Physics 4310 Homework #9

5 problems

Solutions

▷ **1.**

Prove that if f is an eigenfunction of L_z with eigenvalue μ, then $L_\pm f$ is also an eigenfunction of L_z, but with eigenvalue $\mu \pm \hbar$.

Answer:

We know that $L_z f = \mu f$. We want to calculate $L_z (L_\pm f)$. Now $[L_z, L_\pm] = L_z L_\pm - L_\pm L_z$ so

$$\begin{aligned} L_z L_\pm f &= [L_z, L_\pm]f + L_\pm L_z f \\ &= ([L_z, L_x]f \pm i[L_z, L_y]f) + L_\pm \mu f \\ &= (i\hbar L_y f \mp \hbar L_x f) + \mu L_\pm f \\ &= \pm\hbar(L_x \pm iL_y)f + \mu L_\pm f \\ L_z(L_\pm f) &= (\mu \pm \hbar)(L_\pm f) \end{aligned}$$

which is what we wanted to show.

▷ **2.**

The raising and lowering operators for spin are $S_\pm = S_x \pm iS_y$. Using the S_z matrix notation from Macintyre,

(a) …prove that applying the raising operator to $|\uparrow\rangle$, or the lowering operator to $|\downarrow\rangle$, gives you zero

(b) show that $S_+|\downarrow\rangle \propto |\uparrow\rangle$ and $S_-|\uparrow\rangle \propto |\downarrow\rangle$

Answer:

In matrix notation,

$$S_+ = S_x + iS_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

and

$$S_- = S_x - iS_y = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

(a)

$$S_+|\uparrow\rangle = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad S_-|\downarrow\rangle = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b)

$$S_+|\downarrow\rangle = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2|\uparrow\rangle$$

$$S_-|\uparrow\rangle = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2|\downarrow\rangle$$

▷ **3.**
For a pair of spin–1/2 particles, prove that $s = 1$ for $|\uparrow\uparrow\rangle$ and $s = 0$ for $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. Use the fact that $S^2\chi = \hbar^2 s(s+1)\chi$, and write $S^2 = (\vec{S}_1 + \vec{S}_2)\cdot(\vec{S}_1 + \vec{S}_2)$.

Answer:

$$S^2 = (\vec{S}_1 + \vec{S}_2)\cdot(\vec{S}_1 + \vec{S}_2) = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$= S_1^2 + S_2^2 + 2(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z})$$

Now let's use matrix notation to figure out what S_x and S_y do to different one-particle states.

$$S_x|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|\downarrow\rangle \qquad S_x|\downarrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}|\uparrow\rangle$$

$$S_y|\uparrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ i \end{pmatrix} = i\frac{\hbar}{2}|\downarrow\rangle \qquad S_y|\downarrow\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\frac{\hbar}{2}|\uparrow\rangle$$

Also $S^2\chi = \hbar^2\frac{1}{2}\frac{3}{2}\chi = \frac{3}{4}\chi$. So

$$\begin{aligned}
S^2|\uparrow\uparrow\rangle &= S_1^2|\uparrow\uparrow\rangle + S_2^2|\uparrow\uparrow\rangle + 2S_{1x}S_{2x}|\uparrow\uparrow\rangle + 2S_{1y}S_{2y}|\uparrow\uparrow\rangle + 2S_{1z}S_{2z}|\uparrow\uparrow\rangle \\
&= \frac{3}{4}\hbar^2|\uparrow\uparrow\rangle + \frac{3}{4}\hbar^2|\uparrow\uparrow\rangle + 2\frac{\hbar^2}{4}|\downarrow\downarrow\rangle + 2\frac{\hbar^2}{4}(i^2)|\downarrow\downarrow\rangle + 2\frac{\hbar^2}{4}|\uparrow\uparrow\rangle \\
&= \left(\frac{3}{4} + \frac{3}{4} + \frac{1}{2}\right)\hbar^2|\uparrow\uparrow\rangle + \left(\frac{1}{2} - \frac{1}{2}\right)\hbar^2|\downarrow\downarrow\rangle \\
&= 2\hbar^2|\uparrow\uparrow\rangle
\end{aligned}$$

So $|\uparrow\uparrow\rangle$ is an eigenstate of S^2 with eigenvalue $2 = 1(1+1)$ corresponding to $s = 1$.

Now let's calculate $S^2|\uparrow\rangle\downarrow$:

$$\begin{aligned}
S^2|\uparrow\downarrow\rangle &= S_1^2|\uparrow\downarrow\rangle + S_2^2|\uparrow\downarrow\rangle + 2S_{1x}S_{2x}|\uparrow\downarrow\rangle + 2S_{1y}S_{2y}|\uparrow\downarrow\rangle + 2S_{1z}S_{2z}|\uparrow\downarrow\rangle \\
&= \frac{3}{4}\hbar^2|\uparrow\downarrow\rangle + \frac{3}{4}\hbar^2|\uparrow\downarrow\rangle + 2\frac{\hbar^2}{4}|\downarrow\uparrow\rangle + 2\frac{\hbar^2}{4}(-i^2)|\downarrow\uparrow\rangle - 2\frac{\hbar^2}{4}|\uparrow\downarrow\rangle \\
&= \left(\frac{3}{4} + \frac{3}{4} - \frac{1}{2}\right)\hbar^2|\uparrow\downarrow\rangle + \left(\frac{1}{2} + \frac{1}{2}\right)\hbar^2|\downarrow\uparrow\rangle \\
&= \hbar^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)
\end{aligned}$$

Similarly, $S^2|\downarrow\uparrow\rangle = \hbar^2(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$ which is the same. Thus

$$S^2(|\uparrow\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) = 0$$

and so $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is an eigenstate of S^2 with eigenvalue $0 = 0(0+1)$ corresponding to a spin state of $s = 0$.

▷ **4.**

Consider a spin–1/2 particle and a spin–3/2 particle. Their total angular momentum is measured to be $s = 1$ and $m = 0$.

(a) What is the probability that the first particle is spin-up ($m_1 = +1/2$)?

(b) What other value(s) could s take?

Answer:

(a) We can use Clebsch-Gordon coefficients to solve this problem. We can write the state $s = 1$, $m = 0$ as

$$\left|\begin{matrix} s=1 \\ m=0 \end{matrix}\right\rangle = \sum_{m_1+m_2=0} C^{s_1=\frac{1}{2}, s_2=\frac{3}{2}, s=1}_{m_1, m_2, m=0} \left|\begin{matrix} s_1 = 1/2 & s_2 = 3/2 \\ m_1 & m_2 \end{matrix}\right\rangle$$

If the first particle has $m_1 = 1/2$ then the second particle has $m_2 = -1/2$ because $m_1 + m_2 = m = 0$. The probability of this happening is

$$P = \left|\left\langle\begin{matrix} s_1 = 1/2 & s_2 = 3/2 \\ m_1 = 1/2 & m_2 = -1/2 \end{matrix}\middle|\begin{matrix} s=1 \\ m=0 \end{matrix}\right\rangle\right|^2 = \left|C^{s_1=\frac{1}{2}, s_2=\frac{3}{2}, s=1}_{m_1=1/2, m_2=-1/2, m=0}\right|^2$$

According to the table, this coefficient is $\frac{1}{\sqrt{2}}$, so the probability is $\boxed{P = \frac{1}{2}}$.

(b) s can range from $s_1 + s_2 = 2$ to $|s_1 - s_2| = 1$, so s can also equal 2.

▷ **5.**

(Griffiths 5.1) Typically, the interaction potential only depends on the vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$. In that case the Schrodinger equation separates, if we change variables from $\vec{r}_1, \vec{r}_2$ to

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad \text{and} \quad \vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

(the latter is the center of mass).

(a) If

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass of the system, show that

$$\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{\mu}{m_2}\vec{r} \qquad \nabla_1 = \frac{\mu}{m_2}\nabla_R + \nabla_r \quad \text{and} \quad \nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

(b) Show that the energy eigenstate equation (aka the time-independent Schrodinger equation) becomes

$$-\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec{r})\psi = E\psi$$

(c) Separate the variables, letting $\psi(\vec{R}, \vec{r}) = \psi_R(\vec{R})\psi_r(\vec{r})$. Show that ψ_R satisfies the one-particle Schrodinger equation, with the *total* mass, potential zero, and some energy E_R. Show that ψ_r satisfies the one-particle Scrondinger equation with the *reduced* mass, potential $V(\vec{r})$, and some energy E_r, so that $E = E_R + E_r$.

What this tells us is that the center of mass moves like a free particle, and the relative motion is the same as if we had a single particle with the reduced mass, subject to the potential V. We can do the same thing in classical mechanics, reducing the two-body problem to an equivalent one-body problem.

Answer:

(a) Let's start with the first two:

$$\begin{aligned}
\vec{R} + \frac{\mu}{m_1}\vec{r} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} + \frac{1}{m_1}\frac{m_1 m_2}{m_1 + m_2}(\vec{r}_1 - \vec{r}_2) \\
&= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_2(\vec{r}_1 - \vec{r}_2)}{m_1 + m_2} \\
&= \frac{(m_1 + m_2)\vec{r}_1 + (m_2 - m_2)\vec{r}_2}{m_1 + m_2} \\
&= \vec{r}_1
\end{aligned}$$

Now to do the gradients I need some sort of chain rule to convert from one to another. Let's think in terms of 1D first:

$$\begin{aligned}\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial X}\frac{\partial X}{\partial x_1} + \frac{\partial}{\partial x}\frac{\partial x}{\partial x_1} \\ &= \frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x} \\ &= \frac{\mu}{m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\end{aligned}$$

Same goes for y and z, and so

$$\begin{aligned}\nabla_{r_1} &= \frac{\partial}{\partial x_1}\hat{x} + \frac{\partial}{\partial y_1}\hat{y} + \frac{\partial}{\partial z_1}\hat{z} \\ &= \frac{\mu}{m_2}\left(\frac{\partial}{\partial X}\hat{x} + \frac{\partial}{\partial Y}\hat{y} + \frac{\partial}{\partial Z}\hat{z}\right) + \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \\ &= \frac{\mu}{m_2}\nabla_R + \nabla_r\end{aligned}$$

The only difference with ∇_2 is that $\frac{\partial x}{\partial x_2} = -1$, so that ∇_r picks up a negative sign:

$$\nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

(b) We start with the kinetic energy part of the Hamiltonian, in terms of $\vec{r}_1$ and $\vec{r}_2$:

$$\begin{aligned}H_K &= -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 \\ &= -\frac{\hbar^2}{2m_1}\left[\frac{\mu}{m_2}\nabla_R + \nabla_r\right]^2 - \frac{\hbar^2}{2m_2}\left[\frac{\mu}{m_1}\nabla_R - \nabla_r\right]^2 \\ &= -\frac{\hbar^2}{2m_1}\left[\frac{m_1^2}{(m_1+m_2)^2}\nabla_R^2 + \nabla_r^2 + 2\frac{m_1}{m_1+m_2}\nabla_R\nabla_r\right] \\ &\quad - \frac{\hbar^2}{2m_2}\left[\frac{m_2^2}{(m_1+m_2)^2}\nabla_R^2 + \nabla_r^2 - 2\frac{m_2}{m_1+m_2}\nabla_R\nabla_r\right] \\ &= -\frac{\hbar^2}{2}\left[\frac{m_1}{(m_1+m_2)^2} + \frac{m_2}{(m_1+m_2)^2}\right]\nabla_R^2 - \frac{\hbar^2}{2}\left[\frac{1}{m_1} + \frac{1}{m_2}\right]\nabla_r^2 - \frac{\hbar^2}{m_1+m_2}(1-1)\nabla_R\nabla_r \\ &= -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2}\frac{m_1+m_2}{m_1m_2}\nabla_r^2 \\ &= -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2\end{aligned}$$

The total Hamiltonian is just this plus the potential $V(\vec{r})$, so

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec{r})\psi = E\psi$$

(c) Writing $\psi = \psi_R(\vec{R})\psi_r(\vec{r})$, we have

$$\begin{aligned}
-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R\psi_r - \frac{\hbar^2}{2\mu}\nabla_r^2\psi_R\psi_r + V(\vec{r})\psi_R\psi_r &= E\psi_R\psi_r \\
-\psi_r\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R - \psi_R\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r})\psi_R\psi_r &= E\psi_R\psi_r \\
-\frac{1}{\psi_R}\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R - \frac{1}{\psi_r}\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r}) &= E
\end{aligned}$$

where I divided both sides by $\psi_R\psi_r$. Isolating the term that depends on $\vec{R}$ gives us

$$\frac{1}{\psi_R}\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R = -\frac{1}{\psi_r}\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r}) - E = \lambda$$

where λ is the separation constant.

Multiplying the R equation by $-\psi_R$ gives us

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R = -\lambda\psi_R$$

and so ψ_R obeys the Schrodinger equation with no potential, the total mass, and energy $E_R = -\lambda$. Multiplying the r equation by ψ_r and substituting $\lambda = -E_R$ gives us

$$-\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r})\psi_r - E = -E_R \implies -\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r + V(\vec{r})\psi_r = (E - E_R)$$

So ψ_r satisfies the Schrodinger equation with reduced mass μ, potential $V(\vec{r})$, and energy $E - E_R \equiv E_r$.