Physics 4310 Homework #7

5 problems

Solutions

▷ **1.**

Show that the eigenstates of the momentum operator p in one dimension are travelling waves with the deBroglie wavelength

$$\lambda = \frac{2\pi\hbar}{p}$$

Hint: If you're a little rusty with waves, remember that the definition of wavelength is the smallest λ for which $\psi(x) = \psi(x+\lambda)$ for all x.

Answer:

The momentum operator is $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$. What function is left unchanged by a derivative? An exponential, of course. If we write $\psi(x) = e^{ikx}$. The momentum of this wavefunction is

$$p\psi(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}e^{ikx} = \frac{\hbar i k}{i}e^{ikx} = \hbar k e^{ikx}$$

and so the momentum is $p = \hbar k \implies k = \frac{p}{\hbar}$, and we can write the eigenstate as

$$\psi(x) = e^{ipx/\hbar}$$

The wavelength λ satisfies

$$e^{ipx/\hbar} = e^{ip(x+\lambda)/\hbar} = e^{ipx/\hbar}e^{ip\lambda/\hbar} \implies e^{ip\lambda/\hbar} = 1$$

The smallest value of λ for which this is true (other than zero) is

$$\frac{p\lambda}{\hbar} = 2\pi \implies \lambda = \frac{2\pi\hbar}{p}$$

which is the deBroglie wavelength.

▷ **2.**

Use the equation

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{dQ}{dt}\right\rangle$$

to find $\frac{d\langle p\rangle}{dt}$.

Answer:

The momentum operator $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$ has no explicit time dependence, so $\frac{dp}{dt} = 0$. The standard Hamiltonian is the kinetic and potential energies

$$H = \frac{1}{2m}p^2 + V$$

and so

$$
\begin{aligned}
[H,p] &= [\frac{1}{2m}p^2,p]+[V,p] \\
&= 0 + V\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}V \qquad ([p^2,p]=0) \\
&= [H,p]f(x) = V\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(Vf) \\
&= -i\hbar\left[Vf' - (V'f+Vf')\right] \\
&= i\hbar V'(x)f(x) \\
\implies [H,p] &= i\hbar\frac{\partial V(x)}{\partial x}
\end{aligned}
$$

and thus

$$
\frac{\partial \langle p\rangle}{\partial t} = \frac{i}{\hbar}\left\langle i\hbar\frac{\partial V(x)}{\partial x}\right\rangle = -\left\langle\frac{\partial V(x)}{\partial x}\right\rangle
$$

What does this mean? In classical terms, the negative derivative of the potential energy is the force: $-\left\langle\frac{\partial V(x)}{\partial x}\right\rangle = \langle F\rangle$. (You may be more familiar with this as $F=-\nabla U$.) Assuming the mass is constant, $\frac{\partial\langle p\rangle}{\partial t} = \frac{\partial}{\partial t}(mv) = m\frac{\partial v}{\partial t} = ma$. Thus our result is Newton's Second Law: $m\langle a\rangle = \langle F\rangle$.

▷ **3.**

(Griffiths 4.2) Use separation of variables in *cartesian* coordinates to solve the infinite *cubical* well (a "particle in a box"):

$$
V(x,y,z) = \begin{cases} 0 & \text{if } x,\ y, \text{ and } z \text{ are all between 0 and } a \\ \infty & \text{otherwise} \end{cases}
$$

(a) Find the energy eigenstates, and their corresponding eigenvalues.
(b) Call the *distinct* energies $E_1 < E_2 < E_3 < \ldots$. Find E_1, E_2, E_3, E_4, E_5, and E_6. Determine their degeneracies—that is, the number of different states that share the same energy. (In one dimension, degenerate bound states do not occur (Problem 2.45), but in three dimensions they are very common.)

Answer:

(a) The energy eigenstate equation in Cartesian coordinates is

$$
-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right) = E\psi
$$

Let's write $\psi = X(x)Y(y)Z(z)$, $\frac{2mE}{\hbar^2} = k^2$, so that

$$X''YZ + XY''Z + XYZ'' = -k^2XYZ$$
$$\implies \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2$$

where I divided both sides by XYZ. If I put all the x stuff over on one side, I have

$$\frac{X''}{X} = -k^2 - \frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda^2$$

with $-\lambda^2$ the separation variable. Solving

$$X'' = -\lambda^2 X \implies X = A\sin\lambda x + B\cos\lambda x$$

Boundary conditions say that $X(0) = X(a) = 0$, so $B = 0$, and $\lambda a = n_x\pi \implies \lambda = \frac{n_x\pi}{a}$ where $n_x = 1, 2, 3, \ldots$.

Next we separate the other equation

$$\frac{Y''}{Y} + \frac{Z''}{Z} = -k^2 + \lambda^2$$
$$-\frac{Y''}{Y} = \frac{Z''}{Z} + k^2 - \lambda^2 = \mu^2$$

where μ^2 is the second separation variable. I solve the y equation to get

$$\frac{Y''}{Y} = -\mu^2 \implies Y'' = -\mu^2 Y \implies Y = C\sin\mu y + D\cos\mu y$$

and again $D = 0$ and $\mu = \frac{n_y\pi}{a}$ for some positive integer n_y.

Lastly, we have

$$\frac{Z'}{Z} = \mu^2 + \lambda^2 - k^2 \implies Z = F\sin\nu z$$

where $-\nu^2 = \mu^2 + \lambda^2 - k^2$, and the boundary condition requires that $\nu = \frac{n_z\pi}{a}$. The definition of ν^2 gives us

$$k^2 = \lambda^2 + \mu^2 + \nu^2 = (n_x^2 + n_y^2 + n_z^2)\frac{\pi}{a} = \frac{2mE}{\hbar^2}$$
$$\implies \boxed{E_{n_x,n_y,n_z} = \frac{\pi\hbar^2}{2ma}(n_x^2 + n_y^2 + n_z^2)}$$

These are the energy eigenvalues, and the corresponding eigenstates are

$$\psi(x,y,z) = X(x)Y(y)Z(z) = A\sin\frac{n_x\pi x}{a}\sin\frac{n_y\pi y}{a}\sin\frac{n_z\pi z}{a}$$

The normalization constant can be gotten via

$$
\begin{aligned}
1 &= \int_0^a \int_0^a \int_0^a A^2 \sin^2 \frac{n_x \pi x}{a} \sin^2 \frac{n_y \pi y}{a} \sin^2 \frac{n_z \pi z}{a} \, dx \, dy \, dz \\
&= A^2 \left(\int_0^a \sin^2 \frac{n_x \pi x}{a} \, dx \right) \left(\int_0^a \sin^2 \frac{n_y \pi y}{a} \, dy \right) \left(\int_0^a \sin^2 \frac{n_z \pi z}{a} \, dz \right)
\end{aligned}
$$

Now

$$\int_0^a \sin^2 \frac{n_x \pi x}{a} \, dx = \frac{1}{2} a - \frac{1}{4} a \frac{\sin 2\pi n_x}{n_x \pi} = \frac{1}{2} a$$

since $\sin 2\pi n_x = 0$ for integer n_x. The other two integrals are the same, so

$$1 = A^2 \left(\frac{a}{2} \right)^3 \implies A = \left(\frac{2}{a} \right)^{3/2}$$

and the energy eigenstates

$$\psi(x,y,z) = \left(\frac{2}{a} \right)^{3/2} \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{a} \sin \frac{n_z \pi z}{a}$$

(b) To find the first few energy eigenvalues, we start with the smallest values of (n_x, n_y, n_z) and work upwards. Note that E is symmetric under interchange of the n's, so we can assume $n_x \geq n_y \geq n_z$ with no loss of generality.

(n_x, n_y, n_z)	$n_x^2 + n_y^2 + n_z^2$	Degeneracy
(1,1,1)	3	1
(2,1,1)	6	3
(2,2,1)	9	3
(3,1,1)	11	3
(2,2,2)	12	1
(3,2,1)	14	6

Thus the first six energy eigenstates are $3\frac{\pi\hbar^2}{2ma}$,$6\frac{\pi\hbar^2}{2ma}$,$9\frac{\pi\hbar^2}{2ma}$,$11\frac{\pi\hbar^2}{2ma}$,$12\frac{\pi\hbar^2}{2ma}$, and $14\frac{\pi\hbar^2}{2ma}$. The table includes the degeneracy of each value, which I calculated by figuring out how many different ways I can rearrange the values of n_x, n_y, and n_z. For instance, there is only one way to rearrange $(1,1,1)$, but three ways to rearrange (2,1,1)—(211, 121, and 112), and six ways to rearrange (3,2,1)—(321, 312, 213, 231, 123, and 132).

▷ **4.**

(a) Find Y_0^0 and Y_2^1 Show that they are normalized and orthogonal.
(b) Find Y_l^l. Check that it satisfies the angular equation

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\phi^2}=-l(l+1)\sin^2\theta\, Y$$

Answer:
The general formula is

$$Y_l^m=\epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos\theta)$$

where $\epsilon=(-1)^m$ for $m>0$ (i.e. -1 for odd m)
(a)

$$Y_0^0=+\sqrt{\frac{1}{4\pi}\frac{0!}{0!}}e^{i0}P_0^0(\cos\theta)=\frac{1}{\sqrt{4\pi}}$$

since $P_0^0=1$. And

$$\begin{aligned}Y_2^1&=-\sqrt{\frac{4+1}{4\pi}\frac{(2-1)!}{(2+1)!}}e^{i\phi}P_2^1(\cos\theta)\\&=-\sqrt{\frac{5}{4\pi}\frac{1!}{3!}}e^{i\phi}(3\sin\theta\cos\theta)\\&=-\sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta\end{aligned}$$

To prove they are normalized and orthogonal, we need to calculate three integrals:

$$\begin{aligned}\langle Y_0^0|Y_0^0\rangle&=\int_0^{2\pi}\int_0^{\pi}Y_0^{0*}(\theta,\phi)Y_0^0(\theta,\phi)\,\sin\theta\,d\theta\,d\phi\\&=\int_0^{2\pi}\int_0^{\pi}\frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{4\pi}}\,\sin\theta\,d\theta\,d\phi\\&=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\\&=\frac{1}{4\pi}(2\pi)\left[-\cos\theta\right]_0^{\pi}=1\end{aligned}$$

$$
\begin{aligned}
\langle Y_2^1 | Y_2^1 \rangle &= \int_0^{2\pi} \int_0^{\pi} Y_2^{1*}(\theta, \phi) Y_2^1(\theta, \phi) \, \sin\theta \, d\theta \, d\phi \\
&= \frac{15}{8\pi} \int_0^{2\pi} \int_0^{\pi} (e^{-i\phi} \sin\theta \cos\theta)(e^{i\phi} \sin\theta \cos\theta) \, \sin\theta \, d\theta \, d\phi \\
&= \frac{15}{8\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta \\
&= \frac{15}{8\pi} (2\pi) \left(\frac{4}{15} \right) = 1
\end{aligned}
$$

And

$$
\begin{aligned}
\langle Y_0^0 | Y_2^1 \rangle &= \int_0^{2\pi} \int_0^{\pi} Y_0^{0*}(\theta, \phi) Y_2^1(\theta, \phi) \, \sin\theta \, d\theta \, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{\sqrt{4\pi}} \left(-\sqrt{\frac{15}{8\pi}} e^{i\phi} \sin\theta \cos\theta\right) \, \sin\theta \, d\theta \, d\phi \\
&= -\sqrt{\frac{15}{32\pi^2}} \int_0^{2\pi} e^{i\phi} \, d\phi \int_0^{\pi} \sin^2\theta \cos\theta \, d\theta
\end{aligned}
$$

The θ integral is zero, and so the two functions are orthogonal.

(b) Setting $m = l$ gives us

$$
Y_l^l(\theta, \phi) = (-1)^l \sqrt{\frac{2l+1}{4\pi} \frac{(l-l)!}{(l+l)!}} e^{il\phi} P_l^l(\cos\theta) = (-1)^l \sqrt{\frac{2l+1}{4\pi(2l)!}} e^{il\phi} P_l^l(\cos\theta)
$$

Next we need $P_l^l(\cos\theta)$:

$$
\begin{aligned}
P_l^l(x) &= (1-x^2)^{l/2} \left(\frac{d}{dx} \right)^l \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l \\
&= \frac{1}{2^l l!} (1-x^2)^{l/2} \left(\frac{d}{dx} \right)^{2l} (x^{2l} + \dots)
\end{aligned}
$$

The additional terms in the polynomial will become zero when we take $2l$ derivatives of it. Notice that $\frac{d}{dx}x^{2l} = 2lx^{2l-1}$, $\frac{d^2}{dx^2}x^{2l} = 2l(2l-1)x^{2l-2}$, and so forth, so that $\left(\frac{d}{dx}\right)^{2l} x^{2l} = (2l)!$. Thus

$$\begin{aligned}
Y_l^l(\theta,\phi) &= (-1)^l\sqrt{\frac{2l+1}{4\pi(2l)!}}e^{il\phi}\frac{1}{2^l l!}(1-\cos^2\theta)^{l/2}(2l)! \\
&= (-1)^l\frac{1}{2^l l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}}e^{il\phi}\sin^l\theta \\
&= \boxed{\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}\sin^l\theta}
\end{aligned}$$

Let's check a couple:

$$\begin{aligned}
Y_0^0 &= \frac{1}{1}\sqrt{\frac{1!}{4\pi}}\sin^0\theta = \frac{1}{\sqrt{4\pi}} \\
Y_1^1 &= -\frac{1}{2}\sqrt{\frac{3!}{4\pi}}e^{i\phi}\sin\theta = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta \\
Y_2^2 &= +\frac{1}{2^2(2!)}\sqrt{\frac{5!}{4\pi}}e^{2i\phi}\sin^2\theta = \sqrt{\frac{120}{64(4\pi)}}e^{2i\phi}\sin^2\theta = \sqrt{\frac{15}{32\pi}}e^{2i\phi}\sin^2\theta
\end{aligned}$$

which all match Table 4.3 in Griffiths.

▷ **5.**

For the infinite spherical well of radius $a = 10^{-10}\,\mathrm{m}$, find the energy eigenvalue (in eV) corresponding to $n = 3$, $l = 2$, and $m = 1$, assuming we have an electron ($m = 0.511 \times 10^6\,\mathrm{eV/c^2}$).

Answer:

We can get so caught up in abstractions in this class that it can be nice to throw in a couple real numbers from time to time. According to (Eq. 4.50), $E_{nl} = \frac{\hbar^2}{2ma^2}\beta_{nl}^2$, and so

$$E_{32} = \frac{\hbar^2}{2ma^2}\beta_{32}^2$$

You can get the spherical Bessel zeroes in several places, such as my notes from Friday where you'll see that $\beta_{32} = 3.70$. Given that $a = 1 \times 10^{-10}\,\mathrm{m}$, $\hbar = 6.6 \times 10^{-16}\,\mathrm{eV\cdot s}$, and $m = 0.511 \times 10^6\,\mathrm{eV/c^2}$, we have

$$\begin{aligned}
E_{32} &= \frac{(6.6\times 10^{-16}\,\mathrm{eV\cdot s})^2}{2(0.511\times 10^6\,\mathrm{eV}/(3\times 10^8\,\mathrm{m/s})^2)(10^{-10}\,\mathrm{m})^2}(3.70)^2 \\
&= 3.83\,\mathrm{eV}(3.70)^2 = \boxed{52.5\,\mathrm{eV}}
\end{aligned}$$