Physics 370 Homework #2

5 problems

Solutions

▷ **1.**

Let us use the Lorentz transformations to solve problem 1 from last week. To be specific: a rocket containing clock R, moving at $0.6c$, passes clock X, when clocks R and X read zero. The rocket then passes clock Y, a distance d from clock X in the ground's frame. The rocket's clock reads 8 s when it passes clock Y. We will consider three events:

A. the moment R passes X (which is also when R and X read zero)

B. the moment R passes Y

C. the moment clock Y reads zero

Here is a table showing the position and time for these three events, in the frame of the ground (unprimed) and the frame of the rocket (primed). Let's work in the units where $c = 1$ and lengths are measured in seconds.

A. R passes X	B. R passes Y	C. Y reads zero
$x_A = 0$	$x_B = d$	$x_C = d$
$t_A = 0$	$t_B =$	$t_C = 0$
$x'_A = 0$	$x'_B =$	$x'_C =$
$t'_A = 0$	$t'_B =$	$t'_C =$

(a) Find γ.
(b) What is x'_B, the position of event B in the rocket's frame?
(c) Which is true? Is $t_B = 8\,\mathrm{s}$ or $t'_B = 8\,\mathrm{s}$?
(d) Find d.
(e) Find the rest of the quantities in the table. (Don't turn in this sheet, recopy the table in your homework please.)
(f) Find the time on clock Y when the rocket passes it (at event B).

Answer:____

(a) $\gamma = \dfrac{1}{\sqrt{1-v^2}} = \dfrac{1}{\sqrt{1-0.6^2}} = \dfrac{5}{4}$
(b) Event B occurs at the position of the rocket, so $x'_B = 0$.
(c) The value of $8\,\mathrm{s}$ is the time the clock reads at event B; this is the time measured in the

rocket's frame, so $t'_B = 8\,\mathrm{s}$.

(d) The speed of the rocket in the ground frame is

$$0.6 = \frac{\Delta x}{\Delta t} = \frac{x_B - x_A}{t_B - t_A} = \frac{d-0}{t_B - 0} = \frac{d}{t_B}$$

I need to calculate t_B using the Lorentz transformation:

$$t_B = \gamma(t'_B + vx'_B) = \gamma t'_B = \frac{5}{4}(8\,\mathrm{s}) = 10\,\mathrm{s}$$

Therefore

$$d = (0.6)(10\,\mathrm{s}) = 6\,\mathrm{s}$$

(e) We have enough values to find the rest:

$$x'_C = \gamma(x_C - vt_C) = \gamma d = \frac{5}{4}(6\,\mathrm{s}) = 7.5\,\mathrm{s}$$

$$t'_C = \gamma(t_C - vx_C) = -v\gamma d = -4.5\,\mathrm{s}$$

A. R passes X	B. R passes Y	C. Y reads zero
$x_A = 0$	$x_B = 6\,\mathrm{s}$	$x_C = 6\,\mathrm{s}$
$t_A = 0$	$t_B = 10\,\mathrm{s}$	$t_C = 0$
$x'_A = 0$	$x'_B = 0$	$x'_C = 7.5\,\mathrm{s}$
$t'_A = 0$	$t'_B = 8\,\mathrm{s}$	$t'_C = -4.5\,\mathrm{s}$

(f) The time on clock Y at event B is the time that passes from event C to event B, in the clock's frame, which is $t_B - t_C = 10\,\mathrm{s}$.

▷ **2.**

Suppose a person in a rocket flies by you at $v = 0.5c$. A person inside the rocket fires a gun which is able to deliver bullets that move at half the speed of light (in the gun's own frame, of course). What is the bullet's speed in your frame if
(a) it is fired in the direction of the rocket's motion
(b) it is fired in the direction opposite the rocket's motion
(c) it is fired in a direction perpendicular to the rocket's motion

Answer:______________________________

(a) The velocity addition equation says

$$u_x = \frac{v + u'_x}{1 + vu'_x}$$

assuming the rocket is moving in the x direction. If v and u'_x point in the same direction then

$$u_x = \frac{0.5+0.5}{1+(0.5)(0.5)} = \frac{1}{1.25} = \boxed{0.8}$$

(b) In this case

$$u_x = \frac{0.5-0.5}{1-(0.5)(0.5)} = \boxed{0}$$

Yes, the bullet would appear motionless, even at relativistic speeds!
(c) Here we need the x and y components of the speed:

$$u_x = \frac{u'_x + v}{1+vu'_x} = \frac{0+0.5}{1+0} = 0.5$$

$$u_y = \frac{u'_y}{\gamma(1+u'_x v)} = \frac{0.5}{\frac{1}{\sqrt{1-0.5^2}}(1+0(0.5))} = 0.433$$

The magnitude of this velocity is $|\vec{u}| = \sqrt{0.5^2 + 0.433^2} = \boxed{0.66}$

▷ **3.**
Given an object moving with subluminal speed $\vec{u} = u_x\hat{x} + u_y\hat{y}$ in one frame (so that $|u| < 1$) in one frame. Find the magnitude $|\vec{u}'|$ of that speed in a frame moving with speed $v > 0$ along the $+x$–axis, and prove that $|\vec{u}'| < 1$ for all values of $u_x, u_y, v < 1$. (This is just algebra, but it requires moderately clever algebra.)

Answer:
According to the velocity addition equations,

$$u'_x = \frac{u_x - v}{1-u_x v} \quad \text{and} \quad u'_y = \frac{u_y}{\gamma(1-u_x v)}$$

Then

$$\begin{aligned}
(u')^2 &= \left(\frac{u_x - v}{1-u_x v}\right)^2 + \left(\frac{u_y}{\gamma(1-u_x v)}\right)^2 \\
&= \frac{1}{(1-u_x v)^2}\left[(u_x - v)^2 + \frac{1}{\gamma^2}u_y^2\right] \\
&= \frac{1}{(1-u_x v)^2}\left[u_x^2 + v^2 - 2u_x v + (1-v^2)u_y^2\right] \\
&= \frac{1}{(1-u_x v)^2}\left[u_x^2 + u_y^2 + v^2 - 2u_x v - v^2 u_y^2\right]
\end{aligned}$$

Now note that $u_x^2 + u_y^2 = u^2$, and $u_y^2 = u^2 - u_x^2$:

$$
\begin{aligned}
&= \frac{1}{(1-u_x v)^2}\left[u^2 + v^2 - 2u_x v - v^2(u^2 - u_x^2)\right] \\
&= \frac{1}{(1-u_x v)^2}\left[u^2 + v^2 - v^2u^2 - 1 + 1 - 2u_x v + v^2 u_x^2\right] \qquad \text{I added } -1+1 \text{ here} \\
&= \frac{1}{(1-u_x v)^2}\left[-(1-u^2)(1-v^2) + (1-u_x v)^2\right] \qquad \text{factoring} \\
(u')^2 &= 1 - \frac{(1-u^2)(1-v^2)}{(1-u_x v)^2}
\end{aligned}
$$

Because $u^2, v^2 < 1$, the second term of this expression is positive, and so $(u')^2 < 1$. **Q.E.D.**

It's worth checking a couple special cases. If $u = u_x = 0$, then the object is stationary in the first frame, and so its speed in the second frame is

$$|u'|^2 = 1 - \frac{1(1-v^2)}{1^2} = 1 - (1-v^2) = v^2$$

or just v. If $u = 1$ (light), then the second term is zero, and so $(u')^2 = 1$.

There's a less coordinate-based form of this equation, if we note that we can write the speed of the second frame as $\vec{v} = v\hat{x}$, so that $u_x v = \vec{u} \cdot \vec{v}$. Thus

$$|u'|^2 = 1 - \frac{(1-u^2)(1-v^2)}{(1-\vec{u}\cdot\vec{v})^2}$$

▷ **4.**

In the Earth's frame of reference, someone snaps their fingers: ten nanoseconds and 5 feet away, a firecracker goes off. (We will use the approximation $1\,\text{ft} \approx 1\,\text{light} - \text{nanosecond}$, because isn't that cool how close they are?).

(a) What is the spacetime interval Δs between these two events?

(b) To an observer moving at $0.1c$ between the fingers and the firecracker, it took 20 seconds for the firecracker to go off after the snap. How far does the firecracker appear to be from the fingers in this person's timeframe, in feet?

Answer:______________________________

(a) The spacetime interval between the two events is

$$\Delta s = \sqrt{(\Delta t)^2 - (\Delta x)^2} = \sqrt{10^2 - 5^2} = \boxed{8.66\,\text{ns}}$$

(b) The spacetime interval is the same in all frames, and so

$$
\begin{aligned}
(\Delta s)^2 &= (\Delta t)^2 - (\Delta x)^2 \\
(8.66\,\mathrm{ns})^2 &= (20\,\mathrm{ns})^2 - (\Delta x)^2 \\
\implies \Delta x &= \sqrt{400\,(\mathrm{ns})^2 - 75\,(\mathrm{ns})^2} \\
&= \boxed{18\,\mathrm{ft}}
\end{aligned}
$$

▷ **5.**
A spaceship orbits the Earth at a constant speed $v = 0.1c$, so that it takes $5\,\mathrm{s}$ to go around once, according to clocks on Earth. A clock on the spaceship is visible on Earth, and during one complete orbit that clock goes from $t = 0$ to $t = T$. Find T.

Answer:________________________________

What we're asked to find is the *proper time interval* of the clock, which is given by

$$\Delta\tau = \int_{t_i}^{t_f} \sqrt{1 - v^2}\, dt$$

where time and speed are measured in the same frame (the Earth's frame, in this case). Because v is constant, we can simplify the integral.

$$
\begin{aligned}
\Delta\tau &= \int_0^{5\,\mathrm{s}} \sqrt{1 - (0.1)^2}\, dt \\
&= \sqrt{0.99} \int_0^{5\,\mathrm{s}} dt \\
&= (5\,\mathrm{s})\sqrt{0.99} = \boxed{4.97\,\mathrm{s}}
\end{aligned}
$$

The clock in the spaceship moves slightly slower than if the spaceship is stationary.