Identical Particles

$$\psi(\vec{r}_1, \vec{r}_2) = \pm \psi(\vec{r}_2, \vec{r}_1)$$

+ : bosons
symmetric

− : fermions,
antisymmetric

e.g. Calculating $\langle (x_1 - x_2)^2 \rangle$

Distinguishable Particles

$$\psi(x_1, x_2) = \psi_a(x_1)\,\psi_b(x_2)$$

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

$$\langle x_1^2 \rangle = \langle \psi | x_1^2 | \psi \rangle$$

$$= \langle \psi_a(x_1)\psi_b(x_2) | x_1^2 | \psi_a(x_1)\,\psi_b(x_2) \rangle$$

$$= \langle \psi_a(x_1) | x_1^2 | \psi_a(x_1) \rangle \ \langle \psi_b(x_2) | \psi_b(x_2) \rangle$$

$$= \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \langle \psi_a(x_1)\psi_b(x_2) | x_1 x_2 | \psi_a(x_1)\psi_b(x_2) \rangle$$

$$= \langle x \rangle_a \ \langle x \rangle_b$$

$$= \langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$$

$$\psi_\pm(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)\right]$$

$$\langle x_1^2\rangle = \tfrac{1}{2}\langle \psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1) | x_1^2 | \psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)\rangle$$

Suppose ψ_a & ψ_b are orthogonal

$$= \tfrac{1}{2}\langle \psi_a(x_1)\psi_b(x_2) | x_1^2 | \psi_a(x_1)\psi_b(x_2)\rangle$$

$$\pm \tfrac{1}{2}\langle \psi_a(x_2)\psi_b(x_1) | x_1^2 | \psi_a(x_1)\psi_b(x_2)\rangle \leftarrow \langle\psi_b(x_1)|x_1^2|\psi_a(x_1)\rangle \underbrace{\langle\psi_a(x_2)|\psi_b(x_2)\rangle}_{0 \text{ b/c orthogonal}}$$

$$\pm \tfrac{1}{2}\langle \psi_a(x_1)\psi_b(x_2) | x_1^2 | \psi_a(x_2)\psi_b(x_1)\rangle \leftarrow 0$$

$$+ \tfrac{1}{2}\langle \psi_a(x_2)\psi_b(x_1) | x_1^2 | \psi_a(x_2)\psi_b(x_1)\rangle$$

$$= \tfrac{1}{2}\left[\langle x^2\rangle_a + \langle x^2\rangle_b\right]$$

$$\langle x_2^2\rangle = \tfrac{1}{2}\left[\langle x^2\rangle_b + \langle x^2\rangle_a\right]$$

$$\langle x_1 x_2\rangle = \langle x\rangle_a \langle x\rangle_b \pm \Big|\overbrace{\langle\psi_a(x)|x|\psi_b(x)\rangle}^{\langle x\rangle_{ab}}\Big|^2$$

$$\langle (x_1-x_2)^2\rangle = \underbrace{\langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b}_{\text{same as for distinguishable}} \mp 2|\langle x\rangle_{ab}|^2$$

for bosons, $\langle (x_1-x_2)^2\rangle$ will be smaller

bosons tend to be closer together

for fermions, spacing is larger

fermions tend to spread apart

"exchange force" or exchange interaction e.g. Hydrogen molecule

$\oplus \quad \ominus \quad \ominus \quad \oplus$ ← if electrons are in the middle, they will pull the protons inward and hold molecule together

vs

$\ominus \quad \oplus \quad \oplus \quad \ominus$

if electrons are on the outside, protons will push each other apart

for electrons to end up in the middle, they need to act like bosons

their spatial wavefunction $\psi(r_1, r_2)$ is symmetric

Overall wavefunction is $\psi(r_1, r_2)\, \chi(s_1, s_2)$

We can have symmetric ψ

if we have an antisymmetric χ

so that overall wavefunction is antisymmetric as befitting fermions

$\chi(s_1, s_2)$ is one of four possibilities

$$\text{symmetric} \begin{cases} \uparrow\uparrow \\ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \downarrow\downarrow \end{cases}$$

$$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

"singlet state" is antisymmetric

$\therefore$ electrons in a H_2 molecule are in the singlet state

$s = 0 \quad m = 0$