Physics 4310 Homework #2

3 problems

Solutions

▷ **1.**

Given $A \doteq \begin{pmatrix} 1 & 2 \\ 3i & 4 \end{pmatrix}$ and $|\psi\rangle \doteq \begin{pmatrix} i \\ -1 \end{pmatrix}$, find the matrix representations of

(a) $A|\psi\rangle$
(b) $\langle\psi|A$
(c) $\langle\psi|A^\dagger$

Answer:

(a)

$$A|\psi\rangle \doteq \begin{pmatrix} 1 & 2 \\ 3i & 4 \end{pmatrix} \begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} i-2 \\ -3-4 \end{pmatrix} = \begin{pmatrix} i-2 \\ -7 \end{pmatrix}$$

(b) Remember that when you write the bra of a ket, you take the complex conjugate. The operator is still A, however, so you don't do anything to that.

$$\langle\psi|A \doteq \begin{pmatrix} -i & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3i & 4 \end{pmatrix} = \begin{pmatrix} (-i-3i) & (-2i-4) \end{pmatrix} = \begin{pmatrix} -4i & -2i-4 \end{pmatrix}$$

Notice that this is *not* the bra-version of $A|\psi\rangle$.

(c)

$$\langle\psi|A^\dagger \doteq \begin{pmatrix} -i & -1 \end{pmatrix} \begin{pmatrix} 1 & -3i \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -i-2 & -3-4 \end{pmatrix} = \begin{pmatrix} -i-2 & -7 \end{pmatrix}$$

Notice that this *is* the bra-version of $A|\psi\rangle$.

▷ **2.**

Suppose B is a Hermitian operator which is represented in the S_z basis by the matrix

$$\begin{pmatrix} 1 & 2-i \\ a & 2 \end{pmatrix}$$

(a) What is a?
(b) If I apply this measurement to a system in arbitrary state $|\psi\rangle$, what are the possible outcomes of this measurement? What could the final state of the system be?
(c) Find the probability of the outcomes, if $|\psi\rangle = |\uparrow\rangle$.

Answer:____

(a) B is Hermitian, which means $B^\dagger = B$. Since $B^\dagger = \begin{pmatrix} 1 & a^* \\ 2+i & 2 \end{pmatrix}$, it follows that $\boxed{a = 2+i}$.

(b) The possible outcomes of the measurement are the eigenvalues λ_i of B, and the final state of the system would be the corresponding eigenvector.

We can find the eigenvalues by using the formula $\det(B - \lambda I) = 0$ and solving for λ:

$$\begin{aligned}
0 &= \begin{vmatrix} 1-\lambda & 2-i \\ 2+i & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)(2-\lambda) - (2-i)(2+i) = (2 - 3\lambda + \lambda^2) - 5 \\
\implies 0 &= \lambda^2 - 3\lambda - 3 \\
\implies \lambda_\pm &= \frac{3 \pm \sqrt{9+12}}{2} = \frac{3}{2} \pm \frac{\sqrt{21}}{2}
\end{aligned}$$

Once we have the eigenvalues, we can solve the "eigenequation" $B|v_i\rangle = \lambda_i|v_i\rangle$ for the eigenvectors $|v_i\rangle$. I might assume that $|v_i\rangle \doteq \begin{pmatrix} c \\ b \end{pmatrix}$, but I know that you can multiply an eigenvector by any number and it will remain an eigenvector (with the same eigenvalue), so I'm going to use this freedom to assume that $c = 1$. (The only risk is that $c = 0$, but if that's true I'll end up with some sort of error and I'll know to try $c = 0$ next.) Given that assumption, we write

$$\begin{aligned}
\begin{pmatrix} 1 & 2-i \\ 2+i & 2 \end{pmatrix} \begin{pmatrix} 1 \\ c_\pm \end{pmatrix} &= \lambda_\pm \begin{pmatrix} 1 \\ c_\pm \end{pmatrix} \\
\begin{pmatrix} 1 + (2-i)c_\pm \\ 2 + i + 2c_\pm \end{pmatrix} &= \begin{pmatrix} \lambda_\pm \\ \lambda_\pm c_\pm \end{pmatrix}
\end{aligned}$$

This gives us two equations, but both must give the same answer so I can choose either one and solve for $c_\pm$. The bottom one looks a little nicer because $c_\pm$ is not multiplied by i (which means the denominator will be real):

$$
\begin{aligned}
2+i+2c_\pm &= \lambda_\pm c_\pm \\
\implies 2+i &= (\lambda_\pm - 2)c_\pm \\
\implies c_\pm &= \frac{2+i}{\lambda_\pm - 2} \\
&= \frac{2+i}{\frac{1}{2}(3\pm\sqrt{21})-2} \\
c_\pm &= \frac{4+2i}{-1\pm\sqrt{21}}
\end{aligned}
$$

Thus the eigenvalues and eigenvectors of this operator are

$$\lambda_\pm = \frac{1}{2}(3\pm\sqrt{21}) \qquad |v_\pm\rangle = \begin{pmatrix} -1\pm\sqrt{21} \\ 4+2i \end{pmatrix}$$

(I multiplied the vector top and bottom by the denominator of c to make it prettier.)

(c) The probability that $|\uparrow\rangle$ will give response $\lambda_\pm$ is $|\langle\uparrow|v_\pm\rangle|^2$, but only if $v_\pm$ is normalized. To normalize, we need its magnitude $\sqrt{\langle v_\pm|v_\pm\rangle}$ first:

$$
\begin{aligned}
\langle v_\pm|v_\pm\rangle &= \begin{pmatrix} -1\pm\sqrt{21} & 4-2i \end{pmatrix} \begin{pmatrix} -1\pm\sqrt{21} \\ 4+2i \end{pmatrix} \\
&= (1+21\mp 2\sqrt{21}) + (16+4) \\
&= 42\mp 2\sqrt{21} \\
\sqrt{\langle v_\pm|v_\pm\rangle} &= \sqrt{42\mp 2\sqrt{21}} \equiv N_\pm
\end{aligned}
$$

I'm going to call that $N_\pm$ so I don't have to keep writing it. Thus the normalized version of $|v_\pm\rangle \doteq \frac{1}{N}\begin{pmatrix} -1\pm\sqrt{21} \\ 4+2i \end{pmatrix}$, and thus

$$\langle\uparrow|v_\pm\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{N_\pm} \begin{pmatrix} -1\pm\sqrt{21} \\ 4+2i \end{pmatrix} = \frac{-1\pm\sqrt{21}}{N_\pm}$$

$$\mathcal{P}_\pm = |\langle\uparrow|v_\pm\rangle|^2 = \left(\frac{-1\pm\sqrt{21}}{\sqrt{42\mp 2\sqrt{21}}}\right)^2 = \frac{22\mp 2\sqrt{21}}{42\mp 2\sqrt{21}}$$

(I really don't blame you if you switched to Mathematica long before this; I'm doing the math by hand as penance for giving a problem with such messy values.) I'm going to now multiply top and bottom by $(42 \pm 2\sqrt{21})$ to move the square root up to the top:

$$
\begin{aligned}
\mathcal{P}_\pm &= \frac{22 \mp 2\sqrt{21}}{42 \mp 2\sqrt{21}} \cdot \frac{42 \pm 2\sqrt{21}}{42 \pm 2\sqrt{21}} \\
&= \frac{(22 \mp 2\sqrt{21})(42 \pm 2\sqrt{21})}{42^2 - 4(21)} \\
&= \frac{924 \pm 4\sqrt{21} \mp 84\sqrt{21} - 84}{1764 - 84} \\
&= \frac{840 \mp 80\sqrt{21}}{1680} \\
&= \frac{1}{2} \mp \frac{1}{21}\sqrt{21} \\
&= \frac{1}{2} \mp \frac{1}{\sqrt{21}}
\end{aligned}
$$

Notice that $\mathcal{P}_+ + \mathcal{P}_- = 1$ which shows that we didn't flub it up. Hurrah!

Numerically, the percentages are $\mathcal{P}_+ = 28\%$ and $\mathcal{P}_- = 72\%$. (And if you didn't use Mathematica, you should have at *least* gone to decimal representations if you're not a masochist.)

▷ **3.**
Consider the operator $S_{\hat{n}}$ for a Stern-Gerlach device oriented along the vector

$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

(a) Prove that it is represented (in the S_z basis) by the matrix

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

(b) Prove that $\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\phi}|\downarrow\rangle$ is an eigenvector of $S_{\hat{n}}$, and find the corresponding eigenvalue.
(c) For what values of θ and ϕ does $S_{\hat{n}} = S_y$?

Answer:______________________________________

(a) We discussed in class that we can define $\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$, and write

$$\begin{aligned}
S_{\hat{n}} &= \vec{S}\cdot\hat{n} \\
&= S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta \\
&= \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sin\theta\sin\phi + \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos\theta \\
&= \frac{\hbar}{2}\begin{pmatrix}0 & \sin\theta\cos\phi\\ \sin\theta\cos\phi & 0\end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix}0 & -i\sin\theta\sin\phi\\ i\sin\theta\sin\phi & 0\end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix}\cos\theta & 0\\ 0 & -\cos\theta\end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta(\cos\phi - i\sin\phi)\\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta\end{pmatrix} \\
&= \begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}
\end{aligned}$$

where I used $e^{i\phi} = \cos\phi + i\sin\phi$.

(b) If we're given the eigenvector, finding the eigenvalue is simply a matter of solving the "eigenequation":

$$\begin{aligned}
&\frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix} = \lambda\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix} \\
\implies &\frac{\hbar}{2}\begin{pmatrix}\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\\ \sin\theta\cos\frac{\theta}{2}e^{i\phi} - \cos\theta\sin\frac{\theta}{2}e^{i\phi}\end{pmatrix} = \lambda\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}
\end{aligned}$$

We have to solve both equations for λ, and if they agree then this is an eigenvector. The first equation is

$$\begin{aligned}
\frac{\hbar}{2}(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}) &= \lambda\cos\frac{\theta}{2} \\
\implies \lambda &= \frac{\hbar}{2}\left(\cos\theta + \sin\theta\tan\frac{\theta}{2}\right) \\
\implies \lambda &= \frac{\hbar}{2}\left(\cos\theta + \sin\theta\frac{1-\cos\theta}{\sin\theta}\right) \\
\implies \lambda &= \frac{\hbar}{2}\left(\cos\theta + 1 - \cos\theta\right) = \boxed{\frac{\hbar}{2}}
\end{aligned}$$

where I used a half-angle formula for the tangent. The second equation is

$$\begin{aligned}
\frac{\hbar}{2}\left(\sin\theta\cos\frac{\theta}{2}e^{i\phi} - \cos\theta\sin\frac{\theta}{2}e^{i\phi}\right) &= \lambda\sin\frac{\theta}{2}e^{i\phi} \\
\implies \lambda &= \frac{\hbar}{2}\left(\sin\theta\cot\frac{\theta}{2} - \cos\theta\right) \\
\implies \quad &= \frac{\hbar}{2}\left(\sin\theta\frac{1+\cos\theta}{\sin\theta} - \cos\theta\right) \\
&= \frac{\hbar}{2}\left(1+\cos\theta-\cos\theta\right) = \boxed{\frac{\hbar}{2}}
\end{aligned}$$

The two answers agree, and so it is an eigenvector, with eigenvalue $\lambda = \frac{\hbar}{2}$ (just like $|\uparrow\rangle$, $|\odot\rangle$, and $|\rightarrow\rangle$).

(c) We want to know when the vector $\hat{n}$ points along the $\hat{y}$ axis. Looking at the diagram in McIntyre, we want $\theta = \frac{\pi}{2}$ to put the vector on the equator, and then $\phi = \frac{\pi}{2}$ for it to point in the y direction. Let's see if that works:

$$\begin{aligned}
S_{\hat{n}} &= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix}\cos\frac{\pi}{2} & \sin\frac{\pi}{2}e^{-i\pi/2} \\ \sin\theta e^{i\pi/2} & -\cos\frac{\pi}{2}\end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} = S_y
\end{aligned}$$

This in fact the matrix representation of S_y, so our angle-figuring was correct.