Physics 4310 Homework #4

5 problems

Solutions

▷ **1.**

Suppose a spin–1/2 particle starts with the initial state $|\psi(0)\rangle = |\odot\rangle$. It is placed in a magnetic field $\vec{B} = B_0\hat{z}$.

(a) What is $|\psi(t)\rangle$?

(b) Find $\langle S_x\rangle$ as a function of time.

Answer:____________________________________

For an electron in the magnetic field $\vec{B} = B_0\hat{z}$, the Hamiltonian is $H = \omega S_z$ whre $\omega = eB/m_e$. This has energy eigenstates $|E_+\rangle = |\uparrow\rangle$ and $|E_-\rangle = |\downarrow\rangle$, with eigenvalues $E_\pm = \pm\frac{1}{2}\hbar\omega$.

(a) To find $|\psi(t)\rangle$ we first need to write $|\psi(0)\rangle$ in terms of the energy eigenstates. We know that $|\odot\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, and so

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|E_+\rangle + |E_-\rangle\right)$$

Now, because the Hamiltonian is time-independent, we can add in the Schrodinger factors $e^{-iE_\pm t/\hbar} = e^{\mp i\omega t/2}$:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}|E_+\rangle + e^{i\omega t/2}|E_-\rangle\right)$$

$$= frc\sqrt{2}e^{-i\omega t/2}\left(|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle\right)$$

(I factored out a common phase which is irrelevant to the state of the system.) In matrix notation (in the $\uparrow\downarrow$ basis)

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t/2}\begin{pmatrix}1\\ e^{i\omega t}\end{pmatrix}$$

(b) The average value of $\langle S_x\rangle = \langle\psi(t)|S_x|\psi(t)\rangle$. Remembering that the bra involves taking a complex conjugate, we write

$$\begin{aligned}
\langle\psi(t)|S_x|\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{+i\omega t/2}\begin{pmatrix}1 & e^{-i\omega t}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t/2}\begin{pmatrix}1\\ e^{i\omega t}\end{pmatrix}\\
&= \frac{\hbar}{4}\begin{pmatrix}1 & e^{-i\omega t}\end{pmatrix}\begin{pmatrix}e^{i\omega t}\\ 1\end{pmatrix}\\
&= \frac{\hbar}{4}(e^{i\omega t}+e^{-i\omega t})\\
&= \frac{\hbar}{2}\cos\omega t
\end{aligned}$$

(remembering that $2\cos\theta = e^{i\theta}+e^{-i\theta}$). The average value of S_x fluctuates between $\pm\frac{\hbar}{2}$, just as it would if the vector were spinning with frequency ω.

▷ **2.**
Consider an electron that starts in the state $|\uparrow\rangle$, in a magnetic field of $B_0 = 0.1\,\mathrm{T}$ pointing upward. A secondary magnetic field of $B_1 = 0.001\,\mathrm{T}$ is applied to the electron, perpendicular to the initial field, that rotates around the z axis with angular frequency. That is,

$$\vec{B} = B_0\hat{z} + B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$$

when the field is turned on.
(a) What should the frequency of rotation ω be to maximize the probability that the spin will flip to $|\downarrow\rangle$?
(b) Given the answer in part (a), what is the shortest amount of time that B_1 should be turned on, to guarantee that the spin flips?

Answer:

(a) The probability that the spin will flip is

$$\mathcal{P} = \frac{\omega_1^2}{(\Delta\omega)^2+\omega_1^2}\sin^2\left(\frac{\sqrt{(\Delta\omega)^2+\omega_1^2}}{2}t\right)$$

where $\Delta\omega = \omega-\omega_0$, $\omega_0 = \frac{eB_0}{m}$, and $\omega_1 = \frac{eB_1}{m}$. The probability is maximized when $\Delta\omega = 0 \implies \omega = \omega_0$. With these numbers,

$$\omega_0 = \frac{eB_0}{m} = \frac{(1.6\times10^{-19}\,\mathrm{C})(0.1\,\mathrm{T})}{(9.11\times10^{-31}\,\mathrm{kg})} = \boxed{1.8\times10^{10}\,\mathrm{Hz}} = 18\times10^{GHz}$$

(b) At time $t=0$, the $\sin^2$ function in the probability is equal to 0. The spin is guaranteed to flip when the $\sin^2$ is 1, or when

$$\begin{aligned}\frac{\sqrt{(\Delta\omega)^2+\omega_1^2}}{2}t &= \frac{\pi}{2}\\ \implies t &= \frac{\pi}{\omega_1}\\ &= \frac{\pi}{\frac{eB_1}{m}} = \frac{m\pi}{eB_1}\\ &= \frac{(9.11\times 10^{-31}\,\mathrm{kg})\pi}{(1.6\times 10^{-19}\,\mathrm{C})(0.001)}\\ &= \boxed{1.8\times 10^{-8}\,\mathrm{s}} = 1.8\,\mathrm{ns}\end{aligned}$$

Notice that the larger B_1 is, the shorter this time is.

▷ **3.**
Define the wavefunctions

$$\psi_n(x) = \begin{cases} C_n(1-x^n) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider the two functions $|\psi_2\rangle$ and $|\psi_4\rangle$, specifically.
(a) Find C_2 and C_4 so that the two functions are normalized.
(b) Find $\langle\psi_2|\psi_4\rangle$. Are the functions orthogonal?
(c) Find the average value $\langle x\rangle$ and $\langle p\rangle$ for both functions.

Answer:

(a) The normalization condition is

$$\begin{aligned}1 &= \langle\psi_n||\psi_n\rangle\\ &= \int_{-1}^{1} C_n^*(1-x^n)C_n(1-x^n)\,dx\\ &= |C_n|^2\int_{-1}^{1} 1-2x^n+x^{2n}\,dx\\ &= |C_n|^2\left[x-\frac{2}{n+1}x^{n+1}+\frac{1}{2n+1}x^{2n+1}\right]_{-1}^{1}\end{aligned}$$

The phase of C_n doesn't matter, so let's assume it is positive. If $n = 2$,

$$1 = C_2^2 \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = C_2^2 \left(2 - \frac{4}{3} + \frac{2}{5}\right) = \frac{16}{15}C_2^2$$

$$\implies C_2 = \sqrt{\frac{15}{16}} = \boxed{\frac{\sqrt{15}}{4}}$$

and for $n = 4$,

$$1 = C_4^2 \left[x - \frac{2}{5}x^5 + \frac{1}{9}x^9\right]_{-1}^{1} = C_4^2 \left(2 - \frac{4}{5} + \frac{2}{9}\right) = \frac{64}{45}C_4^2$$

$$\implies C_4 = \sqrt{\frac{45}{64}} = \boxed{\frac{\sqrt{45}}{8}}$$

(b)

$$\begin{aligned}
\langle \psi_2 | \psi_4 \rangle &= \int_{-1}^{1} \frac{\sqrt{15}}{4}(1 - x^2)\frac{\sqrt{45}}{8}(1 - x^4)\, dx \\
&= \frac{15\sqrt{3}}{32} \int_{-1}^{1} 1 - x^2 - x^4 + x^6 \, dx \\
&= \frac{15\sqrt{3}}{32} \left[x - \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7\right]_{-1}^{1} \\
&= \frac{15\sqrt{3}}{32} \left[2 - \frac{2}{3} - \frac{2}{5} + \frac{2}{7}\right] \\
&= \frac{15\sqrt{3}}{32}\frac{128}{105} \\
&= \boxed{\frac{4}{7}\sqrt{3}}
\end{aligned}$$

It isn't zero, so the two wavefunctions aren't orthogonal.

(c) The average value $\langle x \rangle$ is

$$\begin{aligned}
\langle x \rangle &= \langle \psi_n | x | \psi_n \rangle \\
&= \int_{-1}^{1} C_n^2 (1 - x^n) x (1 - x^n)\, dx \\
&= C_n^2 \int_{-1}^{1} x - 2x^{n+1} + x^{2n+1}\, dx \\
&= C_n^2 \left[\frac{1}{2} x^2 - \frac{2}{n+2} x^{n+2} + \frac{1}{2n+2} x^{2n+2} \right]_{-1}^{1}
\end{aligned}$$

Because n is even, each term inside the brackets is even, and so $\langle x \rangle = 0$ for any even n.

Using the operator $p = \frac{\hbar}{i} \frac{d}{dx}$, we have

$$\begin{aligned}
\langle p \rangle &= \langle \psi_n | p | \psi_n \rangle \\
&= C_n^2 \frac{\hbar}{-i} \frac{\hbar}{i} \int_{-1}^{1} (1 - x^n) \frac{d}{dx} (1 - x^n)\, dx \\
&= C_n^2 \hbar^2 \int_{-1}^{1} (1 - x^n)(n x^{n-1})\, dx \\
&= C_n^2 \hbar^2 \int_{-1}^{1} n x^{n-1} - n x^{2n-1}\, dx \\
&= C_n^2 \hbar^2 \left[\frac{n}{n} x^n - \frac{n}{2n} x^{2n} \right]_{-1}^{1}
\end{aligned}$$

Again, because n is even, this is equal to zero as well.

▷ **4.**

In the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V\psi = E\psi$$

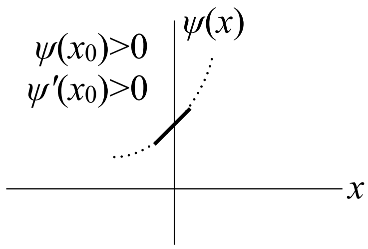
Show that the eigenvalues E must exceed the minimum value of $V(x)$, for every normalizable solution to the time-independent Schrödinger equation. Hint: Isolate $\frac{d^2\psi}{dx^2}$, and show that, if $E < V_{\min}$, then ψ and ψ'' have the same sign. Argue that such a function cannot be normalized.

Answer:

Taking the hint, we rewrite the equation in terms of the second derivative:

$$\frac{d^2\psi}{dx^2} = +\frac{2m}{\hbar^2}(V - E)\psi$$

If E is smaller than the minimum value of V, then $V - E > 0$ for all values of x, which means that ψ and ψ'' always have the same sign. Now, if ψ is going to be normalizable, then $\psi(x)$ must go to zero as $x \to \pm\infty$.



Consider the case where, at some point $x = x_0$, the wavefunction and its first derivative are both positive, as shown. In order for $\psi(x)$ to go to zero for $x > x_0$, the slope must go from positive to negative at some point to the right of x_0. But the second derivative is always positive, which means that as x increases, the positive slope at x_0 can only get more and more positive, and can never go negative. Thus the wavefunction does not approach zero as x goes to $+\infty$, and so it is unnormalizable. Similar arguments can be made for the other possible combinations of positive and negative $\psi(x_0)$ and $\psi'(x_0)$ (i.e. by flipping this picture vertically or horizontally), and so the only way that a wavefunction can be normalizable is if $\psi(x)$ and $\psi''(x)$ have different signs at least some of the time, and that can only happen here if $E > V(x)$ for at least some values of x. Therefore, E is greater than the minimum value of $V(x)$. **Q.E.D.**

▷ **5.**

A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A[\psi_1(x) + \psi_2(x)]$$

(a) Normalize $\Psi(x, 0)$; that is, find A.
(b) Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$. Express the latter as a sinusoidal function of time. To simplify, let $\omega = \pi^2\hbar/2ma^2$.
(c) Compute $\langle x\rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? The amplitude? (If your amplitude is greater than $a/2$, then the particle has left the well which is impossible; try again.)
(d) Compute $\langle p\rangle$.
(e) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H.

Answer:

(a) Assuming $\psi_n(x)$ are orthonormal, $\Psi(x,0)$ is normalized if

$$\begin{aligned} 1 &= \langle \Psi(x,0)|\Psi(x,0)\rangle = A^2[\langle\psi_1| + \langle\psi_2|][|\psi_1\rangle + |\psi_2\rangle] \\ 1 &= A^2[2] \\ \implies A &= \frac{1}{\sqrt{2}} \end{aligned}$$

(I assumed that A is positive because the overall phase doesn't matter.)

(b) $\Psi(x,0)$ is written in the energy basis, so we only need to add the Schrodinger factors, where $E_n = n^2 E_1$ (where $E_1 = \pi^2\hbar^2/2ma^2 = \hbar\omega$). Thus

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[e^{-i\omega t}\psi_1(x) + e^{-4i\omega t}\psi_2(x)\right]$$

and

$$\begin{aligned} |\Psi(x,t)|^2 &= \Psi^*(x,t)\Psi(x,t) \\ &= \frac{1}{2}\left[e^{i\omega t}\psi_1^*(x) + e^{4i\omega t}\psi_2^*(x)\right]\left[e^{-i\omega t}\psi_1(x) + e^{-4i\omega t}\psi_2(x)\right] \\ &= \frac{1}{2}\left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + e^{3i\omega t}\psi_2^*(x)\psi_1(x) + e^{-3i\omega t}\psi_1(x)\psi_2^*(x)\right] \end{aligned}$$

Because $\psi_n(x)$ is real for the infinite square well, $\psi_n^*(x) = \psi_n(x)$, and

$$\begin{aligned} |\Psi(x,t)|^2 &= \frac{1}{2}\left[\psi_1(x)^2 + \psi_2(x)^2\right] + \psi_1(x)\psi_2(x)\frac{e^{3i\omega t} + e^{-3i\omega t}}{2} \\ &= \frac{1}{2}\left[\psi_1(x)^2 + \psi_2(x)^2\right] + \psi_1(x)\psi_2(x)\cos 3\omega t \end{aligned}$$

Filling in the form $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$,

$$\Psi(x,t) = \frac{1}{\sqrt{a}}\left[e^{-i\omega t}\sin\left(\frac{\pi x}{a}\right) + e^{-4i\omega t}\sin\left(\frac{2\pi x}{a}\right)\right]$$

and

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\right] + \frac{2}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos 3\omega t$$

(c) The average position is

$$\begin{aligned}
\langle x\rangle &= \langle\Psi(x,t)|x|\Psi(x,t)\rangle \\
&= \frac{1}{2}\left(\langle\psi_1|e^{+i\omega t}+\langle\psi_2|e^{+4i\omega t}\right)x\left(|\psi_1\rangle e^{-i\omega t}+|\psi_2\rangle e^{-4i\omega t}\right) \\
&= \frac{1}{2}\left(\langle\psi_1|x|\psi_1\rangle+\langle\psi_2|x|\psi_2\rangle+e^{3i\omega t}\langle\psi_2|x|\psi_1\rangle+e^{-3i\omega t}\langle\psi_1|x|\psi_2\rangle\right) \\
&= \frac{1}{2}\langle\psi_1|x|\psi_1\rangle+\frac{1}{2}\langle\psi_2|x|\psi_2\rangle+\cos 3\omega t\langle\psi_1|x|\psi_2\rangle
\end{aligned}$$

where I use the fact $\langle\psi_1|x|\psi_2\rangle = \langle\psi_2|x|\psi_1\rangle$ because the operator and the eigenfunctions are all real. Now we need to find the matrix elements $\langle\psi_m|x|\psi_n\rangle$. Remember that $\psi_n = \frac{2}{a}\sin\left(\frac{n\pi x}{a}\right)$. I'm going to ask Mathematica to do the integrals, if y'all don't mind:

$$\begin{aligned}
\langle\psi_1|x|\psi_1\rangle &= \int_0^a\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right)x\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right)dx = \frac{a}{2} \\
\langle\psi_2|x|\psi_2\rangle &= \frac{2}{a}\int_0^a \sin^2\frac{2\pi x}{a}x\,dx = \frac{a}{2} \\
\langle\psi_1|x|\psi_2\rangle &= \frac{2}{a}\int_0^a \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}x\,dx = -\frac{16}{9\pi^2}a
\end{aligned}$$

(The first two should make sense, given that the energy eigenstates are symmetric in the well around $x = a/2$.) Thus

$$\langle x\rangle = \frac{1}{2}\frac{a}{2}+\frac{1}{2}\frac{a}{2}-\frac{16}{9\pi^2}a\cos 3\omega t$$

The angular frequency of the position's oscillation is 3ω, and the amplitude is $\frac{16}{9\pi^2}a = 0.18a$. This is less than $\frac{a}{2}$, so the particle's position expectation value oscillates around the center of the well, but stays inside the well: good!

(d) Similar to the previous step, the expectation value of the momentum is

$$\langle p\rangle = \frac{1}{2}\langle\psi_1|x|\psi_1\rangle+\frac{1}{2}\langle\psi_2|x|\psi_2\rangle+\frac{1}{2}\langle\psi_1|x|\psi_2\rangle e^{-3i\omega t}+\frac{1}{2}\langle\psi_2|x|\psi_1\rangle e^{+3i\omega t}$$

(I'm not going to assume that the last two terms can be combined, because the momentum operator isn't real. Again, we need the matrix elements:

$$
\begin{aligned}
\langle\psi_m|p|\psi_n\rangle &= \frac{2}{a}\int_0^a \sin\frac{m\pi x}{a}\left(\frac{\hbar}{i}\frac{d}{dx}\right)\sin\frac{n\pi x}{a}\,dx \\
&= -i\hbar\frac{2}{a}\int_0^a \sin\frac{m\pi x}{a}\frac{n\pi}{a}\cos\frac{n\pi x}{a}\,dx \\
&= -i\frac{2\pi n\hbar}{a^2}\int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}\,dx
\end{aligned}
$$

Mathematica tells me that

$$
\begin{aligned}
\langle\psi_1|p|\psi_1\rangle &= \langle\psi_2|p|\psi_2\rangle = 0 \\
\langle\psi_1|p|\psi_2\rangle &= -\frac{2a}{3\pi}\left(\frac{-4\pi i\hbar}{a^2}\right) = \frac{8i\hbar}{3a} \\
\langle\psi_2|p|\psi_1\rangle &= \frac{4a}{3\pi}\left(\frac{-2\pi i\hbar}{a^2}\right) = \frac{-8i\hbar}{3a}
\end{aligned}
$$

and so

$$
\begin{aligned}
\langle p\rangle &= \frac{4i\hbar}{3a}e^{-3i\omega t} - \frac{4i\hbar}{3a}e^{3i\omega t} \\
&= \frac{e^{3i\omega t} - e^{-3i\omega t}}{2i}\frac{8\hbar}{3a} \\
&= \frac{8\hbar}{3a}\sin 3\omega t
\end{aligned}
$$

(e) If you measure the energy, you will get either $E = E_1$ or $E = E_2 = 4E_1$, each with probability 50%. The expectation value is

$$
\begin{aligned}
\langle H\rangle = \langle\Psi(x,t)|H|\Psi(x,t)\rangle &= \frac{1}{2}\langle\psi_1|H|\psi_1\rangle + \frac{1}{2}\langle\psi_2|H|\psi_2\rangle + \frac{1}{2}\langle\psi_1|H|\psi_2\rangle e^{-3i\omega t} + \frac{1}{2}\langle\psi_2|H|\psi_1\rangle e^{+3i\omega t} \\
&= \frac{1}{2}\langle\psi_1|E_1|\psi_1\rangle + \frac{1}{2}\langle\psi_2|E_2|\psi_2\rangle + \frac{1}{2}\langle\psi_1|E_2|\psi_2\rangle e^{-3i\omega t} + \frac{1}{2}\langle\psi_2|E_1|\psi_1\rangle e^{+3i\omega t} \\
&= \frac{1}{2}E_1\langle\psi_1||\psi_1\rangle + \frac{1}{2}E_2\langle\psi_2||\psi_2\rangle + \frac{1}{2}E_2\langle\psi_1||\psi_2\rangle e^{-3i\omega t} + \frac{1}{2}E_1\langle\psi_2||\psi_1\rangle e^{+3i\omega t} \\
&= \frac{1}{2}E_1 + \frac{1}{2}E_2 \\
&= \frac{1}{2}(E_1 + 4E_1) = \boxed{\frac{5}{2}E_1}
\end{aligned}
$$

halfway in-between the two energies, as one might expect. (Though interesting that it doesn't have any time dependence.)