$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x \mp ip)$$

if $H\psi = E\psi$ $\quad H(a_+\psi) = (E+\hbar\omega)\psi \quad H(a_-\psi) = (E-\hbar\omega)\psi$

$$a_-\psi_0 = 0 \qquad \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega/2\hbar\, x^2}$$

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n \psi_0(x) \qquad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\langle x^2\rangle = \langle \psi_n | x^2 | \psi_n\rangle$$

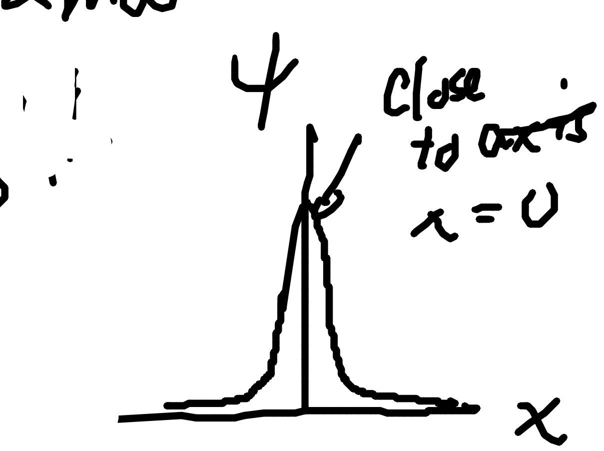
$$\boxed{\begin{aligned} a_+\psi_n &= \sqrt{n+1}\,\psi_{n+1} \\ a_-\psi_n &= \sqrt{n}\,\psi_{n-1} \end{aligned}}$$

$$= \int \psi_n^* \frac{\hbar}{2m\omega}(a_+ + a_-)^2 \psi_n \, dx$$

$$= \frac{\hbar}{2m\omega}\int \psi_n^* (a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)\psi_n \, dx$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

small $\langle x^2\rangle$



big $\langle x^2\rangle$

$$\langle x^2\rangle \propto (2n+1)$$

Analytic Method $\quad -\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad K = \frac{E}{\frac{1}{2}\hbar\omega}$$

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$

At large ξ, $\quad \psi'' \approx \xi^2\psi \rightarrow \psi(\xi) \approx A e^{-\xi^2/2} + B e^{+\xi^2/2}$

$$\frac{d}{d\xi} e^{-\xi^2/2} = -\xi e^{-\xi^2/2}$$

$$\frac{d^2}{d\xi^2} e^{-\xi^2/2} = \Big[\underbrace{-e^{-\xi^2/2}}_{\text{irrelevant}} + \underbrace{\xi^2 e^{-\xi^2/2}}_{\text{big}}\Big]$$

So $\psi(\xi) = h(\xi)\, e^{-\xi^2/2}$

$$\frac{d^2\psi}{d\xi^2} = \left(\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (\xi^2-1)h\right)\cancel{e^{-\xi^2/2}} = (\xi^2 - K)h\,\cancel{e^{-\xi^2/2}}$$

$$h'' - 2\xi h' + (K-1)h = 0$$

$$h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{j=0}^{\infty} a_j \xi^j$$

$$h'(\xi) = \sum_{j=0}^{\infty} j a_j \xi^{j-1}$$

$$h''(\xi) = \sum_{j=0}^{\infty} j(j-1)a_j \xi^{j-2} = \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}\xi^j$$

$$h'' - 2\xi h' + (K-1)h = 0$$

$$h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{j=0}^{\infty} a_j \xi^j$$

$$h'(\xi) = \sum_{j=0}^{\infty} j a_j \xi^{j-1}$$

$$h''(\xi) = \sum_{j=0}^{\infty} j(j-1) a_j \xi^{j-2} = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} \xi^j$$

$$\sum_{j=0}^{\infty}\left((j+2)(j+1) a_{j+2} \xi^j - 2\xi j a_j \xi^{j-1} + (K-1) a_j \xi^j \right) = 0$$

$$\sum_{j=0}^{\infty}\left((j+2)(j+1) a_{j+2} - 2j a_j + (K-1) a_j \right)\xi^j = 0$$

$$\rightarrow \quad (j+2)(j+1) a_{j+2} - 2j a_j + (K-1) a_j = 0$$

$$a_{j+2} = a_j \frac{2j+1-K}{(j+1)(j+2)}$$

if $j \gg 1$

$$a_{j+2} \approx \frac{2}{j} a_j$$

$$a_{j+4} \approx \frac{2}{(j+2)} a_{j+2} = \frac{4}{j(j+2)} a_j$$

$$a_{j+6} \approx \frac{8}{j(j+2)(j+4)} a_j$$

$\sum \frac{1}{n!} x^n = e^x$

$$\longrightarrow a_j \sim \frac{C}{(j/2)!}$$

$$h(\xi) = C\sum \frac{1}{(j/2)!} \xi^j = C \sum \frac{1}{j!} \xi^{2j} = e^{\xi^2}$$

$$\psi(\xi) = h(\xi) e^{-\xi^2/2} \sim e^{\xi^2/2} \xrightarrow[\xi\to\infty]{} \infty$$

not normalizable

$$a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)} a_j$$

if $a_j = 0$, then ~~seq~~ sequence dies ($a_{j+2n} = 0 \ \forall n \in 1,2,3,\ldots$)

Set $K = 2n+1$ for some value of n

e.g. $n = 2 \rightarrow 5$

$$a_2 = \frac{2(0)+1-\overset{5}{K}}{(0+1)(0+2)} a_0 = \frac{1-\overset{5}{K}}{2} a_0$$

$$a_4 = \frac{2(2)+1-5}{3\cdot 4} a_2 = 0\, a_2 = 0$$

how about $a_3 = \frac{2(1)+1-5}{(1+1)(1+2)} a_1 = \frac{3-5}{6} a_1$

$$a_5 = \frac{2(3)+1-5}{(3+1)(3+2)} a_3 = \frac{2}{20} a_3 \quad \text{etc} \ldots$$

the a_1, a_3, a_5 sequence doesn't end if $n=2$.
so choose $a_1 = 0$.

$K = 2n+1$ if n even, $a_1 = 0$
if n odd, $a_0 = 0$

$$K = \frac{E}{\frac{1}{2}\hbar\omega} = 2n+1$$

$$\rightarrow E = \frac{1}{2}\hbar\omega + n\hbar\omega.$$

~~These~~ Energy eigenstates are parametrized by $n = 0,1,2,\ldots$

$$\psi_n(\xi) = h_n(\xi) e^{-\xi^2/2} \qquad E_n = (n+\tfrac{1}{2})\hbar\omega$$

$h_n(\xi)$ are Hermite polynomials $H_n(\xi)$

Free Particle

$V(x) = 0.$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \longrightarrow \psi = A\sin kx + B\cos kx$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \text{wavenumber} = \frac{2\pi}{\lambda}$$

$$\psi_{(x)} = Ae^{ikx} + Be^{-ikx} \qquad (\text{different } A, B)$$

Any energy values are allowed, any value of k

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\frac{E}{\hbar} = \frac{\hbar k^2}{2m} = k\left(\frac{\hbar k}{2m}\right)$$

$$\Psi(x,t) = \left(Ae^{ikx} + Be^{-ikx}\right)e^{-iEt/\hbar}$$

$$= Ae^{ik(x - \frac{\hbar k}{2m}t)} + Be^{-ik(x + \frac{\hbar k}{2m}t)}$$

this term is a function of $x - vt$ where $v = \frac{\hbar k}{2m}$

this is a pulse moving to the right with speed v

pulse moving to the left with speed v.

pulse maintains its shape

$$\Psi_k(x,t) = Ae^{i(kx - \frac{\hbar k^2}{2m}t)}$$

$$k = \pm\frac{\sqrt{2mE}}{\hbar} \qquad \begin{array}{l} k>0 \text{ right} \\ k<0 \text{ left} \end{array}$$