Physics 370 Exam 2 v1 Solutions

November 10–18, 2016

This is a take-home exam, due on Friday, November 18th in class. You may use the following resources:

- *Your textbook (no other books)*
- *Your notes (including any reference sheets you may have put together)*
- *A calculator*
- *Items posted on the class website*
- *Scratch paper*
- *A reference for physical constants and integrals (including Google)*
- *A symbolic algebra system such as Mathematica, Maple, or WolframAlpha for doing integrals*
- *Your instructor (though he may be coy)*

No other resources may be used without consulting with the instructor first.
You may attach additional pages to this exam to show your work (in fact, it's encouraged if the derivation is long). Be sure that those pages are clearly labelled to indicate which problems they refer to.
You have a lot of time, so take the time to do neat work, please.
Remember that every question has partial credit available. Don't leave anything blank.

Good luck!

$\boxed{4}$ 1. Circle all of the following wavefunctions that could describe a real, physical system in the range $-\infty < x < \infty$. A is an arbitrary constant.

Ae^{2ix} $\quad$ Ae^{-3x} $\quad$ Ax^2 $\quad$ Ae^{-x^2} $\quad$ $\boxed{Ae^{-x^2}}$ $\quad$ None of these

This is the only function which goes to zero at $\pm\infty$.

2. Consider an infinite square well that runs from $x = 0$ to $x = 2$. At $t = 0$, a system has wavefunction
$$\Psi(x,0) = \frac{4}{5}\sin\pi x + \frac{3}{5}\sin 2\pi x$$

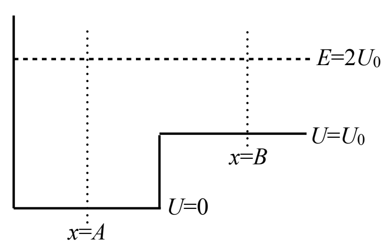
$\boxed{4}$ (a) Find $\Psi(x,t)$.

The eigenstates of an infinite square well with $L = 2$ is $\psi_n(x) = \sin\frac{n\pi x}{2}$ with $E_n = \frac{n^2\pi^2\hbar^2}{8m}$; the time-dependence of the eigenstate is $\psi_n(x,t) = \psi_n(x)e^{-i\frac{E_n}{\hbar}t}$. We can write
$$\Psi(x,0) = \frac{4}{5}\psi_2(x) + \frac{3}{5}\psi_4(x)$$
and so
$$\Psi(x,t) = \frac{4}{5}\sin\pi x\, e^{-i\frac{4\pi^2\hbar}{8m}t} + \frac{3}{5}\sin 2\pi x\, e^{-i\frac{16\pi^2\hbar}{8m}t}$$

$\boxed{4}$ (b) Find the probability that this particle will be found in the middle half ($0.5 < x < 1.5$) of the well, at time $t = 0$.
$$P = \int_{0.5}^{1.5}\left|\frac{4}{5}\sin\pi x + \frac{3}{5}\sin 2\pi x\right|^2 dx = 9\%$$
So small! The particle is most likely to be found towards the edges of the well.

$\boxed{4}$ 3. __C__ The figure shows a step-like potential. A particle with energy $2U_0$ is travelling through the potential. What is the ratio λ_A/λ_B in the wavefunction's wavelengths at points $x = A$ and $x = B$?

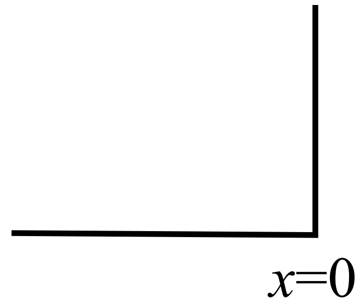
A) $\frac{1}{4}$ **B)** $\frac{1}{2}$ **C)** $\frac{1}{\sqrt{2}}$ **D)** 1
E) $\sqrt{2}$ **F)** 2 **G)** 4



$$\frac{\lambda_A}{\lambda_B} = \frac{k_B}{k_A} = \sqrt{\frac{KE_B}{KE_A}} = \sqrt{\frac{U_0}{2U_0}} = \frac{1}{\sqrt{2}}$$

4. A particle travels to the right with energy E and interacts with the L–shaped potential

$$U = \begin{cases} 0, & x < 0 \\ \infty, & x > 0 \end{cases}$$



$\boxed{2}$ (a) **B** Is the energy quantized?

A) Yes **B)** No

This is an unbound state.

$\boxed{4}$ (b) Write the wavefunction $\psi(x)$ for $x < 0$ and for $x > 0$. It will include one arbitrary constant.

The wavefunction is zero for $x > 0$ because of the infinite potential wall. For $x < 0$, we can write

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

where $k = \sqrt{2mE}/\hbar$. This must satisfy the boundary condition $\psi(0) = 0 = Ae^0 + Be^0 = A + B \implies B = -A$. Thus

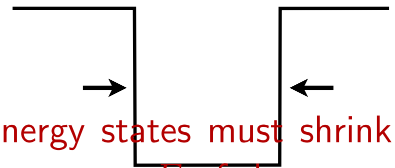
$$\psi(x) = \begin{cases} A(e^{ikx} - e^{-ikx}), & x < 0 \\ 0, x > 0 \end{cases}$$

$\boxed{4}$ 5. A particle in an infinite square well jumps from the $n = 4$ state to the $n = 2$ state, releasing a photon. What is the wavelength of that photon?

The photon must have energy $\Delta E = \frac{4^2\pi^2\hbar^2}{2mL^2} - \frac{2^2\pi^2\hbar^2}{2mL^2} = \frac{3h^2}{2mL^2}$ (where I used $\pi\hbar = \frac{1}{2}h$). A photon with this energy has wavelength

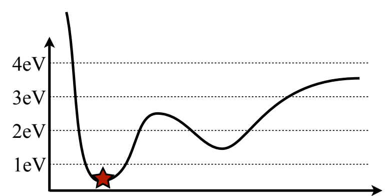
$$\lambda = \frac{hc}{\Delta E} = hc\frac{2mL^2}{3h^2} = \boxed{\frac{2mcL^2}{3h}}$$

$\boxed{4}$ 6. If the walls of a finite square well are moved closer together but not changed in height, the well will hold progressively fewer bound states. Explain why.



As the well is squeezed, the wavelength of the energy states must shrink to fit, and their wavenumbers k will grow. But the energy E of the states are proportional to k^2, so the states increase in energy, and eventually the topmost state is pushed up above the finite well, and no longer exists as a bound state.

7. The figure shows a potential $U(x)$. Assume the potential goes to infinity on the left-hand side, and asymptotically approaches 3.5 eV on the right.



$\boxed{4}$ (a) For which of the following energies would a particle be unbound in this potential? Circle all that apply.

1 eV   2 eV   3 eV $\boxed{4\,\text{eV}}$ None of these.

$\boxed{2}$ (b) Mark the position on the graph where an unbound particle would move the fastest.

$\boxed{4}$ (c) For which energies would the particle be able to tunnel?

1 eV $\boxed{2\,\text{eV}}$ 3 eV   4 eV   None of these.

$\boxed{4}$ 8. A wavefunction of a particle is

$$\psi(x) = \begin{cases} A(1-x^4), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

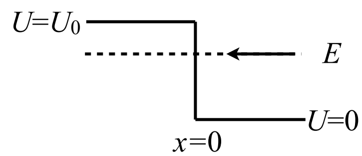
Find the normalization constant A. (Assume it is real and positive.)

$$1 = \int_{-1}^{1} A^2(1-x^4)^2\,dx = A^2 \int_{-1}^{1} (1 - 2x^4 + x^8)\,dx$$

$$= A^2 \left[x - \frac{2}{5}x^5 + \frac{1}{9}x^9 \right]_{-1}^{1}$$

$$1 = 1.42A^2 \implies A = \frac{1}{\sqrt{1.42}} = \boxed{0.839}$$

$\boxed{4}$ 9. A particle moving to the left with energy E runs into a potential step as shown. The wavefunction, including any reflection and transmission, is



$$\psi(x) = \begin{cases} \psi_L(x), & x < 0 \\ \psi_R(x), & x > 0 \end{cases}$$

Write expressions for $\psi_L(x)$ and $\psi_R(x)$. You may use the following notation as a shortcut.

$k = \frac{\sqrt{2mE}}{\hbar}$ $\quad k_- = \frac{\sqrt{2m(E-U_0)}}{\hbar}$ $\quad k_+ = \frac{\sqrt{2m(E+U_0)}}{\hbar}$ $\quad \alpha = \frac{\sqrt{2m(U_0-E)}}{\hbar}$

$$\psi_R(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_L(x) = Ce^{+\alpha x} = Ce^{-ik_- x}$$ (Both are equivalent.)

10. In cylindrical polar coordinates, the z–component of the angular momentum operator is

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

It has eigenfunctions $\frac{1}{\sqrt{2\pi}}e^{in\phi}$ where n is an integer, and $0 < \phi < 2\pi$. Let

$$\psi(\phi) = \frac{1}{\sqrt{4\pi}}(e^{2i\phi} + e^{i\phi})$$

which is a normalized wavefunction.

$\boxed{4}$ (a) Find the corresponding eigenvalue of $\frac{1}{\sqrt{2\pi}}e^{in\phi}$.

$$-i\hbar\frac{\partial}{\partial\phi}\frac{1}{\sqrt{2\pi}}e^{in\phi} = -i\hbar in\frac{1}{\sqrt{2\pi}}e^{in\phi} = \boxed{n\hbar}\left(\frac{1}{\sqrt{2\pi}}e^{in\phi}\right)$$

Note that $\hat{L}_z e^{in\phi} = n\hbar e^{in\phi}$ as well.

$\boxed{2}$ (b) **F** True or false: $\psi(\phi)$ is an eigenfunction of $\hat{L}_z$.

$\boxed{4}$ (c) Find the expectation value $\langle L_z\rangle$ of $\psi(\phi)$.

$$\begin{aligned}\langle L_z\rangle &= \frac{1}{4\pi}\int_0^{2\pi}(e^{-2i\phi} + e^{-i\phi})\hat{L}_z(e^{2i\phi} + e^{i\phi})\,d\phi \\ &= \frac{1}{4\pi}\int_0^{2\pi}(e^{-2i\phi} + e^{-i\phi})(2\hbar e^{2i\phi} + \hbar e^{i\phi})\,d\phi \\ &= \frac{\hbar}{4\pi}\int_0^{2\pi}(2 + 1 + 2e^{i\phi} + e^{-i\phi})\,d\phi = \boxed{\frac{3}{2}\hbar}\end{aligned}$$

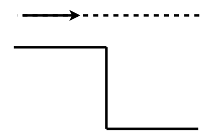
$\boxed{4}$ (d) Find the standard deviation ΔL_z of $\psi(\phi)$.

Note that $\hat{L}_z^2 e^{in\phi} = \hat{L}_z(n\hbar e^{in\phi}) = n^2\hbar^2 e^{in\phi}$.

$$\begin{aligned}\langle L_z^2\rangle &= \frac{1}{4\pi}\int_0^{2\pi}(e^{-2i\phi} + e^{-i\phi})\hat{L}_z^2(e^{2i\phi} + e^{i\phi})\,d\phi \\ &= \frac{1}{4\pi}\int_0^{2\pi}(e^{-2i\phi} + e^{-i\phi})(4\hbar^2 e^{2i\phi} + \hbar^2 e^{i\phi})\,d\phi = \frac{5}{2}\hbar^2\end{aligned}$$

$$\Delta L_z = \sqrt{\langle L_z^2\rangle - \langle L_z\rangle^2} = \sqrt{\frac{5}{2}\hbar^2 - \left(\frac{3}{2}\hbar\right)^2} = \boxed{\frac{1}{2}\hbar}$$

$\boxed{4}$ 11. An incident unbound particle described by the wavefunction Ae^{ikx} passes across a potential step, and its speed doubles. What is the transmission probability T?

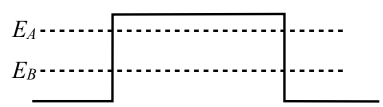


$$T = \frac{4kk'}{(k+k')^2}$$

But $\hbar k = p$, so $\frac{k}{k'} = \frac{p}{p'} = \frac{v}{v'}$; since $v' = 2v$, $k' = 2k$. Thus

$$T = \frac{4k(2k)}{(k+2k)^2} = \frac{8k^2}{9k^2} = \boxed{\frac{8}{9}}$$

12. The figure shows a wide barrier.



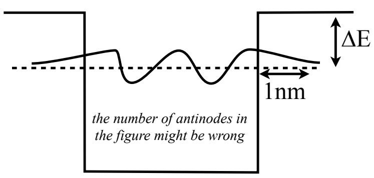
$\boxed{3}$ (a) **A** An electron is more likely to tunnel through this barrier if it has energy

A) E_A **B)** E_B **C)** Both the same

$\boxed{3}$ (b) **B** If the energy is the same, what type of particle is more likely to tunnel through the barrier?

A) proton **B)** electron **C)** both the same

$\boxed{4}$ 13. An electron is trapped in a finite square well, and penetrates by 1 nm into the walls. How much energy ΔE does the particle need to escape to infinity?



We're given $\delta = \frac{\hbar}{\sqrt{2m\Delta E}} = 10^{-9}\,\mathrm{m}$, so

$$\Delta E = \frac{\hbar^2}{2m\delta^2} = \boxed{6.1 \times 10^{-21}\,\mathrm{J}} = 0.039\,\mathrm{eV}$$

$\boxed{4}$ 14. The phase velocity of a wave is $v_p(k) = k^2 + 1$. Find the group velocity $v_g(k)$.

$$v_p = \frac{\omega}{k} \implies \omega = kv_p = k^3 + k$$

$$v_g = \frac{\partial \omega}{\partial k} = \boxed{3k^2 + 1}$$