Projection Operator

suppose $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$

$$\langle\uparrow|\psi\rangle = a$$

$$\langle\uparrow|\psi\rangle\,|\uparrow\rangle = a|\uparrow\rangle$$

$$= \underbrace{(|\uparrow\rangle\langle\uparrow|)}_{\substack{\text{this is the}\\ \text{projection operator}\\ P_\uparrow}}|\psi\rangle = a|\uparrow\rangle$$

$$P_\downarrow = |\downarrow\rangle\langle\downarrow|$$

$$P_\downarrow|\psi\rangle = b|\downarrow\rangle$$

Eigenstates of $P_\uparrow$

$P_\uparrow|\uparrow\rangle = |\uparrow\rangle$ eigenvalue $= 1$

$P_\uparrow|\downarrow\rangle = 0|\downarrow\rangle$ eigenvalue $= 0$.

In matrix form

$$|\uparrow\rangle\langle\uparrow| \doteq \begin{pmatrix}1\\0\end{pmatrix}(1\;0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$|\downarrow\rangle\langle\downarrow| = \begin{pmatrix}0\\1\end{pmatrix}(0\;1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = I.$$

$$P_\uparrow + P_\downarrow = I$$

identity operator

$$P_\uparrow|\psi\rangle + P_\downarrow|\psi\rangle$$

$$= a|\uparrow\rangle + b|\downarrow\rangle = |\psi\rangle$$

Probability of measuring $|\uparrow\rangle$, ~~given~~ given $|\psi\rangle$:

$$\mathcal{P}_\uparrow = |\langle\uparrow|\psi\rangle|^2 \qquad |a|^2 = a^*a \qquad |\psi\rangle \rightarrow \boxed{S_z}$$

$$= \langle\uparrow|\psi\rangle^*\langle\uparrow|\psi\rangle$$

$$= \langle\psi\underbrace{|\uparrow\rangle\langle\uparrow|}\psi\rangle$$

$$\boxed{\mathcal{P}_\uparrow = \langle\psi|P_\uparrow|\psi\rangle}$$

~~After~~ a measurement of A on $|\psi\rangle$ will yield the result a_n with probability $\langle\psi|P_{a_n}|\psi\rangle$

After the measurement, the system will be in the new state

$$|\psi'\rangle = \frac{P_{a_n}|\psi\rangle}{\sqrt{\langle\psi|P_{a_n}|\psi\rangle}}$$

e.g. if $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ and I measure with S_z and I get $\uparrow$ as a result,

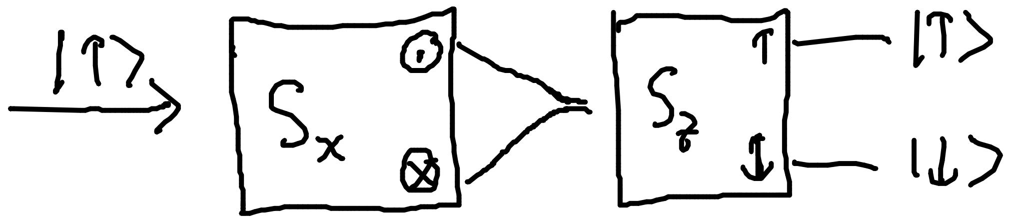
$$|\psi'\rangle = \frac{P_\uparrow|\psi\rangle}{\sqrt{\langle\psi|P_\uparrow|\psi\rangle}} = \frac{a|\uparrow\rangle}{\sqrt{|a|^2}} = \frac{a}{|a|}|\uparrow\rangle$$

$\langle\psi|P_\uparrow|\psi\rangle$

$\langle\psi|a|\uparrow\rangle$

$a\langle\psi|\uparrow\rangle$

$aa^* = |a|^2$



Outcomes of S_x measurement

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\odot\rangle + |\otimes\rangle)$$

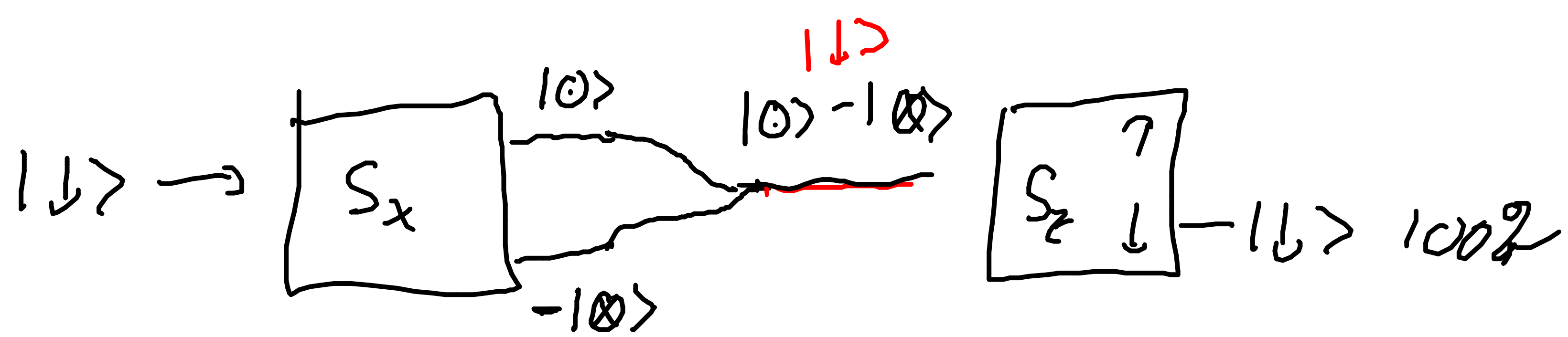
$$|\downarrow\rangle = \frac{1}{\sqrt{2}}(|\odot\rangle - |\otimes\rangle)$$

$\odot$: $$\frac{P_\odot |\uparrow\rangle}{\sqrt{\langle\uparrow|P_\odot|\uparrow\rangle}} = |\odot\rangle$$

$\otimes$: $$\frac{P_\otimes |\uparrow\rangle}{\sqrt{\langle\uparrow|P_\otimes|\uparrow\rangle}} = |\otimes\rangle$$

When I combine them, the combination state that enters S_z is sum : $|\odot\rangle + |\otimes\rangle$

if I feed $|\downarrow\rangle$ into this apparatus



For this to work, the relative phases of the outcomes of S_x must be preserved
(e.g. same length of time to reach S_z, same environment, etc.)

If relative phase is randomized,
outcome is a mixture of $|\odot\rangle$ & $|\otimes\rangle$
not $|\odot\rangle + |\otimes\rangle$ superposition
& result is 50%-50%.

Average Measurement Results

$$|\psi\rangle \longrightarrow \boxed{Z \;\; {\uparrow \atop \downarrow}} \quad \begin{matrix} P_\uparrow = |\langle\uparrow|\psi\rangle|^2 \\ P_\downarrow = |\langle\downarrow|\psi\rangle|^2 \end{matrix}$$

If only one particle goes in $S_z = \frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.

If I have a bunch of $|\psi\rangle$'s

we can calculate

average S_z $\quad \langle S_z\rangle = +\frac{\hbar}{2} P_\uparrow + -\frac{\hbar}{2} P_\downarrow$

$$\langle S_z\rangle = +\frac{\hbar}{2}|\langle\uparrow|\psi\rangle|^2 - \frac{\hbar}{2}|\langle\downarrow|\psi\rangle|^2$$

$$= +\frac{\hbar}{2}\langle\psi|\uparrow\rangle\langle\uparrow|\psi\rangle - \frac{\hbar}{2}\langle\psi|\downarrow\rangle\langle\downarrow|\psi\rangle$$

$$= \langle\psi|\Big[\underbrace{+\tfrac{\hbar}{2}|\uparrow\rangle}\langle\uparrow| \underbrace{-\tfrac{\hbar}{2}|\downarrow\rangle}\langle\downarrow|\Big]|\psi\rangle$$

$$= \langle\psi|\Big[S_z|\uparrow\rangle\langle\uparrow| + S_z|\downarrow\rangle\langle\downarrow|\Big]|\psi\rangle$$

$$= \langle\psi|S_z\underbrace{\Big[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\Big]}_{\text{identity}\,:\,P_\uparrow + P_\downarrow}|\psi\rangle$$

when measuring $|\psi\rangle$ $\quad \langle S_z\rangle = \langle\psi|S_z|\psi\rangle$

e.g. if $|\psi\rangle = |\uparrow\rangle$

$$\langle S_z\rangle = \langle\uparrow|S_z|\uparrow\rangle$$

$$= \langle\uparrow|\tfrac{\hbar}{2}|\uparrow\rangle$$

$$= \tfrac{\hbar}{2}\langle\uparrow|\uparrow\rangle = \tfrac{\hbar}{2}$$

if $|\psi\rangle = |\odot\rangle$,

$|\odot\rangle \to \boxed{S_z}$

$$\langle S_z\rangle = \langle\odot|S_z|\odot\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{\hbar}{4}(1-1) = 0.$$

50%, $+\frac{\hbar}{2}$ $\qquad$ 50%, $-\frac{\hbar}{2}$