Finding the surface area of an n-dimensional hypersphere $S(r) = S_n r^{n-1}$

Calculate a total mass...

...using Cartesian coordinates

...using spherical shells

$$\int \rho(r) dx_1 dx_2 \dots dx_n = \int \rho(r) S(r) dr = S_n \int_0^R r^{n-1} \rho(r) dr$$

Let $\rho(r) = e^{-r^2}$ and integrate over all space $(R \to \infty)$

$$\int_{-\infty}^{\infty} e^{-r^2} dx_1 dx_2 \dots dx_n = S_n \int_{0}^{\infty} r^{n-1} e^{-r^2} dr$$

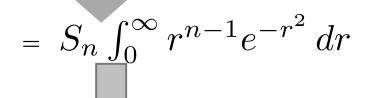
Expand r in terms of the coordinates.

$$\int_{-\infty}^{\infty} e^{-x_1^2} e^{-x_2^2} \dots e^{-x_n^2} dx_1 dx_2 \dots dx_n$$

Each integral is identical.

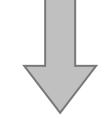
$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^n$$

Evaluate the integral



Define the gamma function.

$$\Gamma(m) \equiv 2 \int_0^\infty e^{-r^2} r^{2m-1} dr$$



$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma(1) = 1$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \qquad \Gamma(2) = 1$$

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$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi} \qquad \Gamma(3) = 2$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi} \qquad \Gamma(4) = 6$$

$$\Gamma(m+1) = m\Gamma(m) \quad n! = \Gamma(n+1)$$

$$\left(\sqrt{\pi}\right)^n = \frac{1}{2} S_n \Gamma\left(\frac{n}{2}\right)$$

$$S_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \longrightarrow S(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{n-1}$$