Physics 4310 Homework #3

3 problems

Solutions

Note: Please feel free to use software to calculate eigenvectors and eigenvalues etc.

▷ **1.**

In the ↑↓ spin–1/2 basis, consider the two operators

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$$

(a) Find the commutator $[A, B]$.

(b) What does the result to (a) say about the eigenvectors of A and B? Confirm this.

(c) Suppose we measure a number of particles in state $|\uparrow\rangle$, using A and B. Find the average values $\langle A\rangle$ and $\langle B\rangle$ from these measurements.

(d) Use the uncertainty principle to find the lower bound on $\Delta A\, \Delta B$, for the same set of particles in state $|\uparrow\rangle$.

(e) What is the lower bound on $\Delta A\, \Delta B$ if the particles' state is one of the eigenvectors of A? You can either do the calculation, or make a clever argument for your answer.

Answer:

(a)

$$\begin{aligned}
[A, B] &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} - \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 - i & 2i + 2 \\ 2 - 2i & i + 4 \end{pmatrix} - \begin{pmatrix} 2 + i & 2 + 2i \\ -2i + 2 & -i + 4 \end{pmatrix} \\
&= \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix} = -2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

(b) The two operators don't commute, which means they must have different eigenvectors. Mathematica confirms that the eigenvectors of A are $|\uparrow\rangle \pm |\downarrow\rangle$ (*i.e.* $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$), and the eigenvectors of B are $\pm|\uparrow\rangle + i|\downarrow\rangle$, which are clearly different.

(c)

$$\begin{aligned}\langle A\rangle &= \langle\uparrow|A|\uparrow\rangle \\ &= (1\ 0)\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} \\ &= (1\ 0)\begin{pmatrix}2\\ 1\end{pmatrix} = \boxed{2}\end{aligned}$$

$$\langle B\rangle = \langle\uparrow|B|\uparrow\rangle = (1\ 0)\begin{pmatrix}2 & i\\ -i & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (1\ 0)\begin{pmatrix}2\\ -i\end{pmatrix} = \boxed{2}$$

(d) The uncertainty principle says that

$$\Delta A\,\Delta B \geq \left|\left(\frac{1}{2i}\langle[A,B]\rangle\right)\right|$$

Now

$$\begin{aligned}\frac{1}{2i}\langle[A,B]\rangle &= \frac{1}{2i}\langle\uparrow|[A,B]|\uparrow\rangle \\ &= \frac{1}{2i}(1\ 0)\begin{pmatrix}-2i & 0\\ 0 & 2i\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2i}(1\ 0)\begin{pmatrix}-2i\\ 0\end{pmatrix} = \frac{1}{2i}(-2i) \\ &= -1\end{aligned}$$

and so

$$\Delta A\,\Delta B \geq |-1| = \boxed{1}$$

(e) If we're measuring an eigenstate of A, then the A measurement will always give the same answer, and so $\Delta A = 0$. Thus the lower bound on $\Delta A\,\Delta B$ must be $\boxed{\text{zero}}$.

▷ **2.**

Consider a two-state quantum system with a Hamiltonian $H \doteq \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$.

Another physical observable A is described by the operator $A \doteq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ where a is real and positive. Let the initial state of the system be $|\psi(0)\rangle = |a_1\rangle$, the eigenstate of A corresponding to the larger of the two eigenvalues of A.

(a) Find $|\psi(t)\rangle$.

(b) What is the frequency of oscillation (i.e. the Bohr frequency) of $\langle A\rangle$?

Answer:

The eigenvectors and eigenvalues of A are

$$|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_1 = +a \qquad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda_2 = -a$$

(Thanks Mathematica!) If $|\psi(0)\rangle = |a_1\rangle$, then to find $|\psi(t)\rangle$ the first step is to write $|a_1\rangle$ in terms of energy eigenvalues. But the matrices we're writing are in the energy basis (we can tell because the Hamiltonian is diagonal), so

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}|E_2\rangle$$

Now we multiply both terms by the corresponding "Schrodinger factor":

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}|E_1\rangle + \frac{1}{\sqrt{2}}e^{-iE_2t/\hbar}|E_2\rangle$$

We can write this in matrix form as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}\begin{pmatrix} 1 \\ e^{-i\omega t} \end{pmatrix} \text{ where } \omega = (E_2 - E_1)/\hbar$$

(c) Chances are pretty good that the ω I wrote above is the frequency, but let's prove it by finding the expectation value $\langle A\rangle$ for $|\psi(t)\rangle$:

$$
\begin{aligned}
\langle A\rangle &= \langle\psi(t)|A|\psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}e^{+iE_1t/\hbar}\begin{pmatrix}1 & e^{+i\omega t}\end{pmatrix}\begin{pmatrix}0 & a\\ a & 0\end{pmatrix}\frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}\begin{pmatrix}1\\ e^{-i\omega t}\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1 & e^{i\omega t}\end{pmatrix}\begin{pmatrix}ae^{-i\omega t}\\ a\end{pmatrix} \\
&= \frac{1}{2}(ae^{-i\omega t}+ae^{i\omega t}) \\
&= a\frac{e^{i\omega t}+e^{-i\omega t}}{2} = a\cos\omega t
\end{aligned}
$$

So yes indeed, $\langle A\rangle$ oscillates between $-a$ and a with frequency $\boxed{\omega = \frac{E_2 - E_1}{\hbar}}$.

▷ **3.**

A quantum mechanical system starts out in the state

$$|\psi(0)\rangle = C(3|a_1\rangle + 4|a_2\rangle)$$

where $|a_i\rangle$ are the normalized eigenstates of the operator A corresponding to the eigenvalues a_i. In this $|a_i\rangle$ basis, the Hamiltonian of this system is represented by the matrix

$$H \doteq E_0\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

(a) If you measure the energy of this system, what values are possible, and what are the probabilities of measuring those values?
(b) Calculate the expectation value $\langle A\rangle$ as a function of time.

Answer:

(a) The possible energy values are the eigenvalues of the Hamiltonian. These are $\boxed{E_1 = 3E_0 \text{ and } E_2 = E_0}$. The corresponding energy eigenvectors (in the a_1, a_2 basis) are

$$|E_1\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \quad \text{and} \quad |E_2\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$$

The probability that the system in state $|\psi(0)\rangle$ has energy E_1 is

$$\mathcal{P} = |\langle E_1|\psi(0)\rangle|^2$$

Remember that we need to use normalized versions of both vectors for this to work: we need to divide $\psi(0)$ by

$$\sqrt{\langle\psi(0)|\psi(0)\rangle} = \sqrt{9C^2 + 16C^2} = 5C$$

(I'm assuming that C is real and positive, because if it has a phase other than 1, it would just be an overall phase of $|\psi(0)\rangle$ which wouldn't affect the calculation at all.) Thus the normalized $|\psi(0)\rangle = \frac{3}{5}|a_1\rangle + \frac{4}{5}|a_2\rangle$, and

$$\begin{aligned}
\mathcal{P}(3E_0) &= |\langle E_1|\psi(0)\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}3/5 \\ 4/5\end{pmatrix}\right|^2 \\
&= \frac{1}{2}\left|\frac{3}{5} + \frac{4}{5}\right|^2 \\
&= \frac{1}{2}\left(\frac{49}{25}\right) = \boxed{98\%}
\end{aligned}$$

and of course the probability of it having energy E_0 is $\boxed{2\%}$.

(b) To find $|\psi(t)\rangle$, we first have to write $|\psi(0)\rangle$ in the energy basis. Now we know that

$$|E_1\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle + |a_2\rangle) \quad \text{and} \quad |E_2\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle - |a_2\rangle)$$

If we add the two together, we get

$$|E_1\rangle + |E_2\rangle = \frac{1}{\sqrt{2}}(2|a_1\rangle) \implies |a_1\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$$

and if we subtract them, we get

$$|a_2\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)$$

Thus we can write

$$\begin{aligned}
|\psi(0)\rangle &= \frac{1}{5}(3|a_1\rangle + 4|a_2\rangle) \\
&= \frac{1}{5}(3\frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle) + 4\frac{1}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)) \\
&= \frac{1}{5\sqrt{2}}(7|E_1\rangle - |E_2\rangle)
\end{aligned}$$

To find the time dependence, we add in the Schrodinger factors:

$$|\psi(t)\rangle = \frac{1}{5\sqrt{2}} \left(7e^{-3iE_0t/\hbar}|E_1\rangle - e^{-iE_0t/\hbar}|E_2\rangle\right)$$

It's *really* handy now to go back and write this in the a_1, a_2 basis. We know that $|E_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|E_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, so

$$|\psi(t)\rangle = \frac{1}{5\sqrt{2}}\left(7e^{-3iE_0t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - e^{-iE_0t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right) = \frac{1}{10}e^{-3iE_0t/\hbar}\begin{pmatrix}7-e^{+2iE_0t/\hbar}\\7+e^{+2iE_0t/\hbar}\end{pmatrix}$$

where I factored out an overall phase $e^{-3iE_0t/\hbar}$.

Now we're asked to find the expectation value of A for this state. In the a_1, a_2 basis, $A = \begin{pmatrix}a_1 & 0\\0 & a_2\end{pmatrix}$ (try it if you're not sure), so

$$\begin{aligned}
\langle A\rangle &= \langle\psi(t)|A|\psi(t)\rangle \\
&= \frac{1}{10}e^{+3iE_0t/\hbar}\left(7-e^{-2iE_0t/\hbar} \quad 7+e^{-2iE_0t/\hbar}\right)\begin{pmatrix}a_1 & 0\\0 & a_2\end{pmatrix}\frac{1}{10}e^{-3iE_0t/\hbar}\begin{pmatrix}7-e^{+2iE_0t/\hbar}\\7+e^{+2iE_0t/\hbar}\end{pmatrix} \\
&= \frac{1}{100}\left(7-e^{-2iE_0t/\hbar} \quad 7+e^{-2iE_0t/\hbar}\right)\begin{pmatrix}a_1(7-e^{+2iE_0t/\hbar})\\a_2(7+e^{+2iE_0t/\hbar})\end{pmatrix} \\
&= \frac{1}{100}\left(a_1(7-e^{+2iE_0t/\hbar})(7-e^{-2iE_0t/\hbar}) + a_2(7+e^{+2iE_0t/\hbar})(7+e^{-2iE_0t/\hbar})\right) \\
&= \frac{1}{100}\left(a_1(49+1-7(e^{2iE_0t/\hbar}+e^{-2iE_0t/\hbar})) + a_2(49+1+7(e^{2iE_0t/\hbar}+e^{-2iE_0t/\hbar}))\right) \\
&= \frac{1}{100}\left(a_1(50-14\cos(2E_0t/\hbar)) + a_2(50+14\cos(2E_0t/\hbar))\right) \\
&= \boxed{\frac{1}{2}(a_1+a_2) + 0.28(a_2-a_1)\cos(2E_0t/\hbar)}
\end{aligned}$$

So $\langle A\rangle$ oscillates around the average value of its eigenvalues with a frequency of $\omega = 2E_0/\hbar$.