Free Particle

$V(x) = 0 \qquad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$

$$k = \pm\frac{\sqrt{2mE}}{\hbar} \quad \text{continuous}$$

$$\Psi_k(x,t) = Ae^{i(kx - \frac{\hbar k^2}{2m}t)} \qquad \begin{matrix} k>0 & \text{right} \\ k<0 & \text{left} \end{matrix}$$

$$\int_{-\infty}^{\infty} |\Psi_k(x,t)|^2\,dx = \int_{-\infty}^{\infty} |A|^2\,dx = \infty$$

These energy eigenfunctions are <u>not</u> normalizable

$\therefore$ do not describe physical systems

But actual states of system $\psi(x)$ can be written as linear combinations of those.

$$\Psi(x,t) = \int_{-\infty}^{\infty} \overbrace{e^{i(kx - \frac{\hbar k^2}{2m}t)}}^{|E_k\rangle}\ \overbrace{\frac{1}{\sqrt{2\pi}}\phi(k)\,dk}^{c_k}$$

For right choices of $\phi(k)$, this can be normalized

wave packet: a combination of $|E_k\rangle$ for a range of k.

if $|\psi\rangle = \sum_k c_k |E_k\rangle$

how can I find c_j?

$$\langle E_j|\psi\rangle = \sum_k c_k \langle E_j|E_k\rangle$$
$$= c_j$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int \phi(k)e^{ikx}\,dk$$

$$\langle E_k|\Psi\rangle = \int_{-\infty}^{\infty} e^{-ikx}\,\Psi(x,0)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-ikx}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k')e^{ik'x}\,dk'\right)dx$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\phi(k')\left(\int_{-\infty}^{\infty} e^{-ikx}e^{+ik'x}\,dx\right)dk'$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{+ik'x}\,dx = \delta(k-k')$$

$$= \int_{-\infty}^{\infty} \phi(k')\,\delta(k-k')\,dk'$$

$$= \phi(k)$$

$$\Psi(x,t) = e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} = e^{ik\left(x - \frac{\hbar k}{2m}t\right)} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

has form $f(x - vt)$ $\qquad V_Q = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$

according to the wavefunction

This energy E must be kinetic energy

$$E = \frac{1}{2}mv^2 \longrightarrow V_c = \sqrt{\frac{2E}{m}} = 2V_Q$$

Discrepancy:

group velocity vs phase velocity

$$V_{phase} = \frac{\omega}{k}$$

$$e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} \qquad \omega = \frac{\hbar k^2}{2m}$$

$$\hookrightarrow e^{i(kx - \omega t)}$$

for this particular problem

dispersion relation

universal

$$V_{phase} = \frac{\omega}{k} = \frac{1}{k}\frac{\hbar k^2}{2m} = \frac{\hbar k}{2m}$$

$$V_{group} = \frac{d\omega}{dk} = 2\frac{\hbar k}{2m} = \frac{\hbar k}{m} = 2V_{phase}$$