Generally, write $A \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the S_z basis

$$A|\uparrow\rangle \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \doteq a|\uparrow\rangle + c|\downarrow\rangle$$

$\langle\uparrow|A|\uparrow\rangle = a$ $\qquad$ $\langle\downarrow|A|\uparrow\rangle = c$

$\langle\uparrow|A|\downarrow\rangle = b$ $\qquad$ $\langle\downarrow|A|\downarrow\rangle = d$

$$A = \begin{array}{c} \\ \langle\uparrow| \\ \langle\downarrow| \end{array} \begin{array}{c} \begin{array}{cc} |\uparrow\rangle & |\downarrow\rangle \end{array} \\ \begin{pmatrix} \langle\uparrow|A|\uparrow\rangle & \langle\uparrow|A|\downarrow\rangle \\ \langle\downarrow|A|\uparrow\rangle & \langle\downarrow|A|\downarrow\rangle \end{pmatrix} \end{array}$$

$\langle\cdot|A|\cdot\rangle$ is a "matrix element"

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suppose we're given a matrix for an operator. How can we determine the possible measurement outcomes?

e.g. $S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in S_z basis

$|\psi\rangle \longrightarrow \boxed{S_x}$

possible outcomes are eigenvalues & eigenvectors of S_x.

Eigenvalues λ of S_x: & eigenvectors $\vec{v}$.

$$S_x\vec{v} = \lambda\vec{v}.$$

eigenvalues $\det(S_x - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\lambda = \pm\frac{\hbar}{2}$$

$$\underbrace{\cancel{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_x}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\vec{v}} = \underbrace{\pm\cancel{\frac{\hbar}{2}}}_{\lambda}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\vec{v}}$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$$

$b = \pm a$ $\qquad$ eigenvectors: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$|\odot\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

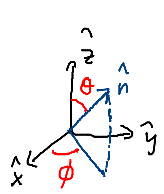
$$|\otimes\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad$ $S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad$ $S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ← Pauli spin matrices

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $|\odot\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $|\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$

$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ $|\otimes\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad$ $|\leftarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

General spin matrix



$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$S_{\hat{n}} = (S_x\hat{x} + S_y\hat{y} + S_z\hat{z})\cdot\hat{n}$$

has eigenvectors $|+_{\hat{n}}\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\phi}|\downarrow\rangle$

$\qquad\qquad\qquad\quad |-_{\hat{n}}\rangle = \sin\frac{\theta}{2}|\uparrow\rangle - \cos\frac{\theta}{2}e^{i\phi}|\downarrow\rangle$

Any normalized vector $a|\uparrow\rangle + b|\downarrow\rangle$ is equivalent to either of these forms.

Hermitian Operators

suppose $A|\psi\rangle = |\phi\rangle$

$$\langle\psi|A = \langle\xi| \neq \langle\phi|$$

$$\langle\phi| = \langle\psi|A^\dagger$$

$A^\dagger$ is the Hermitian adjoint of A
turns bra of $|\psi\rangle$ to bra of $A|\psi\rangle$.

Let $|\phi\rangle = A|\psi\rangle$ & $|\beta\rangle$ be some other ket

$$\langle\phi|\beta\rangle = \langle\beta|\phi\rangle^*$$

$$\langle\psi|A^\dagger|\beta\rangle = \langle\beta|A|\psi\rangle^*$$

$$\begin{pmatrix} \langle\uparrow|A|\uparrow\rangle & \langle\uparrow|A|\downarrow\rangle \\ \langle\downarrow|A|\uparrow\rangle & \langle\downarrow|A|\downarrow\rangle \end{pmatrix}$$

complex conjugate of the transpose of A.

e.g. $\begin{pmatrix} 1 & 3+i \\ 5-i & 2 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 5+i \\ 3-i & 2 \end{pmatrix}$ $\qquad A^\dagger = (A^T)^*$

if $A^\dagger = A$ $\qquad$ i.e. if $A|\psi\rangle = |\phi\rangle$
$\qquad\qquad\qquad\qquad\quad \langle\psi|A = \langle\phi|$

then A is an Hermitian operator

In QM, all operators corresponding to physical observables are Hermitian.

e.g. $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hermitian operators – have real eigenvalues
eigenvectors form a complete set of basis states