Ideal Gas

2 particles in a 3D box with volume V

$$\Omega = \Omega_{pos}\,\Omega_{mom}$$

$$\Omega_{pos} \propto V^2$$

total (kinetic) energy of both particles conserved

$$p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + p_{2x}^2 + p_{2y}^2 + p_{2z}^2 = 2mU$$

$\Omega_{mom} \propto$ surface area of a 6-dimensional sphere with radius $\sqrt{2mU}$

Surface area of an n-dim sphere, radius R : $S_n(R) = \dfrac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} R^{n-1}$

Gamma Function

$$\Gamma(m) = 2\int_0^\infty e^{-r^2} r^{2m-1}\,dr$$

$$\Gamma(1) = 2\int_0^\infty e^{-r^2} r\,dr \overset{\substack{u=r^2\\ du=2r\,dr}}{=} 1$$

$$\Gamma(m+1) = m\Gamma(m)$$

$$m! = m(m-1)!$$

$\rightarrow \boxed{\Gamma(m) = (m-1)!}$ when m is a positive integer

$$\Gamma(1) = 0! = 1$$

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2$$

$$\Gamma(4) = 3\Gamma(3) = 6$$

$\vdots$

$\Gamma(x)$ defined for all real numbers.

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-r^2} r^{2(\frac{1}{2})-1}\,dr = 2\int_0^\infty e^{-r^2}\,dr$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4}$$

$$\boxed{\begin{aligned}&\Gamma(m) = (m-1)!\\ &\Gamma(m+1) = m\Gamma(m)\\ &\Gamma(\tfrac{1}{2}) = \sqrt{\pi}\end{aligned}}$$

$$S_n(R) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} R^{n-1}$$

$$S_3(R) = \frac{2\pi^{3/2}}{\underbrace{\sqrt{\pi}/2}_{\Gamma(\frac{3}{2})}} R^2 = \frac{2\pi}{1/2} R^2 = 4\pi R^2$$

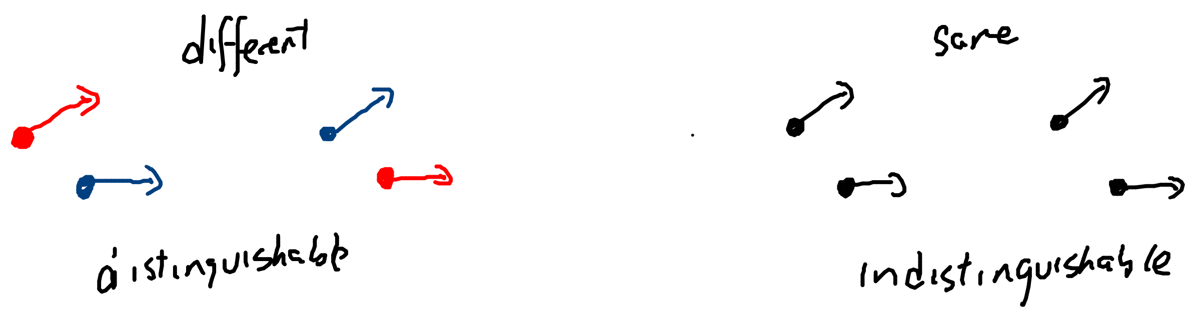
$$S_2(R) = \frac{2\pi^{2/2}}{\Gamma(\frac{2}{2})} R = \frac{2\pi}{1} R = 2\pi R \quad \text{circumference}$$

$$S_1(R) = \frac{2\cancel{\pi^{1/2}}}{\cancel{\Gamma(\frac{1}{2})}} 1 = 2.$$

$$S_6(R) = \frac{2\pi^{6/2}}{\Gamma(\frac{6}{2})} R^{6-1} = \frac{2\pi^3}{2} R^5$$

$$S_6(R) = \pi^3 R^5 \rightarrow \Omega_{mom} \propto \pi^3 (2mU)^{5/2}$$

2 distinguishable particles $\Omega = \dfrac{V^2 \pi^3 (2mU)^{5/2}}{h^6}$ ← $h = \Delta x \Delta p$



2 indistinguishable particles $\Omega = \dfrac{1}{2!} \dfrac{V^2 \pi^3 (2mU)^{5/2}}{h^6}$

N indistinguishable particles

$$\Omega = \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{2\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}\right)} (2mU)^{\frac{3N-1}{2}}$$

if $N \gg 1$, $\ln \Gamma\left(\frac{3N}{2}\right) \simeq \ln\left(\frac{3N}{2} - 1\right)! \approx \ln\left(\frac{3N}{2}\right)! = \frac{3N}{2}\ln\frac{3N}{2} - \frac{3N}{2}$

$$\Gamma\left(\frac{3N}{2}\right) \approx \left(\frac{3N/2}{e}\right)^{3N/2}$$

$N \gg 1$ $\quad \Omega \approx f(N)\, V^N U^{\frac{3N}{2}}$ $\quad f(N) = \dfrac{1}{N!} \dfrac{1}{h^{3N}} \dfrac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}\right)} (2m)^{3N/2}$

if N is constant, this is constant

- Ω usually shows up in ratios, so overall constants cancel out

e.g. Probability that all molecules are in left half of the room?

$$P = \frac{\Omega(\text{left half})}{\Omega(\text{all})} = \frac{\cancel{f(N)}\left(\frac{V}{2}\right)^N \cancel{U^{3N/2}}}{\cancel{f(N)}\, V^N \cancel{U^{3N/2}}} = \frac{1}{2^N}$$

$$S = k \ln \Omega$$

$$S = k\left[\ln f(N) + N \ln V + \frac{3}{2} N \ln U\right]$$

$$\boxed{S = kN\left[\ln \frac{V}{N} + \frac{3}{2} \ln \frac{U}{N} + C\right]}$$

$$C = \frac{5}{2} + \frac{3}{2} \ln \frac{4\pi m}{3h^2}$$

Sackur–Tetrode equation

→ entropy of an ideal gas

S increases with N, V, U