Physics 3410 Homework #11

3 problems

Solutions

▷ **1.**
Consider a system of fermions at $T = 300\,\mathrm{K}$. What is the probability that an energy microstate is occupied, if its energy is $0.03\,\mathrm{eV}$ above μ?

Answer:
First note that $\beta = 1/kT = 1/(8.617 \times 10^{-5}\,\mathrm{eV/K})(300\,\mathrm{K}) = 38.7\,/\mathrm{eV}$. In Gibbs statistics, the probability that a system is in a particular state is the ratio of that state's Gibbs factor, to the grand partition function. In this case, the "state" in question is where one particle ($N = 1$) occupies a microstate $E = \mu + 0.03\,\mathrm{eV}$ (and thus has energy NE). The Gibbs factor is then

$$e^{-\beta(E-\mu)N} = e^{-(38.7\,/\mathrm{eV})(\mu+0.03\,\mathrm{eV}-\mu)1} = 0.313$$

There is only one other state, where the microstate is completely empty and $N = 0$: this has a Gibbs factor of $e^0 = 1$. Thus the grand partition function is $\mathcal{Z} = 1.313$ and the probability that the microstate is occupied is

$$P = \frac{0.313}{1.313} = \boxed{24\%}$$

Note that we're using "state" to mean two different things here. There's the microstate which is either occupied by a fermion or not: it plays the same role as the binding site on hemoglobin in that problem. Then there's the "state of the microstate": that is, whether it is empty or not. It's this *"state of the microstate" (or* state *for short) which has a Gibbs factor, and the grand partition function is calculated as the sum over all of these states. If you prefer, replace* microstate *everywhere with* energy level*, and see if that makes more sense.*

That's one way to do it. It's more general, which is why I started with it.

The other way is to remember that, for fermions, the average occupancy $\bar{n}$ of a microstate is the probability that it is occupied (because it can't be any larger than 1, because of the Pauli Exclusion Principle). We find $\bar{n}$ using the Fermi-Dirac distribuion:

$$\bar{n} = \frac{1}{e^{\beta(E-\mu)} + 1} = \frac{1}{e^{(38.7\,/\mathrm{eV})(0.03\,\mathrm{eV})} + 1} = 24\%$$

▷ **2.**
Consider a system of bosons at $T = 250\,\mathrm{K}$. Consider an energy microstate with energy $0.03\,\mathrm{eV}$ above μ.
(a) How many particles would you expect to find in this microstate, on average?
(b) What is the probability that exactly two particles are in this microstate?

Answer:

First note that $\beta = \frac{1}{(8.617\times10^{-5}\,\text{eV/K})(250\,\text{K})} = 46.4\,/\text{eV}$.

(a) Part (a) is asking for the average occupancy, given by the Bose-Einstein distribution:

$$\bar{n} = \frac{1}{e^{\beta(E-\mu)}-1} = \frac{1}{e^{(46.4\,/\text{eV})(\mu+0.03\,\text{eV}-\mu)}-1} = \frac{1}{3.02} = \boxed{0.33}$$

This suggests that this microstate is low-occupancy.

(b) We want to calculate the probability of the state where exactly two particles exist in this microstate. This, as in question 1, is the ratio of the state's Gibbs factor, to the grand partition function. Now the grand partition function of a system of bosons is already known to us:

$$\mathcal{Z} = \frac{1}{1-e^{-\beta(E-\mu)}} = \frac{1}{1-e^{-(46.4\,/\text{eV})(0.03\,\text{eV})}} = 2.7$$

This particular state has $N = 2$, and a total energy of $2E = 2(\mu + 0.03\,\text{eV})$, so the Gibbs factor is

$$e^{-\beta(E-\mu)N} = e^{-(46.4\,/\text{eV})(0.03\,\text{eV})2} = 0.40$$

Thus the probability that there are exactly two particles in this microstate is

$$P = \frac{0.40}{2.7} = \boxed{15\%}$$

▷ **3.**

Consider a system of two particles. Each particle can be in one of three possible microstates: one ground state ($E = 0$) and two excited states (both with $E = \mathcal{E}$).

(a) List the possible states that this system can be in if both particles are bosons. (For example, one such state has both particles in the ground state.) Then write the *partition function* Z of this system as a function of β.

(b) Now do the same thing if the particles are *fermions*.

Answer:

Let G be the ground state, and A & B be the two excited states.

(a) The possible states are $\{GG, GA, GB, AA, BB, AB\}$. (This is essentially the "put 2 balls in 3 boxes" problem, where the balls are particles and the boxes are microstates. Thus there are $\binom{N+q-1}{q} = \binom{3+2-1}{2} = \binom{4}{2} = \frac{4!}{2!2!} = 6$ possibilities.) The energy of GG is 0 (with a Boltzmann factor of $e^{-\beta(0)} = 1$), the energy of GA and GB are $\mathcal{E}$, and the energy of the other three are $2\mathcal{E}$. Therefore the partition function is

$$\boxed{Z = 1 + 2e^{-\beta\mathcal{E}} + 3e^{-2\beta\mathcal{E}}}$$

I didn't specify a temperature so we'll have to leave it at that.

(b) For fermions, three of the states above are impossible, and we're left with the states $\{GA, GB, AB\}$. (This is a straight-up combination problem: choose two spots for two particles. $\binom{3}{2} = 3$.) Two of them have energy $\mathcal{E}$ and the third has energy $\mathcal{E}$, so

$$\boxed{Z = 2e^{-\beta\mathcal{E}} + e^{-2\beta\mathcal{E}}}$$