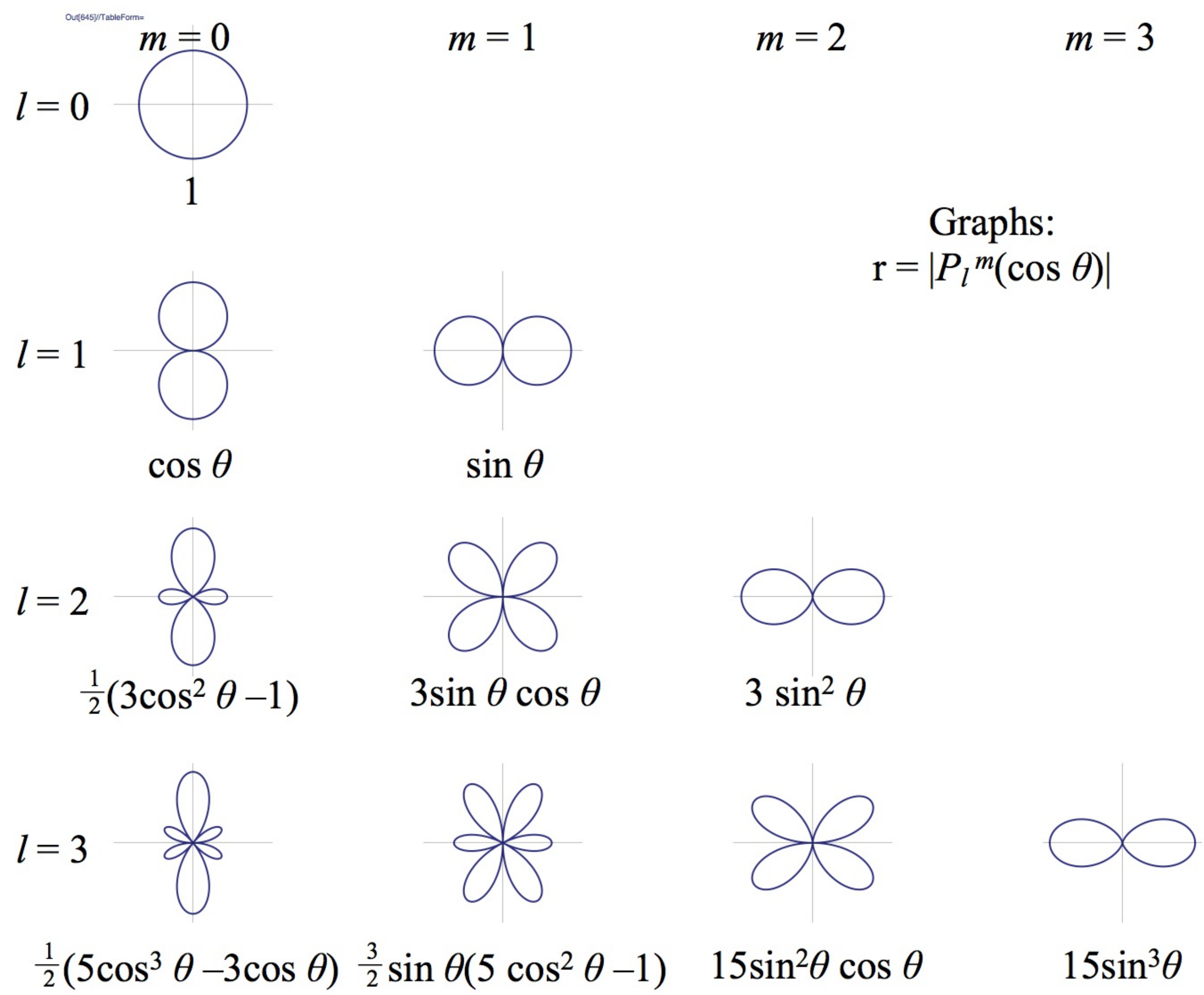
$$\Psi(\vec{r}) = R(r)\,\Theta(\theta)\,\Phi(\phi) \qquad H\Psi = E\Psi$$

$$\Theta(\theta)\,\Phi(\phi) = Y_l^m(\theta,\phi) = A\,P_l^m(\cos\theta)\,e^{im\phi}$$



$Y_l^m(\theta,\phi)$	$m=0$	$m=\pm1$	$m=\pm2$	$m=\pm3$
$l=0$	$\sqrt{\frac{1}{4\pi}}$			
$l=1$	$\sqrt{\frac{3}{4\pi}}\cos\theta$	$\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$		
$l=2$	$\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$	$\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}$	$\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}$	
$l=3$	$\sqrt{\frac{7}{16\pi}}(5\cos^3\theta-3\cos\theta)$	$\mp\sqrt{\frac{21}{64\pi}}\sin\theta(5\cos^2\theta-1)e^{\pm i\phi}$	$\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{\pm 2i\phi}$	$\mp\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{\pm 3i\phi}$

Radial Part

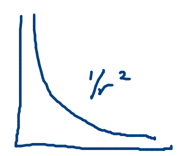
$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right] = \ell(\ell+1)$$

$$u(r) = r\,R(r)$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \underbrace{\left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]}_{\text{effective potential}} u = E u$$

centrifugal term

pushes system outward



eg. Infinite Sperical Well

$$V(r) = \begin{cases} 0, & r < a \\ \infty, & r > a \end{cases}$$

$r<a$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}u = Eu$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\frac{d^2u}{dr^2} = \left[\frac{\ell(\ell+1)}{r^2} - k^2\right]u$$

If $\ell = 0$,

$$u(r) = A\sin kr + B\cos kr$$

$$R(r) = A\frac{\sin kr}{r} + B\frac{\cos kr}{r}$$

$B=0$ or $R(0) = \infty$

Boundary condition: $R(a) = 0$

$$A\frac{\sin ka}{a} = 0$$

$$\rightarrow ka = n\pi \qquad n \in \mathbb{Z}$$

$$\rightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad n = 1, 2, 3, \dots$$

$$\psi_{n,0,0} = \frac{A\sin\frac{n\pi}{a}r}{r}\,Y_0^0(\theta,\phi) = A\frac{\sin\frac{n\pi}{a}r}{r}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} |\psi_{n,0,0}|^2 \, r^2\sin\theta \, dr\, d\theta\, d\phi = 1$$

For r alone

$$\int_0^{\infty} |R(r)|^2 r^2\, dr$$

$$\rightarrow \int_0^{\infty} |rR|^2\, dr$$

$$\rightarrow \int_0^{\infty} |u|^2\, dr = 1$$

General solution to $u'' = \left[\frac{\ell(\ell+1)}{r^2} - k^2\right] u$

is

$$u(r) = A r\, j_\ell(kr) + B r\, n_\ell(kr)$$

spherical Bessel function

spherical Neumann function – blows up at $r=0$.

$B = 0$.

unless particle can't reach $r=0$.

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{\partial}{\partial x}\right)^\ell \frac{\sin x}{x}$$

$$j_\ell(z) = \frac{\sqrt{\pi/2}}{\sqrt{z}} J_{\ell+\frac{1}{2}}(z)$$

(normal) Bessel functions

Boundary conditions: $j_\ell(ka) = 0$

$\beta_{n\ell}$ is the nth zero of the ℓth spherical Bessel function

Mathematica: $\boxed{\text{BesselJZero}[\ell + \tfrac{1}{2}, n]}$

$$ka = \beta_{n\ell}$$

$$k = \frac{\beta_{n\ell}}{a}$$

$$\rightarrow \quad E_{n\ell} = \frac{\hbar^2 \beta_{n\ell}^2}{2ma^2}$$

Spherical Bessel Zeroes $\beta_{n,l}$