problem 1

(a)

X_t, Y_t i.i.d and independent $P(X_t=0)=P(X_t=1)=\frac{1}{2}$

$P(Y_t=1)=P(Y_t=-1)=\frac{1}{2}$, $Z_t = X_t(1-X_{t-1})\cdot Y_t$

$$P(Z_t=k) = \begin{cases} \frac{6}{8} & k=0 \\ \frac{1}{8} & k=1 \\ \frac{1}{8} & k=-1 \\ 0 & \text{else} \end{cases}$$

Show that Z_t is White Noise:

$$E[Z_t] = E[X_t(1-X_{t-1})\,Y_t]$$

$$= \underbrace{E(X_t)}_{\frac{1}{2}}\,\underbrace{E[1-X_{t-1}]}_{\frac{1}{2}}\cdot\underbrace{E[Y_t]}_{=0}$$

$$=0$$

$$E[Z_t^2] = E[X_t^2 (1-X_t)^2 Y_t^2]$$

$$= \underbrace{E[X_t^2]}_{\frac{1}{2}} \cdot \underbrace{E[(1-X_{t-1})^2]}_{\frac{1}{2}} \cdot \underbrace{E[Y_t^2]}_{1}$$

$$= \frac{1}{4}$$

$$E[Z_t \cdot Z_s] = E[X_t \cdot X_s (1-X_{t-1})(1-X_{s-1}) Y_t \cdot Y_s]$$

$$= E[X_t X_s (1-X_{t-1})(1-X_{s-1})] \cdot \underbrace{E[Y_s \cdot Y_t]}_{= \begin{cases} 0 & s \neq t \\ 1 & s = t \end{cases}} = \begin{cases} 0 & s \neq t \\ & s = t \end{cases}$$

Show Z_t and Z_{t-1} are not independent

For independent events A and B. It holds that

$P(A \cap B) = P(A) \cdot P(B)$. For us: $P(Z_t = 1, Z_{t-1} = 1)$

$$= \underbrace{P(z_t=1 \mid z_{t-1}=1)}_{=0} \cdot \underbrace{P(z_{t-1}=1)}_{\frac{1}{8}} = 0 \neq P(z_t=1) \cdot P(z_{t-1}=1) = \frac{1}{64}$$

ⓑ $\varepsilon_t \overset{iid}{\sim} N(0,1)$, (i) $y_t = a + bt + \varepsilon_t$, $E[y_t] = a + bt$

$Var(y_t) = Var(\varepsilon_t) = 1$

$\xrightarrow{}$ AC function

$$\gamma(s,t) = Cov(y_t, y_s)$$

$$= Cov(a+bt+\varepsilon_t, a+bs+\varepsilon_s)$$

$$= Cov(\varepsilon_t, \varepsilon_s) = \begin{cases} 1 & s=t \\ 0 & s \neq t \end{cases}$$

ii-) $y_t = \varepsilon_t \cdot \varepsilon_{t-1}$, $E[y_t] = 0 \cdot 0 = 0$, $Var(\varepsilon_t^2, \varepsilon_{t-1}^2) = 1 \cdot 1 = 1$

$$\gamma(s,t) = Cov(\varepsilon_t \varepsilon_{t-1} \varepsilon_s \varepsilon_{s-1}) = \underbrace{E[\varepsilon_t \varepsilon_{t-1} \varepsilon_s \varepsilon_{s-1}]} =$$

$$= \begin{cases} 1 & s=t \\ E[\varepsilon_s^2]\, E[\underbrace{\varepsilon_t}_{=0}] = 0 \quad E[\varepsilon_{t-2}] & s=t-1 \\ 0 & s=t+1 \\ 0 & \text{else} \end{cases}$$

iii) $Y_t = \Delta \varepsilon_t$, $E[Y_t] = 0$, $Var(\Delta \varepsilon_t) = Var(\varepsilon_t) + Var(\varepsilon_{t-1})$

$= 2$

$$\gamma(s,t) = cov(\varepsilon_t - \varepsilon_{t-1}, \varepsilon_s - \varepsilon_{s-1}) = Cov(\varepsilon_t, \varepsilon_s) + Cov(\varepsilon_{t-1}, \varepsilon_{s-1})$$

$$- cov(\varepsilon_{t-1}, \varepsilon_s) - cov(\varepsilon_t, \varepsilon_{s-1}) =$$

$$= \begin{cases} 2 & s = t \\ -1 & s = t-1 \text{ or } t+1 \\ 0 & \text{else} \end{cases}$$

$Y_t \sim N(0,2)$

$$(Y_t, Y_{t-1}, Y_1) \sim N(0, \Sigma), \quad \Sigma = \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}$$

(iv) $Y_t = \cos(\varphi t)\varepsilon_t + \sin(\varphi t)\varepsilon_{t-2}$

$E[Y_t] = 0$, $Var(Y_t) = \cos^2(\varphi t) + \sin^2(\varphi t) = 1$

$$\gamma(s,t) = Cov(\cos(\varphi t)\cdot\varepsilon_t + \sin(\varphi t)\varepsilon_{t-2}, \cos(\varphi s)\varepsilon_s + \sin(\varphi s)\cdot\varepsilon_{s-2})$$

$$= \begin{cases} 1 & s = t \\ 0 & s = t+1 \text{ or } t-1 \\ \cos(\phi t) + \sin(\phi s) & s = t+2 \text{ or } s = t-2 \\ 0 & \text{else} \end{cases}$$

V-) $y_t = \theta_0 \cdot \varepsilon_t + \cdots\cdots + \theta_q \varepsilon_{t-q}$, $E[y_t] = 0$

$$\mathrm{Var}(y_t) = \theta_0^2 + \cdots\cdots + \theta_q^2$$

$$\mathrm{cov}(y_t, y_s) = \mathrm{cov}\left(\sum_{j=0}^{q} \theta_j \varepsilon_{t-j}, \sum_{j'=0}^{q} \theta_{j'} \varepsilon_{s-j'}\right)$$

$$= \sum_{j=0}^{q} \sum_{j'=0}^{q} \theta_j \theta_{j'} \underbrace{\mathrm{cov}(\varepsilon_{t-j}, \varepsilon_{s-j'})}_{= \mathbb{1}\{t-j = s-j'\} = \mathbb{1}\{j' = j-t+s\} = \delta_{j', j-t+s}} =$$

$$= \sum_{j=0}^{q-(t-s)} \theta_j \theta_{j \overbrace{-t+s}^{=-h}} = \sum_{j=0}^{q-h} \theta_{j+h} \theta_j$$

Problem 2

(a) Borel-Cantelli Lemma: Let A_n be a set of sets

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 0$$

$$P(\underbrace{S_n=0 \text{ infinitly many}}_{\limsup_{n\to\infty}\{S_n=0\}} \text{ time})$$

Here $A_n = \{S_n = 0\}$. $P(S_{2n+1}=0)=0$

$$P(S_{2n}=0) = \underbrace{\binom{2n}{n}}_{\#\text{ of possibility}} \underbrace{p^n}_{\text{prob to go up } n\text{-time}} \underbrace{q^n}_{\text{prob to go down } n\text{-time}}$$

$$= 4^n (pt)^n = (4pt)^n \quad , \quad pq < \frac{1}{4} \text{ for } p \neq q \neq \frac{1}{2}$$

$$= \left(\frac{4}{4+\delta}\right)^4 \text{ where } \delta > 0 \Rightarrow \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} \left(\frac{4}{4+\delta}\right)^i < \infty$$

geometric sum

$$\Rightarrow P(S_n = 0 \ \infty \text{ many times}) = 0$$

Borell Cantelli

(b) $P(S_{2n}=0) = \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n = \frac{(2n)!}{n!\,n!} \cdot \left(\frac{1}{4}\right)^n = \cdots \approx \sqrt{\frac{1}{2\pi n}}$

$$\Longrightarrow E\left[\sum_{n=1}^{\infty} \#\{S_n=0\}\right] = \sum_{n=1}^{\infty} P(\{S_n=0\}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi \cdot n}} = \infty$$

$$\sum_{n=-1}^{\infty} n^{\alpha} = \begin{cases} <\infty & \alpha < -1 \\ \infty & \alpha \geq -1 \end{cases}$$

Let a be the probability of returning to 0 when starting at 0

$B_k = \{S_n = 0 \ k \text{ times}\}$ is distributed geometrically

$$E\left[\sum_{n=1}^{\infty} \#\{S_n=0\}\right] = \sum_{k=1}^{\infty} k \cdot a^{k-1}(1-a) = \frac{a}{1-a} = \infty \Rightarrow a = 1$$

problem 3

If λ is a r-fold, $r \in \mathbb{N}$, root of P_W, then

$$\{t^{r-1}\lambda^t\}_{t\in\mathbb{R}} \in Ker(W)$$

λ is a 2-fold root of $P_W \Rightarrow P_W(\lambda)=0,\ P'_W(\lambda)=0$

$$\Rightarrow \tilde{P}_W = \lambda^{t-q} \cdot P_W$$

$\tilde{P}_W(\lambda)=0$ and $\tilde{P}_W(\lambda)=0$. $\tilde{P}_W(\lambda) = \lambda^{-t-1}(w_q + \cdots\cdots + w_p\lambda^{p+q})$

$$= w_q\lambda^{-t-q} + \cdots\cdots + w_p\lambda^{p-t}$$

$$\tilde{P}_W(\lambda) = (-t-q)\lambda^{-t-q-1} w_q + \cdots\cdots + (p-t)\lambda^{p-t-1} \cdot w_p = 0 \ \Big|\cdot\lambda$$

$$\Rightarrow \sum_{j=-q}^{p} (-t+q)\lambda^{-t+q} w_j = \sum_{j=q}^{p} B^{-j} w_j \underbrace{(-t\cdot\lambda^{-t})}_{\{-t\lambda^{-t}\}_{t\in\mathbb{Z}} \in Ker(W)} = 0$$