Statistical Depth Functions

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17 May, 2024

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Outline

A *depth function* quantifies how *central* a point $\boldsymbol{x} \in \mathscr{X}$ is with respect to a distribution F.

This induces a *center-outwards* ordering on the space $\mathscr{X}$.

We want $D\colon \mathbb{R}^d \times \mathscr{F} \to \mathbb{R}$ to be bounded, non-negative, continuous, and satisfy the following properties.

P1. **Affine invariance:** $D(A\mathbf{x} + b, F_{A\mathbf{x}+b}) = D(\mathbf{x}, F_X)$.

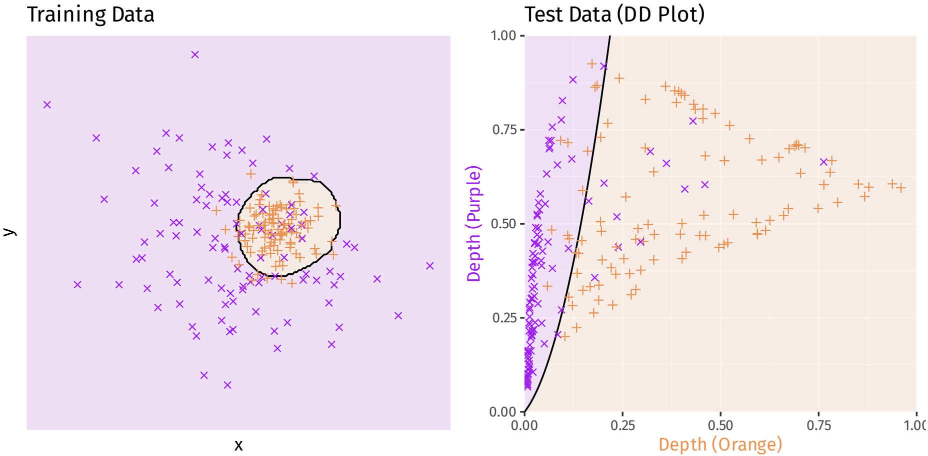
P2. **Maximality at centre:** $D(\theta, F_X) = \sup_{\mathbf{x}\in\mathbb{R}^d} D(\mathbf{x}, F)$.

P3. **Monotonicity along rays:** $D(\mathbf{x}, F) \leq D(\theta + \alpha(\mathbf{x} - \theta), F)$.

P4. **Vanish at infinity:** $D(\mathbf{x}, F) \to 0$ as $\|\mathbf{x}\| \to \infty$.

Zuo, Y., & Serfling, R. (2000) General notions of statistical depth function

The DD classifier



Depth Functions for Functional Data

Depth Functions in Banach spaces $\mathscr{X}$

Let $\mathscr{X}$ be a class of functions of the form $\boldsymbol{x}\colon [0,1] \to \mathbb{R}^d$, equipped with a norm $\|\cdot\|$. We typically choose $L^2[0,1]$ or $\mathcal{C}[0,1]$.

We want to generalize the Zuo-Serfling properties (P1-4) in this setting, for depth functions $D\colon \mathscr{X} \times \mathscr{F} \to \mathbb{R}$.

Gijbels, I., & Nagy, S. (2017) On a General Definition of Depth for Functional Data

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Properties $P3$ (Monotonicity along rays) and $P4$ (Vanish at infinity) carry over naturally.

Non-degeneracy

P0. **Non-degeneracy:** $\inf_{x \in \mathscr{X}} D(x, F) < \sup_{x \in \mathscr{X}} D(x, F)$.

The naïve generalization of the halfspace/Tukey depth

$$D_H(x, F) = \inf_{v \in \mathscr{X}^*} P_{X \sim F}(v^*(X) \leq v^*(x)),$$

is degenerate for a wide class of distributions $\mathscr{F}$. For instance, $\mathscr{X} = \mathcal{C}[0, 1]$, Gaussian processes with positive definite covariance kernels.

Chakraborty, A., & and Chaudhuri, P. (2014) On data depth in infinite dimensional spaces

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This also applies to the functional analogue of the projection depth.

Non-degeneracy

The functional analogue of the spatial depth

$$D_{Sp}(x, F) = 1 - \left\| \mathbb{E}_{X \sim F} \left[\frac{x - X}{\|x - X\|_2} \right] \right\|_2,$$

does not suffer from degeneracy.

Chakraborty, A., & and Chaudhuri, P. (2014) The spatial distribution in infinite dimensional spaces and related quantiles and depths

Affine invariance

P1S. **Scalar-affine invariance:** For $a, b \in \mathbb{R}$ with a non-zero and $\mathbf{x} \in \mathscr{X}$,

$$D(a\mathbf{x} + b, F_{a\mathbf{X}+b}) = D(\mathbf{x}, F_{\mathbf{X}}).$$

P1F. **Function-affine invariance:** For $a, b, x \in \mathscr{X}$, with $ax \in \mathscr{X}$,

$$D(ax + b, F_{aX+b}) = D(x, F_X).$$

Maximality at center

We say that F_X is symmetric about $\boldsymbol{\theta} \in \mathscr{X}$ if for all $\varphi \in \mathscr{X}^*$, we have $\varphi(X)$ symmetric about $\varphi(\boldsymbol{\theta})$.

P2C. **Maximality at center of central symmetry:** For $F \in \mathscr{X}$ centrally symmetric about $\boldsymbol{\theta} \in \mathscr{X}$, $D(\boldsymbol{\theta}, F) = \sup_{x \in \mathscr{X}} D(x, F)$.

P2H. **Maximality at center of halfspace symmetry:** For $F \in \mathscr{X}$ halfspace symmetric about $\boldsymbol{\theta} \in \mathscr{X}$, $D(\boldsymbol{\theta}, F) = \sup_{x \in \mathscr{X}} D(x, F)$.

The Integrated and Infimal Depths

$$D_{FM}(x, F_X) = \int_{[0,1]} D(x(t), F_{X(t)})\, w(t)\, dt.$$

$$D_{Inf}(x, F_X) = \inf_{t \in [0,1]} D(x(t), F_{X(t)}).$$

The J-th order Integrated and Infimal Depths

$$D^J_{FM}(x, F_X) = \int_{[0,1]^J} D((x(t_1), \ldots, x(t_J))^\top, F_{(X(t_1),\ldots,X(t_J))^\top})\, w(\mathbf{t})\, d\mathbf{t}.$$

$$D^J_{Inf}(x, F_X) = \inf_{\mathbf{t} \in [0,1]^J} D((x(t_1), \ldots, x(t_J))^\top, F_{(X(t_1),\ldots,X(t_J))^\top}).$$

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These J-th order depths carry information about the derivatives of the curves, of orders $0, \dots, J-1$.

The Band Depth

$$D_B^J(x, F) = \sum_{j=2}^{J} P_{X_i \overset{\text{iid}}{\sim} F}(x \in \operatorname{conv}(X_1, \ldots, X_j)).$$

This is the proportion of j-tuples of curves, for $2 \leq j \leq J$, which *completely* envelope x.

The band depth becomes degenerate for $\mathscr{X} = \mathcal{C}[0,1]$, Feller processes X (e.g. Brownian motion) with $P(X_0 = 0) = 1$ and each X_t for $t > 1$ non-atomic and symmetric about 0.

The Modified Band Depth

Define the *enveloping time*

$$\mathrm{ET}(x; x_1, \dots, x_j) = m_1(\{t \in [0,1] \colon x(t) \in \mathrm{conv}(x_1(t), \dots, x_j(t))\})$$

The modified band depth is defined as

$$D_{\mathrm{MBD}}(x, F) = \sum_{j=2}^{J} \mathbb{E}_{X_i \overset{\mathrm{iid}}{\sim} F} \left[\mathrm{ET}(x; X_1, \dots, X_j) \right].$$

The Half-Region Depth

We say that $\mathbf{y}$ is in the hypograph (resp. epigraph) of $\mathbf{x}$, denoted $\mathbf{y} \in H_{\mathbf{x}}$ (resp. $E_{\mathbf{x}}$), if $\mathbf{y}(t) \leq \mathbf{x}(t)$ (resp. $\geq$) for all $t \in [0, 1]$.

The half-region depth is defined as

$$D_{HR}(\mathbf{x}, F) = \min \left\{ P_F(H_{\mathbf{x}}),\ P_F(E_{\mathbf{x}}) \right\}.$$

This suffers from the same degeneracy problems as the band depth.

The Modified Half-Region Depth

Define the Modified Hypograph (MHI) and Epigraph (MEI) Indices as

$$\mathrm{MHI}_F(x) = \mathbb{E}_{X \in F}[m_1\{t \in [0,1] \colon x(t) \geq X(t)\}],$$
$$\mathrm{MEI}_F(x) = \mathbb{E}_{X \in F}[m_1\{t \in [0,1] \colon x(t) \leq X(t)\}].$$

The modified half-region depth is defined as

$$D_{MHR}(x, F) = \min\left\{\mathrm{MHI}_F(x),\ \mathrm{MEI}_F(x)\right\}.$$

Partially Observed Functional Data

Suppose that $X \sim F_X$ is not observed on the entire interval $[0,1]$, but rather on some random compact subinterval $O \sim Q$ (independent of X).

Given a dataset $\mathscr{D} = \{(X_i, O_i)\}_{i=1}^n$ where $(X_i, O_i) \overset{\text{iid}}{\sim} F_X \times Q$, we keep track of the indices observed at time $t \in [0,1]$ as $\mathscr{J}(t) = \{j \colon t \in O_j\}$, as well as their number $q(t) = |\mathscr{J}(t)|$.

The Partially Observed Integrated Functional Depth (POIFD)

We may define a depth function in this setting via

$$D_{POIFD}((\mathbf{X}, o), F_{\mathbf{X}} \times Q) = \int_{o} D(\mathbf{X}(t), F_{\mathbf{X}(t)})\, w_o(t)\, dt,$$

where $w_o(t) = q(t) / \int_0 q(t)\, dt$.

The functional reconstruction problem

Given (X, O), can we estimate X on $M = [0,1] \setminus O$?

We may search for a reconstruction operator $\mathcal{R}\colon L^2(O) \to L^2(M)$ that minimizes the mean integrated prediction squared error loss $\mathbb{E}[\|X_M - \mathcal{R}(X_O)\|^2]$. In this setup, the best predictor is the conditional expectation $\mathbb{E}[X_M \mid X_O]$.

We may also search for a continuous linear reconstruction operator $\mathscr{A}$, by estimating terms of the Karhunen-Loéve expansion of X.

The functional reconstruction problem

Another approach is to take a convex linear combination of curves from a suitable curve envelope with indices $\mathscr{I}$.

The enveloping curves $\mathscr{I}$ may be chosen so that (X, O) is as deep as possible inside the curve envelope.

Additionally, we want $\mathscr{I}$ to envelope (X, O) for as long as possible (in the sense of the enveloping time ET), and contain as many near curves (in an appropriately modified norm $\|\cdot\|'$) to (X, O) as possible.

Outlier detection for Functional Data

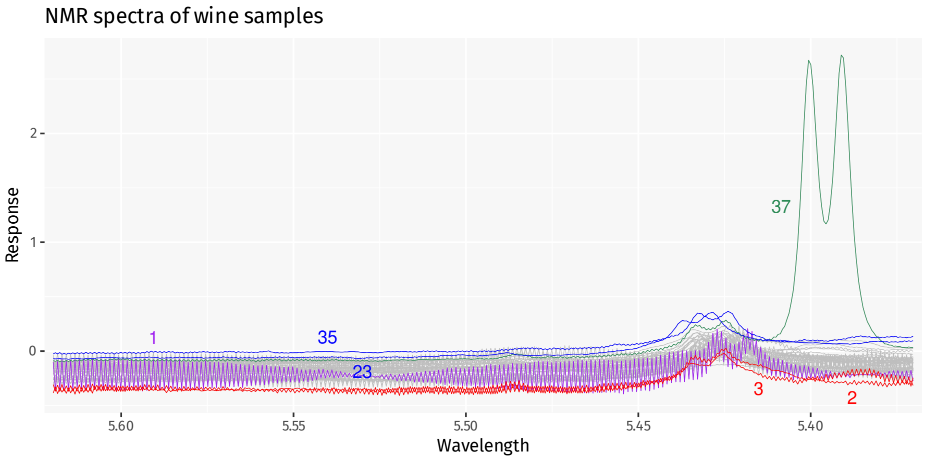
A naïve outlier detection scheme

Given data $\mathscr{D} = \{\mathbf{x}_i\}_{i=1}^{n}$, we may extract ranks $r_i = R(\mathbf{x}_i, \hat{F}_n)$.

For instance, we may choose

$$R(\mathbf{x}, \hat{F}_n) \; = \; \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(D(\mathbf{x}_i, \hat{F}_n) \leq D(\mathbf{x}, \hat{F}_n)).$$

Declare those $\mathbf{x}_i$ with unusually high ranks r_i as outliers, say greater than a cutoff $Q_3 + 1.5\,\mathrm{IQR}$.


NMR spectra of wine samples
Response
Wavelength
2
1
0
5.60
5.55
5.50
5.45
5.40
37
1
35
23
3
2

Functional outliers

A curve $x \colon [0, 1] \to \mathbb{R}$ may exhibit outlying behaviour within a body of curves in many ways.

- **Isolated outlier:** Significant deviation over a short interval.
- **Persistent outlier:** Deviation over a large/entire interval.
  - Shape
  - Shift
  - Amplitude

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For a shape outlier, the slices $x(t)$ may all seem inconspicuous in the marginals $F_{X(t)}$.

Shape outliers and derivatives

One way of incorporating shape information of a curve $\boldsymbol{x}$ is to bundle it with its derivatives $\boldsymbol{x}^{(j)}$.

$$\int_{[0,1]} D((\boldsymbol{x}^{(0)}(t), \ldots, \boldsymbol{x}^{(J)}(t))^\top, F_{(\boldsymbol{x}^{(0)}(t), \ldots, \boldsymbol{x}^{(J)}(t))^\top})\, w(t)\, dt.$$

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$$\int_{[0,1]} D((x^{(0)}(t), \dots, x^{(J)}(t))^\top, F_{(X^{(0)}(t), \dots, X^{(J)}(t))^\top})\, w(t)\, dt.$$

This suffers from errors in approximating derivatives, and the assumption of differentiability in the first place.

Shape outliers and the J-th order Integrated depth

We say that a curve $\boldsymbol{x}$ is a J-th order outlier with respect to $F_{\boldsymbol{X}}$ if there exists $\boldsymbol{t} \in [0,1]^J$ such that the vector $(\boldsymbol{x}(t_1), \ldots, \boldsymbol{x}(t_J))^\top$ is outlying with respect to $F_{(\boldsymbol{X}(t_1),\ldots,\boldsymbol{X}(t_J))^\top}$.

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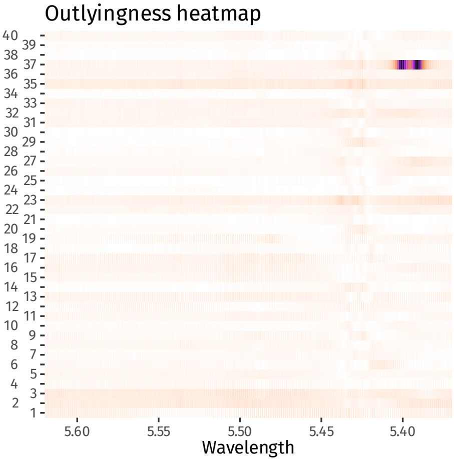
- This process looks at points of the form $(\boldsymbol{x}(t), \boldsymbol{x}(t+h), \dots)$, thus encoding information about the derivatives.
- One may choose the weight function $w(\cdot)$ to put emphasis on the diagonal.

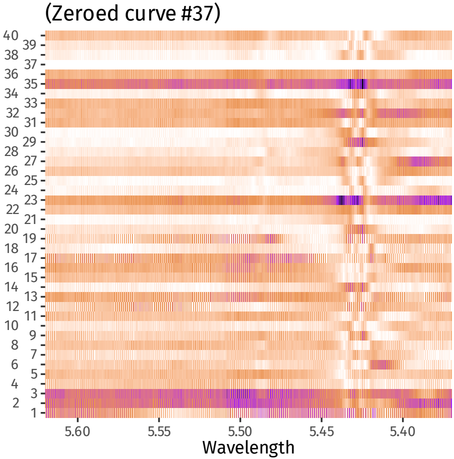
Consider an outlyingness function $O(\mathbf{x}(t))$ which measures the outlyingness of $\mathbf{x}(t)$ with respect to $F_{X(t)}$. For instance, we may choose

$$O(\mathbf{x}(t)) = \frac{\mathbf{x}(t) - \mathrm{med}(X(t))}{\mathrm{MAD}(X(t))}.$$

Then, we may define a depth function

$$D(\mathbf{x}, F_X) = \int_{[0,1]} (1 + |O(\mathbf{x}(t))|)^{-1} \, dt.$$


Outlyingness heatmap
40
39
38
37
36
35
34
33
32
31
30
29
28
27
26
25
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
5.60
5.55
5.50
5.45
5.40
Wavelength


(Zeroed curve #37)
40
39
38
37
36
35
34
33
32
31
30
29
28
27
26
25
24
23
22
21
20
19
18
17
16
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12
11
10
9
8
7
6
5
4
3
2
1
5.60
5.55
5.50
5.45
5.40
Wavelength

Define

$$\widetilde{MO}(x, F_X) = \int_{[0,1]} |O(x(t))| \, dt$$

Then, Cauchy-Schwarz gives

$$D(x, F_X) \cdot (1 + \widetilde{MO}(x, F_X)) \geq 1,$$

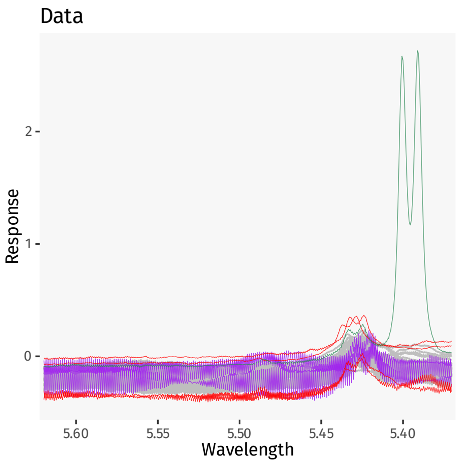
with equality when $O(x(\cdot))$ remains constant over time.

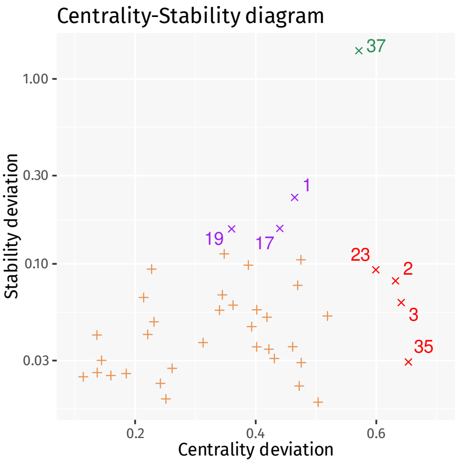
Any sudden deviation in outlyingness will be detected by the *stability deviation*

$$\Delta S = (1 + \widetilde{MO}(\mathbf{x}, F)) - \frac{1}{D(\mathbf{x}, F)}.$$

The *centrality deviation* is measured as

$$\Delta C = 1 - D(\mathbf{x}, F).$$

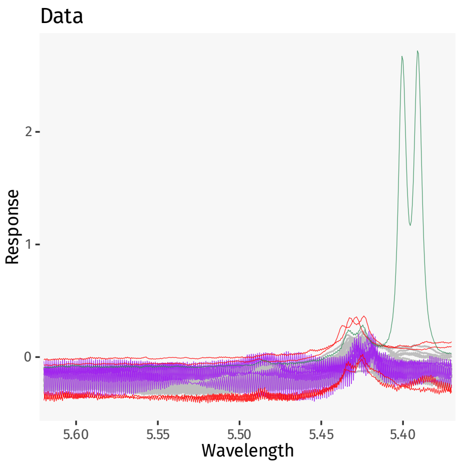

Data
Response
2
1
0
5.60
5.55
5.50
5.45
5.40
Wavelength


Centrality-Stability diagram
Stability deviation
1.00
0.30
0.10
0.03
0.2
0.4
0.6
Centrality deviation
37
1
19
17
23
2
3
35

The MO-VO scheme

We may measure the variability in outlyingness over time more simply via

$$\begin{aligned}
\mathrm{MO}(\mathbf{x}, F) &= \int_{[0,1]} O(\mathbf{x}(t))\, dt, \\
\mathrm{VO}(\mathbf{x}, F) &= \int_{[0,1]} \|O(\mathbf{x}(t)) - \mathrm{MO}(\mathbf{x}, F)\|^2\, dt.
\end{aligned}$$


Data
Response
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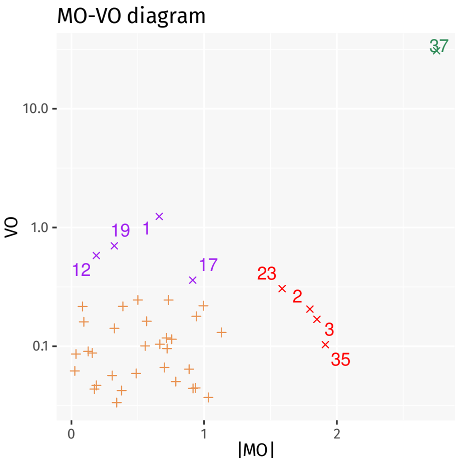
5.60
5.55
5.50
5.45
5.40
Wavelength


MO-VO diagram
VO
10.0
1.0
0.1
0
1
2
|MO|
37
19
1
12
17
23
2
3
35

The Outliergram

Given a dataset of curves $\mathscr{D} = \{\mathbf{x}_i\}_{i=1}^n$, the distances

$$d_i = a_0 + a_1 \operatorname{MEI}(\mathbf{x}_i) + a_2 n^2 \operatorname{MEI}(\mathbf{x}_i)^2 - \operatorname{MBD}(\mathbf{x}_i),$$

where $a_0 = a_2 = -2/n(n+1)$, $a_1 = 2(n+1)/(n-1)$, are indicative of shape outlyingness.

Thus, one may declare $\mathbf{x}_i$ as an outlier if d_i exceeds a cutoff such as $Q_3 + 1.5 \operatorname{IQR}$.

The numbers d_i are always positive!

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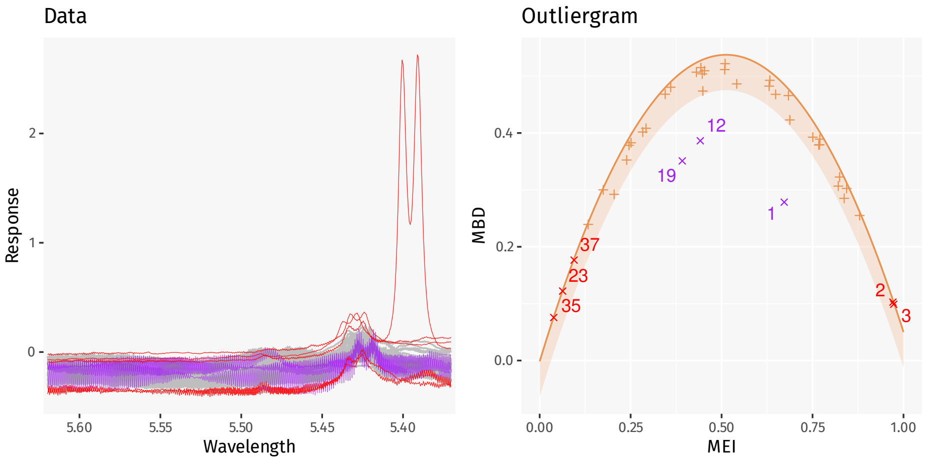
Data
Response
Wavelength
2
1
0
5.60
5.55
5.50
5.45
5.40
Outliergram
MBD
MEI
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
12
19
1
37
23
35
2
3

Local Depth Functions

Elliptical distributions

We say that a distribution F is elliptical if it admits a density of the form

$$f_X(x) = c|\Sigma|^{-1/2} h\left((x-\boldsymbol{\mu})^\top \Sigma^{-1}(x-\boldsymbol{\mu})\right)$$

for some strictly decreasing function h. Write $F \in \mathrm{Ell}(h; \boldsymbol{\mu}, \Sigma)$.

An affine invariant depth function continuous in x uniquely determines F within $\mathrm{Ell}(h; \cdot, \cdot)$. The depth and density contours coincide.

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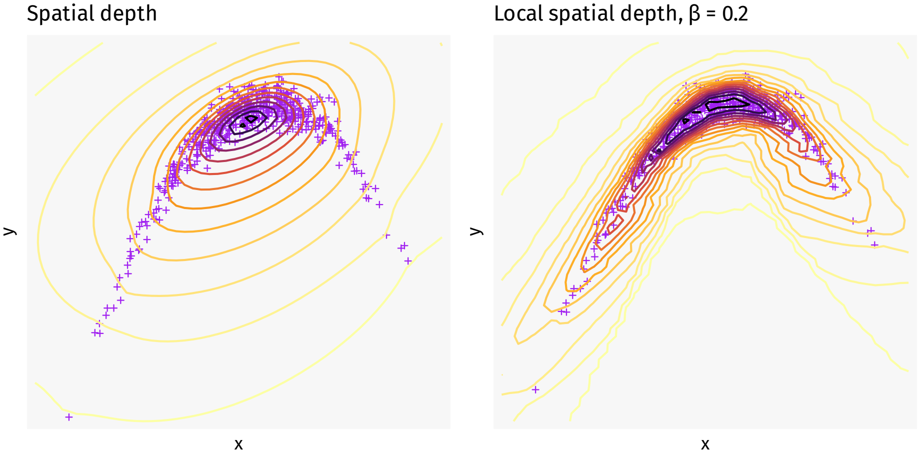
- The whitened random variable $\boldsymbol{Z} = \Sigma^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})$ has density $f_Z(\boldsymbol{z}) \propto h(\|\boldsymbol{z}\|^2)$.
- The halfspace, simplicial, projection depths satisfy this property.
- In general, depths such as the halfspace depth always produce convex central regions.


Spatial depth
y
x
Local spatial depth, β = 0.2
y
x

Local Depth neighbourhoods

Given $\boldsymbol{x} \in \mathscr{X}$, we may symmetrize F_X as

$$F_X^{\boldsymbol{x}} = \frac{1}{2} F_X + \frac{1}{2} F_{2\boldsymbol{x}-X}.$$

The probability-β depth-based neighbourhood of $\boldsymbol{x}$ in F_X is simply the β-th central region of $F_X^{\boldsymbol{x}}$. This is denoted by $N_\beta^{\boldsymbol{x}}(F_X)$.

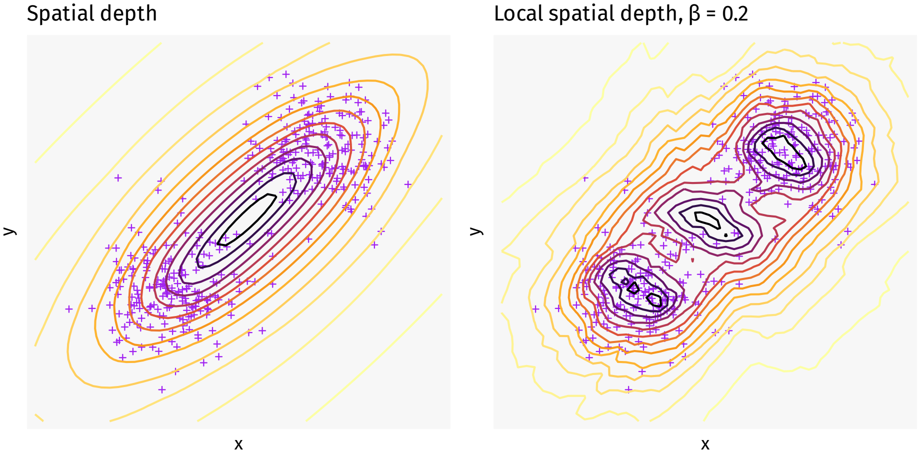
Local Depth

Let $F_\beta^{\mathbf{x}}$ denote the distribution $F_{\mathbf{X}}$ conditioned on $N_\beta^{\mathbf{x}}(F_{\mathbf{X}})$.

The local depth function at locality level $\beta \in (0, 1]$ is defined as

$$LD(\mathbf{x}, F_{\mathbf{X}}) = D(\mathbf{x}, F_\beta^{\mathbf{x}}).$$

When $\beta = 1$, we have $LD_1 = D$.


Spatial depth
y
x
Local spatial depth, β = 0.2
y
x

Local Depth based Regression

Let $\widetilde{F}^{\mathbf{x}}_{\beta}$ denote the distribution $F^{\mathbf{x}}_{\mathbf{X}}$ conditioned on $N^{\mathbf{x}}_{\beta}(F_{\mathbf{X}})$. Note that this is angularly symmetric about $\mathbf{x}$.

Given $\mathbf{x} \in \mathscr{X}$, we may define a local depth kernel, centered at $\mathbf{x}$, via

$$K^{\mathbf{x}}_{\beta} \colon N^{\mathbf{x}}_{\beta}(F_{\mathbf{X}}) \to \mathbb{R}, \qquad \mathbf{z} \mapsto D(\mathbf{z}, \widetilde{F}^{\mathbf{x}}_{\beta}).$$

Extend this to $\mathscr{X}$ by setting $K^{\mathbf{x}}_{\beta}(\cdot) = 0$ outside $N^{\mathbf{x}}_{\beta}(F_{\mathbf{X}})$.

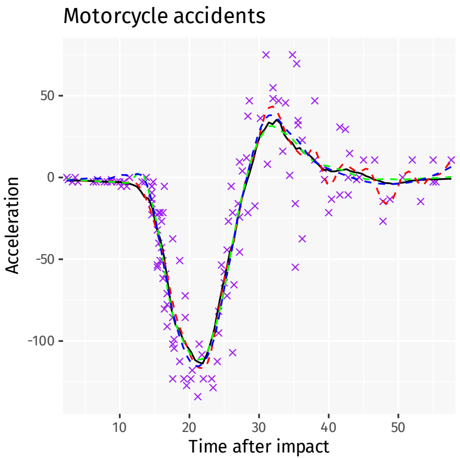
Local Depth based Regression

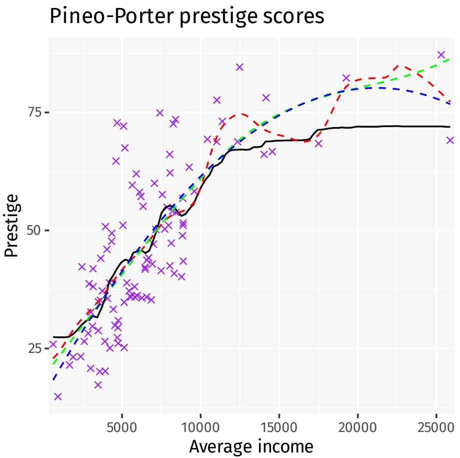
We propose a linear estimator of the form

$$\hat{y}_\beta(x) = \sum_i w_i(x) y_i, \qquad w_i(x) = \frac{K_\beta^x(x_i)}{\sum_j K_\beta^x(x_j)}.$$

This may be interpreted as a weighted KNN estimator, or a variable bandwidth kernel estimator.

We only have one tuning parameter $\beta \in (0, 1]$.


Motorcycle accidents
Acceleration
50
0
-50
-100
10
20
30
40
50
Time after impact


Pineo-Porter prestige scores
Prestige
75
50
25
5000
10000
15000
20000
25000
Average income

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