Quiz 1: CS 215

Name: ______________________________ **Roll Number:** ____________________

Attempt all five questions. Each question carries 10 points for a total of 50. You have a time of 80 minutes for this quiz.

1. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables, and let $M = \frac{1}{n}\sum_{i=1}^n X_i$. Then prove that for any i from 1 to n, M and $X_i - M$ are uncorrelated. [10 points]
   Solution: We need to prove that $E(M, X_i - M) = E(M)E(X_i - M)$. Now $E(X_i - M) = E(X_i - \sum_{j=1}^n X_j/n) = E(NX_i - \sum_{j=1}^n X_j) = nE(X_i) - nE(X_i) = 0$ since the variables are identically distributed. So we need to show that $E(M, X_i - M) = 0$.
   We have $E(M, X_i - M) = E(MX_i) - E(M^2) = E(\frac{1}{n}\sum_j X_j, X_i) - E(\frac{1}{n^2}(\sum_j X_j)^2) = \frac{1}{n}E(X_i^2) - \frac{1}{n^2}E(\sum_j X_j^2 + \sum_i \sum_{i \neq j} X_i X_j) = \frac{1}{n}E(X_i^2) - nE(X_i^2)/n^2 = 0$. This is because the variables are i.i.d.

2. For a random variable X with moment generating function $\phi_X(t)$ and cumulative distribution function $F_X(x)$, prove that $F_X(x) \leq e^{-tx}\phi_X(t)$ for $t < 0$, and $1 - F_X(x) \leq e^{-tx}\phi_X(t)$ for $t > 0$. [10 points]
   Solution: For the second part, we have $1 - F_X(x) = P(X \geq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}} = e^{-tx}\phi_X(t)$. The first inequality follows from Markov's inequality if $t > 0$. For the first part, $F_X(x) = P(X \leq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}} = e^{-tx}\phi_X(t)$ for $t < 0$ using Markov's inequality again.

3. Consider a sequence of independent Bernoulli trials $X_1, X_2, ...$ each with success probability p. Let N be a random variable that denotes the trial number of the first success. Derive an expression for $P(N > n)$ and $E(N)$. [5+5=10 points]
   Solution: We have $P(N = n) = p(1-p)^{n-1}$ since the first $n-1$ trials were failures and the n^{th} trial was a success. $N > n$ implies the first n trials are all failures. Hence $P(N > n) = (1-p)^n$. Hence $P(N \leq n) = 1 - (1-p)^n$. Also we have $E(N) = \sum_{n=1}^\infty np(1-p)^{n-1} = p\sum_{n=1}^\infty n(1-p)^{n-1} = p\sum_{n=1}^\infty -\frac{d}{dp}(1-p)^n = -p\frac{d}{dp}\sum_{n=0}^\infty (1-p)^n = -p\frac{d}{dp}(1/p) = \frac{1}{p}$.

4. The exponential distribution has a pdf which is given as $f(x) = \lambda e^{-\lambda x}$ where $x \in [0, \infty)$. If X is an exponential random variable with $\lambda > 0$, then derive the pmf of floor(X)andceil(X). Recall that ceil(X) is the smallest integer greater than or equal to X and floor(X) is the largest integer less than or equal to X. [10 points]
   Solution: $P(\text{floor}(X) = n) = P(n \leq X < n+1) = F_X(n+1) - F_X(n) = (1 - e^{-\lambda(n+1)}) - (1 - e^{-\lambda n}) = e^{-\lambda n}(1 - e^{-\lambda})$.
   $P(\text{ceil}(X) = n) = P(n-1 \leq X < n) = F_X(n) - F_X(n-1) = (1 - e^{-\lambda(n)}) - (1 - e^{-\lambda(n-1)}) = e^{-\lambda(n-1)}(1 - e^{-\lambda})$.

5. Let $X_1, X_2, ..., X_n$ be iid random variables from a uniform distribution in the interval $[0, a]$. Let $Y = \max(X_1, X_2, ..., X_n)$. Determine $E(Y)$ and $\text{Var}(Y)$. [10 points]
   Solution: $P(Y \leq y) = \prod_{i=1}^n P(X_i \leq y) = (\frac{y}{a})^n$. Therefore $f_Y(y) = \frac{ny^{n-1}}{a^n}$.
   $E(Y) = \int_0^a y\frac{ny^{n-1}}{a^n}dy = \frac{na}{n+1}$.

$E(Y^2) = \int_0^a y^2 \frac{ny^{n-1}}{a^n} dy = \frac{na^2}{n+2}$.

$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{na^2}{n+2} - (\frac{na}{n+1})^2 = \frac{na^2}{(n+2)(n+1)^2}$.