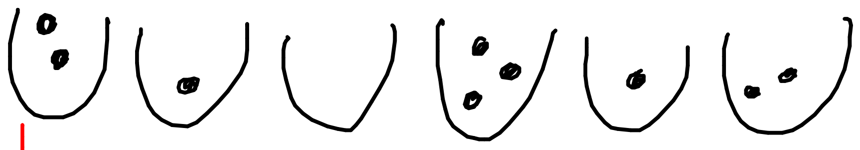
Einstein Solid

N boxes
q quanta
of energy



$N = 6$
$q = 9$

trick

If system is isolated, q is fixed
How many accessible microstates?
→ How many ways can I put q balls in N boxes?

• • | • | | • • • | • | • •

$N-1$ lines
q dots

one-to-one correspondence
between arrangements of lines & dots, & microstates

$$\Omega = \binom{N-1+q}{q} = \frac{(N-1+q)!}{(N-1)!\,q!}$$

$N = 6$
$q = 9$

$$\Omega = \binom{14}{9} = \frac{14!}{9!\,5!} = \frac{14 \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 14 \cdot 13 \cdot 11 = 2002$$

$$\boxed{\text{Einstein Solid w/ } N \text{ oscillators \& } q \text{ energy} \qquad \Omega = \binom{N+q-1}{q}}$$

Why a solid?

& A solid (in 3D)
with N atoms
has $3N$ springs →
$3N$ oscillators

If $N, q \gg 1$ $\qquad \Omega = \frac{(N+q-1)!}{(N-1)!\, q!}$

Generally,

$a, b \gg 1$, $a-b \gg 1$ $\qquad \binom{a}{b} = \frac{a!}{b!\,(a-b)!}$

$$\ln \binom{a}{b} = \ln a! - \ln b! - \ln (a-b)!$$

$$\approx (a \ln a - a) - (b \ln b - b) - \left[(a-b)\ln(a-b) - (a-b)\right]$$

$$= a \ln a - b \ln b - (a-b)\ln(a-b)$$

$$\cancel{-a + b +(a-b)} \quad 0.$$

$$= a(\ln a - \ln a-b) - b(\ln b - \ln a-b)$$

$$\underbrace{\simeq a \ln \frac{a}{a-b} - b \ln \frac{b}{a-b}}$$

$$\ln \Omega = \binom{N+q-1}{q} \approx (N+q-1) \ln \frac{N+q-1}{N-1} - q \ln \frac{q}{N-1}$$

$$\approx (N+q) \ln \frac{N+q}{N} - q \ln \frac{q}{N}$$

OR $\quad N \ln \frac{N+q}{N} + q \ln \frac{N+q}{q}$.

OR $\quad N \ln\left(1 + \frac{q}{N}\right) + q \ln\left(1 + \frac{N}{q}\right)$

Case 1: $q \gg N \gg 1$  high-temperature limit

$\frac{N}{q}$ is small

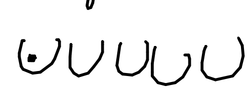
$\ln(1+\epsilon) \simeq \epsilon$ $\quad \rightarrow \ln\left(1 + \frac{N}{q}\right) \approx \frac{N}{q}$

$$\ln\left(1 + \frac{q}{N}\right) \approx \ln \frac{q}{N}$$

$$\ln \Omega = N \ln \frac{q}{N} + \cancel{q} \frac{N}{\cancel{q}} = N\left(\ln \frac{q}{N} + \overset{\ln e}{\cancel{1}}\right)$$

$$\ln \Omega = N \ln \frac{eq}{N}$$

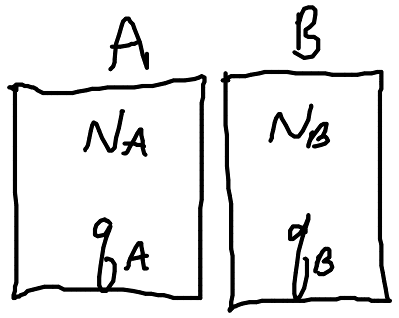
Einstein solid $q \gg N \gg 1$ $\quad \boxed{\Omega = \left(\frac{eq}{N}\right)^N}$ $\quad$ VLN very sensitive to fluctuations in q & N

Case 2: $N \gg q \gg 1$ $\quad$ low-temperature 

Case 1 with N & q interchanged

$$\Omega = \left(\frac{eN}{q}\right)^q.$$

all of these microstates are equally likely if N & q are fixed, Two Einstein solids in contact



$N = N_A + N_B$ all constant

$q = q_A + q_B$ $= q - q_A$

constant   variable

Energy can flow between solids at a slower rate than it does within each solid

If q_A is a certain value (independent so long as q_A is fixed)

$$\Omega(q_A) = \Omega_A(q_A)\,\Omega_B(q_B) =$$

$$= \binom{N_A + q_A - 1}{q_A}\binom{N_B + q_B - 1}{q_B}$$

Now, suppose q_A can change.

Probability that q_A has a particular value?

q_A defines a macrostate of system

$$P(q_A) = \frac{\Omega(q_A)}{\Omega_{all}} \qquad \Omega_{all} = \sum_{q_A=0}^{q} \Omega(q_A)$$

$$= \binom{N+q-1}{q}$$

$$P(q_A) = \frac{\binom{N_A + q_A - 1}{q_A}\binom{N_B + q_B - 1}{q_B}}{\binom{N+q-1}{q}}$$

What macrostate is most likely?