Sample Physics 3410 Final Exam Solutions

May 2, 2014

$\boxed{3}$ 1. Suppose I have some air in a box. In which of the following scenarios…

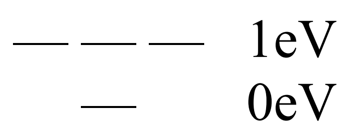
A) The box is sealed and insulated
B) The box is sealed, but heat can flow through the walls
C) The box is open

…would I use

(a) **B** Boltzmann statistics (and Z)?

(b) **C** Gibbs statistics (and $\mathcal{Z}$)?

$\boxed{3}$ 2. **A** The figure shows the four possible microstates of a system, which is in contact with a thermal reservoir at room temperature. The most likely energy of this system at room temperature is $E = 0\,\text{eV}$. What would I do to make $E = 1\,\text{eV}$ the most likely energy?



A) increase the temperature of the reservoir
B) decrease the temperature of the reservoir
C) Nothing; $E = 0\,\text{eV}$ is always the most likely energy of the system

$\boxed{3}$ 3. **A** The partition function of an ideal gas of point particles is

$$Z = \frac{1}{N!}\left(\frac{V}{v_Q}\right)^N$$

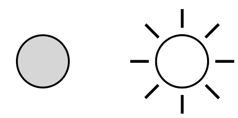
in which of the following scenarios?
A) $V = 1000 v_Q$ **B)** $v_Q = 1000 V$ **C)** The expression is valid in both cases.

4. A certain particle can exist in one of two states. When the particle is "off", it has no energy. When the particle is "on", it has energy $\mathcal{E}$. This particle can exchange energy with a thermal reservoir with temperature given by β.

$\boxed{1}$ (a) What is the Boltzmann factor for the "off" state (as a function of β and $\mathcal{E}$)?

$$e^{-\beta E} = 1$$

$\boxed{1}$ (b) What is the Boltzmann factor for the "on" state?



$$e^{-\beta\mathcal{E}}$$

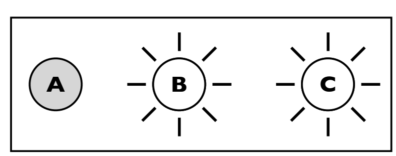
$\boxed{2}$ (c) Write the partition function $Z(\beta, \mathcal{E})$ of this system.

$$Z = 1 + e^{-\beta\mathcal{E}}$$

$\boxed{2}$ (d) Write an expression for the probability $P(\beta, \mathcal{E})$ that the particle is off.

$$\frac{1}{1 + e^{-\beta\mathcal{E}}}$$

$\boxed{2}$ (e) Now suppose I have 3 of these particles in a row: they are *distinguishable* by their position. Find the partition function of the system made up of all three.



Partition functions are multiplicative:

$$Z = (1 + e^{-\beta\mathcal{E}})^3$$

5. A system has partition function $Z = AV^{2N}(kT)^3$ where A is some constant (to make the units work out right).

$\boxed{3}$ (a) Find the average energy U (or $\langle E\rangle$) of the system, as a function of N, V, and T.

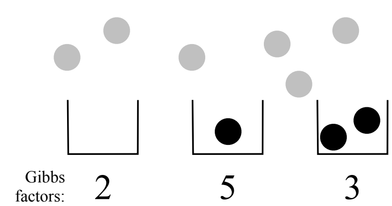
$$\langle E\rangle = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\partial}{\partial \beta}\ln\left(AV^{2N}\beta^{-3}\right)$$
$$= -\frac{\partial}{\partial \beta}(-3\ln\beta) = \frac{3}{\beta} = \boxed{3kT}$$

$\boxed{3}$ (b) Find the pressure P of the system as a function of N, V, and T.

$$P = -\frac{\partial F}{\partial V} = -\frac{\partial}{\partial V}(-kT\ln Z) = kT\frac{\partial}{\partial V}\ln(AV^{2N}(kT)^3)$$
$$= kT\frac{\partial}{\partial V}(2N\ln V)$$
$$= \boxed{kT\frac{2N}{V}}$$

6. A system in contact with a reservoir can be occupied by 0, 1, or 2 particles. The figure shows the *Gibbs factors* for those three states.

$\boxed{3}$ (a) What is the grand partition function $\mathcal{Z}$?



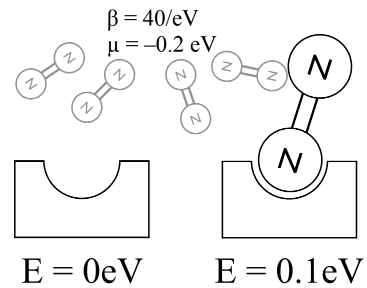
$2 + 5 + 3 = 10$

$\boxed{3}$ (b) Find the average occupancy $\bar{n}$ of this system.

Let p_n be the probability of the state with occupancy n: $p_0 = \frac{2}{10} = 0.2$, $p_1 = 0.5$, $p_2 = 0.3$.

$$\bar{n} = 0p_0 + 1p_1 + 2p_2 = 0 + 1(0.5) + 2(0.3) = \boxed{1.1}$$

7. A particular molecule has a binding site which can be occupied by a nitrogen molecule. If the site is empty, it has an energy $E = 0\,\text{eV}$. If the site is occupied by a nitrogen molecule, it has an energy $E = 0.1\,\text{eV}$. This molecule is in contact with the air, where the temperature is $\beta = 40\,/\text{eV}$ and the chemical potential of nitrogen in the air is $\mu = -0.2\,\text{eV}$.



$\boxed{2}$ (a) Write the Gibbs factor for the empty state, as a number.

$$e^{-\beta(E-\mu N)} = e^{-(40)(0--0.2(0))} = \boxed{1}$$

$\boxed{2}$ (b) Write the Gibbs factor for the occupied state, as a number.

$$e^{-\beta(E-\mu N)} = e^{-(40)(0.1-(-0.2)(1))} = e^{-40(0.3)} = e^{-12} = 6{\times}10^{-6}$$

$\boxed{2}$ (c) __**A**__ At any given moment, the binding site is more likely to be
A) empty **B)** occupied

8. Suppose I have two particles, and each can exist in one of three microstates (A, B, or C). If these were macroscopic particles, the entire system would have 9 possible states: AA, AB, AC, BA, BB, BC, CA, CB, and CC. Find the number of possible states if the particles are

$\boxed{2}$ (a) indistinguishable bosons.

AA, AB, AC, BB, BC, CC: $\boxed{6}$

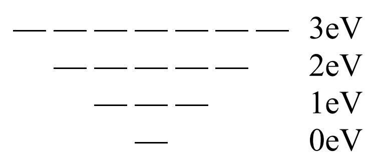
$\boxed{2}$ (b) indistinguishable fermions.

AB, AC, BC: $\boxed{3}$

$\boxed{3}$ 9. Suppose I have a gas of photons (bosons with $\mu = 0$) at temperature $\beta = 40\,/\text{eV}$. About how many photons would you expect to find with energy 0.001 eV? *(Note: this does not require any esoteric knowledge of photons or light, other than what has been stated in this problem.)*

$$\bar{n}(E = 0.001) = \frac{1}{e^{\beta(E-\mu)} - 1} = \frac{1}{e^{(40)(0.001-0)} - 1} = \boxed{24.5}$$

$\boxed{3}$ 10. __**B**__ The figure shows the lowest possible microstates for a particular type of fermion. If I have a collection of 3 such fermions, what is their Fermi energy $\mathcal{E}_F$?
A) 0 eV **B)** 1 eV **C)** 2 eV **D)** 3 eV



11. Suppose 1000 J flows into an ideal gas at temperature $T = 300\,\text{K}$ and heat capacity $C_V = 100\,\text{J/K}$.

$\boxed{3}$ (a) If the temperature is held constant, what is the change ΔS of the gas's entropy?

If the temperature is held constant, then the change in entropy is

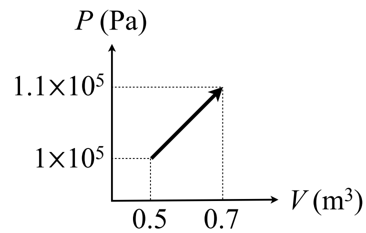
$$\Delta S = \frac{Q}{T} = \frac{1000\,\text{J}}{300\,\text{K}} = \boxed{3.33\,\text{J/K}}$$

$\boxed{3}$ (b) If the volume is held constant, what is the change ΔT in the gas's temperature?

At constant volume,

$$Q = C_V \Delta T \implies \Delta T = \frac{Q}{C_V} = \frac{1000\,\text{J}}{100\,\text{J/K}} = \boxed{10\,\text{K}}$$

12. The figure shows a process performed on an ideal gas of point particles in three dimensions.



[1] (a) **C** How many degrees of freedom f does each particle have?
A) 1 **B)** 2 **C)** 3 **D)** 5 **E)** 7

[3] (b) Find the change in the gas's internal energy ΔU. (Hint: you don't need N but you do need f, which you can figure out.)

$U = \frac{f}{2}NkT = \frac{f}{2}PV$, so

$$\Delta U = \frac{f}{2}\Delta(PV) = \frac{3}{2}\left[(1.1\times 10^5)(0.7) - (1\times 10^5)(0.5)\right] = \boxed{40.5\,\mathrm{kJ}}$$

[2] (c) **B** Work is flowing
A) into this system **B)** out of this system **C)** neither

[2] (d) **A** Heat is flowing
A) into this system **B)** out of this system **C)** neither
D) it cannot be determined

[3] 13. Give an example of a non-quasistatic process.

Slam a piston into a gas.

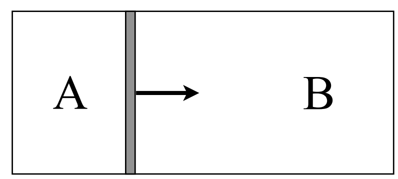
$\boxed{3}$ 14. What is the entropy S of an Einstein solid of 5 oscillators with 3 energy quanta? Give me a *number*, like "2 J/K". (Hint: it's not 2 J/K). Note: 5 and 3 are not large!

$$\Omega = \binom{N+q-1}{q} = \binom{5+3-1}{3} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

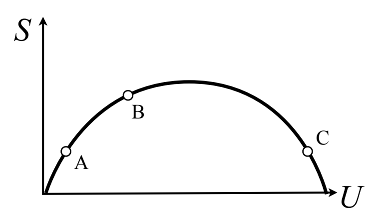
$$S = k \ln \Omega = (1.38 \times 10^{-23}) \ln 35 = \boxed{4.9 \times 10^{-23}\,\mathrm{J/K}}$$

$\boxed{2}$ 15. __**A**__ A box is filled with a gas, and is divided into two compartments (A and B) by a perfect piston. When the piston is released, an imbalance of pressure causes the piston to move to the right. Everything is insulated and sealed so no heat or particles can enter or leave either compartment. As the divider moves, which compartment full of gas *gains* entropy?

A) A **B)** B **C)** Both do **D)** Neither do



16. The figure shows the entropy S of a system as a function of its internal energy U. Suppose I have three such systems, A, B, and C, whose entropies and energies are indicated by dots on the graph. A and C have the same entropy.



$\boxed{3}$ (a) __**B**__ If I place A and B next to each other so that they can exchange heat, what happens?

A) Heat flows from A to B
B) Heat flows from B to A
C) No heat flows between them

$\boxed{2}$ (b) __**B**__ Instead, suppose I place A and C next to each other. What happens?

A) Heat flows from A to C
B) Heat flows from C to A
C) No heat flows between them

$\boxed{2}$ 17. __**B**__ If I want to compress a gas without changing its entropy, I should do it
A) rather slowly **B)** rather quickly
(Please feel free to explain.)

18. A heat engine has an efficiency of 20%. Suppose I want the engine to do 100 J of work.

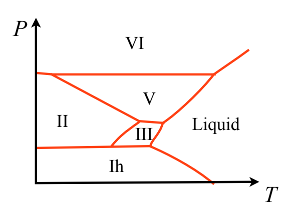
$\boxed{3}$ (a) How much heat must the engine draw from the hot reservoir?

$$\eta = 0.2 = \frac{W}{Q_{in}} \implies Q_{in} = \frac{W}{0.2} = \boxed{500\,\text{J}}$$

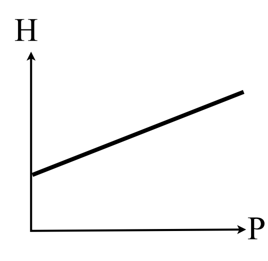
$\boxed{3}$ (b) How much heat must the engine dump into the cold reservoir?

$$Q_{out} = Q_{in} - W = \boxed{400\,\text{J}}$$

$\boxed{3}$ 19. __**C**__ Ice (i.e. solid water) can exist in a number of different phases. The phase that is most prevalent for a given temperature and pressure is the one with the lowest
A) enthalpy **B)** entropy **C)** Gibbs free energy **D)** volume



$\boxed{3}$ 20. __**D**__ If I plot the enthalpy of a system as a function of pressure, the slope of the line is the system's
A) chemical potential **B)** entropy
C) temperature **D)** volume



21. Suppose I have an ideal gas with the following parameters:

$$\begin{array}{lll} T = 300\,\text{K} & P = 10^5\,\text{Pa} & \mu = -3\times 10^{-18}\,\text{J} \\ S = 20\,\text{J/K} & V = 0.004\,\text{m}^3 & N = 10^{23} \end{array}$$

and an internal energy of $U = 5000\,\text{J}$.

$\boxed{2}$ (a) What is the Helmholtz free energy F?

$$F = U - TS = 5000\,\text{J} - (300\,\text{K})(20\,\text{J/K}) = \boxed{-1000\,\text{J}}$$

$\boxed{3}$ (b) Now suppose I add 10^{18} particles to the gas, and increase its temperature by 2 degrees. (Assume these are small changes.) What is the Helmholtz free energy after I'm done? (Or you can tell me the change ΔF instead, but specify which.)

$$\begin{aligned} dF &= -S\,dT - P\,dV + \mu\,dN \\ &= -(20\,\text{J/K})(+2\,\text{K}) - 0 + (-3\times 10^{-18}\,\text{J})(+10^{18}) \\ &= -40\,\text{J} + 0 - 3\,\text{J} = -43\,\text{J} \end{aligned}$$

The Helmholtz free energy is

$$F = -1000 - 43 = \boxed{-1043\,\text{J}}$$

$\boxed{3}$ 22. The entropy of a (non-ideal) gas has the functional form

$$S(V, N, T) = ANkT \ln V$$

where A is a constant. Find $\left(\frac{\partial P}{\partial T}\right)_{N,V}$.

Use the Maxwell relation

$$\left(\frac{\partial P}{\partial T}\right)_{N,V} = \left(\frac{\partial S}{\partial V}\right)_{N,T}$$

(The sign is positive because $dF = -S\,dT - P\,dV + \mu\,dN$.) Therefore

$$\left(\frac{\partial P}{\partial T}\right)_{N,V} = \frac{\partial}{\partial V} ANkT \ln V = \boxed{\frac{ANkT}{V}}$$