$$-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

$$= \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]\psi = E\psi$$

$$a_-a_+ = \frac{1}{2m\hbar\omega}\left[p^2 + (m\omega x)^2 - im\omega[x,p]\right]$$

$$[x,p] = i\hbar$$

$$a_-a_+ = \frac{1}{\hbar\omega}H + \frac{1}{2} \qquad a_+a_- = \frac{1}{\hbar\omega}H - \frac{1}{2}$$

Suppose ψ satisfies $H\psi = E\psi$

then $H(a_-\psi) = (E-\hbar\omega)(a_-\psi)$

so $a_-\psi$ also an energy eigenstate, eigenvalue $E-\hbar\omega$

also $H(a_+\psi) = (E+\hbar\omega)(a_+\psi)$

So given one energy eigenstate, can construct a ladder of others: $(a_-)^n\psi$ & $(a_+)^n\psi$ $n=0,1,2,\ldots$

w/ energy $E-n\hbar\omega$ $\quad E+n\hbar\omega$

BUT ~~every~~ energy $E \geq 0$ because $\min(V) = 0$.

$\therefore$ exists a ground state ψ_0 such that $a_-\psi_0 = 0$

$$\frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x)\psi_0 = 0$$

$$\rightarrow \hbar\frac{d\psi_0}{dx} + m\omega x\psi_0 = 0$$

$$\rightarrow \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega/2\hbar\, x^2}$$

$$H\psi_0 = \hbar\omega\left[a_+(a_-\psi_0) + \frac{1}{2}\psi_0\right]$$

$$= 0 + \frac{1}{2}\hbar\omega\psi_0$$

$\therefore$ ground state ψ_0 has energy $E = \frac{1}{2}\hbar\omega$

Possible Energy eigenvalues are $\frac{1}{2}\hbar\omega + n\hbar\omega$ $\quad n=0,1,2,\ldots$

$$\psi_n(x) = (a_+)^n\psi_0(x).$$