

# Ideal Gas

2 particles in a 3D box with volume  $V$

$$\Omega = \Omega_{\text{pos}} \Omega_{\text{mom}}$$

$$\Omega_{\text{pos}} \propto V^2$$

total (kinetic)  
energy of  
both particles  
conserved

$$p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + p_{2x}^2 + p_{2y}^2 + p_{2z}^2 = 2mU$$

$\Omega_{\text{mom}} \propto$  surface area of a 6-dimensional sphere with radius  $\sqrt{2mU}$

Surface area  
of an  $n$ -dim  
sphere,  
radius  $R$

$$S_n(R) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} R^{n-1}$$

Gamma Function

$$\Gamma(m) = 2 \int_0^\infty e^{-r^2} r^{2m-1} dr$$

$$\Gamma(1) = 2 \int_0^\infty e^{-r^2} r dr \stackrel{u=r^2, du=2r dr}{=} 1$$

$$\Gamma(m+1) = m \Gamma(m)$$

$$m! = m(m-1)!$$

$$\rightarrow \boxed{\Gamma(m) = (m-1)!} \quad \text{when } m \text{ is a positive integer}$$

$$\Gamma(1) = 0! = 1$$

$$\Gamma(2) = 1 \Gamma(1) = 1$$

$$\Gamma(3) = 2 \Gamma(2) = 2$$

$$\Gamma(4) = 3 \Gamma(3) = 6$$

$\Gamma(x)$  defined for all real numbers.

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-r^2} r^{2(\frac{1}{2})-1} dr = 2 \int_0^\infty e^{-r^2} dr$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(m) = (m-1)!$$

$$\Gamma(m+1) = m \Gamma(m)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$S_n(R) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} R^{n-1}$$

$$S_3(R) = \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})} R^2 = \frac{2\pi}{\frac{1}{2}} R^2 = 4\pi R^2$$

$\Gamma(\frac{3}{2}) \rightarrow \frac{\sqrt{\pi}}{2}$

$$S_2(R) = \frac{2\pi^{2/2}}{\Gamma(\frac{2}{2})} R = \frac{2\pi}{1} R = 2\pi R \quad \text{circumference}$$

$$S_1(R) = \frac{2\pi^{1/2}}{\Gamma(\frac{1}{2})} 1 = 2$$

$$S_6(R) = \frac{2\pi^{6/2}}{\Gamma(\frac{6}{2})} R^{6-1} = \frac{2\pi^3}{2} R^5$$

$$S_6(R) = \pi^3 R^5 \rightarrow \Omega_{\text{mom}} \propto \pi^3 (2mU)^{5/2}$$

2 distinguishable particles  $\Omega = \frac{V^2 \pi^3 (2mU)^{5/2}}{h^6}$   $\leftarrow h = \Delta x \Delta p$



2 indistinguishable particles  $\Omega = \frac{1}{2!} \frac{V^2 \pi^3 (2mU)^{5/2}}{h^6}$

$N$  indistinguishable particles  $\Omega = \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{2 \pi^{3N/2}}{\Gamma(\frac{3N}{2})} (2mU)^{\frac{3N-1}{2}}$

if  $N \gg 1$ ,  $\ln \Gamma(\frac{3N}{2}) \approx \ln(\frac{3N}{2} - 1)! \approx \ln(\frac{3N}{2})! = \frac{3N}{2} \ln \frac{3N}{2} - \frac{3N}{2}$   
 $\Gamma(\frac{3N}{2}) \approx \left(\frac{3N/2}{e}\right)^{3N/2}$

$N \gg 1$   $\Omega \approx f(N) \frac{V^N}{h^{3N}} U^{\frac{3N}{2}}$   
 $f(N) = \frac{1}{N!} \frac{1}{h^{3N}} \frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})} (2m)^{3N/2}$   
 $\leftarrow$  if  $N$  is constant, this is constant  $\rightarrow$

$\Omega$  usually shows up in ratios,  
 so overall constants cancel out

e.g. Probability that all molecules are in left half of the room?

$$P = \frac{\Omega(\text{left half})}{\Omega(\text{all})} = \frac{\cancel{f(N)} \left(\frac{V}{2}\right)^N U^{3N/2}}{\cancel{f(N)} V^N U^{3N/2}} = \frac{1}{2^N}$$

$$S = k \ln \Omega$$

$$S = k \left[ \ln f(N) + N \ln \bar{V} + \frac{3}{2} N \ln U \right]$$

$$S = kN \left[ \ln \frac{\bar{V}}{N} + \frac{3}{2} \ln \frac{U}{N} + C \right]$$

$$C = \frac{5}{2} + \frac{3}{2} \ln \frac{4\pi m}{3h^2}$$

Sackur-Tetrode equation  
 → entropy of an ideal gas

$S$  increases with  $N, \bar{V}, U$