Physics 3410 Homework #5

5 problems

Solutions

▷ **1.**

Find an expression for the temperature of a paramagnet with N dipoles and U dipoles pointing upward (with an external magnetic field pointing down). What is the temperature when half the dipoles point upward?

Answer:

Let's measure energy in units of μB, so that U is the energy of the paramagnet. The number of ways you can have U dipoles pointing upward is a combination:

$$
\begin{aligned}
\Omega(N, U) &= \binom{N}{U} \\
\implies S &= k \ln \Omega = k \ln \binom{N}{U} \\
\implies \frac{1}{T} &= \frac{\partial S}{\partial U} = k \frac{\partial}{\partial U} \ln \binom{N}{U} \\
&= k \frac{\partial}{\partial U} \left[\ln N! - \ln U! - \ln(N-U)! \right] \\
&= k \frac{\partial}{\partial U} \left[(N \ln N - N) - (U \ln U - U) - ((N-U)\ln(N-U) - (N-U)) \right] \\
&= k \frac{\partial}{\partial U} \left[N \ln N - U \ln U - (N-U)\ln(N-U) \right] \\
&= k \left[-(\ln U + 1) - (-\ln(N-U) - 1) \right] \\
&= k \left[-\ln U - 1 + \ln(N-U) + 1 \right] \\
&= k \ln \frac{N-U}{U} \qquad \text{From a previous week's homework} \\
\implies T &= \boxed{\left(k \ln \frac{N-U}{U} \right)^{-1}}
\end{aligned}
$$

When half the dipoles point upward, $U = \frac{1}{2}N$, and $\frac{N-U}{U} = \frac{N-\frac{1}{2}N}{\frac{1}{2}N} = 1$, so

$$
T = \left(k \ln \frac{N - \frac{1}{2}N}{\frac{1}{2}N} \right)^{-1} = (k \ln 1)^{-1} = 0^{-1} = \boxed{\infty}
$$

We'll be talking more about the paramagnet in class, as it is a bizarre little system.

▷ **2.**
A container of water begins with a heat capacity of $500\,\mathrm{J/K}$ (constant) and a temperature of $70^\circ\,\mathrm{C}$. It cools in a room at $10^\circ\,\mathrm{C}$ until it too reaches that temperature; the temperature of the room does not change.
(a) What is the change in the entropy of the room during the cooling? (It's not zero.)
(b) What is the change in the entropy of the water? (It's not zero either.)
(c) What is the net change in the entropy of the Universe? Does it obey the Second Law?

Answer:
The cooling is definitely slow and quasistatic, so the change in entropy for a small bit of heat is $dS = \dfrac{Q}{T}$. The water drops by $\Delta T = 60\,\mathrm{K}$, which means that it must lose $Q = C_V \Delta T = (500\,\mathrm{J/K})(60\,\mathrm{K}) = 30\,\mathrm{kJ}$ of energy to the room.

(a) The temperature of the room is a constant $T = 10^\circ\,\mathrm{C} = 283\,\mathrm{K}$ throughout the cooling, so the change in entropy of the room is

$$\Delta S = +\frac{30\,\mathrm{kJ}}{283\,\mathrm{K}} = \boxed{+106\,\mathrm{J/K}}$$

It's positive because heat flows into the room.

(b) The water's temperature is not constant, so to find the total change of entropy we need to integrate:

$$\begin{aligned}
dS &= \frac{Q}{T} = \frac{C_V\,dT}{T} \\
\Longrightarrow \quad \Delta S &= \int_{T_i}^{T_f} \frac{(500\,\mathrm{J/K})dT}{T} \\
&= (500\,\mathrm{J/K})\,[\ln T]_{343\,\mathrm{K}}^{283\,\mathrm{K}} \\
&= (500\,\mathrm{J/K})\ln\left(\frac{283}{343}\right) = \boxed{-96\,\mathrm{J/K}}
\end{aligned}$$

The water loses entropy because heat leaves it.

(c) The net change of entropy of the universe is the sum of the two, or $\boxed{10\,\mathrm{J/K}}$. This does obey the Second Law, which says that entropy can be created but not destroyed.

▷ **3.**
Consider an ideal gas with N particles that increases in volume by 10%, at a constant temperature T (and energy U).
(a) What is the change in the gas's entropy?
(b) What is the work done by the gas on the environment?
(c) What is the heat flow Q? Does it flow in or out? How is it related to the entropy?

Answer:

(a) The entropy of an ideal gas, according to the Sackur-Tetrode equation, is

$$S = kN \left[\ln \frac{V}{N} + \frac{3}{2} \ln \frac{U}{N} + C \right]$$

If U and N remain constant, then only the first term changes, and we have

$$\begin{aligned} \Delta S = S_f - S_i &= kN \left[\ln \frac{V_f}{N} - \ln \frac{V_i}{N} \right] \\ &= kN \ln \frac{V_f}{V_i} \end{aligned}$$

In this case, $V_f = 1.1 V_i$ (10% bigger), so

$$\Delta S = \boxed{kN \ln 1.1} = 0.095 kN$$

(b) The work done on the gas during expansion is given by

$$W = -\int_{V_i}^{V_f} P \, dV$$

Now P is not constant during this expansion, but temperature is. However, we know that

$PV = NkT$, so $P = \frac{NkT}{V}$, and NkT *is* a constant. Thus

$$\begin{aligned} W &= -\int_{V_i}^{V_f} \frac{NkT}{V}\,dV \\ &= -NkT \int_{V_i}^{V_f} \frac{dV}{V} \\ &= -NkT \ln \frac{V_f}{V_i} \\ &= \boxed{-NkT \ln 1.1} \end{aligned}$$

(c) Since the total energy U is constant, and $\Delta U = W + Q$, the heat that flows into the gas during this expansion is $Q = -W = \boxed{+NkT \ln 1.1}$. This is positive, so heat flows $\boxed{\text{into}}$ the gas. (An expanding gas loses energy by doing work on the environment, so it must take heat in to maintain a constant U.) Notice that

$$Q = T\Delta S$$

here (as it normally does).

▷ **4.**
Given the thermodynamic identity:

$$dU = T\,dS - P\,dV + \mu\,dN$$

(a) Find $\left(\frac{\partial U}{\partial V}\right)_{S,N}$
(b) Suppose a system at standard temperature and pressure contracts from $0.50\,\mathrm{m}^3$ to $0.48\,\mathrm{m}^3$, and the energy increases by 2 J. What is the change in the system's entropy?

Answer:________________________________

(a) At constant S and N, $dS = dN = 0$, and

$$dU = -P\,dV \implies \frac{dU}{dV} = -P$$

and so

$$\left(\frac{\partial U}{\partial V}\right)_{S,N} = -P$$

(b) We can use the thermodynamic identity to find dS, the change in the entropy:

$$\begin{aligned}
dS &= \frac{1}{T}dU + \frac{P}{T}dV \\
&= \frac{1}{300\,\mathrm{K}}(2\,\mathrm{J}) + \frac{10^5\,\mathrm{Pa}}{300\,\mathrm{K}}(-0.02\,\mathrm{m}^3) \\
&= 6.67\times 10^{-3}\,\mathrm{J/K} - 6.67\,\mathrm{J/K} \\
&= \boxed{-6.667\,\mathrm{J/K}}
\end{aligned}$$

▷ **5.**

If $a = b\ln c$ and $b = \ln(cd)$, find $\left(\frac{\partial a}{\partial c}\right)_b$ and $\left(\frac{\partial a}{\partial c}\right)_d$ in terms of b and c.

Answer:____________________________________

$$\frac{da}{dc} = \frac{db}{dc}\ln c + b\frac{d\ln c}{dc} = \frac{db}{dc}\ln c + \frac{b}{c}$$

$$\left(\frac{\partial a}{\partial c}\right)_b = \boxed{\frac{b}{c}}$$

$$\left(\frac{\partial b}{\partial c}\right)_d = \frac{1}{c}$$

$$\left(\frac{\partial a}{\partial c}\right)_d = \boxed{\frac{\ln c}{c} + \frac{b}{c}}$$