Complex Identities

Satvik Saha

Problem 1 Prove the following, for ($n > 1$).

$$C \;=\; \sum_{k=0}^{n-1} \cos\left(\phi + \frac{2k\pi}{n}\right) \;=\; 0$$

$$S \;=\; \sum_{k=0}^{n-1} \sin\left(\phi + \frac{2k\pi}{n}\right) \;=\; 0$$

where $n \in \mathbb{N}$ and $\phi \in \mathbb{R}$.

Solution Note that $e^{2\pi i} = 1$. Let z be an n^{th} root of unity, satisfying $z^n = 1$.

$$\begin{aligned} z^n - 1 \;&=\; 0 \\ (z-1)(z^{n-1} + z^{n-2} + \cdots + 1) \;&=\; 0 \end{aligned}$$

Let z be defined as $e^{2\pi i/n}$. Thus, $z \neq 1$,

$$\sum_{k=0}^{n-1} z^k \;=\; 0 \tag{1}$$

Using Euler's Formula, $e^{\varphi i} \;=\; \cos\varphi + i\sin\varphi$, we can write

$$\begin{aligned} e^{\phi i} \cdot z^k \;=\; e^{\phi i} \cdot e^{2k\pi i/n} \;&=\; \cos\left(\phi + \frac{2k\pi}{n}\right) + i\sin\left(\phi + \frac{2k\pi}{n}\right) \\ e^{\phi i} \cdot \sum_{k=0}^{n-1} z^k \;&=\; \sum_{k=0}^{n-1} \cos\left(\phi + \frac{2k\pi}{n}\right) + i\sum_{k=0}^{n-1} \sin\left(\phi + \frac{2k\pi}{n}\right) \end{aligned}$$

Using (1), this reduces to

$$\boxed{0 \;=\; C \;+\; iS}$$

Comparing the real and imaginary parts of the above, we can conclude that $C = S = 0$.

Problem 2 Let $a + b + c = 0$. Prove the following, where $\omega^3 - 1 = 0$.

$$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = 27abc$$

Solution Note the following identities.

$$\begin{aligned} 1 + \omega + \omega^2 &= 0 && (2) \\ x^3 + y^3 &= (x + y)(x\omega^2 + y\omega)(x\omega + y\omega^2) && (3) \end{aligned}$$

Let $x = a + b\omega + c\omega^2$ and $y = a + b\omega^2 + c\omega$. Clearly,

$$\begin{aligned} x + y &= 2a + b(\omega + \omega^2) + c(\omega^2 + \omega) \\ &= 2a - b - c \\ &= 3a \\ x\omega^2 + y\omega &= a(\omega^2 + \omega) + 2b + c(\omega + \omega^2) \\ &= -a + 2b + -c \\ &= 3b \\ x\omega + y\omega^2 &= a(\omega + \omega^2) + b(\omega^2 + \omega) + 2c \\ &= -a - b + 2c \\ &= 3c \end{aligned}$$

Using (3), we can write

$$x^3 + y^3 \ = \ (3a)(3b)(3c) \ = \ 27abc$$

which, with change in notation, is the desired result.