EXERCISE 13:

P1:

, $X_t = a_1 X_{t-1} + a_2 X_{t-2} + a_3 X_{t-3} + \varepsilon_t + c$

Yule Walker
$\Rightarrow$ $\underline{\gamma} = \underline{\underline{\Gamma}}\, \underline{\alpha}$, $\begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \gamma(3) \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$\overset{K}{\Rightarrow} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0{,}0777 \\ 0{,}0630 \\ 0{,}0777 \end{pmatrix}$

to find $\Leftrightarrow$ $E(X_t) = \alpha_1 E(X_{t-1}) + \alpha_2 E(X_{t-2}) + \alpha_3 E(X_{t-3}) + c$

$\Rightarrow E(X_t) = \dfrac{c}{1 + \alpha_1 - \alpha_2 + \alpha_3}$ $\Leftrightarrow$ $c = \mu(1 - a_1 - a_2 - a_3) = 25.72$

c) $\rho(4) = \dfrac{\gamma(4)}{\gamma_0} = 0{,}02$, $\rho(5) = \dfrac{\gamma(5)}{\gamma_0} = 0{,}01$ $\Rightarrow$ Yule walker

$\gamma(4) = a_1 \gamma(3) + a_2 \gamma(2) + a_3 \gamma(1) = 0{,}19$

$\gamma(5) = a_1 \gamma(4) + a_2 \gamma(3) + a_3 \gamma(2) = 0{,}14$

b) By taking variance on both sides we get
$\underline{\text{variance}}$
$\text{Var}(\cdot)$

$\sigma_\varepsilon^2 = \underbrace{\gamma(0)}_{\sigma^2 = 10} - \underbrace{\underline{\gamma}_3^T \underline{\underline{\Gamma}}_3 \underline{\gamma}_3}_{} = 9.81$

Problem 4: $AR(2) = X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t$, $\varepsilon_t \sim WN(0, \sigma^2)$

In order to be stationary, the roots of the characteristic polynomial $AR(2)$, have to be greater than 1, in absolute value.

Characteristic polynomial (corresponding to the linear filter)

$$1 - \phi_1 z - \phi_2 z^2 = f(z) \stackrel{!}{=} 0$$

$$z_{1/2} = \frac{-\alpha_1 \mp \sqrt{\alpha_1^2 + 4\alpha_2}}{2 \cdot \alpha_2}$$

, for $a, b, c \in \mathbb{C}$; $|a| > 1$, $|b| > 1$, $|ab| > 1$

We require $|z_1| > 1$ and $|z_2| > 1 \Rightarrow |z_1 \cdot z_2| > 1$

$$1 < |z_1 \cdot z_2| = \left|\frac{1}{4\alpha_2^2}\left(\alpha_1^2 - [\alpha_1^2 + 4\alpha_2]\right)\right| = \frac{1}{\alpha_2}$$

$\Leftrightarrow \left|\frac{1}{-\alpha_2}\right| > 1 \Leftrightarrow \alpha_2 \in (-1, 1)$, Now Yule-Walker and $|\rho(h)| < 1$ $\forall h \in \mathbb{Z}$.

YW: $\gamma = \Gamma \alpha$, more explicitly $\gamma(1) = \alpha_1 \gamma(0) + \alpha_2 \gamma(1)$

$\gamma(h) = \alpha_1 \gamma(h-1) + \alpha_2 \gamma(h-2)$

Dividing by $\gamma(0) \Rightarrow \rho(1) = \alpha_1 + \alpha_2 \cdot \rho(1)$

$\rho(h) = \alpha_1 \rho(h-1) + \alpha_2 \rho(h-2)$

$\Rightarrow \rho(1) = \frac{\alpha_1}{1-\alpha_2}$ and $\rho(2) = \frac{\alpha_1^2}{1-\alpha_2} + \alpha_2$

From $|\rho(1)| \leq 1$ we get $\left|\frac{\alpha_1}{1-\alpha_2}\right| \leq 1 \Leftrightarrow -1 + \alpha_2 \leq \alpha_1 \leq 1-$

G

$\iff \boxed{\alpha_2 - \alpha_1 \le 1}$ and $\alpha_1 + \alpha_2 \le 1$

↳ need to get rid of equalities here

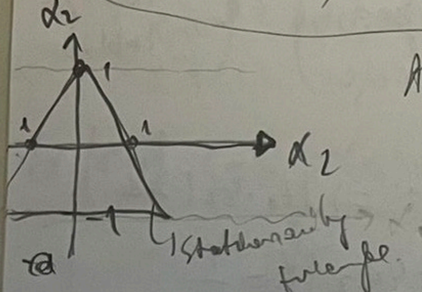
For $\alpha_2 - \alpha_1 = 1$ and $\alpha_1 + \alpha_2 = 1$ $\iff |\rho(1)| = 1$

we set $z_1 = 1 \not> 1 \Rightarrow$

take $\alpha_1 = 1 - \alpha_2$

we have to exclude the $= $

$\Rightarrow \boxed{\alpha_2 - \alpha_1 < 1, \ \alpha_1 + \alpha_2 < 1}$



AR(1) process $\Rightarrow X_t = \phi X_{t-1} + \varepsilon_t$

$\Rightarrow (1 - \phi B) X_t = \varepsilon_t$

$1 - \phi z = 0 \Rightarrow z = \frac{1}{\phi} \Rightarrow |\phi| < 1$

Problem 2: ARMA(1,1) $= X_t = \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$

$\iff (1 - \phi B) X_t = (1 + \theta B) \varepsilon_t$

$(1 + \theta B)^{-1} \underset{\substack{\text{condition} \\ \theta < 1}}{=} (1 - (-\theta) B)^{-1} = \sum_{j=0}^{\infty} (-\theta)^j B^j$

$\varepsilon_t = \left(\sum_{j=0}^{\infty} (-\theta)^j B^j \right) (X_t - \phi X_{t-1}) = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} - \phi \sum_{j=0}^{\infty} (-\theta)^j X_{t-1-j}$