I've been talking about $\psi(x)$.

I could also talk about $\psi(p)$ instead.

e.g. What is the probability that a particle has a momentum between -1 & 1.

e.g. What is probability that a particle has a position between -1 & 1.

$$P = \int_{-1}^{1} |\psi(x)|^2 \, dx$$

$$P = \int_{-1}^{1} |\psi(p)|^2 \, dp$$

$$\psi(x) = \langle x | \psi \rangle$$

$$\psi(p) = \langle p | \psi \rangle = \int_{-\infty}^{\infty} \psi_p^*(x) \psi(x) \, dx$$

ψ_p

I need ψ_p in terms of x.

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$\hat{p}\, \psi_p(x) = p\, \psi_p(x)$$

$$\int_{-\infty}^{\infty} |\psi_p|^2 \, dx = 1$$

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) \, dx$$

Chapter 4: QM in three dimensions

(spherical coordinates)

$$i\hbar \frac{d\Psi}{dt} = H\Psi$$

in 1D: $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$

3D, $H = -\frac{\hbar^2}{2m}\nabla^2 + V$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\int |\Psi|^2 \, d\vec{r} = 1 \quad \text{normalization}$$

If $\psi_n(\vec{r})$ are energy eigenstates,

i.e. $H\psi_n(\vec{r}) = E_n\psi_n(\vec{r})$

then any wavefunction $\Psi(\vec{r},t) = \sum_n c_n \psi_n(\vec{r}) e^{-iE_nt/\hbar}$

assuming H is time-independent

$x = r\cos\phi\ \sin\theta$
$y = r\sin\phi\ \sin\theta$
$z = r\cos\theta$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

Laplacian in spherical coordinates

Energy eigenstates satisfy $\quad H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

To solve, use separation of variables

$$\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

$$\Phi'' = -m^2\Phi \qquad \phi\ \varphi$$

$$\rightarrow \Phi = Ae^{im\phi} + Be^{-im\phi}$$

Let m be $+$ or $-$

Because $\phi + 2\pi \Leftrightarrow \phi$

$$\Phi(\phi+2\pi) = \Phi(\phi)$$

$$\Phi(\phi+2\pi) = e^{im\phi}e^{im2\pi} = e^{im\phi}$$

m must be an integer

$$\frac{1}{\Theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \ell(\ell+1)\sin^2\theta\right] = m^2$$

$$\Theta(\theta) = A\,P_\ell^m(\cos\theta)$$

Associated Legendre function

$$P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}P_\ell(x)$$

Legendre polynomial

$$P_\ell(x) = \frac{1}{2^\ell\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell$$

$P_\ell^m(x)$ blows up in the $-1\leq x\leq 1$ range unless ℓ is an integer, $\ell\geq 0$.

$$P_\ell^m(x) = \frac{1}{2^\ell\ell!}(1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{\ell+|m|}(x^2-1)^\ell$$

I can only take 2ℓ derivatives of $(x^2-1)^\ell$ before I get zero, and so

$$\ell + |m| \leq 2\ell$$
$$|m| \leq \ell$$

$$m = \underbrace{-\ell, -\ell+1, -\ell+2,}\ \ldots, \ell-1, \ell$$

$$Y_\ell^m(\theta,\phi) = A\,e^{im\phi}P_\ell^m(\cos\theta) = \Theta(\theta)\Phi(\phi)$$

$$A = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} \times \begin{cases}(-1)^m & m\geq 0\\ 1 & m\leq 0\end{cases}$$

Spherical harmonic functions

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m\, Y_{\ell'}^{m'}\ \sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\delta_{mm'}$$

they are orthogonal