"fundamental assumption of statistical mechanics":

In thermal equilibrium,
all distinct states w/ same energy
are equally probable.

e.g. two particles in one of 3 states $\overline{A}$ $\overline{B}$ $\overline{C}$

if distinguishable, 9 states as a pair

AA AB AC BA BB BC CA CB CC

if bosons, 6

AA AB AC BB BC CC

really $\frac{1}{\sqrt{2}}(AB+BA)$ etc.

if fermions

AB AC BC

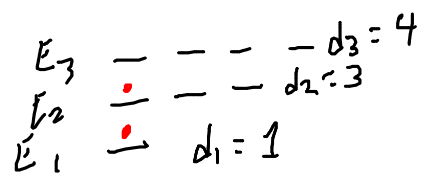
really $\frac{1}{\sqrt{2}}(BC-CB)$

Probability that particle 1 is A & particle 2 is B

fermions $\frac{1}{3}$ bosons $\frac{1}{6}$ dist. $\frac{1}{9}$

N particles which can be in one-particle energy states $E_1, E_2, E_3 \ldots$ with degeneracy $d_1, d_2, d_3, \ldots$

e.g.




$Q(1,1,0) = 3$

put $N_1$ into $E_1$, $N_2$ into $E_2$, etc.

How many ways can I do this.

$$Q(N_1, N_2, N_3, \ldots)$$

## Distinguishable

How to put $N_1$ particles into $E_1$

choose your $N_1$ particles $\rightarrow \binom{N}{N_1} d_1^{N_1} \leftarrow$ each chooses its own slot

$$Q(N_1, N_2, N_3, \ldots) = \binom{N}{N_1} d_1^{N_1} \binom{N-N_1}{N_2} d_2^{N_2} \binom{N-N_1-N_2}{N_3} d_3^{N_3} \cdots$$

$$\frac{N!}{N_1!\cancel{(N-N_1)!}} \quad \frac{\cancel{(N-N_1)!}}{N_2!\cancel{(N-N_1-N_2)!}} \quad \frac{\cancel{(N-N_1-N_2)!}}{N_3!(N-N_1-N_2-N_3)!} \cdots d_1^{N_1} d_2^{N_2} d_3^{N_3} \cdots$$

$$\frac{N!}{N_1!N_2!N_3!\cdots} d_1^{N_1} d_2^{N_2} d_3^{N_3} \cdots$$

$$Q(N_1, N_2 \ldots) = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$$

## Fermions

put $N_1$ particles into $d_1$ slots in row $E_1$

$$\binom{d_1}{N_1}$$

$$Q = \prod_{n=1}^{\infty} \binom{d_n}{N_n} = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n-N_n)!}$$

## Bosons

put $N_1$ particles into $d_1$ slots, but slots can have more than 1

$$\binom{N+d-1}{N}$$

$$Q = \prod_{n=1}^{\infty} \frac{(N_n+d_n-1)!}{N_n!(d_n-1)!}$$

Distinguishable $\quad Q = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$

Fermions $\quad Q = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n-N_n)!}$

Bosons $\quad Q = \prod_{n=1}^{\infty} \frac{(N_n+d_n-1)!}{N_n!(d_n-1)!}$

Suppose entire set of $N$ particles has energy $E$

want to maximize $Q(N_1, N_2, N_3, \dots)$

under constraints $\quad N = \sum_{n=1}^{\infty} N_n \qquad E = \sum_{n=1}^{\infty} N_n E_n$

maximize

$$G = \ln Q + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$$

$$0 = \frac{\partial G}{\partial \alpha} \rightarrow 0 = N - \sum_{n=1}^{\infty} N_n \rightarrow \sum_{n=1}^{\infty} N_n = N$$

$$\frac{\partial G}{\partial N_j} = 0$$

Distinguishable

$N \gg 1$

$$G \simeq \ln N! + \sum_{n=1}^{\infty}\left[N_n \ln d_n - (N_n \ln N_n - N_n) - \alpha N_n - \beta E_n N_n\right] + \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_j} = \ln d_j - \left(\ln N_j + \cancel{\frac{N_j}{N_j}} - \cancel{1}\right) - \alpha - \beta E_j$$

$$\rightarrow N_j = d_j e^{-(\alpha + \beta E_j)} = \frac{d_j}{e^{\alpha+\beta E_j}}$$

Fermions

$N \gg 1,\ d \gg 1$

$d \gg N$

$$G \simeq \sum_{n=1}^{\infty}\Big(d_n \ln d_n - N_n \ln N_n - (d_n - N_n)\ln(d_n - N_n) - \alpha N_n - \beta E_n N_n\Big) + \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_j} = -\ln N_j + \ln(d_j - N_j) - \alpha - \beta E_j = 0$$

$$\rightarrow N_j = \frac{d_j}{e^{(\alpha+\beta E_j)} + 1}$$

Bosons

$N \gg 1$

$d \gg 1$

$$G \simeq \sum_{n=1}^{\infty}\Big[(N_n + d_n)\ln(N_n + d_n) - N_n \ln N_n - d_n \ln d_n - \alpha N_n - \beta E_n N_n\Big] + \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_j} = \ln(N_j + d_j) - \ln N_j - \alpha - \beta E_j = 0$$

$$\rightarrow N_j = \frac{d_j}{e^{(\alpha+\beta E_j)} - 1}$$

General

$$N_j = \frac{d_j}{e^{\alpha+\beta E_j} + \begin{cases} 1, & \text{fermions} \\ 0, & \text{disting.} \\ -1, & \text{bosons} \end{cases}}$$

for fermions

$N_j \leq d_j$

denominator $\geq 1$

$\beta = \frac{1}{kT} \qquad \alpha = -\frac{\mu}{kT}$

bosons

denominator can be 0.

$N_j$ can be big as you like.

$$N_j = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + \begin{cases} 1 \\ 0 \\ -1 \end{cases}}$$

Fermi-Dirac

Maxwell-Boltzmann

Bose-Einstein

$$n(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/kT} + -}$$