$X \sim N(\mu, \sigma^2)$

$Y = a + bX$

$E(Y) = a + b\mu$ ✓

$Var(Y) = b^2 \, Var(X) = b^2 \sigma^2$ ✓

$F_Y(y) = P(Y \le y) = P(a + bX \le y)$ $\quad b > 0$

$= P(X \le (y-a)/b)$

$= F_X\left(\frac{y-a}{b}\right)$

If $b < 0$

$P(Y \le y) = P(a + bX \le y) = P(bX \le y - a)$

$= P(X \ge (y-a)/b)$

$= 1 - F_X\left(\frac{y-a}{b}\right)$

If $b > 0 \quad F_Y(y) = F_X\left(\frac{y-a}{b}\right)$

$\longrightarrow f_Y(y) = f_X\left(\frac{y-a}{b}\right) \frac{1}{|b|}$

If $b < 0$, $F_Y(y) = 1 - F_X\left(\frac{y-a}{b}\right)$

$f_Y(y) = -f_X\left(\frac{y-a}{b}\right) \frac{1}{b}$

$= f_X\left(\frac{y-a}{b}\right) \frac{1}{|b|}$

For all $b \neq 0$, $f_Y(y) = e^{-\left(\frac{y-a}{b} - \mu\right)^2 / 2\sigma^2} \Big/ \sigma\sqrt{2\pi}\,|b|$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma|b|} e^{-\left(\frac{y-a}{b} - \mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma|b|} e^{-(y-a-\mu b)^2 / 2\sigma^2 b^2}$$

$$\cancel{\sim} = N(a+b\mu, b^2\sigma^2)$$

$$X \sim N(0,1)$$

$$\phi_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x^2 - 2tx)\right] dx$$

$$= (2\pi)^{-1/2} \left(\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x^2 - 2tx + t^2)\right] dx \right) e^{t^2/2}$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x-t)^2\right] \Big/ \sqrt{2\pi} \; dx$$

$$\downarrow$$

$$N(t, 1)$$

$$= e^{t^2/2}$$

$$Y = \mu + X \sim N(\mu, 1)$$

$$\phi_Y(t) = e^{\mu t} \phi_X(t) = e^{\mu t} e^{t^2/2}$$

$$Y = \mu + aX \sim N(\mu, a^2)$$

$$\phi_Y(t) = e^{\mu t} \phi_X(at) = e^{\mu t} e^{(at)^2/2}$$

$$\phi_Y'(t) = e^{\mu t + a^2 t^2/2} \left(\mu + \frac{a^2 \, 2t}{2}\right) \quad \phi_Y'(0) = \mu$$

$$\phi_Y(t) = e^{\mu t + a^2 t^2/2}$$

$$\phi_Y'(t) = e^{\mu t + a^2 t^2/2} (\mu + a^2 t)$$

$$\phi_Y''(t) = e^{\mu t + a^2 t^2/2} (a^2)$$

$$+ (\mu + a^2 t)^2 \, e^{\mu t + a^2 t^2/2}$$

$$\phi_Y''(0) = a^2 + \mu^2 = E(Y^2)$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$= a^2 + \mu^2 - (\mu)^2 = a^2$$