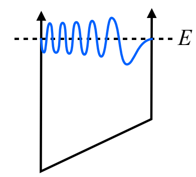
Physics 370 Homework #7

5 problems

Solutions

▷ **1.**

Consider a particle bound in an infinite well where the potential inside isn't constant, as shown in the picture. Suppose the particle is in a fairly high energy state, so that its wave function stretches across the entire well and its wavenumber is fairly high. Decide how, if at all, the particle's wavelength should vary across the potential, and sketch a plausible wavefunction.



Answer:

As you move to the right, the kinetic energy (which is $E - U$) decreases, as does the momentum and the wavenumber k, and so the wavelength increases at the right side of the well. The figure shows a rough idea.

▷ **2.**

Show that $\psi(x) = Ae^{ikx} + Be^{-ikx}$ is a solution to the infinite-well Schrodinger equation, find the relationship between A and B, and show that the energy $E = \frac{\hbar^2 k^2}{2m}$ is quantized as usual.

Answer:

The Schrodinger equation for the infinite well potential is

$$-\frac{\hbar^2}{2m}\psi'' = E\psi$$

The second derivative of our wavefunction is

$$\psi''(x) = -Ak^2 e^{ikx} - Bk^2 e^{-ikx} = -k^2\psi(x)$$

and so

$$-\frac{\hbar^2}{2m}(-k^2)\psi(x) = E\psi(x) \implies E = \frac{\hbar^2 k^2}{2m}$$

To find the coefficients, we need to use the boundary conditions $\psi(0) = \psi(L) = 0$. First,

$$0 = \psi(0) = A + B \implies B = -A$$

so that $\psi(x) = A\left[e^{ikx} - e^{-ikx}\right]$. Second,

$$0 = \psi(L) = A\left[e^{ikL} - e^{-ikL}\right] \implies e^{ikL} = e^{-ikL} \implies e^{2ikL} = 1$$

This equation is true if $2kL = 2\pi n$ for some integer value of n, and so

$$2kL = 2\pi n \implies k = \frac{\pi n}{L} \implies E = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

as we found before.

(Solutions of the form $e^{\pm ikx}$ are *travelling waves*, and hopefully you've seen in the past that standing waves on a string are combinations of two travelling waves moving in opposite directions.)

▷ **3.**
An electron in the $n = 4$ state of a 5 nm wide infinite well makes a transition to the ground state, giving off energy in the form of a photon. What is the photon's wavelength?

Answer:
The energy of a particle in an infinite well is $E = n^2E_1$, where E_1 is the ground-state energy

$$E_1 = \frac{\hbar^2\pi^2}{2mL^2} = \frac{(6.626 \times 10^{-34}\,\mathrm{J/Hz})^2\pi^2}{2(9.11 \times 10^{-31}\,\mathrm{kg})(5 \times 10^{-9}\,\mathrm{m})^2} = 2.41 \times 10^{-21}\,\mathrm{J}$$

If the electron transitions from $n = 4$ to $n = 1$, then it releases a photon with energy

$$\Delta E = E_1(4^2 - 1^2) = 15E_1 = 3.62 \times 10^{-20}\,\mathrm{J}$$

and wavelength is related to energy as

$$E = \frac{hc}{\lambda} \implies \lambda = \frac{hc}{E} = \frac{(6.626 \times 10^{-34}\,\mathrm{J/Hz})(3 \times 10^8\,\mathrm{m/s})}{3.62 \times 10^{-20}\,\mathrm{J}} = 5.49 \times 10^{-6}\,\mathrm{m}$$

or $5491\,\mathrm{nm}$, which is in the infrared.

▷ **4.**
What is the probability that a particle in the first excited ($n = 2$) state of an infinite well would be found in the middle third of the well? How does this compare with your classical expectation?

Answer:
The wavefunction of the first excited state is

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$$

The probability that the particle is in the middle third of the well is

$$P = \int_{L/3}^{2L/3} |\psi_2(x)|^2\,dx$$

$$
\begin{aligned}
&= \frac{2}{L}\int_{L/3}^{2L/3} \sin^2 \frac{2\pi x}{L}\,dx \qquad \text{Let } u = \tfrac{2\pi}{L}x \\
&= \frac{2}{L}\int_{2\pi/3}^{4\pi/3} \sin^2 u \,\frac{du}{2\pi/L} \\
&= \frac{1}{\pi}\int_{2\pi/3}^{4\pi/3} \sin^2 u\,du \\
&= \frac{1}{\pi}\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_{2\pi/3}^{4\pi/3} \\
&= \frac{1}{\pi}\left[\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) - \frac{1}{4}\left(\sin\frac{8\pi}{3} - \sin\frac{4\pi}{3}\right)\right] \\
&= \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = \boxed{0.196}
\end{aligned}
$$

The classical expectation is for the probability to be $\frac{1}{3}$; it is lower in this case because there is a node in the center of the well.

▷ **5.**
A tiny $1\,\mu$g particle is in a $1\,$cm wide enclosure and takes a year to bounce from one end to the other and back.
(a) How many nodes are there in its enclosure?
(b) How would your answer change if the particle were more massive, or moving faster?

Answer:
The distance between nodes is $\frac{1}{2}\lambda$, and the wavelength of the particle is $\lambda = \frac{h}{p}$, so if the box has width L it will contain

$$\frac{L}{\frac{1}{2}\lambda} = \frac{2L}{(h/p)} = \frac{2Lp}{h}\ \text{nodes}$$

(a) In this case, the particle has a speed of

$$v = \frac{0.01\,\text{m}}{1\,\text{yr}} = \frac{0.01\,\text{m}}{3.16\times 10^{7}\,\text{s}} = 6.34\times 10^{-10}\,\text{m/s}$$

and its momentum is $p = mv = (10^{-9}\,\text{kg})(6.34\times 10^{-10}\,\text{m/s}) = 6.34\times 10^{-19}\,\text{kg}\cdot\text{m/s}$. Thus there will be

$$\frac{2Lp}{h} = \frac{2(0.01\,\text{m})(6.34\times 10^{-19}\,\text{kg}\cdot\text{m/s})}{(6.626\times 10^{-34}\,\text{J/Hz})} = \boxed{1.9\times 10^{13}\,\text{nodes}}$$

This is large to reflect the fact that this is not at all wave-like in behavior.
(b) If the particle were more massive or moving faster (as you would expect in an everyday situation), its momentum would be larger and there would be even more nodes.