HW1 - CS754

1

1. We know that $\delta_{2s} \propto S$ (also the upperbound: $\delta_{2s} \leq \mu(s-1)$). So, as S increases, δ_{2s} will increase. Also as m is staying constant, the chances of any 2S columns of measuring matrix being linearly independent decreases as the length of column is fixed and S is increasing.

   The constant terms (C_1, C_2) are increasing functions of δ_{2s}, so they increases as S increases

   So from the above statements:

   - δ_{2s} increases as S increases
   - C_1, C_2 increases as δ_{2s} increases.

   Also, if we change m. ϵ will have a tight lower bound thus resulting in larger $C_2\epsilon$ term.

   Overall they increases in such a way that the overall term increases if S increases
2. m might not be present in the error bound but it is hidden in various terms.
   - The second term contains ϵ which depends on m. It depends on m and the distribution of the noise. For eg.,
     - For η_i ~ Uniform(-r, r), i.e. uniform random noise which known error r, $\epsilon \geq r^2 m$
     - For η ~ N(0,σ^2) with known σ, $\epsilon \geq 9m\sigma^2$
   - Theorem 3 holds only when the matrix satisfies RIP of order 2S. Now if we vary m, δ_{2s} will change and C_1, C_2 will change accordingly.
   - As m decreases, since $m > 2S$ is necessary for RIP, chances of matrix satisfying RIP will decrease.
   - For a sensing matrix Φ constructed, the matrix A will obey the RIP of order S with overwhelming probability provided that the number of rows $m \geq CS\log(n/S)$.

   From the above mentioned points we can conclude that even if the error bound appears to be independent of m, terms in the error bound indirectly depends on m.
3. Theorem 3 is more useful than 3A as it has more relaxed condition on δ_{2s}.
   - When $\delta_{2s} < 0.3$, reconstruction will be similar in both the cases
   - When $\delta_{2s} > 0.3$:
     - If $\delta_{2s} > 0.41$: In this case both the theorem will not be applicable.
     - If $0.3 < \delta_{2s} < 0.41$: In this case, theorem 3A will not be applicable. But we can use theorem 3. The C_1, C_2 error bound terms are increasing functions of δ_{2s}, so the error might me more but we can reconstruct the image. Also since we can apply theorem 3, we can manipulate other terms (decreasing m will increase δ_{2s}) and can compress the signal more and reconstruct from it.
4. If we set $\epsilon = 0$ in BP, even if the noise vector η has non-zero magnitude. The reconstruction won't give result as expected:
   - As discussed in the 1st part of this ques, if we increase S, $1/\sqrt{2} * ||y - \Phi\Psi\theta||_2^2$ term will decrease. therefore we need the $C_2\epsilon$ term to avoid any such discrepancy in our theorem.
   - To get a better reconstruction, we must choose $\epsilon > \kappa$ where, $||\eta||_2^2 \leq \kappa$. And η has non-zero magnitude therefore choosing $\epsilon = 0$ will give poor reconstruction result
   - If we take $\epsilon = 0$, reconstruction algorithms will take more time to converge (or might even fail to converge).

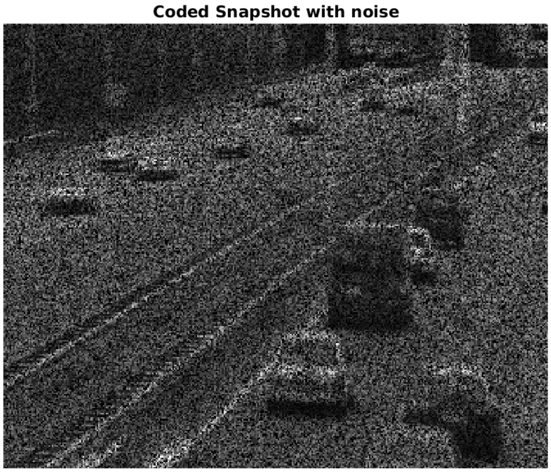
2

Usage instructions:

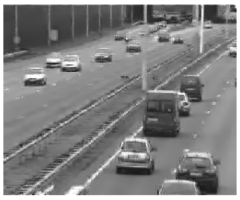
- There are two files:
  - **q2.m**: Contains most the code for this question
  - **omp.m**: Contains the implementation of OMP algorithm
- For T=3,5. We run the algorithm for full image (on matlab online both took around 3 minutes)
- For T=7(cars) and T=5(flame). We run the algorithm for 240 x 120 patch (took around 2-4 minutes)
- Uploaded code reads 240 x 120 patch
- T = (on line 7)
- For changing the video, change video file name on line 8
- To run the code, you can either type q2 or press run

- All these images are included in the images folder

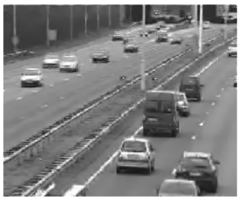
Cars.avi (T=3)



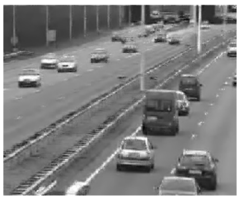
Original


Reconstructed

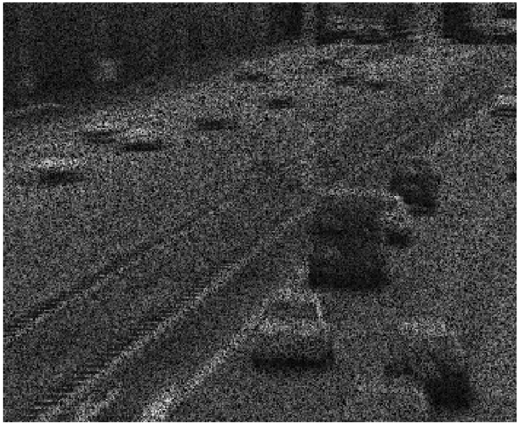
Original

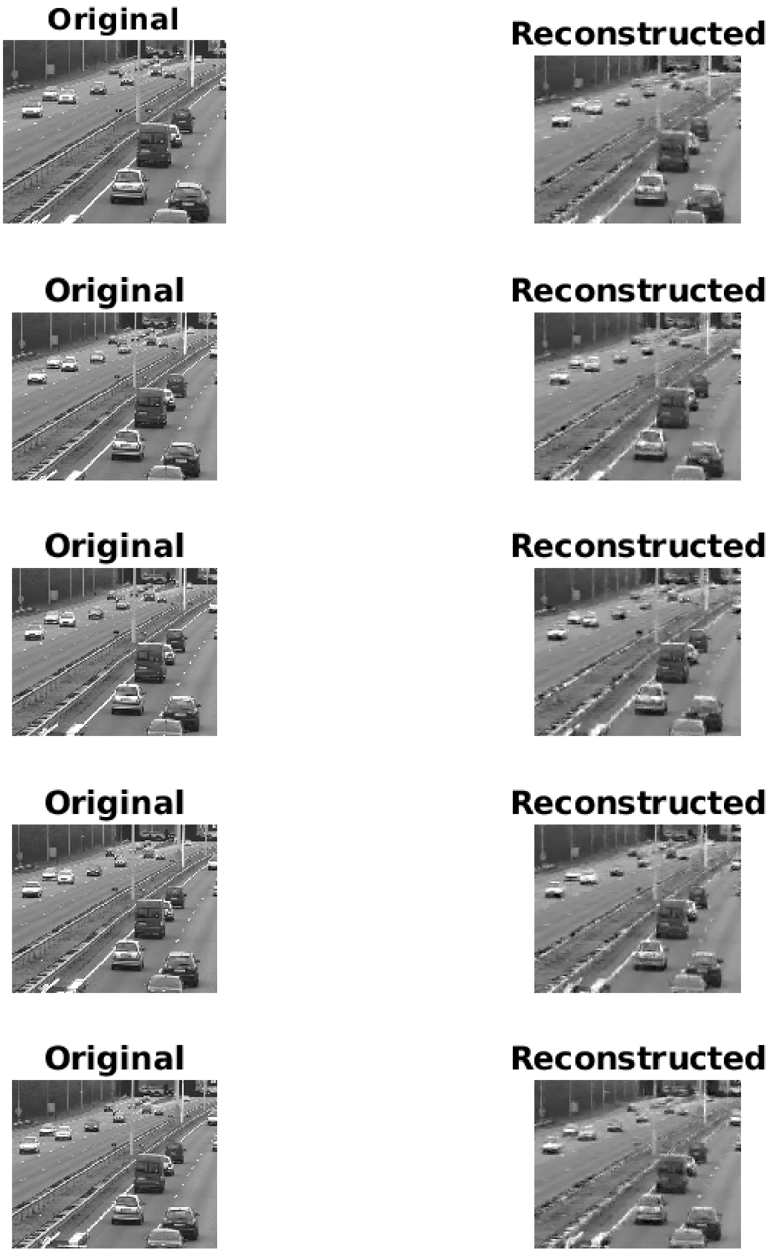

Reconstructed

Original

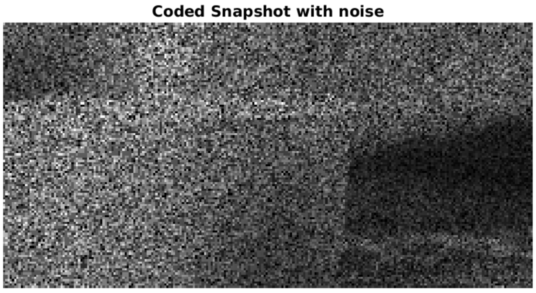

Reconstructed

Cars.avi (T=5)

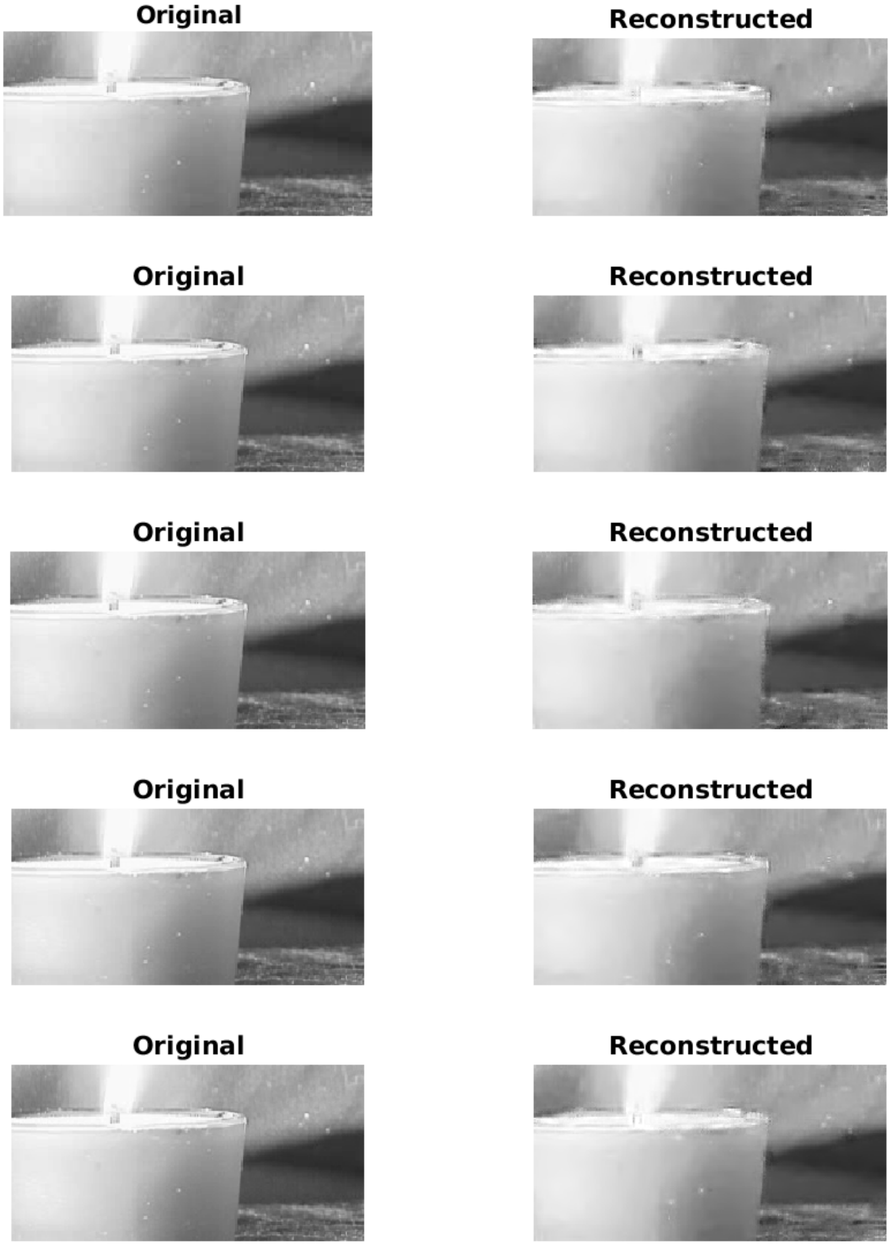

Coded Snapshot with noise



Flames.avi (T=5)



Coded Snapshot with noise



Original

Reconstructed

Original

Reconstructed

Original

Reconstructed

Original

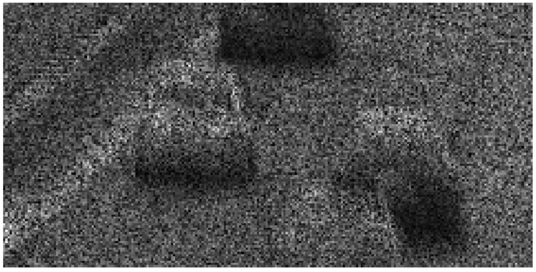
Reconstructed

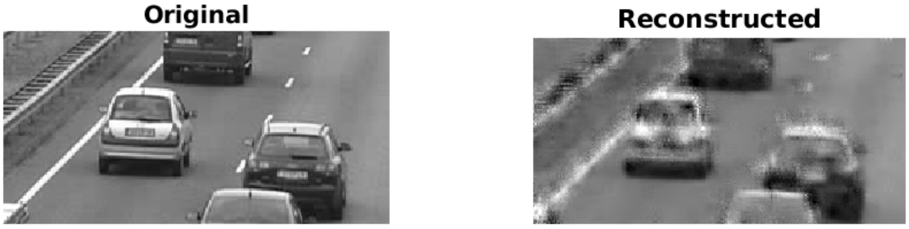
Original

Reconstructed

Cars.avi (T=7)

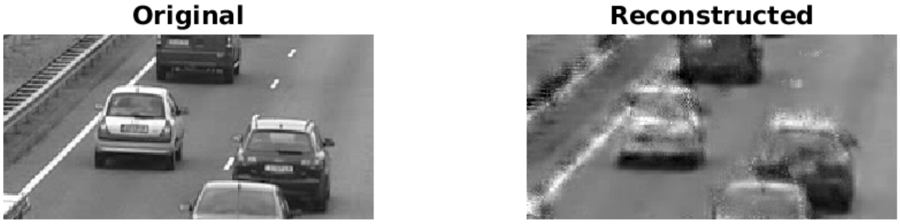
Coded Snapshot with noise





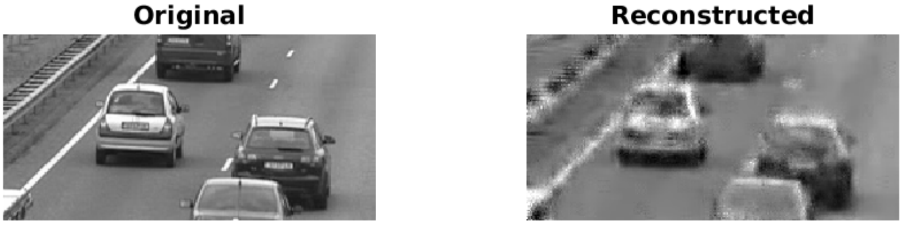
Original

Reconstructed



Original

Reconstructed

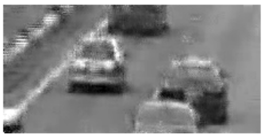


Original

Reconstructed

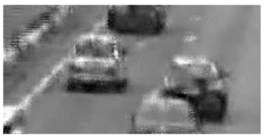
Original

Reconstructed



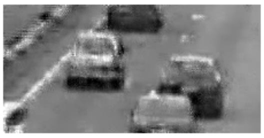
Original

Reconstructed



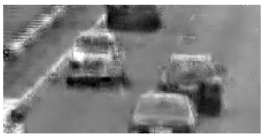
Original

Reconstructed



Original

Reconstructed



c)

We used the equation provided in the question to get an another equation **Ax = b**

$$E_u = \Sigma_{t=1}^{T} C_t \cdot F_t$$

E is the coded image (I for our matlab code). We divided it into 8 x 8 x T patches.

For each patch of 8 x 8, we define:

- $E^{i,j}$ be the patch of size 8 x 8 x T with top left corner at i, j.
- $F^{i,j} = (\psi \otimes \psi) * \theta$, where θ is sparse.
- $C^{i,j}$ be the patch from the binary code C_t with top left corner at i, j.

$$E^{i,j} = \Sigma_{t=1}^{T} F_t^{i,j} \cdot C_t^{i,j}$$

Now, we can represent this as **Ax = b** form:

- $b = E^{i,j}$
- $A = [diag(C_1^{i,j})\ diag(C_2^{i,j})\ diag(C_3^{i,j})]$ ...... for T = 3
- **x** is the vectorised form of video

d)

- Consider $y = \Phi f + \eta = \Phi \Psi \theta + \eta$
- If for each i = 1 to m, η_i is $N(0, \sigma^2)$, with known σ.
- The squared magnitude of the vector η is a chi-square random variable.

- Hence with very high probability, the magnitude of η will lie within 3 SD from the mean, i.e. $\epsilon \geq 9m\sigma^2$.
- Therefore, we set $\epsilon \geq 9 * 64 * 2 * 2 = 2304$.

e) Root mean squared error between reconstructed and original data:

- RMSE for T=□5 (flames) is 0.033017
- RMSE for T=3 (cars) is 0.11745
- RMSE for T=5 (cars) is 0.15479
- RMSE for T=7 (cars) is 0.19776

These data are for full image reconstruction.

3

$$\mu(\Phi, \Psi) = \sqrt{n} \max_{i,j\ \epsilon\{0,1,\ldots,n-1\}} |\Phi^{i^t} \Psi_j|$$

Any $\mathbf{g}\ \epsilon\ \mathbb{R}^n$ can be expressed as $\mathbf{g} = \sum_{k=1}^{n} \alpha_k \Psi_k$ as Ψ is an orthonormal basis.

$$\mu(\mathbf{g}, \boldsymbol{\Psi}) = \sqrt{n} max_{i\epsilon\{0,1,..,n-1\}} |\mathbf{g}^T \Psi_i|$$

$$\mathbf{g^T} = \sum_{k=1}^{n-1} \alpha_k \Psi_k^T$$

$$\mu(\mathbf{g}, \boldsymbol{\Psi}) = \sqrt{n} max_{i\epsilon\{0,1,..,n-1\}} |\sum_{k=1}^{n-1} \alpha_k \Psi_k^T \Psi_i|$$

$$\Psi_j^T \Psi_i = 0\ \forall i \neq j\ \text{ else} = 1$$

$$\mu(\mathbf{g}, \boldsymbol{\Psi}) = \sqrt{n} max_{i\epsilon\{0,1,..,n-1\}} |\alpha_i|$$

$$\text{As g is a unit vector, } ||\mathbf{g}|| = 1$$

$$\mathbf{gg^T = 1}$$

$$\sum_{k=0}^{n-1} \alpha_k \Psi_k \sum_{k=0}^{n-1} \alpha_k \Psi_k^T = 1$$

$$\sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \alpha_k \alpha_l \Psi_k^T \Psi_l = 1$$

$$\sum_{k=0}^{n-1} \alpha_k^2 = 1$$

$$\text{So, } \mu(\mathbf{g}, \boldsymbol{\Psi}) = \sqrt{n} max_{i\epsilon\{0,1,..,n-1\}} \frac{|\alpha_i|}{\sum_{k=0}^{n-1} \alpha_k^2}$$

> Claim: $max|\alpha_i|$ is minimum when all $|\alpha_i|$s are equal and $|\alpha_i| = \sqrt{k}/\sqrt{n}$ if $\sum_{i=0}^{n-1} \alpha_i^2 = k$
>
> Base Case: n=1, trivially true as $\alpha_0^2 = 1$
>
> Induction Hypothesis: Let this hold true for n-1
>
> Proof: w.l.o.g let's say $|\alpha_0|$ is maximum among the α_is (It doesn't matter which index is chosen because addition is associative).
>
> Lets say $\sum_{i=1}^{n-1} \alpha_i^2 = k$

> So $\alpha_0^2 = 1 - k$ we want to minimize α_0^2
>
> α_0^2 should greater than other elements so in order to ensure that we want $max_{i=1}^{n-1} \alpha_i$ to be minimum which will be when $\alpha_1^2 = \alpha_2^2 = ... = \alpha_{n-1}^2 = k/(n-1)$ [By induction Hypothesis]
>
> We want to maximise k to minimise α_0^2
>
> $$\alpha_0^2 \geq k/(n-1) \\ 1 - k \geq k/(n-1) \\ kn/(n-1) \leq 1 \\ k \leq (n-1)/n \\ \alpha_0^2 = 1 - k \geq 1 - (n-1)/n \\ \alpha_0^2 \geq 1/n$$
>
> So $\min \alpha_0^2 = 1/n, \alpha_1^2 = \alpha_2^2 = ... = \alpha_{n-1}^2 = k/(n-1) = (1 - 1/n)/(n-1) = 1/n$
>
> Therefore our claim holds.

So $\min \mu(\mathbf{g}, \mathbf{\Psi}) = \min \sqrt{n} max_{i \epsilon \{0,1,..,n-1\}} |\alpha_i| = \sqrt{n} \times 1/\sqrt{n} = 1$

And it is attained when $|\alpha_0| = |\alpha_1| = ... = |\alpha_{n-1}| = 1/\sqrt{n}$

So minimal value of coherence is attained when $\mathbf{g} = \sqrt{1/n} \sum_{k=0}^{n-1} \pm \Psi_k$

4

a) We can't estimate $\mathbf{x}$ uniquely as we need to know the magnitude and position of the non-zero element but we only have one equation which is $\alpha_i x_i = y$ where α_i is the i^{th} element of ϕ. So we don't know x as well as i. But if we are provided with i then we can determine $\mathbf{x}$ uniquely by $x_i = \frac{y}{\alpha_i}$ and $x_j = 0, \forall j \neq i$.

b) Yes we can determine $\mathbf{x}$ uniquely now we have a measurement vector $\mathbf{y}$ which is 2×1 in dimensions. So $\mathbf{y} = \alpha_\mathbf{i} x_i$ where α_i is the i^{th} element of ϕ. If we have unique $\frac{\alpha_{1,i}}{\alpha_{2,i}}$ for each i then we can determine i using $\frac{\alpha_{1,i} x_i}{\alpha_{2,i} x_i}$,i.e., $\frac{y_1}{y_2}$ which is unique for each i. Now $x_i = y_1/\alpha_{1,i}$ and $x_j = 0 \; \forall j \neq i.$

c) As we have m=3, so $\phi = 3 \times n$.

We have $\mathbf{y} = \mathbf{\Phi x}$ where $||x||_0 = 2$

$$\mathbf{\Phi = [V_1 ... V_n]; V_i = 3 \times 1} \\ \mathbf{\Phi x = x_1 V_1 + ... + x_n V_n} \\ \mathbf{\Phi x = x_i V_i + x_j V_j}$$

Suppose we have an $x^{'}$ which also satisfies this equation. So $\mathbf{\Phi(x - x^{'}) = 0}$.

We know $\max(||\mathbf{x} - \mathbf{x}^{'}||_0) = 4$.

$$\mathbf{\Phi(x - x^{'})} = \sum_\mathbf{i} \alpha_\mathbf{i} \mathbf{V_i}$$

If these 4 $\mathbf{V_i}$s are not linearly independent then there would be α_is other than zeros which satisfies this equation (which means we can't uniquely estimate x) so we have to make sure these are linearly independent to obtain a unique x. But as $\mathbf{V_i} \; \epsilon \; \mathbb{R}^\mathbf{3}$ so any 3 independent $\mathbf{V_i}$s can represent any vector in $\mathbb{R}^3$ which means 4 such vectors can never be linearly independent.

A special instance of Φ exists for which this problem doesn't arise, that is, when it doesn't have 4 vectors, i.e., we resort to $\Phi : 2 \times 3$ and the three columns are linearly independent. In short we want $n = 3$, and at this instance of Φ can uniquely determine $\mathbf{x}$ as now $\max(||\mathbf{x} - \mathbf{x}^{'}||_0) = 3$.

d) The same problem doesn't arise for m=4, so our matrix should be such that any 4 columns of Φ are linearly independent.

> Algorithm: Check for all combinations of i,j ϵ [n] and i<j. Such that y can be represented as a linear combination of $\mathbf{V_i}$ and $\mathbf{V_j}$ if it is possible then x_i = coefficient of $\mathbf{V_i}$, x_j = coefficient of $\mathbf{V_j}$ and rest of the elements of $\mathbf{x}$ are zero. The $\mathbf{x}$ obtained will be unique due to our choice of Φ.
>
> It is possible to determine the coefficients of $\mathbf{V_i}$ and $\mathbf{V_j}$ as it will give >=2 equations for x_i and x_j. There are always >=2 equations as if there were only 1 equation then that would mean $\mathbf{V_i}$ and $\mathbf{V_j}$ are not linearly independent which is not possible.

Existence of such a Φ:

> Four vectors in $\mathbb{R}^4$ are linearly dependent iff they line in a hyperplane.
>
> Consider the following process for building Φ (we are considering it as a set of 4×1 vectors). We can start with the empty set, and choose any three linearly independent vectors $v_1, v_2, v_3 \ \epsilon \ \mathbb{R}^4$ and add them to S. Then choose a fourth vector v_4 to add to S, we must make sure it is not int the unique hyperplane containing (i.e. spanned by) v_1, v_2, v_3. Thus v_4 can be any vector in $\mathbb{R}^4 \setminus span(v_1, v_2, v_3)$.
>
> Similarly, if at some stage $\Phi = \{v_1, \ldots, v_k\}$, we can add to Φ any vector v_{k+1} in $\mathbb{R}^4 \setminus \cup_{x_a,x_b,x_c} span(x_a, x_b, x_c)$. $\cup_{x_a,x_b,x_c} span(x_a, x_b, x_c)$ is a finite union of hyperplanes, so it can never be all of $\mathbb{R}^4$. In this way we can construct Φ for any n with the desired property.

5

a)

Here we have used a passive coded aperture for low power space-time compressive measurement. Coding is implemented by a chrome-on-glass binary transmission mask in an intermediate image plane. Modulation of the image data stream ia done by harmonic oscillations of this mask.

While in Hitomi camera we used an active coded aperture. Here an LCoS device is used to achieve pixel wise exposure control, this is our active aperture. In this device a CMOS chip controls the voltage on square reflective aluminium electrodes buried just below the chip surface, each controlling one pixel. Each pixel can only have a single bump (on-time) during one camera exposure. The start and end times of the bump are controlled.

Hardware designs used in both are also somewhat different. As for CACTI the mask is kept between the lens and image sensor. While for Hitomi, the LCoS is essentially a reflector, so first the light gets reflected by the device and then it reaches image sensor sing a polarising beam splitter.

Both architectures make use of coded aperture which are implemented very differently (CACTI: Mechanical translation of passive aperture, Hitomi: Active aperture) and use CCD array for image sensing.

b)

$$C_f = ||\mathbf{g} - \mathbf{H}\mathbf{f}||^2 + \lambda\Omega(\mathbf{f})$$

where $\Omega(\mathbf{f})$ and λ are the regularizer and regularization weights. We employ a Total Variation (TV) regularizer given by

$$\Omega(\mathbf{f}) = \sum_{k}^{N_F} \sum_{i,j}^{N} \sqrt{(f_{i+1,j,k} - f_{i,j,k})^2 + (f_{i,j+1,k} - f_{i,j,k})^2}$$

which penalises estimates with sharp spatial gradients. TV regularization has been chosen since many natural scenes are well described by sparse gradients.

C_f is our cost function for $TwIST$ reconstruction.

$$\mathbf{f_e} = \arg\min_{\mathbf{f}} C_f$$

$f, f_e \,\epsilon\, \Re^{\sqrt{N} \times \sqrt{N} \times N_F}$; f is our data cube which contains N_F images of N pixels.

We have flattened out f as a column vector so $f, f_e\, : NN_F \times 1$

$\mathbf{T} \,\epsilon\, \Re^{\sqrt{N} \times \sqrt{N} \times N_F}$ is our time varying spatial transmission pattern which uniquely codes each of the N_F temporal channels of f prior to integrating into one detector image $\mathbf{g} \,\epsilon\, \Re^{\sqrt{N} \times \sqrt{N}}$ during acquisition.

g is our detector image which is measured as

$$g_{i,j} = \sum_{k=1}^{N_F} T_{i,j,k} f_{i,j,k} + n_{i,j}$$

where $n_{i,j}$ represents imaging noise at the $(i,j)^{th}$ pixel.

We can represent this using vector multiplication as

$$\mathbf{g} = \mathbf{Hf} + \mathbf{n}$$

$\mathbf{g} : N \times 1 \qquad \mathbf{H} : N \times NN_f \qquad \mathbf{f} : NN_f \times 1 \qquad \mathbf{n} : N \times 1$

$\mathbf{H}$ is the system's discrete forward matrix that accounts for sampling factors including the optical impulse response, pixel sampling function, and time varying transmission function. The forward matrix is a 2-D representation of the 3-D transmission function $\mathbf{T}$:

$$\mathbf{H_k} := diag[T_{1,1,k}\; T_{2,1,k} \;\ldots\; T_{\sqrt{N},\sqrt{N},k}],\; k = 1, \ldots, N_F;$$
$$\mathbf{H} := [\mathbf{H_1}\; \mathbf{H_2} \;\ldots\; \mathbf{H_{N_F}}],$$

where $\mathbf{H_k} \,\epsilon\, \Re^{N \times N}$ is a matrix containing the entries of $\mathbf{T_k}$, binary matrix for coding k^{th} temporal channel, along its diagonals and $\mathbf{H}$ is a concatenation of all $\mathbf{H_k}$, $k \,\epsilon\, [N_F]$.

6

a)

High-speed flow microscopy using compressed sensing with ultrafast laser pulses

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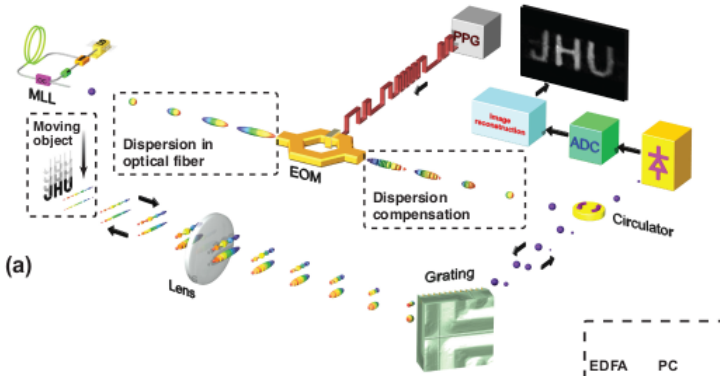
Link to the paper

b)

The principle of operation of the CHiRP-CS imaging system is to modulate pseudorandom patterns at an ultrahigh rate onto the optical spectra of broadband mode-locked laser pulses and then utilize these spectral patterns to create structured illumination of an object.

Light collected from the object is directed onto a single-pixel high-speed photodetector and the energy of each returned laser pulse in recorded continuously by a synchronized real-time ADC (Analog-to-Digital Converter).

Spectral patterning is accomplished using chirp processing in optical fiber. A passively mode-locked erbium-doped fiber laser is used in conjunction with a fiber amplifier. Dispersive spectrum-to-time mapping is then performed in a dispersion compensating fiber. Pattern modulation is achieved with a pulse pattern generator.



Broadband laser pulses are dispersed in optical fiber to accomplish spectrum-to-time mapping. Each pulse is modulated with a unique ultrahigh-rate pseudorandom binary pattern and then re-compressed in fiber (Dispersion compensation) to an ultrashort duration before passing through a 1D wavelength-to-space mapping diffraction grating and lens that focuses the spectral pattern onto the object plane, providing structured illumination of the object flow. The output pulse energy traveling back through the spatial disperser to the photodiode and ADC represents an optically-computed inner product between the pseudorandom pattern and the object. The image is reconstructed via a sparsity-driven optimization from sub-Nyquist compressive measurements.

c)

The coefficient $\alpha \,\epsilon\, \mathbb{R}^{N_b \times M_b}$ of x under some sparsifying transform $\tilde{\Psi}(.)$ defined by

$$\alpha = \tilde{\Psi}(x)$$

should be sparse or compressible.

The recovery process estimates the set of sparse coefficients $\{\alpha^k\}_{k=1}^p$ of the patch set $\{x^k\}_{k=1}^p$ covering the entire image. Denoting $\{\tilde{\alpha}^k\}_{k=1}^p$ as the sparse coefficients of the patches $\{\bar{x}^k\}_{k=1}^p$ extracted from original image $\bar{G} \,\epsilon\, \mathbb{R}^{N \times M}$, the 1D compressive measurement process can be written as

$$y_j = \Phi_j[P(\Psi(\bar{\alpha}^k)_{k=1}^p)]_j, \forall j = 1, \ldots, M$$

where Ψ(.) is the inverse sparsifying trasform of $\tilde{\Psi}$(.) satisfying $\bar{x}^k = \Psi(\bar{\alpha}^k), \forall k = 1, \ldots, p$; P(.) is the operator that combines the set of image patches $\{\bar{x}^k\}_{k=1}^p$ back to the original image, i.e., $\bar{G} = P(\Psi(\bar{\alpha}^k)_{k=1}^p)$; $\Phi_j \,\epsilon\, \mathbb{R}^{m \times N}, \forall j = 1, \ldots, M$, is the local pseudorandom sensing matrix used to measure row $\bar{g}_j$ of $\bar{G}$ and y_j is the corresponding measurement vector. Given the set of measurement vectors and sensing matrices $\{(y_j, \Phi_j)\}_{j=1}^M$, we propose to obtain the sparse coefficients from the following optimization problem