$\{\eta_t\}_{t \in \{1,\dots,T\}}$ $\quad E(\eta_t) = 0$

$Cov(\eta, \eta^T) = \Sigma \neq I_{T\times T}$ in general.

Let $V \in \mathbb{R}^{T\times T}$ s.t. $V\Sigma V^T = I_{T\times T}$

Def.: $\Psi = V \cdot \eta$

(a) $Var(\Psi) = Cov(\Psi, \Psi^T) = Cov(V\eta, \eta^T V^T)$

$= V\, Cov(\eta, \eta^T)\, V^T = V\Sigma V^T = I_{T\times T}$

(b) $V\Sigma V^T = I_{T\times T}$ $\quad | \cdot (V^{-1})$ from left

$\cdot (V^T)^{-1}$ from right

$\Sigma = V^{-1} \cdot (V^T)^{-1}$ (Note "T" & "-1" commute

$\Leftrightarrow \Sigma = (V^T V)^{-1}$

$\Leftrightarrow \Sigma^{-1} = V^T V$

(c) $X = A \cdot \beta + \eta$ $\quad | \cdot (V)$ from left

$\underbrace{VX}_{X^*} = \underbrace{VA}_{A^*} \cdot \beta + \underbrace{V\eta}_{\eta^*}$ now the "new" errors η^* are uncorrelated !

OLS
$\Rightarrow \hat{\beta} = (A^{*T} A^*)^{-1} A^{*T} \cdot X^*$

$= (A^T \underbrace{V^T V}_{\Sigma^{-1}} A)^{-1} A^T \underbrace{V^T V}_{\Sigma^{-1}} X = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} X$

(d) For the transformed model Eq., all cond. for Gauss-Markov are fulfilled
$\Rightarrow \hat{\vec{\beta}}$ is BLUE.

(e) $d^2(A\beta, X) = (A\beta - X)^T \Sigma^{-1} (A\beta - X)$

$= (A\beta)^T \Sigma^{-1} A\beta + X^T \Sigma^{-1} X - (A\beta)^T \Sigma^{-1} X - \underbrace{X^T \Sigma^{-1} A\beta}_{\text{scalar! so we can transpose}}$

$= \beta^T A^T \Sigma^{-1} A\beta + X^T \Sigma^{-1} X - 2\beta^T A^T \Sigma^{-1} X$

Now we want to minimise with respect to β:

$\vec{\nabla}_\beta = d^2(A\beta, X) = 2A^T \Sigma^{-1} A\beta - 2A^T \Sigma^{-1} X \overset{!}{=} 0$

$\Leftrightarrow \hat{\beta} = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} X$

P3:

$f: \{1, \ldots, T\} \to \mathbb{R}$

$$f(t) = \sum_{k=-\frac{(T-1)}{2}}^{\frac{T-1}{2}} c_k e^{i 2\pi \frac{k}{T} t} \qquad T \text{ odd}$$

$$f(t) = \sum_{k=-\frac{T}{2}+1}^{T/2} c_k e^{i 2\pi \frac{k}{T} t} \qquad T \text{ even}$$

Helpful formula: $\displaystyle \sum_{t=1}^{n} e^{\frac{i 2\pi k t}{n}} = \begin{cases} n & k=0 \\ 0 & k \neq 0 \end{cases}$

$$c_k = \frac{1}{T} \sum_{t=1}^{T} f(t) e^{-i 2\pi \frac{k}{N} t}$$

(a) With Eulers formula, $e^{ix} = \cos(x) + i \sin(x)$, we see that the repr. are equiv.

(b) T even:

$$f(t) = \sum_{k=-\frac{T}{2}+1}^{T/2} \underbrace{\frac{1}{T} \sum_{t'=1}^{T} f(t') e^{-i 2\pi \frac{k}{T} t'}}_{c_k} e^{i 2\pi \frac{k}{T} t}$$

$$= \frac{1}{T} \sum_{k=-\frac{T}{2}+1}^{T/2} \sum_{t'=1}^{T} f(t') e^{i 2\pi \frac{k}{T} (t-t')}$$

$$= \frac{1}{T} \sum_{t'=1}^{T} f(t')\, T \cdot \delta_{tt'} = f(t)$$

for T odd same calculation.

(c)

T even

$$f(t) = \sum_{k=-\frac{T}{2}+1}^{T/2} c_k\, e^{i 2\pi \frac{k}{T} t}$$

$$= \sum_{k=-\frac{T}{2}+1}^{T/2} c_k \cos(2\pi \tfrac{k}{T} t) + i c_k \sin(2\pi \tfrac{k}{T} t)$$

$$\overset{!}{=} \sum_{k=0}^{\frac{T}{2}} a_k \cos(2\pi \tfrac{k}{T} t) + b_k \sin(2\pi \tfrac{k}{T} t)$$

$\Rightarrow c_0 = a_0, \quad c_{\frac{T}{2}} = a_{\frac{T}{2}}$ since $\cos \to 1$ & $\sin \to 0$

for $k \in \{1, \dots, \frac{T}{2} - 1\}$:

$$\begin{pmatrix} c_k + c_{-k} = a_k \\ i c_k - i c_{-k} = b_k \end{pmatrix} \Rightarrow \begin{pmatrix} 2 i c_k = i a_k + b_k \\ \Leftrightarrow c_k = \frac{-a_k + i b_k}{2} \end{pmatrix}$$

$$\Rightarrow -2i c_{-k} = -i a_k + b_k$$

$$c_{-k} = \frac{-a_k - i b_k}{2}$$