

Suppose I have two masses with masses m_i and initial velocities u_i . They collide with a coefficient of restitution e . What are the final velocities?

First we have to switch to the center of mass frame, where

$$v_{com} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

In the COM frame, we have

$$u'_1 = u_1 - \left[\frac{m_1}{m_1 + m_2} u_1 + \frac{m_2}{m_1 + m_2} u_2 \right] = \frac{m_2}{m_1 + m_2} u_1 - \frac{m_2}{m_1 + m_2} u_2 = \frac{\mu}{m_1} (u_1 - u_2)$$

and

$$u'_2 = \frac{\mu}{m_2} (u_2 - u_1)$$

The kinetic energy is

$$KE_i = \frac{1}{2} m_1 \frac{\mu^2}{m_1^2} (u_1 - u_2)^2 + \frac{1}{2} m_2 \frac{\mu^2}{m_2^2} (u_1 - u_2)^2 = \frac{1}{2} \mu^2 (u_1 - u_2)^2 \left[\frac{1}{m_1} + \frac{1}{m_2} \right] = \frac{1}{2} \mu (u_1 - u_2)^2$$

The final velocities in this frame are $v'_1 = \frac{\mu}{m_1} (v_1 - v_2)$ and $v'_2 = \frac{\mu}{m_2} (v_2 - v_1)$, and the kinetic energy is $KE_f = \frac{1}{2} \mu (v_1 - v_2)^2$. Given the coefficient of restitution e ,

$$\frac{1}{2} \mu (v_1 - v_2)^2 = e^2 \frac{1}{2} \mu (u_1 - u_2)^2 \implies (v_1 - v_2)^2 = e^2 (u_1 - u_2)^2 \implies v_1 - v_2 = \pm e (u_1 - u_2)$$

We must also have

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

Solving the first equation for v_1 and substituting:

$$\begin{aligned} m_1 (v_2 \pm e(u_1 - u_2)) + m_2 v_2 &= m_1 u_1 + m_2 u_2 \\ (m_1 + m_2) v_2 &= u_1 (m_1 \mp e m_1) + u_2 (m_2 \pm e m_1) \\ \implies v_2 &= u_1 \frac{m_1}{m_1 + m_2} (1 \mp e) + u_2 \frac{m_2 \pm e m_1}{m_1 + m_2} \\ &= \frac{m_1 u_1 + m_2 u_2 \pm e m_1 (u_2 - u_1)}{m_1 + m_2} \\ &= v_{com} \pm e \frac{m_1}{m_1 + m_2} (u_2 - u_1) \end{aligned}$$

and similarly

$$v_1 = v_{com} \pm e \frac{m_2}{m_1 + m_2} (u_1 - u_2)$$

although I'm not sure if the \pm in the two equations are connected; I'd need to investigate.

If $e = 1$, then

$$v_2 = u_2 \frac{m_2 + m_1}{m_1 + m_2} = u_2 \quad \text{or} \quad v_2 = u_1 \frac{2m_1}{m_1 + m_2} + u_2 \frac{m_2 - m_1}{m_2 + m_1}$$

If $e = 0$, then