Sheet 2

problem 1

Let y_t, $t = -4, \dots, 0, \dots, 4$ be a TS with realizations $y = (9.7, \dots, 17.4)$ We assume $y_t = m_t + s_t + \varepsilon_t$

1-) We assume $m_t = \beta_0 + \beta_1 \cdot t$ & want to estimate β_0 & β_1

Model: $y_t = \beta_0 + \beta_1 \cdot t + u_t = (1\ t) \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + u_t$

$$\begin{pmatrix} y_{-4} \\ \vdots \\ y_0 \\ \vdots \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ \vdots & \vdots \\ 1 & 4 \end{pmatrix} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}_{=\beta} + \begin{pmatrix} u_{-4} \\ \vdots \\ u_4 \end{pmatrix}$$

$\overset{OLS}{\Longrightarrow} \hat{\beta} = (A^T A)^{-1} A^T y$ (I)

Here we use

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \cdot \bar{t}$ and $\hat{\beta}_1 = \dfrac{\overline{ty} - \bar{t}\bar{y}}{\overline{t^2} - \bar{t}^2}$ (#) from the lecture

$\bar{y} = 10.7$, $\bar{t} = 0$, $\overline{t^2} = \sum_{t=-4}^{4} t^2 = \frac{2}{9} \sum_{t=1}^{4} t^2 = \frac{60}{9}$

$\overline{ty} = \frac{1}{9} \sum_{t=-4}^{4} t \cdot y = \frac{65.1}{9} \Rightarrow \hat{\beta}_0 = 10.7$, $\hat{\beta}_1 \approx 1.1$

$\hat{m}_t = \hat{\beta}_0 + \hat{\beta}_1 \cdot t \rightarrow (6.3, 7.4, 8.5, 9.6, 10.7, 11.8, 12.1, 14.0, 15.1)$

$y_t^{\text{demand}} = y_t - \hat{m}_t \rightarrow (\underset{S_1}{\underline{3.6}}, \underset{S_2}{\underline{-4.6}}, \underset{S_3}{\underline{1.1}}, \underset{S_1}{\underline{0.8}}, \underset{S_2}{\underline{-5.0}}, \underset{S_3}{\underline{3.9}}, \underset{S_1}{\underline{3.0}}, \underset{S_2}{\underline{-5.2}}, \underset{S_3}{\underline{2.1}})$

2-) Model: $Y_t = \delta_1 \cdot S_{1,t} + \cdots + \delta_3 \cdot S_{3,t} + \varepsilon_t$, $\sum_{i=1}^{3} \delta_i = 0$, $\hat{\delta} = \bar{y} - \bar{y}_{\ell=i}$

S_1 means season 1 , S_2 means season 2 and so on

Season	1	2	3
	3.6	-4.6	1.1
	0.8	-5.0	3.9
	3.0	-5.2	2.1
average	2.5	-4.9	2.4

$$\Rightarrow \hat{\delta}_i = \begin{cases} -2.5 & i=1 \\ 4.9 & i=2 \\ -2.4 & i=3 \end{cases}$$

problem 2

Let $\{X_t\}_{t\in\{1,\dots,T\}}$ be a TS with $X_t = \beta_0 + \beta_1 \cdot t + \varepsilon_t$

$$\operatorname{cov}(\varepsilon_t, \varepsilon_{t-h}) = \begin{cases} \sigma^2 & h=0 \\ \rho\cdot\sigma^2 & h=\pm 1 \\ 0 & |h|\geq 2 \end{cases}$$

$E(\varepsilon_t) = 0$

a-) $E(\hat{\beta})$ using (I)

$$E\left((A^TA)^{-1}A^Ty\right)$$

$$= E\left((A^TA)^{-1}A^T.(A\beta+\varepsilon)\right)$$

$$= E\left(\underbrace{(A^TA)^{-1}A^T.A}_{I}.\beta+(A^TA)^{-1}A^T\varepsilon\right)$$

$$= E\left(\beta+(A^TA)^{-1}A^T\varepsilon\right)$$

$$= \beta + \underbrace{E\left((A^TA)^{-1}A^T\varepsilon\right)}_{=(A^TA)^{-1}A^T.\underbrace{E(\varepsilon)}_{=0}}$$

$$= \beta$$

b-) $Var(\hat{\beta})$

$$\left[= \begin{pmatrix} Var(\hat{\beta}_0) & Cov(\hat{\beta}_0,\hat{\beta}_1) \\ \bullet & Var(\hat{\beta}_1) \end{pmatrix} \right] = Cov(\hat{\beta},\hat{\beta}) = Cov\left((A^TA)^{-1}A^T(A\beta+\varepsilon), \ldots\right)$$

$$= cov\left((A^TA)^{-1}A\varepsilon, \varepsilon^TA(A^TA)^{-1}\right)$$

$$= (A^TA)^{-1}A^T\, Cov(\varepsilon,\varepsilon^T)\, A\,(A^TA)^{-1}$$

$$Cov(\varepsilon,\varepsilon^T) = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & & & \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & & \\ & \rho\sigma^2 & \sigma^2 & \ddots & \\ & & \rho\sigma^2 & \ddots & \rho\sigma^2 \\ & & & \rho\sigma^2 & \sigma^2 \end{pmatrix}$$

$$= \sigma^2.I_{T\times T} + \rho\sigma^2\begin{pmatrix} 0 & 1 & \cdots & & 0 \\ 1 & \ddots & & & \vdots \\ 0 & & \ddots & & 1 \\ 0 & & & \ddots & \\ 0 & \cdots & & 1 & 0 \end{pmatrix}$$

$$= (A^T A)^{-1} A^T \left(\sigma^2 I_{T \times T} + \rho\sigma^2 \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & \vdots \\ 0 & 1 & \cdots & 0 \; 1 \\ 0 & 0 & & \\ 0 & 0 & \cdots & 1 \; 0 \end{pmatrix} \right) A (A^T A)^{-1}$$

$$= \underbrace{\sigma^2 (A^T A)^{-1}}_{\text{Variance without Autocorrelation.}} + \underbrace{\rho\sigma^2 (A^T A)^{-1} A^T (\quad) A (A^T A)^{-1}}_{\text{Influence of Autocorrelation}}$$

$\Rightarrow$ The variance of Estimator $\hat{\beta}$ is influenced

$$A^T \cdot A = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & T \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & T \end{pmatrix} = \begin{pmatrix} T & \frac{T(T+1)}{2} \\ \sum_{t=1}^{T} t = \frac{T(T+1)}{2} & \sum_{t=1}^{T} t^2 = \frac{T(T+2)(2T+2)}{6} \end{pmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{\frac{1}{T} \sum_{t=1}^{T} (t-\bar{t})^2} \begin{pmatrix} (T+1)(2T+1) & -(T+1) \\ -(T+1) & 1 \end{pmatrix}$$

$$\mathrm{Var}(\hat{\beta}_0) = \frac{2\sigma^2}{T^2(T-1)} \left(T(2T+1) + \rho 4 (T^2 - 2T - 2) \right) \xrightarrow{T \to \infty} 0$$

$$\mathrm{Var}(\hat{\beta}_1) = \frac{12\sigma^2}{(T-1)T(T+1)} \left(1 + \rho \cdot 2 \cdot \frac{T-2}{T+1} \right) \xrightarrow{T \to \infty} 0$$

$\Rightarrow$ Together with unbiasedness we conclude that

$\hat{\beta}_0$ & $\hat{\beta}_1$ are consistent.

problem 4

Autocorrelation

$\rho : \mathbb{Z} \to \mathbb{R}$, st $\hat{\rho}(h) = \dfrac{\frac{1}{T}\sum_{t=1}^{T-h}(y_{t+h}-\bar{y})(y_t-\bar{y})}{\frac{1}{T}\sum_{t=1}^{T}(y_t-\bar{y})^2}$

wlog $h > 0$

a-) $|\hat{\rho}(h)| \leq 1$, Cauchy-Swartz : let x, y be vectors from vector space with an inner product $\langle \cdot, \cdot \rangle$ then $|\langle x \cdot y \rangle|^2 \leq \langle x, y \rangle \langle y, y \rangle$

$$|\rho(h)|^2 \leq \frac{\langle \vec{y}_{t+h}^{\,T-h}, \vec{y}_t^{\,T-h} \rangle^2}{\langle y_t^T, y_t^T \rangle^2}$$

$$\vec{y}_{t+h}^{\,T-h} = \begin{pmatrix} y_{t+h}-\bar{y} \\ \\ \\ y_{T-h+h}-\bar{y} \end{pmatrix}, \quad y_t = \begin{pmatrix} y_t - \bar{y} \\ \vdots \\ y_{T-h}-\bar{y} \end{pmatrix}$$

$$|\rho(h)|^2 \leq \frac{\langle \vec{y}_{t+h}^{\,T-h}, \vec{y}_t^{\,T-h} \rangle^2}{\langle y_t^T, y_t^T \rangle^2} \underset{C\text{-}S}{\leq} \frac{\langle \vec{y}_{t+h}^{\,T-h}, \vec{y}_{t+h}^{\,T-h} \rangle \langle \vec{y}_t^{\,T-h}, \vec{y}_t^{\,T-h} \rangle}{\langle y_t^T, y_t^T \rangle \langle y_t^T, y_t^T \rangle} \leq$$

$$\underbrace{\frac{\langle y_t^T, y_t^T \rangle \langle y_t^T, y_t^T \rangle}{\langle y_t^T, y_t^T \rangle \langle y_t^T, y_t^T \rangle}}_{=1}$$