Physics 370 Homework #6

6 problems

Solutions

▷ **1.**
Prove that the force-less Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}$$

has the solution

$$Ae^{i(kx-\omega t)}$$

and derive the relationship between k and ω.

Answer:
Call $\Psi(x,t) = Ae^{ikx}e^{-i\omega t}$, the plane wave. The derivatives of it are

$$\frac{\partial\Psi}{\partial t} = Ae^{ikx}(-i\omega)e^{-i\omega t} = -i\omega(Ae^{ikx}e^{-i\omega t}) = -i\omega\Psi$$

$$\frac{\partial\Psi}{\partial x} = Ae^{-i\omega t}ike^{ikx} = ik(Ae^{ikx}e^{-i\omega t}) = ik\Psi$$

and so

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial}{\partial x}(ik\Psi) = (ik)^2\Psi = -k^2\Psi$$

Putting these into the Schrodinger equation gives us

$$-\frac{\hbar^2}{2m}(-k^2\Psi) = i\hbar(-i\omega\Psi)$$

$$\frac{\hbar^2k^2}{2m}\Psi = \hbar\omega\Psi$$

$$\implies \frac{\hbar^2k^2}{2m} = \hbar\omega$$

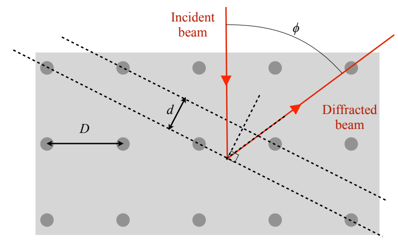
Thus the plane wave is a solution to the Schrodinger equation, so long as

$$\omega = \frac{\hbar k^2}{2m}$$

or, since $E = \hbar\omega$ and $p = \hbar k$, if

$$E = \frac{p^2}{2m} = KE$$

▷ **2.**



Atoms in a crystal form atomic planes at many different angles with respect to the surface. The figure shows the behaviors of representative incident and scattered waves in the Davisson-Germer experiment. A beam of electrons accelerated through 54 V is directed normally at a nickel surface, and strong reflection is detected only at an angle ϕ of 50°. Using the Bragg law, show that this implies a spacing D of nickel atoms on the surface in agreement with the known value of 0.22 nm.

Answer:

The Bragg Law says we get constructive interference when

$$2d\sin\theta = n\lambda$$

where θ is the angle from the atomic plane to the incident beam: $\theta = 90° - 25° = 65°$ in this case. For the wavelength, we use the deBroglie wavelength $\lambda = h/p$. The momentum (and the corresponding kinetic energy) come from the electric potential energy of accelerating the electron through a potential difference:

$$eV = KE = \frac{p^2}{2m} \implies p = \sqrt{2meV}$$

which means that the wavelength is

$$\lambda = \frac{h}{\sqrt{2meV}} = \frac{6.626\times10^{-34}}{sqrt2(9.11\times10^{-31})(1.6\times10^{-19})(54)} = 1.67\times10^{-10}\,\mathrm{m}$$

and therefore

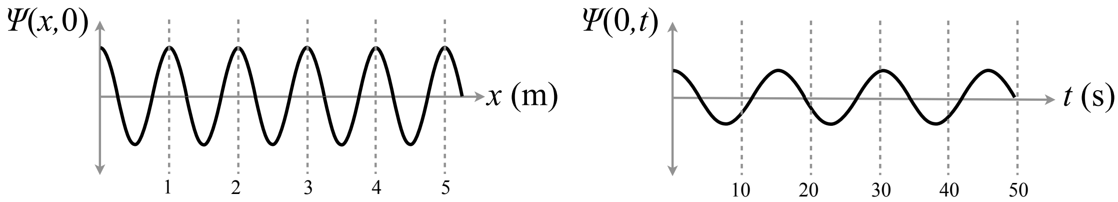
$$d = n\frac{\lambda}{2\sin\theta} = n\frac{1.67\times10^{-10}}{2\sin 65°} = (9.21\times10^{-11}\,\mathrm{m})n$$

where n is an integer. The problem says that strong reflection is detected "only at" this angle, so it's reasonable to suppose that this is the $n=1$ case.

To get D, we notice that it is the hypotenuse of a right triangle with angle 25° opposite d. Thus

$$d = D\sin 25° \implies D = \frac{d}{\sin 25°} = 2.2\times10^{-10}\,\mathrm{m} = 0.22\,\mathrm{nm}$$

Q.E.D.



▷ **3.**

The following two graphs show a snapshot of a wave at $t = 0$, and the amplitude of a point on the wave at $x = 0$. Find the wave's

(a) wavenumber k

(b) angular velocity ω

(c) phase velocity v

Answer:________________________________

(a) The wavenumber is the number of radians the wave passes through in $1\,\mathrm{m}$. From the first graph, we see that the wave goes through one entire cycle in $1\,\mathrm{m}$, so $k = \boxed{2\pi\,/\mathrm{m}}$. Alternatively, we could note that $\lambda = 1\,\mathrm{m}$, so $k = \frac{2\pi}{\lambda} = \frac{2\pi}{1\,\mathrm{m}} = 2\pi\,/\mathrm{m}$.

(b) Similarly, the angular frequency is the number of radians the wave goes through in $1\,\mathrm{s}$. This wave goes through 3.25 cycles, or $(3.25)(2\pi) = 6.5\pi$ radians, so

$$\omega = \frac{6.5\pi\,\mathrm{rad}}{50\,\mathrm{s}} = \boxed{0.408\,\mathrm{rad/s}}$$

(c) The phase velocity is

$$v = \lambda f = \frac{2\pi}{k}\frac{\omega}{2\pi} = \frac{\omega}{k}$$

In this case

$$v = \frac{0.408\,\mathrm{rad/s}}{2\pi\,\mathrm{rad/m}} = \boxed{0.065\,\mathrm{m/s}}$$

▷ **4.**

When you look at an object, we can establish its location by, at best, $550\,\mathrm{nm}$ (the wavelength of light).

(a) What is the minimum uncertainty in a nickel's velocity? (A nickel has a mass of $5\,\mathrm{g}$.)

(b) If the average momentum of the nickel is zero, then $p \sim \Delta p$. What does the nickel's wavelength equal in that case? (Note that it isn't ∞, which we might expect if $\lambda = h/p$ and $p = 0$.)

Answer:________________________________

(a) The uncertainty in the nickel's position is $\Delta x = 550 \times 10^{-9}\,\mathrm{m}$, and so the minimum uncertainty in its momentum is

$$\Delta p = \frac{\hbar}{2}\frac{1}{\Delta x} = \frac{1.055 \times 10^{-34}\,\mathrm{J \cdot s}}{2}\frac{1}{550 \times 10^{-9}\,\mathrm{m}} = 9.59 \times 10^{-29}\,\mathrm{kg \cdot m/s}$$

The uncertainty in its velocity is

$$\Delta v = \frac{1}{m}\Delta p = \frac{9.59 \times 10^{-29}\,\mathrm{kg \cdot m/s}}{0.005\,\mathrm{kg}} = \boxed{1.92 \times 10^{-26}\,\mathrm{m/s}}$$

(b) If $p = \Delta p = 9.59 \times 10^{-29}\,\mathrm{kg \cdot m/s}$, then

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\mathrm{J \cdot s}}{9.59 \times 10^{-29}} = \boxed{6.91 \times 10^{-6}\,\mathrm{m}}$$

or more generally

$$\lambda = \frac{h}{p} = \frac{h}{\hbar/2\Delta x} = \frac{h}{h/4\pi\Delta x} = 4\pi\Delta x$$

I must admit, this is surprisingly large to me! Would a nickel act as a wave when confronted with a slit which was a micron across? Hmm. Well, the size of a nickel is much larger than that, so it wouldn't even fit through. Also, note that this is the largest λ could be, assuming that $\Delta x \Delta p = \frac{\hbar}{2}$. Handwave handwave. At least it isn't ∞. :)

▷ **5.**
Calculate, by hand, the mean and standard deviation of these two sets of numbers. Which has the larger standard deviation?
(a) $0, 2, 5, 9$
(b) $3, 3, 4, 6$

Answer:

(a) The mean is $\frac{1}{4}(0+2+5+9) = 4$, and the standard deviation is

$$\sigma_a = \sqrt{\frac{(0-4)^2 + (2-4)^2 + (5-4)^2 + (9-4)^2}{4}} = \sqrt{11.5} = \boxed{3.39}$$

(b) The mean here is $\frac{1}{4}(3+3+4+6) = 4$ as well, and the standard deviation is

$$\sigma_a = \sqrt{\frac{(3-4)^2 + (3-4)^2 + (4-4)^2 + (6-4)^2}{4}} = \sqrt{1.5} = \boxed{1.22}$$

This set has the smaller standard deviation, because the numbers are closer together.

▷ **6.**

Consider the function

$$f(x) = \begin{cases} x, & -L < x < L \\ 0, & \text{otherwise} \end{cases}$$

This function can be written as an integral:

$$f(x) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk$$

(a) Find $A(k)$.
(b) Find the value $k_{\max} > 0$ which maximizes $A(k)$. Show that it is inversely proportional to L. (Note: this function has many maxima and minima. A graph may help you find the true maximum.)

Answer:____________________________________

(a) The formula for $A(k)$ is

$$\begin{aligned}
A(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \\
&= \frac{1}{2\pi} \int_{-L}^{L} x e^{-ikx}\, dx \qquad \text{Let } \lambda = -ik \\
&= \frac{1}{2\pi} \int_{-L}^{L} x e^{\lambda x}\, dx \qquad x e^{\lambda x} = \tfrac{d}{d\lambda} e^{\lambda x} \\
&= \frac{1}{2\pi} \frac{d}{d\lambda} \int_{-L}^{L} e^{\lambda x}\, dx \\
&= \frac{1}{2\pi} \frac{d}{d\lambda} \left[\frac{1}{\lambda} e^{\lambda x} \right]_{-L}^{L} \\
&= \frac{1}{2\pi} \left[-\frac{1}{\lambda^2} e^{\lambda x} + \frac{x}{\lambda} e^{\lambda x} \right]_{-L}^{L} \\
&= \frac{1}{2\pi} \left[-\frac{1}{(-ik)^2} e^{-ikx} + \frac{x}{-ik} e^{-ikx} \right]_{-L}^{L} \\
&= \frac{1}{2\pi} \left[\frac{1}{k^2} (e^{-ikL} - e^{ikL}) + \frac{i}{k} (L e^{-ikL} + L e^{+ikL}) \right] \\
&= -\frac{i}{\pi k^2} \sin kL + \frac{iL}{\pi k} \cos kL \\
&= \boxed{\frac{i}{\pi k^2} \left[kL \cos kL - \sin kL \right]}
\end{aligned}$$

(b) The value of k which maximizes $A(k)$ must satisfy $\frac{dA}{dk} = 0$. So

$$0 = \frac{dA}{dk} = -\frac{2i}{\pi k^3}\left[kL\cos kL - \sin kL\right] + \frac{i}{\pi k^2}\left[L\cos kL - kL^2\sin kL - L\cos kL\right]$$

Instead of trying to solve this, let's write $u = kL$, so that $L = u/k$. Then

$$\begin{aligned}
0 &= -\frac{2i}{\pi k^3}\left[u\cos u - \sin u\right] + \frac{i}{\pi k^2}\left[-k\frac{u^2}{k^2}\sin u\right] \\
0 &= -\frac{2i}{\pi k^3}\left[u\cos u - \sin u\right] + \frac{i}{\pi k^3}\left[-u^2\sin u\right] \\
&= -2u\cos u + 2\sin u - u^2\sin u
\end{aligned}$$

(assuming that k is finite and not zero, I multiplied through by $-i\pi k^3$). Now if I solve this for u, I will get some number, call it $u_{\max}$. But $u_{\max} = k_{\max}L$ and so $k_{\max} = \frac{u_{\max}}{L}$ and thus $k_{\max}$ is inversely proportional to L.

Why does this matter? Because $A(k)$ has the form of a wavepacket with an envelope of $\frac{1}{k^2}$ and an oscillatory wave inside. The distance of the highest point from $k = 0$ is proportional to the width of the wavepacket and thus the uncertainty Δk in the position's wavenumber. Because this is inversely proportional to L, which is itself a measure of the uncertainty in the position, we have that $\Delta x \propto \frac{1}{\Delta k}$ and thus $x \propto \frac{1}{\Delta p}$, which is what the Heisenberg uncertainty principle would predict.

The actual solution, by the way, can be had by solving numerically. It is $u = \pm 2.08158$, and so $k_{\max} = \pm\frac{2.08158}{L}$.