Physics 370 Homework #10

5 problems

Due by Wednesday, November 9

▷ **1.**

A *paramagnet*, in its simplest formulation, is a collection of magnetic dipoles, each with magnetic dipole moment $\vec{\mu}$. When placed in a magnetic field $\vec{B} = B\hat{z}$, each dipole has energy $U = -\vec{\mu}\cdot\vec{B} = -\mu_z B$. Suppose the z–component of the dipole moment is *quantized*, so that it can only take one of two values: $\mu_z = +\mu_0$ (call this "up") and $\mu_z = -\mu_0$ ("down"). If there are $N_\uparrow$ dipoles that point up, and $N_\downarrow$ that point down, then the total energy is

$$E = -N_\uparrow \mu_0 B + N_\downarrow \mu_0 B = (N_\downarrow - N_\uparrow)\mu_0 B$$

(a) Rewrite the energy formula in terms of the total number of dipoles N and the number $N_\downarrow$ that point down, and then solve that equation for $N_\downarrow$. (Note that, if N is constant, $N_\downarrow$ parametrizes the energy.)
(b) Find the multiplicity Ω of the macrostate where there are $N_\downarrow$ dipoles pointing down. (This is also an energy macrostate.)
(c) Write an expression for the entropy as a function of $N_\downarrow$, and use Stirling's approximation $\ln x! \approx x\ln x - x$ $(x \gg 1)$ to eliminate the factorials from the expression assuming that the number of dipoles is large.
(d) Show that

$$kT = 2\mu_0 B \left(\ln \frac{N_\uparrow}{N_\downarrow}\right)^{-1}$$

(e) Find the temperature of the paramagnet when $N_\downarrow$ is $\frac{1}{4}N$ and $\frac{1}{2}N$.

▷ **2.**

The entropy of an *ideal gas* (point particles that do not interact with one another) has the form

$$S = kN\left[\ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln\left(\frac{E}{N}\right) + C\right]$$

where C is a constant. Find an expression of the ideal gas's energy as a function of temperature $E(T)$.

▷ **3.**

In a large number of distinguishable harmonic oscillators, how high does the temperature have to be for the probability of occupying the ground state to be less than $\frac{1}{2}$?

▷ **4.**

In class we showed that a system in contact with a thermal reservoir T has an average energy

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{1}{kT^2}\frac{1}{Z}\frac{\partial Z}{\partial T}$$

where $\beta = \frac{1}{kT}$ and Z is the *partition function* of a system, defined as

$$Z = \sum_s e^{-E_s/kT}$$

summed over all possible microstates of the system.

(a) The possible microstates of a harmonic oscillator have energy $E_j = \left(j + \frac{1}{2}\right)\hbar\omega_0$, where $j = 0, 1, 2, \ldots$. Find the partition function of a harmonic oscillator, and do out the sum so you have a closed expression. (Hint: the sum is a power series.)

(b) Find the average energy of the harmonic oscillator.

(c) What is the energy approximately in the low-temperature limit $kT \ll \hbar\omega_0$ and the high-temperature limit $kT \gg \hbar\omega_0$?

▷ **5.**

Find the *most probable* speed of an ideal gas, according to the Maxwell Speed Distribution; that is, find the mode of the distribution.