Uncertainty Principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta A \Delta B \geq \left| \frac{1}{2i} \langle [A,B] \rangle \right|$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

$$i\hbar \frac{d\psi}{dt} = H\psi$$

$$\frac{d}{dt}\langle Q \rangle = \frac{d}{dt}\langle \psi | Q | \psi \rangle$$

$$= \left\langle \frac{d\psi}{dt} \right| Q |\psi\rangle + \langle \psi | \frac{dQ}{dt} |\psi\rangle + \langle \psi | Q \left| \frac{d\psi}{dt} \right\rangle$$

$$= +\frac{i}{\hbar} \langle H\psi | Q |\psi\rangle + \langle \psi | \frac{dQ}{dt} |\psi\rangle - \frac{i}{\hbar} \langle \psi | Q | H\psi \rangle$$

$$= + \langle \psi | \frac{dQ}{dt} |\psi\rangle + \frac{i}{\hbar} \langle \psi | HQ |\psi\rangle - \frac{i}{\hbar} \langle \psi | QH |\psi\rangle$$

$$\frac{d}{dt}\langle Q \rangle = \left\langle \frac{dQ}{dt} \right\rangle + \frac{i}{\hbar} \langle \psi | [H,Q] |\psi\rangle$$

$$\frac{d}{dt}\langle Q \rangle = \left\langle \frac{dQ}{dt} \right\rangle + \frac{i}{\hbar} \langle [H,Q] \rangle$$

If Q does not have any explicit time dependence,

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar} \langle [H,Q] \rangle$$

If [H,Q] = 0, then <Q> is time-independent

e.g. What is d/dt <X>

$$[H,X] = [-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x), x]$$

$$= -\frac{\hbar^2}{2m}[\frac{d^2}{dx^2}, x]$$

$$[\frac{d^2}{dx^2}, x]f(x) = \frac{d^2}{dx^2}xf - x\frac{d^2}{dx^2}f$$

$$[\frac{d^2}{dx^2}, x]f(x) = (2f' + xf'') - xf''$$

$$[\frac{d^2}{dx^2}, x] = 2\frac{d}{dx}$$

$$\frac{d\langle X \rangle}{dt} = \frac{i}{\hbar}\frac{-\hbar^2}{2m} 2 \left\langle \frac{d}{dx} \right\rangle$$

$$p = -i\hbar \frac{d}{dx}$$

$$\frac{d\langle X \rangle}{dt} = \frac{\langle p \rangle}{m}$$

$$\sigma_H \sigma_Q \geq \left| \frac{1}{2i} \langle [H, Q] \rangle \right|$$

$$\sigma_H \sigma_Q \geq \left| \frac{1}{2i} \frac{\hbar}{i} \frac{d \langle Q \rangle}{dt} \right|$$

assuming <dQ/dt> = 0

$$\Delta E \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d \langle Q \rangle}{dt} \right|$$

$$\Delta E \frac{\sigma_Q}{\left| d \langle Q \rangle / dt \right|} \geq \frac{\hbar}{2}$$

Δt?

$$\sigma_Q = \left| \frac{d \langle Q \rangle}{dt} \right| \Delta t$$

Δt is the time it takes for Q to change by one standard deviation

$\Delta E \Delta t \geq h/2$: so if Δt is small, then the system is changing quickly and ΔE must be big
A rapidly changing system has a very uncertain energy
On the other hand, if ΔE=0, Δt=∞: energy eigenstates are stable

e.g. Δ particle has lifetime 1e-23 s before it disintegrates
$\Delta E \Delta t = \Delta m$ c^2 $\Delta t \geq$ hbar/2: mass is not well-defined!
Δm = 1.75e-20 kg