$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x \mp ip)$$

if $H\psi = E\psi$ $\quad H(a_+\psi) = (E+\hbar\omega)\psi \quad H(a_-\psi) = (E-\hbar\omega)\psi$

$$a_-\psi_0 = 0 \qquad \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega/2\hbar\, x^2}$$

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0(x) \qquad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\boxed{\begin{aligned} a_+\psi_n &= \sqrt{n+1}\,\psi_{n+1} \\ a_-\psi_n &= \sqrt{n}\,\psi_{n-1} \end{aligned}}$$

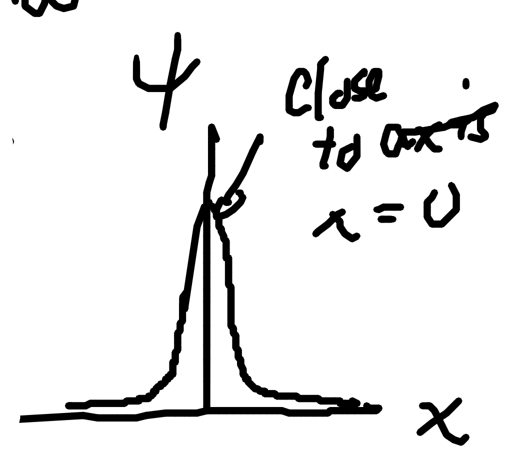
$$\langle x^2\rangle = \langle \psi_n | x^2 | \psi_n\rangle$$

$$= \int \psi_n^* \frac{\hbar}{2m\omega}(a_+ + a_-)^2 \psi_n \, dx$$

$$= \frac{\hbar}{2m\omega}\int \psi_n^*(a_+a_+ + a_+a_- + a_-a_+ + a_-a_-)\psi_n \, dx$$

$$= \frac{\hbar}{2m\omega}(2n+1)$$

small $\langle x^2\rangle$



big $\langle x^2\rangle$

$$\langle x^2\rangle \propto (2n+1)$$

Analytic Method $\quad -\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2x^2\psi = E\psi$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\,x \qquad K = \frac{E}{\frac{1}{2}\hbar\omega}$$

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$

At large ξ, $\quad \psi'' \approx \xi^2\psi \rightarrow \psi(\xi) \approx Ae^{-\xi^2/2} + Be^{+\xi^2/2}$

$$\frac{d}{d\xi}e^{-\xi^2/2} = -\xi e^{-\xi^2/2}$$

$$\frac{d^2}{d\xi^2}e^{-\xi^2/2} = \Big[\underbrace{-e^{-\xi^2/2}}_{\text{irrelevant}} + \underbrace{\xi^2 e^{-\xi^2/2}}_{\text{big}}\Big]$$

So $\psi(\xi) = h(\xi)e^{-\xi^2/2}$

$$\frac{d^2\psi}{d\xi^2} = \left(\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (\xi^2-1)h\right)\cancel{e^{-\xi^2/2}} = (\xi^2-K)h\,\cancel{e^{-\xi^2/2}}$$

$$h'' - 2\xi h' + (K-1)h = 0$$

$$h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots = \sum_{j=0}^{\infty} a_j\xi^j$$

$$h'(\xi) = \sum_{j=0}^{\infty} j a_j \xi^{j-1}$$

$$h''(\xi) = \sum_{j=0}^{\infty} j(j-1)a_j\xi^{j-2} = \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}\xi^j$$