Graph Theory - Lecture notes.

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Disclaimers and Warnings

- These notes are highly incomplete, not well-written, not well 'latexed' and not meant to be taken seriously like a book.
- The notes might fail on many aesthetic counts and the author has put little effort to ensure that the notes are more pleasing to the eye.
- These are written for the introductory course on graph theory to second year undergraduate students. `http://www.isibang.ac.in/~adean/infsys/database/Bmath/GT.html`
- These are compendium of some of the class material (mostly definitions and result statements) to serve as a pointer to students. I hope to add proof details over the years.
- These are mostly a subset of the material covered in class.
- All figures are taken from various sources and sorry for not acknowledging them all.
- I am thankful to students of the course in the years 2018, 2019 and 2022 for pointing out errors, inaccuracies and also listening to my lectures :-)
- Unless mentioned, the proofs or results are all borrowed from some book or the other. Few of them are listed in the bibliography.
- Do not look for any new results or proofs or something else fancy here. Just stuff arranged according to my own teaching convenience.
- I have tried to mention some related research-level questions and open problems at the end of each chapter.
- The main references are [Bollobas 2013, Van Lint and Wilson, West 2001, Diestel 2000, Jukna 2011, Sudakov 2019].
- Most of the notes were prepared in 2018 and 2019 but after noting similarity with [Sudakov 2019], I have added some results from [Sudakov 2019] as they fit well and also re-organized the notes.

Chapter 1

Introduction to Graphs

1.1 Definition and some motivating examples

Some notation : $[n] = \{1, \ldots, n\}$, $V \times V$ the usual set product, $\binom{V}{2}$ denote unordered pairs of distinct elements in V.

Definition 1.1.1. *(Graph). A (simple) graph G consists of a finite vertex set $V := V(G)$ and an edge set $E := E(G) \subset \binom{V}{2}$.*

Remark 1.1.2. *We shall occasionally consider graphs on countably many vertices. But we shall assume such graphs are* **locally-finite** *i.e., vertices occur only in finitely many edge-pairs.*

For a vertex set V, we represent edges as $(v, w), v, w \in V$. We also write $v \sim w$ to denote that $(v, w) \in E$. A very common pictorial representation of graphs is as follows : Vertices are represented as points on plane and edges are lines / curves between the two vertices. See Figure 1.1[1]. As an exercise, explicitly define the graphs based on these representations.

See these two talks by Hugo Touchette (`http://www.physics.sun.ac.za/~htouchette/archive/talks/networks-short1.pdf`) and (`http://www.physics.sun.ac.za/~htouchette/archive/talks/complexnetworks1.pdf`) for many examples and applications of graphs in real-life problems. We shall briefly touch upon these in our examples as well.

The following two variants shall be hinted upon in our motivation and some examples but we shall not discuss the same for most of the course.

Remark 1.1.3 (Two variants)**.**

Directed graphs: *These are graphs with directed edges or equivalently the edge-pairs are ordered*

Multi-graphs: *These are graphs with multiple edges between vertices including self-loops.*

[1] All of the figures in these notes are not mine and taken from the internet

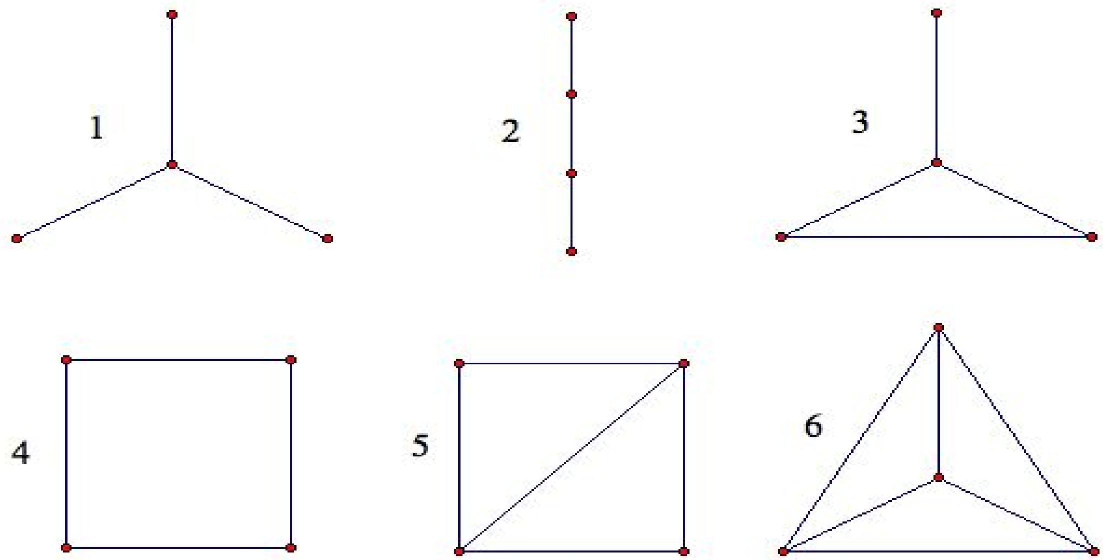


Figure 1.1: Some representations of graphs on vertex sets of cardinalities 3 and 4.

1.1.1 Popular examples of graphs

Let us see some popular examples of graphs that illustrate the use of graphs in different contexts.

Example 1.1.4. *(Facebook graph) V is the set of Facebook users and an edge is places between two vertices if they are friends of each other. See Figure 1.2*

Example 1.1.5. *(Road networks) V is the set of cities and an edge represents roads/trains/air-routes between the cities. See Figure 1.3*

Example 1.1.6. *Collaboration graph V is the set of all mathematicians who have published (say listed on mathscinet) and an edge represents that two mathematicians have collaborated on a paper together.*

Example 1.1.7. *(Complexity of Shakespeare's plays) V represents the characters in a shakespeare play and an edge between two characters means that both appeared in a scene together. See Figures 1.5 and 1.6 for the graph of Othello and Macbeth. See* `http://www.martingrandjean.ch/network-visualization-shakespeare/` *for more details. Network density is the ratio $|E|/|V|$ where $|.|$ denotes the cardinality of a set.*

More examples of networks abound in biology (`https://en.wikipedia.org/wiki/Biological_network`), chemical reaction networks (`https://en.wikipedia.org/wiki/Chemical_reaction_network_theory`) et al. We shall later give a more historical perspective with some more examples.

Definition 1.1.8. *(Weighted graphs) A graph G with a weight function $w : E \to \mathbb{R}$.*

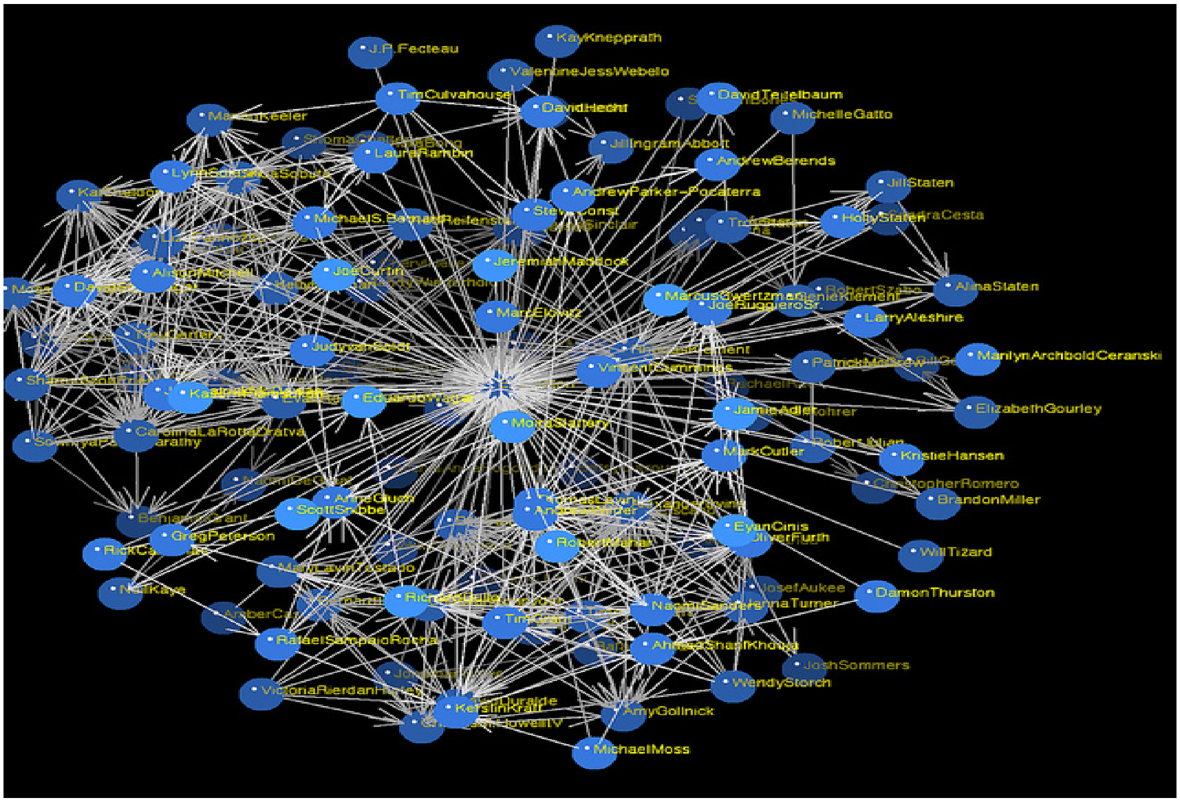


Figure 1.2: A snapshot of facebook graph

Often the weights are non-negative.

Example 1.1.9. *(Traffic Networks) G is the road network with the weight w denoting average traffic in a day. See Figure 1.7*

Example 1.1.10. *(Football graph) This is an example of weighted directed graph. Let the vertex set be the 11 players in a team and directed edge from i to j represents that player i has passed to player j. Associated with such a directed edge (i, j), a weight $w(i, j)$ that denotes the number of passes from player i to player j. Now see the football graph from the France vs Croatia final in 2018 WC in Figure 1.8. More such graphs are available* `https://grafos-da-bola.netlify.app/` *Each vertex is also weighed with its degree and colored accordingly. As one can observe that although Croatia has more number of passes but France's pass are well distributed.*

As another example, see the football graph from the Spain vs Holland final in 2010 WC in Figure 1.9 Spain's passing game is very evident in the graph

This illustrates best the appealing visualization offered by graphs and also gives a good way to analyse such effects in sports and other domains.

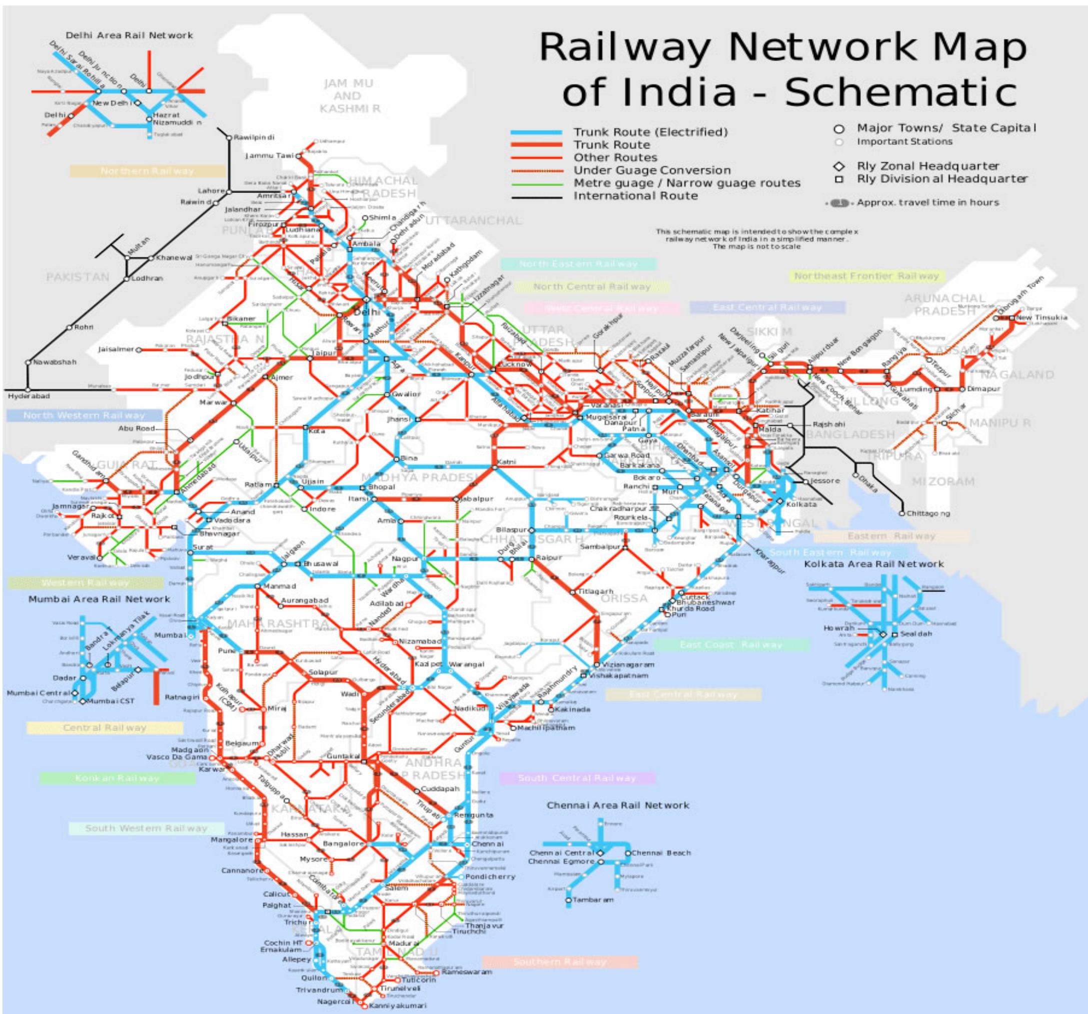


Figure 1.3: Indian railway network

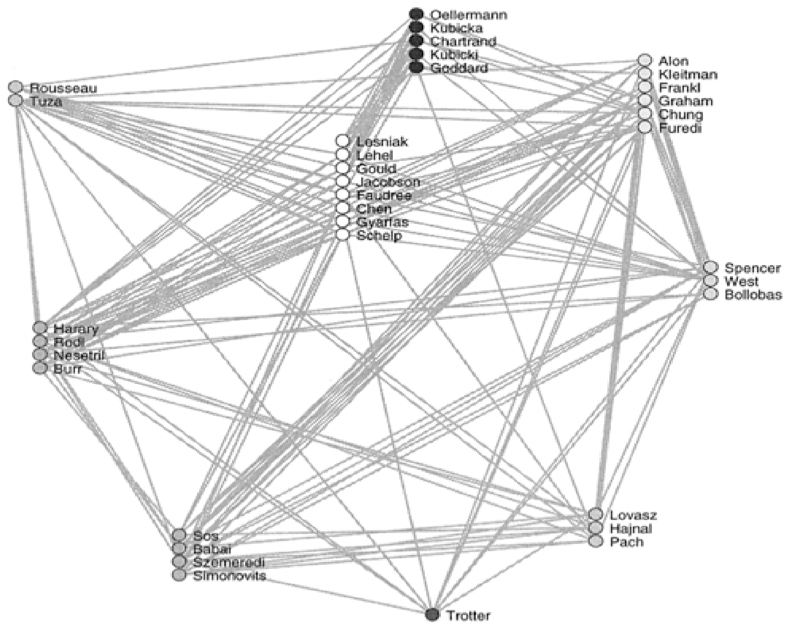


Figure 1.4: Collaboration graph of mathematicians based on mathscinet

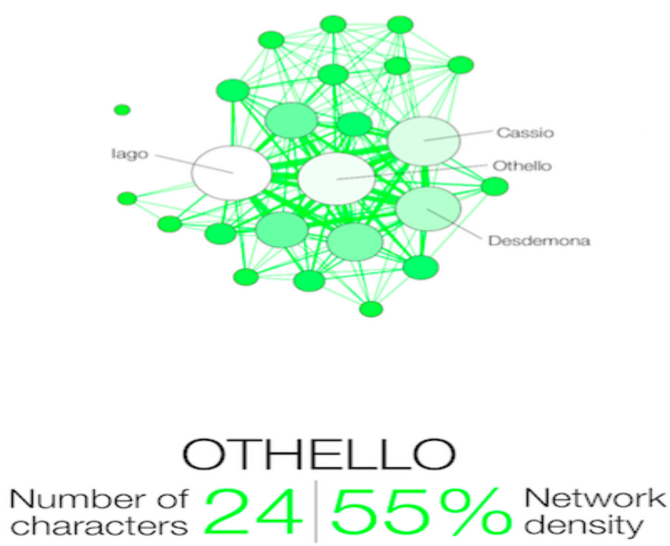


Figure 1.5: Othello characters graph

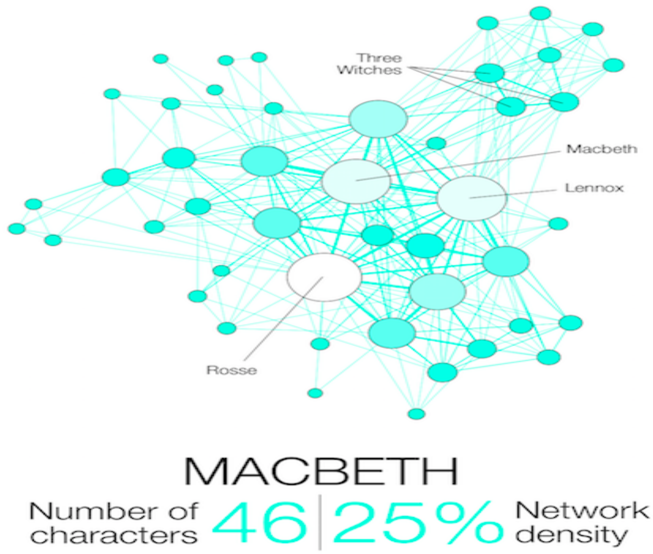


Figure 1.6: Macbeth characters graph

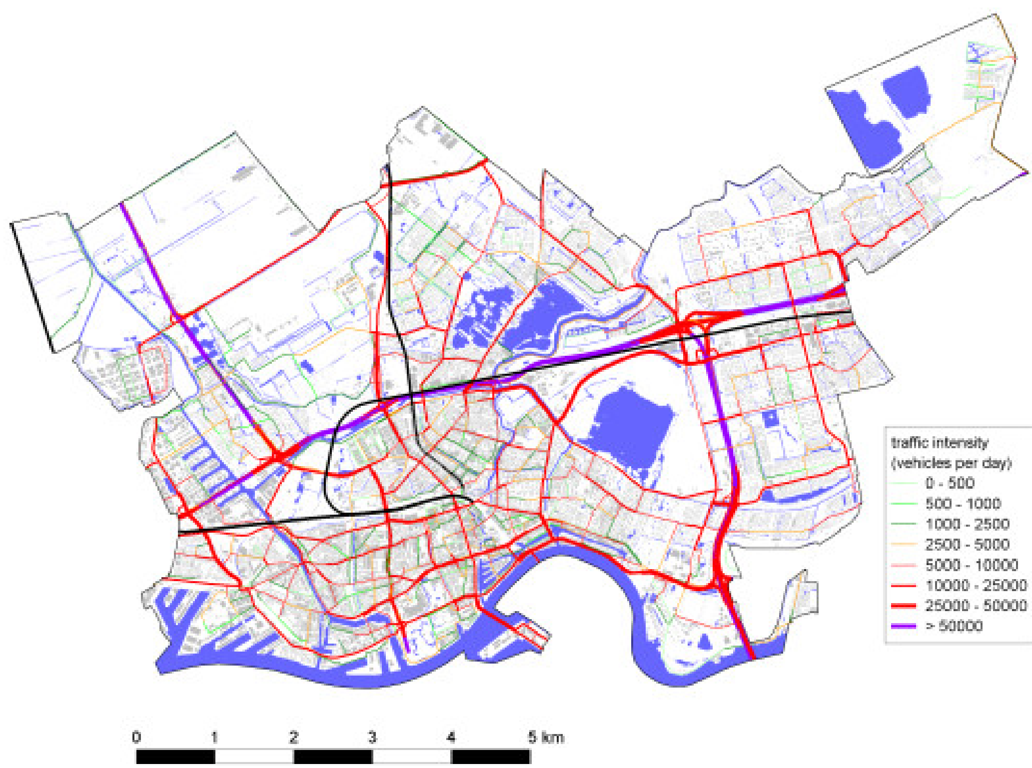


Figure 1.7: A road network with traffic density

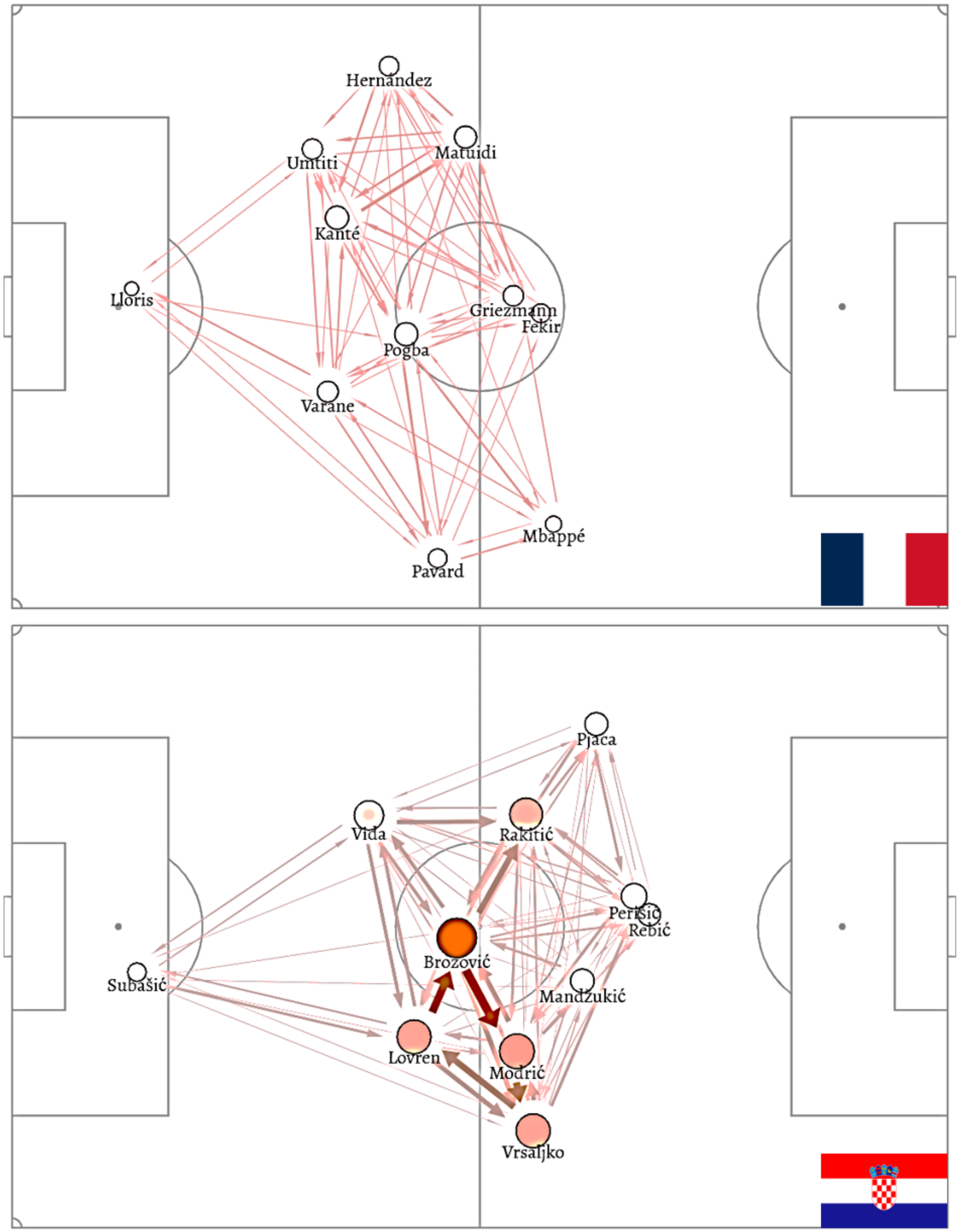


Figure 1.8: Football graph from the Spain vs Croatia 2018 WC Final.

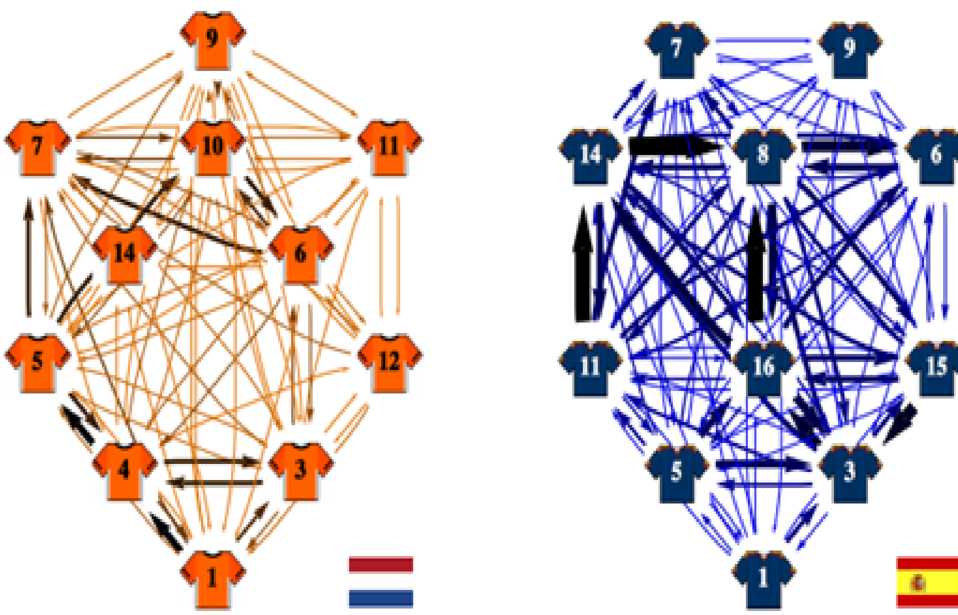


Figure 1.9: Football graph from the Spain vs Holland 2010 WC Final.

1.2 Some history and more motivation

Example 1.2.1. *(Konigsberg Problem. Euler, 1736) The problem was to find a path starting at any point*

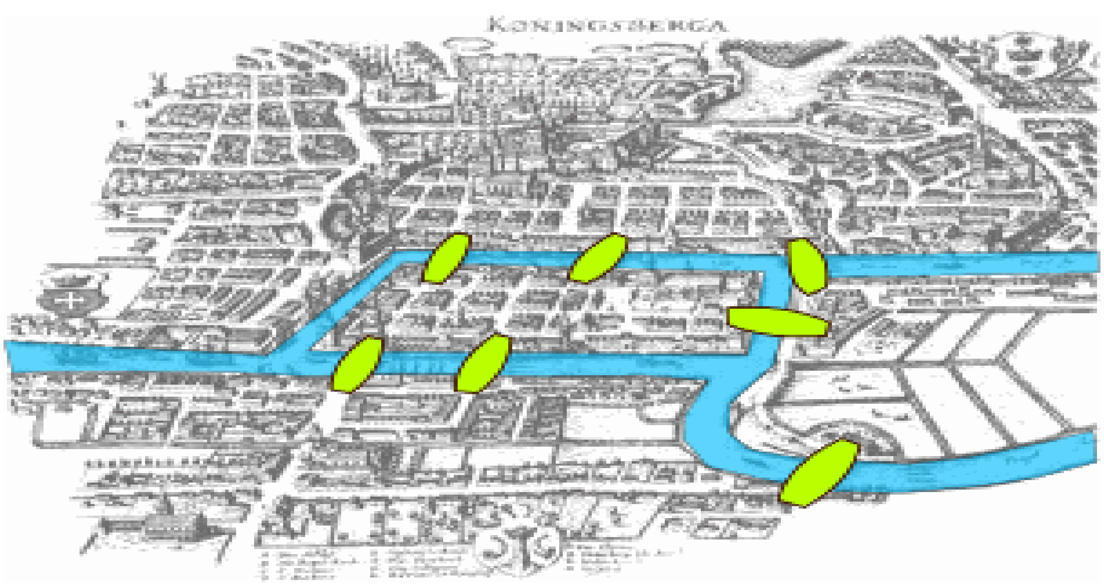


Figure 1.10: Seven bridges of Konigsberg on the river Pregel.

that traverses through all the bridges exactly once and return to the starting point (See figure 1.10). After many attempts in vain, Euler showed that this is not possible. This problem is considered the birth of both probability and topology. We shall see Euler's solution later.

Example 1.2.2. *(Electrical Networks. Kirchoff, 1847) Electrical networks can be represented as weighted directed graphs with current and resistance viewed as weights; See Figure 1.11. This formalism can explain Kirchoff's and Ohm's laws. This connection between graphs and electrical networks is highly useful not only for graph theory and electrical networks but also used in random walks and algebraic graph theory. It is not entirely inaccurate to talk of Kirchoff's formalism also as discrete cohomology.*

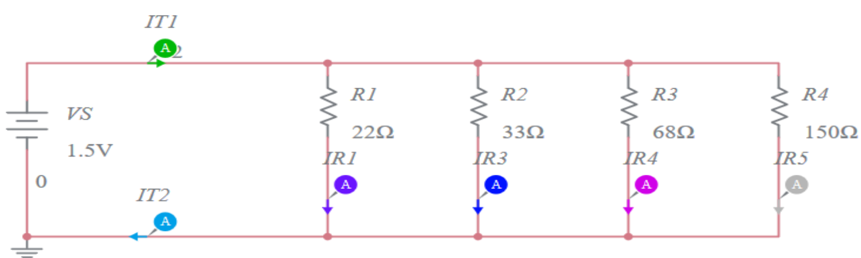


Figure 1.11: Electrical network as graphs.

Example 1.2.3. *(Chemical Isomers. Cayley, 1857) Atoms were represented by vertices and bonds between atoms by an edge. Such a representation was used to understand the structure of molecules. We shall see Cayley's tree enumeration formula which was used in enumeration of number of chemical isomers of a compound. See Figure 1.12.*

Example 1.2.4. *(Tour of cities. Hamilton, 1859) Given a set of cities and roads between them, find a path that starts at one city, visits all the cities exactly once and returns to the starting city. Can we guarantee such a path exists for all road networks ?*

Example 1.2.5. *(Four colour theorem) Suppose we take a map of the world and assume that countries are contiguous land masses. Let countries be vertices and edges are drawn between two countries share a boundary. Can we colour countries such that neighbouring countries have different colours ? What is the minimum number of colours required for the same ?*

A more general question, can any graph be drawn as a map i.e., can any graph be drawn on the plane such that edges do not cross each other ?

Example 1.2.6. *(A number theory graph) Here are two examples of graphs that were introduced towards a proof of Chowla's conjecture. Define $G_m = (V_m = \{m, \dots, 2m\}, E_m)$ where are $n \sim n + p$ for a prime*

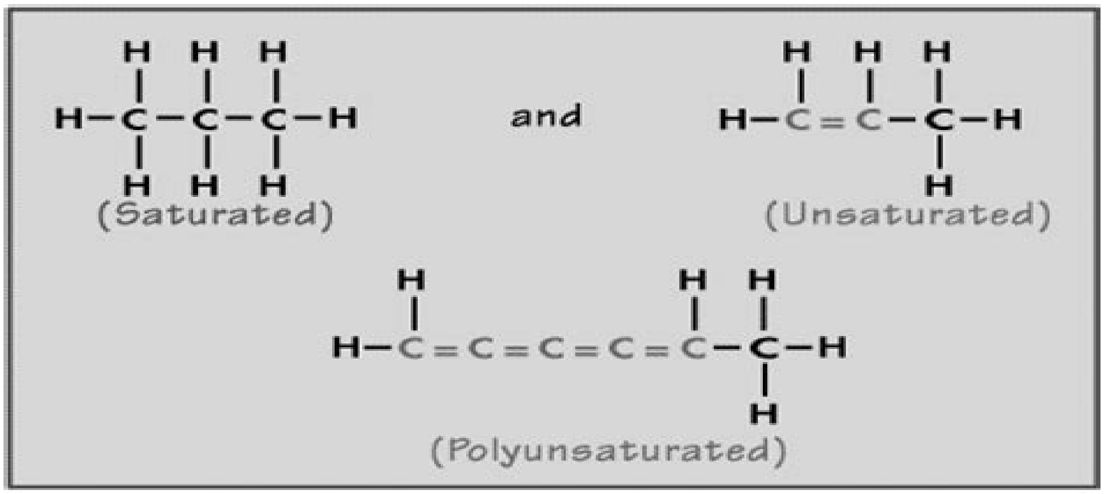


Figure 1.12: Hydrocarbons represented as graphs.

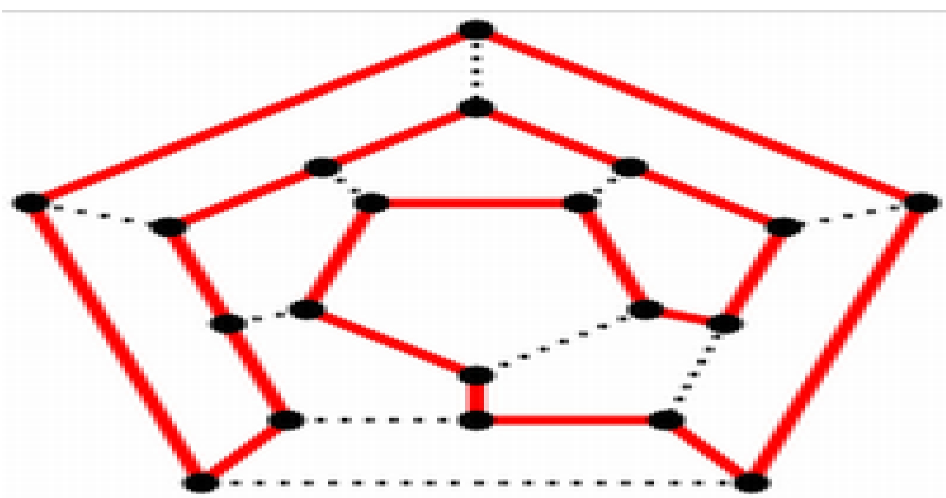


Figure 1.13: Hamilton cycle of a graph

p such that $p|n$ (i.e., p divides n). The interest is in the number of walks of length k (asymptotically in k, m) as $m \to \infty$ and also eigenvalues of certain matrices associated to G_m. This was defined by T. Tao. To approximate this graph better, Helfgott and Radizwiłł defined the following weighted graph. $G'_m = (V_m == \{m, \ldots, 2m\}, E'_m)$ where edges are $n \sim n+p$ for a prime p and $w(n, n+p) = 1/p$.

For a popular exposition on Chowla conjecture and this graph, see `https://www.quantamagazine.org/mathematicians-outwit-hidden-number-conspiracy-20220103/` and [Helfgott 2022] for more details.

1.3 Tentative Course overview :

Already, we have seen some historic questions that shall be discussed in the course. Now, we shall see something more specific that will be covered in the course.

1.3.1 Flows, Matchings and Games on Graphs

Example 1.3.1 (Maximum traffic flow)**.** *Consider the traffic network (weighted graphs) and take a starting point (source) and ending point (sink). What is the maximum amount of traffic that can flow from source to the sink in one instance ?*

The solution is very famously known as the max flow-min cut theorem and has many applications.

Example 1.3.2 (Hostel room allocation)**.** *There are n rooms and m students. Each student gives a list of rooms acceptable to them. Can the warden allot rooms to students such that each student gets at least one room in their list ?*

Hall's marriage theorem shall give a deceptively simple sufficient and necessary condition for the same. Hall's marriage theorem shall be proved using max flow-min cut theorem.

Example 1.3.3 (Hide and Seek game)**.** *Consider an area with horizontal main roads and vertical cross roads (See the grid in Figure 1.14). There are safe houses at certain intersections, marked by crosses in the figure. A robber choses to hide at one of the safe houses. A cop wants to find the main road or the cross road in which the robber stays. What is the best strategy for the cop to succeed ? What is the best strategy for robber to defeat the cop ? We shall exhibit a strategy for both using Hall's marriage theorem.*

1.3.2 Graphs and matrices

A graph G on V can be represented as a $V \times V$ matrix with 1 in the column (v, w) iff (v, w) is an edge in G. One can represent weighted and directed graphs as well. What does rank and nullity mean here ? Are there other matrices that encode the properties of graphs ? This viewpoint shall connect Kirchoff's formalism with cohomology theory.

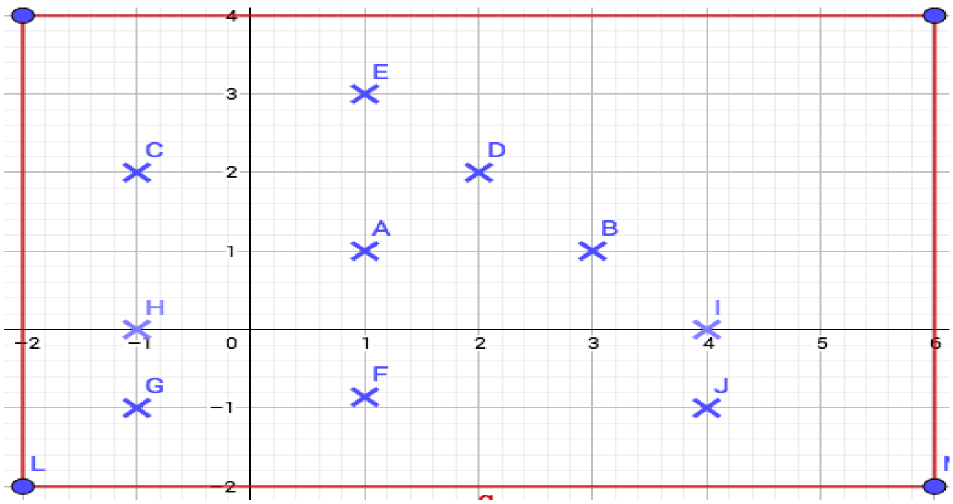


Figure 1.14: An example of a hide and seek scenario.

1.3.3 Random graphs and probabilistic method.

If time permits, we shall mention random graphs. We fix $[n]$ as the vertex set and choose edges at random i.e., at each edge toss a coin and place the edge if it lands heads. What are the properties of these graphs ?

Probabilistic method : Suppose we want to show that there exists graphs with a certain property, let us show that the random graph satisfies the property with positive probability. Thus, the set of graphs satisfying the property is non-empty. This approach was pioneered by Erdös and is still remarkably successful in proving various results.

I shall try to emphasize various mathematical topics (such as cohomology) that show up in their simplest incarnation in graph theory and also, many mathematical tools such as linear algebra, analysis and probability are used to study graphs. A very readable survey on the growing importance of graph theory is [Lovasz 2011].

Chapter 2

The very basics

By a graph, we shall always refer to finite simple graphs. Occasionally for examples or to illustrate we shall mention locally-finite, infinite graphs.

Exercise* denotes open ended exercises or slightly difficult exercises. **Exercise(A)** denote exrecise problems that will appear in assignments.

2.1 Some useful classes of graphs :

Example 2.1.1 (Complete graph). *$K_n : V = [n], E = \binom{V}{2}$.*

Example 2.1.2 (Cayley graphs). *$H, +$ be a finite group with a finite set of generators S such that $S = -S$ (*symmetric*) and $0 \notin S$ (where 0 is the identity). The* Cayley graph *G is defined with vertex set $V = H$ and $x \sim y$ if $x - y \in S$. Since S is symmetric, $y - x \in S$ iff $x - y \in S$ and so abelianness of the group does not matter.*

Example 2.1.3 (Intersection graph). *Let $S_1, \ldots, S_n$ be subsets of a set S. Define G with $V = [n]$ and $i \sim j$ if $S_i \cap S_j \neq \emptyset$.*

Example 2.1.4 (Delaunay graph). *$P \subset \mathbb{R}^d, d \geq 1$ - finite distinct set of points. For $y \in \mathbb{R}^d, d(y, P) := \min_{x \in P} |x - y|$. Define $C_x := \{y : d(y, P) = |x - y|\}, x \in P$. Delaunay graph is the intersection graph on P with intersecting sets $C_x, x \in P$. C_x is called as the* voronoi cell *of x.*

Example 2.1.5 (Euclidean Lattices). *Let $B_r(x)$ be the closed ball of radius r centered at x. The d-dimensional integer lattice is the intersection graph formed with $\mathbb{Z}^d$ as vertex set and $B_{1/2}(z), z \in \mathbb{Z}^d$ as the intersecting sets. Alternatively, $z \sim z'$ if $\sum_{i=1}^d |z_i - z'_i| = 1$. Show that Euclidean lattices are Cayley graphs. Find the generators S.*

Exercise(A) 2.1.6. *Show that every graph is an intersection graph.*

2.1.1 Some graph constructions

Example 2.1.7 (Bi-partite graph)**.** *Graphs (V, E) such that $V = V_1 \sqcup V_2$ and $E \subset V_1 \times V_2$.*

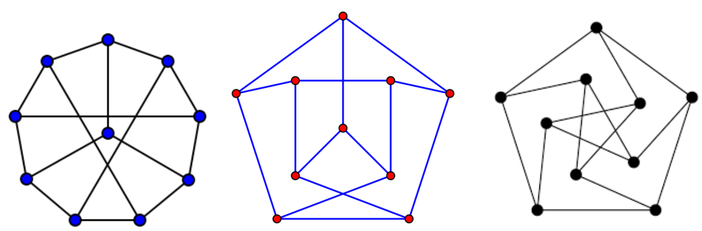
Example 2.1.8 (Complementary graph)**.** *Let $G = (V, E)$ be a graph. The complementary graph is $G^c = (V, \binom{[V]}{2} - E)$.*

Example 2.1.9 (Line graph)**.** *Let $G = (V, E)$ be a graph. The line graph is $L(G)$ with vertex set $V = E$ and $e_1 \sim e_2$ is they are adjacent in G.*

Example 2.1.10 (Petersen graph)**.** *The vertex set of the graph is $\binom{[5]}{2}$ and $\{i, j\} \sim \{k, l\}$ if $\{i, j\} \cap \{k, l\} = \emptyset$.*

Exercise(A) 2.1.11. *Show that the Petersen graph is the complement of the line graph of K_5.*

Exercise(A) 2.1.12. *Are the following three graphs isomorphic to Petersen graph ?*



Exercise* 2.1.13. *Can you find the number of edges in a Line graph $L(G)$ in terms of the number of edges in G ?*

2.2 Some basic notions

We shall assume that all our graphs are finite unless mentioned otherwise. In some examples, we shall illustrate things using infinite graphs but all our results are for finite graphs only.

Fix a graph G with vertex set V and edge set E. We say v, w are neighbours if $v \sim w$.

Definition 2.2.1 (Graph homomorphism and isomorphism)**.** *Suppose G, H are two graphs. A function $\phi : V(G) \to V(H)$ is said to be a* graph homomorphism *if $x \sim y$ implies $\phi(x) \sim \phi(y)$. ϕ is said to be an* isomorphism *if ϕ is a bijection and $x \sim y$ iff $\phi(x) \sim \phi(y)$ i.e., ϕ, ϕ^{-1} are graph homomorphisms. G and H are* isomorphic *$(G \cong H)$ if there exists an isomorphism between G and H. An* automorphism *is an isomorphism $\phi : G \to G$.*

Exercise(A) 2.2.2. *The set of all automorphisms of G is called $Aut(G)$. Define a binary operation on $Aut(G)$ as followss : For $g, f \in Aut(G)$, $g.f = g \circ f$ i.e., the composition operation. Is $Aut(G)$ a group ?*

Essentially, G and H are the same graphs upto re-labelling. H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. H is an *induced subgraph* of G if H is a subgraph of G and if $v, w \in H$ such that $(v, w) \in E(G)$ then $(v, w) \in E(H)$. Alternatively, H is an induced subgraph if H is a subgraph and $E(H) = \binom{V(H)}{2} \cap E(G)$.

Exercise 2.2.3. *H is an induced subgraph of G iff H is the maximal (w.r.t. edge inclusion) subgraph in G with vertex set $V(H)$.*

Example 2.2.4 (Trivial homomorphisms)**.** *The identity automorphism is a trivial homomorphism ; If $H \subset G$, then the inclusion map from $V(H)$ to $V(G)$ gives rise to a homomorphism.*

Lemma 2.2.5. *Show that there exists a homomorphism from G to K_2 iff G is bi-partite.*

Proof. If there exists an homomorphism $\phi : G \to K_2$, observe that $V = \phi^{-1}(1) \sqcup \phi^{-1}(2)$ and this defines the correct partition and shows that G is bi-partite. If G is bi-partite with $V = V_1 \sqcup V_2$ then defining $\phi : G \to K_2$ as $V_1 \mapsto 1, V_2 \mapsto 2$ gives the required homomorphism. □

Exercise(A) 2.2.6. *Define a notion of k-partite graphs and characterize k-partite graphs G using homomorphisms.*

Exercise* 2.2.7. *Denote by $Hom^*(H, G)$ to be the number of injective homomorphisms from H to G. Let $|V(H)| = k$. Show that*

$$|Hom^*(H, G)| = \sum_{(v_1,\dots,v_k)\in V(G)^k}^{\neq} \mathbf{1}[H \subset < \{v_1, \dots, v_k\} >],$$

where $\sum^{\neq}$ denotes that the sum is over distinct elements and $< v_1, \dots, v_k >$ is the induced subgraph on the vertices $v_1, \dots, v_k$.

See here `http://www.cs.elte.hu/~lovasz/problems.pdf` for open problems about graph homomorphisms.

Definition 2.2.8 (Some notions)**.** *Let $\mathcal{G}$ be the set of all finite graphs.*

- Graph property : *A set $\mathcal{P} \subset \mathcal{G}$ is said to be a* graph property *if $G_1 \in \mathcal{P}$ and $G_1 \cong G_2$, then $G_2 \in \mathcal{P}$.*
- Graph invariant : *$\phi : \mathcal{G} \to \mathbb{R}$ is a* graph invariant *if $\phi(G_1) = \phi(G_2)$ whenever $G_1 \cong G_2$. Equivalently ϕ is a graph invariant if $\phi^{-1}(r)$ is a graph property for every $r \in \mathbb{R}$.*
- Spanning subgraph : *H is a* spanning subgraph *if $V(H) = V(G)$.*
- Neighbourhood : *If $v \in V$, the neighbourhood of v, $N_v := \{w : w \sim v\}$.*
- Degree : *$d_v := |N_v|$.*
- *v is isolated if $d_v = 0$.*

- Regular graph : *G is d-regular if $d_v = d_w = d$ for all v, w.*
- Minimum degree : $\delta(G) := \min\{d_v : v \in V\}$.
- Maximum degree : $\Delta(G) := \max\{d_v : v \in V\}$.
- Average degree : $d(G) := |V|^{-1} \sum_v d_v$.
- Edge density : $\epsilon(G) := |E|/|V|$.

Prove that $\delta(G) \leq d(G) \leq \Delta(G)$.

Exercise 2.2.9. *Which of the graphs in Section 3.1 are regular and what are their average degrees ? Can you compute $Aut(G)$ for these examples ?*

Exercise* 2.2.10. *What can you say about the Minimum degree, maximum degree and average degree of Complementary and Line graphs given those of the original graph ?*

Lemma 2.2.11. *$\sum_v d_v$ is even. Thus the number of odd-degree vertices is even and $d(G) = 2\epsilon(G)$.*

The proof is a straightforward double-counting argument.

Exercise(A) 2.2.12. *Suppose G is a 3-regular graph on 10 vertices such that any two non-adjacent vertices have exactly one common neighbour. Is G isomorphic to Petersen graph ?*

Definition 2.2.13. *(Walk, Trail , Path and Cycle) An ordered set of vertices $P = v_0 \dots v_k$ is said to be* a walk from v_0 to v_k *if $v_i \sim v_{i+1}$. A walk $P = v_0 \dots v_k$ is said to be* a trail from v_0 to v_k *if (v_i, v_{i+1}) are distinct for all i i.e., no repeated edges. A trail $P = v_0 \dots v_k$ is said to be* a path from v_0 to v_k *if v_i's are distinct i.e., no repeated vertices. A walk is* closed *if $v_k = v_0$. A walk or trail is said to be* closed *if $v_k = v_0$ and else* open*. A closed trail is also called as a* circuit*. A circuit $C = v_0 \dots v_{k-1} v_0$ with no repetition of intermediate vertices is called a* cycle *i.e., $v_0, \dots, v_{k-1}$ are distinct. In other words, $v_0 \dots v_{k-1}$ is a path and $v_{k-1} \sim v_0$.*

Path as defined above is also referred to as *simple path* or *self-avoiding walk.* For $v \neq w$, we say that v is connected to w (denoted by $v \to w$) if there exists a path from v to w. We shall always assume that $v \to v$. Show that $\to$ induces an equivalence relation. Define the component of v as $C_v := \{w : v \to w\}$. If $v \to w$, then $C_v = C_w$.

Exercise 2.2.14. *For $v \neq w$, show that there exists a walk from v to w iff there exists a trail from v to w iff there exists a path from v to w.*

Exercise(A) 2.2.15. *Show that $\to$ partitions V into equivalence classes and the equivalence class of v is C_v. Show that C_v is the maximal connected subgraph containing v.*

The equivalence classes induced by $\to$ are called as *connected components* and the number of connected components are denoted by $\beta_0(G)$.

Lemma 2.2.16. *If G has n vertices and m edges then $\beta_0(G) \geq n - m$.*

Proof. We prove using induction on m. Start with a graph on n vertices and no edges. It has n components. Add edges one by one. Addition of an edge decreases $\beta_0(G)$ by at most 1 and hence proved. $\square$

Exercise 2.2.17. *Show that $\beta_0(G) = 1$ iff for all $v \neq w \in G$, $v \to w$.*

We call the graph to be *connected* if $\beta_0(G) = 1$.

Exercise 2.2.18. *Show that $\delta(G), \Delta(G), d(G), \beta_0(G)$ are all graph invariants.*

Exercise(A) 2.2.19. *Let G be a simple graph with at least two vertices. Show that G must contain at least two vertices with the same degree.*

2.3 Graphs and matrices : A little peek

We shall now give a brief hint about utility of matrices and linear algebra in answering graph-theoretic questions.

Definition 2.3.1 (Adjacency matrix.)**.** *Let G be a graph on n vertices. For simplicity, assume $V = [n]$. The adjacency matrix $A := A(G) := (A(i,j))_{1 \leq i,j \leq n}$ of a simple finite graph is defined as follows : $A(i,j) = \mathbf{1}[i \sim j]$. The definition can be appropriately extended for multi-graphs.*

A is a symmetric matrix and hence has real eigenvalues.

Lemma 2.3.2. *Let G be a graph on n vertices and A be its adjaceny matrix. Show that $A^l(i,j)$ is the number of walks of length l from i to j.*

Proof. By definition, $A^l(i,j) = \sum_{i_1,\ldots,i_{l-1}} A(i,i_1)A(i_1,i_2)\ldots A(i_{l-1},j)$ and since A is a $0-1$ valued matrix, we get that $A(i,i_1)A(i_1,i_2)\ldots A(i_{l-1},j) \in \{0,1\}$. The proof is complete by noting that $A(i,i_1)A(i_1,i_2)\ldots A(i_{l-1},j) = 1$ if $i \sim i_1 \sim i_2 \ldots i_{l-1} \sim j$ i.e., $ii_1 \ldots i_{l-1}j$ is a walk of length l. $\square$

Exercise(A) 2.3.3. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Show that the number of closed walks of length l in G is $\sum_{i=1}^n \lambda_i^l$. Here, we count the walk $v_1, \ldots, v_{l-1}, v_1$ and $v_2, \ldots, v_{l-1}, v_1, v_2$ as distinct walks.*

Exercise(A) 2.3.4. *Count the number of closed walks of length l in the complete graph K_n.*

Lemma 2.3.5. *Let G be a connected graph on vertex set $[n]$. If $d(i,j) = m$, then $I, A, \ldots, A^m$ are linearly independent.*

Proof. Assume $i \neq j$. Since there is no path from i to j of length less than m, $A^k(i,j) = 0$ for all $k < m$ and $A^m(i,j) > 0$. Thus, if $I, A, \ldots, A^m$ are linearly dependent with coefficients $c_0, \ldots, c_m$, then by the above observation and positivity of entries of A^k for all k, we have that $c_m = 0$ i.e.,$I, A, \ldots, A^{m-1}$ are linearly dependent. Since $d(i,j) = m$, there exists j' such that $d(i,j') = m - 1$. Now applying the above argument recursively, we get that $c_0 = c_1 = \ldots = c_m = 0$. $\square$

Define length of a path P as the number of edges in the path P. We denote it by $l(P)$. The distance between two vertices is defined as $d(u,v) = \inf\{l(P) : P \text{ is a path from } u \text{ to } v\}$. Further defined $diam(G) = \max\{d(u,v) : u, v \in V\}$.

Corollary 2.3.6. *Let G be a connected graph with k distinct eigenvalues. Then $k > diam(G)$.*

Proof. Let $d = diam(G)$. Recall that *the minimal polynomial* of a matrix A is the monic polynomial Q of least degree such that $Q(A) = 0$. By Lemma 2.3.5, we have that $deg(Q) > d$. The proof is complete by observing that the number of distinct eigenvalues of A is at least $deg(Q)$. □

2.4 ***Graphs as metric spaces***

Let G be a connected weighted graph with strictly positive edge-weights i.e., $w(e) > 0$ for all $e \in E$. or un-weighted graphs, set $w \equiv 1$. We now shall view graphs as metric spaces. The weight/length of a path or walk $P = v_0 \dots v_k$ is $w(P) := \sum_{i=0}^{k-1} w((v_i, v_{i+1}))$. Define the distance between two vertices $v \neq w$ as follows :

$$d_G(v,w) := \inf\{w(P) : P \text{ is a path from } v \text{ to } w\}.$$

Set $d_G(v,v) = 0$ for all $v \in V$. Show that $d_G(v,w) = d_G(w,v)$ and further $d_G(v,w) = 0$ implies that $v = w$.

Exercise* 2.4.1 (Graph metric)**.** *Show that (V, d_G) is a metric space.*

Even if G is not connected, we have that d_G satisfies the three axioms of a metric space. Further, define the *diameter of a graph* as

$$diam(G) = \max\{d_G(v,w) : v, w \in G\}.$$

Given a vertex v and $r > 0$, set $B_r(v) := \{w : d_G(v,w) \leq r\}$, the ball of radius r at v. For un-weighted graphs show that $|B_n(v)| \leq \sum_{i=0}^{n} \Delta(G)^i$.

Exercise* 2.4.2. *Let G be the Cayley graph of a free group generated by a finite symmetric set of generators $S = \{s_1, -s_1, \dots, s_n, -s_n\}$. Compute $|B_n(e)|$ for all n.*

Exercise 2.4.3. *Can you compute $|B_n(O)|$ for $\mathbb{L}_d$ for $d \geq 2$?*

For (un-weighted) graphs, let $\Pi_n(v)$ be the set of self-avoiding walks of length exactly n from v. it holds that $|\Pi_n(v)| \leq \Delta(G)(\Delta(G)-1)^n$ where $n \in \mathbb{N}$. Thus for $\mathbb{L}^d$, we have that $|\Pi_n(O))| \leq 2d(2d-1)^n$.

Exercise* 2.4.4. *Is it true that there exists a κ_d such that as $n \to \infty$, we have that $|\Pi_n(O)|^{1/n} \to \kappa_d \in [0, \infty)$?* Hint : Use that $\log |\Pi_n(O)|$ is sub-additive.

2.5 ***Some questions***

Question 2.5.1. *Instead of counting* $|B_n(0)|$, *consider the following counting. Let* $f_d(r) = |\{(n_1, \dots, n_d) \in \mathbb{Z}^d : \sum_{i=1}^d n_i^2 \leq r^2 :\}|$ *be the number of lattice points within* r *radius ball in* $\mathbb{R}^d$. *Show that* $f_d(r)/r^d \to$ *a constant. To determine exact asymptotics of* $f_2(r)$ *is known as the Gauss circle problem.*

Here are two questions from geometric group theory which can be stated in the language of graph theory. For more on this fascinating subject, refer to [Clay and Margalit 2017].

Question 2.5.2. *Suppose* G *is the Cayley graph of a finitely generated countable group such that* $n^{-d}|B_n(e)| \to \infty$ *for all* $d \in (0, \infty)$. *Is it true that* $n^{-a} \log |B_n(e)| \to \infty$ *for some* $a > 0$ *?*

Question 2.5.3. *Can you construct a finitely generated group such that its Cayley graph has the following growth property for some* $a < 0.7$ *?*

$$0 < \liminf_{n\to\infty} n^{-a} \log |B_n(e)| \leq \limsup_{n\to\infty} n^{-a} \log |B_n(e)| < \infty.$$

Now some questions on self-avoiding walks.

Question 2.5.4. *Can you calculate* κ_d *for any* $d \geq 2$ *?*

Note : If you have solutions for any of the above four questions, please consider submitting them here. It was recently shown by Duminil-Copin and Smirnov that $\kappa = \sqrt{2+\sqrt{2}}$ for the hexagonal lattice (see Figure 2.5). In listing problems in IMO which can lead to research problems, Stanislav Smirnov mentions this ([Smirnov 2011]).

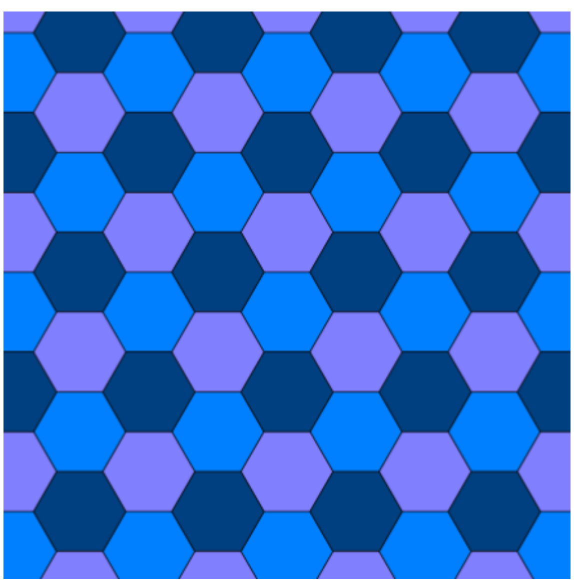


Figure 2.1: Hexagonal lattice

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