Telescoping Sums

Satvik Saha

A sum in which subsequent terms cancel each other, leaving only its initial and final terms is called a *telescoping sum*.

$$
\begin{aligned}
S_n &= \sum_{k=0}^{n-1} (T_{k+1} - T_k) \\
&= (T_1 - T_0) + (T_2 - T_1) + (T_3 - T_2) + \ldots (T_n - T_{n-1}) \\
&= T_n - T_0
\end{aligned}
$$

Problem 1 Calculate the given sum.

$$
S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}
$$

Solution Define the sequences

$$
\begin{aligned}
T_k &= \frac{1}{k \cdot (k+1)} \\
F_k &= \frac{1}{k}
\end{aligned}
$$

Observe that we can write

$$
\begin{aligned}
T_k &= \frac{(k+1) - k}{k \cdot (k+1)} \\
&= \frac{1}{k} - \frac{1}{k+1} \\
&= F_k - F_{k+1}
\end{aligned}
$$

Finally

$$
\begin{aligned}
\sum_{k=1}^{n} T_k &= \sum_{k=1}^{n} (F_k - F_{k+1}) \\
&= F_1 - F_{n+1}
\end{aligned}
$$

$$
\boxed{S_n = 1 - \frac{1}{n+1}}
$$

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Problem 2 Calculate the given sum.

$$S_n = \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{(2n+1)(2n+3)(2n+5)}$$

Solution Define the expression for each term and manipulate its factors.

$$
\begin{aligned}
T_k &= \frac{1}{(2k+1)(2k+3)(2k+5)} \\
&= \frac{1}{4(2k+3)} \left(\frac{1}{2k+1} - \frac{1}{2k+5} \right) \\
&= \frac{1}{4} \left(\frac{1}{(2k+1)(2k+3)} - \frac{1}{(2k+3)(2k+5)} \right) \\
&= \frac{1}{8} \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right) - \frac{1}{8} \left(\frac{1}{2k+3} - \frac{1}{2k+5} \right)
\end{aligned}
$$

Define the sequence

$$
\begin{aligned}
F_k &= \frac{1}{8(2k+1)} \\
T_k &= (F_k - F_{k+1}) - (F_{k+1} - F_{k+2})
\end{aligned}
$$

Finally

$$
\begin{aligned}
\sum_{k=0}^{n} T_k &= \sum_{k=0}^{n} (F_k - F_{k+1}) - \sum_{k=0}^{n} (F_{k+1} - F_{k+2}) \\
&= (F_0 - F_{n+1}) - (F_1 - F_{n+2})
\end{aligned}
$$

$$\boxed{S_n = \frac{1}{12} - \frac{1}{4(2n+3)(2n+5)}}$$

Problem 3   Calculate the given sum, to n terms

$$S_n = (1) \cdot 0! + (3) \cdot 1! + (7) \cdot 2! + \ldots + (n^2 - n + 1) \cdot n!$$

Solution   Define the sequences

$$T_k = (k^2 + k + 1) \cdot k!$$
$$F_k = k \cdot k!$$

Note that $n! = n \cdot (n-1)!$

$$
\begin{aligned}
F_{k+1} &= (k+1) \cdot (k+1)! \\
&= (k+1) \cdot (k+1) \cdot k! \\
&= (k^2 + 2k + 1) \cdot k! \\
&= (k^2 + k + 1) \cdot k! + k \cdot k! \\
&= T_k + F_k \\
\boxed{T_k} &\boxed{= F_{k+1} - F_k}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_{k+1} - F_k) \\
&= F_n - F_0 \\
\boxed{S_n = n \cdot n!}
\end{aligned}
$$

Problem 4 Calculate the given sum.

$$S_n = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^{n-1}}{x^{2^{n-1}}+1}$$

Solution Define the sequences

$$T_k = \frac{2^k}{x^{2^k}+1}$$

$$F_k = \frac{2^k}{x^{2^k}-1}$$

Observe

$$
\begin{aligned}
F_k - T_k &= 2^k \cdot \left(\frac{1}{x^{2^k}-1} - \frac{1}{x^{2^k}+1} \right) \\
&= 2^k \cdot \left(\frac{2}{x^{2^k \cdot 2}-1} \right) \\
&= \frac{2^{k+1}}{x^{2^{k+1}}-1} \\
&= F_{k+1} \\
\boxed{T_k = F_k - F_{k+1}}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{k=0}^{n-1} T_k &= \sum_{k=0}^{n-1} (F_k - F_{k+1}) \\
&= F_0 - F_n \\
\boxed{S_n = \frac{1}{x-1} - \frac{2^n}{x^{2^n}-1}}
\end{aligned}
$$

Note that for $(x > 1)$, L'Hospital's Rule yields

$$\lim_{n \to \infty} \left(\frac{2^n}{x^{2^n}-1} \right) = \lim_{n \to \infty} \left(\frac{2^n \ln 2}{2^n \ln 2 \cdot x^{2^n} \ln x} \right) = 0$$

$$\boxed{S_\infty = \frac{1}{x-1}} \qquad\qquad (x > 1)$$

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Problem 5 Define the function $f : \mathbb{N} \to \mathbb{R}$ as follows.

$$f(n) \;=\; \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}$$

Calculate the sum

$$S_N \;=\; f(1) + f(2) + f(3) + \cdots + f(N)$$

Solution Rationalize and rearrange the expression for $f(n)$ as follows.

$$
\begin{aligned}
f(n) \;&=\; \frac{(4n + \sqrt{2n+1}\sqrt{2n-1})(\sqrt{2n+1} - \sqrt{2n-1})}{(2n+1) - (2n-1)} \\
&=\; \frac{1}{2}\left(4n\sqrt{2n+1} - 4n\sqrt{2n-1} + (2n+1)\sqrt{2n-1} - (2n-1)\sqrt{2n+1}\right) \\
&=\; \frac{1}{2}\left((4n - 2n + 1)\sqrt{2n+1} - (4n - 2n - 1)\sqrt{2n-1}\right) \\
&=\; \frac{1}{2}\left((2n+1)^{\frac{3}{2}} - (2n-1)^{\frac{3}{2}}\right)
\end{aligned}
$$

Define the function $g : \mathbb{N} \to \mathbb{R}$ as follows.

$$
\begin{aligned}
g(n) \;&=\; \frac{1}{2} \cdot (2n+1)^{\frac{3}{2}} \\
\implies\; g(n-1) \;&=\; \frac{1}{2} \cdot (2n-1)^{\frac{3}{2}}
\end{aligned}
$$

Finally

$$
\begin{aligned}
\sum_{n=1}^{N} f(n) \;&=\; \sum_{n=1}^{N} (g(n) - g(n-1)) \\
&=\; g(N) - g(0)
\end{aligned}
$$

$$\boxed{\,S_N \;=\; \frac{1}{2}(2N+1)^{\frac{3}{2}} - \frac{1}{2}\,}$$

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