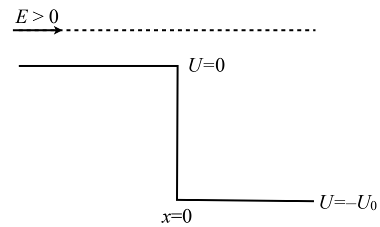
Physics 370 Homework #9

6 problems

Solutions

▷ **1.**

Suppose an incident wave Ae^{ikx} with energy $E > 0$ passes a potential which drops down to $U = -U_0$. (Notice that the constant U_0 is positive here.)



(a) Find the reflection and transmission coefficients as a function of $\mathcal{E} = \frac{E}{U_0}$.

(b) Graph R and T as a function of $\mathcal{E}$, and on the same graph the result of the step up,

$$R = \left(\frac{\sqrt{\mathcal{E}} - \sqrt{\mathcal{E}-1}}{\sqrt{\mathcal{E}} + \sqrt{\mathcal{E}-1}}\right)^2 \quad \text{and} \quad T = \frac{4\sqrt{\mathcal{E}(\mathcal{E}-1)}}{(\sqrt{\mathcal{E}} + \sqrt{\mathcal{E}-1})^2}$$

(c) Find the reflection probability for a 5 eV electron encountering a step in which the potential drops by 2 eV.

Answer:

(a) The reflection coefficient for a step-up when $E > U_0$ is

$$R = \frac{(\sqrt{E} - \sqrt{E-U_0})^2}{(\sqrt{E} + \sqrt{E-U_0})^2}$$

The solution for the step down is the same, if we replace U_0 with $-U_0$. Thus

$$\begin{aligned} R &= \frac{(\sqrt{E} - \sqrt{E+U_0})^2}{(\sqrt{E} + \sqrt{E+U_0})^2} \qquad \text{Let } E = \mathcal{E}U_0 \\ &= \frac{(\sqrt{\mathcal{E}U_0} - \sqrt{\mathcal{E}U_0+U_0})^2}{(\sqrt{\mathcal{E}U_0} + \sqrt{\mathcal{E}U_0+U_0})^2} \\ &= \frac{(\sqrt{\mathcal{E}} - \sqrt{\mathcal{E}+1})^2}{(\sqrt{\mathcal{E}} + \sqrt{\mathcal{E}+1})^2} \end{aligned}$$

Similarly,

$$T = 4\frac{\sqrt{\mathcal{E}(\mathcal{E}+1)}}{(\sqrt{\mathcal{E}} + \sqrt{\mathcal{E}+1})^2}$$

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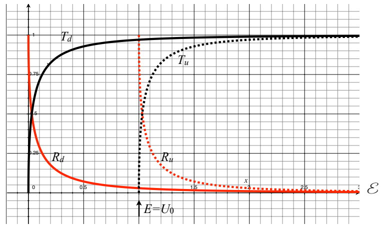
(b) Here is a graph of all four. The only difference between the two is that, for the step up, the graphs start



at $E = U_0$ (because when $E < U_0$ there is a different solution), while for the step down, the graphs start at $E = 0$.

(c) In this case $\mathcal{E} = \frac{5\,\text{eV}}{2\,\text{eV}} = 2.5$, and

$$R = \left(\frac{\sqrt{2.5} - \sqrt{2.5+1}}{\sqrt{2.5} + \sqrt{2.5+1}} \right)^2 = 0.007 = 0.7\%$$

▷ **2.**
A beam of particles of energy E and incident upon a potential step of $U_0 = \frac{5}{4}E$ is described by a wave-function

$$\psi_{inc} = 1e^{ikx}$$

(a) Determine completely the reflected wave, and the wave inside the step, by enforcing the required continuity conditions to obtain their (possibly complex) amplitudes.
(b) Verify by explicit calculation that $R = 1$.

Answer:

(a) We have $E < U_0$, here so

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{-\alpha x} & x > 0 \end{cases}$$

According to the text, we have the conditions $A + B = C$ and $ik(A - B) = -\alpha C$. We are told that $A = 1$, so we need to solve for B and C. The first equation says that $C = 1 + B$, and so

$$ik(1 - B) = -\alpha(1 + B) \implies B(-ik + \alpha) = -\alpha - ik \implies B = \frac{ik + \alpha}{ik - \alpha}$$

We know that $k = \frac{\sqrt{2mE}}{\hbar}$ and

$$\alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} = \frac{\sqrt{2m(\frac{5}{4}E - E)}}{\hbar} = \frac{\sqrt{2m(\frac{1}{4}E)}}{\hbar}$$

and so

$$B = \frac{ik + \alpha}{ik - \alpha} = \frac{i\sqrt{E} + \sqrt{E/4}}{i\sqrt{E} - \sqrt{E/4}} = \frac{2i+1}{2i-1}$$

and

$$C = A + B = 1 + \frac{2i+1}{2i-1} = \frac{(2i-1)+(2i+1)}{2i-1} = \frac{4i}{2i-1}$$

Thus

$$\psi(x) = \begin{cases} e^{ikx} + \frac{2i+1}{2i-1}e^{-ikx} & x < 0 \\ \frac{4i}{2i-1}e^{-\alpha x} & x > 0 \end{cases}$$

(b) The reflection coefficient is

$$R = \frac{B^*B}{A^*A} = B^*B = \left(\frac{-2i+1}{-2i-1}\right)\left(\frac{2i+1}{2i-1}\right) = \frac{(1-2i)(1+2i)}{(1+2i)(1-2i)} = 1$$

▷ **3.**
It is shown in Section 6.1 that for the $E < U_0$ potential step,

$$B = -\frac{\alpha + ik}{\alpha - ik}A$$

(a) Use this to calculate the probability density to the left of the step $P_L(x) = |\psi_L(x)|^2$. You should get a typical standing wave pattern.
(b) What is the largest value that $P_L(x)$ takes?

Answer:____________________________________

(a) The wavefunction on the left is

$$\psi_L(x) = Ae^{ikx} - \frac{\alpha + ik}{\alpha - ik}Ae^{-ikx}$$

The probability density, setting $A = 1$ (because it doesn't really matter), is

$$\begin{aligned} P_L(x) = \psi_L^*(x)\psi_L(x) &= \left(e^{ikx} - \frac{\alpha + ik}{\alpha - ik}e^{-ikx}\right)^*\left(e^{ikx} - \frac{\alpha + ik}{\alpha - ik}e^{-ikx}\right) \\ &= \left(e^{-ikx} - \frac{\alpha - ik}{\alpha + ik}e^{ikx}\right)\left(e^{ikx} - \frac{\alpha + ik}{\alpha - ik}e^{-ikx}\right) \\ &= 1 - \frac{\alpha + ik}{\alpha - ik}e^{-2ikx} - \frac{\alpha - ik}{\alpha + ik}e^{2ikx} + \left(\frac{\alpha - ik}{\alpha + ik}\right)\left(\frac{\alpha + ik}{\alpha - ik}\right) \end{aligned}$$

We've already shown that $\frac{\alpha-ik}{\alpha+ik}$ has magnitude 1; therefore we can write it as a phase $e^{i\phi}$. Thus

$$P_L(x) = 2 - e^{-i\phi}e^{-2ikx} - e^{i\phi}e^{2ikx} = 2\left(1 - \cos(2kx + \phi)\right)$$

Thus we have a sinusoidal standing-like wave on the left.

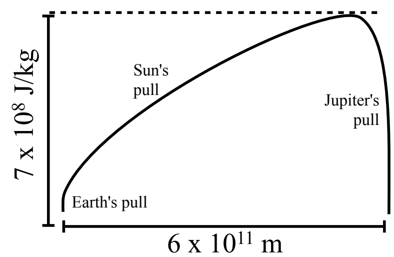
We can calculate the phase by noting that

$$\cos\phi + i\sin\phi = \left(\frac{\alpha - ik}{\alpha + ik}\right)\left(\frac{\alpha - ik}{\alpha - ik}\right) = \frac{\alpha^2 - k^2 - 2i\alpha k}{\alpha^2 + k^2} \implies \phi = \cos^{-1}\frac{\alpha^2 - k^2}{\alpha^2 + k^2}$$

(b) The largest value it takes is when the cosine is -1 and the probability is $2(2) = 4|A^2|$: there are locations that are 4 times more likely to hold the particle than if the particle weren't reflected at all.

▷ **4.**

The gravitational potential energy of a 1 kg object is plotted versus position from Earth's surface to the surface of Jupiter. Mostly it is due to the Sun, but there are downturns at each end due to the attractions to the two planets. Make the crude approximation that this is a rectangular barrier with the same average height (you can estimate this by eye). If a 65 kg person jumps upwards at 4 m/s on Earth, what is the probability that they will tunnel to Jupiter? (See equation 6.18).



Answer:

This is a very wide barrier, so we can use the transmission probability of a wide barrier

$$T \approx 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0-E)}/\hbar}$$

Here $L = 6 \times 10^{11}$ m is the width of the barrier. The height of the barrier is $U_0 = (3 \times 10^8\,\mathrm{J/kg})(65\,\mathrm{kg}) = 195 \times 10^8$ J, and the energy of the person is $E = \frac{1}{2}(65\,\mathrm{kg})(4\,\mathrm{m/s})^2 = 2080$ J. Thus the exponent in the exponential is

$$\begin{aligned} -2L\frac{\sqrt{2m(U_0 - E)}}{\hbar} &= -2(6 \times 10^{11})\frac{\sqrt{2(65)(195 \times 10^8 - 2080)}}{6.626 \times 10^{-34}} \\ &= -1.81 \times 10^{45}\sqrt{2.54 \times 10^{12}} = -2.88 \times 10^{51} \end{aligned}$$

Now $e^{-2.88\times10^{51}}$ is not just a small number, it's a *very* small number: so small, that if we multiply it by the coefficient

$$16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) = 16\frac{2080}{195 \times 10^8}\left(1 - \frac{2080}{195e8}\right) = 1.7 \times 10^{-6} = e^{-13.3}$$

we get

$$e^{-13.3}e^{-2.88\times10^{51}} = e^{-2.88\times10^{51}-13.3} \approx e^{-2.88\times10^{51}}$$

That is, the coefficient doesn't matter at all. Since $e = 10^{0.434}$, we can also write it as

$$T = \left(10^{0.434}\right)^{-2.88\times10^{51}} = 10^{-1.25\times10^{51}}$$

so the probability is $0.$ followed by 10^{51} zeroes and a 1.

▷ **5.**

For wavelengths less than about 1 cm, the dispersion relation for waves on the surface of water is $\omega = \sqrt{(\gamma/\rho)k^3}$, where γ and ρ are the surface tension and density of water. Given $\gamma = 0.072\,\mathrm{N/m}$ and $\rho = 10^3\,\mathrm{kg/m^3}$, calculate the phase and group velocities for a wave of 5 mm wavelength.

Answer:

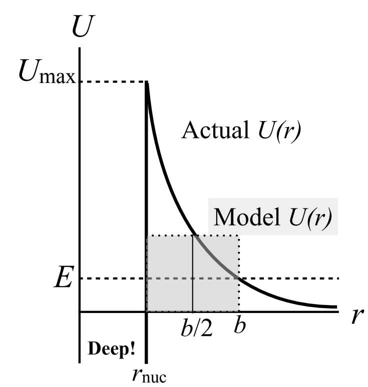
Note that $\gamma/\rho = 0.072\,\mathrm{N/m}/10^3\,\mathrm{kg/m^3} = 7.2\times10^{-5}\,\mathrm{m^3/s^2}$, and a $0.005\,\mathrm{m}$ wavelength wave has $k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.005\,\mathrm{m}} = 1257\,/\mathrm{m}$. The phase velocity is

$$v_p = \frac{\omega}{k} = \frac{\sqrt{(\gamma/\rho)k^3}}{k} = \sqrt{\frac{\gamma}{\rho}}k^{1/2} = \sqrt{7.2\times10^{-5}\,\mathrm{m^3/s^2}}(1257\,/\mathrm{m})^{1/2} = \boxed{0.3\,\mathrm{m/s}}$$

The group velocity is

$$v_g = \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\sqrt{\frac{\gamma}{\rho}}k^{3/2} = \frac{3}{2}\sqrt{\frac{\gamma}{\rho}}k^{1/2} = \frac{3}{2}v_p = \boxed{0.45\,\mathrm{m/s}}$$

▷ **6.**



Fusion in the Sun: Without tunneling, our Sun would fail us. The source of its energy is nuclear fusion, and a crucial step is the fusion of a hydrogen nucleus (a proton) and a deuterium nucleus (a proton and a neutron). When these nuclei get close enough, their short-range attraction via the strong force overcomes their Coulomb repulsion. This allows them to stick together, resulting in a reduced total mass/internal energy and a consequent release of kinetic energy. However, the Sun's temperature is simply too low to ensure that nuclei move fast enough to overcome their repulsion.

(a) By equating the average thermal kinetic energy that the nuclei would have when distant, $\frac{3}{2}k_BT$, and the Coulomb potential energy they would have when 2 fm apart, roughly the separation at which they stick, show that a temperature of about a billion Kelvin would be needed.

(b) The Sun's core is only about 10^7 K. If nuclei can't make it "over the top", they must tunnel. Consider the following model, illustrated in the figure: One nucleus is fixed at the origin, while the other approaches from far away with energy E. As r decreases, the Coulomb potential energy increases, until the separation r is roughly the nuclear radius r_{nuc}, whereupon the potential energy is $U_{\max}$ and then quickly drops down a very deep "hole" as the strong-force attraction takes over. Given that $E \ll U_{\max}$, the point b where tunneling must begin will be very large compared with r_{nuc}, so we approximate the barrier's width L as simply b. Its height U_0, we approximate by the Coulomb potential evaluated at $b/2$. Finally, let the energy be $E = 4(\frac{3}{2}k_BT)$ which is a reasonable limit, given the natural range of speeds in a thermodynamic system. Combining these approximations, show that the exponential factor in the wide-barrier tunneling probability (see 6.18) is

$$\exp\left[\frac{-e^2}{(4\pi\epsilon_0)\hbar}\sqrt{\frac{4m}{3k_BT}}\right]$$

(The e above is the proton charge, not 2.781828.)
(c) Using the proton mass for m, evaluate this factor for a temperature of 10^7 K. Then evaluate it at 3000 K, about that of a hot flame. Discuss the consequences.

Answer:____________________________________

(a) The Coulomb potential energy of two protons ($q = e = 1.6 \times 10^{-19}\,\mathrm{C}$) a distance of $d = 2\,\mathrm{fm} = 2 \times 10^{-15}\,\mathrm{m}$ apart is

$$U = k\frac{qq}{d} = (9 \times 10^9)\frac{(1.6 \times 10^{-19})^2}{2 \times 10^{-15}} = 1.15 \times 10^{-13}\,\mathrm{J}$$

If this energy is going to be supplied by thermal kinetic energy $\frac{3}{2}k_BT$, then

$$\frac{3}{2}k_BT = 1.15 \times 10^{-13} \implies T = \frac{2}{3}\frac{(1.15 \times 10^{-13}}{1.38 \times 10^{-23})} = 5.6 \times 10^9\,\mathrm{K}$$

or more than a billion degrees! The Sun is only 1% as hot as that.

(b) So the wide-barrier tunneling approximation is

$$T \approx 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0-E)}/\hbar}$$

and its exponential is

$$e^{-2L\sqrt{2m(U_0-E)}/\hbar} \equiv e^{-X}$$

If $U(r) = k\frac{e^2}{r}$ is the potential of the two charges, then we approximate our barrier with a rectangle with width $L = b$, where $U(b) = E$, and height $U_0 = U(b/2)$. Solving for b;

$$U(b) = E \implies k\frac{e^2}{b} = E \implies b = \frac{ke^2}{E}$$

and

$$U_0 = U(b/2) = k\frac{e^2}{(ke^2/2E)} = 2E$$

Thus

$$
\begin{aligned}
X &= \frac{2L}{\hbar}\sqrt{2m(U_0 - E)} = -\frac{2}{\hbar}\frac{ke^2}{E}\sqrt{2m(2E - E)} \\
&= \frac{ke^2}{\hbar}\frac{2\sqrt{2mE}}{E} \\
&= \frac{2ke^2}{\hbar}\sqrt{\frac{2m}{E}} \\
&= \frac{e^2}{4\pi\epsilon_0\hbar}2\sqrt{\frac{2m}{6k_BT}} \\
&= \frac{e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{4m}{3k_BT}}
\end{aligned}
$$

Since $E/U_0 = \frac{1}{2}$

$$
T = 16(1/2)(1/2)\exp\left[-X\right] = 4\exp\left[-\frac{e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{4m}{3k_BT}}\right]
$$

Q.E.D.

(c) Note that

$$
X = \frac{e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{4m}{3k_B}}T^{-1/2} = \frac{(9\times10^9)(1.6\times10^{-19})^2}{(1.055\times10^{-34})}\sqrt{\frac{4(1.67\times10^{-27})}{3(1.38\times10^{-23})}}T^{-1/2} = 27741T^{-1/2}
$$

If $T = 10^7\,\mathrm{K}$, the temperature of the Sun, then $X = 8.77$, and the transmission probability is

$$
T = 4e^{-8.77} = 0.0006 = 0.06\%
$$

which is a small but not infinitesimal number of protons fusing. For $T = 3000\,\mathrm{K}$, on the other hand, $X = 506$ and $T = 4e^{-506} = 7\times10^{-220}$ which is practically zero: fusion doesn't occur in candle flames.