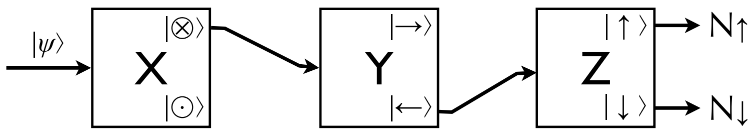
Physics 4310 Exam 1 Solutions

$\boxed{3}$ 1. **B** A large number of particles with arbitrary state $|\psi\rangle$ are run through three Stern-Gerlach apparatuses, as shown. $N_\uparrow$ particles leave the final S_z apparatus in the state $|\uparrow\rangle$ and $N_\downarrow$ particles leave in the state $|\downarrow\rangle$. Which of the following is true?
A) $N_\uparrow = 0$ **B)** $N_\uparrow \approx N_\downarrow$ **C)** $N_\downarrow = 0$



2. In the spin–1/2 system,

$\boxed{2}$ (a) **A** What is $\left|\langle \leftarrow \mid \rightarrow \rangle\right|^2$?
A) 0 **B)** $\frac{1}{2}$ **C)** $\frac{1}{\sqrt{2}}$ **D)** 1

$\boxed{2}$ (b) **B** What is $\left|\langle \downarrow \mid \leftarrow \rangle\right|^2$?
A) 0 **B)** $\frac{1}{2}$ **C)** $\frac{1}{\sqrt{2}}$ **D)** 1

$\boxed{3}$ 3. If $P_\rightarrow$ is the projection operator for $|\rightarrow\rangle$, what is $P_\rightarrow\Big(3\,|\rightarrow\rangle - 2\,|\leftarrow\rangle\Big)$?

$$3\,|\rightarrow\rangle$$

$\boxed{3}$ 4. If $|\psi\rangle = |\uparrow\rangle - i\,|\downarrow\rangle$ and $|\phi\rangle = 2i\,|\uparrow\rangle + 3\,|\downarrow\rangle$, find $\langle \psi \mid \phi \rangle$.

$$\langle \psi \mid \phi \rangle = (\langle\uparrow| + i\,\langle\downarrow|)(2i\,|\uparrow\rangle + 3\,|\downarrow\rangle)$$
$$= 2i + 3i = \boxed{5i}$$

5. Consider the vector $|\psi\rangle = 3i\,|{\uparrow}\rangle - |{\downarrow}\rangle$.

$\boxed{3}$ (a) Normalize $|\psi\rangle$.

Let $|\psi'\rangle = C\,|\psi\rangle$ be normalized, and C be real. We want

$$1 = \langle \psi' \mid \psi' \rangle = C^2(-3i\,\langle{\uparrow}| - \langle{\downarrow}|)(3i\,|{\uparrow}\rangle - |{\downarrow}\rangle) = C^2(9+1) = 10C^2$$

and so $C = \frac{1}{\sqrt{10}}$, and the normalized ket is

$$|\psi'\rangle = \frac{1}{\sqrt{10}}\left(3i\,|{\uparrow}\rangle - |{\downarrow}\rangle\right)$$

$\boxed{2}$ (b) Write $|\psi\rangle$ as a matrix in the S_z basis (normalized or not, as you like).

$$\begin{pmatrix} 3i \\ -1 \end{pmatrix}$$

$\boxed{2}$ (c) Write $\langle\psi|$ as a matrix in the S_z basis (normalized or not, as you like).

$$\begin{pmatrix} -3i & -1 \end{pmatrix}$$

6. A system's Hamiltonian has three energy states: $E_1 = 1\,\mathrm{J}$, $E_2 = 2\,\mathrm{J}$, and $E_3 = 3\,\mathrm{J}$. The system is in the state $|\psi\rangle = \frac{1}{2}|E_1\rangle - |E_2\rangle$, and its energy is then measured.

$\boxed{3}$ (a) **A** What is the probability that the measurement will return the value $1\,\mathrm{J}$?
A) $\frac{1}{5} = 20\%$ **B)** $\frac{1}{4} = 25\%$ **C)** $\frac{1}{\sqrt{5}} = 45\%$ **D)** $\frac{1}{2} = 50\%$

We need to use the normalized ket to calculate probabilities. Write $|\psi'\rangle = C|\psi\rangle$ as the normalized vector, and we want

$$1 = \langle \psi' \mid \psi' \rangle = C^2(\frac{1}{4} + 1) = \frac{5}{4}C^2 \implies C = \frac{2}{\sqrt{5}}$$

or $|\psi'\rangle = \frac{1}{\sqrt{5}}|E_1\rangle - \frac{2}{\sqrt{5}}|E_2\rangle$. Now the probability of getting E_1 is

$$P = \left|\langle E_1 \mid \psi' \rangle\right|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \boxed{\frac{1}{5}}$$

$\boxed{3}$ (b) What is the probability that the measurement will return the value $3\,\mathrm{J}$?

We can see that $\langle E_3 \mid \psi' \rangle = 0$, so the probability of returning E_3 is $\boxed{\text{zero}}$.

$\boxed{3}$ 7. **B** $S_x|\uparrow\rangle$ is equal to
A) $\frac{\hbar}{2}|\uparrow\rangle$ **B)** $\frac{\hbar}{2}|\downarrow\rangle$ **C)** $\frac{\hbar}{2}|\odot\rangle$ **D)** $\frac{\hbar}{2}|\otimes\rangle$ **E)** $\frac{\hbar}{2}$ **F)** 0 **G)** $-\frac{\hbar}{2}$

Matrices make this easiest. In the S_z basis:

$$S_x|\uparrow\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \doteq \frac{\hbar}{2}|\downarrow\rangle$$

$\boxed{3}$ 8. **B** Which of the following matrices could represent a measurement?
A) $\begin{pmatrix} i & 3 \\ -3 & -i \end{pmatrix}$ **B)** $\begin{pmatrix} 2 & -i+2 \\ i+2 & 2 \end{pmatrix}$ **C)** $\begin{pmatrix} i & 1-2i \\ 1-2i & -i \end{pmatrix}$ **D)** None of these (Explain)

B is the only one that's Hermitian, where $A = A^\dagger$.

9. Consider the operator $A \doteq \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ in the S_z basis. Note that $A^2 \doteq \begin{pmatrix} 10 & 9 \\ 9 & 13 \end{pmatrix}$.

$\boxed{3}$ (a) Find the expectation value $\langle A \rangle$ for the state $|\psi\rangle = |{\uparrow}\rangle$.

Since $|{\uparrow}\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\langle A \rangle = \langle\psi| A |\psi\rangle = (1\ 0) \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0) \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \boxed{1}$$

(Or you might remember that the upper-left corner of the matrix is $\langle{\uparrow}| A |{\uparrow}\rangle$ and read the answer off automatically.)

$\boxed{3}$ (b) Find the standard deviation ΔA for the state $|\psi\rangle = |{\uparrow}\rangle$.

The standard deviation is

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

We know that $\langle A \rangle = 1$, and we can see that $\langle A^2 \rangle = 10$ (upper-left corner), so

$$\Delta A = \sqrt{10 - (1)^2} = \sqrt{9} = \boxed{3}$$

$\boxed{2}$ 10. __D__ Suppose the spin of a particle is $|{\nearrow}\rangle$, halfway between $\uparrow$ and $\rightarrow$. What is the probability that $|{\nearrow}\rangle$, when measured by S_z, will give a value of $+\frac{\hbar}{2}$?
A) 0% **B)** 15% **C)** 50% **D)** 85% **E)** 100%

It's not 100%, and it's greater than 50% because $|{\rightarrow}\rangle$ has a probability of 50%. So 85% is the only logical answer.

$\boxed{3}$ 11. **B** I measured a particular state $|\psi\rangle$ using Stern-Gerlach equipment, and I got the following results:

$$\Delta S_y = \frac{\hbar}{2} \qquad \Delta S_z = \frac{2\hbar}{5} \qquad \langle S_x \rangle = \frac{2\hbar}{5}$$

Does this satisfy the uncertainty principle? (It's possible my results are wrong!) Prove your result.
A) yes **B)** yes, barely (= instead of >) **C)** no

According to the Uncertainty Principle,

$$\Delta S_y \Delta S_z \geq \left| \frac{1}{2i} \langle [S_y, S_z] \rangle \right|$$

Now $[S_y, S_z] = i\hbar S_x$, so we have

$$\begin{aligned}
\Delta S_y \Delta S_z &\geq \left| \frac{1}{2i} \langle [S_y, S_z] \rangle \right| \\
&\geq \left| \frac{1}{2i} i\hbar \langle S_x \rangle \right| \\
&\geq \frac{\hbar}{2} |\langle S_x \rangle| \\
\implies \left(\frac{\hbar}{2}\right)\left(\frac{2\hbar}{5}\right) &\geq \frac{\hbar}{2}\left(\frac{2\hbar}{5}\right)
\end{aligned}$$

Thus the inequality is satisfied as an equality.

$\boxed{3}$ 12. **A** If the Hamiltonian has an eigenstate $|E_1\rangle = \frac{3}{5}|\uparrow\rangle - \frac{4}{5}i|\downarrow\rangle$ with corresponding eigenvalue E_1, what is the probability that $|\uparrow\rangle$ has energy E_1 (when its energy is measured)?
A) 36% **B)** 40% **C)** 60% **D)** 64% **E)** 80%

The probability that $|\uparrow\rangle$ has energy E_1 is $P = \left|\langle E_1 \mid \uparrow \rangle\right|^2$. But $\langle E_1 \mid \uparrow \rangle = \langle \uparrow \mid E_1 \rangle^*$, and so the probability is the same with the terms reversed. Thus

$$P = \left|\langle \uparrow \mid E_1 \rangle\right|^2 = \left|\frac{3}{5}\right|^2 = \frac{9}{25} = 36\%$$

13. If $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ represents a measurement in a spin-1 system:

$\boxed{3}$ (a) __C__ If a particle in an arbitrary state $|\psi\rangle$ is measured by this operator, which of the following states could the particle be after it is measured by A? You can ignore normalization.

A) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ **B)** $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **C)** $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

The particle must be in one of A's eigenstates. I hope you tested each one to see which was the eigenvector, rather than solving for the eigenvectors directly.

$$\begin{pmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$$

Only the last one fits the form $Av = \lambda v$, and so it is the only eigenstate of the three.

$\boxed{3}$ (b) __C__ If the particle comes out in the state given by your answer to part (a), what value will A return?

A) 1 **B)** 4 **C)** 5 **D)** 17 **E)** $\hbar/2$

This is clear from the answer to part (a).

$\boxed{3}$ 14. A particle in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. It is placed in a magnetic field so that its Hamiltonian has the eigenvectors

$$H|\uparrow\rangle = E_0|\uparrow\rangle \quad \text{and} \quad H|\downarrow\rangle = 4E_0|\downarrow\rangle$$

. Write $|\psi(t)\rangle$.

$|\psi(0)\rangle$ is already written in terms of the energy basis $|\uparrow\rangle$ and $|\downarrow\rangle$, so we merely apply the Schrodinger factors:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-iE_0t/\hbar}|\uparrow\rangle + e^{-4iE_0t/\hbar}|\downarrow\rangle\right]$$

$\boxed{3}$ 15. Given the following:

$$\begin{aligned} |\psi\rangle &= |\uparrow\rangle - 2|\downarrow\rangle \\ |a_1\rangle &= \frac{\sqrt{3}}{2}|\uparrow\rangle + \frac{i}{2}|\downarrow\rangle \\ |a_2\rangle &= \frac{1}{2}|\uparrow\rangle - \frac{i\sqrt{3}}{2}|\downarrow\rangle \end{aligned}$$

Write $|\psi\rangle$ in the a_1, a_2 basis. (That is, in the form $|\psi\rangle = c_1|a_1\rangle + c_2|a_2\rangle$.)

In the a_i basis we can write

$$|\psi\rangle = \langle a_1 \mid \psi\rangle |a_1\rangle + \langle a_2 \mid \psi\rangle |a_2\rangle$$

It's faster to write this in matrix form, in the S_z basis:

$$\begin{aligned} \langle a_1 \mid \psi\rangle &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{i}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{\sqrt{3}}{2} + i \\ \langle a_2 \mid \psi\rangle &= \begin{pmatrix} \frac{1}{2} & +\frac{i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{2} - i\sqrt{3} \end{aligned}$$

and so

$$|\psi\rangle = \left(\frac{\sqrt{3}}{2} + i\right)|a_1\rangle + \left(\frac{1}{2} - i\sqrt{3}\right)|a_2\rangle$$

$\boxed{3}$ 16. Evaluate and fully simplify the commutator $[x^2, \frac{d}{dx}]$.

$$\left[x^2, \frac{d}{dx}\right] f = x^2\frac{df}{dx} - \frac{d}{dx}(x^2 f) = x^2\frac{df}{dx} - \left(2xf + x^2\frac{df}{dx}\right) = -2xf$$

and so $[x^2, \frac{d}{dx}] = \boxed{-2x}$

$\boxed{3}$ 17. If

$$\psi(x) = \begin{cases} x^2 - 1, & -1 \le x \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

find the expectation value of the momentum $\langle p \rangle$. ($\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$)

The expectation value is

$$\begin{aligned} \langle p \rangle &= \langle \psi | \, p \, | \psi \rangle \\ &= \int_{-1}^{1} \psi^* \frac{\hbar}{i}\frac{d}{dx}\psi \, dx \\ &= -i\hbar \int_{-1}^{1} (x^2 - 1)\frac{d}{dx}(x^2 - 1)\, dx \\ &= -i\hbar \int_{-1}^{1} (x^2 - 1)(2x)\, dx \end{aligned}$$

This is the integral of an odd function, so $\boxed{\langle p \rangle = 0}$.

$\boxed{3}$ 18. Explain why $\psi(x) = e^{-x}$ is not a valid wavefunction—*i.e.* why it can't describe the state of a system—over the range $-\infty < x < \infty$.

It's not normalizable.

$$\int_{-\infty}^{\infty} \left|e^{-x}\right|^2 dx = \int_{-\infty}^{\infty} e^{-2x}\, dx = -\frac{1}{2}\left[e^{-2x}\right]_{-\infty}^{\infty} = -\frac{1}{2}\left[0 - \infty\right] = \infty$$