Quiz Answers (2. Descriptive time series analysis)

1. Empirical Autocovariance and Autocorrelation Functions

Question: What do the empirical autocovariance and autocorrelation functions measure? What is the difference between the two?

Answer:

- **Empirical Autocovariance Function:** This function measures the linear dependence between values of the time series at different times, specifically how a time series value at one time point relates to another at a different time point (lagged values). It gives a raw measure of the relationship between the observations at different lags.
- **Empirical Autocorrelation Function (ACF):** This is a normalized version of the autocovariance function. It measures the strength and direction of a linear relationship between lagged values of the time series, but it is normalized to ensure that the values fall between -1 and 1. The ACF shows how each lagged value is correlated with the current value of the series.
- **Difference:** The primary difference is that while the autocovariance function measures the covariance (which is unbounded), the autocorrelation function standardizes this measure to be within a fixed range, making it easier to interpret.

2. ACF Plot of a Trending Time Series

Question: What do you expect to see in the ACF plot of a trending time series?

Answer: In the ACF plot of a trending time series, you would expect to see autocorrelation values that remain high and decrease slowly as the lag increases. This is because in a trending series, values are not only related to their immediate predecessors but also to values further back in time due to the persistent trend. Therefore, the ACF decays slowly, indicating a long memory in the data.

3. Transformation to Eliminate a Linear Trend

Question: What transformation would you use to eliminate a linear trend in the data?

Answer: To eliminate a linear trend in the data, you would typically use the first differencing transformation. This involves subtracting the previous observation from the current observation:

$$\Delta y_t = y_t - y_{t-1}$$

First differencing removes the linear trend by stabilizing the mean of the series, making it more stationary.

4. Transforming an Exponential Trend

Question: How would you transform an exponential trend into a linear one?

Answer: To transform an exponential trend into a linear one, you would apply a logarithmic transformation to the data. If the data y_t follows an exponential trend, applying the natural log:

$$\log(y_t)$$

will linearize the trend, allowing you to apply linear modeling techniques.

5. Interpretation of Log-Differences

Question: What is the interpretation of a time series in log-differences and why is it a popular transformation?

Answer: Log-differences of a time series are interpreted as the approximate percentage change in the series. This transformation is popular because it stabilizes the variance (especially in financial time series) and converts multiplicative relationships into additive ones, making the data more amenable to linear analysis. For small changes, the log-difference approximates the growth rate:

$$\Delta \log(y_t) = \log(y_t) - \log(y_{t-1})$$

This is particularly useful in economic and financial contexts where percentage changes are more meaningful than absolute changes.

Quiz Answers (3. Time series decomposition)

1. Difference Between Additive and Multiplicative Components Models

Question: What is the difference between an additive and a multiplicative components model? When would you use the latter one?

Answer:

- **Additive Model:** In an additive model, the time series is decomposed into components that are added together. The model is of the form:

  $$y_t = m_t + s_t + c_t + u_t$$

  where m_t is the trend, s_t is the seasonal component, c_t is the cyclical component, and u_t is the irregular component.

- **Multiplicative Model:** In a multiplicative model, the components are multiplied together. The model is of the form:

  $$y_t = m_t \cdot s_t \cdot c_t \cdot u_t$$

  This model is typically used when the variation in the series increases or decreases with the level of the series, implying that the seasonality and trend components interact in a non-additive way.

- **When to Use:** Use the multiplicative model when the components exhibit proportional or percentage changes rather than absolute changes, often seen in economic and financial time series where variability increases with the level of the series.

2. Frisch-Waugh-Lovell Theorem

Question: Why is the Frisch-Waugh-Lovell Theorem useful?

Answer: The Frisch-Waugh-Lovell Theorem is useful because it allows for the simplification of multiple regression analysis. It demonstrates that the effect of a subset of variables in a regression model can be isolated by regressing the dependent variable and the other independent variables separately on the variables of interest. This theorem provides a way to partition the regression into parts, which is particularly useful when dealing with multicollinearity or when needing to understand the contribution of specific variables in a model with many predictors.

3. Autocorrelation in Residuals

Question: Suppose you fit a linear deterministic trend by OLS, but your residual analysis shows evidence of autocorrelation in the residuals. What does this mean for your parameter estimates?

Answer: If there is evidence of autocorrelation in the residuals after fitting a linear deterministic trend by OLS, it implies that the assumption of independent errors is violated. This means that while the OLS estimates of the coefficients remain unbiased, they are no longer efficient (i.e., they do not have the minimum variance). Additionally, the standard errors of the coefficients may be underestimated, leading to overly optimistic significance levels (p-values) and confidence intervals. As a result, the OLS model might incorrectly identify relationships between the variables.

4. Forecasting with Quadratic Time Trend Model

Question: Suppose you have fitted a quadratic time trend model with ℓ seasonal effects (as in slide 25). Write down your forecast for the value y_{T+h}.

Answer: The forecast for y_{T+h} when a quadratic time trend model with ℓ seasonal effects is fitted is given by:

$$y_{T+h} = \hat{m}_{T+h} + \hat{s}_{T+h}$$

where:

$$\hat{m}_{T+h} = \hat{\beta}_0 + \hat{\beta}_1(T+h) + \hat{\beta}_2(T+h)^2$$

and $\hat{s}_{T+h}$ represents the seasonal component, typically a value from the seasonal index corresponding to the period $T+h$.

This formula combines the quadratic trend projection with the seasonal adjustment to provide the forecasted value.

Quiz Answers (4. Modelling periodic fluctuations Part 1)

1. How would you model seasonal fluctuations with a known period?

Answer:

- **Method**: Model seasonal fluctuations using harmonic oscillations (sine and cosine functions).
- **Model**: Use the model
$$y_t = \mu + \alpha \sin(\omega t + \theta) + u_t$$
  - μ: Mean level.
  - α: Amplitude of the oscillation.
  - ω: Angular frequency, $\omega = \frac{2\pi}{\ell}$ where ℓ is the period.
  - θ: Phase shift.
  - u_t: Error term.

2. How would you define a frequency and the corresponding angular frequency? What is the unit of measurement for the latter?

Answer:

- **Frequency** (λ): The number of cycles per unit time.
  - **Unit**: Cycles per unit time (e.g., cycles per year).
  - **Formula**: $\lambda = \frac{1}{\ell}$ where ℓ is the period.
- **Angular Frequency** (ω): The rate of change of the function's phase, measured in radians per unit time.
  - **Unit**: Radians per unit time.
  - **Formula**: $\omega = 2\pi\lambda = \frac{2\pi}{\ell}$.

3. Can you observe monthly cycles in quarterly data?

Answer:

- **No**. Monthly cycles have a period of 1 month, while quarterly data have a period of 3 months. Since the monthly cycle period is shorter than the quarterly data interval, these cycles cannot be observed in quarterly data.

4. For what frequencies do we have desirable properties of $\sin(\omega t)$ and $\cos(\omega t)$? What are these properties? What do they enable us to do?

Answer:

- **Frequencies**: Desirable properties occur for the base frequencies $\lambda_k = \frac{k}{T}$, where $k = 1, 2, \ldots, \frac{T}{2}$.
- **Properties**: Orthogonality and completeness.
  - **Orthogonality**: The integral (or sum) of the product of sine and cosine functions with different frequencies over a period is zero.
$$\sum_{t=1}^{T} \sin\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) = 0$$
  - **Completeness**: Any periodic function can be expressed as a linear combination of these sine and cosine functions.
- **Enables Us To**: Decompose complex signals into simpler sinusoidal components, analyze periodic behavior, and reconstruct signals from their frequency components.

5. Show that the basis sine and cosine functions are orthogonal, i.e.

$$\sum_{t=1}^{T} \sin\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) = 0$$

Answer:

- **Orthogonality Proof**:
  - Consider the product of $\sin\left(2\pi \frac{k}{T} t\right)$ and $\cos\left(2\pi \frac{k}{T} t\right)$:

    $$\sin\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) = \frac{1}{2}\left[\sin\left(2\pi \frac{2k}{T} t\right) + \sin(0)\right]$$

  - Summing over one period T:

    $$\sum_{t=1}^{T} \sin\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) = \frac{1}{2}\sum_{t=1}^{T} \sin\left(2\pi \frac{2k}{T} t\right)$$

  - Since the sine function completes an integer number of cycles over the period T, the sum of the sine function over one period is zero:

    $$\sum_{t=1}^{T} \sin\left(2\pi \frac{2k}{T} t\right) = 0$$

  - Hence:

    $$\sum_{t=1}^{T} \sin\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) = 0$$

6. How would you interpret the coefficient estimates $\hat{\beta}$ and $\hat{\gamma}$ on Slide 18?

Answer:

- **Context**: Slide 18 shows the model:

  $$y_t = \mu + \beta \sin(\omega t) + \gamma \cos(\omega t) + u_t$$

- **Interpretation**:
  - $\hat{\beta}$: The estimated coefficient for the sine term, representing the contribution of the sine component to the overall signal.
  - $\hat{\gamma}$: The estimated coefficient for the cosine term, representing the contribution of the cosine component to the overall signal.

- **Relationship to Amplitude and Phase**:
  - **Amplitude** (α): Calculated as $\alpha = \sqrt{\beta^2 + \gamma^2}$.
  - **Phase Shift** (θ): Calculated as $\theta = \arccos\left(\frac{\beta}{\alpha}\right)$ if $\gamma \geq 0$ or $\theta = 2\pi - \arccos\left(\frac{\beta}{\alpha}\right)$ if $\gamma < 0$.

Quiz Answers (4. Modelling periodic fluctuations Part 2)

1. How would you approach the problem of modelling multiple cycles in time series?

Answer:

- **Method**: Model multiple cycles using a superposition (linear combination) of harmonic oscillations with different amplitudes and frequencies.

- **Model**:

  $$s_t = s_1(t) + s_2(t) + \ldots + s_k(t)$$

  Each component $s_k(t)$ can be represented as a Fourier series, which is a sum of sinusoidal functions (sine and cosine) with different frequencies and amplitudes.

2. What are the Fourier frequencies? What makes them "special"?

Answer:

- **Fourier Frequencies**: These are the specific frequencies at which the sine and cosine functions are evaluated in the Fourier series. They are given by:

$$\lambda_k = \frac{k}{T}, \quad k = 1, 2, \ldots, \left\lfloor \frac{T}{2} \right\rfloor$$

- **Special Characteristics**:
  - **Orthogonality**: Sine and cosine functions at these frequencies are orthogonal to each other, which simplifies the computation and interpretation of their coefficients.
  - **Completeness**: Any periodic function can be expressed as a sum of these sinusoidal functions, allowing for a comprehensive representation of the signal.

3. What is a Fourier series expansion of a sequence?

Answer:

- **Fourier Series Expansion**: This is a way to represent a time series as a sum of sinusoidal functions (sine and cosine) with different frequencies, amplitudes, and phases.

$$y_t = \sum_{k=0}^{\lfloor \frac{T}{2} \rfloor} a_k \cos(\omega_k t) + \sum_{k=1}^{\lfloor \frac{T}{2} \rfloor} b_k \sin(\omega_k t), \quad t = 1, \ldots, T$$

where $\omega_k = \frac{2\pi k}{T}$.

4. How can you uncover the unknown frequencies of multiple cycles from the data in practice?

Answer:

- **Method**: Use the periodogram to identify significant frequencies in the data.
- **Steps**:
  1. Compute the discrete Fourier transform (DFT) of the time series.
  2. Calculate the periodogram, which shows the power of different frequencies.
  3. Identify peaks in the periodogram, which correspond to dominant frequencies in the time series.

5. What is aliasing and how does it arise?

Answer:

- **Aliasing**: This is a phenomenon where higher frequency components of a signal are misrepresented as lower frequencies due to insufficient sampling.
- **Cause**: It arises when the sampling rate is less than twice the highest frequency present in the signal (below the Nyquist rate). Frequencies higher than the Nyquist frequency will appear as lower frequency aliases in the sampled data.

6. What are harmonics and what are they used for?

Answer:

- **Harmonics**: These are integer multiples of a fundamental frequency λ. For a fundamental frequency λ, the harmonics are $2\lambda, 3\lambda, \ldots$.
- **Usage**: Harmonics are used to model periodic fluctuations with the same period as the fundamental frequency. They are essential in representing complex periodic signals, especially in time series with seasonal effects.

7. Simulate some white noise series (e.g., $\varepsilon_t \sim$ i.i.d. $N(0,1)$) and look at their periodogram. What is the dominant frequency? Can you think of an explanation why "white" noise is called this way?

Answer:

- **Simulation**:

```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(0)
N = 500
white_noise = np.random.normal(0, 1, N)

frequencies = np.fft.fftfreq(N)
power = np.abs(np.fft.fft(white_noise))**2 / N

plt.figure(figsize=(10, 6))
plt.plot(frequencies, power)
plt.xlabel('Frequency')
plt.ylabel('Power')
plt.title('Periodogram of White Noise')
plt.show()
```

- **Dominant Frequency**: In white noise, the periodogram will show no dominant frequency; the power will be evenly distributed across all frequencies.
- **Explanation for "White" Noise**: White noise is called this way because it has equal power across all frequencies, similar to how white light contains all colors (frequencies) in equal measure.

Quiz Answers (5. Naive forecasting methods)

1. Examples of Smoothing Methods

Examples:

- **Moving Averages**:
  - **Usage**: Used to smooth out short-term fluctuations and highlight longer-term trends or cycles in data.
  - **Description**: Averages a fixed number of past observations. It can be simple (equal weights) or weighted (different weights).
- **Simple Exponential Smoothing (SES)**:
  - **Usage**: Suitable for short-term forecasting of time series without trend or seasonality.
  - **Description**: Uses exponentially decreasing weights, giving more importance to recent observations.
- **Holt's Linear Method**:
  - **Usage**: Suitable for time series with a trend but no seasonality.
  - **Description**: Extends SES by adding a trend component, using two smoothing parameters (one for the level and one for the trend).
- **Holt-Winters Method**:
  - **Usage**: Suitable for time series with both trend and seasonality.

- **Description**: Extends Holt's method by adding a seasonal component, using three smoothing parameters (level, trend, and seasonality).

Differences:

- **Moving Average**: Fixed number of non-zero weights, does not adapt as new data arrives.
- **Exponential Smoothing**: Uses all past data with exponentially decreasing weights, adapts to new data more effectively.
- **Holt's Method**: Adds a trend component, suitable for data with trends.
- **Holt-Winters Method**: Adds both trend and seasonal components, suitable for data with seasonal patterns.

2. Algorithm of Simple Exponential Smoothing

Algorithm:

1. **Initialization**: Start with an initial forecast, $\hat{y}_1 = y_1$.
2. **Forecast Update**:
$$\hat{y}_{T+1} = \theta\hat{y}_T + (1-\theta)y_T$$
3. **Recursive Calculation**:
$$\hat{y}_{T+1} = (1-\theta)\sum_{j=0}^{T-1}\theta^j y_{T-j} + \theta^T y_0$$

Interpretation of Smoothing Parameter (θ):

- θ controls the weight given to the most recent observation versus past observations.
- $\theta \approx 1$: Stronger smoothing, more weight on the past observations.
- $\theta \approx 0$: Less smoothing, more weight on the most recent observation.

When to Use:

- Use SES for short-term forecasting when there is no clear trend or seasonality in the data.

3. Differences Between Holt's Linear Method and Holt-Winters Approach

Holt's Linear Method:

- **Model**: Adds a trend component.
- **Equations**:
$$\begin{aligned}
y_t &= m_t + b_t \cdot t + \epsilon_t \\
m_t &= \theta y_t + (1-\theta)(m_{t-1} + b_{t-1}) \\
b_t &= \gamma(m_t - m_{t-1}) + (1-\gamma)b_{t-1}
\end{aligned}$$
- **Components**: Level and trend.

Holt-Winters Method:

- **Model**: Adds both trend and seasonal components.
- **Equations**:
$$\begin{aligned}
y_t &= m_t + b_t \cdot t + s_t + \epsilon_t \\
m_t &= \theta(y_t - s_{t-\ell}) + (1-\theta)(m_{t-1} + b_{t-1}) \\
b_t &= \gamma(m_t - m_{t-1}) + (1-\gamma)b_{t-1} \\
s_t &= \delta(y_t - m_t) + (1-\delta)s_{t-\ell}
\end{aligned}$$
- **Components**: Level, trend, and seasonality.

4. Compute a Forecast for the Number of Students

Given Observations: 38, 58, 33, 53, 28, 48, 27

Simple Exponential Smoothing (SES) Calculation: Assume $\theta = 0.5$ (common default value).

- **Initialization**:

$$\hat{y}_1 = y_1 = 38$$

- **Updating Forecasts**:

$$\begin{aligned}
\hat{y}_2 &= \theta\hat{y}_1 + (1-\theta)y_1 = 0.5 \cdot 38 + 0.5 \cdot 38 = 38 \\
\hat{y}_3 &= \theta\hat{y}_2 + (1-\theta)y_2 = 0.5 \cdot 38 + 0.5 \cdot 58 = 48 \\
\hat{y}_4 &= \theta\hat{y}_3 + (1-\theta)y_3 = 0.5 \cdot 48 + 0.5 \cdot 33 = 40.5 \\
\hat{y}_5 &= \theta\hat{y}_4 + (1-\theta)y_4 = 0.5 \cdot 40.5 + 0.5 \cdot 53 = 46.75 \\
\hat{y}_6 &= \theta\hat{y}_5 + (1-\theta)y_5 = 0.5 \cdot 46.75 + 0.5 \cdot 28 = 37.375 \\
\hat{y}_7 &= \theta\hat{y}_6 + (1-\theta)y_6 = 0.5 \cdot 37.375 + 0.5 \cdot 48 = 42.6875 \\
\hat{y}_8 &= \theta\hat{y}_7 + (1-\theta)y_7 = 0.5 \cdot 42.6875 + 0.5 \cdot 27 = 34.84375
\end{aligned}$$

Forecast for Next Tuesday:

$$\hat{y}_8 = 34.84$$

(approximately 35 students)

Quiz Answers (6. Smoothing methods)

1. Smoothing Away Seasonality in Quarterly/Monthly Data

Question: How would you smooth away seasonality in quarterly/monthly data? Write down the corresponding MAs.

Answer: To smooth away seasonality in quarterly data, use a 4-period moving average (MA), and for monthly data, use a 12-period MA. For quarterly data:

$$\hat{m}_t = \frac{1}{4}(y_{t-2} + y_{t-1} + y_t + y_{t+1})$$

For monthly data:

$$\hat{m}_t = \frac{1}{12}\sum_{i=-5}^{6} y_{t+i}$$

These MAs help remove the seasonal fluctuations by averaging the data over one complete seasonal cycle.

2. Local Polynomial Trend Approximation

Question: Explain how local polynomial trend approximation works. What properties do the weights of the resulting MA possess?

Answer: Local polynomial trend approximation fits a polynomial of degree p to a small window around each time point t. The trend at t is estimated by the polynomial's intercept. The resulting MA from this method has weights that are symmetric, add up to one, and the sum of their squared values equals the central weight, which reduces variance while capturing the local trend.

3. Choosing the Order of the Polynomial p

Question: How would you choose the order of the polynomial p?

Answer: The order p should be chosen based on the complexity of the trend you aim to capture. For simpler, linear trends, a lower order (e.g., $p = 1$ or $p = 2$) may suffice. For more complex, nonlinear trends, a higher order (e.g., $p = 3$) may be necessary. It's essential to balance flexibility with the risk of overfitting.

4. Other Smoothing Methods

Question: What other smoothing methods do you know? Describe them briefly.

Answer:

- **Exponential Smoothing:** Applies exponentially decreasing weights to past observations, useful for short-term forecasting.
- **Kernel Smoothing:** Uses a weighted moving average with a kernel function to apply weights, allowing for more flexible smoothing depending on the kernel shape.
- **Smoothing Splines:** Fits piecewise polynomials, ensuring smooth transitions at the boundaries, particularly effective for nonlinear trends.

Quiz Answers (7. Linear Filters - Part 1)

1. What is a linear filter?

Question: What is a linear filter?

Answer: A linear filter is a linear transformation W of a time series y_t into another series z_t. The transformation is defined as:

$$z_t = W y_t = \sum_{j=-k}^{l} w_j y_{t-j} = \sum_{j=-k}^{l} w_j B^j y_t,$$

where w_j are the weights the filter assigns to the observations, y_t is the input series, and z_t is the output series. Linear filters are used to emphasize certain frequencies within the time series and suppress others.

2. Examples of High-Pass and Low-Pass Filters

Question: Give examples of high-pass and low-pass filters.

Answer: Examples of filters include:

- **High-pass filter:** The difference filter $\Delta y_t = y_t - y_{t-1}$, which has weights $w_0 = 1$ and $w_1 = -1$. This filter allows high-frequency components to pass through while attenuating low-frequency components.
- **Low-pass filter:** The moving average filter, which smooths the series by averaging over a window of observations, hence attenuating the high-frequency components and allowing low-frequency components to pass through.

3. Usage of Filter $W(B) = 1 - B^4$

Question: What would you use the filter $W(B) = 1 - B^4$ for?

Answer: The filter $W(B) = 1 - B^4$ is used to remove or differentiate out a seasonal component with a period of 4 from the time series. It acts as a difference filter at lag 4, effectively eliminating any periodicities corresponding to this lag.

4. Filtering Out a Seasonal Component with Period $\ell = 4.5$

Question: How would you filter out a seasonal component with period $\ell = 4.5$?

Answer: To filter out a seasonal component with a non-integer period $\ell = 4.5$, you would generalize the differencing operator to handle fractional lags. This can be approximated by the filter:

$$\Delta_\ell = 1 - (1 - \alpha) B^s - \alpha B^{s+1},$$

where $\ell = s + \alpha$, s is the integer part (in this case, 4), and α is the fractional part (0.5). This approach allows the filter to effectively handle non-integer seasonality.

Quiz Answers (7. Linear Filters - Part 2)

1. What is a convolution of linear filters? What are its properties?

Answer: A convolution of linear filters combines two filters, V and W, by applying one filter to the result of applying the other. Mathematically, if V and W are linear filters with weights v_j and w_j, then their convolution $U = V \circ W$ is another linear filter with weights given by the convolution sum:

$$u_k = \sum_j v_j w_{k-j}.$$

The convolution of linear filters is commutative and associative, meaning the order of convolution does not change the result.

2. What is a z-transform and what do we use it for?

Answer: The z-transform is a mathematical tool used to study linear filters in the context of complex analysis. It transforms a linear filter W with weights w_j into a complex function $f_W(z) = \sum_{j=-\infty}^{\infty} w_j z^j$. The z-transform is particularly useful for analyzing the stability and frequency response of filters, and for determining inverse filters.

3. What is the difference between a z-transform and a characteristic polynomial?

Answer: The z-transform is a complete representation of a linear filter as a complex function, while the characteristic polynomial is a specific polynomial derived from the z-transform by multiplying it by an appropriate power of z to obtain a polynomial with positive powers only. The roots of the characteristic polynomial are crucial for determining the stability and invertibility of the filter.

4. Under what conditions does a causal inverse filter exist?

Answer: A causal inverse filter exists if and only if all roots of the z-transform of the original filter lie outside the complex unit circle (i.e., have absolute value greater than 1). This condition ensures that the inverse filter has absolutely summable weights and thus is causal.

5. What is a null space of a filter? How do we determine its constituent sequences? What is its connection to the existence of an inverse filter?

Answer: The null space (or kernel) of a filter consists of all sequences that the filter maps to zero. For a finite linear filter, the null space can be determined using the roots of its characteristic polynomial. Sequences corresponding to the roots of the characteristic polynomial belong to the null space. An inverse filter is designed to reverse the effect of a filter. If a filter has a non-trivial null space (i.e., there are non-zero sequences that it reduces to zero), then it's impossible to have an inverse filter because you can't reverse something that has been entirely lost or canceled out. For the inverse filter to exist, the original filter must not nullify any non-zero sequences—its null space should only contain the zero sequence.

This is why the null space is crucial in determining whether an inverse filter can exist: a filter with a non-trivial null space can't have an inverse because there's no way to recover the original sequence from a completely nullified output.

6. How can we determine if any sequences pass through the filter unaltered?

Answer: A sequence passes through a filter unaltered if it is part of the invariance set of the filter. The invariance set consists of sequences that are eigenfunctions of the filter with eigenvalue 1. For a polynomial to pass through a filter unaltered, the polynomial must be invariant under the filter's operation. This is related to the roots of the characteristic polynomial: if $z = 1$ is an r-fold root, then the filter leaves polynomials up to degree $r - 1$ invariant.