Physics 370 Homework #12

5 problems

Solutions

▷ **1.**

From the experimental evidence that the force between nucleons has a range of about 1 fm, obtain a rough value (in MeV/c^2) for the mass of the particle exchanged to convey the force, the pion.

Answer:

The range of an exchange particle is roughly $\Delta x = \frac{\hbar}{mc}$ and so it has mass $m = \frac{\hbar}{c\Delta x}$ and rest energy

$$mc^2 = \hbar c \Delta x = (6.58 \times 10^{-16}\,\mathrm{eV \cdot s})(3 \times 10^8\,\mathrm{m/s})(10^{-15}\,\mathrm{m}) = 2 \times 10^8\,\mathrm{eV} = 200\,\mathrm{MeV}$$

Thus one would expect the pion to have a mass of $200\,\mathrm{MeV}/c^2$. This is about 40% larger than the measured value of 135–$140\,\mathrm{MeV}/c^2$, but certainly the right order of magnitude.

▷ **2.**

(a) Show that $\Psi_1(x,t) = Ae^{ikx - i\omega t}$ is a solution of both the Klein-Gordon and the Schrodinger equation.
(b) Show that $\Psi_2(x,t) = Ae^{ikx}e^{i\omega t}$ is a solution of the Klein-Gordon but not of the Schrodinger equation.
(c) Compare the time dependence of $|\Psi|^2$ for Ψ_1 and $\Psi = \Psi_1 + \Psi_2$.

Answer:

Let's do each equation separately.
(a) The Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$

Notice that $\dfrac{\partial \Psi_1}{\partial x} = ik\Psi_1$ and $\dfrac{\partial \Psi_1}{\partial t} = -i\omega\Psi_1$. If we plug $\Psi_1(x,t)$ into the equation,

$$\begin{aligned} -\frac{\hbar^2}{2m}\left(-k^2\Psi_1\right) + V(x)\Psi_1 &= i\hbar(-i\omega\Psi_1) \\ \implies \frac{\hbar^2 k^2}{2m} + V(x) &= \hbar\omega \\ \implies \frac{p^2}{2m} + V(x) &= E \end{aligned}$$

which gives the correct energy relationship.
(b) Notice that $\frac{\partial \Psi_2}{\partial x} = ik\Psi_2$ and $\frac{\partial \Psi_2}{\partial t} = i\omega\Psi_2$ If we replace ω with $-\omega$, then $\frac{\hbar^2 k^2}{2m} + V(x) = -\hbar\omega$ or $\frac{p^2}{2m} + V(x) = -E$ which is not correct.

Now the Klein-Gordon equation is

$$-c^2\hbar^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2} + m^2c^4\Psi(x,t) = -\hbar^2 \frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

(a) Substituting in Ψ_1, then

$$-c^2\hbar^2 \left(-k^2\Psi_2\right) + m^2c^4\Psi_2 = -\hbar^2 \left(-\omega^2\Psi_2\right)$$

$$\Longrightarrow \hbar^2k^2c^2 + m^2c^4 = \hbar^2\omega^2$$

$$\Longrightarrow p^2c^2 + m^2c^4 = E^2$$

which is true.
(b) If we replace ω with $-\omega$, the result is the same.
(c) For definite-energy solutions to the Schrodinger equation, we expect $|\Psi|^2$ to remain constant. Ψ_1 is such a solution, and $|\Psi_1|^2 = |Ae^{ikx}e^{-i\omega t}|^2 = |A|^2$ and so it does. However

$$|\Psi_1 + \Psi_2| = |Ae^{ikx}\left(e^{i\omega t} + e^{-i\omega t}\right)|^2 = |2A\cos\omega t|^2$$

which is time-dependent. That's OK, because $\Psi_1 + \Psi_2$ is not a solution to the Schrodinger equation. However, it *is* a solution to the Klein-Gordon equation, and so we see that $|\Psi|^2$ does not stay constant, in general, for definite-energy solutions to the Klein-Gordon equation.

▷ **3.**
Trying to pull two quarks apart would produce more quarks in groups, or hadrons. Suppose that when the separation reaches 1 fm (the radius of a nucleon), the lightest hadron (a π^0) is created. How much force is involved?

Answer:________________________________
To create a π^0 particle requires $135\,\text{MeV}$ of energy. That energy is being supplied by the work $W = F_{avg}d$ done by the force F_{avg} applied to separate the charges. Thus

$$F_{avg} = \frac{W}{d} = \frac{135 \times 10^6\,\text{eV}}{10^{-15}\,\text{m}} \times \frac{1.6 \times 10^{-19}\,\text{J}}{1\,\text{eV}} = \boxed{21{,}600\,\text{N}}$$

This is a staggering force, equal to the force of gravity on a 2–1/2 ton object. By comparison, the electromagnetic force between two up quarks at that distance is

$$F = (9 \times 10^9)\frac{(2/3)^2(1.6 \times 10^{-19}\,\mathrm{C})^2}{(10^{-15}\,\mathrm{m})^2} = 100\,\mathrm{N}$$

, two hundred times smaller.

▷ **4.**
Draw a Feynman diagram for the interaction

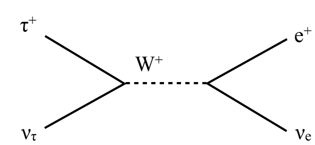
$$\tau^+ \rightarrow e^+ + \nu_e + \bar{\nu_\tau}$$

Prove that the interaction satisfies charge, energy, strangeness, lepton and baryon conservation.

Answer:

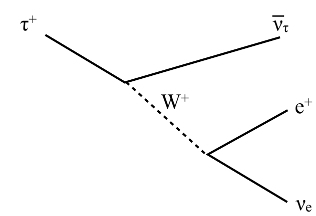
This is similar to the Feynman diagram in Figure 12.12. There is a change in particle type here, which means that this is due to a weak interaction. In the figure in the book, we see that a W boson can turn into a lepton and its corresponding antineutrino, and it can work in reverse, too. Thus, to change an antitauon into a positron, we could have something as shown in (a) here. It must be a W^+ boson to preserve charge conservation.

(a)



This isn't quite the reaction we want, but in a Feynman diagram you can always replace a particle with its antiparticle going backwards in time. If we do that with the tau neutrino, we get the Feynman diagram in (b), which is correct.

(b)



- Charge: On the left side is the tauon with charge $+e$. On the right side is the positron (charge $+e$), and two neutral neutrinos. Thus charge is conserved.
- We must confirm that the rest energy does not increase. The rest energy of the tauon is $1780\,\mathrm{MeV}$. The mass of the electron is $0.511\,\mathrm{MeV}$, the electron neutrino has mass no larger than $10^{-5}\,\mathrm{MeV}$, and the mass of the tau antineutrino is no larger than $20\,\mathrm{MeV}$, so at most the right hand side has a rest energy is $20.5\,\mathrm{MeV} < 1780\,\mathrm{MeV}$. Clearly most of the tauon's energy is converted into kinetic energy.
- None of these particles have strangeness, so strangeness is conserved.

- e^+ has electron lepton number $+1$, ν_e -1, and the tau leptons have electron lepton number 0. Therefore, $0 = +1 - 1 + 0$ which is true. Meanwhile, the antitauon and the tau antineutrino have tau lepton number -1, and so $-1 = 0 + 0 - 1$, also true.
- All of these particles have baryon number 0, so baryon number is conserved.

▷ **5.**
Show that the presence of a positive cosmological constant Ω_Λ in the Friedmann equation must, as R becomes very large, lead to an exponential expansion of the universe.

Answer:
The Friedmann equation is

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 = \frac{\Omega_M}{R^3} + \Omega_\Lambda - \frac{K'}{R^2}$$

where R is the measure of the Universe's size. We can multiply through by R^2 to get

$$\left(\frac{dR}{dt}\right)^2 = \frac{\Omega_M}{R} + \Omega_\Lambda R^2 - K'$$

If R is very large, then $\Omega_\Lambda R^2$ dominates the other two terms on the right, and we have

$$\begin{aligned} \left(\frac{dR}{dt}\right)^2 &\approx \Omega_\Lambda R^2 \\ \implies \frac{dR}{dt} &\approx \pm\sqrt{\Omega_\Lambda} R \\ \implies R(t) &\sim e^{\pm\sqrt{\Omega_\Lambda} t} \end{aligned}$$

If R starts off at 1 at the present time and is later very large, it's rather unlikely that it's in exponential decay, and so we surmise that we have exponential growth instead.