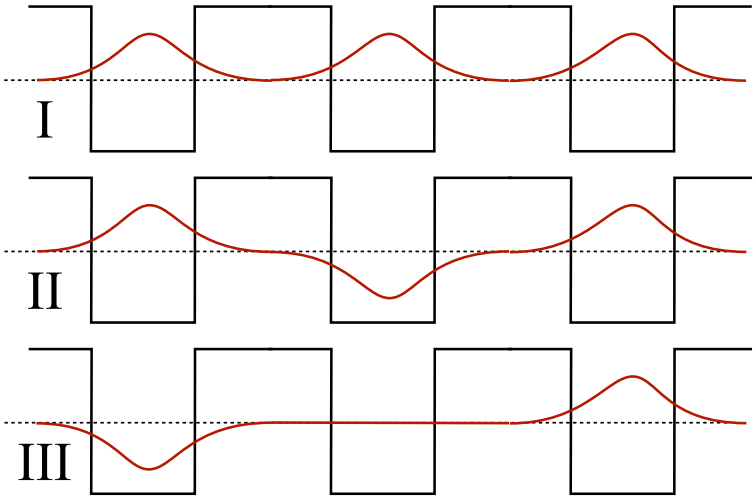
Physics 370 Homework #11
9 problems
Solutions

▷ **1.**
(Harris 10.26) The diagrams below represent the three lowest energy wave functions for three "atoms". As in all truly molecular states we consider, these states are shared among the atoms. At such large atomic separation, however, the energies are practically equal, so an electron would be just as happy occupying any combination.



(a) Identify algebraic combinations of the states (for instance, I+$\frac{1}{2}$II+$\frac{1}{2}$III) that would place the electron in each of the three atoms.
(b) Were the atoms closer together, the energies of states I, II, and III would spread out and an electron would occupy the lowest energy one. Rank them in order of increasing energy as the atoms draw closer together. Explain your reasoning.

Answer:

(a) To have the atom appear in only one of the wells, we need the wavefunctions to cancel out in the other two. For instance, in the state $I - II$ the particle only appears in the middle well, while in the state $I + II$ the particle only appears in the outer wells. Combining this latter state with state III gives us

- Left well: $\frac{1}{2}(I + II) - III$
- Middle well: $\frac{1}{2}(I - II)$
- Right well: $\frac{1}{2}(I + II) + III$

(b) We've seen in this chapter that the kinetic energy is related to the wavenumber (and the number of nodes), and the total energy follows likewise. When the wells come closer together, state I will have zero nodes (the boundaries don't count), state II has 2 nodes, and state III has 1 node. Therefore

$$E_I < E_{III} < E_{II}$$

▷ **2.**

(Harris 10.50) Assuming an interatomic spacing of 0.15 nm, obtain a rough value for the width (in eV) of the $n = 2$ band in a one-dimensional crystal.

Answer:

At the bottom of the $n = 2$ band (referring to Figure 10.25), the wavefunction has a wavelength of $\lambda = 2a$ where $a = 0.15\,\text{nm}$ is the interatomic spacing. At the top of the band, $\lambda = a$. If we assume that the potential energy is constant across the band, then the energies of the bands are given by the kinetic energy formula

$$E = \frac{h^2k^2}{2m} = \frac{h^2}{2m}\left(\frac{2\pi}{\lambda}\right)^2 = \frac{\hbar^2}{2m\lambda^2}$$

and so

$$\begin{aligned}\Delta E &= \frac{\hbar^2}{2ma^2} - \frac{\hbar^2}{2m(2a)^2} = \frac{3\hbar^2}{4(2ma^2)} \\ &= \frac{3(4.1\times10^{-15}\,\text{eV}\cdot\text{s})^2}{8(0.511\times10^6\,\text{eV}/(3\times10^8\,\text{m/s})^2)(0.15\times10^{-9}\,\text{m})^2} \\ &= \boxed{49\,\text{eV}}\end{aligned}$$

This turns out to be an over-estimate because the potential energy at the top of the band is lower than at the bottom of the band, bringing the edges closer together. (This is what creates the gaps in the first place.)

▷ **3.**

(Harris 10.52) Carbon (diamond) and silicon have the same covalent crystal structure, yet diamond is transparent while silicon is opaque to visible light. Argue that this should be the case based only on the difference in band gaps—roughly 5 eV for diamond and 1 eV for silicon.

Answer:

The energy of a visible light photon is

$$E = \frac{hc}{\lambda} = \frac{(4.13\times10^{-15}\,\text{eV}\cdot\text{s})(3\times10^8\,\text{m/s})}{500\times10^{-9}\,\text{m}} = 2.5\,\text{eV}$$

Such a photon provides plenty of energy to bump an electron across the gap for silicon, and so photons are readily absorbed and scattered by silicon. However, photons are less able to be absorbed by diamond because of the much larger band gap, and they pass straight through.

▷ **4.**
(Harris 10.57) Show that for a room-temperature semiconductor with a band gap of 1 eV, a temperature rise of 4 K would raise the conductivity by about 30%.

Answer:
The conductivity is proportional to the number N_{excited} of electrons in the conduction band, and (Eq. 10-11)

$$\sigma \propto N_{\text{excited}} = N_V \frac{k_B T}{\Delta E_V} e^{-E_{\text{gap}}/2k_B T}$$

This term is dominated by the exponential factor, so

$$\frac{\sigma'}{\sigma} = \frac{e^{-E_{\text{gap}}/2k_B T'}}{e^{-E_{\text{gap}}/2k_B T}} = \exp\left[\frac{-E_{\text{gap}}}{2k_B}\left(\frac{1}{T'} - \frac{1}{T}\right)\right]$$
$$= \exp\left[-\frac{E_{\text{gap}}\Delta T}{2k_B(TT')}\right]$$

If we say room temperature is $T = 295\,\text{K}$ and we approximate $TT' \approx T^2$, then

$$\frac{\sigma'}{\sigma} = \exp\left[-\frac{(1\,\text{eV})(4\,\text{K})}{2(8.62\times10^{-5}\,\text{eV/K})(295\,\text{K})^2}\right] = e^{-0.267} = 1.3$$

so $\sigma' = 1.3\sigma$, a 30% increase. **Q.E.D.**

▷ **5.**
(Harris 11.5) The semiempirical binding energy formula has four terms. Suppose we have a nucleus with 18 protons and 22 neutrons. For each term in the formula, indicate (without calculation) whether adding one more proton would cause an increase or a decrease, and explain why it should have this effect. Focus on the underlying idea.

Answer:

- Adding another proton would increase the total **volume** of the nucleus, and thus **increase** the binding energy because there is one more nucleon to be attracted to the other nucleons via the strong force.
- Adding another proton would also increase the total **surface area** of the nucleus, which would **decrease** the second term (which is really just a correction to the first term).

- Adding another proton would increase the charge of the nucleus, thus **decreasing** the **Coulomb** term of the binding energy.
- Adding another proton would decrease the imbalance in the number of protons and neutrons, and so would **increase** the **asymmetry** binding energy term.

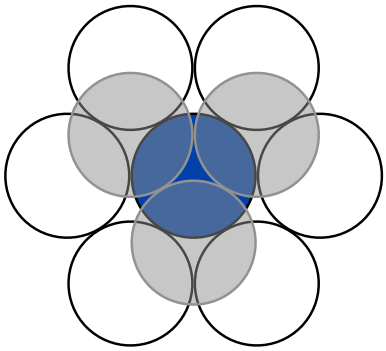
In summary, the volume and asymmetry terms increase, and the surface and Coulomb terms decrease. Whether the new nucleus is more stable or not depends on which pair of terms grows more.

▷ **6.**

(Harris 11.20) By classical, hard-sphere assumptions, what smallest value of A would make one nucleon surrounded? Relate your answer to Figure 11.14.

Answer:

By "hard-sphere assumptions" we mean: if the nucleons were just spheres, how many spheres could you pack around another sphere? The densest packing of spheres is the hexagonal close-packed configuration, which is made up of layers of spheres in hexagonal arrangement. Each sphere is surrounded by six other spheres in its own layer, and three more on top, and three more on bottom, for a total of 12. Thus once $A = 13$ you expect to have at least one sphere completely surrounded. If you look at Figure 11.14, you can see that the binding energy per nucleon climbs quickly until around $A = 13$, because every nucleon can touch every other nucleon. Beyond that point, the binding energy starts to level off, as adding a new nucleon only affects a portion of the nucleons already present instead of all of them.



▷ **7.**

(Harris 11.40) How much energy is released, and what is the daughter nucleus, in the β^+ decay of nitrogen–13?

Answer:

A nitrogen atom has $Z = 7$ protons and $N = 6$ neutrons. When a positron is released, the atomic number of $A = 13$ stays the same, but the nucleus loses one positive charge, as a proton turns into a neutron, so that the daughter nucleus has $Z = 6$ protons and $N = 7$ neutrons: this is carbon–13.

The mass of the initial nitrogen–13 nucleus is $m_i = 13.0057\,\mathrm{u} - 7m_e$ (the second term because there are 7 electrons). The mass of the final carbon–13 nucleus is $m_f = 13.0034\,\mathrm{u} - 6m_e$. A

positron is also created, so the change in mass is

$$\begin{aligned}\Delta m &= [(13.0034\,\mathrm{u} - 6m_e) + m_e] - [13.0057\,\mathrm{u} - 7m_e] \\ &= -0.0023\,\mathrm{u} + 2m_e \\ &= -0.0023\,\mathrm{u} + 2(5.486 \times 10^{-4}\,\mathrm{u}) \\ &= -1.2 \times 10^{-3}\,\mathrm{u}(931.5\,\mathrm{MeV/u}) = \boxed{-1.01\,\mathrm{MeV}}\end{aligned}$$

The change in energy is negative, and so decay is energetically favorable (and in fact, this is how the unstable nitrogen–13 nucleus decays).

▷ **8.**
(Harris 11.50) Given initially 100 g of plutonium–239, how much time must pass for the amount to drop to 1 g?

Answer:
If λ is the decay constant of plutonium–239 and t is the time in question, then

$$N = N_0 e^{-\lambda t} \implies \ln\left(\frac{N}{N_0}\right) = -\lambda t \implies t = -\frac{1}{\lambda}\ln\left(\frac{N}{N_0}\right)$$

If we're given the half-life $T_{1/2}$ instead, then $\lambda = \ln 2/T_{1/2}$ and so

$$t = \frac{T_{1/2}}{\ln 2}\ln\left(\frac{N_0}{N}\right)$$

In this case, $N_0/N = 100$, and $T_{1/2} = 24.11\,\mathrm{kyr}$, so

$$t = (24.11\,\mathrm{kyr})\frac{\ln 100}{\ln 2} = (24.11\,\mathrm{kyr})(6.64) = 160\,\mathrm{kyr} = \boxed{160{,}000\,\mathrm{yr}}$$

Or notice that $\frac{\ln 100}{\ln 2} = \log_2 100$, which is the number of times you can divide 100 by 2 before you get to 1. This is the more straightforward way to calculate decay with half-life. (e.g. if we'd ask how long it takes to go from 4 g to 1 g, the answer would be "two half-lives"). Still, if you prefer this way, that's fine too.

	Volume	Area	Coulomb	Asym.	Volume ÷ A	Area ÷ A	Coulomb ÷ A	Asym. ÷ A	Total ÷A
Ne20									
Fe 56									
U 238									

▷ **9.**
(Harris 11.30) The semiempirical binding energy formula predices the binding energies of neon–20, iron–56, and uranium–238 within about 1%. Fill in the following table (on a separate piece of paper) for the four terms of the formula. Discuss what the table reveals.

Answer:
The semiempirical formula is

$$BE = 15.8A - 17.8A^{2/3} - 0.71\frac{Z(Z-1)}{A^{1/3}} - 23.7\frac{(N-Z)^2}{A}$$

where each term corresponds to a separate column. It's an easy matter to plug the numbers in, and then to divide each of the first four columns by A. (All values are in MeV.) The total binding

	Volume	Area	Coulomb	Asym.	Volume ÷ A	Area ÷ A	Coulomb ÷ A	Asym. ÷ A	Total ÷A
Ne20	**316**	**-131**	**-23.5**	**0**	**15.8**	**-6.56**	**-1.18**	**0**	**8.07**
Fe 56	**885**	**-261**	**-121**	**-6.77**	**15.8**	**-4.65**	**-2.15**	**-0.12**	**8.87**
U 238	**3760**	**-684**	**-959**	**-290**	**15.8**	**-2.87**	**-4.03**	**-1.22**	**7.68**

energy grows with A, and the Coulomb and asymmetry terms grow particularly fast. Considering the per-nucleon numbers, however, we see that all atoms start with the same volume term, and the other terms tend to destabilize the nucleus. The surface area term is most significant for the smaller neon nuclei (because a larger proportion of atoms are on the surface). The destabilizing Coulomb and asymmetry terms, on the other hand, are dominant for the larger uranium nucleus. Iron is the best balance between two much area and too much repulsion/asymmetry, and so is the most stable of the three nuclei.