Olympiad Inequalities

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Theorems

Theorem 1 (Trivial Inequality) *For all $x \in \mathbb{R}$:*

$$x^2 \;\geq\; 0$$

Theorem 2 (QM-AM-GM-HM) *For non-negative real numbers $a_1, a_2, \ldots, a_n$, the following inequality holds:*

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \;\geq\; \frac{a_1 + a_2 + \cdots + a_n}{n} \;\geq\; \sqrt[n]{a_1 a_2 \cdots a_n} \;\geq\; \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 3 (Bernoulli) *For all $r \geq 1$ and $x \geq -1$:*

$$(1 + x)^r \;\geq\; 1 + xr$$

Theorem 4 (Schur) *Let a, b, c be positive real numbers and $r > 0$. The following inequality holds:*

$$a^r(a - b)(a - c) + b^r(b - c)(b - a) + c^r(c - a)(c - b) \;\geq\; 0$$

Theorem 5 (Nesbitt) *Let a, b, c be positive real numbers. The following inequality holds:*

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \;\geq\; \frac{3}{2}$$

Equality holds if $a = b = c$.

Theorem 6 (Titu's Lemma) *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be sequences of non-negative real numbers. The following inequality holds:*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \;\geq\; \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

Equality holds if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Theorem 7 (Cauchy-Schwarz) *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be sequences of non-negative real numbers. The following inequality holds:*

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \;\geq\; (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

Equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Theorem 8 (Hölder) *Let $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), \cdots, (z_1, z_2, \ldots, z_n)$ be sequences of non-negative real numbers, and $\lambda_a, \lambda_b, \cdots, \lambda_z$ be positive reals which sum to 1. The following inequality holds:*

$$(a_1+a_2+\cdots+a_n)^{\lambda_a}(b_1+b_2+\cdots+b_n)^{\lambda_b}\cdots(z_1+z_2+\cdots+z_n)^{\lambda_z} \;\geq\; (a_1^{\lambda_a}b_1^{\lambda_b}\cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a}b_n^{\lambda_b}\cdots z_n^{\lambda_z})$$

Theorem 9 (Rearrangement) *Let $(a_1 \leq a_2 \leq \cdots \leq a_n)$ and $(b_1 \leq b_2 \leq \cdots \leq b_n)$ be sequences of non-decreasing real numbers. For any permutation π of $\{1, 2, \ldots, n\}$, the following inequality holds:*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \;\geq\; a_1b_{\pi(1)} + a_2b_{\pi(2)} + \cdots + a_nb_{\pi(n)} \;\geq\; a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$$

Theorem 10 (Chebyshev) *Let $(a_1 \leq a_2 \leq \cdots \leq a_n)$ and $(b_1 \leq b_2 \leq \cdots \leq b_n)$ be sequences of non-decreasing real numbers. The following inequality holds:*

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \;\geq\; \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \;\geq\; \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}$$

Theorem 11 (Muirhead) *Let the sequence $(a_1, a_2, \ldots, a_n)$ majorize[1] $(b_1, b_2, \ldots, b_n)$. Then, for a sequence of positive reals $(x_1, x_2, \ldots, x_n)$, the following inequality holds:*

$$\sum_{sym} x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \;\geq\; \sum_{sym} x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$$

Here, the sums are taken over all permutations of the n variables.

[1] A sequence $(x_1 \geq x_2 \geq \cdots \geq x_n)$ majorizes $(y_1 \geq y_2 \geq \cdots \geq y_n)$ if $\sum x_i = \sum y_i$ and for all $k = 1, 2, \ldots, n-1$

$$(x_1 + x_2 + \cdots + x_k) \;\geq\; (y_1 + y_2 + \cdots + y_k)$$