Incompatible Observables

generally, $AB \neq BA$

define commutator $[A,B] = AB - BA$
(it's an operator)

if $[A,B] = 0$ $\quad AB - BA = 0 \rightarrow AB = BA$
A & B commute

if A & B commute
let $|a\rangle$ be an eigenvector of A

$$A|a\rangle = \lambda|a\rangle$$

$$BA|a\rangle = \lambda B|a\rangle$$

$$[A,B]=0 \longrightarrow \quad A(B|a\rangle) = \lambda(B|a\rangle)$$

$\therefore$ $B|a\rangle$ is also an eigenvector of A, same eigenvalue.

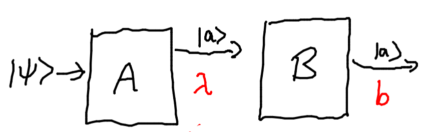
If eigenvalues of A are all unique,
then $B|a\rangle$ is same eigenvector as $|a\rangle$

$$B|a\rangle = b|a\rangle$$

you can always multiply an eigenvector by a constant & it's still an eigenvector.

$|a\rangle$ is an eigenvector of B as well.

$\therefore$ if $[A,B]=0$, they have the same eigenvectors.



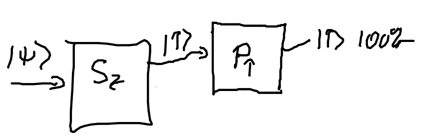
$|a\rangle$ has a value for A & B at the same time.

e.g. S_z and $P_\uparrow$ have common eigenvectors $|\uparrow\rangle$ & $|\downarrow\rangle$

$$S_z P_\uparrow = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$P_\uparrow S_z = \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$\therefore [S_z, P_\uparrow] = 0.$



S_z & S_x don't commute

S_z : $|\uparrow\rangle$ & $|\downarrow\rangle$
S_x : $|0\rangle$ & $|\alpha\rangle$

$$S_z S_x = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$S_x S_z = \frac{\hbar^2}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$[S_z, S_x] = S_z S_x - S_x S_z = \frac{\hbar^2}{4}\begin{pmatrix}0&2\\-2&0\end{pmatrix} \neq 0.$$

don't commute
incompatible

$$= \hbar\frac{\hbar}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$= i\hbar\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$[S_z, S_x] = i\hbar S_y$$

$$[S_x, S_y] = i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$[S^2, S_x] = [S_x^2, S_x] + [S_y^2, S_x] + [S_z^2, S_x]$$

$$= 0 + S_y[S_y, S_x]^* + [S_y, S_x]^* S_y$$

$$+ S_z[S_z, S_x] + [S_z, S_x] S_z$$

$[S_y, S_x]$

$= -[S_x, S_y]$

$= -i\hbar S_z$

$$= S_y(-i\hbar S_z) + -i\hbar S_z S_y$$

$$+ S_z(i\hbar S_y) + i\hbar S_y S_z$$

$$= i\hbar[-S_y S_z - S_z S_y + S_z S_y + S_y S_z]$$

$$= 0.$$

$^*[AB, C]$

$= ABC - CAB$

$= ABC - ACB$
$+ ACB - CAB$

$= A(BC - CB)$
$+ (AC - CA)B$

$= A[B, C] + [A, C]B$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S^2 |\psi\rangle = \frac{3}{4}\hbar^2 |\psi\rangle$$

$$[S^2, S_y] = [S^2, S_z] = 0.$$

$$|\vec{S}| = \frac{\sqrt{3}}{2}\hbar$$

$$S_z = \pm \frac{\hbar}{2}$$

$$|S_z| < |\vec{S}|$$

If $|\psi\rangle$ is not an eigenvector of A, there is some uncertainty in outcome of measuring $|\psi\rangle$ with A.

standard deviation $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

$$\langle A^2\rangle = \langle\psi|AA|\psi\rangle$$

e.g. $|\psi\rangle = |\uparrow\rangle \qquad A = S_z$

$$\langle S_z\rangle = \langle\uparrow|S_z|\uparrow\rangle = \langle\uparrow|\tfrac{\hbar}{2}|\uparrow\rangle = \tfrac{\hbar}{2}\langle\uparrow|\uparrow\rangle = \tfrac{\hbar}{2}.$$

$$\langle S_z^2\rangle = \langle\uparrow|S_z S_z|\uparrow\rangle$$

$$= \langle\uparrow|S_z \tfrac{\hbar}{2}|\uparrow\rangle$$

$$= \tfrac{\hbar}{2}\langle\uparrow|S_z|\uparrow\rangle = \tfrac{\hbar}{2}\langle\uparrow|\tfrac{\hbar}{2}|\uparrow\rangle = \frac{\hbar^2}{4}$$

$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2} = 0.$$