Sheet 11

problem 1:

a-) We have to show that the acf is absolutely summable

$$\sum_{i=0}^{\infty} |\gamma_i| \overset{\text{Insert ACF}}{=} \sum_{i=0}^{\infty} \left| \frac{a^i \cdot \sigma^2}{1-a^2} \right|$$

$$\overset{\text{geom series}}{=} \left| \frac{\sigma^2}{(1-a^2)(1-a)} \right| < \infty$$

b-) Confidence Interval for μ AR(1)-process is stationary $\longrightarrow$ Wold decomposition

$$X_t = \mu + a X_{t-1} + \varepsilon_t$$

$$\Longleftrightarrow (1 - aB) X_t = \mu + \varepsilon_t$$

chapter 7

$$\underset{\text{St. 33}}{\Longleftrightarrow} \quad X_t = \sum_{i=0}^{\infty} a^i B^i \mu + \sum_{i=0}^{\infty} a^i B^i \varepsilon_t$$

$$= \sum_{i=0}^{\infty} \mu a^i + \sum_{i=0}^{\infty} a^i \varepsilon_{t-i}$$

$$\overset{\text{geom series}}{=} \frac{\mu}{1-a} + \sum_{i=0}^{\infty} a^i \varepsilon_{t-i} \quad (WO)$$

$$\longrightarrow E\left[\sum_{i=0}^{\infty} a^i \varepsilon_{t-i}\right] = \sum_{i=0}^{\infty} a^i \underbrace{E[\varepsilon_{t-i}]}_{0} = 0$$

Since the $\sum_{j=-\infty}^{+\infty} |a_i| \overset{\text{geom}}{\underset{\text{series}}{=}} \frac{1}{1-a} < \infty$

Theorem 8.5

$$\Longrightarrow \sqrt{T}\left(\bar{X}_t - \frac{\mu}{1-a}\right) \xrightarrow{D} N\left(0, \frac{\sigma^2}{(1-a)^2}\right)$$

$$\Longrightarrow \sqrt{T}\left(\bar{X}_t - \frac{\mu}{1-a}\right) \frac{1-a}{\sigma} \xrightarrow{D} N(0,1)$$

We know,

$$0.95 = \lim_{t\to\infty} P\left(-1.96 \leq \sqrt{T}\left(\bar{X}_t - \frac{\mu}{1-a}\right) \leq 1.96\right)$$

$$= \lim_{t\to\infty} P\left(\frac{1.96}{\sqrt{T}}\cdot\sigma + \frac{\bar{X}_t}{1-a} \geq \mu \geq -1.96\cdot\frac{\sigma}{\sqrt{T}} + \frac{\bar{X}_t}{1-a}\right)$$

$$0.95 \underset{\text{insert}}{\approx} P\left(\frac{1.96\cdot\sqrt{4}}{\sqrt{100}} + \frac{\bar{X}_t}{0.3} \geq \mu \geq \frac{-1.96\cdot\sqrt{4}}{\sqrt{100}} + \frac{\bar{X}_t}{1-a}\right)$$

$$\Rightarrow CI_{0.95} \approx \left[-\frac{1.96}{5} + \frac{\bar{X}_t}{0.3}, \frac{1.96}{5} + \frac{\bar{X}_t}{0.3}\right]$$

Realized Confidence Interval

$$CI_{0.95,\,\text{realized}} = \left[-\frac{1.95}{5} + \frac{0.157}{0.3}, \frac{1.96}{5} + \frac{0.157}{0.3}\right]$$

$$= [-0.35, 0.44]$$

0 lies in this interval $\Rightarrow$ We don't reject the null

$\boxed{\text{Question 1: b-) Is resolved in next week}}$

Does $\sqrt{T}\left(\bar{X}_t - \frac{\mu}{1-a}\right)$ really converge in distribution to $N\left(0, \frac{\sigma^2}{(1-a)^2}\right)$

Claim: Yes. $\sqrt{T}\left(\bar{X}_t - \frac{\mu}{1-a}\right) \xrightarrow[\text{Theorem 8.5}]{D} N\left(0, \sum_{h=-\infty}^{\infty} \gamma(h)\right)$

$$\sum_{h=-\infty}^{\infty} \gamma(h) \overset{(a)}{=} \sum_{h=-\infty}^{\infty} \frac{a^{|h|}}{1-a^2} = \frac{\sigma^2}{1-a^2} \sum_{h=-\infty}^{\infty} a^h$$

$$= \frac{\sigma^2}{1-a^2}\left[\sum_{h=0}^{\infty} a^h + \underbrace{\sum_{h=-\infty}^{-1} a^{|h|}}\right]$$

$$\sum_{h=1}^{\infty} a^h = \sum_{h=0}^{\infty} a^{h+1} = \sum_{h=0}^{\infty} a^h \cdot a$$

$$= \frac{\sigma^2}{1-a^2}\left[\sum_{h=0}^{\infty} a^h + a\sum_{h=0}^{\infty} a^h\right] = a\sum_{h=0}^{\infty} a^h$$

$$\overset{\text{geom}}{\underset{\text{series}}{=}} \frac{\sigma^2}{1-a^2}\left[\frac{1}{1-a} + \frac{a}{1-a}\right] = \frac{\sigma^2}{1-a}\left[\frac{1+a}{1-a}\right]$$

binomial formula $= \frac{\sigma^2}{(1-a)(1+a)}\left[\frac{1+a}{1-a}\right] = \frac{\sigma^2}{(1-a)^2}$

② R exercise

problem 3

ⓐ $X_t = aX_{t-1} + \varepsilon_t \iff X_t - aX_{t-1} = \varepsilon_t$

$\Rightarrow \varepsilon_t = WX_t$, where W is the linear filter $1 - aB$

$\Rightarrow X_t = W^{-1}\varepsilon_t$, since we know from cpt 7 that such an inverse filter exists and is given by $W^{-1}(B) = \sum_{j=0}^{\infty} a^j B^j$, as $|a| < 1$

Applying this filter

$$X_t = \sum_{j=0}^{\infty} a^j \varepsilon_{t-j}$$ $\Rightarrow$ This is $MA(\infty)$ by definition

b-) Chapter 9, Slides 46-47

c-) $f(w) = \frac{1}{2\pi} \gamma(e^{iw})$

$$\overset{\text{insert}}{=} \frac{\sigma^2}{2\pi} W^{-1}(e^{iw}) W^{-1}(e^{-iw})$$

$$\overset{\text{hint}}{=} \frac{\sigma^2}{2\pi} \frac{1}{(1-ae^{iw})(1-ae^{-iw})}$$

$$= - \frac{\sigma^2}{2\pi(1+a^2-a(e^{-100}+e^{100})} \overset{\text{euler}}{=} \frac{\sigma^2}{2\pi\,(1+a^2-2a\cdot\cos(w))}$$

important Note

$$\sum_{i=0}^{\infty} a^i \underset{\text{series}}{\overset{\text{geom}}{=}} \frac{1}{1-a} \quad \text{if } |a|<1$$

$$\sum_{i=0}^{\infty} a^i \beta^i \neq \frac{1}{1-a\beta}$$