Physics 370 Homework #8

7 problems

Solutions

▷ **1.**

(Harris 5.34) A 50 eV electron is trapped between electrostatic walls 200 eV high. How far does its wave function extend beyond the walls?

Answer:

The distance the wave penetrates into the walls is characterized by

$$\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

In this case, $U_0 = 200\,\mathrm{eV}$ and $E = 50\,\mathrm{eV}$. Also, $m = 511 \times 10^3\,\mathrm{eV/c^2} = \frac{511\times10^3\,\mathrm{eV}}{(3\times10^8\,\mathrm{m/s})^2} = 5.68 \times 10^{-12}\,\mathrm{eV \cdot s^2/m^2}$ and $\hbar = 6.58 \times 10^{-16}\,\mathrm{eV \cdot s}$, so

$$\begin{aligned}\delta &= \frac{(6.58 \times 10^{-16}\,\mathrm{eV \cdot s})}{\sqrt{2(5.68 \times 10^{-12}\,\mathrm{eV\,s^2/m^2})(200\,\mathrm{eV} - 50\,\mathrm{eV})}} \\ &= \boxed{1.6 \times 10^{-11}\,\mathrm{m}}\end{aligned}$$

or $0.016\,\mathrm{nm}$.

▷ **2.**

(Harris 5.52) To a good approximation, the hydrogen chloride molecule HCl behaves vibrationally as a quantum harmonic oscillator of spring constant $480\,\mathrm{N/m}$ and with effective oscillating mass just that of the lighter atom, hydrogen. If it were in its ground vibrational state, wht wavelength photon would be just right to bump this molecule up to its next-higher vibrational energy state?

Answer:

The energy levels of a harmonic oscillator are $E_n = (n + \frac{1}{2})\hbar\omega_0$, where $\omega_0 = \sqrt{\frac{\kappa}{m}}$. (The mass is the mass of a hydrogen atom, $m = 1.67 \times 10^{-27}\,\mathrm{kg}$.) To bump the molecule from the ground state to the next-higer state requires energy

$$\Delta E = \frac{3}{2}\hbar\omega_0 - \frac{1}{2}\hbar\omega_0 = \frac{h}{2\pi}\omega_0$$

The energy of a photon is $E = \frac{hc}{\lambda}$, so the wavelength required is

$$\begin{aligned}
\lambda &= \frac{hc}{\Delta E} = \frac{hc}{(h/2\pi)\omega_0} = \frac{2\pi c}{\omega_0} = 2\pi c\sqrt{\frac{m}{\kappa}} \\
&= 2\pi(3\times 10^{8}\,\mathrm{m/s})\sqrt{\frac{1.67\times 10^{-27}\,\mathrm{kg}}{480\,\mathrm{N/m}}} \\
&= \boxed{3.51\times 10^{-6}\,\mathrm{m}} = 3510\,\mathrm{nm}
\end{aligned}$$

which is infrared radiation.

▷ **3.**

(Harris 5.56-58) Consider a particle in an infinite square well with the wavefunction $\psi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$.

(a) Show that the uncertainty in a particle's position is given by

$$\Delta x = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

(b) Show that the uncertainty in a particle's momentum is $\Delta p = \frac{n\pi\hbar}{L}$.

(c) What is the product of these uncertainties? Discuss the result.

Answer:____

(a) To calculate the uncertainty in position, we need $\langle x\rangle$ and $\langle x^2\rangle$. In general,

$$\begin{aligned}
\langle x^a\rangle &= \int_0^L \psi_n^* x^a \psi_n\,dx \\
&= \frac{2}{L}\int_0^L x^a \sin^2\frac{n\pi x}{L}\,dx \qquad \text{Let } u = n\pi x/L \\
&= \frac{2}{L}\int_0^{n\pi} \frac{u^a}{(n\pi/L)^a}\sin^2 u\,\frac{du}{n\pi/L} \\
&= \frac{2}{L}\left(\frac{L}{n\pi}\right)^{a+1}\int_0^{n\pi} u^a \sin^2 u\,du \\
&= \frac{2L^a}{(n\pi)^{a+1}}\int_0^{n\pi} u^a \sin^2 u\,du
\end{aligned}$$

Using Mathematica and noting that $\cos 2n\pi = 1$ and $\sin 2n\pi = 0$, we find that

$$\begin{aligned}
\langle x \rangle &= \frac{2L}{(n\pi)^2}\frac{1}{8}\left(2n^2\pi^2\right) = \frac{1}{2}L \\
\left\langle x^2 \right\rangle &= \frac{2L^2}{(n\pi)^3}\frac{1}{24}(4n^3\pi^3 - 6n\pi) = L^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)
\end{aligned}$$

and so

$$\begin{aligned}
(\Delta x)^2 &= \left\langle x^2 \right\rangle - \langle x \rangle^2 \\
&= L^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right) - \frac{1}{4}L^2 \\
&= \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)L^2 \\
\implies \Delta x &= L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}
\end{aligned}$$

Q.E.D.

(b) Note that

$$\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} = -i\hbar\sqrt{\frac{2}{L}}\frac{n\pi}{L}\cos\frac{n\pi x}{L},$$

and

$$\hat{p}^2\psi(x) = -\hbar^2\frac{\partial^2}{\partial x^2}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} = \hbar^2\sqrt{\frac{2}{L}}\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L} = \hbar^2\left(\frac{n\pi}{L}\right)^2\psi(x)$$

Thus

$$\begin{aligned}
\langle p \rangle &= \int_0^L \psi^*(x)\hat{p}\psi(x)\,dx = -i\hbar\frac{2}{L}\frac{n\pi}{L}\int_0^L \sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \\
\left\langle p^2 \right\rangle &= \int_0^L \psi^*(x)\hbar^2\left(\frac{n\pi}{L}\right)^2\psi(x)\,dx = \hbar^2\left(\frac{n\pi}{L}\right)^2\int_0^L \psi^*(x)\psi(x)\,dx = \left(\frac{\hbar n\pi}{L}\right)^2 \\
\implies \Delta p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\left(\frac{\hbar n\pi}{L}\right)^2 - 0} = \frac{\hbar n\pi}{L}
\end{aligned}$$

(c)

$$\Delta x \Delta p = \left(L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \right) \frac{\hbar n \pi}{L}$$
$$= \hbar \sqrt{\frac{n^2\pi^2}{12} - \frac{1}{2}}$$

We of course expect this to obey the Heisenberg Uncertainty Principle $\Delta x \Delta p \geq \frac{\hbar}{2}$, which will be true if $\sqrt{\frac{n^2\pi^2}{12} - \frac{1}{2}} \geq \frac{1}{2}$. The left-hand side is smallest when $n = 1$ (the ground state), and is equal to

$$\sqrt{\frac{\pi^2}{12} - \frac{1}{2}} = 0.567$$

which is greater than $\frac{1}{2}$. Thus the uncertainty principle is upheld. The uncertainty increases as one moves to larger energy states.

▷ **4.**
Prove that $\psi(x) = xe^{-\frac{1}{2}b^2x^2}$ is a solution to the Schrodinger equation with a harmonic-well potential

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}\kappa x^2 \psi(x) = E\psi(x)$$

and find b and E in terms of the spring constant κ and the mass m. (Hint: Because derivatives of complex exponentials makes me nervous, I would write $\psi(x) = xf(x)$ where $f(x)$ is the exponential, and I would calculate $f'(x)$ and $f''(x)$ separately.)

Answer:
if you're a little nervous about the chain rule like I am, it's handy to define $f(x) = e^{-\frac{1}{2}b^2x^2}$, and to note that

$$f'(x) = -b^2 x e^{-\frac{1}{2}b^2x^2} = -b^2 x f(x)$$
$$f''(x) = \frac{d}{dx}(-b^2 x f(x)) = -b^2 \left[f(x) + x f'(x) \right]$$
$$= -b^2 \left[f(x) + x \left(-b^2 x f(x) \right) \right]$$
$$= -b^2 \left[1 - b^2 x^2 \right] f(x)$$

and so

$$\psi'(x) = \frac{d}{dx}(xf(x)) = f(x) + xf'(x)$$

$$\psi''(x) = f'(x) + f'(x) + xf''(x) = xf''(x) + 2f'(x)$$

$$= -b^2x\left[1 - b^2x^2\right]f(x) + 2\left[-b^2xf(x)\right]$$

$$= \left[-b^2x + b^4x^3 - 2b^2x\right]f(x)$$

$$= \left[b^4x^3 - 3b^2x\right]f(x)$$

Substituting this into the Schrodinger equation gives us

$$-\frac{\hbar^2}{2m}\left[b^4x^3 - 3b^2x\right]f(x) + \frac{1}{2}\kappa x^2\left[xf(x)\right] = Exf(x)$$

$$x^3\left[-\frac{\hbar^2b^4}{2m} + \frac{1}{2}\kappa\right]f(x) + x\left[\frac{3b^2\hbar^2}{2m} - E\right]f(x) = 0$$

For this to be true for all x, the two coefficients must be zero. Therefore

$$\frac{1}{2}\kappa = \frac{\hbar^2b^4}{2m} \implies b^4 = \frac{m\kappa}{\hbar^2} \implies b = \left(\frac{m\kappa}{\hbar^2}\right)^{1/4}$$

(which is as it's given in the text), and

$$E = \frac{3\hbar^2}{2m}b^2 = \frac{3\hbar^2}{2m}\frac{\sqrt{m\kappa}}{\hbar} = \frac{3}{2}\hbar\sqrt{\frac{\kappa}{m}} = \frac{3}{2}\hbar\omega_0$$

▷ **5.**

Prove that the variance $(\Delta x)^2 = \langle(x - \langle x\rangle)^2\rangle$, can also be written as $\langle x^2\rangle - \langle x\rangle^2$.

Answer:____

The key is to note that $\langle\langle x\rangle\rangle = \langle x\rangle$; that is, $\langle x\rangle$ is just a number that can be pulled out of an average.

$$\begin{aligned}\left\langle(x-\langle x\rangle)^2\right\rangle &= \left\langle x^2 - 2x\langle x\rangle + \langle x\rangle^2\right\rangle \\ &= \left\langle x^2\right\rangle - 2\langle x\rangle\langle x\rangle + \langle x\rangle^2 \\ &= \left\langle x^2\right\rangle - 2\langle x\rangle^2 + \langle x\rangle^2 \\ &= \left\langle x^2\right\rangle - \langle x\rangle^2\end{aligned}$$

Q.E.D.

▷ **6.**
Show that the vector $v = \begin{pmatrix}1\\3\end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}$, and find its eigenvalue.

Answer:
v is an eigenvector if $Av = \lambda v$, so we calculate

$$\begin{pmatrix}2 & 1\\3 & 3\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}2\cdot 1 + 3\cdot 1\\3\cdot 1 + 4\cdot 3\end{pmatrix} = \begin{pmatrix}5\\15\end{pmatrix} = 5\begin{pmatrix}1\\3\end{pmatrix}$$

and so v is an eigenvector of A with eigenvalue $\lambda = 5$.

▷ **7.**
Use integration by parts to show that the expectation value of the momentum of a bound state $\psi(x)$ is zero. (Hints: you can assume that $\psi(x)$ is real with no loss of generality. Your solution must not hold for a free state like e^{ikx}.)

Answer:
The expectation value of the momentum of a wavefunction $\psi(x)$ is

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{\partial}{\partial x}\psi(x)\,dx = -i\hbar\int_{-\infty}^{\infty}\psi(x)\frac{\partial\psi}{\partial x}\,dx$$

if we assume that $\psi(x)$ is real. Integration by parts says that $\int f\,dg = (fg) - \int g\,df$. If we let $f = \psi(x)$ and

$g = \psi(x)$, then $dg = \frac{\partial \psi(x)}{\partial x} dx$, and so

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(x) \frac{\partial \psi(x)}{\partial x}\, dx &= [\psi(x)\psi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(x) \frac{\partial \psi(x)}{\partial x}\, dx \\
2\int_{-\infty}^{\infty} \psi(x) \frac{\partial \psi(x)}{\partial x}\, dx &= \left[|\psi(x)|^2\right]_{-\infty}^{\infty} \\
\implies \langle p \rangle &= -\frac{i\hbar}{2} \left[|\psi(x)|^2\right]_{-\infty}^{\infty}
\end{aligned}$$

If the particle is bound, however, then the probability $|\psi(x)|^2$ of finding it at $\pm\infty$ is zero. Therefore $\langle p \rangle = 0$.