Exercise sheet 10

Exercise 1:

Let X_t and Y_t be indep. time series

with acf γ_x and γ_y

Spectrum of Z_t:

$$f_z(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_z(k) e^{-i\omega k}$$

to rewrite in terms of f_x and f_y, we first write γ_z in terms of γ_x and γ_y

$$\gamma_z(k) = \operatorname{cov}(Z_t, Z_{t-k}) = \operatorname{cov}(X_t + y_t, X_{t-k} + y_{t-k})$$

$$= \underbrace{\operatorname{cov}(X_t, X_{t-k})}_{=\gamma_x(k)} + \underbrace{\operatorname{cov}(X_t, y_{t-h})}_{=0} + \underbrace{\operatorname{cov}(y_t, X_{t-k})}_{=0} +$$

$$+ \underbrace{\operatorname{cov}(y_t, y_{t-k})}_{=\gamma_y(k)}$$

$$= \gamma_x(k) + \gamma_y(k) \Rightarrow f_z(\omega) =$$

$$= \underbrace{\frac{1}{2\pi} \sum \gamma_x(k) e^{-i\omega k}}_{= f_x(\omega)} + \underbrace{\frac{1}{2\pi} \sum \gamma_y(k) e^{-i\omega k}}_{f_y(\omega)}$$

Exercise 2:

a) $\gamma(h) = \dots = \begin{cases} (\psi_0^2 + \psi_1^2)\sigma^2, & h = 0 \\ \psi_0 \psi_1 \sigma^2, & h = \pm 1 \\ 0, & \text{else} \end{cases}$

$$\Rightarrow \rho(h) = \begin{cases} 1, & h = 0 \\ \dfrac{\psi_0 \psi_1}{\psi_0^2 + \psi_1^2}, & h = \pm 1 \\ 0, & \text{else} \end{cases}$$

b) show $|\rho(1)| \leq \frac{1}{2}$

$$-(\psi_0 + \psi_1)^2 \leq 0 \leq (\psi_0 - \psi_1)^2 \qquad +2\psi_0\psi_1$$

$$-\psi_0^2 - 2\psi_0\psi_1 - \psi_1^2 \leq 0 \leq \psi_0^2 - 2\psi_0\psi_1 + \psi_1^2$$

$$\Rightarrow -(\psi_0^2 + \psi_1^2) \leq 2\psi_0\psi_1 \leq \psi_0^2 + \psi_1^2$$

$$\Rightarrow -\frac{1}{2} \leq \frac{\psi_0\psi_1}{\psi_0^2 + \psi_1^2} \leq \frac{1}{2}$$

c) spectrum: $f(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma(k) \underbrace{e^{-i\omega k}}_{\cos + i \sin} =$

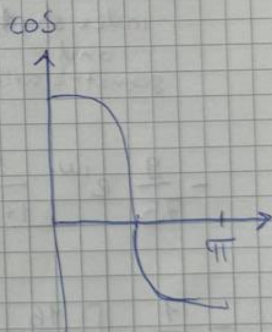
$$= \underbrace{\frac{1}{2\pi} \gamma_0}_{\Rightarrow k=0} + \frac{1}{\pi} \sum_{k=1}^{\infty} \gamma(k) \cos(\omega k)$$

Spectral density:

$$\rho(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \rho(k) e^{-i\omega k} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \rho(k) \cos(\omega k)$$

$$\Rightarrow f(\omega) = \frac{1}{2\pi} (\psi_0^2 + \psi_1^2) \sigma^2 + \frac{1}{\pi} \psi_0 \psi_1 \sigma^2 \cos(\omega)$$

$$\Rightarrow \rho(\omega) = \frac{1}{2\pi} + \frac{1}{\pi} \frac{\psi_0 \psi_1}{\psi_0^2 + \psi_1^2} \cos(\omega)$$



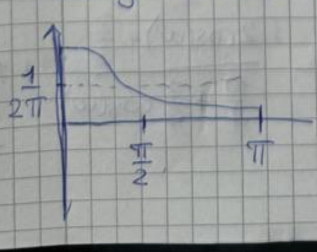
d) $\psi_0 = 1$, $\psi_1 > 0$

Two ways to show this:

(I) $\cos(\omega)$ is decreasing on $[0, \pi]$ and ρ is linear in $\cos(\omega)$

(II) show $\frac{d}{d\omega} \rho(\omega) < 0 \quad \forall \omega \in [0, \pi]$

using $\sin(\omega) > 0 \quad \forall \omega\ [0, \pi]$



For Y_t to be WN, we need

$\psi_0 = 1 \quad \psi_1 = 0$

Problem 3:

a) $$\sum_{t\in \mathbb{Z}} |z_t| \underset{\substack{\text{triangle}\\ \text{inequality}}}{\le} \underbrace{\sum_{t\in \mathbb{Z}} \frac{16}{25}\left(\frac{1}{2}\right)^{|t|}}_{\substack{\text{geometric}\\ \text{series with}\\ (\frac{1}{2})<1}} + \underbrace{\sum_{t\in \mathbb{Z}} \frac{9}{25}\left(\frac{1}{3}\right)^{|t|}}_{\substack{\text{geom. ser.}\\ \text{with}\\ (\frac{1}{3})<1}} < \infty$$

b) $$f(\omega) = \frac{1}{2\pi}\sum_{t\in \mathbb{Z}} z_t e^{-i\omega t} = \frac{1}{2\pi}\Big[\underbrace{\frac{16}{25}\sum_{t=1}^{\infty}\left(\frac{1}{2}\right)^t e^{i\omega t}}_{=-\infty,\dots,-1} +$$

$$+ \underbrace{\frac{16}{25}\sum_{t=1}^{\infty}\left(\frac{1}{2}\right)^t e^{-i\omega t}}_{=1,\dots,+\infty} + \frac{9}{25}\sum_{t=1}^{\infty}\left(-\frac{1}{3}\right)^t e^{i\omega t} + \frac{9}{25}\sum\left(-\frac{1}{3}\right)^t e^{-i\omega t} + 1\Big]$$

$$\underset{\substack{\text{index shift}\\ \text{and}\\ \text{geometric series}}}{= \dots =} \frac{1}{2\pi}\Big[\frac{16}{50} e^{i\omega}\frac{1}{1-\frac{1}{2}e^{i\omega}} + \frac{16}{50} e^{-i\omega}\frac{1}{1-\frac{1}{2}e^{-i\omega}} + 1$$

$$- \frac{9}{75} e^{i\omega}\frac{1}{1+\frac{1}{3}e^{i\omega}} - \frac{9}{75} e^{-i\omega}\frac{1}{1+\frac{1}{3}e^{-i\omega}}\Big]$$

$$= \frac{1}{2\pi}\Big[\frac{16}{50}\,\frac{e^{i\omega}-\frac{1}{2}+e^{-i\omega}-\frac{1}{2}}{1-\frac{1}{2}e^{-i\omega}-\frac{1}{2}e^{i\omega}+\frac{1}{4}} - \frac{9}{75}\,\frac{e^{i\omega}+\frac{1}{3}+e^{-i\omega}+\frac{1}{3}}{1+\frac{1}{3}e^{-i\omega}+\frac{1}{3}e^{i\omega}+\frac{1}{3}} + 1\Big]$$

$$= \frac{1}{2\pi}\Big[\frac{16}{50}\,\frac{2\cos(\omega)-1}{\frac{5}{\omega}-2\cos(\omega)} - \frac{9}{75}\,\frac{2\cos(\omega)+\frac{2}{3}}{\frac{10}{9}+\frac{2}{3}\cos(\omega)} + 1\Big]$$

Note: $\cos(x) = \frac{1}{2}\left(e^{i\omega}+e^{-i\omega}\right)$

Exercise 4:

$$f_1(\omega) = \frac{1-\cos(\omega)+\cos(2\omega)}{2\pi} \Rightarrow Z_{1,t} = \int_{-\pi}^{\pi} f_1(\omega) e^{i\omega t} d\omega \; (=)$$

trick for $s \in \mathbb{Z}$: $\int_{-\pi}^{\pi} e^{i\omega s} d\omega = \int_{-\pi}^{\pi} \cos(\omega s) + i\sin(\omega s) d\omega$

$= \begin{cases} 2\pi & s=0 \\ 0 & \text{else} \end{cases}$ because we integrate over exactly one period.

$$(=) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[e^{i\omega t} - \frac{1}{2} e^{i\omega(t+1)} - \frac{1}{2} e^{i\omega(t-1)} + \frac{1}{2} e^{i\omega(t+2)} + \frac{1}{2} e^{i\omega(t-2)} \right] d\omega =$$

$$= \begin{cases} 1, & t=0 \\ -1/2 & t = \pm 1 \\ 0 & \text{else} \end{cases}$$

$$f_2(\omega) = \frac{1}{2\pi} \Rightarrow Z_{2,t} = \int_{-\pi}^{\pi} f_2(\omega) e^{i\omega t} d\omega =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega t} = \begin{cases} 1, & t=0 \\ 0, & \text{else} \end{cases}$$

$$f_3(\omega) = \frac{1+\cos(4\omega)}{2\pi} \Rightarrow z_{3,t} = \int_{-\pi}^{\pi} f_3(\omega)\, \phi^{i\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[e^{i\omega t} + \frac{1}{2} e^{i\omega(t+4)} + \frac{1}{2} e^{i\omega(t-4)} \right] d\omega =$$

$$= \begin{cases} 1 & , t = 0 \\ 0 & , t = \pm 1, 2, 3 \\ 1/2 & , t = \pm 4 \\ 0 & , \text{else} \end{cases}$$