Problem 18:

$$\overline{W}_1 = \frac{1}{8}B^2 + \frac{1}{4}B + \frac{1}{4} + \frac{1}{4}B^{-1} + \frac{1}{8}B^{-2}$$

$$\Rightarrow f^{\overline{W}_1}(z) = \frac{1}{8}z^2 + \frac{1}{4}z + \frac{1}{4} + \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2}$$

$$p^{\overline{W}_1}(z) = \frac{1}{8}z^4 + \frac{1}{4}z^3 + \frac{1}{4}z^2 + \frac{1}{4}z + \frac{1}{8}$$

$$p^{\overline{W}_1}(z) = 0 \quad \Rightarrow \quad z_1 \overset{\text{sheet}}{=} -1$$

Find other roots of $p^{\overline{W}_1}(z)$:

$$\left(\frac{1}{8}z^4 + \frac{1}{4}z^3 + \frac{1}{4}z^2 + \frac{1}{4}z + \frac{1}{8}\right)(z+1) = \underbrace{\frac{1}{8}z^3 + \frac{1}{8}z^2 + \ldots}$$

$$-\left(\frac{1}{8}z^4 - \frac{1}{8}z^3\right)$$

$$\frac{1}{8}z^3$$

$$-\left(\frac{1}{8}z^3 + \frac{1}{8}z^2\right)$$

$$\frac{1}{8}z^2$$

$$-\left(\frac{1}{8}z^2 + \frac{1}{8}z\right)$$

$$\frac{1}{8}z$$

$$-\left(\frac{1}{8}z + \frac{1}{8}\right)$$

$$\Rightarrow z_2 = -$$

$$\left(\frac{1}{8}z^3 + \frac{1}{8}z^2 + \frac{1}{8}z + \frac{1}{8}\right)(z+1) \overset{\text{pol. div}}{=} \frac{1}{8}z^2 + \frac{1}{8}$$

$$\Rightarrow z_3 = i \qquad z_4 = -i$$

$\Rightarrow \rho^{\bar{w}_1}_{(z)}$ has roots $z_1 = -1,\ z_2 = -1,\ z_3 = i,\ z_4$

Prop. 8.6: If $\lambda = |\lambda|(\cos\varphi + i\sin\varphi)$ is a root of char. pol. of lin filter w, then

$$\{|\lambda|^{-t}\cos(t\varphi)\}_{t\in\mathbb{Z}} \in \mathrm{Ker}(w)$$

$$\{|\lambda|^{-t}\sin(t\cdot\varphi)\}_{t\in\mathbb{Z}} \in \mathrm{Ker}(w)$$

$z_1 = |-1|\cdot\cos(\pi) \qquad z_3 = |1|\cdot\sin\left(\frac{\pi}{2}\right)$

$z_4 = |1|\cdot\sin\left(\frac{3\pi}{2}\right)$

Basis of Kernel:

$$Y_t = \sum_{k=1}^{K_C} |\lambda_k|^{-t}\Big[\cos(t\cdot\varphi_k)(a_{k1}+a_{k2}\cdot t+\ldots+a_{kr_k}t^{r_k-1})$$

$$+\sin(t\cdot\varphi_k)(b_{k1}+b_{k2}\cdot t+\ldots+b_{k\cdot r_k}t^{r_k-1})\Big)$$

$$+\sum_{k=K_c+1}^{K}\lambda_k^{-t}(c_{k1}+c_{k2}\cdot t+\ldots+c_{kr_k}t^{r_k-1}$$

Here: $K_C = \cancel{2}\ 1,\ \varphi_{k=1} = \frac{\pi}{2},\ \left(\varphi_{k=2} = \frac{3\pi}{2}\right)$ complex conjugate, $K = \cancel{4}\ 2$

$$Y_t = 1^{-t}\cdot\sin\left(t\cdot\frac{3\pi}{2}\right)\cdot a_{11} + \sin\left(t\cdot\frac{\pi}{2}\right)\cdot b_{21}$$

$$+(-1)^{-t}(c_{31}+c_{32}\cdot t) + \left(\cos\left(t\cdot\frac{3\pi}{2}\right)+b_{11}+\cancel{\cos}(t\cdot\frac{\pi}{2}\right.$$

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$$\Rightarrow y_t = 1^{-t}\left(\sin\left(t\cdot\frac{\pi}{2}\right)\cdot b_{11} + \cos\left(t\cdot\frac{\pi}{2}\right)\cdot a_{11}\right)$$

$$+ (-1)^{-t}\left(c_{21} + c_{22}\cdot t\right)$$

r_k: Number of the same roots
for example if there are two "-1",
and -1 is reel root, therefore,
$r_k = 2$. If there would have been
one "-1", then the r_k would have

Invariance set:

$$\tilde{W}_1 = W - 1 = \frac{1}{8}B^2 + \frac{1}{4}B - \frac{3}{4} + \frac{1}{4}B^{-1} + \frac{1}{8}B^{-2}$$

$$p^{\tilde{W}_1}(z) = \frac{1}{8}z^4 + \frac{1}{4}z^3 - \frac{3}{4}z^2 + \frac{1}{4}z + \frac{1}{8} \overset{!}{=} 0$$

$$z_1 = 1, \; z_2 = 1 \quad ; \; z^2 + 4z + 1$$

$$z_{3/4} = \frac{-4}{2} \pm \sqrt{4 - 1} = \begin{cases} -0.27 \\ -3.73 \end{cases}$$

$\lambda = 1$ is 2-fold root of $p^{\tilde{W}_1}(z)$

$\Rightarrow$ W_1 leaves band of polyn. order 1 invariant

$$\{Y_t = t\}_{t \in \mathbb{Z}} \in \mathrm{Inv}(W_1)$$

Problem 20:

$$W = 1 - B^5$$

$$V = \frac{1}{5} \sum_{j=-2}^{2} B^j$$

Show that W & V eliminate seas. effect with period length $l = 5$.

To show that calculate Basis of trend of filters. If they eliminate $l=5$, $\cos(\varphi_k \cdot t)$ & $\sin(\varphi_k \cdot t)$ with $\varphi_k = \frac{2\pi \cdot k}{l}$ should be in the Basis.

(i) W: roots of $p^{W}(z)$ are $z_i = e^{i\varphi_i}$

$\varphi_i = 2\pi \frac{i}{5}$ $\quad i = 1, .., 5$ (from Problem 16)

$$l_i = \frac{2\pi}{\varphi_i} = \frac{2\pi}{\frac{2\pi i}{5}} = \frac{5}{i}$$

(ii) $V = \frac{1}{5} \sum_{j=-2}^{2} B^j \Rightarrow p_V = \frac{1}{5}(z^4 + z^3 + z^2 + z + 1)$

trick! $W = (B^4 + B^3 + B^2 + B + 1)(1 - B)$

$\Rightarrow p^W = p^V(z)(1 - z)$ $\quad \Rightarrow p^V(z) = 0$

$\Leftrightarrow \varphi_i = 2\pi \frac{i}{5}, \; i = 1, .., 4$

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Problem 21:

Cond. (a): $\bar{W}$ should leave lin. trends invariant

$\Rightarrow$ The characteristic ~~x~~ polyn. of $\bar{W}$, respectively $\tilde{\bar{W}} = \bar{W} - 1$ has to have two fold root at $z = 1$

$$\Rightarrow p^{\tilde{\bar{W}}}(1) = 0 \ \& \ \underbrace{(p^{\tilde{\bar{W}}})'}_{\text{derivative}}(1) = 0$$

Cond. (b): $\bar{W}$ should <u>only</u>* eliminate seasonal effects with period length $\ell = 4$.

$\Rightarrow p^{\bar{W}}(z)$ has to have roots

$$z_1 = i, \ z_2 = -1, \ z_3 = -i$$

$$\Rightarrow p^{\bar{W}}(z) = c \cdot (z - i)(z+1)(z+i)$$

$$\Rightarrow p^{\tilde{\bar{W}}}(z) = c \cdot (z-i)(z+1)(z+i) - z^4$$

Now $p^{\tilde{\overline{w}}}(z)$ from cond. (b) does not fulfill the requirements of cond. (a) because:

(1) $p^{\tilde{\overline{w}}}(1) \overset{!}{=} 0 \iff c(1-i)(1+1)(1+i) = 1$

$\implies$ fulfilled if $c = \frac{1}{4}$

(2) $\left(p^{\tilde{\overline{w}}}\right)'(1) \overset{!}{=} 0$

$\iff \left(p^{\tilde{\overline{w}}}\right)'(1) = \frac{1}{4}(3+2+1) - q\,1^{q-1} \overset{!}{=} 0$

only fulfilled when $q = \frac{3}{2}$ ↯

since q is the order of the lin. filter, it has to be an integer.

* It is meant that $\overline{w}$ should <u>only</u> eliminate seasonal effects with $l = 4$ and <u>no</u> other functions whatsoever, e.g. $(-1)^t(C_{11} + C_{12}t)$ should also not be eliminated.