(1)

(a) As we don't have any book already, any book we pick will be a ~~different~~ unique colour, thus only 1 book needs to be picked.

So $X_1 = 1$

When $i-1$ books picked, probability of picking a book with different colour = ~~Probability~~ $\frac{\text{No. of unpicked books}}{\text{Total no. of books}}$

$$= \frac{n-(i-1)}{n}$$

$$= \frac{n-i+1}{n}$$

(b) X_i is a geometric random variable, by definition p (parameter) should be the probability of picking a book with a color that has not been picked.

$$\therefore \; p = \frac{n-i+1}{n} \quad \{\text{parameter for } X_i\}$$

(c) $P(X_i = k) = (1-p)^{k-1} p$ {Let X_i be the geometric random variable}

$$E(X_i) = \sum_{k=0}^{\infty} k \, P(X_i = k)$$

$$= \sum_{k=1}^{\infty} kp(1-p)^{k-1} \quad \text{—— (I)}$$

Now multiplying both sides by $1-p$ we get,

$$(1-p)\,E(X_i) = \sum_{k=1}^{\infty} kp(1-p)^{k} \quad \text{—— (II)}$$

Eq^n (I) is

$$E(X_i) = p(1-p)^0 + 2p(1-p)^1 + 3p(1-p)^2 + \cdots \infty$$

Eq^n (II) is $(1-p)E(X_i) = \qquad p(1-p)^1 + 2p(1-p)^2 + \cdots \infty$

Subtracting the above 2 eq^{n}s,

$$(1-(1-p))E(X_i) = p(1-p)^0 + p(1-p) + p(1-p)^2 + \cdots \infty$$

$$\Rightarrow \quad pE(X_i) = p\{1 + (1-p) + (1-p)^2 + \cdots \infty\}$$

RHS has an infinite G.P sum with ratio $(1-p)$

$$\therefore \cancel{p}E(X_i) = \cancel{p} \times \frac{1}{1-(1-p)}$$

$$\Rightarrow E(X_i) = 1/p$$

Now,

$$E(X_i^2) = \sum_{k=1}^{\infty} k^2 \, P(X_i = k)$$

$$= \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1}$$

Now,

$$E(X_i^2) = p(1-p)^0 + 2^2 p(1-p) + \cdots + k^2 p(1-p)^{k-1} + \cdots \infty$$

$$(1-p)E(X_i^2) = \qquad + 1^2 p(1-p) + \cdots + (k-1)^2 p(1-p)^{k-1} + \cdots \infty$$

Subtracting above 2 eq^{n}s,

$$pE(X_i^2) = p + (2^2-1^2)p(1-p) + (3^2-2^2)(1-p)^2 p + \cdots + (k^2-(k-1)^2)p(1-p)^{k-1} + \cdots \infty$$

$$\Rightarrow pE(X_i^2) = p + (2\times2-1)p(1-p) + (2\times3-1)p(1-p)^2 + \cdots + (2k-1)p(1-p)^{k-1} + \cdots \infty$$

$\Rightarrow p\,E(X_i^2) = (2\times1-1)p + (2\times2-1)p(1-p) + \dots + (2k-1)p(1-p)^{k-1} + \dots\infty$

$\Rightarrow (1-p)\,p\,E(X_i^2) = \qquad (2\times1-1)p(1-p) + \dots + (2k-3)p(1-p)^{k-1} + \dots\infty$

$\Rightarrow$ Subtracting above 2 eq^{n}s,

$$p^2E(X_i^2) = p + 2p(1-p) + 2p(1-p)^2 + \dots + 2p(1-p)^{k-1} + \dots + \infty$$

$$\Rightarrow p^2E(X_i^2) = \{ p + 2p((1-p) + (1-p)^2 + \dots + \infty) \}$$

↳ A infinite G.P sum with ratio $1-p$ and first term $(1-p)$.

$$\Rightarrow p^2E(X_i^2) = p + \frac{2p\times(1-p)}{(1-(1-p))}$$

$$\Rightarrow p^2E(X_i^2) = p + \frac{2\cancel{p}(1-p)}{\cancel{p}}$$

$$= p + 2 - 2p$$

$$\Rightarrow E(X_i^2) = \frac{2-p}{p^2}$$

$$Var(X_i) = E(X_i^2) - (E(X_i))^2$$

$$= \frac{2-p}{p^2} - 1/p^2$$

$$= \frac{1-p}{p^2}$$

d) $E(X^{(n)}) = E(X_1 + X_2 + \cdots + X_n)$

$$= \sum_{i=1}^{n} E(X_i) \quad \{\text{since } X_i\text{s are independent R.V}\}$$

$$= \sum_{i=1}^{n} 1/p_i$$

From (b) we know $p_i = \frac{n-i+1}{n}$

$$\Rightarrow E(X^{(n)}) = \sum_{i=1}^{n} \frac{n}{n-i+1}$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

The above cannot be simplified to a closed form, but for large enough n or $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \log n$

Sorry

e) $\operatorname{Var}(X^{(n)}) = \sum_{i=1}^{n} \operatorname{Var}(X_i) \quad \{\because X_i\text{s are independent}\}$

$$= \sum_{i=1}^{n} \frac{1 - \frac{n-i+1}{n}}{\left(\frac{n-i+1}{n}\right)^2}$$

$$= \sum_{i=1}^{n} \frac{\frac{i-1}{n}}{\frac{(n-i+1)^2}{n^2}}$$

$$= \sum_{i=1}^{n} \frac{n(i-1)}{(n-i+1)^2}$$

$$\operatorname{Var}(X^{(n)}) = \sum_{i=1}^{n} \frac{n(i-1)}{(n-i+1)^2} < \sum_{i=1}^{n} \frac{n^2}{(n-i+1)^2}$$

$$\mathrm{Var}(X^{(n)}) < \sum_{i=1}^{n} \frac{n^2}{(n-i+1)^2}$$

$$< n^2 \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right\}$$

$$< n^2 \left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty \right\}$$

$$< \frac{n^2 \pi^2}{6}$$

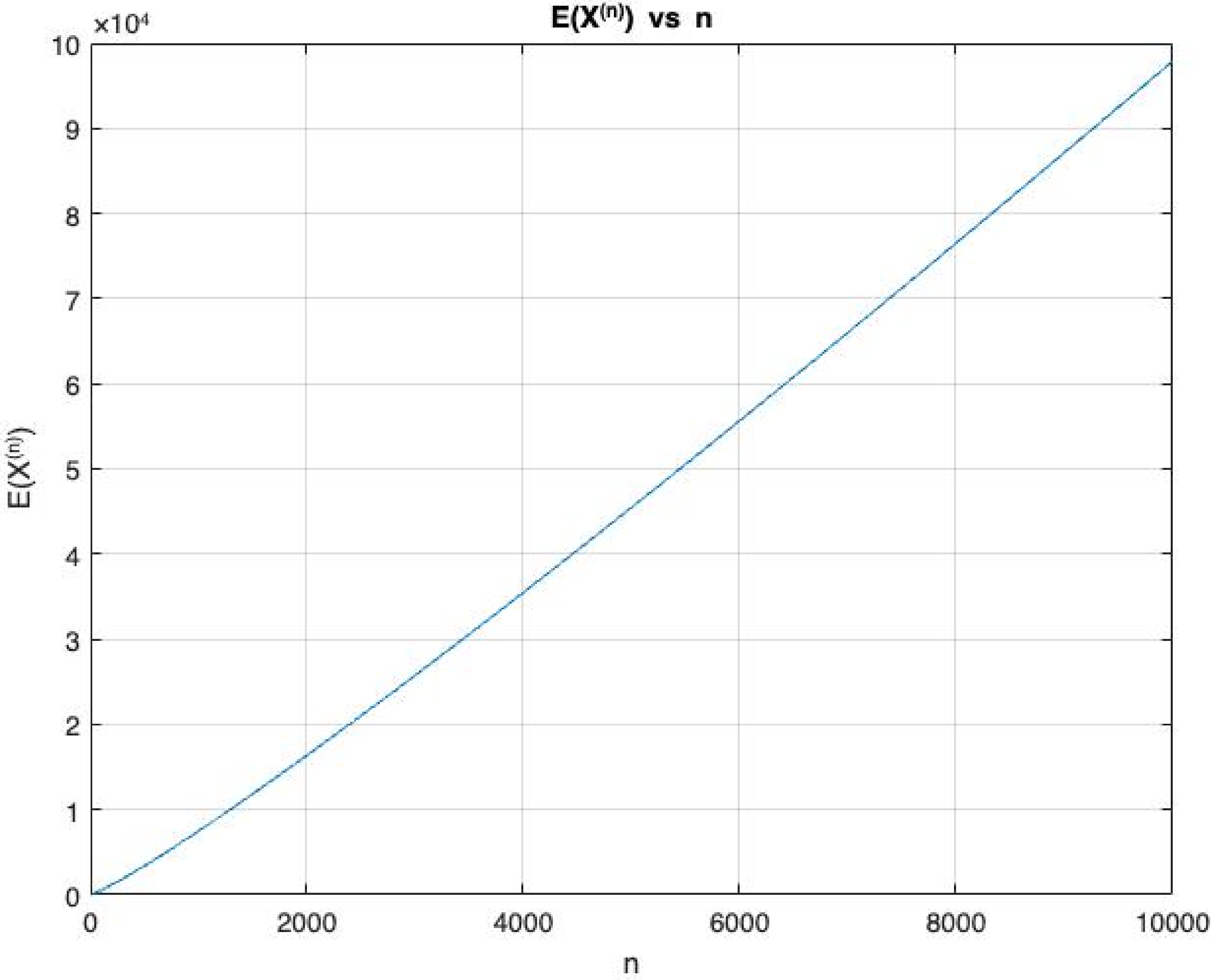
(f) $E(X^{(n)}) = n(1 + 1/2 + 1/3 + \cdots + 1/n)$

for large n, $1 + 1/2 + 1/3 + \cdots + 1/n$ approximates to ~~$1/n$~~ log
{by approximating it to $\int_1^n \frac{1}{x} dx$ }

So $E(X^{(n)})$ is bounded by $n \log n$.

Therefore if $E(X^{(n)}) = \Theta(f(n))$,

$f(n) = n \log n$



E(X^(n)) vs n
×10^4
10
9
8
7
6
5
4
3
2
1
0
0
2000
4000
6000
8000
10000
n
E(X^(n))