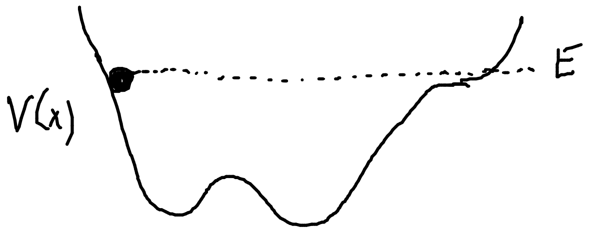
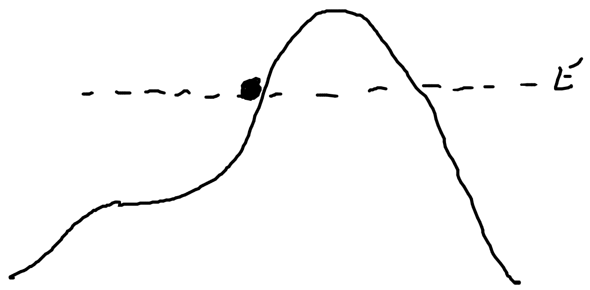
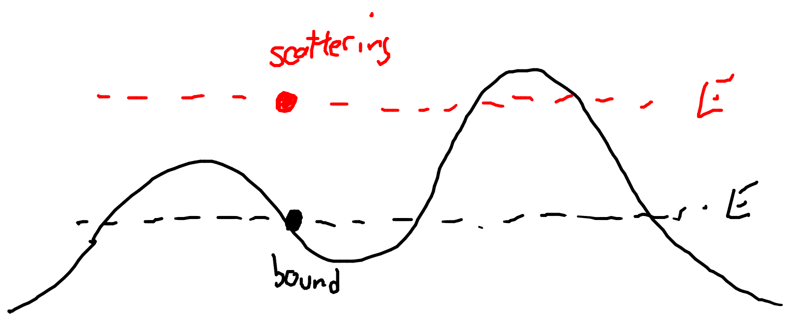
Bound States & Scattering States



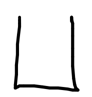
A particle in this potential is <u>bound</u>: it can't go to infinity

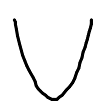


This particle can escape to ∞: <u>scattering</u> <u>state</u>



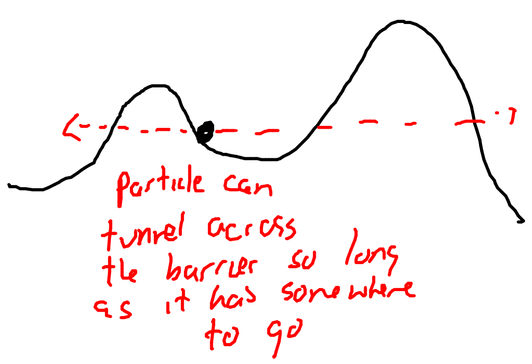
Some potentials can allow both

Infinite square well: bound only 

Harmonic oscillator: bound only 

Free Particle: Scattering only

In quantum mechanics, distinction is even starker because of tunneling



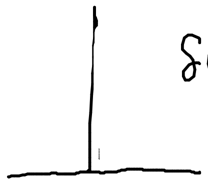
if $E < V(-\infty)$ & $E < V(\infty)$: bound state

If $E > V(-\infty)$ or $E > V(\infty)$: scattering state

bound states: discrete set of possible energy values & eigenstates
standing waves $\sin kx$ $\cos kx$

scattering states: continuous energy spectrum.
travelling waves e^{ikx} e^{-ikx}
non-normalizable energy eigenstates

Dirac Delta Function



$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{with } \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$\delta(x-a)$ spikes at $x = a$

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\, dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a)$$

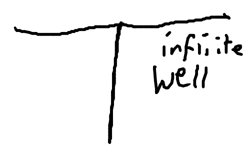
Position eigenstates

eg. $|x_0\rangle = \delta(x-x_0)$

$$\psi(x) = \langle x | \psi \rangle$$

$$\psi(x_0) = \langle x_0 | \psi \rangle = \int_{-\infty}^{\infty} \delta(x-x_0)\,\psi(x)\, dx = \psi(x_0)$$

Consider $V(x) = -\alpha\,\delta(x)$



$$-\frac{\hbar^2}{2m}\psi'' - \alpha\,\delta(x)\,\psi = E\psi$$

$$\psi'' + \frac{2m\alpha}{\hbar^2}\delta(x)\,\psi = -\frac{2mE}{\hbar^2}\psi$$

if $x \neq 0$, $\psi'' = -\frac{2mE}{\hbar^2}\psi$ $\quad k = \frac{\sqrt{2mE}}{\hbar}$ $\quad \psi'' = -k^2\psi$

solution $\psi = Ae^{-ikx} + Be^{ikx}$

Bound States: $E < 0$ $\quad k = \frac{\sqrt{2mE}}{\hbar}$ is imaginary

$$\kappa = ik$$

$$\psi = Ae^{-\kappa x} + Be^{\kappa x}$$

if $x < 0$, $\psi = Be^{\kappa x}$ $\qquad$ if $x > 0$, $\psi = Ae^{-\kappa x}$

~~to~~ or else ψ blows up

at $x \to 0_-$, $\psi = B$ $\qquad$ as $x \to 0_+$, $\psi = A$

ψ must be continuous

$\therefore B = A$

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x < 0 \\ Be^{-\kappa x}, & x > 0 \end{cases}$$