$|\psi\rangle$

All states of a QM system

form a vector space

$|a\rangle + |b\rangle = |b\rangle + |a\rangle$

$c|a\rangle$

<u>Hilbert space</u>: vector space with an inner product or dot product

Spin-$\frac{1}{2}$ system forms a 2D Hilbert space

Hilbert spaces have a complete orthonormal basis

e.g. 3D spatial vector space (normal vectors $\nearrow$)

$\hat{x}, \hat{y}, \hat{z}$ form a complete o.n. basis

$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{x}\cdot\hat{z} = 0$, orthogonal

$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$ normal

$\forall \vec{A}$, $\vec{A} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ completeness

In spin-$\frac{1}{2}$ system, $|\uparrow\rangle$ & $|\downarrow\rangle$ form a basis (or $|\odot\rangle$ and $|\otimes\rangle$)

all $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ for some a & b complex numbers

Inner product is written $\langle\phi|\psi\rangle$

$|\psi\rangle$ ket $\langle\phi|$ bra

orthogonality $\langle\uparrow|\downarrow\rangle = 0$.

normalization $\langle\uparrow|\uparrow\rangle = 1 = \langle\downarrow|\downarrow\rangle$

$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ $(3+5i)^* = 3-5i$

$\langle\psi| = a^*\langle\uparrow| + b^*\langle\downarrow|$

$$\langle\psi|\psi\rangle = aa^*\overset{1}{\langle\uparrow|\uparrow\rangle} + a^*b\overset{0}{\langle\uparrow|\downarrow\rangle} + ab^*\overset{0}{\langle\downarrow|\uparrow\rangle} + bb^*\overset{1}{\langle\downarrow|\downarrow\rangle}$$
$$= aa^* + bb^*$$
$$= |a|^2 + |b|^2$$

Note $\langle\uparrow|\psi\rangle = a$ & $\langle\downarrow|\psi\rangle = b$

$$|\psi\rangle = \langle\uparrow|\psi\rangle|\uparrow\rangle + \langle\downarrow|\psi\rangle|\downarrow\rangle$$

$$\langle\psi|\uparrow\rangle = \left(a^*\langle\uparrow| + b^*\langle\downarrow|\right)|\uparrow\rangle$$
$$= a^*\langle\uparrow|\uparrow\rangle + b^*\langle\downarrow|\uparrow\rangle$$
$$= a^* = \langle\uparrow|\psi\rangle^*$$

In general $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

Postulate 1: The state of a QM system including all the information you can know about it, is represented by a normalized ket $|\psi\rangle$

$$\langle\psi|\psi\rangle = 1 = |a|^2 + |b|^2$$

Why normalized?

e.g. take $|\psi\rangle$ and apply the S_z measurement.

The outcome will be one of a orthonormal basis associated with S_z. $|\uparrow\rangle$ & $|\downarrow\rangle$

Probability of coming out $\uparrow$ is $|\langle\uparrow|\psi\rangle|^2 = |a|^2$

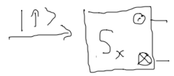
" " " " $\downarrow$ is $|\langle\downarrow|\psi\rangle|^2 = |b|^2$

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 100\% = 1$$ because there are no other outcomes

$\langle\uparrow|\psi\rangle$ is called the probability amplitude

it can be complex

probability $|\langle\uparrow|\psi\rangle|^2$ is real and nonnegative



$\frac{1}{2} = |\langle\odot|\uparrow\rangle|^2$

$\frac{1}{2} = |\langle\otimes|\uparrow\rangle|^2$

$$|\odot\rangle = a|\uparrow\rangle + b|\downarrow\rangle \qquad \text{complete basis}$$

$$|\otimes\rangle = c|\uparrow\rangle + d|\downarrow\rangle$$

$$|\langle\odot|\odot\rangle|^2 = |a|^2 + |b|^2 = 1 \quad \& \quad |c|^2 + |d|^2 = 1$$

$$|\langle\odot|\uparrow\rangle|^2 = \frac{1}{2} = |a|^2$$

$\langle\odot|\uparrow\rangle$

$(a^*\langle\uparrow| + b^*\langle\downarrow|)|\uparrow\rangle$

a^*

$$|b|^2 = \frac{1}{2} = |c|^2 = |d|^2$$

$$a = |a|e^{i\alpha} \qquad b = |b|e^{i\beta}$$

$$|\odot\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle e^{i\alpha} + |\downarrow\rangle e^{i\beta}\right)$$

$$|\otimes\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle e^{i\gamma} + |\downarrow\rangle e^{i\delta}\right)$$

- overall phase of a ket doesn't matter

$$|\odot\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + e^{i\beta}|\downarrow\rangle\right) \qquad \alpha = 0$$

$$|\otimes\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + e^{i\delta}|\downarrow\rangle\right)$$

$$\langle\odot|\otimes\rangle = \frac{1}{2}\left(1 + e^{i(\beta-\delta)}\right) = 0$$

$$e^{i\beta} = -e^{i\delta}$$

I get to choose $\beta = 0$.

For $|\odot\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

$|\otimes\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$