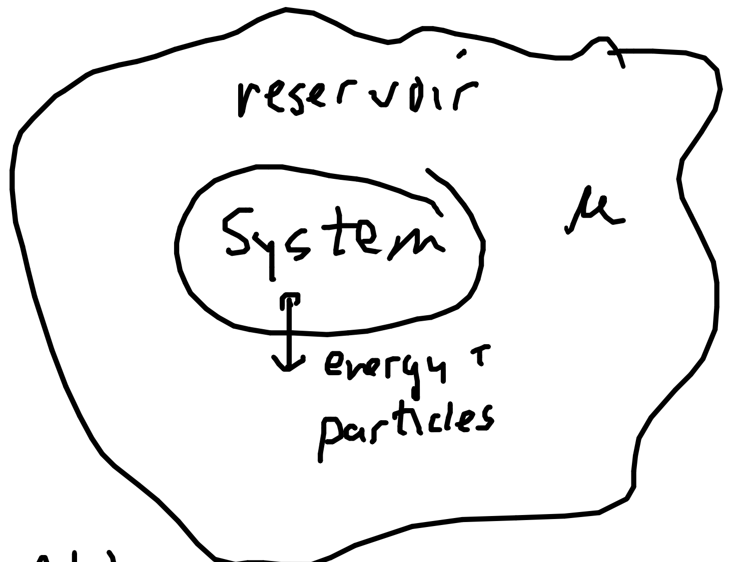
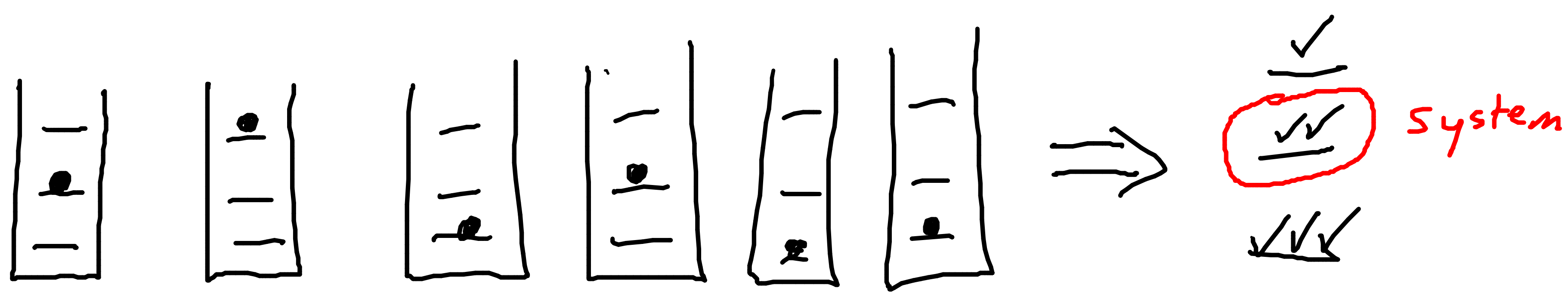
Gibbs statistics

$$P_s = \frac{e^{-\beta(E_s - \mu N_s)}}{\mathcal{Z}}$$



$$\mathcal{Z} = \sum_s e^{-\beta(E_s - \mu N_s)}$$

Identical Particles



Consider all particles in one energy microstate to be the "system"

Particles enter or leave system by changing energy. by sharing energy with other particles.

Define occupancy n of state = # particles in that state

Let ε be the energy of our chosen microstate

Let μ be the "chemical potential" of the reservoir. i.e. particles not in system

If $\mu << \varepsilon$ then μ is "small"

low density $\rightarrow$ non-interacting particles

<u>Non-interacting particles</u>

How many particles are in state ε?

$$\langle n \rangle = \underbrace{\text{Probability of being in } \varepsilon}_{\frac{1}{Z_1} e^{-\beta\varepsilon}} \times N$$

for entire set of particles

$$Z = \frac{1}{N!} Z_1^N$$

$$\mu = -kT \frac{\partial \ln Z}{\partial N}$$

$$= -kT \frac{\partial}{\partial N} \left[N \ln Z_1 - (N \ln N - N) \right]$$

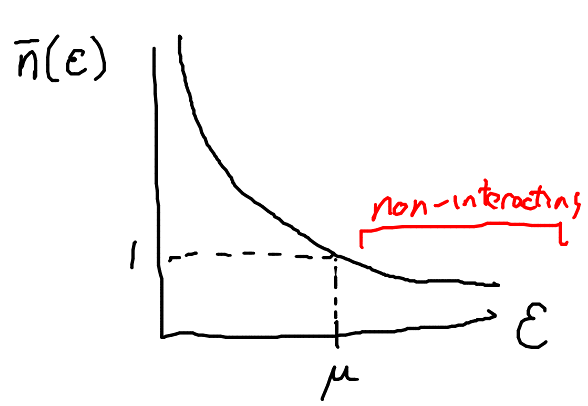
$$= -kT \left[\ln Z_1 - \ln N \right]$$

$$\mu = -kT \ln \frac{Z_1}{N}$$

$$\rightarrow Z_1 = N e^{-\beta\mu}$$

$$\langle n \rangle = \frac{N e^{-\beta\varepsilon}}{Z_1} = \frac{N e^{-\beta\varepsilon}}{N e^{-\beta\mu}} = e^{-\beta(\varepsilon - \mu)}$$

$$\bar{n} = \langle n \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}} \quad \text{when } \mu << \varepsilon.$$



$\bar{n} < 1$

mostly empty

If $\varepsilon < \mu$ then many particles want to be in the system. Need a different model.

Using Gibbs Statistics,

"states": # of particles n in the system (i.e. the microstate $\mathcal{E}$)

$$\mathcal{Z} = \sum_s e^{-\beta(E_s - \mu N_s)}$$ $N_s = n \quad E_s = n\mathcal{E}$

$$= \sum_{n=0} e^{-\beta(n\mathcal{E} - \mu n)} = \sum_n e^{-n\beta(\mathcal{E}-\mu)}$$

Bosons $\mathcal{Z} = \sum_{n=0}^{N} e^{-n\beta(\mathcal{E}-\mu)}$

Fermions $\mathcal{Z} = \sum_{n=0}^{1} e^{-n\beta(\mathcal{E}-\mu)}$ Pauli exclusion principle

Let $x = \beta(\mathcal{E}-\mu)$

Bosons: $\mathcal{Z}_B = \sum_{n=0}^{\infty} \underbrace{e^{-nx}}_{(e^{-x})^n} = \frac{1}{1-e^{-x}}$

Fermions: $\mathcal{Z}_F = \sum_{n=0}^{1} e^{-nx} = 1 + e^{-x}$

Average occupancy

$$\bar{n} = \frac{1}{\mathcal{Z}} \sum_n n e^{-nx} = \frac{1}{\mathcal{Z}} \sum_n \left(-\frac{\partial}{\partial x} e^{-nx}\right)$$

$$= -\frac{1}{\mathcal{Z}} \frac{\partial}{\partial x} \sum_n e^{-nx} = -\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial x}$$

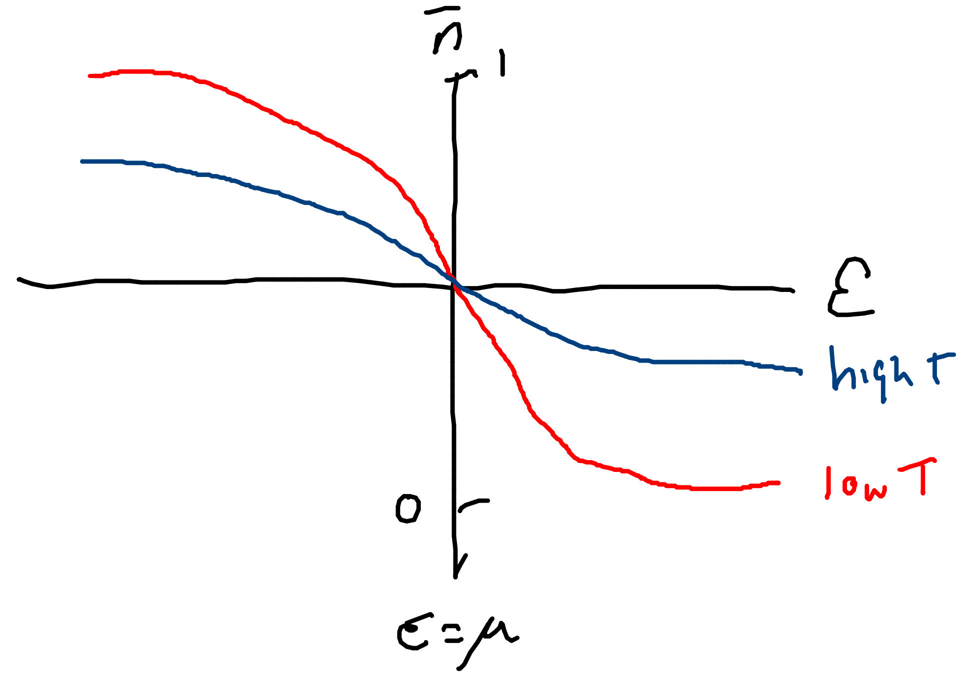
Fermions

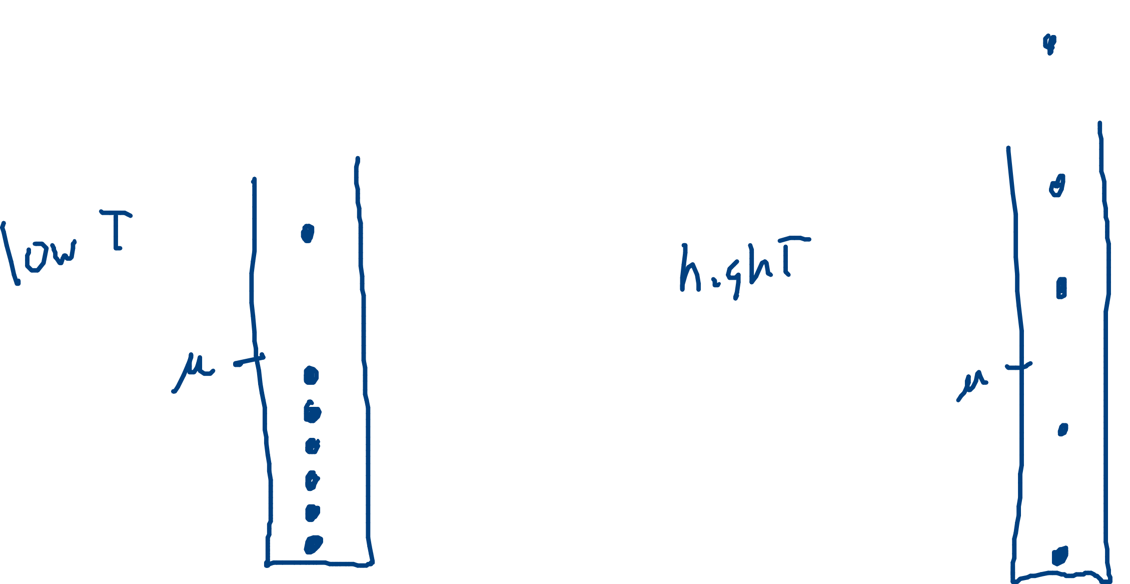
$$\mathcal{Z}_F = 1 + e^{-x} \qquad \frac{\partial \mathcal{Z}}{\partial x} = -e^{-x}$$

$$\bar{n} = -\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial x} = -\frac{-e^{-x}}{1+e^{-x}} = \frac{1}{e^{x}+1}$$

$$\bar{n}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$ Fermi-Dirac Distribution

If $\varepsilon = \mu$, $\bar{n} = \frac{1}{2}$





Bosons

$$\mathcal{Z}_B = (1-e^{-x})^{-1}$$

$$\frac{\partial \mathcal{Z}}{\partial x} = -(1-e^{-x})^{-2}\left(-(e^{-x})\right)$$

$$= -\frac{e^{-x}}{(1-e^{-x})^2} = -e^{-x}\mathcal{Z}^2$$

$$\bar{n} = -\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial x} = +\frac{1}{\cancel{\mathcal{Z}}}\left(+e^{-x}\mathcal{Z}^{\cancel{2}}\right)$$

$$= \frac{e^{-x} \times e^{x}}{1-e^{-x} \times e^{x}} = \frac{1}{e^{x}-1}$$

$$\bar{n} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$ Bose-Einstein distribution

if $\varepsilon \gg \mu$ $\bar{n}$ very small

if $\varepsilon < \mu$ $\bar{n} < 0$ impossible!

ε always $\geq \mu$

$\varepsilon \approx \mu$, x small $\quad \dfrac{1}{e^{x}-1} \approx \dfrac{1}{1+x-1} = \dfrac{1}{x}$ huge!

μ is determined by $\sum_{s} \bar{n}_s = N$

(s: every microstate)