Physics 4310 Homework #1

4 problems

Solutions

▷ **1.**
The possible outcomes of a S_y analyzer are

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \quad \text{and} \quad |\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle)$$

Prove the following:
(a) Prove these vectors are normalized. (That is: $\langle\leftarrow | \leftarrow\rangle = \langle\rightarrow | \rightarrow\rangle = 1$.)
(b) Prove they are orthogonal: $\langle\leftarrow | \rightarrow\rangle = 0$
(c) Prove that, if passed through a S_z analyzer, they will come out as $|\uparrow\rangle$ with probability 1/2.
(d) Prove that, if passed through a S_x analyzer, they will come out as $|\odot\rangle$ with probability 1/2.

Answer:

(a) Remember that the bra of a corresponding ket involves taking the complex conjugate:

$$\begin{aligned}
\langle\leftarrow | \leftarrow\rangle &= \frac{1}{\sqrt{2}}\left(\langle\uparrow| - i\langle\downarrow|\right)\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right) \\
&= \frac{1}{2}\left(\langle\uparrow | \uparrow\rangle - i\langle\downarrow | \uparrow\rangle + i\langle\uparrow | \downarrow\rangle + \langle\downarrow | \downarrow\rangle\right) \\
&= \frac{1}{2}\left(1 - i0 + i0 + 1\right) = 1
\end{aligned}$$

The calculation to show that $\langle\rightarrow | \rightarrow\rangle = 1$ is nearly identical.

(b) Orthogonality:

$$\langle\leftarrow | \rightarrow\rangle = \frac{1}{\sqrt{2}}\left(\langle\uparrow| - i\langle\downarrow|\right)\frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(\langle\uparrow | \uparrow\rangle - i\langle\downarrow | \uparrow\rangle - i\langle\uparrow | \downarrow\rangle - \langle\downarrow | \downarrow\rangle\right)$$

$$= \frac{1}{2}\left(1 - i0 - i0 - 1\right) = 0$$

(c) The probability that $|{\leftarrow}\rangle$ will come out of an S_z analyzer with outcome $|{\uparrow}\rangle$ is $\mathcal{P} = |\langle\uparrow\,|\leftarrow\rangle|$. However,

$$\begin{aligned}
\langle\uparrow\,|\leftarrow\rangle &= \langle\uparrow|\frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + i|{\downarrow}\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\langle\uparrow\,|\uparrow\rangle + i\langle\uparrow\,|\downarrow\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(1+0\right) = \frac{1}{\sqrt{2}}
\end{aligned}$$

and therefore $\mathcal{P} = |1/\sqrt{2}|^2 = 1/2$. **Q.E.D.**

The calculation with $|{\rightarrow}\rangle$ is nearly identical.

(d) The probability that $|{\leftarrow}\rangle$ will come out of S_x as $|{\odot}\rangle$ is $\mathcal{P} = |\langle\odot|\leftarrow\rangle|^2$. We can use the fact that $|{\odot}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + |{\downarrow}\rangle)$:

$$\begin{aligned}
\langle\odot|\leftarrow\rangle &= \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right)\frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + i|{\downarrow}\rangle\right) \\
&= \frac{1}{2}\left(\langle\uparrow\,|\uparrow\rangle + \langle\downarrow\,|\uparrow\rangle + i\langle\uparrow\,|\downarrow\rangle + i\langle\downarrow\,|\downarrow\rangle\right) \\
&= \frac{1}{2}\left(1+i\right)
\end{aligned}$$

and so the probability is

$$\mathcal{P} = |\frac{1}{2}\left(1+i\right)|^2 = \frac{1}{4}(1^2+1^2) = \frac{1}{2}$$

▷ **2.**
In the spin–1/2 quantum system, consider the ket $|\psi\rangle = -3|{\uparrow}\rangle + 4i|{\downarrow}\rangle$.
(a) Normalize the ket.
(b) Find the probability that, if the ket were fed into an S_z analyzer, it would give a result of $S_z = +\frac{\hbar}{2}$.
(c) Find the probability that, if the ket were fed into an S_x analyzer, it would come out as $|{\otimes}\rangle$.
(d) Write the normalized ket as a column vector in the S_z basis.
(e) Write the normalized ket as a column vector in the S_x basis.

Answer:____

(a) The magnitude of $|\psi\rangle$ is

$$|\psi|^2 = \langle\psi|\psi\rangle = (-3\langle\uparrow| - 4i\langle\downarrow|)(-3|\uparrow\rangle + 4i|\downarrow\rangle) = 9 + 16 = 25$$

and so the normalized ket is

$$|\psi\rangle = \frac{1}{\sqrt{25}}(-3|\uparrow\rangle + 4i|\downarrow\rangle) = \boxed{-\frac{3}{5}|\uparrow\rangle + \frac{4i}{5}|\downarrow\rangle}$$

(b) The analyzer will give a result of $S_z = +\frac{\hbar}{2}$ if it comes out as $|\uparrow\rangle$; the probability of that happening is

$$\mathcal{P} = |\langle\uparrow|\psi\rangle|^2 \left(\frac{3}{5}\right)^2 = 9/25 = \boxed{36\%}$$

(c) To find the probability that it will come out as $|\otimes\rangle$ with

$$\langle\otimes|\psi\rangle = \frac{1}{\sqrt{2}}(\langle\uparrow| - \langle\downarrow|)(-\frac{3}{5}|\uparrow\rangle + \frac{4i}{5}|\downarrow\rangle)$$
$$= -\frac{3}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}}$$

and the probability is

$$\mathcal{P} = |\langle\otimes|\psi\rangle|^2 = \frac{9}{50} + \frac{16}{50} = \frac{25}{50} = \boxed{50\%}$$

(d) Because I have it $|\psi\rangle = -\frac{3}{5}|\uparrow\rangle + \frac{4i}{5}|\downarrow\rangle$,

$$|\psi\rangle \doteq \boxed{\begin{pmatrix} -3/5 \\ 4i/5 \end{pmatrix}}$$

(e) In the S_x basis $|\otimes\rangle, |\odot\rangle$, the vector $|\psi\rangle$ can be written

$$|\psi\rangle \doteq \begin{pmatrix} \langle\odot|\psi\rangle \\ \langle\otimes|\psi\rangle \end{pmatrix}$$

We can calculate these elements by using the expansion of $\langle\odot|$ and $\langle\otimes|$ in the S_z basis.

$$\langle\odot| = \frac{1}{\sqrt{2}}\langle\uparrow| + \frac{1}{\sqrt{2}}\langle\downarrow| \quad \text{and} \quad \langle\otimes| = \frac{1}{\sqrt{2}}\langle\uparrow| - \frac{1}{\sqrt{2}}\langle\downarrow|$$

$$\begin{aligned}
\langle \odot|\psi\rangle &= \left(\frac{1}{\sqrt{2}}\langle\uparrow| + \frac{1}{\sqrt{2}}\langle\downarrow|\right)\left(-\frac{3}{5}|\uparrow\rangle + \frac{4i}{5}|\downarrow\rangle\right) \\
&= -\frac{3}{5\sqrt{2}} + \frac{4i}{5\sqrt{2}} = \frac{-3+4i}{5\sqrt{2}} \\
\langle \otimes|\psi\rangle &= \left(\frac{1}{\sqrt{2}}\langle\uparrow| - \frac{1}{\sqrt{2}}\langle\downarrow|\right)\left(-\frac{3}{5}|\uparrow\rangle + \frac{4i}{5}|\downarrow\rangle\right) \\
&= -\frac{3}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}} = \frac{-3-4i}{5\sqrt{2}}
\end{aligned}$$

Thus, in the S_x basis, we can write

$$|\psi\rangle \doteq \boxed{\frac{1}{5\sqrt{2}}\begin{pmatrix} -3+4i \\ -3-4i \end{pmatrix}}$$

▷ **3.**
Show that a change in the overall phase of a quantum state vector does not change the probability of obtaining a particular result in a measurement. To do this, consider how the probability is affected by changing the state $|\psi\rangle$ to the state $e^{i\delta}|\psi\rangle$.

Answer:
Suppose I have an operator A with eigenvectors $|a_i\rangle$. If the state $e^{i\delta}|\psi\rangle$ is measured with this operator, the probability of getting outcome a_i is

$$\begin{aligned}
\mathcal{P} &= |\langle a_i|e^{i\delta}\psi\rangle|^2 \\
&= \langle a_i|e^{i\delta}\psi\rangle^* \langle a_i|e^{i\delta}\psi\rangle \\
&= e^{-i\delta}\langle a_i|\psi\rangle^* e^{i\delta}\langle a_i|\psi\rangle \\
&= \langle a_i|\psi\rangle^*\langle a_i|\psi\rangle = |\langle a_i|\psi\rangle|^2
\end{aligned}$$

Thus the probability is independent of the overall phase δ of the state.

▷ **4.**
Prove the *Cauchy-Schwarz inequality*:

$$|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

(Ed: Sorry Cauchy!)
Hint: Consider the vector

$$|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle$$

and the fact that $\langle\gamma|\gamma\rangle \geq 0$.

Answer:

Let's use the hint, and calculate $\langle\gamma|\gamma\rangle$. Note that

$$\langle\gamma| = \langle\beta| - \frac{\langle\alpha|\beta\rangle^*}{\langle\alpha|\alpha\rangle}\langle\alpha|$$

and $\langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle$.

$$\begin{aligned}
\langle\gamma|\gamma\rangle &= \left(\langle\beta| - \frac{\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle}\langle\alpha|\right)\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) \\
&= \langle\beta|\beta\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle}\langle\alpha|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\beta|\alpha\rangle + \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle\langle\alpha|\alpha\rangle}\langle\alpha|\alpha\rangle \\
&= \langle\beta|\beta\rangle - \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} - \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle} + \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} \\
&= \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle}
\end{aligned}$$

because $\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = |\langle\alpha|\beta\rangle|^2$. Because this is an inner product, it must be nonnegative, so

$$\begin{aligned}
0 &\leq \langle\gamma|\gamma\rangle \\
&\leq \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle}
\end{aligned}$$

$$|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

Q.E.D.