Variable force on a Moving Body

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Question A body of mass 2.5 kg, is at rest at the position $x = 4$ at $t = 0$. It experiences a force given by the expression $\vec{F} = -10x + 20\hat{i}$. Express the position of the body as a function of time.

Solution We have $\vec{F} = m\vec{a} = -10x + 20\hat{i}$ and $m = 2.5$. Thus

$$\vec{a} \;=\; -4x + 8\,\hat{i}$$

Changing notation to scalars and replacing a with x''

$$\boxed{x'' \;+\; 4x \;-\; 8 \;=\; 0} \tag{1}$$

This is a second order non-homogeneous double differential equation.
Let the solutions to (1) be of the form

$$x(t) \;=\; x_C(t) + x_P(t) \tag{2}$$

where $x_C(t)$ is the solution to $x'' + 4x = 0$.

Make the substitution $x = e^{\lambda t}$, for some constant λ. Thus

$$\begin{aligned}
\frac{d^2}{dt^2}e^{\lambda t} + 4e^{\lambda t} &= 0 \\
\lambda^2 e^{\lambda t} + 4e^{\lambda t} &= 0 \\
\lambda^2 + 4 &= 0 \\
\lambda &= \pm 2i
\end{aligned}$$

We have $x_C(t) = c_1x_1 + c_2x_2$ for each value of x corresponding with a value of λ, i.e. , $x_1 = e^{2it}$ and $x_2 = e^{-2it}$. Thus

$$x_C(t) \;=\; c_1e^{2it} + c_2e^{-2it}$$

Applying Euler's Formula, $e^{ix} = \cos x + i \sin x$, we have

$$\begin{aligned} x_C(t) &= c_1(\cos(2t) + isin(2t)) + c_2(\cos(2t) - i\sin(2t)) \\ &= (c_1 + c_2)\cos(2t) + i(c_1 - c_2)\sin(2t) \\ &= k_1 \cos(2t) + k_2 \sin(2t) \end{aligned} \tag{3}$$

Let $x_P(t) = a_1 \implies x_p''(t) = 0$ for some constant a_1. Substituting this in (1) gives

$$\begin{aligned} x_P''(t) + 4x_P(t) &= 8 \\ x_P(t) &= 2 \end{aligned} \tag{4}$$

Thus, from (1), (3) and (4)

$$\boxed{x(t) \ = \ k_1 \cos(2t) \ + \ k_2 \sin(2t) \ + \ 2} \tag{5}$$

Note that we know that at $t = 0$, $x = 4$ and $v = x' = 0$. Thus

$$\begin{aligned} x(0) &= k_1 + 2 = 4 \\ k_1 &= 2 \end{aligned}$$

And

$$\begin{aligned} x'(t) &= -2k_1 \sin(2t) + 2k_2 \cos(2t) \\ x'(0) &= 2k_2 = 0 \\ k_2 &= 0 \end{aligned}$$

Finally, from (5), we can write

$$\boxed{x(t) \ = \ 2\cos(2t) \ + \ 2}$$