Finding the surface area of an n-dimensional hypersphere $S(r) = S_n r^{n-1}$

Calculate a total mass…

...using Cartesian coordinates *...using spherical shells*

$$\int \rho(r) dx_1\, dx_2\, \ldots\, dx_n = \int \rho(r) S(r)\, dr = S_n \int_0^R r^{n-1} \rho(r)\, dr$$

Let $\rho(r) = e^{-r^2}$ and integrate over all space $(R \to \infty)$

$$\int_{-\infty}^{\infty} e^{-r^2} dx_1\, dx_2\, \ldots\, dx_n = S_n \int_0^\infty r^{n-1} e^{-r^2}\, dr$$

Expand r in terms of the coordinates.

$$\int_{-\infty}^{\infty} e^{-x_1^2} e^{-x_2^2} \ldots e^{-x_n^2} dx_1\, dx_2\, \ldots\, dx_n$$

Each integral is identical.

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^n$$

Evaluate the integral

Define the gamma function.

$$\Gamma(m) \equiv 2 \int_0^\infty e^{-r^2} r^{2m-1}\, dr$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$	$\Gamma(1) = 1$
$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$	$\Gamma(2) = 1$
$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$	$\Gamma(3) = 2$
$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$	$\Gamma(4) = 6$
$\Gamma(m+1) = m\Gamma(m)$	$n! = \Gamma(n+1)$

$$(\sqrt{\pi})^n = \frac{1}{2} S_n \Gamma\left(\frac{n}{2}\right)$$

$$S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \quad \Rightarrow \quad S(r) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$$