Topology

19 Jan 22

Lecture 1

1. Introduction

Trying to understand "distance" (nearness).

Set theory. Abstract platform on which all concepts are understood.

Groups – Set + binary operations ✓

Rings – Set + two binary operations ✓ with distributivity law
($a \cdot (b+c) = a \cdot b + a \cdot c$)

S . a set consists of elements.

. functions between sets. $f : S \to T$ (domain: S, range: T)
assigning a unique value in T to every element of S.

$\mathrm{Im}(f)$ = all elements of T of the type $f(s)$ for some s in S.

. injective functions are those where no information is lost. ie $f(x) = f(y)$ iff $x = y$.

. surjective functions are those which hit every elt. of T. ie. $\forall t \in T \ \exists s \in S$ s.t. $f(s) = t$.

. Isomorphism is a function that is both inj. & surjective. We think $S = T$ if $\exists \ \varphi : S \to T$ s.t. φ is 1-1 (inj.) & on-to (surj.).

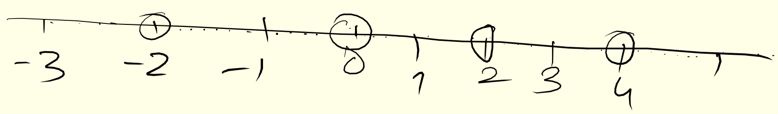
Cardinality of a set.

If is the number of elements of a set. S - set, $|S|$ denotes its cardinality.

Def^n:- ① We say that $|S| \leq |T|$ iff $\exists$ an inj. map $f: S \to T$. ② $|S| = |T|$ iff $\exists$ an iso. bet^n two.

ex. $\mathbb{Z}$ set of integers has $\leq$ card. than set of even integers.

$$\mathbb{Z} \to 2\mathbb{Z} = \{ m \mid 2 \mid m \,;\, m \in \mathbb{Z} \}.$$
$$n \to 2n.$$



clearly $|2\mathbb{Z}| \leq |\mathbb{Z}|$

$m \mapsto m$ identity map is 1-1.

Thm: (Schröder-Bernstein) ① If $|M| \leq |N|$ & $|M| \geq |N|$ then $|M| = |N|$. → ② $\forall M, N$, either $|M| \leq |N|$ or $|N| \leq |M|$

i.e. If $\exists$ 1-1 maps $\varphi: M \to N$ & $\psi: N \to M$ then $\exists$ an iso. $f: M \to N$.

Pf. Observe that given $m = m_0 \in M$ we can construct a sequence

$$m = m_0 \to \overset{\varphi(m_0)}{n_0} \to \overset{\psi(n_0)}{m_1} \to \overset{\varphi(m_1)}{n_1} \to \overset{\psi(n_1)}{m_2}$$

But if $m_0 \in \operatorname{Im}(\psi)$ then extend it to the left.

$$\cdots \to m_{-1} \xrightarrow{\varphi} n_{-1} \xrightarrow{\psi} m_0 \xrightarrow{\varphi} \cdots$$

→ Therefore any $m \in M$ & $n \in N$ is in a unique such sequence.

$$m_{-k} \to \cdots \to n_{-1} \to m_0 \to n_1 \to m_1 \to \cdots$$

Define $f: M \to N$ as follows:

- for $m \in M$ define $\boxed{f(m) = \varphi(m)}$ if the sequence to which m belongs starts with an elt. of M or is infinite on the left.

- Otherwise $f(m) = n$ where $\psi(n) = m$.

Case 2 $\quad n_{-k} \to m_{-k+1} \to \cdots - n_{-1} \to \underset{m}{\overset{\|}{m_0}} \to \cdots\cdots$

Case 1 $\quad m_{-k} \to n_{-k} \to \cdots$

Exercise : Check that f is 1-1 & on-to both.

QED.

—— x —— x ——

Method of induction.

$$m_0 = m \longrightarrow \phi(m) = n_0 \xrightarrow{\psi} m_1$$

having constructed up to i^{th} stage if last elt is in N apply ψ to get $i+1^{th}$ elt.

if i^{th} elt is in M then apply ϕ to get $i+1^{th}$ elt.

$\Rightarrow$ Sequence can be constructed.

$$\begin{tikzcd} m_0 \arrow[r, "\phi"] & n_0 \arrow[l, bend left, "\psi"] \end{tikzcd}$$

—— x —— x —— x ——

→ Cantor's diagonalisation trick says that

$$|\mathbb{Q}| = |\mathbb{Z}| = |2\mathbb{Z}| = |n\mathbb{Z}| \text{ for any } n.$$

→ $|\mathbb{R}| > |\mathbb{Z}|$. (fact).

Cardinal numbers are distinct cardinalities.

$0, 1, 2, 3, \ldots, \infty = \aleph_0 = |\mathbb{Z}|$.

$\aleph_1 = |\mathbb{R}|$.

—— x ——

Power set of a set. X - set $\mathcal{P}(X)$ - power set of X.

$\mathcal{P}(X) =$ set of all subsets of X.

$|X| = n$ then $|\mathcal{P}(X)| = 2^n$. $\quad |\mathcal{P}(X)| > |X|$.

what happens if $|X| \neq n$ for some n.

<u>Cantor's Theorem:-</u> $\quad |\mathcal{P}(X)| > |X| \quad \forall X$.

<u>pf:-</u> $\quad |X| \leq |\mathcal{P}(X)|$ clear -

$x_0 \in X$ fixed.

$x \mapsto \{x\}$. $\rightarrow$ simplest injection.

$x \rightarrow \{x, x_0\}$.

if $|\mathcal{P}(X)| \not> |X|$ then $\exists$ an injection $|\mathcal{P}(X)| \rightarrow |X|$

$\therefore$ therefore a bijection say $g : X \rightarrow \mathcal{P}(X)$.

$x \rightarrow S_x \subset X$.

define $A \subseteq X$ by $A = \{a \in X \mid a \notin g(a)\}$.

$\underset{X.}{\cap}$ $\quad$ $\overset{A}{}$ (may be empty !)

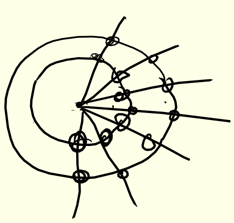
$\therefore \exists \; x_0 \in X$ st. $g(x_0) = A$.

logic $\longleftrightarrow$ $\begin{cases} \text{if } x_0 \in A \; ; \; x_0 \in g(x_0) & \text{contradiction!} \\ \text{if } x_0 \notin A \Rightarrow x_0 \notin g(x_0) & \text{contradiction!} \end{cases}$

$\therefore$ we can not determine if $x_0 \in A$ or not.

This contradiction occured because we assumed the existence of g! $\quad \therefore \nexists g$. $\quad$ QED.

—x—x—x—



End of part 1.

—x—x—x—

$H \subset G \; ; \; G/H \subset \mathcal{P}(G)$.

cosets of H.

$A \in \mathcal{P}(X) \Leftrightarrow A \subseteq X$.

Axiom of choice :- ... { If one takes product of non-empty sets, then the product is non-empty. } ...

Cartesian product. X, Y two sets

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$

what happens if the collection is infinite?

$X_1, X_2, \ldots, X_n, \ldots = \{X_i\}_{i \in \mathbb{N}}$.

$$\prod_{i=1}^{\infty} X_i = \{ (x_i)_{i \in \mathbb{N}} \mid x_i \in X_i \}.$$

If $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is collection of sets.

what is $\prod_{\alpha \in \mathcal{A}} X_\alpha$?

* Note a sequence $x_1, x_2, \ldots, x_i, x_{i+1}, \ldots$ with $x_i \in X_i$ can * be thought of as a map

$$f : \mathbb{N} \longrightarrow \bigsqcup_{i=1}^{\infty} X_i \quad \text{s.t.}$$

$$f(i) \in X_i$$

$$\therefore \boxed{\prod_{\alpha \in \mathcal{A}} X_\alpha = \{ f : \mathcal{A} \to \bigsqcup_{\alpha \in \mathcal{A}} X_\alpha \mid f(\alpha) \in X_\alpha \ \forall \alpha \in \mathcal{A} \}}$$

$\therefore \prod_{\alpha \in \alpha} X_\alpha \neq \phi$ iff given $X_\alpha \neq \phi \ \forall \alpha$, $\exists$ at least one function $f : \alpha \to \bigsqcup X_\alpha$ s.t. $f(\alpha) \in X_\alpha$.

↳ choosing an elt. of X_α.

—x—x—x—

Zorn's lemma

Order on a set.

" generalisation of comparing two elements"

S a set $\leq$ is an order if.

$\left.\begin{matrix} 2 \leq 3 \\ 3 \leq 3 \\ 4 \not\leq 3 \end{matrix}\right\}$ etc.

$$\left\{ \begin{matrix} s \leq s \quad \forall s. \\ s \leq t, \ t \leq r \Rightarrow s \leq r. \\ s \leq t, \ t \leq s \text{ iff } s = t. \end{matrix} \right\}$$

An order is called total order if any two elements can be compared. i.e given $s_1, s_2 \in S$ either $s_1 \leq s_2$ or $s_2 \leq s_1$

Defn

A chain in an ordered set is a subset that is totally ordered (under the same order!)

S a set. $\mathcal{P}(S) = T$. $\leq$ on T is as follows:

$s_1 \leq s_2$ iff $s_1 \subseteq s_2$.

example $T_1 = \{ \{1\}, \{1,2\}, \{1,2,4\}, \{1,2,4,25\} \}$

$\{1,2,4,25,60\}$

$T_1 \subset \mathcal{P}(\mathbb{Z})$

Zorn's lemma: In a partially ordered set if every chain is bounded by an element then $\exists$ an element that is maximal.

ie. $\exists\, \alpha$ s.t. $\nexists$ any $\beta \neq \alpha$ s.t. $\beta \geq \alpha$.

* $T_1 = \{\underbrace{\{2\}, \{1,3\}}_{\text{not comparable.}}\} \to$ Not a chain.

* Every chain is bounded means bounded above, ie. given any chain $T \subset S$ $\exists\, m_T \in S$ s.t. $m_T \geq a \;\; \forall a \in T$.