Physics 4310 Homework #10
3 problems
Solutions

▷ **1.**
Write the wavefunction $\psi(x_1, x_2)$ of two non-interacting particles in a harmonic oscillator (see chapter 2.3, particularly equations 2.59 and 2.62) in the lowest possible energy eigenstate, if the particles are
(a) …distinguishable particles
(b) …bosons
(c) …fermions.

Answer:
Let $\psi_n(x)$ be the nth energy eigenstate of the harmonic oscillator, with $n = 0, 1, 2, \ldots$.
(a) For distinguishable particles, both particles will be in the ground state:

$$\begin{aligned}\psi(x_1, x_2) &= \psi_0(x_1)\psi_0(x_2) \\ &= \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}\end{aligned}$$

(b) Bosons will also both be in the ground state. The wavefunction must be symmetric, but the wavefunction from part (a) already is, so the answer is the same.

$$\psi(x_1, x_2) = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}$$

(c) For fermions, one particle will be in the ground state and one will be in ψ_1. The wavefunction must be antisymmetric, so

$$\begin{aligned}\psi(x_1, x_2) &= \frac{1}{\sqrt{2}}\left(\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)\right) \\ &= \sqrt{\frac{m\omega}{2\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\left(e^{-\frac{m\omega}{2\hbar}x_1^2} x_2 e^{-\frac{m\omega}{2\hbar}x_2^2} - e^{-\frac{m\omega}{2\hbar}x_2^2} x_1 e^{-\frac{m\omega}{2\hbar}x_1^2}\right) \\ &= \frac{m\omega}{\hbar\sqrt{\pi}} e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}\left(x_2 - x_1\right)\end{aligned}$$

▷ **2.**

Suppose I have two noninteracting particles of mass m in the infinite square well. How much larger is the square separation distance $\langle (x_1 - x_2)^2 \rangle$ if they are identical fermions, than if they are distinguishable particles?

Answer:

The wavefunctions in the infinite square well are

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

The square separation of two fermions is (Eq. 5.22)

$$\left\langle (\Delta x)^2 \right\rangle_f = \left\langle (\Delta x)^2 \right\rangle_d + 2|\langle x \rangle_{ab}|^2$$

where the first term is the square separation for distinguishable particles. Clearly we're looking for the second term. Now

$$\begin{aligned}\langle x \rangle_{ab} &= \int x \psi_a^*(x) \psi_b(x)\, dx \\ &= \frac{2}{a} \int_0^a x \sin\left(\frac{n_1 \pi}{a}x\right) \sin\left(\frac{n_2 \pi}{a}x\right)\, dx\end{aligned}$$

(The problem didn't specify what states the fermions are in, so let's do the general case with n_1 and n_2 specifying the two states. If you chose specific values of n_1 and n_2, that's fine.)

Plugging this into Mathematica, and replacing $\sin n_i \pi$ with zero and $\cos n_i \pi$ with $(-1)^{n_i}$, gives us

$$\langle x \rangle_{ab} = \frac{4a}{\pi^2} \frac{n_1 n_2}{(n_1^2 - n_2^2)^2} (-1 + (-1)^{n_1 + n_2})$$

If $n_1 + n_2$ is even (including if it's zero), then the last factor is zero and the whole thing is zero. This corresponds to a situation where the single-particle wavefunctions are both odd or both even. If $n_1 + n_2$ is odd, then the last factor is -2, and we have

$$\langle x \rangle_{ab} = -\frac{8 a n_1 n_2}{\pi^2 (n_1^2 - n_2^2)^2}$$

Thus the square separation increase for fermions is

$$\left\langle (\Delta x)^2 \right\rangle_f - \left\langle (\Delta x)^2 \right\rangle_d = \begin{cases} 0 & n_1 + n_2 \,\text{even} \\ \frac{128 a^2 n_1^2 n_2^2}{\pi^4 (n_1^2 - n_2^2)^4} & n_1 + n_2 \,\text{odd} \end{cases}$$

For the ground state, $n_1 = 1$ and $n_2 = 2$, so $n_1 + n_2$ is odd and we have the difference

$$\frac{128 a^2 (1)^2 (2^2)}{\pi^4 (1^2 - 2^2)^4} = \frac{512}{(3\pi)^4} a^2 = 0.065 a^2$$

which is about $(a/4)^2$, a rather substantial portion of the infinite square well. (Remember that the particles can't be farther than a apart.)

▷ **3.**

Consider the potential of evenly-spaced delta functions $V(x) = \alpha \sum_{j=0}^{N-1} \delta(x - ja)$ in a system with periodic boundary conditions $\psi(x + Na) = \psi(x)$. In class we said/will say that

$$\psi(x) = A \sin kx + B \cos kx, \qquad 0 < x < a$$

and that $\psi(x + a) = e^{iKa}\psi(x)$ where $K = \frac{2\pi n}{Na}$ for some integer n.

(a) Write the wavefunction $\psi(x)$ for the region $-a < x < 0$. *(This is in the book but it's worth working it out on your own, with the book as reference.)*

(b) Use the boundary conditions at $x = 0$ to show that

$$\cos Ka = \cos z + \beta \frac{\sin z}{z}$$

where $\beta = \frac{m\alpha a}{\hbar^2}$ and $z = ka$.

Answer:

(a) Call the wavefunction mentioned above ψ_R. We know that $\psi(x) = e^{-iKa}\psi(x + a)$, so the wavefunction between $-a$ and 0 is $\psi_R(x + a)$ multiplied by the exponential:

$$\psi_L(x) = Ae^{-iKa} \sin k(x + a) + Be^{-iKa} \cos k(x + a), \ -a < x < 0$$

(b) The wavefunction must be continuous at $x = 0$, so

$$\psi_L(0) = \psi_R(0)$$

$$Ae^{-iKa} \sin ka + Be^{-iKa} \cos ka = B$$

$$\implies A = B \frac{1 - e^{-iKa} \cos z}{e^{-iKa} \sin z}$$