Sheet 12

① Ω: sample space

Random variable $X: \Omega \to \mathbb{R}$

Stochastic Process $Y: \Omega \xrightarrow[\text{Measurable}]{} \mathcal{F}$

Probably Not exam Relevant

$\mathcal{F}$ is a space of real valued functions defined on some index set I.

In TSA typically: $I = \mathbb{Z}$ or $I = \mathbb{N}$

a-) $I = \{t \in \mathbb{R}, t > 0\} = \mathbb{R}^+$

b-) At a fixed time point t $\quad Y_t' = \varepsilon_t \sim N(0,1)$

For the next parts, we have to assume i.i.d ness of the ε_t's c-) $F_{t_1, \ldots, t_n}(y_1, \ldots, y_n) \underset{\substack{\text{i.i.d} \\ \text{ness}}}{=} F_{N(0,1)}(y_1) \cdot \ldots \cdot F_{N(0,1)}(y_n)$

d-) Now we take y_k

$$\lim_{y_t \to \infty} F_{t_1, \ldots, t_k}(y_1, \ldots, y_k) \underset{\text{i.i.d}}{=} F_{N(0,1)}(y_1) \cdot \ldots \cdot F_{N(0,1)}(y_{k-1})$$

$$\lim_{y_k \to \infty} F_N(0,1)(y_k) :$$

$= 1$ since F is a cdf

$\Rightarrow$ The process is consistent

Problem 2

a-) $X_t = \Phi_1 X_{t-1} + \Phi_2 X_{t-2} + \varepsilon_t$

$\Leftrightarrow$

$X_t - \Phi_1 X_{t-1} - \Phi_2 X_{t-2} = \varepsilon_t$

$\Leftrightarrow$

$W X_t = \varepsilon_t \ , \ W := (1 - \Phi_1 B - \Phi_2 B^2)$

b-) $\mathrm{Cov}(X_t, X_{t-h}) \overset{\text{hint}}{=} E[X_t, X_{t-h}]$

$$\overset{\text{Definition}}{=} E[(\Phi_1 X_{t-1} + \Phi_2 X_{t-2} + \varepsilon_t) X_{t-h}]$$

$$\overset{\substack{\text{expectation} \\ \text{rules}}}{=} \Phi_1 E[X_{t-1} X_{t-h}] + \Phi_2 E[X_{t-2} X_{t-h}] + \underbrace{E[\varepsilon_t X_{t-h}]}_{=0 \text{ , if } h>0}$$

$$\overset{\text{Stationarity}}{=} \Phi_1 \gamma(h-1) + \Phi_2 \gamma(h-2) + \sigma^2 \big|_{\{h=0\}} \Big\} \text{ means when } h \text{ is } 0$$

$$\rho(h) = \frac{\Phi_1 \gamma(h-1) + \Phi_2 \gamma(h-2) + \sigma^2 \big|_{\{h=0\}}}{\Phi_1 \gamma(1) + \Phi_2 \gamma(2) + \sigma^2}$$

c-) AR(2) - filter : $W \overset{\text{a-)}}{=} (1 - \Phi_1 B - \Phi_2 B^2)$

characteristic polynomial of W:

$$\Gamma(z) = 1 - \Phi_1 z - \Phi_2 z^2 \text{ , } z \in \mathbb{C}$$

[cf cpt 7 slide 29]

Roots: $z^* = \dfrac{\Phi_1 + \sqrt{\Phi_1^2 + 4\Phi_2}}{-2\Phi_2}$ (cf cpt 9 slide 19)

Problem 3

a-) $X_t = \Phi_1 X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$

$$\iff X_t - \Phi X_{t-1} = \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\iff W(B) X_t = V(B) \varepsilon_{t-1} \,, \quad W := 1 - \Phi B$$

$$V(B) := 1 + \theta B$$

$$\iff X_t \overset{\text{chapter 7}}{\underset{\text{slide 33}}{=}} \left(\sum_{j=0}^{\infty} \Phi^j B^j \right) \left(1 + \theta B \right) \varepsilon_t$$

$$\overset{\text{lin}}{\underset{\text{operators}}{=}} \Big[\underbrace{\sum_{j=0}^{\infty} \Phi^j B^j}_{= 1 + \sum_{j=1}^{\infty} \Phi^j \cdot B^j} + \underbrace{\sum_{j=0}^{\infty} \Phi^j \theta B^{j+1}}_{= \theta \sum_{j=1}^{\infty} \Phi^{j-1} B^j} \Big] \varepsilon_t$$

$$\overset{\text{lin}}{\underset{\text{operator}}{=}} \Big[1 + \sum_{j=1}^{\infty} B^j \left[\Phi^j + \Phi^{j-1} \theta \right] \Big] \varepsilon_t$$

with $V_0 = 1$

$V_i = (\Phi + \theta) \Phi^{j-1}$

iid $i > 0$

$$= \varepsilon_t + \sum_{j=1}^{\infty} \Phi^{j-1} (\Phi + \theta) \varepsilon_{t-j}$$

$$= \sum_{j=0}^{\infty} V_j \varepsilon_{t-j}$$

$$= \varepsilon_t + (\Phi + \theta) \sum_{j=1}^{\infty} \Phi^{j-1} \underbrace{\varepsilon_{t+1}}_{\text{MA-}\infty} + 0 = \sum_{j=0}^{\infty} V_j \varepsilon_{t-j}$$

b-) $E[X_t] = E\left[\sum_{j=0}^{\infty} \nu_j \varepsilon_{t-j}\right] = \sum_{j=0}^{\infty} \nu_j \underbrace{E[\varepsilon_{t-j}]}_{=0} = 0$

$$Var(X_t) = Var\left(\sum_{j=0}^{\infty} \nu_j \cdot \varepsilon_{t-j}\right) \overset{\varepsilon_t \sim WN}{=} \sum_{j=0}^{\infty} \nu_j^2 \cdot Var(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{\infty} \nu_j^2 \cdot \sigma^2$$

$$\sum_{j=0}^{\infty} \nu_j^2 = 1 + \sum_{j=1}^{\infty} \left(\phi^j (1 + \phi^{-1}\theta)\right)^2$$

$$= \vdots$$

$$= \frac{1 + \theta^2 + 2\phi\theta}{1 - \phi^2}$$

$$\Rightarrow Var(X_t) = \frac{1 + \theta^2 + 2\phi\theta}{1 - \phi^2} \cdot \sigma^2$$

Given $k > 0$

$$Cov(X_t, X_{t+k}) = Cov\left(\sum_{j=0}^{\infty} \nu_j \varepsilon_{t-j}, \sum_{j=0}^{\infty} \nu_j \varepsilon_{t+k-j}\right)$$

$$\overset{WN}{=} \sum_{j=0}^{\infty} v_j v_{j+k} \cdot \sigma^2$$

$$= \sigma^2 \left(v_k + \sum_{j=1}^{\infty} v_j v_{j+k} \right)$$

$$= \sigma^2 \left(v_k + \sum_{j=1}^{\infty} \left(\phi^j (1+\phi^{-1}\theta) \phi^{j+k} (1+\phi^{-1}\theta) \right) \right)$$

$$= \sigma^2 \left(v_k + \sum_{j=1}^{\infty} \phi^{j-1} (\phi+\theta)^2 \phi^{j+k-1} \right. = \ldots = $$

$$= \frac{\sigma^2 \cdot \phi^k + \theta\phi^{k-1} + \phi^{k+1} + \theta + \phi^k \theta^2}{1-\phi^2}$$

→ This is important for exam

$\boxed{\text{problem 4}}$

① $X_t = \frac{4}{5} X_{t-1} - \frac{3}{20} + \varepsilon_t - \frac{3}{10} \varepsilon_{t-1}$

$\Longleftrightarrow$

$\left(1 - \frac{4}{5}B\right) X_t = \frac{-3}{20} + \left(1 - \frac{3}{10}B\right) \varepsilon_t$

$\Rightarrow$ ARMA(1,1) with intercept

$\left(1 - \frac{4}{5}B\right)$ as well as $\left(1 - \frac{3}{10}B\right)$ are invertible

(Since $\left|\frac{4}{5}\right| < 1$ and $\left|\frac{3}{10}\right| < 1$) (of chapter 7)

$\Rightarrow X_t$ is causal and invertible

2 $Y_t = Y_{t-1} - \frac{1}{2} Y_{t-2} + \varepsilon_t + \varepsilon_{t-1}$

$\Leftrightarrow \underbrace{\left(1 - B + \frac{1}{2}B^2\right)}_{}Y_t = \underbrace{(1-B)}_{}\varepsilon_t$

$\rightarrow$ z-transform

$\rightarrow$ z-transform

$f(z) = 1 - z = 0$

$f(z) = 1 - z + \frac{1}{2}z^2 \overset{!}{=} 0$

$\Rightarrow z = 1$

$\Rightarrow z^2 - 2z + 2 = 0$

$\rightarrow$ not invertable

$\Leftrightarrow z_{\frac{1}{2}} = -\frac{2}{2} \pm \sqrt{1-2} \rightarrow$ not invertible

$= 1 \pm i \Rightarrow |z_{\frac{1}{2}}| > 1 \Rightarrow$ AR-part is invertible, Y_t is causal,

but not invertible