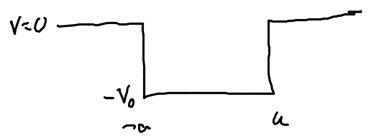
Finite Square Well



$$V(x) = \begin{cases} -V_0, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$$

Potential is even, symmetric around $x=0$

$\therefore$ solutions are either even or odd functions of x.

Suppose $\psi(x)$ is even $\psi(x) = \psi(-x)$

Bound states

$E<0$

$$\psi(x) = \begin{cases} F e^{-\kappa x} & x>a \\ D\cos \ell x & 0<x<a \\ \psi(-x) & x<0 \end{cases} \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar} \qquad \ell = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

Boundary Conditions

$\psi_+(a) = \psi_-(a)$

① $F e^{-\kappa a} = D\cos \ell a$

$\psi_+'(a) = \psi_-'(a)$

② $-\kappa F e^{-\kappa a} = -D\ell \sin \ell a$

$\frac{②}{①} \Rightarrow \quad \kappa = \ell \tan \ell a$

Solve graphically

$z = \ell a \qquad z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$

$\kappa^2 + \ell^2 = -\frac{2m\cancel{E}}{\hbar^2} + \frac{2m(\cancel{E}+V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2} = \frac{z_0^2}{a^2}$

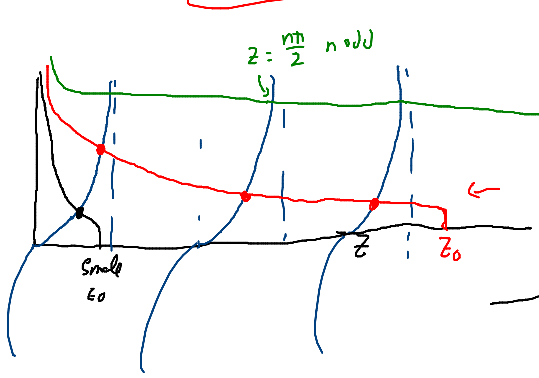
$z_0^2 - z^2 = a^2(\kappa^2+\ell^2) - \ell^2a^2 = \kappa^2 a^2$

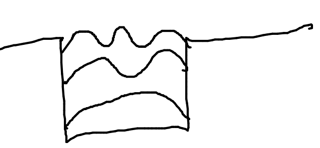
$a\kappa = \sqrt{z_0^2 - z^2} = \kappa\frac{z}{\ell} \rightarrow \frac{\kappa}{\ell} = \frac{\sqrt{z_0^2-z^2}}{z}$

$= \sqrt{\frac{z_0^2}{z^2} - 1}$

$\frac{\kappa}{\ell} = \frac{\ell}{\cancel{\ell}} \tan \ell a$

$$\boxed{\sqrt{\frac{z_0^2}{z^2} - 1} = \tan z}$$





If z_0 is very large,

$z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$ a deep well or a wide well

$z \approx \frac{n\pi}{2}$, n odd

$\ell a = \frac{n\pi}{2} \qquad \frac{\sqrt{2m(E+V_0)}}{\hbar} a = \frac{n\pi}{2} \qquad E + V_0 = \frac{n^2\pi^2\hbar^2}{(2a)^2\, 2m}$