Physics 3410 Homework #4

5 problems

Solutions

▷ **1.**

Find the entropy of

(a) an N–state paramagnet with energy $U = N_\uparrow$, if N and U are both large.

(b) an N–oscillator Einstein solid with energy q, when both variables are large numbers.

Answer:

(a) An N–state paramagnet with U up arrows has $\Omega = \binom{N}{U}$ microstates, so has entropy

$$S = k \ln \binom{N}{U} = k\left[N \ln N - U \ln U - (N-U)\ln(N-U)\right]$$

$$= \boxed{k\left[N \ln \frac{N}{N-U} + U \ln \frac{N-U}{U}\right]}$$

(b) The Einstein solid has $\Omega = \binom{N+q-1}{q}$, so

$$S = k \ln \Omega = k \ln(N+q-1)! - k\ln(N-1)! - k \ln q!$$

$$\approx k\left[(N+q-1)\ln(N+q-1) - (N+q-1)\right] - k\left[(N-1)\ln(N-1) - (N-1)\right] - k\left[q \ln q - q\right]$$

$$= k(N-1)\ln\left(\frac{N+q-1}{N-1}\right) + kq \ln\left(\frac{N+q-1}{q}\right)$$

$$\approx \boxed{kN \ln \frac{N+q}{N} + kq \ln \frac{N+q}{q}}$$

▷ **2.**

Two three-oscillator Einstein solids ($N_A = N_B = 3$) can exchange energy with each other; together they have energy $q = 5$.

(a) What is the probability that $q_A = 2$?

(b) What is the probability that $q_A = 5$?

Answer:___________________________________

(a) The probability is

$$P(q_A = 2) = \frac{\Omega(q_A = 2)}{\Omega_{all}}$$

Ω_{all} is the number of ways that energy can be distributed across both solids regardless of how many quanta are in A. To figure that out, we can think of both solids as one giant solid with $N = N_A + N_B = 6$ and $q = 5$; thus

$$\Omega_{all} = \binom{5+6-1}{5} = \frac{10!}{5!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \boxed{252}$$

$\Omega(q_A = 2)$ is the number of ways both solids can arrange themselves so that 2 units of energy are in A (and $q - q_A = 3$ units of energy in B). This is

$$\Omega(q_A = 2) = \Omega_A \Omega_B = \binom{2+3-1}{2}\binom{3+3-1}{3} = \frac{4!}{2!2!}\frac{5!}{3!2!} = (6)(10) = 60$$

and so the probability that A ends up with 2 units of energy is

$$P = \frac{60}{252} = \boxed{24\%}$$

(b) Ω_{all} is the same in this case, and

$$\Omega(q_A = 5) = \Omega_A \Omega_B = \binom{5+3-1}{5}\binom{0+3-1}{0} = \frac{7!}{5!2!}(1) = 21$$

so the probability that all the energy in solid A is

$$P = \frac{21}{252} = 8\%$$

This is less likely, but not incredibly so. (Small systems are much more likely to be found away from equilibrium.)

▷ **3.**

Two Einstein solids are in contact with each other, the first having N_A oscillators and the second N_B oscillators. The total energy of both is q, with solid A having energy q_A and solid B having energy q_B. (In other words, same as in class.) In the high-temperature limit $q_A \gg N_A \gg 1$ and $q_B \gg N_B \gg 1$, prove that the most likely energy macrostate q_A is the one where

$$\frac{q_A}{q_B} = \frac{N_A}{N_B}$$

Hints: Find the value of q_A which maximizes the multiplicity $\Omega(q_A)$. Use the chain rule, the product rule, and the fact that $\frac{\partial q_A}{\partial q_B} = -1$. And if you end up seeing $N_A - 1$, resist the urge to drop the 1.

Answer:____________________________________
The most likely value of q_A is the one which maximizes

$$\Omega(q_A) = \Omega_A \Omega_B$$

where Ω_A is the multiplicity of solid A assuming it has energy q_A, and Ω_B is the multiplicity of solid B with energy $q_B = q - q_A$. In the high-temperature limit, we can use the approximation $\Omega = (eq/N)^N$, so

$$\begin{aligned}
\Omega &= \left(\frac{eq_A}{N_A}\right)^{N_A} \left(\frac{eq_B}{N_B}\right)^{N_B} \\
&= \left(\frac{e}{N_A}\right)^{N_A} \left(\frac{e}{N_B}\right)^{N_B} q_A^{N_A}(q-q_A)^{N_B}
\end{aligned}$$

This is maximized where

$$\begin{aligned}
0 &= \frac{\partial \Omega}{\partial q_A} \\
&= \frac{\partial}{\partial q_A}\left[\left(\frac{e}{N_A}\right)^{N_A} \left(\frac{e}{N_B}\right)^{N_B} q_A^{N_A}(q-q_A)^{N_B}\right] \\
&= \left(\frac{e}{N_A}\right)^{N_A} \left(\frac{e}{N_B}\right)^{N_B} \frac{\partial}{\partial q_A}\left[q_A^{N_A}(q-q_A)^{N_B}\right]
\end{aligned}$$

I can divide out the first two factors to get.

$$\begin{aligned}
0 &= \frac{\partial}{\partial q_A}\left[q_A^{N_A}(q-q_A)^{N_B}\right] \\
&= \frac{\partial q_A^{N_A}}{\partial q_A}(q-q_A)^{N_B} + q_A^{N_A}\frac{\partial (q-q_A)^{N_B}}{\partial q_A} \\
&= \left(N_A q_A^{N_A-1}\right)(q-q_A)^{N_B} + q_A^{N_A}\left(-N_B(q-q_A)^{N_B-1}\right) \\
&= N_A \frac{q_A^{N_A}}{q_A}(q-q_A)^{N_B} - N_B q_A^{N_A}\frac{(q-q_A)^{N_B}}{q-q_A} \\
&= q_A^{N_A}(q-q_A)^{N_B}\left[\frac{N_A}{q_A} - \frac{N_B}{q_B}\right]
\end{aligned}$$

The first two factors can't be zero, so

$$\frac{N_A}{q_A} = \frac{N_B}{q_B} \implies \frac{q_A}{q_B} = \frac{N_A}{N_B}$$

Q.E.D.

▷ **4.**

What is the surface area of a 12–dimensional sphere with radius $R = 2\,\mathrm{m}$?

Answer:____________________________________

The formula for the surface area of an n–sphere is

$$S_n(R) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} R^{n-1}$$

where in this case $n = 12$. Since $\Gamma(n) = (n-1)!$, $\Gamma\left(\frac{12}{2}\right) = 5! = 120$, and so

$$S_{12}(2\,\mathrm{m}) = \frac{2\pi^6}{120}(2\,\mathrm{m})^{11} = \frac{512\pi^6}{15}\,\mathrm{m}^{11} = \boxed{3.28 \times 10^4\,\mathrm{m}^{11}}$$

(Not often you get an answer in meters to the eleventh power! :))

▷ **5.**

Consider an ideal gas with $N = 10^{23}$ particles and an internal energy of $U = 100\,\mathrm{J}$. If the gas triples in volume, but its entropy remains constant, what is the internal energy after it expands?

Answer:____________________________________

The Sackur-Tetrode equation for the ideal gas says that

$$S = kN\left[\ln\frac{V}{N} + \frac{3}{2}\ln\frac{U}{N} + C\right]$$

During the expansion, the volume increases from V to $3V$, and suppose the energy changes from U to αU. Then

$$\begin{aligned} S &= kN\left[\ln\frac{3V}{N} + \frac{3}{2}\ln\frac{\alpha U}{N} + C\right] \\ &= kN\left[\ln\frac{V}{N} + \ln 3 + \frac{3}{2}\ln\frac{U}{N} + \frac{3}{2}\ln\alpha + C\right] \end{aligned}$$

The only way the entropy can remain constant is if the two new terms, $\ln 3$ and $\frac{3}{2}\ln\alpha$, cancel:

$$\ln 3 = -\frac{3}{2}\ln\alpha \implies \ln\alpha = -\frac{3}{2}\ln 3 \implies \ln\alpha = \ln 3^{-3/2}$$

$$\implies \alpha = 3^{-3/2} = \frac{1}{3\sqrt{3}} = 0.19$$

Thus the new energy after the expansion is

$$U' = \alpha(100\,\mathrm{J}) = \boxed{19\,\mathrm{J}}$$