Problem 11:

$$Y_t = m_t + s_t + u_t$$

$$m_t = m_{t-1} + b_{t-1} + v_t \Leftrightarrow m_t - m_{t-1} = b_{t-1} + v_t$$

$$b_t = b_{t-1} + w_t$$

$$s_t = s_{t-l} + e_t$$

$$\hat{Y}_T(h) = \hat{m}_T + h\hat{b}_T + \hat{s}_{T+h-l} \qquad 1 \le h \le l$$

$$= \hat{m}_T + h\hat{b}_T + \hat{s}_{T+h-2l} \qquad l < h \le 2l$$

$$\hat{m}_T = (1-\alpha)(Y_T - \hat{s}_{5-l}) + \alpha(\hat{m}_{T-1} + \hat{b}_{T-1})$$

$$\hat{b}_T = (1-\beta)(\hat{m}_T - \hat{m}_{T-1}) + \beta\hat{b}_{T-1}$$

$$\hat{s}_T = (1-\delta)(Y_T - \hat{m}_T) + \delta\hat{s}_{T-l}$$

Forecast $\hat{Y}_t$, $t = 5, \ldots, T$, $Y_1, Y_2, Y_3, Y_4 = \bar{Y}_4$

$s_1 = Y_1 - \bar{Y}_4$, $s_2 = Y_2 - \bar{Y}_4, \ldots$

$b_t = 0$, $t = 1, \ldots, 4$

$$\hat{Y}_5 = \hat{Y}_4(1) = \hat{m}_4 + \hat{b}_4 + \hat{s}_{5-4}$$

update $\hat{m}, \hat{b}, \hat{s}$

$$\hat{m}_5 = (1-\alpha)(Y_5 - \hat{s}_{5-4}) + \alpha(\hat{m}_4 + \hat{b}_4)$$

$$\hat{b}_5 = (1-\beta)(\hat{m}_5 - \hat{m}_{5-1}) + \beta\hat{b}_{5-1}$$

$$\hat{s}_5 = (1-\delta)(Y_5 - \hat{m}_5) + \delta\hat{s}_{5-4}$$

Problem 12:

Approximate Y_t with a local polynomial trend with window $t-k, .., 0, .., t+k$. That means the trend at t is estimated using values $Y_{t-k}, .., Y_t, .., Y_{t+k}$.

$$\boxed{Y_i = \beta_0 + \beta_1 \cdot i + \beta_2 \cdot i^2 + u_i, \quad i = t-k, .., t+k}$$

$$Y_{t+i} = \beta_{0t} + \beta_{1t} \cdot i + \beta_{2t} \cdot i^2, \quad i = -k, .., 0, .., k$$

the mov. avg. representation can be obtained by estimating β_{0t}, then

$$\hat{Y}_t = \hat{\beta}_{0t} = \sum_{j=-k}^{k} \theta_j \, Y_{t-j},$$

$\rightarrow$ Find θ_j by estimating $\hat{\beta}_{0t}$. Stack equations,

$$\underbrace{\begin{pmatrix} Y_{t-k} \\ \vdots \\ Y_t \\ \vdots \\ Y_{t+k} \end{pmatrix}}_{Y_t^{(k)}} = \underbrace{\begin{pmatrix} 1 & -k & (-k)^2 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 1 & k & k^2 \end{pmatrix}}_{=:Z} \cdot \underbrace{\begin{pmatrix} \beta_{0t} \\ \beta_{1t} \\ \beta_{2t} \end{pmatrix}}_{\beta_t} + \begin{pmatrix} u_{t-k} \\ \vdots \\ u_{t+k} \end{pmatrix}$$

$$\hat{\beta}_t = (Z^T \cdot Z)^{-1} \cdot Z^T \, Y_t^{(k)} \overset{\text{normal eq.}}{\Longleftrightarrow} (Z^T Z)\,\hat{\beta}_t = Z^T \cdot Y_t^{(k)}$$

$$Z^T \cdot Z = \begin{pmatrix} 1 & \cdots & 1 \\ -k & \cdots & k \\ (-k)^2 & \cdots & k^2 \end{pmatrix} \begin{pmatrix} 1 & -k & (-k)^2 \\ \vdots & \vdots & \vdots \\ 1 & k & k^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2k+1 & 0 & \sum_{j=-k}^{k} j^2 \\ 0 & \sum_{j=-k}^{k} j^2 & \sum_{j=-k}^{k} j^3 \\ \sum_{j=-k}^{k} j^2 & \sum_{j=-k}^{k} j^3 & \sum_{j=-k}^{k} j^4 \end{pmatrix}$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\sum_{i=-k}^{k} i^3 = 0$$

$$\sum_{i=-k}^{k} i^4 =$$

$$= \begin{pmatrix} 2k+1 & 0 & \frac{k(k+1)(2k+1)}{3} \\ 0 & \frac{k(k+1)(2k+1)}{3} & 0 \\ \frac{k(k+1)(2k+1)}{3} & 0 & \frac{k(k+1)(2k+1)(3k^2+3k-1)}{15} \end{pmatrix}$$

$$\sum_{i=1}^{k} i^4 = \frac{1}{4+1} \sum_{j=0}^{4} (-1)^j \binom{4+1}{j} B_j \, k^{4+1-j}$$

$$= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

$$Z^T \cdot y_t^{(k)} = \begin{pmatrix} 1 & \cdots & 1 \\ -k & \cdots & k \\ (-k)^2 & \cdots & k^2 \end{pmatrix} \begin{pmatrix} y_{t-k} \\ \vdots \\ y_{t+k} \end{pmatrix} = \begin{pmatrix} \sum_{j=-k}^{k} y_{t+j} \\ \sum_{j=-k}^{k} j \, y_{t+k} \\ \sum_{j=-k}^{k} j^2 \, y_{t+j} \end{pmatrix}$$

$$A \mathrel{\hat=} Z^T Z, \quad x \mathrel{\hat=} \beta, \quad b \mathrel{\hat=} Z^T \cdot y_t^{(k)}$$

$$\Rightarrow \det(A) = (2k+1) \left(\frac{k(k+1)(2k+1)}{3} \cdot \frac{k(k+1)(2k+1)(3k^2+3k-1)}{15} \right)$$

$$- \left(\frac{k(k+1)(2k+1)}{3} \right)^3$$

$$\det(A_1) = \det \begin{pmatrix} \sum_{j=-h}^{h} Y_{t+j} & 0 & \frac{h(h+1)(2h+1)}{3} \\ \sum_{j=-h}^{h} j\, Y_{t+j} & \frac{h(h+1)(2h+1)}{3} & 0 \\ \sum_{j=-h}^{h} j^2\, Y_{t+j} & 0 & \frac{h(h+1)(2h+1)(3h^2+3h-1)}{15} \end{pmatrix}$$

$$= \left(\sum_{j=-h}^{h} Y_{t+j}\right)\left(\frac{h(h+1)(2h+1)}{3}\right)\left(\underbrace{\frac{h(h+1)(2h+1)(3h^2+3h-1)}{15}}_{g(h)}\right)$$

$$- \left(\underbrace{\frac{h(h+1)(2h+1)}{3}}_{g_2(h)}\right)^2 \sum_{j=-h}^{h} j^2\, Y_{t+j} = \frac{h(h+1)(2h+1)}{3}\left[g(h)\sum Y_{t+j} - g_2(h)\sum j^2 \ldots\right.$$

$$\Rightarrow \hat{\beta}_{0t} = \frac{\det(A_1)}{\det(A)} = \frac{g_2(h)^2\left[\frac{(3h^2+3h-1)}{5}\sum Y_{t+j} - \sum j^2 Y_{t+j}\right]}{g_2(h)^2\left[(2h+1)\frac{(3h^2+3h-1)}{5} - \frac{h(h+1)(2h+1)}{3}\right]}$$

$$= \sum_{j=-h}^{h} \frac{\left(\frac{(3h^2+3h-1)}{5} - j^2\right) Y_{t+j}}{\frac{(2h+1)}{15}\left[\underbrace{9h^2+9h-3-5h^2-5h}_{=4h^2-4h-3 \;=\; (2h-1)(2h+3)}\right]}$$

$$= \sum_{j=-h}^{h} \frac{3(3h^2+3h-1)-15j^2}{(2h+1)(2h-1)(2h+3)}\, Y_{t+j}$$

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Problem B:

$$y_{t+i} = \beta_{0t} + \beta_{1t} \cdot i + \beta_{2t} \cdot i^2 + u_{t+i},$$

$$i = -2, -1, 0, 1, 2 \Leftrightarrow 2h+1 = 5$$

$$\underbrace{\begin{pmatrix} y_{t-2} \\ \vdots \\ \vdots \\ y_{t+2} \end{pmatrix}}_{y_t^{(2)}} = \underbrace{\begin{pmatrix} 1 & -2 & 4 \\ \vdots & -1 & 1 \\ \vdots & 0 & 0 \\ & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}_{=Z} \underbrace{\begin{pmatrix} \beta_{0t} \\ \beta_{1t} \\ \beta_{2t} \end{pmatrix}}_{\beta_t} + \begin{pmatrix} u_{t-2} \\ \vdots \\ u_{t+2} \end{pmatrix}$$

$$\hat{\beta}_t = \underbrace{(Z^T \cdot Z)^{-1} Z^T}\cdot y_t^{(2)}$$

$$Z^T \cdot Z = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & -1 & -2 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & +2 & 1 \\ 1 & +1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{pmatrix}$$

$$(Z^T \cdot Z)^{-1} = \frac{1}{10 \cdot (5 \cdot 34 - 100)} \begin{pmatrix} 340 & 0 & -100 \\ 0 & 5 \cdot 34 - 100 & 0 \\ -100 & 0 & 50 \end{pmatrix}$$

$$= \frac{1}{700}$$

$$= \frac{1}{35} \begin{pmatrix} 17 & 0 & -5 \\ 0 & 7/2 & 0 \\ -5 & 0 & 5/2 \end{pmatrix}$$

$$(Z^T \cdot Z)^{-1} \cdot Z^T = \frac{1}{35} \begin{pmatrix} 17 & 0 & -5 \\ 0 & 7/2 & 0 \\ -5 & 0 & 5/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix}$$

$$= \frac{1}{35} \begin{pmatrix} 17-20 & 17-5 & \cdots \\ & \vdots & \\ & \vdots & \end{pmatrix} = \frac{1}{35} \begin{pmatrix} -3 & 12 & 17 & 12 & -3 \\ & & \vdots & & \end{pmatrix}$$

$$\Rightarrow \hat{\beta}_{0t} = -\frac{3}{35} Y_{t-2} + \frac{12}{35} Y_{t-1} + \frac{17}{35} Y_t + \frac{12}{35} Y_{t+1} - \frac{3}{35}$$

(b) $\Theta_0 = \frac{17}{35} \approx \frac{1}{2}$

(c) For $2h+1 = 7$ and $p = 3$ we had in the lecture $\underline{\Theta_0 = \frac{1}{3}}$. According to Fullerton 6.2, the result is the same for $p=2$ if $2h+1 > p$. $\in$

(d) $\hat{Y}_t = \overline{Y_5} \Rightarrow \Theta_0 = \frac{1}{5}$