Summation Identities

Satvik Saha

General manipulations

$$\begin{aligned}
\sum_{n=s}^{t} C \cdot f(n) &= C \cdot \sum_{n=s}^{t} f(n) \\
\sum_{n=s}^{t} f(n) \pm \sum_{n=s}^{t} g(n) &= \sum_{n=s}^{t} [f(n) \pm g(n)] \\
\sum_{n=s}^{t} f(n) &= \sum_{n=s+k}^{t+k} f(n-k) \\
\sum_{n=a}^{b} f(n) + \sum_{n=b+1}^{c} f(n) &= \sum_{n=a}^{c} f(n) \\
\sum_{i=k_0}^{k} \sum_{j=l_0}^{l} a_{i,j} &= \sum_{j=l_0}^{l} \sum_{i=k_0}^{k} a_{i,j} \\
\sum_{i=s}^{m} \sum_{j=t}^{n} a_i b_j &= \left(\sum_{i=s}^{m} a_i \right) \cdot \sum_{j=t}^{n} b_j
\end{aligned}$$

$$\left(\sum_{k=0}^{n} a_k \right) \cdot \left(\sum_{k=0}^{n} b_k \right) = \sum_{k=0}^{2n} \sum_{i=0}^{k} a_i b_{k-i} - \sum_{k=0}^{n-1} \left(a_k \sum_{i=n+1}^{2n-k} b_i + b_k \sum_{i=n+1}^{2n-k} a_i \right)$$

Polynomial expressions

$$\begin{aligned}
\sum_{k=m}^{n} 1 &= n+1-m \\
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{k=1}^{n} k^3 &= \left[\frac{n(n+1)}{2}\right]^2 \\
\sum_{k=1}^{n} k^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\
\sum_{k=0}^{n-1} a^k &= \frac{a^n-1}{a-1} \\
\sum_{k=0}^{n-1} ka^k &= \frac{a-na^n+(n-1)a^{n+1}}{(a+1)^2}
\end{aligned}$$

Binomial coefficients and Factorials

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} &= 2^n \\
\sum_{i=0}^{n} \binom{n}{i}^2 &= \binom{2n}{i} \\
\sum_{i=0}^{n} \binom{m+i-1}{i} &= \binom{m+n}{n} \\
\sum_{i=1}^{n} i\binom{n}{i} &= n2^{n-1} \\
\sum_{i=0}^{n} i! \cdot \binom{n}{i} &= \lfloor n! \cdot e \rfloor \\
\sum_{i=0}^{n} \binom{i}{k} &= \binom{n+1}{k+1} \\
\sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i &= (a+b)^n
\end{aligned}$$