Physics 4310 Homework #5

5 problems

Solutions

▷ **1.**

Evaluate $a_+\psi_1$ to find the $n = 2$ eigenstate of the harmonic oscillator. Give the correct normalization (which you can find in the text).

Answer:

From the textbook (2.62), the first excited eigenstate is

$$\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2} = Axe^{-qx^2}$$

where we'll define $A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}$ and $q = \frac{m\omega}{2\hbar}$ for simplicity.

The raising operator is

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) = \frac{1}{\sqrt{2\hbar m\omega}}(-\hbar\frac{d}{dx} + m\omega x)$$

where I used $p = \frac{\hbar}{i}\frac{d}{dx}$. Thus

$$\begin{aligned}
a_+\psi_1 &= \frac{1}{\sqrt{2\hbar m\omega}}(-\hbar\frac{d}{dx} + m\omega x)Axe^{-qx^2} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}A\left[-\hbar\frac{d}{dx}xe^{-qx^2} + m\omega x^2e^{-qx^2}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}A\left[-\hbar\left(e^{-qx^2} + x(-2qx)e^{-qx^2}\right) + m\omega x^2e^{-qx^2}\right] \\
&= \frac{1}{\sqrt{2\hbar m\omega}}A\left[-\hbar + (m\omega + 2q\hbar)x^2\right]e^{-qx^2} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\left[-\hbar + 2m\omega x^2\right]e^{-qx^2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-1 + \frac{2m\omega}{\hbar}x^2\right]e^{-\frac{m\omega}{2\hbar}x^2}
\end{aligned}$$

Now Eq. 2.66 says

$$a_+\psi_1 = \sqrt{2}\psi_2 \implies \psi_2 = \frac{1}{\sqrt{2}}a_+\psi_1 = \boxed{\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-1 + \frac{2m\omega}{\hbar}x^2\right]e^{-\frac{m\omega}{2\hbar}x^2}}$$

▷ **2.**

Considering the harmonic oscillator again.

(a) Show that the average position $\langle x \rangle$ and momentum $\langle p \rangle$ of every energy eigenstate ψ_n is zero. (Hint: Use Eq. 2.69, where the x and p operators are written in terms of the raising and lowering operators.)

(b) Find the average kinetic energy $\langle T \rangle$ of the nth eigenstate ψ_n.

Answer:

(a) The average position of an eigenstate ψ_n is

$$\langle x \rangle = \langle \psi_n | x | \psi_n \rangle = \int_{-\infty}^{\infty} \psi_n^* x \psi_n \, dx$$

Equation 2.69 gives us $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$, so

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left[\int \psi_n^* a_+ \psi_n \, dx + \int \psi_n^* a_- \psi_n \, dx \right]$$

From Eq. 2.66, we know that $a_+ \psi_n = \sqrt{n+1}\psi_{n+1}$ and $a_- \psi_n = \sqrt{n}\psi_{n-1}$, so

$$\begin{aligned}
\langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \left[\int \psi_n^* \sqrt{n+1}\psi_{n+1} \, dx + \int \psi_n^* \sqrt{n}\psi_{n-1} \, dx \right] \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n+1} \int \psi_n^* \psi_{n+1} \, dx + \sqrt{n} \int \psi_n^* \psi_{n-1} \, dx \right]
\end{aligned}$$

But the solutions ψ_n are orthogonal, so both integrals are equal to zero. Thus $\langle x \rangle = 0$.

Similarly, because $p = i\sqrt{\frac{hm\omega}{2}}(a_+ - a_-)$,

$$\langle p \rangle = i\sqrt{\frac{hm\omega}{2}} \left[\sqrt{n+1} \int \psi_n^* \psi_{n+1} \, dx - \sqrt{n} \int \psi_n^* \psi_{n-1} \, dx \right] = 0$$

as well.

(b) Kinetic energy is

$$T = \frac{1}{2m} p^2 = \frac{1}{2m} \left(i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \right)^2 = -\frac{\hbar\omega}{4}(a_+^2 + a_-^2 - a_- a_+ - a_+ a_-)$$

and so the average kinetic energy is

$$\langle T \rangle = -\frac{\hbar\omega}{4} \int \psi_n^* (a_+^2 + a_-^2 - a_- a_+ - a_+ a_-) \psi_n \, dx$$

Now

$$a_+^2\psi_n = a_+(\sqrt{n+1}\psi_{n+1}) = \sqrt{(n+1)(n+2)}\psi_{n+2}$$

$$a_-^2\psi_n = a_-(\sqrt{n}\psi_{n-1}) = \sqrt{n(n-1)}\psi_{n-2}$$

$$a_-a_+\psi_n = a_-(\sqrt{n+1}\psi_{n+1}) = (n+1)\psi_n$$

$$a_+a_-\psi_n = a_+(\sqrt{n}\psi_{n-1}) = n\psi_n$$

and so

$$\begin{aligned}
\langle T\rangle = &-\frac{\hbar\omega}{4}\Big[\sqrt{(n+1)(n+2)}\int_{-\infty}^{\infty}\psi_n^*\psi_{n+2}\,dx + \sqrt{n(n-1)}\int_{-\infty}^{\infty}\psi_n^*\psi_{n-2}\,dx \\
&-(n+1)\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx - n\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx\Big] \\
&= \frac{\hbar\omega}{4}(2n+1) = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)
\end{aligned}$$

which is half the total energy $\langle E\rangle$.

▷ **3.**
A particle is in the ground state of the harmonic oscillator with frequency ω, when suddenly the spring constant quadruples, so that $\omega' = 2\omega$, without initially changing the wave function.

(a) What is the probability that a measurement of the energy would still return the value $\hbar\omega/2$?
(b) Show that the probability that it returns a value of $\hbar\omega$ is 0.943.

Answer:____________________________________

(a) The possible energy values after the change is

$$\hbar\omega'\left(n+\frac{1}{2}\right) = \hbar(2\omega)\left(n+\frac{1}{2}\right) = \hbar\omega(2n+1).$$

The lowest energy is $\hbar\omega$, so the probability that the measurement will give a result of $\hbar\omega/2$ is $\boxed{\text{zero}}$.

(b) The wavefunction of the state is

$$\psi(x) = \psi_0(x,\omega) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Under the new value of ω', an energy of $\hbar\omega$ corresponds to the ground state, with eigenstate

$$\psi_0(x, \omega' = 2\omega) = \left(\frac{2m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{\hbar}x^2}$$

and the probability that the wavefunction ends up in the ground state is

$$\begin{aligned}
P &= |\langle\psi_0(x,\omega')||\psi(x)\rangle|^2 \\
&= \Big|\int_{-\infty}^{\infty} \left(\frac{2m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{\hbar}x^2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\, dx\Big|^2 \\
&= \left|2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{-3m\omega}{2\hbar}x^2}\, dx\right|^2
\end{aligned}$$

Let's define $y = \sqrt{\frac{3m\omega}{2\hbar}}x$, so that $dy = \sqrt{\frac{3m\omega}{2\hbar}}\, dx$. Then

$$\begin{aligned}
P &= \frac{\sqrt{2}m\omega}{\pi\hbar}\Big|\int_{-\infty}^{\infty} e^{-y^2}\sqrt{\frac{2\hbar}{3m\omega}}\, dy\Big|^2 \\
&= \frac{2\sqrt{2}}{3\pi}\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)^2 \\
&= \frac{2\sqrt{2}}{3\pi}\left(\sqrt{\pi}\right)^2 \\
&= \frac{2\sqrt{2}}{3} = \boxed{0.942}
\end{aligned}$$

Q.E.D.

▷ **4.**
In the analytic derivation of the harmonic oscillator, use the recursion relation

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)}a_j$$

to work out $H_5(\xi)$ and $H_6(\xi)$. To fix the overall constant, invoke the convention that the coefficient of the highest power of ξ is 2^n.

Answer:
The Hermite polynomials $H_n(\xi)$ are proportional to the functions

$$h_n(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots$$

where only the even or odd coefficients are nonzero.

If $n = 5$, then the recursion relation in the problem only terminates when j is odd: that is $a_{5+2} = \frac{-2(5-5)}{(5+1)(5+2)} a_5 = 0$. So we set $a_0 = 0$ (so that all the even terms are zero). In terms of a_1, we then have

$$a_3 = a_{1+2} = \frac{-2(5-1)}{(1+1)(1+2)} a_1 = \frac{-2(4)}{2(3)} a_1 = -\frac{4}{3} a_1$$

$$a_5 = a_{3+2} = \frac{-2(5-3)}{(3+1)(3+2)} a_3 = \frac{-2(2)}{4(5)} \left(-\frac{4}{3} a_1\right) = \frac{4}{15} a_1$$

Thus

$$h_5(\xi) = a_1 \left[\xi - \frac{4}{3}\xi^3 + \frac{4}{15}\xi^5\right]$$

The Hermite polynomials are defined so that H_n has the lead term (ξ^5 in this case) has a coefficient 2^n. To get $\frac{4}{15} a_1 = 32 \implies a_1 = 120$, and so

$$\boxed{H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi}$$

Now if $n = 6$, then we set $a_1 = 0$ and write the even coefficients in terms of a_0:

$$a_2 = a_{0+2} = \frac{-2(6-0)}{(0+1)(0+2)} a_0 = -6a_0$$

$$a_4 = a_{2+2} = \frac{-2(6-2)}{(2+1)(2+2)} (-6a_0) = 4a_0$$

$$a_6 = a_{4+2} = \frac{-2(6-4)}{(4+1)(4+2)} (4a_0) = -\frac{8}{15} a_0$$

We want $-\frac{8}{15} a_0 = 64 \implies a_0 = -120$. Thus

$$H_6(\xi) = a_0 - 6a_0\xi^2 + 4a_0\xi^4 - \frac{8}{15} a_0 \xi^6 = \boxed{-120 + 720\xi^2 - 480\xi^4 + 64\xi^6}$$

▷ **5.**

The Gaussian wave packet. A free particle has the initial wave function

$$\Psi(x, 0) = Ae^{-ax^2}$$

where A and a are constants (and a is real and positive).
(a) Find A to normalize $\Psi(x, 0)$
(b) Write an expression for $\Psi(x, t)$.
(c) Find a simple expression for $|\Psi(x, t)|^2$ (that is, do out all the integrals). You can use software to do out the integral, or see Griffiths problem 2.22 for a hint of how to do the integral by hand. You may find it helpful to use the change of variables $\xi = \sqrt{a}x$ and $\tau = \frac{2\hbar a}{m} t$, or something similar, to get all the constants out of the way as you do the math.
(d) Describe what $|\Psi(x, t)|^2$ does over time, qualitatively. You can figure this out by graphing the function at a few different times. You needn't include the graphs with your homework, though.

Answer:____________________________________

(a) To normalize, we need

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2} \, dx = |A|^2 \sqrt{\frac{\pi}{2a}}$$

and so $\boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}}$.

(b) The time-dependent wavefunction is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \hbar k^2 t/2m)} \, dk$$

where

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} \, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} \, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \left[\sqrt{\frac{\pi}{a}} e^{-k^2/4a}\right] \\
&= \left(\frac{2a(\pi^2)}{(4\pi^2)\pi a^2}\right)^{1/4} e^{-k^2/4a} \\
&= \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}
\end{aligned}$$

and so

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a} e^{ikx} e^{-i\hbar k^2 t/2m} \, dk \\
&= \frac{1}{(2\pi)^{3/4} a^{1/4}} \int_{-\infty}^{\infty} e^{-k^2(1/4a + i\hbar t/2m)} e^{ikx} \, dk \\
&= \frac{1}{(2\pi)^{3/4} a^{1/4}} 2\sqrt{\pi} a^{1/2} \frac{e^{-\frac{ax^2}{1+2i\hbar t/m}}}{\sqrt{1 + 2ia\hbar t/m}} \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-\frac{ax^2}{1+2i\hbar t/m}}}{\sqrt{1 + 2ia\hbar t/m}}
\end{aligned}$$

(c)

$$\begin{aligned}
|\Psi(x,t)|^2 &= \Psi^*\Psi \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-\frac{ax^2}{1-2i\hbar t/m}}}{\sqrt{1-2ia\hbar t/m}} \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-\frac{ax^2}{1+2i\hbar t/m}}}{\sqrt{1+2ia\hbar t/m}} \\
&= \sqrt{\frac{2a}{\pi}} \frac{\exp\left[-ax^2\left(\frac{1}{1-2i\hbar t/m}+\frac{1}{1+2i\hbar t/m}\right)\right]}{\sqrt{(1-2ia\hbar t/m)(1+2ia\hbar t/m)}} \\
&= \sqrt{\frac{2a}{\pi}} \frac{\exp\left[-ax^2\left(\frac{2}{(1-2i\hbar t/m)(1+2i\hbar t/m)}\right)\right]}{\sqrt{(1-2ia\hbar t/m)(1+2ia\hbar t/m)}} \\
&= \sqrt{\frac{2a}{\pi}} \frac{\exp\left[-ax^2\left(\frac{2}{1+4\hbar^2 t^2/m^2}\right)\right]}{\sqrt{1+4a^2\hbar^2 t^2/m^2}}
\end{aligned}$$

(d) I made the substitution $m = \hbar = a = 1$ to make some qualitative graphs. We see that the wave-packet spreads as time progresses: the uncertainty in the particle's position increases with time.