Q1 The negative log likelihood is given as:

$$\mathcal{L}(\{x_i\}_{i=1}^{n} | \mu) = \frac{1}{n}\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2}$$

$$\longrightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

(a) $v = a\mu + b \longrightarrow \mu = \frac{v-b}{a}$

$$\mathcal{L}(\{x_i\}_{i=1}^{n} | v) = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(x_i - \frac{v-b}{a}\right)^2}{\sigma^2}$$

$$\longrightarrow \frac{2}{\sigma^2 n}\sum_{i=1}^{n}\left(x_i - \frac{v-b}{a}\right)(-1) = 0$$

$$\therefore \frac{\hat{v}-b}{a} = \frac{1}{n}\sum_{i=1}^{n} x_i \longrightarrow \hat{v} = \frac{1}{n}\sum_{i=1}^{n} a x_i + b$$

$$= a(\hat{\mu} + b)$$

$$= g(\hat{\mu})$$

$$E(\hat{v}) = E(a\hat{\mu}+b) = a\mu + b = v$$

$\therefore$ this is an unbiased estimator

(b) $v = \mu^2 \longrightarrow \mu = \sqrt{v} \quad (\mu > 0)$

$$\mathcal{L}(\{x_i\}_{i=1}^{n} | v) = \frac{1}{n}\sum_{i=1}^{n} \frac{(x_i - \sqrt{v})^2}{\sigma^2}$$

$$\longrightarrow \frac{2}{n\sigma^2}\sum_{i=1}^{n} \frac{\cancel{2}(x_i - \sqrt{v})(-1)}{2(\sqrt{v})} = 0$$

$$\longrightarrow \sqrt{\hat{v}} = \frac{1}{n}\sum_{i=1}^{n} x_i \longrightarrow \hat{v} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 = \hat{\mu}^2$$

$$= g(\hat{\mu})$$

$$E(\hat{v}) = E(\hat{\mu}^2) = Var(\hat{\mu}) + \left(E(\hat{\mu})\right)^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

$$\neq \mu^2$$

So this is a biased estimator

Q2a) We have $v = F^{-1}(u)$.

Now $F_V(V \leq y) = P_V(F^{-1}(U) \leq y)$

$= P(U \leq F(y))$

$= F(y)$ since U is a uniform random variable for which we always have $P(U \leq u) = u$, $u \in [0,1]$

Thus V has the same distribution given by F.

b) $$P(D \geq d) = P\left\{ \max_x \left| \frac{\sum_i 1(Y_i \leq x)}{n} - F(x) \right| \geq d \right\}$$

$$= P\left\{ \max_x \left| \sum_i \frac{1(F(Y_i) \leq F(x))}{n} - F(x) \right| \geq d \right\}$$

$$= P\left\{ \max_x \left| \sum_i \frac{1(U_i \leq F(x))}{n} - F(x) \right| \geq d \right\}$$

b) $Y = \min \{X_i\}_{i=1}^{n}$ where $X_i \sim$ Geometric (p).

Then $P(Y \geq y) = \prod_{i=1}^{n} P(X_i \geq y)$

$$= \left((1-p)^y\right)^n$$

$$= \left[(1-p)^n\right]^y$$

$\therefore P(Y < y) = 1 - \left[(1-p)^n\right]^y$

which is the CDF of a geometric r.v. with parameter $1-(1-p)^n$.

Q5 The joint likelihood given n independent samples is

$$L\left(\{x_i\}_{i=1}^{n} \middle| \mu, b\right) = \prod_{i=1}^{n} \frac{1}{2b} \exp\left(-\frac{|x_i - \mu|}{\cancel{2}b}\right)$$

$$LL\left(\{x_i\}_{i=1}^{n} \middle| \mu, b\right) = \sum_{i=1}^{n} \left[-\frac{|x_i-\mu|}{b} - \log(2b)\right]$$

The maximum likelihood estimate for μ is ~~dec~~ precisely the value that minimizes

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The maximum likelihood estimate for μ is ~~[illegible]~~ precisely the value that minimizes

$$E(\mu) = \sum_{i=1}^{n} |x_i - \mu|$$

and $\hat{\mu} = \text{median}(\{x_i\}_{i=1}^{n}) = X_z$ where $z = \left\lfloor \frac{n}{2} \right\rfloor$

If n is odd, $\hat{\mu}$ is unique, otherwise $\hat{\mu}$ is not uniquely defined.

Now $P(X_z \leq x) = P(N_x \geq z)$

where $N_x = \sum_{i=1}^{n} \mathbb{1}(X_i \leq x)$

$$\therefore P(X_z \leq x) = \sum_{j=z}^{n} C(n,j)\left(F_X(x)\right)^j \left(1 - F_X(x)\right)^{n-j}$$

$$\cancel{\therefore f_{X_z}(x) = \sum_{j=z}^{n} C(n,j)\, j \left(F_X(x)\right)^{j-1} f_X(x) \quad (n-j)\left(1 - F_X(x)\right)^{n-j-1}\left(-f_X(x)\right)}$$

$$f_{X_z}(x) = \sum_{j=z}^{n} C(n,j)\left(F_X(x)\right)^j (n-j)\left(1 - F_X(x)\right)^{n-j-1}\left(-f_X(x)\right) + C(n,j)\, j\left(F_X(x)\right)^{j-1} f_X(x)\left(1 - F_X(x)\right)^{n-j}$$

$$= \sum_{j=z}^{n} C(n,j) f_X(x)\, F_X(x)^{j-1}\left(1 - F_X(x)\right)^{n-j-1} \left(j - n F_X(x)\right)$$

Q6

The loglikelihood is given as

$$JLL = \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{(x_i^{(j)} - \mu_j)^2}{2\sigma_j^2} - \frac{1}{2}\log \sigma_j^2$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left\{\frac{(x_i^{(j)} - \mu_j)^2}{2\sigma^2} - \frac{1}{2}\log \sigma^2\right\}$$

Let $N = \sum_{j=1}^{k} n_j$.

Then $\hat{\sigma}^2 = \dfrac{\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(x_i^{(j)} - \bar{x}^{(j)}\right)^2}{N}$ obtained by taking $\dfrac{\partial JLL}{\partial(\sigma^2)} = 0$

$$E(\hat{\sigma}^2) = \frac{1}{N}\sum_{j=1}^{k}\sum_{i=1}^{n_j} E\left[\left(x_i^{(j)}\right)^2\right] + E\left[\left(\bar{x}^{(j)}\right)^2\right] - 2E\left(x_i^{(j)}\,\bar{x}^{(j)}\right)$$

$$= \frac{1}{N}\left[\sum_{j=1}^{k}\left(\sum_{i=1}^{n_j} E\left(\left(x_i^{(j)}\right)^2\right) - E\left(\left(\bar{x}^{(j)}\right)^2\right)\right)\right]$$

$$= \frac{1}{N}\sum_{j=1}^{k} n_j(\sigma^2 + \mu_j^2) - n_j\left(\frac{\sigma^2}{n_j} + \mu_j^2\right)$$

$$= \frac{\sigma^2\left(\sum_{j=1}^{k} n_j - 1\right)}{N} = \frac{\sigma^2(N-k)}{N}$$

$\neq \sigma^2$ $\therefore$ this is a biased estimator

However an estimate

$$\tilde{\sigma}^2 = \frac{N}{N-k} \hat{\sigma}^2 \text{ is unbiased.}$$

Now consider the case where $\mu_2, \ldots, \mu_k$ are known, and μ_1 alone is unknown.

Then the MLE is

$$\hat{\sigma}^2 = \frac{N-1}{N} \sigma^2$$

and the appropriate correction is

$$\tilde{\sigma}^2 = \frac{N}{N-1} \hat{\sigma}^2.$$