The Range of a Projctile

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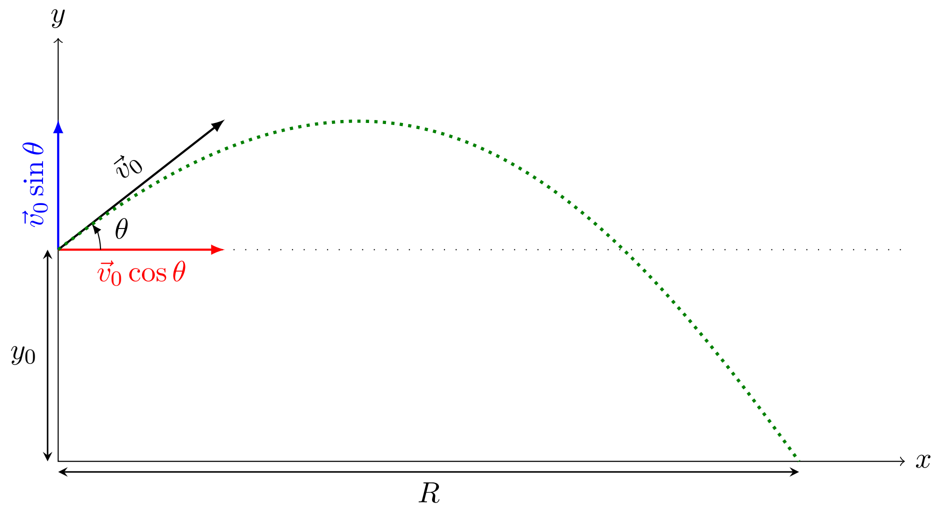

Consider the equations of motion of a projectile, launched at an elevation θ from a height y_0, experiencing uniform acceleration $-g$ along the y-axis.

$$x(t) = v_0 \cos\theta \tag{1}$$

$$y(t) = y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2 \tag{2}$$

When the projectile hits the ground, we see that $y(t) = 0$. Let this time be $tflight$ and the corresponding horizontal displacement be R.

$$\begin{aligned} 0 &= y_0 + v_0 \sin\theta - \frac{1}{2} g t^2 \\ t_{flight} &= \frac{1}{g}(v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2 g y_0}) \end{aligned}$$

$$R = \frac{1}{g}(v_0 \cos\theta)(v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2 g y_0}) \tag{3}$$

For $R = R_{max}$, we have $\frac{d}{d\theta}R = 0$. Solving the resultant equations, we have

$$\boxed{\theta_{max} \;=\; \cos^{-1}\sqrt{\frac{2gy_0 + v_0^2}{2gy_0 + 2v_0^2}}} \tag{4}$$

Note that setting y_0 to 0 simply reduces θ_{max} to $cos^{-1}\frac{1}{\sqrt{2}} = 45°$.