Physics 4310 Homework #6

4 problems

Solutions

▷ **1.**

Evaluate the following integrals. (These are easy, but you need to be a *little* careful.)

(a) $\int_{-3}^{+1} (x^3 - 3x^2 + 2x - 1)\delta(x+2)\,dx$

(b) $\int_{0}^{\infty} [\cos(3x) + 2]\,\delta(x-\pi)\,dx$

(c) $\int_{-1}^{+1} \exp(|x| + 3)\delta(x-2)\,dx$

Answer:

(a) The delta function is only nonzero when $x + 2 = 0 \implies x = -2$, and so we might as well replace all the x's in the other factor by -2, and pull it out of the integral:

$$\int_{-3}^{+1} (x^3 - 3x^2 + 2x - 1)\delta(x+2)\,dx = ((-2)^3 - 3(-2)^2 + 2(-2) - 1)\int_{-3}^{+1} \delta(x+2)\,dx$$

$$= (-8 - 12 - 4 - 1)(1) = \boxed{-25}$$

(b) The delta function is nonzero when $x = \pi$, so we write

$$(\cos 3\pi + 2)\int_{0}^{\infty} \delta(x-\pi)\,dx = (-1+2)(1) = \boxed{1}$$

(c) The delta function is nonzero when $x = 2$, so we write

$$\exp(|2| + 3)\int_{-1}^{+1} \delta(x-2)\,dx$$

Because the integral does not include $x = 2$, however, the integral is $\boxed{\text{zero}}$.

▷ **2.**

Delta functions are actually "distributions" because they live under integral signs. Two distributions ($D_1(x)$ and $D_2(x)$) are said to be equal if

$$\int_{-\infty}^{\infty} D_1(x)f(x)\,dx = \int_{-\infty}^{\infty} D_2(x)f(x)\,dx$$

for every (ordinary) function $f(x)$.
(a) Show that

$$\delta(cx) = \frac{1}{|c|}\delta(x)$$

where c is a real constant. (Be sure to check the case where c is negative.)
(b) Let $\theta(x)$ be the *step function*:

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Show that $\frac{d\theta}{dx} = \delta(x)$. Hint: Use integration by parts, along with the definition of "equal distributions" above.

Answer:

(a) We need to show that

$$\int \delta(cx) f(x)\, dx = \int \frac{1}{|c|}\delta(x) f(x)\, dx$$

The second integral is easily done:

$$\int_{-\infty}^{\infty} \frac{1}{|c|}\delta(x) f(x)\, dx = \frac{1}{|c|} f(0)$$

Start with the first integral, and let $y = cx$, so that $dy = c\, dx$. Then

$$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = \int_{x=-\infty}^{x=\infty} \delta(y) f(y/c)\, \frac{dy}{c} = \frac{1}{c} f(0) \int_{x=-\infty}^{x=\infty} \delta(y)\, dy$$

Now if $c > 0$, then when $x = -\infty$, $y = -\infty$, and the same for the other limit of integration. Then we can write this integral as

$$\frac{1}{c} f(0) \int_{y=-\infty}^{y=+\infty} \delta(y)\, dy = \frac{1}{c} f(0)(1) = \frac{1}{c} f(0)$$

and because $c = |c|$ when $c > 0$, we can write this as $\frac{1}{|c|} f(0)$. However, if $c < 0$, then the limits of integration are reversed, because $y = +\infty$ when $x = -\infty$, and to swap the limits back to their natural order requires a change in sign. Thus we have

$$\frac{1}{c} f(0) \int_{+\infty}^{-\infty} \delta(y)\, dy = \frac{1}{c} f(0) \left(- \int_{-\infty}^{\infty} \delta(y)\, dy \right) = -\frac{1}{c} f(0)$$

and because $c < 0$, we can write $-c = |c|$, and the integral is $\frac{1}{|c|} f(0)$ as well. Thus we've shown that $\delta(cx) = \frac{1}{|c|}\delta(x)$.

Note that because $\int \delta(x)\, dx = 1$, the delta function $\delta(x)$ must have dimensions of $1/x$, and thus we could have gotten pretty close to the correct answer using dimensional analysis.

(b) We need to show that

$$\int_{-\infty}^{\infty} \frac{d\theta}{dx} f(x)\, dx = \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

for all $f(x)$. To do that, we use integration by parts: remember that $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$. In our first integral, we set $dv = \frac{d\theta}{dx}\, dx$ and $u = f(x)$. We then have

$$v = \int dv = \int \frac{d\theta}{dx}\, dx = \theta(x) \quad \text{and} \quad du = \frac{df(x)}{dx}\, dx$$

In order to keep the mathematicians happy, let's do the integrals from $-L$ to L (with $L > 0$, and then take the limit as $L \to \infty$ at the end of the calculation. Thus

$$\begin{aligned} \int_{-L}^{L} \frac{d\theta}{dx} f(x)\, dx &= [f(x)\theta(x)]_{-L}^{L} - \int_{-L}^{L} \theta(x) \frac{df(x)}{dx}\, dx \\ &= [f(L)\theta(L) - f(-L)\theta(-L)] - \int_{0}^{L} \frac{df(x)}{dx}\, dx \end{aligned}$$

In the last step, I used the fact that $\theta(x) = 0$ for $x < 0$ in order to change the limits of integration. I know that $\theta(L) = 1$ and $\theta(-L) = 0$, so the integral becomes

$$\begin{aligned} \int_{-L}^{L} \frac{d\theta}{dx} f(x)\, dx &= [f(L) - 0] - [f(x)]_0^L \\ &= f(L) - [f(L) - f(0)] \\ &= f(0) \end{aligned}$$

Now I can take the limit $L \to \infty$ safely, and this is what I wanted to show! Thus $\frac{d\theta}{dx} = \delta(x)$. (You could have handwaved this as a reasonable result, because the step function has zero slope everywhere except at $x = 0$ where it has infinite slope. However, it was still possible that $\theta'(x) = 2\delta(x)$, or $\delta(x^2)$, or any number of other possibilities.

▷ **3.**
In class we discussed the *even* (symmetric) bound state wave functions for the finite square well. I want you to analyze the *odd* (i.e. antisymmetric) bound state wave functions now. Derive the transcendental equation for the allowed energies, and solve it graphically. Is there always at least one even bound antisymmetric state?

Answer:
The antisymmetric equivalent to the equation 2.151 is

$$\psi(x) = \begin{cases} Fe^{-\kappa x} & x > a \\ D\sin(lx) & 0 < x < a \\ -\psi(-x) & x < 0 \end{cases}$$

The continuity of $\psi(x)$ at $x = a$ gives us

$$Fe^{-\kappa a} = D\sin(la)$$

and the continuity of $\frac{d\psi}{dx}$ has

$$-\kappa Fe^{-\kappa a} = lD\cos(la)$$

Dividing the second equation by the first gives us

$$\kappa = -l\cot(la)$$

We can use the same variable substitution as in the textbook (2.155):

$$z = la \quad \text{and} \quad z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$$

and

$$\sqrt{z_0^2 - z^2} = \kappa a \implies \kappa/l = \sqrt{(z_0/z)^2 - 1}$$

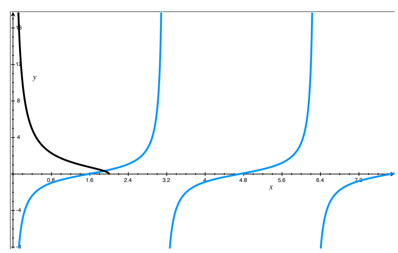
and so

$$\boxed{-\cot z = \sqrt{(z_0/z)^2 - 1}}$$

for the antisymmetric solutions.

The figure shows the graphs of $-\cot z$ (in blue) and $\sqrt{(z_0/z)^2 - 1}$ (in black). The black curve intercepts the horizontal axis at $z = z_0$, and if $z_0 < \frac{\pi}{2}$, then the black curve will hit the axis without crossing any of the blue lines, and there is no solution, and no bound antisymmetric state when

$$\frac{a}{\hbar}\sqrt{2mV_0} < \frac{\pi}{2} \implies V_0 < \frac{\pi^2\hbar^2}{8ma^2}$$



▷ **4.**
Consider the "step" potential:

$$V(x) = \begin{cases} 0, & x \le 0 \\ V_0, & x > 0 \end{cases}$$

(a) Calculate the reflection coefficient for the case $E < V_0$, assuming the incident wave comes in from the left (as in the finite square well).
(b) Calculate the reflection coefficient for the case $E > V_0$.

Answer:
On the left side, we have the energy eigenequation

$$\psi_-'' = -k^2\psi_- \quad \text{where} \quad k = \frac{\sqrt{2mE}}{\hbar}$$

which has the solutions

$$\psi_-(x) = Ae^{ikx} + Be^{-ikx}$$

On the right side, we have the energy eigenequation

$$\psi_+'' = -\chi^2 \psi_+ \quad \text{where} \quad \chi = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

with the solutions

$$\psi_+(x) = F e^{i\chi x} + G e^{-i\chi x}$$

(Note that χ is imaginary if $E < V_0$.) We have the boundary conditions $\psi_+(0) = \psi_-(0)$ and $\psi_+'(0) = \psi_-'(0)$.

$$\psi_+(0) = \psi_-(0) \implies F + G = A + B$$

$$\psi_+'(0) = \psi_-'(0) \implies (A - B)k = (F - G)\chi$$

The reflection coefficient is $R = \frac{|B|^2}{|A|^2}$. Now we need to consider the cases separately:

(a) When $E < V_0$, χ is imaginary; write it as $\chi = iX = i\sqrt{2m(V_0 - E)}/\hbar$. Then the wavefunction on the right-hand side is $\psi_+(x) = F e^{-Xx} + G e^{+Xx}$. However, $G = 0$ or else the wavefunction is unnormalizable. Thus we can write the two boundary conditions as

$$F = A + B \quad \text{and} \quad (A - B)\frac{k}{\chi} = F \implies A + B = (A - B)\frac{k}{\chi}$$

$$\implies B\left(1 + \frac{k}{\chi}\right) = A\left(\frac{k}{\chi} - 1\right)$$

$$\implies \frac{B}{A} = \frac{k - \chi}{k + \chi} = \frac{k - iX}{k + iX}$$

$$\implies R = \frac{|B|^2}{|A|^2} = \frac{BB^*}{AA^*} = \frac{(k - iX)(k + iX)}{(k + iX)(k - iX)} = \boxed{100\%}$$

This is as it should be: there is no place to the right of the origin where the system can exist because $V > E$ everywhere there.

(b) When $E > V_0$ then χ is real. Just as in the well problems for class, we set $G = 0$ because we're only interested in a wave which comes in from the left. The first part of the calculation is the same as in (a):

$$
\begin{aligned}
\frac{B}{A} &= \frac{k-\chi}{k+\chi} \\
\implies R &= \frac{|B|^2}{|A|^2} = \frac{(k-\chi)^2}{(k+\chi)^2} \\
&= \frac{(\sqrt{E}-\sqrt{E-V_0})^2}{(\sqrt{E}+\sqrt{E-V_0})^2} \\
&= \frac{E+(E-V_0)-2\sqrt{E(E-V_0)}}{E+(E-V_0)+2\sqrt{E(E-V_0)}} \\
&= \boxed{\frac{2E-V_0-2\sqrt{E(E-V_0)}}{2E-V_0+2\sqrt{E(E-V_0)}}}
\end{aligned}
$$

This figure shows R as a function of E/V_0. When $E = V_0$, then $R = 1$ and the wave is perfectly reflected backwards (just as when $E < V_0$). As E gets larger, the reflection probability drops. As $E \to \infty$, the reflection coefficient drops to zero. Using a second-order Taylor expansion, you can show that $R \approx \frac{V_0^2}{16E^2}$ for large E.