Two spin-$\frac{1}{2}$ particles

each $s_1 = s_2 = \frac{1}{2}$

$m_1 = \pm\frac{1}{2}$ $\quad$ $m_2 = \pm\frac{1}{2}$

$-: \downarrow \quad +: \uparrow$

Together they can have angular momentum characterized by s & m

$m = m_1 + m_2$ always

$+\frac{1}{2} \; +\frac{1}{2} = 1$

$+\frac{1}{2} \; -\frac{1}{2} = 0$ $\qquad -1 \leq m \leq 1$

$-\frac{1}{2} \; +\frac{1}{2} = 0$ $\qquad -s \leq m \leq s$

$-\frac{1}{2} \; -\frac{1}{2} = -1$

Four possible states

$S^2 \uparrow\uparrow = \hbar^2 s(s+1) \uparrow\uparrow$
where $s=1$

$\uparrow\uparrow$ $\qquad s=1 \quad m=1$

$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$ $\qquad s=1 \quad m=0$

$\downarrow\downarrow$ $\qquad s=1 \quad m=-1$

S_+, S_-

triplet

$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ $\qquad s=0 \quad m=0$

singlet

are all orthonormal

e.g. $\langle \uparrow\uparrow | \frac{1}{\sqrt{2}} \uparrow\downarrow - \downarrow\uparrow \rangle$

$= \frac{1}{\sqrt{2}} \left[\langle \uparrow\uparrow | \uparrow\downarrow \rangle - \langle \uparrow\uparrow | \downarrow\uparrow \rangle \right]$

$= \frac{1}{\sqrt{2}} \left[\langle \uparrow | \uparrow \rangle \langle \uparrow | \downarrow \rangle - \langle \uparrow | \downarrow \rangle \langle \uparrow | \uparrow \rangle \right] = 0.$

In general,
given two particles s_1, m_1 & s_2, m_2

$$m = m_1 + m_2$$

$$s = s_1 + s_2,\ s_1 + s_2 - 1,\ \dots,\ |s_1 - s_2|$$

e.g. spin $\frac{3}{2}$ particle & a spin 1 particle
together I can have $s = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$

$$-s \leq m \leq s$$

$\underline{s = \frac{5}{2}}$

$m = \frac{5}{2} = m_1 + m_2$

$m = \frac{3}{2}$

$m = \frac{1}{2}$

$m = -\frac{1}{2}$

$m = -\frac{3}{2}$

$m = -\frac{5}{2}$

$\underline{s = \frac{1}{2}}$

$m = \frac{1}{2} = m_1 + m_2$

$m = -\frac{1}{2} = m_1 + m_2$

The actual states can be found using
Clebsch-Gordon coefficients

$$|s\ m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 m_1\rangle |s_2 m_2\rangle$$

$s_1 = 2 \quad s_2 = 1$

e.g.

$$|\overset{s}{3}\ \overset{m}{0}\rangle = \frac{1}{\sqrt{5}} |2\ 1\rangle |1\ {-1}\rangle + \sqrt{\frac{3}{5}} |2\ 0\rangle |1\ 0\rangle + \frac{1}{\sqrt{5}} |2\ {-1}\rangle |1\ 1\rangle$$

Suppose I measure both particles together
& get $s = 3$, $m = 0$
What is probability that $m_1 = m_2 = 0$?

$$P = \frac{3}{5}.$$

Chapter 5

Suppose we have two particles $\vec{r}_1 \bullet \quad \bullet \vec{r}_2$

$\Psi(\vec{r}_1, \vec{r}_2, t)$ describes both combined

$$i\hbar \frac{d\Psi}{dt} = H\Psi \quad \text{as usual}$$

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Psi(r_1, r_2, t)|^2 \, d\vec{r}_1 \, d\vec{r}_2 = 1 \quad \text{as usual}$$

Suppose particle 1 is in state ψ_a
& particle 2 is in state ψ_b

$$\Psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)$$

only works if I can tell them apart

If particles are identical, then $\psi_b(\vec{r}_1)\psi_a(\vec{r}_2)$ is just as likely

Identical particles: $\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right]$

$+$: bosons, integer spin

$-$: fermions, half-integer spin

if particle 1 & particle 2 are both in state a

bosons: $\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_a(\vec{r}_2) + \psi_a(\vec{r}_2)\psi_a(\vec{r}_1)$
$= \psi_a(\vec{r}_1)\psi_a(\vec{r}_2)$

fermions $\psi = \psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_a(\vec{r}_1)$
$= 0$

Pauli exclusion principle: 2 identical fermions cannot occupy the same state

Let P be exchange operator

$$P\,\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

$P^2 = \mathbb{1}$ $\rightarrow$ P has eigenvalues of ± 1

$[P, H] = 0$ for identical particles

because H must treat particles identically - symmetric under interchange of labels

Eigenstates of H are eigenstates of P

$$P\psi(\vec{r}_1, \vec{r}_2) = \pm\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

for identical particles, the energy eigenstates satisfy $\psi(\vec{r}_1, \vec{r}_2) = \pm\psi(\vec{r}_2, \vec{r}_1)$

boson eigenstates are symmetric

fermion eigenstates are antisymmetric.