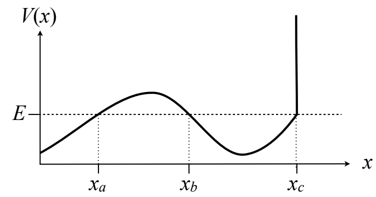
Physics 4310 Exam 2 Solutions

April 15, 2016

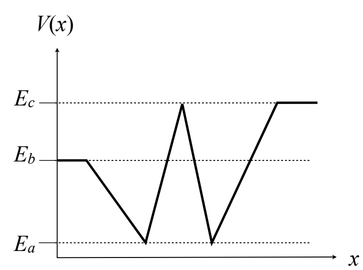
$\boxed{3}$ 1. Given the potential $V(x)$ and a particle with energy E as shown, in which regions does $\psi(x) = 0$? [O/X]

$\cancel{x < x_a}$ $\quad$ $\cancel{x_a < x < x_b}$ $\quad$ $\cancel{x_b < x < x_c}$ $\quad$ $\boxed{x > x_c}$



$\boxed{3}$ 2. For which ranges of E does this potential have scattering state solutions? [O/X]

$\cancel{E < E_a}$ $\quad$ $\cancel{E_a < E < E_b}$ $\quad$ $\boxed{E_b < E < E_c}$ $\quad$ $\boxed{E > E_c}$



$\boxed{3}$ 3. A particle in the potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a \\ \infty, & \text{otherwise} \end{cases}$$

has, at time $t = 0$, the wavefunction

$$\Psi(x, 0) = \frac{1}{\sqrt{a}} \left[\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right) \right]$$

Write $\Psi(x, t)$.

The wavefunction has the form $\Psi(x, 0) = \frac{1}{\sqrt{2}}[\psi_1 - \psi_3]$ where ψ_n are the energy eigenstates for this potential, and so introducting time-dependence is as simple as introducing Schrodinger factors $e^{-iE_n t/\hbar}$. Note that $\frac{1}{\hbar}E_n = \frac{n^2\pi^2\hbar}{2ma^2}$:

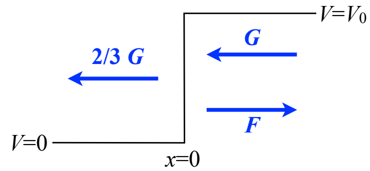
$$\Psi(x, t) = \frac{1}{\sqrt{a}} \left[\sin\left(\frac{\pi x}{a}\right) e^{-i\frac{\pi^2\hbar}{2ma^2}t} - \sin\left(\frac{3\pi x}{a}\right) e^{-i\frac{9\pi^2\hbar}{2ma^2}t} \right]$$

$\boxed{3}$ 4. A delta-function well $V(x) = -\alpha\delta(x)$ $(\alpha > 0)$ has exactly one bound-state solution: $E = -\frac{m\alpha^2}{2\hbar^2}$. For what values of α does it also have the solution $E = +\frac{m\alpha^2}{2\hbar^2}$?

When $E > 0$, the solutions to the delta-function well are scattering states, and all values of energy are allowed. Thus it has this solution for $\boxed{\text{all values of } \alpha}$.

5. One energy eigenstate of the Hamiltonian with potential

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & x > 0 \end{cases}$$



has the form

$$\psi(x) = \begin{cases} \frac{2}{3}Ge^{-ikx} & x < 0 \\ Fe^{+ik'x} + Ge^{-ik'x} & x > 0 \end{cases}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$, $k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$, and $E = \frac{4}{3}V_0$.

$\boxed{3}$ (a) Find F in terms of G.

The wavefunction must be continuous at $x = 0$, and so

$$\frac{2}{3}G = F + G \implies \boxed{F = -\frac{1}{3}G}$$

$\boxed{2}$ (b) **A** This solution $\psi(x)$ models an incident ray travelling to the ..., along with its reflection and transmission at $x = 0$.

A) left $\leftarrow$ **B)** right $\rightarrow$

Remembering that e^{ikx} signifies a wave to the right, and e^{-ikx} one to the left, I've labelled the different waves on the figure above, in blue.

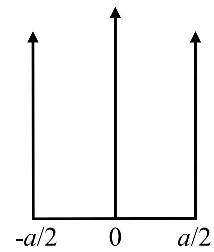
$\boxed{3}$ (c) Find the reflection coefficient R. (The transmission coefficient is a little trickier.)

The incident wave is G and the reflected wave is G so

$$R = \frac{|F|^2}{|G|^2} = \boxed{\frac{1}{9}}$$

6. Consider an infinite square well with a delta barrier in the middle.

$$V(x) = \begin{cases} \alpha\delta(x), & -\frac{a}{2} < x < \frac{a}{2} \\ \infty, & \text{otherwise} \end{cases}$$



3 (a) Write wavefunctions $\psi_L(x)$ and $\psi_R(x)$ for either side of the delta barrier, as a function of $k = \frac{\sqrt{2mE}}{\hbar}$. I recommend writing ψ_L in terms of $x + \frac{a}{2}$ and ψ_R in terms of $x - \frac{a}{2}$.

The wavefunctions on both sides are going to be combinations of sines and cosines. Using my hint, I write

$$\psi_L(x) = A \sin k \left(x + \frac{a}{2} \right) \qquad \psi_R(x) = B \sin k \left(x - \frac{a}{2} \right)$$

Writing them in terms of $x \pm a/2$ guarantees that $\psi_L(-a/2)$ and $\psi_R(a/2)$ are zero as they should be.

3 (b) Use boundary conditions to find an equation for k. It will be transcendental and have a tangent in it; don't try to solve it.

The boundary conditions at $x = 0$ are

$$\psi_L(0) = \psi_R(0) \implies A \sin \frac{ka}{2} = B \sin \left(-\frac{ka}{2} \right) = -B \sin \frac{ka}{2} \implies B = -A$$

$$\psi_R'(0) - \psi_L'(0) = +\frac{2m\alpha}{\hbar^2} \psi(0)$$

$$Bk \cos(-ka/2) - Ak \cos(ka/2) = \frac{2m\alpha}{\hbar^2} A \sin(ka/2)$$

$$-2Ak \cos \frac{ka}{2} = \frac{2m\alpha}{\hbar^2} A \sin \frac{ka}{2} \implies \boxed{k = -\frac{m\alpha}{\hbar^2} \tan \frac{ka}{2}}$$

2 (c) **A** The overall eigenstates $\psi_n(x) = \begin{cases} \psi_L(x) & -\frac{a}{2} < x < 0 \\ \psi_R(x) & 0 < x < \frac{a}{2} \end{cases}$ are

A) all even ($\psi(x) = \psi(-x)$) **B)** all odd ($\psi(x) = -\psi(-x)$)
C) Alternate between even and odd

$$\psi_R(-x) = -A \sin k \left(-x - \frac{a}{2} \right) = A \sin k \left(x + \frac{a}{2} \right) = \psi_L(x)$$

$\boxed{3}$ 7. For the harmonic oscillator, the operator $N = a_+a_-$ is Hermitian, and it commutes with the Hamiltonian, so the energy eigenstates ψ_n of H are also eigenstates of N. What is the eigenvalue of N that corresponds to ψ_n?

$$a_+a_-\psi_n = a_+\left(\sqrt{n}\psi_{n-1}\right) = \sqrt{n}\left(a_+\psi_{n-1}\right) = \sqrt{n}\sqrt{n}\psi_n = n\psi_n$$

so the eigenvalue is $\boxed{n}$. a_+a_- is often called the "number operator" for obvious reasons.

Meanwhile

$$a_-a_+\psi_n = a_-\left(\sqrt{n+1}\psi_{n+1}\right) = (n+1)\psi_n$$

$\boxed{3}$ 8. Calculate $\langle xp\rangle$ for a harmonic oscillator in energy eigenstate ψ_n. Use raising and lowering operators. (Warning: this isn't a Hermitian operator so your answer may seem a little... *imaginary*.)

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \quad \text{and} \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\implies xp = i\frac{\hbar}{2}(a_+a_+ - a_-a_- + a_-a_+ - a_+a_-)$$

Using the number operator from the last problem:

$$\langle\psi_n|xp|\psi_n\rangle = i\frac{\hbar}{2}\left[\langle\psi_n|a_+a_+|\psi_n\rangle - \langle\psi_n|a_-a_-|\psi_n\rangle + \langle\psi_n|a_-a_+|\psi_n\rangle - \langle\psi_n|a_+a_-|\psi_n\rangle\right]$$

$$\langle xp\rangle = i\frac{\hbar}{2}\left[0 - 0 + (n+1)\langle\psi_n||\psi_n\rangle - n\langle\psi_n||\psi_n\rangle\right] = \boxed{i\frac{\hbar}{2}}$$

9. Consider a wavefunction

$$\psi(x) = \sum_{j=0}^{\infty} c_j x^j$$

where $x > 0$. The coefficients obey the recursion relation

$$c_{j+1} = \frac{j-K}{(j+1)(j+2)} c_j$$

$\boxed{3}$ (a) Explain why K must be an integer. Be specific.

For large j, $c_{j+1} \sim \frac{1}{j} c_j$, so $c_j \sim \frac{1}{j!} c_0$, and $\psi(x) \sim \sum_{j=0}^{\infty} \frac{1}{j!} x^j \approx e^x$, which blows up as $x \to \infty$. Wavefunctions aren't allowed to do that. The only way out of this is for the series to terminate, which happens if $K = j$ for some integer j.

$\boxed{3}$ (b) If $K = 2$, what is $\psi(x)$ (in terms of c_0)?

$$c_1 = \frac{0-2}{(0+1)(0+2)} c_0 = -\frac{2}{2} c_0 = -c_0$$

$$c_2 = \frac{1-2}{(1+1)(1+2)} c_1 = -\frac{1}{6}(-c_0) = \frac{1}{6} c_0$$

$$\Longrightarrow \psi(x) = \boxed{c_0 \left[1 - x + \frac{1}{6} x^2\right]}$$

$\boxed{3}$ 10. **D** Consider the wavefunction

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - ck^4 t)}\, dk$$

for some constant c. What is v_g/v_p, the ratio of its group velocity to its phase velocity?

A) 1 **B)** 2 **C)** 3 **D)** 4 **E)** c

$\omega = ck^4$, and

$$\frac{v_g}{v_p} = \frac{\frac{d\omega}{dk}}{\omega/k} = \frac{4ck^3}{ck^3} = 4$$

$\boxed{3}$ 11. Find the normalized ground state wavefunction $\psi(r,\theta,\phi)$ for a harmonic oscillator in three dimensions and no angular momentum, with potential

$$V(x) = \frac{1}{2}kr^2$$

Because there is no angular momentum, $l = m = 0$, and the wavefunction is $\psi(r,\theta,\phi) = R(r)Y_0^0(\theta,\phi)$. The function $u(r) = rR(r)$ satisfies the equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{1}{2}kr^2u(r) = Eu(r)$$

(the centrifugal term is zero), which is the same equation as the harmonic oscillator in one dimension, so the ground state has

$$u(r) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}r^2}$$

Putting everything together (and noting $Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$) gives us

$$\psi(r,\theta,\phi) = [\sqrt{2}]\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{1}{4\pi}\right)^{1/2}\frac{1}{r}e^{-\frac{m\omega}{2\hbar}r^2}$$

The $\sqrt{2}$ is necessary to normalize the wavefunction, but I won't penalize you if you missed it. I think it's because the 1D harmonic oscillator goes from $-\infty < x < \infty$ while the 3D version goes from $0 < r < \infty$.

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12. Suppose particle A is a spin–3 particle, and particle B is a spin–1 particle. Their total m is $m = +1$.

$\boxed{3}$ (a) Which of these are the possible values for the total spin s? **[O/X]**

$\cancel{-2}$ $\quad$ $\cancel{-1}$ $\quad$ $\cancel{0}$ $\quad$ $\cancel{1}$ $\quad$ $\boxed{2}$ $\quad$ $\boxed{3}$ $\quad$ $\boxed{4}$ $\quad$ $\cancel{5}$

$|s_1 - s_2| \leq s \leq s_1 + s_2$

$\boxed{3}$ (b) Which of the following could be the values of m_A and m_B? **[O/X]**

$\cancel{\begin{matrix} m_A = 3 \\ m_B = -2 \end{matrix}}$ $\quad$ $\boxed{\begin{matrix} m_A = 1 \\ m_B = 0 \end{matrix}}$ $\quad$ $\cancel{\begin{matrix} m_A = -2 \\ m_B = 1 \end{matrix}}$ $\quad$ $\cancel{\begin{matrix} m_A = 1 \\ m_B = 1 \end{matrix}}$

$m_A + m_B = m$ and $|m_A| \leq s_A$.

$\boxed{3}$ 13. Suppose a spin–1/2 particle is placed in a constant magnetic field $\vec{B} = B\hat{x}$, where B is a constant. Its Hamiltonian is $H = -\vec{\mu} \cdot \vec{B}$ or

$$H = -\gamma B S_x$$

Find $\dfrac{d\langle S_y \rangle}{dt}$ for the $|\uparrow\rangle$ state (with $m = 1/2$).

$$\begin{aligned}
\frac{d\langle S_y \rangle}{dt} &= \frac{i}{\hbar}\langle [H, S_y] \rangle \\
&= \frac{i}{\hbar}(-\gamma B)\langle [S_x, S_y] \rangle \\
&= -\frac{i\gamma B}{\hbar}\langle i\hbar S_z \rangle \\
&= \gamma B \langle S_z \rangle = \boxed{\gamma B \frac{\hbar}{2}}
\end{aligned}$$

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14. Write the following without the L operator.

$\boxed{3}$ (a) $L_z Y_3^1 =$

$$= 1\hbar Y_3^1 = \boxed{\hbar Y_3^1}$$

$\boxed{3}$ (b) $L^2 Y_3^1 =$

$$= 3(3+1)\hbar^2 Y_3^1 = \boxed{12\hbar^2 Y_3^1}$$

$\boxed{3}$ 15. **D** If $\vec{L}$ is the usual angular momentum operator, then $(L_x - iL_y)Y_3^1(\theta, \phi)$ is equal to which of these (not counting normalization):

A) 0
B) $Y_2^0(\theta, \phi)$
C) $Y_2^1(\theta, \phi)$
D) $Y_3^0(\theta, \phi)$
E) $Y_3^1(\theta, \phi)$
F) $Y_2^1(\theta, \phi)$
G) $Y_3^2(\theta, \phi)$

$L_x - iL_y = L_-$ is the lowering operator, which reduces m by 1 but keeps l the same.

$\boxed{3}$ 16. For a hydrogen atom, which of the following are legitimate sets of values for n, l, & m? **[O/X]**

~~$n = 5$, $l = 5$, $m = 0$~~	~~$n = 6$, $l = 1/2$, $m = 1/2$~~	$\boxed{n = 7,\ l = 6,\ m = -5}$	~~$n = 8$, $l = 2$, $m = 3$~~

$l < n$, $|m| \leq l$, and l must be an integer.