$\bar{x}$ = sample mean s = std. dev. of the samples > 0

$S_k = \{ x_i \mid |x_i - \bar{x}| < ks \}$

$N(S_k) = |S_k| \longrightarrow \dfrac{|S_k|}{n} > 1 - \dfrac{1}{k^2}$ to prove

PROOF:

$$(n-1)s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i \in S_k} (x_i - \bar{x})^2 + \sum_{i \notin S_k} (x_i - \bar{x})^2$$

$$\geq \sum_{i \notin S_k} (x_i - \bar{x})^2$$

$$\geq \sum_{i \notin S_k} k^2 s^2 = k^2 s^2 (n - |S_k|)$$

$$(n-1)\cancel{s^2} \geq k^2 \cancel{s^2} (n - |S_k|)$$

$$\frac{n-1}{nk^2} \geq \frac{n - |S_k|}{n} = 1 - \frac{|S_k|}{n}$$

$$\frac{1}{k^2} - \frac{1}{nk^2} \geq 1 - \frac{|S_k|}{n}$$

$$\frac{|S_k|}{n} \geq 1 - \frac{1}{k^2} + \frac{1}{nk^2} > 1 - \frac{1}{k^2}$$

Proof of one-sided CI (CCI)

$\{x_i\}_{i=1}^N \qquad y_i = x_i - \bar{x} \qquad k > 0 \qquad s > 0$

$$\sum_{i=1}^{n} (y_i + b)^2 \geq \sum_{i:\, y_i \geq ks} (y_i + b)^2 \quad \text{for any } b > 0$$

$$\geq \sum_{i:\, y_i \geq ks} (ks + b)^2 = |S_k| (ks + b)^2 \longrightarrow ①$$

$$\sum_{i=1}^{n} (y_i + b)^2 = \sum_{i=1}^{n} \left(y_i^2 + 2 y_i b + b^2\right)$$

$$= \sum_{i=1}^{n} y_i^2 + 2b \cancel{\sum_{i=1}^{n} y_i} + nb^2$$

$\longrightarrow \sum_i (x_i - \bar{x}) = \sum_i x_i - n\bar{x} = 0$

$$= (n-1) s^2 + nb^2 \longrightarrow ②$$

$$(n-1)s^2 + nb^2 \geq |S_k| (ks + b)^2$$

$$|S_k| \leq \frac{(n-1)s^2 + nb^2}{(ks+b)^2} \longrightarrow \frac{|S_k|}{n} < \frac{s^2 + b^2}{(ks+b)^2} \leq \frac{1}{k^2+1}$$

Choose b s.t. RHS is minimized

$$f(b) = \frac{s^2 + b^2}{(ks+b)^2} \qquad f'(b) = \frac{(ks+b)^2 2b - (s^2+b^2) 2 (ks+b)}{(ks+b)^4}$$

$$b = s/k$$

$$\text{RHS} = (s^2 + s^2/k^2) / (ks + s/k)^2 = \frac{1}{k^2+1}$$