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DAI - Quiz 1

1) $P(F_1, F_1) = p^2$ {All of them are statistically independent}

$P(F_1, F_2) = p(1-p)$

$P(F_2, F_2) = (1-p)^2$

$$JL = (P_{F_1F_1})^{n_1} \times (P_{F_1F_2})^{n_2} \times (P_{F_2F_2})^{n-n_1-n_2}$$

$$= p^{2n_1} \times p^{n_2}(1-p)^{n_2} \times (1-p)^{2(n-n_1-n_2)}$$

$$JL = p^{2n_1+n_2} \times (1-p)^{2n-2n_1-n_2}$$

$$JLL = (2n_1+n_2)\ln p + (2n-2n_1-n_2)\ln(1-p)$$

$$\frac{\partial(JLL)}{\partial p} = \frac{2n_1+n_2}{p} + \frac{2n-2n_1-n_2}{1-p} \times (-1)$$

$$\frac{\partial(JLL)}{\partial p} = 0$$ (for max estimation)

$$\frac{1-\hat{p}}{\hat{p}} = \frac{2n-2n_1-n_2}{2n_1+n_2}$$

$$\frac{1}{\hat{p}} = \frac{\cancel{2n_1}+\cancel{n_2}+2n-\cancel{2n_1}-\cancel{n_2}}{2n_1+n_2}$$

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$$\boxed{\hat{p} = \frac{2n_1 + n_2}{2n}}$$

2) for a discrete RV, x,

$$E(x) = \sum_i x_i \, P(x = x_i).$$

Therefore,

$$E(Y) = \sum_{i=0}^{\infty} i \cdot P(Y = i).$$

$\{$ for $i = 0$, $i \cdot P(Y = i) = 0 \}$

$$= \sum_{i=1}^{\infty} i \cdot P(Y = i)$$

$$= \sum_{i=1}^{\infty} \Big\{ \sum_{j=1}^{i} P(Y = i) \Big\}$$

$$= \sum_{j=1}^{\infty} \Big\{ \sum_{i=j}^{\infty} P(Y = i) \Big\}$$

{ Square integral transform. Both the integrals are half the area of the square with diagonal $i = j$ }.

$$= \sum_{i=1}^{\infty} P(Y > i-1)$$

for $i' = i-1$,

$$\sum_{i'=0}^{\infty} P(Y > i').$$

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Therefore,

(c) $E(Y) = \sum_{i=0}^{\infty} P(Y > i)$.

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3) $P(X_1 = x_1) = f_{X_1}(x_1)$

$P(X_2 = x_2) = f_{X_2}(x_2)$

CDF : $F_{X_1X_2}(x) = P(X_1X_2 \le x)$

$= P(X_1X_2 \le x)$

$= P(X_1X_2 \le x,\ X_1 \ge 0)$

$+ P(X_1X_2 \le x,\ X_1 \le 0)$

$= P(X_2 \le x/X_1,\ X_1 \ge 0)$

$+ P(X_2 \ge x/X_1,\ X_1 \le 0)$

$$\boxed{\text{CDF} = F_{X_1X_2}(x) = \int_0^{\infty}\int_{-\infty}^{x/t} f_{X_1}(t)\, f_{X_2}(x/t)\, dx\, dt + \int_{-\infty}^{0}\int_{x/t}^{\infty} f_{X_1}(t)\, f_{X_2}(x/t)\, dx\, dt}$$

PDF :

$$\frac{\partial}{\partial x} F_{X_1X_2}(x)$$

$$= \int_0^{\infty} \frac{\partial}{\partial x}\left\{ \int_{-\infty}^{x/t} f_{X_1}(t)\, f_{X_2}(x/t)\, dx \right\} dt$$

$$- \int_{-\infty}^{0} \frac{\partial}{\partial x}\left\{ \int_{x/t}^{\infty} f_{X_1}(t)\, f_{X_2}(x/t)\, dx \right\} dt$$

$$= \int_0^{\infty} f_{X_1}(t)\, f_{X_2}(x/t)\, \frac{dt}{t} - \int_{-\infty}^{0} f_{X_1}(t)\, f_{X_2}(x/t)\, \frac{dt}{t}$$

$$= \int_0^{\infty} f_{X_1}(t)\, f_{X_2}(x/t)\, \frac{dt}{t} + \int_{-\infty}^{0} f_{X_1}(t)\, f_{X_2}(x/t)\, \frac{dt}{|t|}$$

PDF: $\boxed{f_{X_1X_2}(x) = \int_{-\infty}^{\infty} f_{X_1}(t)\, f_{X_2}(x/t)\, \frac{dt}{|t|}}$

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4) $$u(x) = -\sum_{i=1}^{k} P_i \log P_i$$

$$\because \sum_{i=1}^{k} P_i = 1,$$

$$P_k = 1 - \sum_{i=1}^{k-1} P_i$$

$$u(x) = -\sum_{i=1}^{k-1} P_i \log P_i - P_k \log P_k$$

$$\frac{\partial u(x)}{\partial P_i} = -\log P_i - \frac{P_i}{P_i} - \frac{\partial P_k}{\partial P_i}(\log P_k)$$

$$- P_k \frac{\partial \log P_k}{\partial P_i}$$

$$= -\log P_i - \cancel{1} - (-1)\log P_k$$

$$+ \cancel{P_k} \times \frac{1}{\cancel{P_k}} \times \cancel{(-1)}$$

$$= -\log P_i + \log P_k = 0$$

$$\Rightarrow P_i = P_k.$$

$$\Rightarrow P_1 = P_2 \ldots P_k = \frac{1}{k}$$

for $p = \{\underline{\frac{1}{k}, \frac{1}{k} \cdots \frac{1}{k}}\}$, $\boxed{\{\forall i \; P_i = \frac{1}{k}\}}$

u will be maximum

$$u(x) = -k \times \frac{1}{k} \log \frac{1}{k}$$

$$= \boxed{\log k.}$$

now,

$$\frac{\partial u(x)}{\partial P_i} = -\log P_i + \log P_k$$

$$\frac{\partial^2 u(x)}{\partial P_i^2} = \frac{-1}{P_i} + \frac{1}{P_k}(-1)$$

$$= -\left(\frac{1}{P_i} + \frac{1}{P_k}\right) < 0.$$

(P_i is +ve).

sign of $\frac{\partial^2 u(x)}{\partial^2 P_i}$ is <u>negative</u>.

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PMF will be least for

$p_1 = 1$,

and $\forall i \in \{2, k\}$ $P_i = 0$.

$H(x) = -1 \log 1 + 0.$

$= 0.$

Minimum value = $\underline{0}$

for $P = \{1, 0 \ldots 0\}$.

or any other combination with one of the $P_i = 1$ & all other 0.

5) $\rho = 0$,

$$f_Z(z, \rho) = 1 \qquad 0 < z < 1$$

$$= 0 \qquad \text{otherwise}$$

$\rho = 1$,

$$f_Z(z, \rho) = \frac{1}{2\sqrt{z}} \qquad 0 < z < 1$$

$$0 \qquad \text{otherwise.}$$

ρ can be either 0 or 1.

$$JL = 1 \qquad (\rho = 0)$$

$$= \left(\frac{1}{2\sqrt{z}}\right)^n \qquad (\rho = 1).$$
$$(0 < z < 1).$$

$$0 < z < 1$$

$$0 < \sqrt{z} < 1$$

$$\frac{1}{\sqrt{z}} > 1$$

$$\frac{1}{2\sqrt{z}} > \frac{1}{2}.$$

$$JL \geq \frac{1}{2^n}$$

But $JL \leq 1.$ { ~~Maxi~~ JL is probabil of happening of data }

So, for $\boxed{\rho = 0}$

JL will be maximum $(=1)$ for any given distr. of $z_1, z_2 \ldots z_n$.