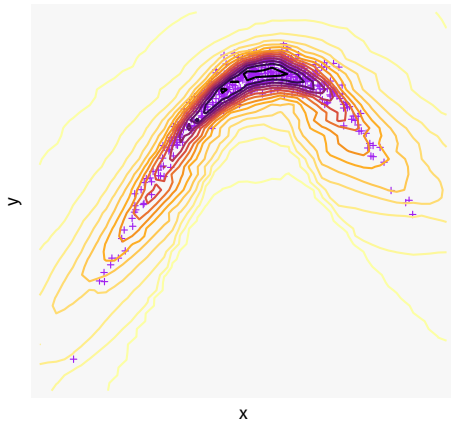
An affine invariant depth function continuous in  $\mathbf{x}$  uniquely determines  $F$  within  $\text{Ell}(\mathbf{h}, \cdot, \cdot)$ . The depth and density contours coincide.

- The whitened random variable  $\mathbf{Z} = \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$  has density  $f_{\mathbf{Z}}(\mathbf{z}) \propto h(\|\mathbf{z}\|^2)$ .
- The halfspace, simplicial, projection depths satisfy this property.
- In general, depths such as the halfspace depth always produce convex central regions.

Spatial depth



Local spatial depth,  $\beta = 0.2$





# Local Depth neighbourhoods

Given  $\mathbf{x} \in \mathcal{X}$ , we may symmetrize  $F_{\mathbf{X}}$  as

$$F_{\mathbf{X}}^{\mathbf{x}} = \frac{1}{2}F_{\mathbf{X}} + \frac{1}{2}F_{2\mathbf{x}-\mathbf{X}}.$$

The probability- $\beta$  depth-based neighbourhood of  $\mathbf{x}$  in  $F_{\mathbf{X}}$  is simply the  $\beta$ -th central region of  $F_{\mathbf{X}}^{\mathbf{x}}$ . This is denoted by  $N_{\beta}^{\mathbf{x}}(F_{\mathbf{X}})$ .

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Paindaveine, D., & Van Bever, G. (2013) From depth to local depth: A focus on centrality

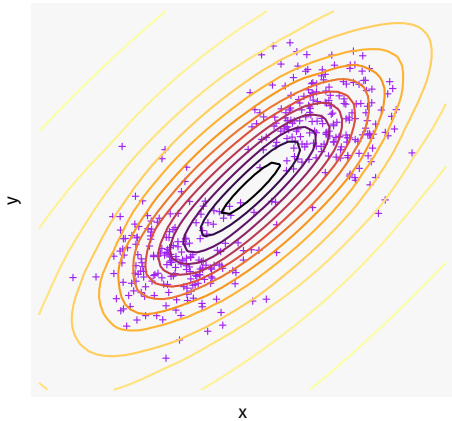
Let  $F_\beta^{\mathbf{x}}$  denote the distribution  $F_X$  conditioned on  $N_\beta^{\mathbf{x}}(F_X)$ .

The local depth function at locality level  $\beta \in (0, 1]$  is defined as

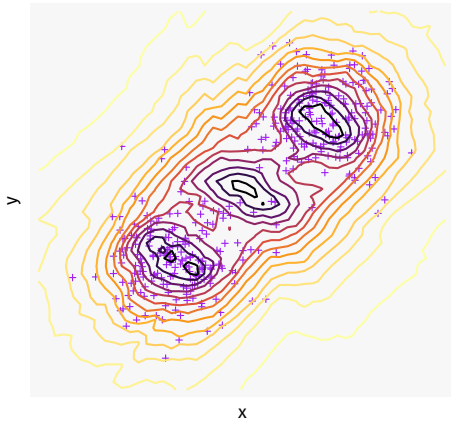
$$LD(\mathbf{x}, F_X) = D(\mathbf{x}, F_\beta^{\mathbf{x}}).$$

When  $\beta = 1$ , we have  $LD_1 = D$ .

Spatial depth



Local spatial depth,  $\beta = 0.2$



# Local Depth based Regression

Let  $\tilde{F}_\beta^{\mathbf{x}}$  denote the distribution  $F_X^{\mathbf{x}}$  conditioned on  $N_\beta^{\mathbf{x}}(F_X)$ . Note that this is angularly symmetric about  $\mathbf{x}$ .

Given  $\mathbf{x} \in \mathcal{X}$ , we may define a local depth kernel, centered at  $\mathbf{x}$ , via

$$K_\beta^{\mathbf{x}}: N_\beta^{\mathbf{x}}(F_X) \rightarrow \mathbb{R}, \quad \mathbf{z} \mapsto D(\mathbf{z}, \tilde{F}_\beta^{\mathbf{x}}).$$

Extend this to  $\mathcal{X}$  by setting  $K_\beta^{\mathbf{x}}(\cdot) = 0$  outside  $N_\beta^{\mathbf{x}}(F_X)$ .

# Local Depth based Regression

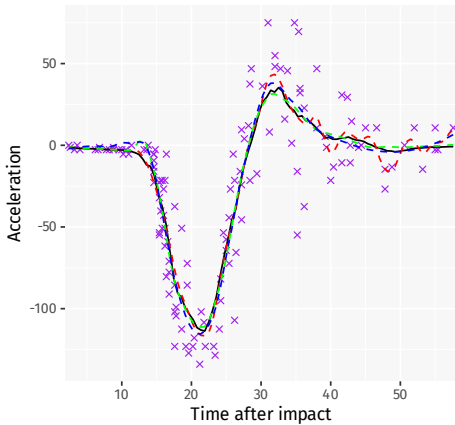
We propose a linear estimator of the form

$$\hat{y}_\beta(\mathbf{x}) = \sum_i w_i(\mathbf{x}) y_i, \quad w_i(\mathbf{x}) = \frac{K_\beta^\mathbf{x}(\mathbf{x}_i)}{\sum_j K_\beta^\mathbf{x}(\mathbf{x}_j)}.$$

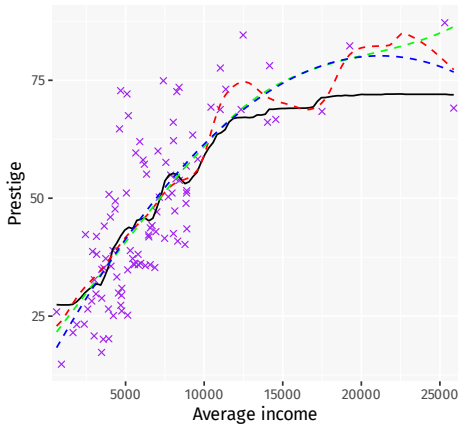
This may be interpreted as a weighted KNN estimator, or a variable bandwidth kernel estimator.

We only have one tuning parameter  $\beta \in (0, 1]$ .

Motorcycle accidents

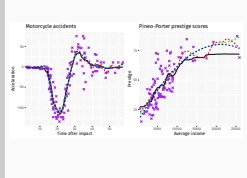


Pineo-Porter prestige scores



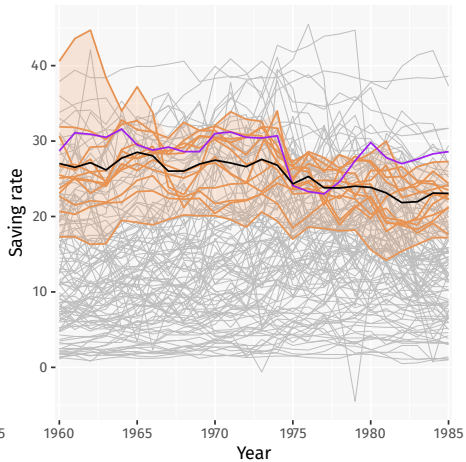
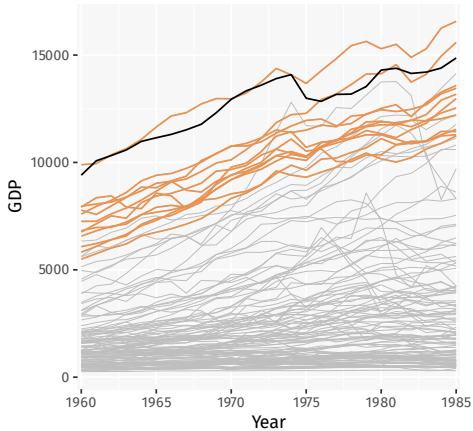
# Statistical Depth Functions

## └ Local Depth Functions



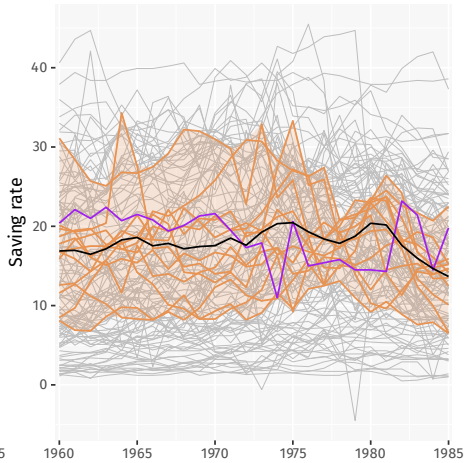
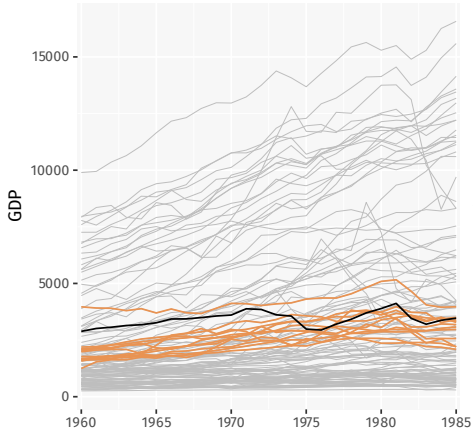
The methods used are local depth based regression (black), Nadaraya-Watson kernel (red), local linear (green) and quadratic (blue).

## Switzerland

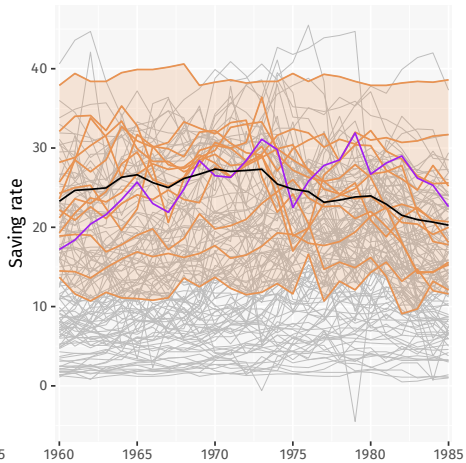
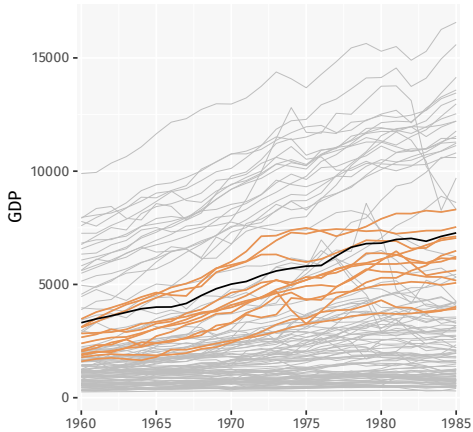




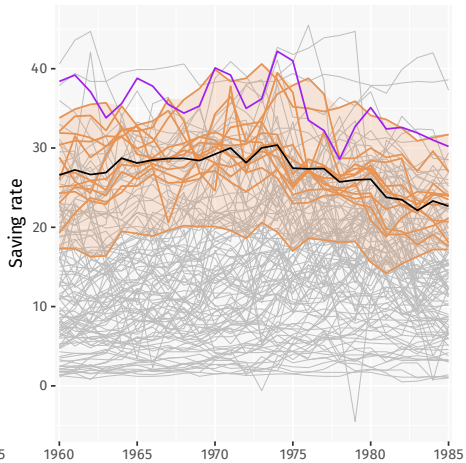
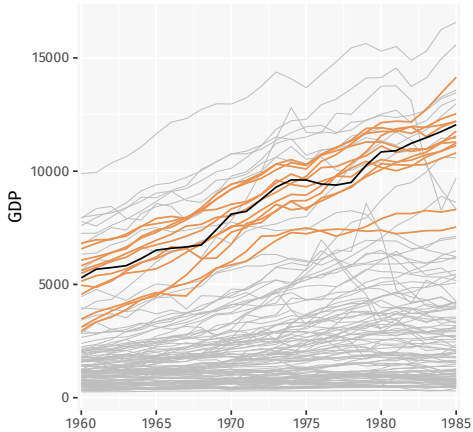
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## Ireland



## Finland



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