Physics 3410 Homework #3

4 problems

Solutions

▷ **1.**

Simplify

$$\frac{\pi(2N+3)^{2N+1}}{N^{3/2}}$$

assuming that N is a large number.

Answer:

N is a large number, so any normal-sized number added to it can be removed.

$$\frac{\pi(2N)^{2N}}{N^{3/2}}$$

Now $(2N)^{2N}$ is a very large number, and so any normal-sized or large number multiplying it can be removed. Thus

$$\frac{\pi(2N+3)^{2N+1}}{N^{3/2}} \approx \boxed{(2N)^{2N}}$$

▷ **2.**

Suppose I have 400 A's, 300 B's, 200 C's, and 1 D. How many ways can I rearrange them? Use Stirling's Approximation.

Answer:

There are 901 letters to rearrange, and with the duplicates

$$\Omega = \frac{901!}{400!300!200!}$$

Take a logarithm and use Stirling's approximation

$$\begin{aligned}
\ln \Omega &= \ln 901! - \ln 400! - \ln 300! - \ln 200! \\
&= (901 \ln 901 - 901) - (400 \ln 400 - 400) - (300 \ln 300 - 300) - (200 \ln 200 - 200) \\
&= 901 \ln 901 - 400 \ln 400 - 300 \ln 300 - 200 \ln 200 - 1 \\
&= 961.6 \\
\implies \Omega &= e^{961.6}
\end{aligned}$$

I'm happy to accept this as the answer, but let me go a little farther. Now my calculator doesn't want to evaluate this, so instead I'm going to find the base-10 logarithm:

$$\log_{10}\Omega = \frac{\ln\Omega}{\ln 10} = \frac{961.6}{\ln 10} = 417.618$$

$$\implies \Omega = 10^{0.618}10^{417} = \boxed{4.1\times 10^{417}}$$

▷ **3.**

Consider a paramagnet with 6 dipoles, in the energy macrostate $U = 3$.
(a) What is the multiplicity of this macrostate?
(b) What is the probability that the paramagnet has three adjacent spins pointing upward, if it's in this macrostate? (Enumerating the possible microstates might be easiest.)
(c) Suppose that U can change freely (because energy can flow in or out of the solid). What is the maximum amount of energy that can be stored in this paramagnet? Which value of U has the largest multiplicity?

Answer:______________________________

(a) There are 6 dipoles, and $U = 3$ means three point up. There are $\binom{6}{3}$ ways.

$$\Omega(U=3) = \binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = \boxed{20}$$

(b) Here are the microstates that satisfy this condition:

$$\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow \qquad \downarrow\uparrow\uparrow\uparrow\downarrow\downarrow \qquad \downarrow\downarrow\uparrow\uparrow\uparrow\downarrow \qquad \downarrow\downarrow\downarrow\uparrow\uparrow\uparrow$$

So 4 of the 20 microstates have this property, and the probability that it has this property is

$$P = \frac{4}{20} = \boxed{20\%}$$

(c) The maximum amount of energy in the paramagnet is when all the dipoles point up, so $U = 6$ is the largest energy it can have. Let's find the multiplicity of all the possible energy values:

$$\binom{6}{0} = \binom{6}{6} = 1 \qquad \binom{6}{1} = \binom{6}{5} = 6 \qquad \binom{6}{2} = \binom{6}{4} = \frac{6!}{4!2!} = \frac{6\cdot 5}{2} = 15 \qquad \binom{6}{3} = 20$$

So we see that $\boxed{U=3}$ is the energy state with the largest multiplicity.

▷ **4.**

Consider an Einstein solid with $N = 5$ oscillators, with total energy $q = 4$.
(d) What is the multiplicity of this macrostate?
(e) What is the probability that the first oscillator contains 1 quantum of energy? (i.e. $q_1 = 1$)
(f) Suppose that q can change freely (because energy can flow in or out of the solid). What is the maximum amount of energy that can be stored in this solid? Which value of q has the largest multiplicity?

Answer:______________________________________

(a)

$$\Omega_{all} = \binom{N+q-1}{q} = \binom{5+4-1}{4} = \binom{8}{4} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1} = \boxed{70}$$

(b) The number of microstates with $q_1 = 1$ is equal to the number of ways we can distribute the $q - 1 = 3$ remaining quanta in the other 4 oscillators:

$$\Omega(q_1 = 1) = \binom{4+3-1}{3} = \binom{6}{3} = \frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 20$$

The probability of this occurring is thus

$$P(q_1 = 1) = \frac{\Omega(q_1 = 1)}{\Omega_{all}} = \frac{20}{70} = \boxed{29\%}$$

(c) The number of microstates $\Omega(q)$ is an increasing function of q; it grows without limit as q approaches infinity.