Next week: meet M noon

T 9am

following week: T 9am

W noon

F noon.

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

L_x, L_y, L_z do not commute

$[L^2, L_x] = 0$ etc.

Suppose f $\quad L^2 f = \lambda f \quad L_z f = \mu f$

$L_\pm = L_x \pm i L_y$

$L^2(L_\pm f) = \lambda(L_\pm f) \quad L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$

$L_\pm$ are raising and lowering operators

$\vdots$
λ — $\mu + 2\hbar$
λ — $\mu + \hbar$
λ — μ
λ — $\mu - \hbar$
λ — $\mu - 2\hbar$
$\vdots$

But $L_z^2 \leq L^2$
(that's how vectors work)

so upper & lower limits to μ.

f_b is lowest state

f_t is highest state

$L_- f_b = 0 \qquad L_+ f_t = 0$

Suppose $\quad L_z f_t = \hbar l f_t \qquad L^2 f_t = \lambda f_t$

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y)$$
$$= L_x^2 + L_y^2 \mp i L_x L_y \pm i L_y L_x$$
$$= L^2 - L_z^2 \pm \hbar L_z$$
$$\rightarrow L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

$$\lambda f_t = L^2 f_t = L_- L_+ f_t + L_z^2 f_t + \hbar L_z f_t$$
$$= 0 + \hbar^2 l^2 f_t + \hbar \hbar l f_t$$
$$\rightarrow \lambda = \hbar^2(l^2 + l) = \hbar^2 l(l+1)$$

$L_z f_b = \hbar b f_b$

$L^2 f_b = \hbar^2 b(b-1) f_b =$

$$\hbar^2 b(b-1) = \lambda = \hbar^2 l(l+1)$$
$$\rightarrow b = -l \quad \rightarrow L_z f_b = -\hbar l f_b$$

l — $\hbar l$
$l-1$ — $\hbar l - \hbar$
$m = l-2$ — $\hbar l - 2\hbar$
$\vdots$
$m = -l+2$ — $-\hbar l + 2\hbar$
$m = -l+1$ — $-\hbar l + \hbar$
$m = -l$ — $-\hbar l$

$l - (-l) =$ integer

$2l =$ integer

$l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$L_z f = \mu f \qquad \mu = \hbar m$

Summary: angular momentum eigenfunctions are characterized by $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
& $m = -l, -l+1, \ldots, l$

$$L^2 f = \hbar^2 l(l+1) f \qquad L_z f = \hbar m f$$

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}(\vec{r} \times \vec{\nabla})$$

$$= \frac{\hbar}{i} r\hat{r} \times \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

$$= \frac{\hbar}{i}\left[\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right]$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$L_z = -i\hbar\frac{\partial}{\partial \phi}$$

$$L_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial \theta} - \cos\phi\cot\theta\frac{\partial}{\partial \phi}\right)$$

$$L_y = -i\hbar\left(+\cos\phi\frac{\partial}{\partial \theta} - \sin\phi\cot\theta\frac{\partial}{\partial \phi}\right)$$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right]$$

$$L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$$

$$L_z Y_l^m = \hbar m Y_l^m \qquad l = 0, 1, 2, \ldots$$

$$-l \leq m \leq l$$

Spin $\vec{S}$ $[S_x, S_y] = i\hbar S_z$

$$S_\pm = S_x \pm i S_y$$

$$S^2 \chi_{sm} = \hbar^2 s(s+1) \chi_{sm}$$

$$S_z \chi_{sm} = \hbar m \chi_{sm}$$

χ_{sm} are <u>not</u> spherical harmonics — no θ or ϕ dependence

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

s is <u>fixed</u> for a given particle

e.g. Spin-$\frac{1}{2}$ particle

$s = \frac{1}{2}$ $\quad S^2 \chi_{\frac{1}{2} m} = \hbar^2 \frac{1}{2}\left(\frac{3}{2}\right) \chi_{\frac{1}{2} m} = \frac{3}{4} \hbar^2 \chi_{\frac{1}{2} m}$

$m = -\frac{1}{2}, \frac{1}{2}$ $\quad \chi_{\frac{1}{2}\frac{1}{2}} = \chi_+$

$\downarrow$ or $\uparrow$

$$\chi_{\frac{1}{2}, -\frac{1}{2}} = \chi_-$$

Addition of Angular Momenta

Given 2 spin $\frac{1}{2}$ particles $\underbrace{\uparrow\uparrow \quad \uparrow\downarrow \quad \downarrow\uparrow \quad \downarrow\downarrow}_{\chi_1 \chi_2}$

What is total angular momentum?

Define $\vec{S} = \vec{S}_1 + \vec{S}_2$ $\qquad S_{1z}|\uparrow\downarrow\rangle = \frac{1}{2}\hbar|\uparrow\downarrow\rangle$

$S_z = S_{1z} + S_{2z}$ $\qquad S_{2z}|\uparrow\downarrow\rangle = -\frac{1}{2}\hbar|\uparrow\downarrow\rangle$

$$S_z \chi_1\chi_2 = (S_{1z} + S_{2z})\chi_1\chi_2$$
$$= (S_{1z}\chi_1)\chi_2 + \chi_1(S_{2z}\chi_2)$$
$$= \hbar m_1 \chi_1\chi_2 + \chi_1(\hbar m_2 \chi_2)$$
$$\hbar m \chi_1\chi_2 = \hbar(m_1 + m_2)\chi_1\chi_2$$
$$\rightarrow m = m_1 + m_2$$

$\uparrow\uparrow \quad m = 1$

$\uparrow\downarrow \quad m = 0$] degeneracy! weird!

$\downarrow\uparrow \quad m = 0$

$\downarrow\downarrow \quad m = -1$

$$S_-(\uparrow\uparrow) = (S_x - iS_y) = (S_{1x} + S_{2x} - iS_{1y} - iS_{2y})$$
$$= (S_{1-} + S_{2-})(\uparrow\uparrow)$$
$$= \downarrow\uparrow + \uparrow\downarrow \qquad m=0 \text{ eigenstate}$$

$$\left.\begin{array}{ll} m=1 & \uparrow\uparrow \\ m=0 & \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ m=-1 & \downarrow\downarrow \end{array}\right\} s = 1$$

$s = 0, \; m = 0 \qquad \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$