Telescoping Sums

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A sum in which subsequent terms cancel each other, leaving only its initial and final terms is called a *telescoping sum*.

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1}(T_{k+1} - T_k) \\ &= (T_1 - T_0) + (T_2 - T_1) + (T_3 - T_2) + \ldots (T_n - T_{n-1}) \\ &= T_n - T_0 \end{aligned}$$

Problem 1 Calculate the given sum, to n terms

$$S_n = (1) \cdot 0! + (3) \cdot 1! + (7) \cdot 2! + (13) \cdot 3! + (21) \cdot 4! + \ldots + (n^2 - n + 1) \cdot n!$$

Solution Define the sequences

$$\begin{aligned} T_k &= (k^2 + k + 1) \cdot k! \\ F_k &= k \cdot k! \end{aligned}$$

Note that $n! = n \cdot (n-1)!$

$$\begin{aligned} F_{k+1} &= (k+1) \cdot (k+1)! \\ &= (k+1) \cdot (k+1) \cdot k! \\ &= (k^2 + 2k + 1) \cdot k! \\ &= (k^2 + k + 1) \cdot k! + k \cdot k! \\ &= T_k + F_k \end{aligned}$$

$$\boxed{T_k = F_{k+1} - F_k}$$

Finally,

$$\begin{aligned} \sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1}(F_{k+1} - F_k) \\ &= F_n - F_0 \end{aligned}$$

$$\boxed{S_n = n \cdot n!}$$

Problem 2 Calculate the given sum.

$$S_n = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^{n-1}}{x^{2^{n-1}}+1}$$

Solution Define the sequences

$$\begin{aligned} T_k &= \frac{2^k}{x^{2^k}+1} \\ F_k &= \frac{2^k}{x^{2^k}-1} \end{aligned}$$

Observe

$$\begin{aligned} F_k - T_k &= 2^k \cdot \left(\frac{1}{x^{2^k}-1} - \frac{1}{x^{2^k}+1} \right) \\ &= 2^k \cdot \left(\frac{2}{x^{2^k \cdot 2}-1} \right) \\ &= \frac{2^{k+1}}{x^{2^{k+1}}-1} \\ &= F_{k+1} \\ \boxed{T_k \;=\; F_k - F_{k+1}} \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_k - F_{k+1}) \\ &= F_0 - F_n \\ \boxed{S_n \;=\; \frac{1}{x-1} - \frac{2^n}{x^{2^n}-1}} \end{aligned}$$

Note that for $(x > 1)$, L'Hospital's Rule yields

$$\lim_{n\to\infty} \left(\frac{2^n}{x^{2^n}-1} \right) = \lim_{n\to\infty} \left(\frac{2^n \ln 2}{2^n \ln 2 \cdot x^{2^n} \ln x} \right) = 0$$

$$\boxed{S_\infty = \frac{1}{x-1}} \qquad (x > 1)$$