Hydrogen Atom

$$\psi_{n\ell m} = R(r)\, Y_\ell^m(\theta, \phi)$$

$$R(r) = \frac{u(r)}{r}$$

$$\rho = \kappa r \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar} \qquad E < 0$$

$$u(\rho) = \rho^{\ell+1} e^{-\rho} v(\rho)$$

$$n = j_{max} + \ell + 1 \qquad v(\rho) = \sum_{j=0}^{j_{max}} c_j \rho^j$$

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)} c_j \qquad n = 1, 2, 3, \ldots$$

$$E_n = \frac{E_1}{n^2} \qquad E_1 = -13.6\,\text{eV}$$

$n = 1$ $\qquad n = j_{max} + l + 1$

$j_{max} = 0 \qquad l = 0$

$$v(\rho) = \sum_{j=0}^{0} c_0 \rho^0 = c_0$$

$$u(\rho) = \rho^{0+1} e^{-\rho} c_0 = \rho e^{-\rho} = \frac{r}{a} e^{-r/a}$$

$\kappa = \frac{1}{an}$

$a = \frac{1}{\kappa n}$

$= .529 \times 10^{-10} m$

$$R(r) = \frac{u(r)}{r} = \frac{1}{a} e^{-r/a}$$

$$Y_0^0(\theta, \phi) = 1 \quad \text{not counting normalization}$$

$$\Psi_{nlm} = \Psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |\Psi_{100}|^2 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$n = 2$ $\quad n = j_{max} + \ell + 1 \quad \rightarrow \quad j_{max} + \ell = 1$

either $j_{max} = 1, \ \ell = 0$ $\quad$ or $\quad j_{max} = 0, \ \ell = 1$

$\underline{\ell = 0}$

$$v(\rho) = c_0 + c_1 \rho \qquad c_1 = \frac{2(0+0+1-2)}{(0+1)(0+2\cdot 0+2)} c_0$$

$$= c_0(1-\rho) \qquad\qquad = -c_0$$

$\rho = \frac{r}{na}$

$$u(\rho) = c_0 \rho^1 e^{-\rho}(1-\rho) = c_0 \frac{r}{2a}\left(1-\frac{r}{2a}\right) e^{-r/2a}$$

$$R(r) = \frac{u(r)}{r} = c_0 \frac{1}{2a}\left(1-\frac{r}{2a}\right) e^{-r/2a}$$

$$Y_\ell^m = Y_0^0 = \text{constant}$$

$$\psi_{200} = A\left(1-\frac{r}{2a}\right) e^{-r/2a}$$

$\underline{\ell = 1}$

$$v(\rho) = c_0$$

$$R(r) = \text{const} \times e^{-r/2a}$$

$$Y_1^m \qquad m = -1, 0, 1$$

$$\psi_{21m} = A e^{-r/2a} Y_1^m(\theta, \phi)$$

$n = j_{max} + \ell + 1$

$n - 1 = j_{max} + \ell$

For given n,

$$\ell = 0, 1, 2, \dots, n-1$$

n possible values of ℓ

For given ℓ, how many values of m? $2\ell+1$

Degeneracy

$$d(n) = \sum_{\ell=0}^{n-1} (2\ell+1) = n^2$$

General

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}}\, e^{-r/na} \left(\frac{2r}{na}\right)^{\ell} L^{2\ell+1}_{n-\ell-1}\left(\frac{2r}{na}\right) Y_\ell^m(\theta,\phi)$$

Laguerre polynomial $L_q(x) = e^x \left(\frac{d}{dx}\right)^q \left(e^{-x} x^q\right)$

associated Laguerre polynomial $L^p_{q-p}(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_q(x)$

$n = 1, 2, 3, \ldots$ shell

$\ell = 0, 1, 2, 3 \ldots, n-1$ s, p, d, f, . . .

$m =$

↑ & ↓

s has $m = 0$ $1 \times 2 = 2$ states

p has $m = -1, 0, 1$ $3 \times 2 = 6$ states

d has $m = -2, -1, 0, 1, 2$ $5 \times 2 = 10$ states

Angular Momentum

$[x, x] = 0$
$[x, p_x] = i\hbar$
$[x, p_y] = 0$
etc.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L_x = y p_z - z p_y = -i\hbar\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right)$$

$$L_y = z p_x - x p_z \qquad \text{etc}$$

$$L_z = x p_y - y p_x \qquad \text{etc}$$

$$[L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_y$$

$$[L^2, L_x] = 0 = [L^2, L_y] = [L^2, L_z]$$

Suppose f is an eigenfunction of L^2 & L_z

$$L^2 f = \lambda f \qquad L_z f = \mu f$$

Define $L_\pm \equiv L_x \pm i L_y$

$$[L_\pm, L^2] = 0$$

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

Now $L_\pm f$ is an eigenfunction of L^2 with eigenvalue λ

$$L^2(L_\pm f) = L_\pm L^2 f = L_\pm(\lambda f) = \lambda(L_\pm f)$$

$L_\pm f$ is an eigenfunction of L_z with eigenvalue $\mu \pm \hbar$.