3) a) We have n such (x_i, y_i, z_i) such that

$\forall i \quad z_i = ax_i + by_i + c + \varepsilon_i$ {where a, b, c are params}

and $\varepsilon_i \sim N(0, \sigma^2)$

ε_is are drawn independent random variables. {Noise}

we know $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ and noisy $\{z_i\}_{i=1}^{n}$

We have to determine a, b, c.

Now z_i can be represents as a ~~normal~~ gaussian distribution s.t;

$$z_i \sim N(ax_i + by_i + c, \sigma^2)$$

The joint distribution is the product of distributions, since z_is are independent.

Let p be the zoint distribution,

~~$p(z_i; x_i, a, b, c, y_i) =$~~

$$p(\{z_i\}; \{x_i\}, \{y_i\}, a, b, c) = \prod_{i=1}^{n} N(ax_i + by_i + c, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{e^{-\frac{(z_i - ax_i - by_i - c)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$\log P = \sum_{i=1}^{n} -\frac{(z_i - ax_i - by_i - c)^2}{2\sigma^2} - n\log\sigma - n\log\sqrt{2\pi}$$

~~Diffe~~

Differentiating the above w.r.t. a and setting to 0 we get,

$$\frac{\partial \log P}{\partial a} = \sum_{i=1}^{n} \frac{2x_i(z_i - ax_i - by_i - c)}{2\sigma^2} = 0$$

$$\Rightarrow a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i y_i + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i z_i \quad \text{— (I)}$$

Differentiating P w.r.t. b and setting to 0 we get,

$$\frac{\partial \log P}{\partial b} = \sum_{i=1}^{n} \frac{2y_i(z_i - ax_i - by_i - c)}{2\sigma^2} = 0$$

$$\Rightarrow a\sum_{i=1}^{n} x_i y_i + b\sum_{i=1}^{n} y_i^2 + c\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i z_i \quad \text{— (II)}$$

Differentiating P wrt c and setting it to 0 we get,

$$\frac{\partial \log P}{\partial c} = \sum_{i=1}^{n} \frac{2(z_i - ax_i - by_i - c)}{2\sigma^2} = 0$$

$$\Rightarrow a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i + cn = \sum_{i=1}^{n} z_i \quad \text{— (III)}$$

(I), (II), (III) are the 3 required linear equations.
These equations can be represented by following:-

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i z_i \\ \sum_{i=1}^{n} y_i z_i \\ \sum_{i=1}^{n} z_i \end{bmatrix}$$

(b) Now we have for ~~in~~ n values,

$$\forall i \quad z_i = a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6 + \varepsilon_i$$

{where $a_1, a_2, a_3, a_4, a_5, a_6$ are params and ε_is are normally distributed noise}

Now $z_i \sim N\left(a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6, \sigma^2\right)$

The joint distribution will be the product of distributions since z_is are independent.

$$P\left\{\{z_i\}; \{x_i\}, \{y_i\}, a_1, a_2, a_3, a_4, a_5, a_6\right\} = \prod_{i=1}^{n} N\left(a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6, \sigma^2\right)$$

~~lo~~ $\cancel{= \prod_{i=1}^{n} e^{-(a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + a_6}}$

$$= \prod_{i=1}^{n} \frac{e^{-(z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)^2 / 2\sigma^2}}{\sqrt{2\pi}\,\sigma}$$

$$\log P = \sum_{i=1}^{n} \frac{-(z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)^2}{2\sigma^2} - n\log(\sqrt{2\pi}\sigma)$$

Differentiating P w.r.t. a_1 and setting it to 0,

$$\frac{\partial \log P}{\partial a_1} = + \sum_{i=1}^{n} \frac{2 x_i^2 (z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)}{2\sigma^2} = 0$$

$$\Rightarrow a_1 \sum_{i=1}^{n} x_i^4 + a_2 \sum_{i=1}^{n} x_i^2 y_i^2 + a_3 \sum_{i=1}^{n} x_i^3 y_i + a_4 \sum_{i=1}^{n} x_i^3 + a_5 \sum_{i=1}^{n} x_i^2 y_i + a_6 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 z_i \quad \text{—(1)}$$

Diff. ~~wrt~~ P wrt a_2 and setting it to 0,

$$\frac{\partial \log P}{\partial a_2} = + \sum_{i=1}^{n} \frac{2 y_i^2 (z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)}{2\sigma^2} = 0$$

$$\Rightarrow a_1 \Sigma x_i^2 y_i^2 + a_2 \Sigma y_i^4 + a_3 \Sigma x_i y_i^3 + a_4 \Sigma x_i y_i^2 + a_5 \Sigma y_i^3 + a_6 \Sigma y_i^2$$
$$= \Sigma z_i y_i^2 \quad \text{— (II)}$$

Diff P wrt a_3 and setting it to 0,

$$\frac{\partial \log P}{\partial a_3} = \sum_{i=1}^{n} \frac{2 x_i y_i (z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)}{2\sigma^2} = 0$$

$$\Rightarrow a_1 \Sigma x_i^3 y_i + a_2 \Sigma x_i y_i^3 + a_3 \Sigma x_i^2 y_i^2 + a_4 \Sigma x_i^2 y_i + a_5 \Sigma x_i y_i^2$$
$$+ a_6 \Sigma x_i y_i = \Sigma x_i y_i z_i \quad \text{— (III)}$$

Diff P wrt a_4 and setting it to 0,

$$\frac{\partial \log P}{\partial a_4} = \sum_{i=1}^{n} \frac{2 x_i (z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)}{2\sigma^2} = 0$$

$$\Rightarrow a_1 \Sigma x_i^3 + a_2 \Sigma x_i y_i^2 + a_3 \Sigma x_i^2 y_i + a_4 \Sigma x_i^2 + a_5 \Sigma x_i y_i + a_6 \Sigma x_i$$
$$= \Sigma x_i z_i \quad \text{— (IV)}$$

Diff wrt a_5 and setting it to 0,

$$\frac{\partial \log P}{\partial a_5} = \sum_{i=1}^{n} \frac{2 y_i (z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6)}{2\sigma^2} = 0$$

$$\Rightarrow a_1 \Sigma x_i^2 y_i + a_2 \Sigma y_i^3 + a_3 \Sigma x_i y_i^2 + a_4 \Sigma x_i y_i + a_5 \Sigma y_i^2 + a_6 \Sigma y_i$$
$$= \Sigma y_i z_i \quad \text{— (V)}$$

Diff w.r.t. a_6 and setting it to 0,

$$\frac{\partial \log P}{\partial a_6} = \sum_{i=1}^{n} \frac{2\left(z_i - a_1 x_i^2 - a_2 y_i^2 - a_3 x_i y_i - a_4 x_i - a_5 y_i - a_6\right)}{2\sigma^2} = 0$$

$\cancel{\Rightarrow a_1 x_i^2 + a_2}$

$$\Rightarrow a_1 \sum x_i^2 + a_2 \sum y_i^2 + a_3 \sum x_i y_i + a_4 \sum x_i + a_5 \sum y_i + n a_6 = \sum z_i \quad \text{—— (VI)}$$

Eqns (I), (II), (III), (IV), (V) and (VI) are the required linear equations. Represented as follows in M.V form -

$$\begin{bmatrix} \sum x_i^4 & \sum x_i^2 y_i^2 & \sum x_i^3 y_i & \sum x_i^3 & \sum x_i^2 y_i & \sum x_i^2 \\ \sum x_i^2 y_i^2 & \sum y_i^4 & \sum x_i y_i^3 & \sum x_i y_i^2 & \sum y_i^3 & \sum y_i^2 \\ \sum x_i^3 y_i & \sum x_i y_i^3 & \sum x_i^2 y_i^2 & \sum x_i^2 y_i & \sum x_i y_i^2 & \sum x_i y_i \\ \sum x_i^3 & \sum x_i y_i^2 & \sum x_i^2 y_i & \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i^2 y_i & \sum y_i^3 & \sum x_i y_i^2 & \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i^2 & \sum y_i^2 & \sum x_i y_i & \sum x_i & \sum y_i & n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} \sum x_i^2 z_i \\ \sum z_i y_i^2 \\ \sum x_i y_i z_i \\ \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix}$$

c) The equation of plane predicted from MATLAB code

$$z = 10.0022x + 19.9980y + 29.9516$$

The noise variance calculated is 23.0685