Complex Expressions

Satvik Saha

Problem Let $a + b + c = 0$. Prove the following, where $\omega^3 - 1 = 0$.

$$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = 27abc$$

Solution Note the following identities.

$$\begin{aligned} 1 + \omega + \omega^2 &= 0 \qquad (1) \\ x^3 + y^3 &= (x + y)(x\omega^2 + y\omega)(x\omega + y\omega^2) \qquad (2) \end{aligned}$$

Let $x = a + b\omega + c\omega^2$ and $y = a + b\omega^2 + c\omega$. Clearly,

$$\begin{aligned} x + y &= 2a + b(\omega + \omega^2) + c(\omega^2 + \omega) \\ &= 2a - b - c \\ &= 3a \\ x\omega^2 + y\omega &= a(\omega^2 + \omega) + 2b + c(\omega + \omega^2) \\ &= -a + 2b + -c \\ &= 3b \\ x\omega + y\omega^2 &= a(\omega + \omega^2) + b(\omega^2 + \omega) + 2c \\ &= -a - b + 2c \\ &= 3c \end{aligned}$$

Using (2), we can write

$$x^3 + y^3 = (3a)(3b)(3c) = 27abc$$

which, with change in notation, is the desired result.