Telescoping Sums

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A sum in which subsequent terms cancel each other, leaving only its initial and final terms is called a *telescoping sum*.

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1} (T_{k+1} - T_k) \\ &= (T_1 - T_0) + (T_2 - T_1) + (T_3 - T_2) + \ldots (T_n - T_{n-1}) \\ &= T_n - T_0 \end{aligned}$$

Problem 1 Calculate the given sum.

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

Solution Define the sequences

$$\begin{aligned} T_k &= \frac{1}{k \cdot (k+1)} \\ F_k &= \frac{1}{k} \end{aligned}$$

Observe that we can write

$$T_k = \frac{(k+1) - k}{k \cdot (k+1)} \qquad = \frac{1}{k} - \frac{1}{k+1} = F_k - F_{k+1}$$

Finally

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (F_k - F_{k+1}) \qquad = F_1 - F_{n+1}$$

$$\boxed{S_n = 1 - \frac{1}{n+1}}$$

Problem 2 Calculate the given sum.

$$S_n = \frac{1}{1\cdot 3\cdot 5} + \frac{1}{3\cdot 5\cdot 7} + \frac{1}{5\cdot 7\cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)(2n+5)}$$

Solution Define the expression for each term and manipulate its factors.

$$\begin{aligned} T_k &= \frac{1}{(2k+1)(2k+3)(2k+5)} \\ &= \frac{1}{4(2k+3)}\left(\frac{1}{2k+1} - \frac{1}{2k+5}\right) \\ &= \frac{1}{4}\left(\frac{1}{(2k+1)(2k+3)} - \frac{1}{(2k+3)(2k+5)}\right) \\ &= \frac{1}{8}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) - \frac{1}{8}\left(\frac{1}{2k+3} - \frac{1}{2k+5}\right) \end{aligned}$$

Define the sequence

$$\begin{aligned} F_k &= \frac{1}{8(2k+1)} \\ T_k &= (F_k - F_{k+1}) - (F_{k+1} - F_{k+2}) \end{aligned}$$

Finally

$$\begin{aligned} \sum_{k=0}^{n} T_k &= \sum_{k=0}^{n}(F_k - F_{k+1}) - \sum_{k=0}^{n}(F_{k+1} - F_{k+2}) \\ &= (F_0 - F_{n+1}) - (F_1 - F_{n+2}) \end{aligned}$$

$$\boxed{S_n = \frac{1}{12} - \frac{1}{4(2n+3)(2n+5)}}$$

Problem 3 Calculate the given sum, to n terms

$$S_n \;=\; (1)\cdot 0! \,+\, (3)\cdot 1! \,+\, (7)\cdot 2! \,+\, \ldots \,+\, (n^2 - n + 1)\cdot n!$$

Solution Define the sequences

$$\begin{aligned} T_k &= (k^2 + k + 1)\cdot k! \\ F_k &= k \cdot k! \end{aligned}$$

Note that $n! = n \cdot (n-1)!$

$$\begin{aligned} F_{k+1} &= (k+1)\cdot(k+1)! \\ &= (k+1)\cdot(k+1)\cdot k! \\ &= (k^2 + 2k + 1)\cdot k! \\ &= (k^2 + k + 1)\cdot k! + k\cdot k! \\ &= T_k + F_k \\ \boxed{T_k \;=\; F_{k+1} - F_k} \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_{k+1} - F_k) \\ &= F_n - F_0 \\ \boxed{S_n \;=\; n \cdot n!} \end{aligned}$$

Problem 4 Calculate the given sum.

$$S_n \;=\; \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^{n-1}}{x^{2^{n-1}}+1}$$

Solution Define the sequences

$$\begin{aligned} T_k &= \frac{2^k}{x^{2^k}+1} \\ F_k &= \frac{2^k}{x^{2^k}-1} \end{aligned}$$

Observe

$$\begin{aligned} F_k - T_k &= 2^k \cdot \left(\frac{1}{x^{2^k}-1} - \frac{1}{x^{2^k}+1} \right) \\ &= 2^k \cdot \left(\frac{2}{x^{2^k \cdot 2}-1} \right) \\ &= \frac{2^{k+1}}{x^{2^{k+1}}-1} \\ &= F_{k+1} \\ \boxed{T_k \;=\; F_k - F_{k+1}} \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_k - F_{k+1}) \\ &= F_0 - F_n \\ \boxed{S_n \;=\; \frac{1}{x-1} - \frac{2^n}{x^{2^n}-1}} \end{aligned}$$

Note that for $(x > 1)$, L'Hospital's Rule yields

$$\lim_{n\to\infty} \left(\frac{2^n}{x^{2^n}-1} \right) \;=\; \lim_{n\to\infty} \left(\frac{2^n \ln 2}{2^n \ln 2 \cdot x^{2^n} \ln x} \right) \;=\; 0$$

$$\boxed{S_\infty \;=\; \frac{1}{x-1}} \qquad\qquad (x > 1)$$

Problem 5 Define the function $f : \mathbb{N} \to \mathbb{R}$ as follows.

$$f(n) \;=\; \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}$$

Calculate the sum

$$S_N \;=\; f(1) + f(2) + f(3) + \cdots + f(N)$$

Solution Rationalize and rearrange the expression for $f(n)$ as follows.

$$\begin{aligned}
f(n) \;&=\; \frac{(4n + \sqrt{2n+1}\sqrt{2n-1})(\sqrt{2n+1} - \sqrt{2n-1})}{(2n+1) - (2n-1)} \\
&=\; \frac{1}{2}\left(4n\sqrt{2n+1} - 4n\sqrt{2n-1} + (2n+1)\sqrt{2n-1} - (2n-1)\sqrt{2n+1}\right) \\
&=\; \frac{1}{2}\left((4n - 2n + 1)\sqrt{2n+1} - (4n - 2n - 1)\sqrt{2n-1}\right) \\
&=\; \frac{1}{2}\left((2n+1)^{\frac{3}{2}} - (2n-1)^{\frac{3}{2}}\right)
\end{aligned}$$

Define the function $g : \mathbb{N} \to \mathbb{R}$ as follows.

$$\begin{aligned}
g(n) \;&=\; \frac{1}{2} \cdot (2n+1)^{\frac{3}{2}} \\
\implies g(n-1) \;&=\; \frac{1}{2} \cdot (2n-1)^{\frac{3}{2}}
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{n=1}^{N} f(n) \;&=\; \sum_{n=1}^{N} (g(n) - g(n-1)) \\
&=\; g(N) - g(0)
\end{aligned}$$

$$\boxed{S_N \;=\; \frac{1}{2}(2N+1)^{\frac{3}{2}} \;-\; \frac{1}{2}}$$