Topology

21 Jan 22

Lecture 2

Zorn's lemma

Axiom of choice :- Given $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ a collection of non-empty sets, $\exists$ a function $f: \mathcal{A} \to \bigsqcup_{\alpha \in \mathcal{A}} X_\alpha$ such that $f(\alpha) \in X_\alpha \ \forall \alpha \in \mathcal{A}$.

ie Product of non-empty sets is non-empty.

An partial order on a set S is a relation denoted by $\leq$ s.t.

$$x \leq x \ ; \quad x \leq y \ \& \ y \leq x \iff x = y.$$

$$\text{And} \quad x \leq y \ \& \ y \leq z \Rightarrow x \leq z.$$

A chain in an ordered set is a subset K such that given $x, y \in K$, either $x \leq y$ or $y \leq x$.

Zorn's lemma :- If in an ordered set every chain is bounded above then, $\exists$ a maximal elt.

ie an elt. m s.t. $x \geq m \Rightarrow x = m$.

$\boxed{\text{ex}}$ ① In the set T of all proper subsets of a set S of size n ($2^n - 2$). inclusion is $\leq$.

any $A \in T$ with $|A| = n-1$ is maximal.

② maximal ideals in a ring are max. elts in the set of ideals which are not the whole ring.

Well-ordering let $(S, \leq)$ be an ordered set. It is called well-ordered if ① given $x, y \in S$ we have $x \leq y$ or $y \leq x$.

& ② Every non-empty subset contains a minimum elt. ie $\forall T \subset S, T \neq \phi \Rightarrow \exists t_0 \in T$ s.t. $t \geq t_0 \ \forall t \in T$.

ex. $\boxed{\mathbb{N} \text{ is well ordered}}$ but $\mathbb{Z}$ is not.

Theorem :- (Induction Principle) :- Let S be a well ordered set. Let $A(k)$ be some assertion about $k \in S$.

If we know that "$A(l)$ is true $\forall\, l < k$ $\Rightarrow A(k)$ is true"

Then $A(s)$ is true $\forall\, s \in S$.

example) $A(n) = \{ n < 1000 \}$.
$A(n)$ is true $\forall\, n < 1000$
false $\forall\, n \geq 1000$.

2) $n(n+1) =: A(n)$ $\forall n \in \mathbb{N}$
$\forall\, n \geq 1$

pf. $T_0 \subseteq S$ be s.t. $A(t)$ is not true $\forall\, t \in T_0$

If $T_0 \neq \phi$ then it has a least elt. $t_0 \in T_0$.

$\Rightarrow A(k)$ is true $\forall\, k < t_0$.

$\Rightarrow A(t_0)$ is true. $\Rightarrow t_0 \notin T_0$ contradiction

QED.

"Proof" of Zorn's lemma (using Axiom of choice)

Lemma :- Every chain bounded above $\Rightarrow \exists$ maximal elt.

Assume the statement is false. ie $\forall\, s \in S$ $\exists s_1 \in S$ s.t. $s_1 > s$ (if $m \geq m_1$ & $m \neq m_1$ then we write as $m > m_1$)

Axiom of choice : For every chain $K \subset S$ choose $m(K)$ s.t. $m(K) > k$ $\forall\, k \in K$.

$\rightarrow$ Indexing set is all chains K in S.

Given any K — $X_K = \{ s \in S \mid s > k\ \forall k \in K \}$.

$X_K \neq \phi$ by assumption.

$m :$ Set of all chains $\rightarrow \bigsqcup_{K \in \mathcal{C}} \{\text{strict upper bounds on } K\}$
$\quad$ " $\mathcal{C}$

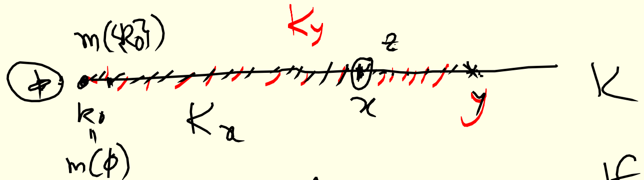
$\phi \rightarrow m(\phi)$ $\quad$ $m(\{s \leq t\}) \rightarrow$ fixed.

$\{s\} \rightarrow m(\{s\})$

Call a chain $K \subset \mathcal{M}$ <u>distinguished</u> if ① K is well-ordered

② $\forall$ "initial part" $K_x = \{k \in K \mid k < x\}$ $x \in K$

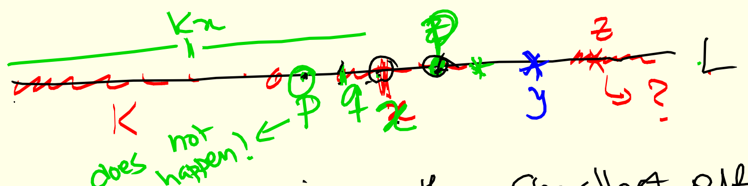
we have $m(K_x) = x$.



<u>Lemma</u> :- If K, L are distinguished chains, then we must have $K = L$; $K_x = L$ for $x \in K$ or $L_y = K$ for some $y \in L$.

<u>proof</u>. Assume that the first two assertions are not true. The subset of L consisting of elts not in K is non-empty.

$\Rightarrow \exists\, y \in L$ s.t. y is the least elt s.t. $y \notin K$.



Assume that the third assertion is also false. i.e $\nexists\, y$ s.t. $L_y = K$.

Since the smallest elt of K & L are same ($= m(\phi)$) there exist smallest $x \in K$ s.t. $x \notin L$ or $K_x \neq L_x$.

(contrapositive of $x \in L$ and $K_x = L_x$)

$\therefore$ we must have $K_x \subset L$ & $K_x \neq L$ by our assumption.

$\Rightarrow p \in L$ Then $p > K_x$; (otherwise $\exists$ $q \in K_x$ s.t. $p < q$ but then $K_q = L_q$ $\Rightarrow p \in K$ cont.)

Clearly $\boxed{K_x = L_p}$ & since K & L are distinguished we have $m(K_x) = x$ in K $\Rightarrow p \in K$ contradiction.

$m(L_p) = p$ in L. $\Rightarrow K_x = L_p$

This proves Zorn's lemma !!

$\boxed{\text{QED}}$

<u>Lemma</u> :- Zorn's lemma $\Rightarrow$ Axiom of choice.

<u>Pf</u>. Let $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ be collection of non-empty set.

$\boxed{\text{TPT}}$ $\exists\, f : \mathcal{A} \to \bigsqcup_{\alpha \in \mathcal{A}} X_\alpha$ s.t $f(\alpha) \in X_\alpha$ $\forall \alpha \in \mathcal{A}$.

Let $S = \{(\Gamma, \phi) \mid \phi : \Gamma \to \bigsqcup_{\alpha \in \mathcal{Q}} X_\alpha$ with $\underline{\Gamma \subseteq \mathcal{Q}}$ & $\phi(\gamma) \in X_\gamma \ \forall \gamma \in \Gamma\}$

$\boxed{\widetilde{TPT}}$ $\exists$ an elt of the type $(\mathcal{Q}, f) \in S$.

Define $\leq$ on S by

$$(\Gamma_1, \phi_1) \leq (\Gamma_2, \phi_2) \text{ iff } \Gamma_1 \subseteq \Gamma_2$$
$$\& \ \phi_2|_{\Gamma_1} = \phi_1$$

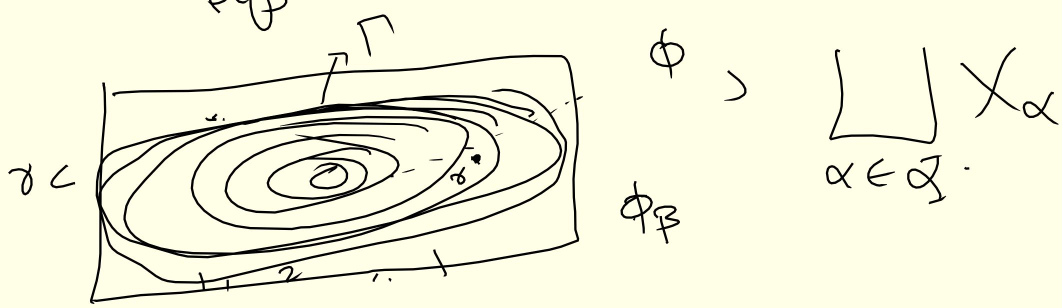
<u>note</u>:- $(\Gamma_1, \phi) \nleq (\Gamma_1, \psi)$ if $\phi \neq \psi$.

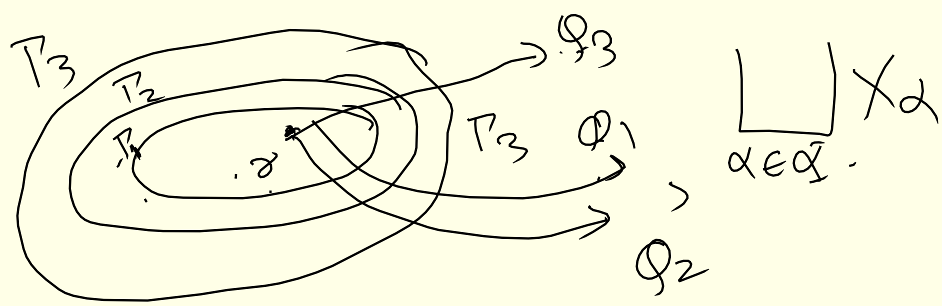
Let K be a chain in S $\quad K = \{(\Gamma_\beta, \phi_\beta)\}_{\beta \in \mathcal{B}}$

s.t. $\beta_1, \beta_2 \in \mathcal{B} \Rightarrow (\Gamma_{\beta_1}, \phi_{\beta_1}) \leq (\Gamma_{\beta_2}, \phi_{\beta_2})$

or other way round.

<u>Define</u> $\Gamma = \bigcup_{\beta \in \mathcal{B}} \Gamma_\beta$, $\Phi(\gamma) = \phi_\beta(\gamma)$ if $\gamma \in \Gamma_\beta$.





$\Phi(\gamma) = \phi_2(\gamma)$
or
$\phi_1(\gamma)$
or $\phi_3(\gamma)$

Φ is well-defined & $\Phi(\gamma) \in X_\gamma \ \forall \gamma \in \Gamma$.

$\Rightarrow \quad (\Gamma, \Phi) \geq (\Gamma_\beta, \phi_\beta) \ \forall \beta \in \mathcal{B}$.

$\Rightarrow$ every chain is bounded above.

$\Rightarrow$ $\exists$ maximal element (S_0, f_0).

Now if $S_0 \subsetneq \mathcal{I}$; then $\exists$ $\alpha \in \mathcal{I} \setminus S_0$.

$S_1 = S_0 \cup \{\alpha\}$, define f to be $f|_{S_0} = f_0$

& $f(\alpha)$ any elt of X_α.

Then $(S_1, f) > (S_0, f_0)$ contradiction.

$\therefore$ $f_0 \in \prod_{\alpha \in \mathcal{I}} X_\alpha$. $\Rightarrow$ axiom of Choice is true!

—x—x—x—

To use Zorn's lemma we

① construct a set S. prove it is non-empty

② construct partial order on S

③ prove that every chain is bounded above.

④ $\exists$ a maximal elt. which usually proves what one wants.

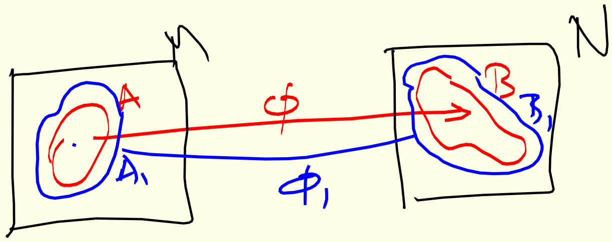
—x—x—x—

Schröder-Bernstein

② Given M, N two sets either $|M| \le |N|$ or $|N| \le |M|$.

($\boxed{\text{Note}}$, the first part of S-B $\Rightarrow$ $|M| \le |N|$ & $|N| \le |M| \Rightarrow |M| = |N|$)

pf. ① $S = \{(A, B, \phi) \mid A \subseteq M, B \subseteq N$ & ϕ - bijection bet^n & A & $B\}$



② $(A_1, B_1, \phi_1) \le (A_2, B_2, \phi_2)$

if $\{A_1 \subseteq A_2$ & $\cancel{B_1 \subseteq B_2}$ & $\phi_2|A_1 = \phi_1\}$

③ Every chain has a max. elt.

$K = \{(A_\alpha, B_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{J}}$ is a chain

then look at $(\bigcup A_\alpha, \bigcup B_\alpha, \bigcup_\alpha \phi_\alpha)$.

ϕ

This bounds K.

$S_1 \xrightarrow{f} S_2$, $S_2 \xrightarrow{f^{-1}} S_1$

④ $\therefore \exists$ a max. elt. (S_1, S_2, f)

If ⓐ $S_1 = M$ or ⓑ $S_2 = N$ then we are done.

ⓐ $f : S_1 \to S_2 \subseteq N$ is injective.

ⓑ $S_2 = N \Rightarrow f^{-1} : S_2 \to S_1 \subseteq M$ is injective.

If $S_1 \subsetneq M$ & $S_2 \subsetneq N$ then choose $m_1 \in M - S_1$ & $n_1 \in N - S_2$.

$$\tilde{S}_1 = S_1 \cup \{m_1\} \qquad \tilde{f} = f \text{ on } S_1$$
$$\tilde{S}_2 = S_2 \cup \{n_1\} \qquad \tilde{f}(m_1) = n_1.$$

$\tilde{f}$ is bijective & $(\tilde{S}_1, \tilde{S}_2, \tilde{f}) > (S_1, S_2, f)$

Contradiction to the maximality.

<u>Theorem</u> :- Every set is well-ordered.

— Exercise —