Intervals in Special Relativity

There are three types of time intervals in special relativity:

- **Coordinate time:** Denoted as Δt, this is the time between two events as measured in a specific *inertial* frame. We suppose that the universe is filled with clocks, and that two clocks are synchronized if and only if, when a light flash travels between them, the time interval registered by the clocks is equal to the distance between them.
- **Proper time:** Denoted as $\Delta\tau$, it is the interval measured by a clock that travels from one event to the other along a particular worldline. It is a frame-independent quantity: everyone can agree on what the clock says at both events. However, it does depend on your choice of worldline. (More on this later.) This time is defined in noninertial frames as well.
- **Spacetime interval:** Denoted as Δs, this the proper time measured by an *inertial* clock that is present at both events. It is frame-independent and *unique*. It is also the coordinate time in the inertial frame that a clock would have to be in to visit both events.

The spacetime interval is given by the metric equation

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

($c = 1$) which was derived in class. (For SI units, divide the distance terms by c^2). If we look at the Lorentz transformations

$$\Delta x = \gamma(\Delta x' + v\Delta t') \quad \text{and} \quad \Delta t = \gamma(\Delta t' + v\Delta x')$$

the spacetime interval in frame S is

$$\begin{aligned}
\Delta s^2 &= \gamma^2 \left[(\Delta t' + v\Delta x')^2 - (\Delta x' + v\Delta t')^2\right] \\
&= \gamma^2 \left[\left((\Delta t')^2 + v^2(\Delta x')^2 + 2v\Delta x'\Delta t'\right) - \left((\Delta x')^2 + v^2(\Delta t')^2 + 2v\Delta t'\Delta x'\right)\right] \\
&= \frac{1}{1-v^2}\left[(1-v^2)(\Delta t')^2 - (1-v^2)(\Delta x')^2\right] \\
&= (\Delta s')^2
\end{aligned}$$

The Lorentz transformations are in fact defined to conserve the spacetime interval in all frames. Notice that $\Delta s \leq \Delta t$ for all inertial frames, and is only equal when $\Delta x = 0$ (that is, in the frame where an object can visit both events).

Proper time can be calculated by breaking the worldline down into a bunch of tiny little straight lines (inertial paths), calculating Δs for each, and then adding them all up.

$$\Delta\tau = \int d\tau = \int ds = \int \sqrt{dt^2 - dx^2}$$

We can evaluate that integral by factoring out a dt from the radical,

$$\Delta\tau = \int \sqrt{1 - \left(\frac{dx}{dt}\right)^2}\, dt = \int_{t_i}^{t_f} \sqrt{1 - v(t)^2}\, dt$$

where $v(t) \geq 0$ is the speed of the object in some inertial frame. This integral is hard to do in general. When the particle is not moving in a particular inertial frame, then $v = 0$ and

$$\Delta\tau = \int_{t_i}^{t_f} dt = \Delta t$$

in the frame where the particle is not moving. (It's also equal to Δs, since the particle's $\Delta x = 0$.)

If the particle's *speed* (not velocity!) is constant, then we can factor the radical out of the integral, and get

$$\Delta\tau = \sqrt{1 - v^2}\, \Delta t = \frac{1}{\gamma}\, \Delta t$$

In other words, we get time dilation. Interestingly, the same formula works even if the particle is moving in a circle (for instance) or along a windy worldline of any sort, just so long as the speed is constant.

Notice that, since $\sqrt{1 - v^2}$ is always less than or equal to one, that $\Delta\tau \leq \int dt = ds$, and is only equal when the particle is not moving. Thus we have the relationship

$$\Delta t \geq \Delta s \geq \Delta\tau$$

Solving the Twin Paradox

The age change of Alice and Bob is their proper time interval $\Delta\tau$. In the Sun's frame, Bob is not moving, so $\Delta\tau_B = \Delta t$ in the Sun's frame. Alice is moving, however, so $\Delta\tau_A < \Delta\tau_B$. (More specifically, if we pretend that Alice instantly turns around so that her speed stays the same, then $\Delta\tau_A = \frac{1}{\gamma}\Delta\tau_B$.

Why can't I do the same calculation in Alice's frame? Because Alice's frame is noninertial! The formula for $\Delta\tau$ only works in an inertial reference frame.

Technically, Bob is not in an inertial reference frame either, mind you! However, he is moving so slowly compared to the Sun (which is itself only approximately inertial) that $\Delta\tau_B$ is not precisely the Δt either, but it's close enough.