$|\psi\rangle \longrightarrow \boxed{S_x}$

two outputs

1) analyzer will report a value
(one of its eigenvalues)

2) particle will change to a different state
(one of its eigenvectors)

States are represented by <u>normalized</u> kets.

$|\psi\rangle \qquad 3|\psi\rangle \qquad (2i+3)|\psi\rangle$

all the same state

if $\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector of A

then $\begin{pmatrix}2\\4\end{pmatrix}$ is as well

or $\begin{pmatrix}-i\\-2i\end{pmatrix}$ etc..

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \lambda\begin{pmatrix}a\\b\end{pmatrix} \rightarrow \text{Let } a=1$$

$$\begin{pmatrix}1 & 3\\3 & 4\end{pmatrix}\begin{pmatrix}1\\c\end{pmatrix} = \lambda\begin{pmatrix}1\\c\end{pmatrix}$$

$$\begin{pmatrix}1+2c\\3+4c\end{pmatrix} = \begin{pmatrix}\lambda\\ \lambda c\end{pmatrix} \rightarrow \begin{matrix}1+2c=\lambda\\ 3+4c=\lambda c\end{matrix} \qquad \begin{pmatrix}1\\ -\end{pmatrix}$$

Normalization: $|\psi\rangle \rightarrow \frac{1}{\sqrt{\langle\psi|\psi\rangle}}|\psi\rangle$

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

$$\langle A \rangle = \langle \psi | A | \psi \rangle \qquad \langle A^2 \rangle = \langle \psi | A A | \psi \rangle$$

if $\quad A = S_z \quad \& \quad |\psi\rangle = |\odot\rangle \qquad\qquad |\odot\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$

$$\langle S_z \rangle = \langle \odot | S_z | \odot \rangle \qquad\qquad |\odot\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \cancel{\frac{1}{\sqrt{2}}\left(\langle \uparrow | + \langle \downarrow | \right)}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\, 0 = 0.$$

$$\langle S_z^2 \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} \frac{\hbar^2}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$\langle S_z^2 \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$$

$$= \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$$

$$S_z = 0 \pm \frac{\hbar}{2}$$

Uncertainty Principle: Suppose two Hermitian operators A & B measuring same state $|\psi\rangle$

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle$$

$$= \langle \psi | (A - \langle A \rangle)(A - \langle A \rangle) | \psi \rangle$$

$$= \langle f | f \rangle \quad \text{if} \quad |f\rangle = (A - \langle A \rangle)|\psi\rangle$$

(A Hermitian)

Ditto: $|g\rangle = (B - \langle B \rangle)|\psi\rangle$

$$(\Delta B)^2 = \langle g | g \rangle$$

Cauchy-Schwarz Inequality: $\underbrace{\langle f|f\rangle}_{(\Delta A)^2}\underbrace{\langle g|g\rangle}_{(\Delta B)^2} \geq |\langle f|g\rangle|^2$

~~If~~ Generally, if z is a complex number

$$|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 \geq (\mathrm{Im}\, z)^2$$

$$|z|^2 \geq \left[\frac{1}{2i}(z - z^*)\right]^2$$

$\mathrm{Im}\,(3-4i)$

$\frac{1}{2i}((3-4i)-(3+4i))$

$\frac{1}{2i}(-8i) = -4$

$$\therefore \quad (\Delta A)^2 (\Delta B)^2 \geq \left[\frac{1}{2i}\left(\langle f|g\rangle - \langle g|f\rangle\right)\right]^2$$

$$\langle f|g\rangle = \langle \psi | (A - \langle A \rangle)(B - \langle B \rangle) | \psi \rangle$$

$$= \langle \psi | AB | \psi \rangle - \langle A \rangle \langle \psi | B | \psi \rangle - \langle B \rangle \langle \psi | A | \psi \rangle + \langle A \rangle \langle B \rangle \langle \psi | \psi \rangle$$

$$= \langle \psi | AB | \psi \rangle - \cancel{\langle A \rangle \langle B \rangle} - \langle B \rangle \langle A \rangle + \cancel{\langle A \rangle \langle B \rangle}$$

$$= \langle AB \rangle - \langle A \rangle \langle B \rangle$$

$$\langle g|f\rangle = \langle BA \rangle - \langle A \rangle \langle B \rangle$$

$$\langle f|g\rangle - \langle g|f\rangle = \langle AB \rangle - \langle BA \rangle$$

$$= \langle AB - BA \rangle = \langle [A,B] \rangle$$

$$(\Delta A)^2 (\Delta B)^2 \geq \left(\frac{1}{2i}\right)^2 \langle [A,B] \rangle^2$$

$$\boxed{\Delta A \, \Delta B \geq \frac{1}{2i}\langle [A,B] \rangle}$$

Uncertainty principle

Correction: there should be absolute value signs around the whole right side of the uncertainty principle formula!
$\Delta A \Delta B \geq | \langle [A,B] \rangle / 2i |$

e.g. if $|\psi\rangle = \cancel{|\hat{z}\rangle}$ $|1\rangle$ $\quad [S_x, S_y]$

$$\Delta S_x \Delta S_y \geq \frac{1}{2i}\langle i\hbar S_z \rangle$$

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2}\langle S_z \rangle$$

if $|\psi\rangle = |1\rangle$ $\quad \langle S_z \rangle = \frac{\hbar}{2}$

$$\therefore \Delta S_x \Delta S_y \geq \frac{\hbar^2}{4}$$

in particular, neither ΔS_x or $\Delta S_y = 0$.

if $|\psi\rangle = |0\rangle$ $\quad \langle S_z \rangle = 0$

$$\Delta S_x \Delta S_y \geq 0.$$