

Telescoping Sums

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A sum in which subsequent terms cancel each other, leaving only its initial and final terms is called a *telescoping sum*.

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1} (T_{k+1} - T_k) \\ &= (T_1 - T_0) + (T_2 - T_1) + (T_3 - T_2) + \dots + (T_n - T_{n-1}) \\ &= T_n - T_0 \end{aligned}$$

Problem 1 Calculate the given sum.

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n \cdot (n+1)}$$

Solution Define the sequences

$$\begin{aligned} T_k &= \frac{1}{k \cdot (k+1)} \\ F_k &= \frac{1}{k} \end{aligned}$$

Observe that we can write

$$\begin{aligned} T_k &= \frac{(k+1) - k}{k \cdot (k+1)} \\ &= \frac{1}{k} - \frac{1}{k+1} \\ &= F_k - F_{k+1} \end{aligned}$$

Finally

$$\begin{aligned} \sum_{k=1}^n T_k &= \sum_{k=1}^n (F_k - F_{k+1}) \\ &= F_1 - F_{n+1} \end{aligned}$$

$$\boxed{S_n = 1 - \frac{1}{n+1}}$$

Problem 2 Calculate the given sum.

$$S_n = \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)(2n+5)}$$

Solution Define the expression for each term and manipulate its factors.

$$\begin{aligned} T_k &= \frac{1}{(2k+1)(2k+3)(2k+5)} \\ &= \frac{1}{4(2k+3)} \left(\frac{1}{2k+1} - \frac{1}{2k+5} \right) \\ &= \frac{1}{4} \left(\frac{1}{(2k+1)(2k+3)} - \frac{1}{(2k+3)(2k+5)} \right) \\ &= \frac{1}{8} \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right) - \frac{1}{8} \left(\frac{1}{2k+3} - \frac{1}{2k+5} \right) \end{aligned}$$

Define the sequence

$$\begin{aligned} F_k &= \frac{1}{8(2k+1)} \\ T_k &= (F_k - F_{k+1}) - (F_{k+1} - F_{k+2}) \end{aligned}$$

Finally

$$\begin{aligned} \sum_{k=0}^n T_k &= \sum_{k=0}^n (F_k - F_{k+1}) - \sum_{k=0}^n (F_{k+1} - F_{k+2}) \\ &= (F_0 - F_{n+1}) - (F_1 - F_{n+2}) \end{aligned}$$

$$\boxed{S_n = \frac{1}{12} - \frac{1}{4(2n+3)(2n+5)}}$$

Problem 3 Calculate the given sum, to n terms

$$S_n = (1) \cdot 0! + (3) \cdot 1! + (7) \cdot 2! + \dots + (n^2 - n + 1) \cdot n!$$

Solution Define the sequences

$$\begin{aligned} T_k &= (k^2 + k + 1) \cdot k! \\ F_k &= k \cdot k! \end{aligned}$$

Note that $n! = n \cdot (n - 1)!$

$$\begin{aligned} F_{k+1} &= (k + 1) \cdot (k + 1)! \\ &= (k + 1) \cdot (k + 1) \cdot k! \\ &= (k^2 + 2k + 1) \cdot k! \\ &= (k^2 + k + 1) \cdot k! + k \cdot k! \\ &= T_k + F_k \end{aligned}$$

$$\boxed{T_k = F_{k+1} - F_k}$$

Finally,

$$\begin{aligned} \sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_{k+1} - F_k) \\ &= F_n - F_0 \end{aligned}$$

$$\boxed{S_n = n \cdot n!}$$

Problem 4 Calculate the given sum.

$$S_n = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^{n-1}}{x^{2^{n-1}}+1}$$

Solution Define the sequences

$$T_k = \frac{2^k}{x^{2^k}+1}$$

$$F_k = \frac{2^k}{x^{2^k}-1}$$

Observe

$$\begin{aligned} F_k - T_k &= 2^k \cdot \left(\frac{1}{x^{2^k}-1} - \frac{1}{x^{2^k}+1} \right) \\ &= 2^k \cdot \left(\frac{2}{x^{2^k \cdot 2} - 1} \right) \\ &= \frac{2^{k+1}}{x^{2^{k+1}} - 1} \\ &= F_{k+1} \end{aligned}$$

$$\boxed{T_k = F_k - F_{k+1}}$$

Finally,

$$\begin{aligned} \sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_k - F_{k+1}) \\ &= F_0 - F_n \end{aligned}$$

$$\boxed{S_n = \frac{1}{x-1} - \frac{2^n}{x^{2^n}-1}}$$

Note that for $(x > 1)$, L'Hospital's Rule yields

$$\lim_{n \rightarrow \infty} \left(\frac{2^n}{x^{2^n}-1} \right) = \lim_{n \rightarrow \infty} \left(\frac{2^n \ln 2}{2^n \ln 2 \cdot x^{2^n} \ln x} \right) = 0$$

$$\boxed{S_\infty = \frac{1}{x-1}}$$

$(x > 1)$

Problem 5 Define the function $f : \mathbb{N} \rightarrow \mathbb{R}$ as follows.

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}$$

Calculate the sum

$$S_N = f(1) + f(2) + f(3) + \cdots + f(N)$$

Solution Rationalize and rearrange the expression for $f(n)$ as follows.

$$\begin{aligned} f(n) &= \frac{(4n + \sqrt{2n+1}\sqrt{2n-1})(\sqrt{2n+1} - \sqrt{2n-1})}{(2n+1) - (2n-1)} \\ &= \frac{1}{2} (4n\sqrt{2n+1} - 4n\sqrt{2n-1} + (2n+1)\sqrt{2n-1} - (2n-1)\sqrt{2n+1}) \\ &= \frac{1}{2} ((4n - 2n + 1)\sqrt{2n+1} - (4n - 2n - 1)\sqrt{2n-1}) \\ &= \frac{1}{2} \left((2n+1)^{\frac{3}{2}} - (2n-1)^{\frac{3}{2}} \right) \end{aligned}$$

Define the function $g : \mathbb{N} \rightarrow \mathbb{R}$ as follows.

$$\begin{aligned} g(n) &= \frac{1}{2} \cdot (2n+1)^{\frac{3}{2}} \\ \implies g(n-1) &= \frac{1}{2} \cdot (2n-1)^{\frac{3}{2}} \end{aligned}$$

Finally

$$\begin{aligned} \sum_{n=1}^N f(n) &= \sum_{n=1}^N (g(n) - g(n-1)) \\ &= g(N) - g(0) \end{aligned}$$

$$\boxed{S_N = \frac{1}{2}(2N+1)^{\frac{3}{2}} - \frac{1}{2}}$$