HW 3 - CS754

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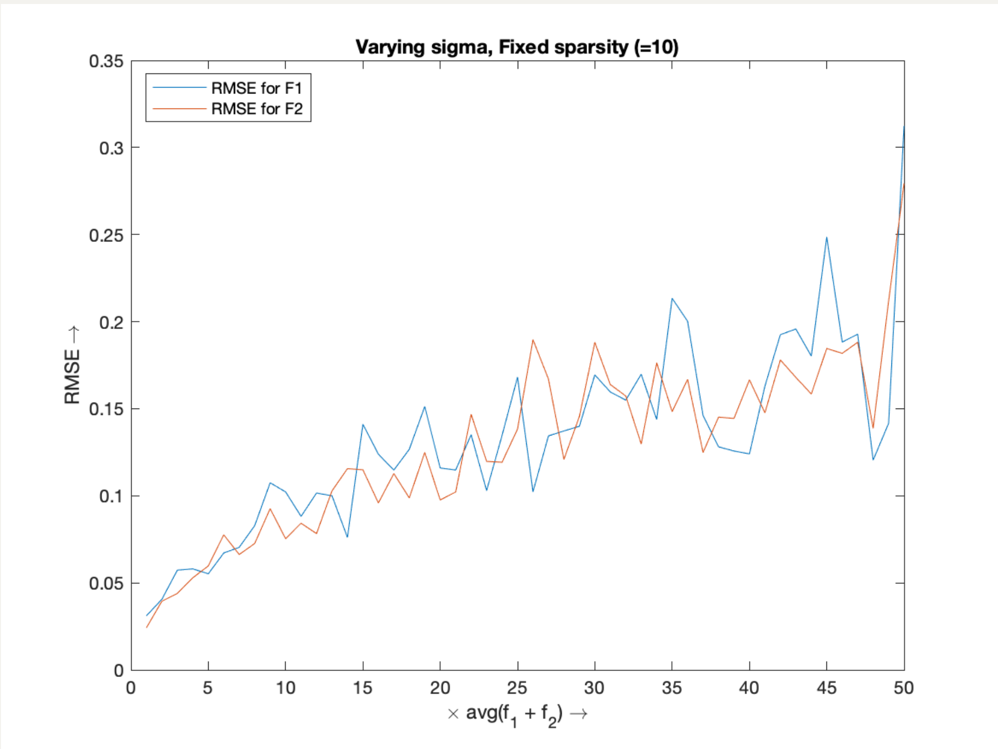
Q1

Technique used for reconstruction:

- Computed a larger basis matrix $A = [A1\ A2]$
- Using the $l1\_ls$ solver in MATLAB, reconstructed the sparse signal $f1, f2$ with basis $A1, A2$ respectively.
- Parameter for the solver was selected using hit an trial method for many values. Out of which we selected the $\epsilon = 10^{-3}$.

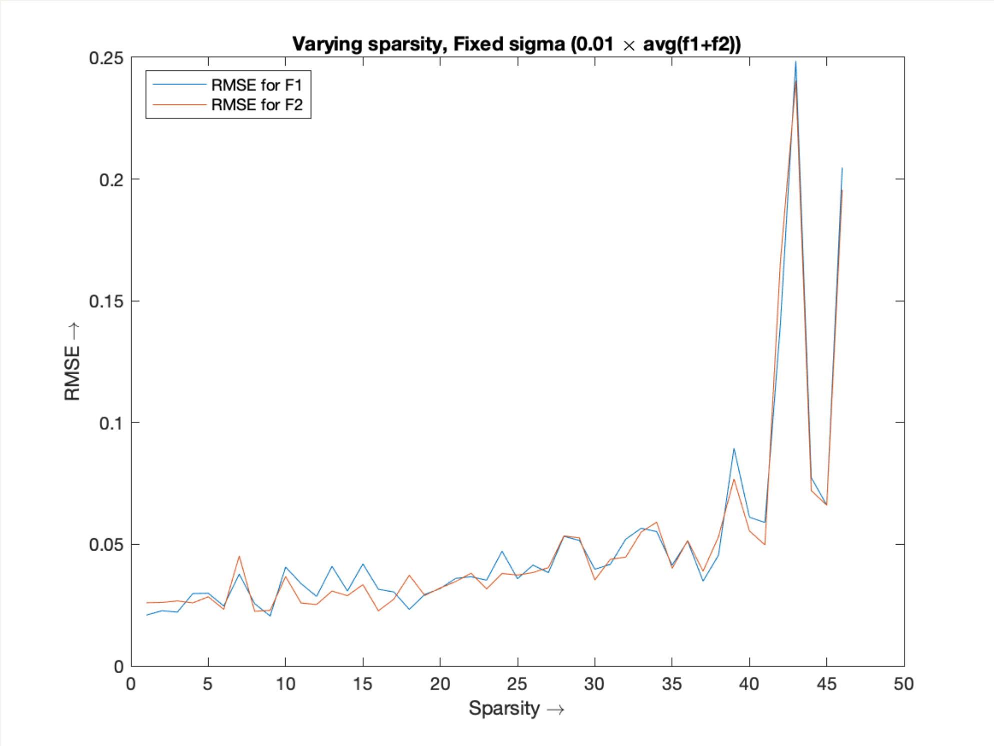
Varying σ and fixed sparsity:

Here, x-axis has the factor τ. Where $\sigma = \tau \times avg(f_1 + f_2)$



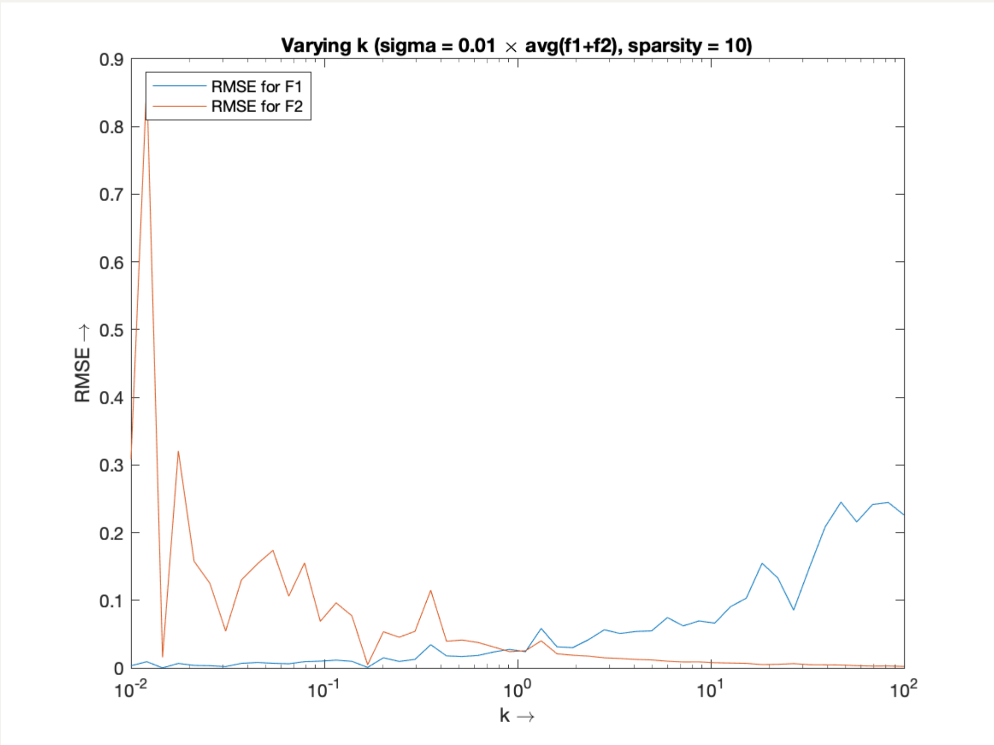
RMSE increases overall with the increase in σ values of the noise.

Varying sparsity and fixed σ



RMSE increases overall with the increase in sparsity of the signal.

varying k, fixed σ and sparsity



With the increase in the factor $(norm(f_2)/norm(f_1))$, the RMSE value for smaller signal is higher. And as observed in this log plot, RMSE value if low around 1 and high around $k \ll 1$ and $k \gg 1$.

Q2

Paper: CoSaMP: Iterative signal recovery from incomplete and inaccurate samples

Author: D. Needell, J. A. Tropp

Journal reference: Appl. Comput. Harmon. Anal., Vol. 26, pp. 301-321, 2008

Algorithm

CoSaMP(Φ, u, s)

Input: Sampling matrix ϕ, noisy sample vector u, sparsity level s.

Output: An s-sparse approximation a of the target signal

1. $a^0 \leftarrow 0$ // Initial approximation
2. $v \leftarrow u$ // Current samples = input samples
3. $k \leftarrow 0$
4. **repeat**
   a. $k \leftarrow k+1$
   b. $y \leftarrow \Phi^* v$ // Form signal proxy
   c. $\Omega \leftarrow supp(y_{2s})$ // Identify large components
   d. $T \leftarrow \Omega \cup supp(a^{k-1})$ // Merge supports
   e. $b|_T \leftarrow \Phi_T^\dagger u$ // Signal estimation by least-squares
   f. $b|_{T^c} \leftarrow 0$
   g. $a \leftarrow b_s$ // Prune to obtain next approx
   h. $v \leftarrow u - \Phi a^k$ // Update current samples

   until haling criterion *true*

Def. *quasi-norm*:

$$||x||_0 = |supp(x)| = |\{j : x_j \neq 0\}|$$

Def.

$$x|_T = x_i \text{ if } i \in T \text{ else } 0$$

$x|_T$ Is treated as an element of the vector space $\mathbb{C}^T$. And the restriction Φ_T of the sampling matrix Φ is defined as the column sub matrix whose columns are listed in the set T.

Def. Pseudo inverse:

$$A^\dagger = (A^* A)^{-1} A^*$$

CoSaMP uses an approach inspired by the restricted isometry property. The sampling matrix $\mathbf{\Phi}$ has restricted isometry constant $\delta_s \ll 1$. For an s-sparse signal x, the vector $\mathbf{y} = \mathbf{\Phi}^* \mathbf{\Phi x}$ serves as a proxy for the signal because the energy in each set of s components of $\mathbf{y}$ approximates the energy in the corresponding s components of x. In particular, the largest s entries of the proxy y point toward the largest s entries of the signal x. Since the samples have the form $\mathbf{u} = \mathbf{\Phi x}$, we can obtain the proxy just by applying the matrix Φ^* to the samples.

Theorem: (CoSaMP). *Suppose that* Φ *is an* $m \times N$ *sampling matrix with RIC* $\delta_{2s} \leq c$. *Let* $\mathbf{u} = \mathbf{\Phi x} + \mathbf{e}$ *be a vector of samples of an arbitrary signal, contaminated with arbitrary noise. For a given precision parameter* η, *the algorithm* CoSaMP *produces a 2s-sparse approximation* $\boldsymbol{a}$ *that satisfies*

$$||\mathbf{x}-\mathbf{a}||_2 \leq C \cdot \max\left\{\eta, \frac{1}{\sqrt{s}}||\mathbf{x}-\mathbf{x_s}||_1 + ||\mathbf{e}||_2\right\}$$

Where $\mathbf{x_s}$ is a best s-sparse approximation to $\mathbf{x}$. The running time is $\mathcal{O}(\mathscr{L} \cdot \log(||\mathbf{x}||_2/\eta))$, where $\mathscr{L}$ bound the cost of a matrix-vector multiply with $\mathbf{\Phi}$ or $\mathbf{\Phi}^*$ The working storage use is $\mathcal{O}(N)$

Q3

Note: D_i is the i^{th} column of dictionary D

- Derivative filter is applied via convolution of an image I with a derivation kernel.

  $\hat{I} = K * I$, we use vectorized I for our dictionary representation .

  This convolution can be represented by a matrix A as convolution is a linear operation, s.t., $vec(\hat{I}) = A.\,vec(I)$

  $vec(I) = DS \Rightarrow A.\,vec(I) = ADS$

  $AD = [AD_1|AD_2|...|AD_n]$

  $\therefore \hat{D} = [AD_1|AD_2|...|AD_n]$

  So our new dictionary $\hat{D}$ is obtained by applying the same derivative filter on the columns of the old dictionary D.

- Rotating an image means re-positioning the cells of that image.

  So a rotated version of a vectorized image I can be computed by multiplying it by a certain matrix, rotated$(I) = \hat{I} = RI$, where each row and column in R have at most 1 non-zero element whose value is 1.

  Rotation by α can be denoted as $I_\alpha = R_\alpha I$, similarly rotation by β can be denoted as $I_\beta = R_\beta I$

  $I = DS$

  $I_\alpha = R_\alpha I = R_\alpha DS$

  $R_\alpha D = [R_\alpha D_1|R_\alpha D_2|...|R_\alpha D_n]$

  $\therefore D^\alpha = [R_\alpha D_1|R_\alpha D_2|...|R_\alpha D_n]$

  Similarly, $D^\beta = [R_\beta D_1|R_\beta D_2|...|R_\beta D_n]$

  As some images are rotated by α while some by β, we need both D^α and D^β.

  $\therefore \hat{D} = [D^\alpha|D^\beta]$

  So our new dictionary $\hat{D}$ is obtained by column concatenation of two dictionaries which are obtained by rotating each column of the old dictionary D (considering it as an image) by α and by β respectively.

- $I^i_{new}(x,y) = \alpha(I^i_{old}(x,y))^2 + \beta(I^i_{old}(x,y)) + \gamma$

  Let the vectorized form of our image be X.

  $X = DS$

  $X_i = D^i.\,S$, where D^i is the i^{th} row of D.

  $X_i^2 = (\sum_j D_{ji}S_j)^2$

  $X_i^2 = \sum_j D_{ji}^2 S_j^2 + 2\sum_{x,y,x>y} D_{xi}D_{yi}S_xS_y$

  $\therefore$ Dictionary D^{sq} for squared signal:

  $D^{sq} = [D^s|D^q]$, where

$D^s = [D_1^2|D_2^2|...|D_n^2]$, D_i^2 is the column vector obtained by squaring elements of D_i

$D^q = 2[D_1D_2|D_1D_3|...|D1D_n|D_2D_3|D_2D_4|...|D_2D_n|...|D_{n-1}D_n]$,

D_iD_j is the column vector obtained by element wise product of D_i and D_j

$aX + b = aDS + b$

$aDS + b = [aD|b\mathbf{1}]S^{'}$, $\mathbf{1}$ is the column vector with every element being 1 and $S^{'}$ is S row concatenated with 1.

$\therefore$ Our new Dictionary $\hat{D}$ is:

$\hat{D} = [\alpha D^{sq}|\beta D|\gamma \mathbf{1}]$

- A blur kernel is applied to the image, so as seen in the first part to obtain the new dictionary $\hat{D}$ we have to apply the same blur kernel to the columns of the old dictionary D.
- I is our image on which convolution is to be applied

  $I^{'} = K * I, K = \sum_i \alpha_i K_i, K_i$ is the i^{th} kernel in the set

  $K * I = \sum_i \alpha_i K_i * I$

  X is the vectorized form of I

  $vec(K_i * I) = A^i.X$, for some matrix A^i

  $X = DS$

  $A^i.X = A^i.DS$

  $A^iD = [A^iD_1|A^iD_2|...|A^iD_n]$

  $D^i = A^iD$

  $\sum_i \alpha_i K_i * I = \sum_i \alpha_i A^i X = \sum_i \alpha_i A^i DS = \sum_i \alpha_i D^i S$

  $\sum_i \alpha_i D^i S = \sum_i D^i(\alpha_i S)$

  $\therefore vec(I^{'}) = \sum_i D^i(\alpha_i S)$

  $\therefore \hat{D} = [D^1|D^2|...|D^C]$, where C is the cardinality of the set of kernels

  So our new Dictionary $\hat{D}$ is concatenation of dictionaries each obtained as described in the previous part for each kernel in the set.

Q4

- The solution to the following optimization problem:

$$\min_{A_r} ||A - A_r||_F^2 \text{ where } \operatorname{rank}(A_r) = r, r \leq \min(m, n), A \in \mathbb{R}^{m \times n}$$

  is given using SVD of A as follows:

$$A_r = \sum_{i=1}^{r} S_{ii} u_i v_i^t, \text{ where } A = USV^T$$

  - This optimization problem occurs in Image Compression.

- Instead of storing mn intensity values, we store $(n+m+1)r$ intensity values where r is the number of stored singular values (or singular vectors). The remaining m-r singular values (and hence their singular vectors) are effectively set to 0.
- This is called as storing a low-rank (rank r) approximation for an image.
- SVD gives the best possible rank-r approximation of any matrix.

-

$$
\begin{gathered}
R^* = \min_R ||A - RB|| \text{ s.t. } R^T R = I \text{ //R is orthonormal} \\
\min_R ||A - RB||_F^2 = \min_A trace((A-RB)^T(A-RB)) \\
= \min_R trace(A^T A - 2A^T RB + B^T B) \\
= \max_R trace(A^T RB) \\
= \max_R trace(RBA^T) \text{ //trace(FG)=trace(GF)} \\
\text{--Let } BA^T = Q \text{, SVD of } Q \text{ gives } Q = UDV^T \\
= \max_R trace(RUDV^T) \\
= \max_R trace(V^T RUD) \\
= \max_R trace(Z(R)D) \text{ where } Z(R) = V^T RU \\
= \max_R \sum_i z_{ii} d_{ii} \le \sum_i d_{ii} \text{ //} Z(R)^T Z(R) = I \\
\text{The maximum is achieved for Z(R)=I,i.e,} \\
V^T RU = I \Rightarrow R = VU^T
\end{gathered}
$$

- This optimization problem arises in Bases learning using Union of Ortho-normal Bases.
- We represent a signal in the following way:

$$
\begin{gathered}
X = AS + \epsilon \\
(A, S) = \min_{A,S} ||X - AS||^2 + \lambda ||S||_1
\end{gathered}
$$

- A is an over-complete dictionary, which is assumed to be a union of ortho-normal bases, in the form

$$
\begin{gathered}
A = [A_1 | A_2 | \dots | A_M] \\
\forall i, 1 \le i \le M, A_i A_i^T = I
\end{gathered}
$$

- This application of SVD is called orthogonal Procrustes problem.

Q5

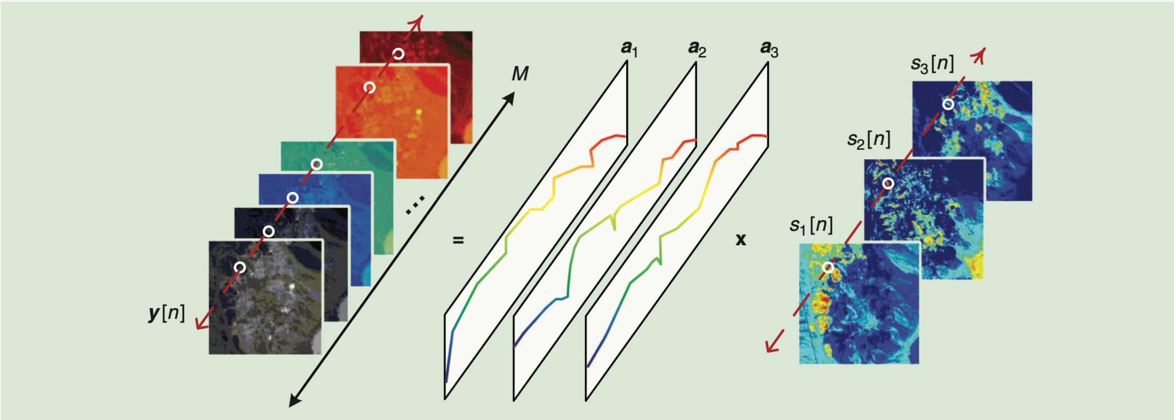
Hyperspectral Unmixing is a procedure that decomposes the measured pixel spectrum of hyperspectral data into a collection of constituent spectral signals and a set of corresponding fractional abundances.

In general, the linear mixture model (LMM) is recognised as an acceptable model for Hyperspectral Unmixing. The LMM is describes as follows. Let $y_m[n]$ denote the hyperspectral camera's measurement at spectral band m and pixel n. Let, $y[n] = [y_1[n], y_2[n], \ldots, y_M[n]]^T \in \mathbb{R}^M$, where M is the number of spectral bands. The LMM is given by:

$$y[n] = \Sigma_{i=1}^{N} \mathbf{a}_i s_i + \mathbf{v}[n] = \mathbf{A}s[n] + \mathbf{v}[n]$$

for $n = 1, \ldots, L$, where,

- each $\mathbf{a}_i \in \mathbb{R}^M, i = 1, \ldots, N$ is called an *endmember signature vector*, which contains the spectral components of a specific material (indexed by i) in the scene.
- N Is the number of endmembers , or material, in the scene. $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_N] \in \mathbb{R}^{M \times N}$ Is called the *endmember matrix*.
- $s_i[n]$ describes the contribution of material i at pixel n. $s[n] = [s_1[n], \ldots, s_N[n]] \in \mathbb{R}^N$ Is called the *abundance vector* at pixel n.
- L Is the number of pixels
- $\mathbf{v}[n] \in \mathbb{R}^M$ is noise.



Non-negative matrix factorisation is posed as a low-rank matrix approximation problem where, given a data matrix $\mathbf{Y} \in \mathbb{R}^{M \times L}$, the task is to find a pair of non-negative matrices $\mathbf{A} \in \mathbb{R}^{M \times N}, \mathbf{S} \in \mathbb{R}^{N \times L}$, with $N < min\{M, L\}$, that solves

$$min_{A \geq 0, S \geq 0} ||\mathbf{Y} - \mathbf{AS}||_F^2$$

In blind HU the connection is that the NMD factors obtained, $\mathbf{A}$ and $\mathbf{S}$, can serve as estimates of the endmembers and abundances.

$$\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_N] \in \mathbb{R}^{M \times N}$$
$$s[n] = [s_1[n], \ldots, s_N[n]] \in \mathbb{R}^N$$

In the HU equation mentioned above, $y[n], s[n]$ are for a pixel at position n. In the NMF equation $\mathbf{S}$ and $\mathbf{Y}$ will be concatenated $s[n]$ and $y[n]$. And so we can reframe hyperspectral unmixing problem into non-negative matrix factorisation problem.

$$min_{A \geq 0, S \in S^L} ||\mathbf{Y} - \mathbf{AS}||_F^2 + \lambda \cdot g(\mathbf{A}) + \mu \cdot h(\mathbf{S})$$

Where, $\mathbf{S}^l = \{\mathbf{S} | s[n] \geq \mathbf{0}, \mathbf{1}^T \mathbf{s}[n] = 1, 1 \leq n \leq L\}$, g and h are regularisers, which vary from one work to another and $\lambda, \mu > 0$ are some constants.

MVC-NMF

$$min_{A \geq 0, S \in S^L} ||\mathbf{Y} - \mathbf{AS}||_F^2 + \lambda \cdot (\text{vol}(B))^2$$

Where, vol(B) is the simplex volume corresponding to $\mathbf{A}$, in which $b_i = C^\dagger(a_i - d)$ for all I This is essentially a variation of the VolMin formulation.

ICE

Iterated constrained endmember (ICE) and sparsity promoting ICE (SPICE) avoid this issue by replacing $(\text{vol}(B))^2$ with a convex surrogate, specifically, $g(\mathbf{A}) = \Sigma_{I=1}^{N-1}\Sigma_{j=i+1}^{N}||a_i - a_j||_2^2$, which is the sum of differences between vertices.

$$min_{A\geq 0, S\in S^L}||\mathbf{Y} - \mathbf{AS}||_F^2 + \Sigma_{I=1}^{N-1}\Sigma_{j=i+1}^{N}||a_i - a_j||_2^2$$

Dictionary Learning

For the abundance regulariser h, the design principle usually follows that of sparsity.

$$min_{A\geq 0, S\in S^L}||\mathbf{Y} - \mathbf{AS}||_F^2 + \mu \cdot ||\mathbf{S}||_{1,1}$$

Where, $||\mathbf{S}||_{1,1} = \Sigma_{n=1}^{L}\Sigma_{I=1}^{N}|s_i[n]|$. The idea is to learn the dictionary $\mathbf{A}$ by joint dictionary and sparse signal optimization.

$\mathbf{L_{1/2}}$-NMF

Similar to the DL optimisation techniques, $L_{1/2}$-NMF uses a non-convex, but stronger sparsity-promoting regulariser based on the $l_{1/2}$ quasi norm. Apart from sparsity, exploitation of spatial contextual information via TV regularisation may be used.

$$min_{A\geq 0, S\in S^L}||\mathbf{Y} - \mathbf{AS}||_F^2 + \mu \cdot ||\mathbf{S}||_{1/2,1/2}^{1/2}$$