Telescoping Sums

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A sum in which subsequent terms cancel each other, leaving only its initial and final terms is called a *telescoping sum*.

$$\begin{aligned} S &= \sum_{k=0}^{n-1}(T_{k+1} - T_k) \\ &= (T_1 - T_0) + (T_2 - T_1) + (T_3 - T_2) + \ldots (T_n - T_{n-1}) \\ &= T_n - T_0 \end{aligned}$$

Problem Calculate the given sum, to n terms

$$S_n = 1(0!) + 3(1!) + 7(2!) + 13(3!) + 21(4!) + \cdots + T_{n-1}$$

where T_k is the k^{th} term of the given series.

Solution Observe

$$T_k = (k^2 + k + 1) \cdot k!$$

Consider the definition

$$F_k = k \cdot k!$$

Note that $n! = n \cdot (n-1)!$

$$\begin{aligned} F_{k+1} &= (k+1) \cdot (k+1)! \\ &= (k+1) \cdot (k+1) \cdot k! \\ &= (k^2 + 2k + 1) \cdot k! \\ &= (k^2 + k + 1) \cdot k! + k \cdot k! \\ &= T_k + F_k \end{aligned}$$

$$\boxed{T_k = F_{k+1} - F_k}$$

The required sum S_n is thus is given by

$$\begin{aligned}
S_n &= \sum_{k=0}^{n-1} T_k \\
&= \sum_{k=0}^{n-1} (F_{k+1} - F_k) \\
&= F_n - F_0 \\
&= n \cdot n! - 0 \cdot 0!
\end{aligned}$$

$$\boxed{S_n = n \cdot n!}$$

Hence, it is obvious that $S_{4000} = 4000 \cdot 4000!$