1-)

Let $k, j \in \mathbb{Z}$ $(*)$, $T \in \mathbb{N}$, $\frac{T}{2} \in \mathbb{N}$ (T is even)

additionally we assume $k, j \in \{0, 1, 2, \ldots, [\frac{T-1}{2}]\}$

$$\sum_{t=1}^{T} \sin(2\pi \tfrac{k}{T} \cdot t) \cdot \cos(2\pi \cdot \tfrac{k}{T} \cdot t) = 0$$

(i) $$\sum_{t=1}^{T} \cos(wt) = \cos\left(\frac{w(T+1)}{T}\right) \frac{\sin(\frac{wT}{2})}{\sin(\frac{w}{2})}$$

(ii) $$\sum_{t=1}^{T} \sin(wt) = \sin \text{———} \,\,\text{''}\,\, \text{———}$$

(iii) (a) $\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

(b) $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

(c) $\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$

$$\sum_{t=1}^{T} \sin(2\pi \cdot \tfrac{k}{T} \cdot t) \cos(2\pi \cdot \tfrac{j}{T} \cdot t) \overset{(iii)(c)}{=} \sum_{t=1}^{T} \left(\sin(2\pi \cdot \tfrac{(k+j)}{T} \cdot t) + \sin(2\pi \cdot \tfrac{(k-j)}{T} \cdot t)\right)$$

$$\overset{(ii)}{=} \frac{1}{2} \sin\left(\frac{\frac{2\pi(k+j)}{T} \cdot (T+1)}{2}\right) \cdot \frac{\sin\left(\frac{\frac{2\pi(k+j)}{T} \cdot T}{2}\right)}{\sin\left(\frac{\frac{2\pi(k+j)}{T}}{2}\right)} = 0$$

$$+ \frac{1}{2} \text{———} (k-j) \text{——} \,\,\text{''}\,\, \text{——} \frac{(k-j)}{(k-j)} \text{———}$$

$$\sum_{t=1}^{T} \sin\left(2\pi \cdot \frac{k}{T} \cdot t\right) \cdot \sin\left(2\pi \cdot \frac{j}{T} \cdot t\right)$$

for $k, j = 0$ or $\frac{T}{2}$ we see that we just sum over zero, so its zero.

(iii)(b)

$$= \sum_{t=1}^{T} \frac{1}{2}\left(\cos\left(2\pi \frac{k-j}{T} \cdot t\right) - \cos\left(2\pi \frac{k+j}{T} \cdot t\right)\right)$$

$$\overset{(i)}{=} \frac{1}{2} \cdot \underbrace{\cos\left(\frac{\frac{2\pi(k-j)}{T}(T+1)}{2}\right)}_{=1 \text{ for } k=j} \cdot \frac{\overbrace{\sin\left(\frac{2\pi(k-j)}{T} \cdot T / 2\right)}^{=0 \text{ for } k=j}}{\underbrace{\sin\left(\frac{2\pi(k-j)}{2T}\right)}_{=0 \text{ for } k=j}}$$

$$\frac{1}{2} \text{ —— } (k+j) \text{ — '' ——} \frac{(k+j)}{(k+j)} \text{——} = \frac{T}{2} \text{ for } k=j$$

$$\sum_{t=1}^{T} \cos\left(2\pi \cdot \frac{k}{T} \cdot t\right) \cdot \cos\left(2\pi \cdot \frac{j}{T} \cdot t\right)$$

(iii)(a)

$$= \sum_{t=1}^{T} \frac{1}{2}\left(\cos\left(2\pi \frac{k+j}{T} \cdot t\right) + \cos\left(2\pi \frac{(k-j)}{T} \cdot t\right)\right)$$

$$= \frac{1}{2} \cos\left(\frac{\frac{2\pi(k+j)}{T}(T+1)}{2}\right) \cdot \frac{\sin\left(\frac{\frac{2\pi(k+j)}{T} \cdot T}{2}\right)}{\sin\left(\frac{2\pi \cdot (k+j)}{2\pi}\right)}$$

$(*)$

$$+ \frac{1}{2} \text{ —— } (k-j) \text{ — '' ——} \frac{(k-j)}{(k-j)} \text{——}$$

$$(*) = \begin{cases} T & \boxed{k = \dot{\varsigma} = 0 \text{ or } \tfrac{1}{2}} \rightarrow \text{Summing } T \text{ times over } 1 \\ \frac{T}{2} & k = \dot{\varsigma} \quad (\text{L'Hopital again}) \\ 0 & k \neq \dot{\varsigma} \quad (\text{Multiples of } \pi \text{ in } \sin) \end{cases}$$

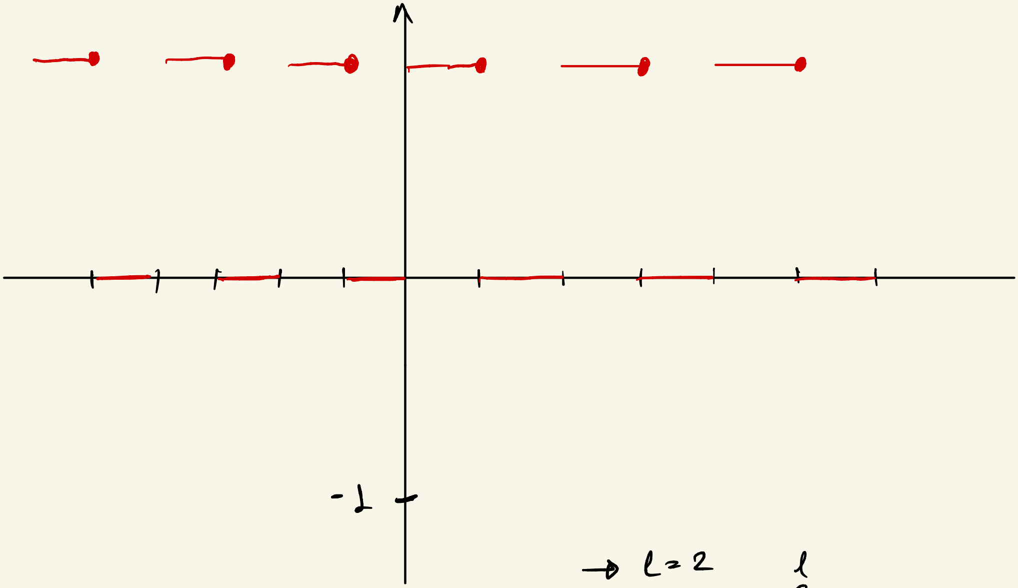
problem 3 : Let $f\ \mathbb{R} \to \mathbb{R}$ be periodicity, $f(t) = f(t+l) = f(t+2l) = \cdots$

the Fourier expansion of order N is given as

$$f_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(2\pi \frac{n}{e} x\right) + b_n \cdot \sin\left(2\pi \cdot \frac{n}{e} x\right)\right)$$

$$f(t) = \begin{cases} 1 & t \in [0,1) + k,\ k \in \{\ldots, -4, -2, 0, 2, 4, \ldots\} \\ 0 & t \in [1,2) + k, \text{ — '' —} \end{cases}$$

(a)



$\rightarrow \ell = 2$

$$a_0 = \frac{1}{\ell} \int_0^{\ell} f(t)\,dt$$

$$= \frac{1}{2} \int_0^{2} \mathbb{1}_{\{t \in [0,1]\}}\, dt$$

$$= \frac{1}{2} \int_0^{1} 1 \cdot dt = \frac{1}{2}$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(t) \cos\left(2\pi \frac{n}{\ell} \cdot t\right) \cdot dt = \int_0^{1} \cos(\pi . n . t) . dt = \left[\frac{1}{\pi . n} \sin(\pi . n . t)\right]_0^1$$

$$= 0$$

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(t) . \sin\left(2\pi . \frac{n}{\ell} . t\right) dt = \int_0^{1} \sin(\pi n . t) dt =$$

$$= \frac{-\cos(\pi \cdot n)}{\pi \cdot n} - \underbrace{\left(- \frac{\cos(0)}{\pi \cdot n}\right)}_{-\frac{1}{\pi \cdot n}} = \frac{1}{\pi \cdot n} \underbrace{\left(1 - \cos(\pi \cdot n)\right)}_{= \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi \cdot n} & n \text{ odd} \end{cases}$$

$$f_N(x) = \frac{1}{2} + \sum_{k=1,3,5,\ldots} \frac{2}{\pi \cdot n} \cdot \sin\left(2\pi \cdot \frac{k}{e} \cdot x\right)$$

problem 4

Let f be $\{1, \ldots, T\} \to \mathbb{R}$, Then we can write f as follows

$$f(t) = a_0 + \sum_{k=1}^{\left[\frac{T}{2}\right]} \left(a_k \cdot \cos\left(2\pi \cdot \frac{k}{T} \cdot t\right) + b_k \cdot \sin\left(2\pi \cdot \frac{k}{T} \cdot t\right)\right)$$

$$f : \{1,2\} \to \{0,1\}, \quad f(t) = \begin{cases} 1 & t = 1 \\ 0 & t = 2 \end{cases}$$

$T = 2$
$\downarrow$
length of
the sequence

$$T = 2 \,, \quad a_0 = \frac{1}{2} \sum_{t=1}^{2} f(t) \cdot \underbrace{\cos\left(\frac{2\pi \cdot 0 \cdot t}{T}\right)}_{=1} = \frac{1}{2}$$

$$a_1 = \frac{1}{2} \cdot \sum_{t=1}^{2} f(t) \cos\left(\frac{2\pi \cdot t}{T}\right)$$

$$= \begin{matrix} \frac{1}{2}\left(\cos(\pi \cdot 1)\right) \rightarrow \text{when } t=1 \\ + \\ \underbrace{\frac{1}{2}\left(\cos(2\pi)\right)}_{=0} \rightarrow \text{when } t=2 \end{matrix} = -\frac{1}{2}$$

$$b_1 = \frac{2}{2} \sum_{t=1}^{T} f(t) \cdot \sin\left(2\pi \cdot \frac{1}{2} \cdot t\right) = 0$$

$$f(t) = \frac{1}{2} + \left(-\frac{1}{2}\right) \cdot \cos\left(2\pi \cdot \frac{1}{2} \cdot t\right) = \begin{cases} 2 & t=1 \\ 0 & t=2 \end{cases}$$