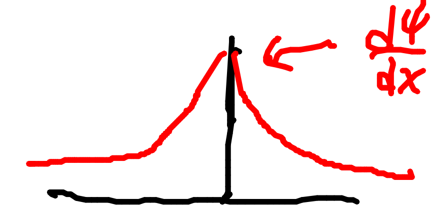
Dirac Delta well

$$-\frac{\hbar^2}{2m}\psi'' - \alpha\delta(x)\psi = E\psi$$

Bound states: $E < 0$

$$\psi(x) = \begin{cases} Be^{\kappa x} & x \le 0 \\ Be^{-\kappa x} & x \ge 0 \end{cases}$$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} = \frac{\sqrt{2m|E|}}{\hbar}$$



$\frac{d\psi}{dx}$ is not continuous

Integrate eigenstate equation from $-\epsilon$ to ϵ

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}\,dx - \alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx = E\int_{-\epsilon}^{\epsilon}\psi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\right]_{-\epsilon}^{\epsilon} - \alpha\,\psi(0) = E\,\psi(0)\,2\epsilon$$

$$\lim_{\epsilon\to 0} \quad -\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}(0_+) - \frac{d\psi}{dx}(0_-)\right] = \alpha\psi(0)$$

$$-\frac{\hbar^2}{2m}\left[(-\cancel{B}\kappa) - (\cancel{B}\kappa)\right] = \alpha\cancel{B}$$

$$\kappa = \frac{m\alpha}{\hbar^2} \qquad \text{but} \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$\rightarrow E = -\frac{m\alpha^2}{2\hbar^2}$$ a single bound state

Scattering State

$$-\frac{\hbar^2}{2m}\psi'' - \alpha\,\delta(x)\,\psi = E\psi$$

for $x<0$, $\psi(x) = Ae^{ikx} + Be^{-ikx}$ $\qquad k = \frac{\sqrt{2mE}}{\hbar}$

$x>0$, $\psi(x) = Fe^{ikx} + Ge^{-ikx}$

At $x=0$, $\psi(x)$ is continuous

$$A+B = F+G$$

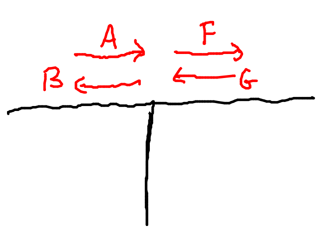
$$\frac{\partial\psi}{\partial x}(0_+) - \frac{\partial\psi}{\partial x}(0_-) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$$ik(F-G) - ik(A-B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$\boxed{\begin{aligned} &F-G = A(1+2i\beta) - B(1-2i\beta) \qquad \beta = \frac{m\alpha}{\hbar^2 k} \\ &F+G = A+B \end{aligned}}$$

e^{ikx}: wave moving to right

e^{-ikx}: wave moving to left



suppose A comes in.

some is transmitted (F)

some is reflected (B)

$G=0$

$$B = \frac{i\beta}{1-i\beta}A \quad \text{reflected}$$

$$F = \frac{1}{1-i\beta}A \quad \text{transmitted}$$

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} \quad \text{reflection coefficient}$$

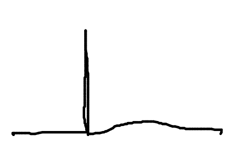
$$T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2} \quad \text{transmission coefficient}$$

notice $R+T=1$

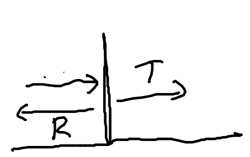
When is R large? When β is large $\quad \beta = \frac{m\alpha}{\hbar^2 k}$

- α is large
- ~~m is large~~
- k is small or E is small $\qquad k = \frac{\sqrt{2mE}}{\hbar}$
- m is large $\quad \beta \sim \sqrt{m}$

How about a delta barrier?



- $\alpha < 0$
- no bound state $\quad E<0$ & $V_{min} = 0$
- scattering states β^2 is same so R & T remain the same



tunnelling probability

$$T = \frac{1}{1+\beta^2}$$

Finite Square Well