TSA Sheet 9

Problem 1

a-) $$E\left[M_t - M_{t-1} \mid \mathcal{F}_{t-1}\right] = \underbrace{E\left[M_t \mid \mathcal{F}_{t-1}\right]}_{\substack{= M_{t-1} \\ \text{(Martingale)}}} - \underbrace{E\left[M_{t-1} \mid \mathcal{F}_{t-1}\right]}_{\substack{= M_{t-1} \\ \text{(measurable with respect to } \mathcal{F}_{t-1})}}$$

$$= M_{t-1} - M_{t-1} = 0$$

b-) $$E\left[(M_t - M_{t-1})(M_{t-h} - M_{t-h-1})\right] \quad \Big| \quad \text{let } h > -$$

$\mathcal{F}_{t-1}$ measurable

$$\overset{\text{LIE}}{=} E\left[E\left[(M_t - M_{t-1})(M_{t-h} - M_{t-h-1}) \mid \mathcal{F}_{t-1}\right]\right]$$

$$= E\left[M_{t-h} - M_{t-h-1}\, \underbrace{E\left[M_t - M_{t-1}\right] \mid \mathcal{F}_{t-1}}_{=0 \text{ (a)}}\right] = 0 \quad \left(\text{analog for } h < 0\right)$$

c-) $E[z_t | \mathcal{F}_{t-1}] = \underbrace{E\left[\sum_{j=1}^{t-1} \varepsilon_j \,\middle|\, \mathcal{F}_{t-1}\right]}_{=\sum_{j=-1}^{t-1} \varepsilon_j} + \underbrace{E[\varepsilon_t | \mathcal{F}_{t-1}]}_{=0}$

(Measurable since ε_t independent $\mathcal{F}_{t-1}$)

d-) $\Delta z_t = \varepsilon_t \quad WN(0, \sigma^2) \Rightarrow$ stationary

e-) $E[\varepsilon_t] = 0$, $Var(\varepsilon_t) = \sigma^2$, $\gamma(h) = \begin{cases} 0, & |h| > 0 \\ \sigma^2 = h = 0 \end{cases}$

f-) Yes, because $\sum_{j=1}^{\infty} |\gamma_j| = \sigma^2 < \infty$ (Chapt 8 slide 42)

g-) $\Delta^2 z_t = \Delta \varepsilon_t \rightarrow$ stationary (sheet 8)

Problem 2: $\{y_t\}$ stationary TS with $E[y_t]=\mu$

and $\gamma_0, \gamma_1, \ldots\ldots$ fullfil $\sum_{h=0}^{\infty} |\gamma_0| < \infty$

a-) $\bar{y}_T = \frac{1}{T} \sum_{t=1}^{T} y_t$,

$$Var(\bar{y}_T) = \frac{1}{T^2}\left[\sum_{t=1}^{T} Var(y_t) + \sum_{\substack{t,t^2 \\ t\neq t}} Cov(y_t, y_t)\right]$$

$$= \frac{1}{T^2}\left[T\gamma_0 + 2\left[(T-1)\gamma_1 + (T-2)\gamma_2 + \cdots + \gamma_{T-1}\right]\right]$$

b-) $$\lim_{T\to\infty} Var(\bar{y}_t) = \lim_{T\to\infty}\left[\frac{1}{T^2}\left[T\gamma_0 + 2\left[(T-1)\gamma_1 + \cdots + \gamma_{T-1}\right]\right]\right]$$

$$\leq \lim_{T\to\infty}\left[\frac{1}{T^2}\left[2\left[T\gamma_0 + [T-1]\gamma_1 + \cdots + \gamma_{T-1}\right]\right)\right.$$

$$\leq \frac{1}{T^2}\cdot 2\sum_{j=0}^{T-1}(T-j)|\gamma_j|$$

$$\leq \frac{2}{T\cancel{2}} \sum_{j=0}^{T-1} \cancel{T} |\gamma_j|$$

$$= \lim_{T\to\infty} \frac{1}{T} \sum_{j=0}^{T-1} |\gamma_j| = \lim_{T\to\infty} \frac{2}{T} \lim_{T\to\infty} \sum_{j=0}^{T-1} |\gamma_j|$$

$O(1)$

$= 0$

$= 0$

c-) $\lim_{T\to\infty} P(|\bar{Y}_T - \mu| > \varepsilon) \leq \lim_{T\to\infty} \frac{Var(\bar{Y}_T)}{\varepsilon^2} = 0$

$\boxed{\text{Problem 3}}$

$\to$ Solution provided by students

a-) $\bar{\rho}_h = \begin{cases} 1 & h = 0 \\ 0.4 & h = \pm 1 \\ -0.8 & h = \pm 2 \\ 0 & |h| \geq 3 \end{cases}$

$\to [-1, 1]$

$\bar{\rho}_{h-1} = 0.4 = \bar{\rho}_{h=-1}$ (sym)

$$A = \begin{bmatrix} 1 & 0.4 & -0.8 \\ 0.4 & 1 & 0.4 \\ -0.8 & 0.4 & 1 \end{bmatrix}$$

$h = \pm 2$
$h = \pm 1$ $\Rightarrow$ Negative eigenvalues hence it is not positive semi definite

it is not stationary

b-) It is not ergodic

c-) $\bar{\rho}_{-2} = \bar{\rho}_2 = 0$

$B = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}$ → positive semidefinite ✓
stationary

d-) Random Walk → Example 1

Example 2

$\{z_t\}$ $\quad z_t = \sum_{i=0}^{\infty} \varepsilon_i$ $\quad$ and

$\boxed{\begin{array}{l}\text{periodic} \\ \varphi_t := \varepsilon_1 \cos(t), \ \varepsilon_1 \sim N(0,1)\end{array}}$

$\boxed{\text{Problem 4}}$ → solution provided by students a-) $E[Y_1] = E[\varepsilon_1] = 0$

$E[Y_2] = E[cY_1 + \varepsilon_2] = cE[Y_1] + E[\varepsilon_2] = 0$

$\vdots$

$E[Y_t] = E[cY_{t-1} + \varepsilon_t] = 0$

$Var(Y_1) = Var(\varepsilon_1) = \sigma^2$

$Var(Y_2) = Var(cY_1 + \varepsilon_2) = c^2 \underbrace{Var(Y_1)}_{=\sigma^2} + \sigma^2 = \sigma^2(c^2+1)$

$= \sigma^2 \sum_{h=0}^{t-1} c^{2h}$

$\vdots$

$Var(Y_t) = \dots$

b-) $\text{Corr}(Y_t, Y_s) = \dfrac{\gamma_{t,s}}{\sigma_s \sigma_t}$

$$\text{Corr}(Y_t, Y_{t-1}) = \frac{\gamma(1)}{\sigma_s \sigma_t} = \frac{c\,\text{Var}(Y_{t-1})}{\sqrt{\sigma_{t-1}^2 \sigma_t^2}} = c \cdot \sqrt{\frac{\text{Var}(Y_{t-1})\,(\text{Var}(Y_{t-1})}{\text{Var}(Y_{t-1})\ \text{Var}(Y_t)}}$$

$$= c\sqrt{\frac{\text{Var}(Y_{t-1})}{\text{Var}(Y_t)}}$$

$$\Rightarrow \text{Corr}(Y_t, Y_{t-h}) = c^h\sqrt{\frac{\text{Var}(Y_{t-1})}{\text{Var}(Y_t}}$$

c-) $$\text{Var}[Y_t] = \sigma^2 \sum_{h=0}^{t-1} c^{2h} = \sigma^2 \sum_{h=0}^{t-1} (c^2)^h$$

$$\xrightarrow{t \to \infty} \sigma^2 \underbrace{\frac{1}{1-c^2}}$$

$$\mathrm{Corr}(Y_t, Y_{t-k}) = c^k \sqrt{\frac{\mathrm{Var}(Y_{t-k})}{\mathrm{Var}(Y_t)}}$$

$$\xrightarrow{t \to \infty} c^k \underbrace{\sqrt{\frac{\sigma^2 \cdot \frac{1}{1-c^2}}{\sigma^2 \cdot \frac{1}{1-c^2}}}}_{1} = c^k$$

Side Remark

$$\lim_{n\to\infty} \sqrt{\frac{a_n}{b_n}} \neq \sqrt{\frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}}$$