Physics 370 Homework #3 v2

6 problems

Solutions

▷ **1.**

Prove that Let u be the velocity of a particle in the x direction in frame S; let u' be the velocity of the particle in frame S' which is moving in the x direction with speed v in S's frame. When $c = 1$, prove that

$$\gamma_{u'} = (1 - uv)\gamma_v\gamma_u$$

Answer:

We want to use the equations

$$\gamma_{u'} = \frac{1}{\sqrt{1 - u'^2}} \quad \text{and} \quad u' = \frac{u - v}{1 - uv}$$

Let's calculate

$$\begin{aligned}
1 - u'^2 &= 1 - \frac{(u-v)^2}{(1-uv)^2} \\
&= \frac{(1-uv)^2 - (u-v)^2}{(1-uv)^2} \\
&= \frac{(1 - 2uv + u^2v^2) - (u^2 + v^2 - 2uv)}{(1-uv)^2} \\
&= \frac{1 + u^2v^2 - u^2 - v^2}{(1-uv)^2} \\
&= \frac{(1-u^2)(1-v^2)}{(1-uv)^2}
\end{aligned}$$

and so

$$\begin{aligned}
\gamma_{u'} &= \frac{1}{\sqrt{1-u'^2}} \\
&= \frac{1 - uv}{\sqrt{1-u^2}\sqrt{1-v^2}} \\
&= (1-uv)\gamma_u\gamma_v \qquad \textbf{Q.E.D.}
\end{aligned}$$

▷ **2.**
What are the momentum, energy, and kinetic energy of a proton moving at $0.8\,c$?

Answer:
We first need

$$\gamma_u = \frac{1}{\sqrt{1-(0.8)^2}} = 1.67$$

The momentum is

$$\begin{aligned} p = \gamma m u &= (1.67)(1.67\times 10^{-27}\,\text{kg})(0.8)(3\times 10^8\,\text{m/s}) \\ &= \boxed{6.69\times 10^{-19}\,\text{kg}\cdot\text{m/s}} \end{aligned}$$

The total energy is

$$E = \gamma m c^2 = (1.67)(1.67\times 10^{-27}\,\text{kg})(3\times 10^8\,\text{m/s})^2 = \boxed{2.51\times 10^{-10}\,\text{J}}$$

The total kinetic energy is

$$KE = (\gamma - 1) m c^2 = (0.67)(1.67\times 10^{-27}\,\text{kg})(3\times 10^8\,\text{m/s})^2 = \boxed{1.0\times 10^{-10}\,\text{J}}$$

▷ **3.**
A spring has a force constant of $18\,\text{N/m}$. If it is compressed $50\,\text{cm}$ from its equilibrium length, how much mass (in kg) will it have gained?

Answer:
The mass increases by the amount of potential energy in the spring, divided by c^2 (for the units to work out right). That is

$$\begin{aligned} \Delta m = \frac{1}{c^2}\Delta PE &= \frac{1}{c^2}\left(\frac{1}{2}k(\Delta x)^2\right) \\ &= \frac{\frac{1}{2}(18\,\text{N/m})(0.5\,\text{m})^2}{(3\times 10^8\,\text{m/s})^2} \\ &= \boxed{2.5\times 10^{-17}\,\text{kg}} \end{aligned}$$

or 25 femtograms, hardly noticeable.

▷ **4.**
Two objects collide with one another. Before the collision, object A has mass 16 (in arbitrary units) and velocity $0.6c$ to the right, and object B has mass 9 and velocity $0.8c$ to the left. After the collision, both objects have the same speed as before, but in the opposite directions.

(a) Prove that the momentum is conserved before and after the collision.
(b) Calculate the change in kinetic energy of the system before and after the collision.
(c) Use the relativistic velocity transformation to find the four velocities in a frame moving to the right at $0.6c$.
(d) Show that the momentum is conserved before and after the collision.
(e) Calculate the change in kinetic energy of the system before and after the collision.

Answer:

(a) Note that $\gamma_{0.6} = \frac{1}{\sqrt{1-(0.6)^2}} = \frac{5}{4}$, and $\gamma_{0.8} = \frac{1}{\sqrt{1-(0.8)^2}} = \frac{5}{3}$. Let $+\hat{x}$ point to the right. The momenta are

$$p_{Ai} = \gamma_{0.6}(16)(+0.6) = +12$$

$$p_{Bi} = \gamma_{0.8}(9)(-0.8) = -12$$

$$p_{Af} = \gamma_{0.6}(16)(-0.6) = -12$$

$$p_{Bf} = \gamma_{0.8}(9)(+0.8) = +12$$

Thus the momentum before the collision is $p_{Ai}+p_{Bi} = 0$ and the momentum after is $p_{Af}+p_{Bf} = 0$

(b) The kinetic energy before the collision (in $c = 1$ terms) is

$$KE = (\gamma_{0.6} - 1)(16) + (\gamma_{0.8} - 1)(9) = \frac{1}{4}(16) + \frac{2}{3}(9) = 10$$

The kinetic energy after the collision is exactly the same. This is an *elastic* collision.

(c) Using the formula $u' = \frac{u+v}{1+uv}$:

$$u'_{Ai} = \frac{0.6 + 0.6}{1 + (0.6)(0.6)} = 0.882$$

$$u'_{Bi} = \frac{-0.8 + 0.6}{1 + (-0.8)(0.6)} = -0.385$$

$$u'_{Af} = \frac{-0.6 + 0.6}{1 + (-0.6)(0.6)} = 0$$

$$u'_{Bf} = \frac{0.8 + 0.6}{1 + (0.8)(0.6)} = 0.946$$

(d) The momenta in this frame is

$$p'_{Ai} = \frac{1}{\sqrt{1 - (0.882)^2}}(16)(+0.882) = 29.9$$

$$p'_{Bi} = \frac{1}{\sqrt{1 - (0.385)^2}}(9)(-0.385) = -3.75$$

$$p'_{Af} = \frac{1}{\sqrt{1 - (0)^2}}(16)(0) = 0$$

$$p'_{Bf} = \frac{1}{\sqrt{1 - (0.946)^2}}(9)(0.946) = 26.3$$

So the initial momentum is $p'_i = 29.9 - 3.75 = 26$ and the final momentum is $p'_f = 26$.

(e) The kinetic energies are

$$
\begin{aligned}
KE'_i &= (\gamma_{Ai} - 1)m_A + (\gamma_{Bi} - 1)m_B \\
&= \left(\frac{1}{\sqrt{1-(0.882)^2}} - 1\right)(16) + \left(\frac{1}{\sqrt{1-(0.385)^2}} - 1\right)(9) \\
&= 18.0 + 0.75 = 19
\end{aligned}
$$

$$
\begin{aligned}
KE'_f &= (\gamma_{Af} - 1)m_A + (\gamma_{Bf} - 1)m_B \\
&= \left(\frac{1}{\sqrt{1-0}} - 1\right)(16) + \left(\frac{1}{\sqrt{1-(0.946)^2}} - 1\right)(9) \\
&= 0 + 0.75 = 19
\end{aligned}
$$

Elastic collisions in one frame are elastic in others as well.

▷ **5.**

Consider a set of N particles with mass m_i and velocity u_i in frame S. The total momentum of the set is

$$p = \sum_{i=1}^{N} \gamma_{u_i} m_i u_i$$

Find the total momentum p' in frame S' which is moving at velocity v with respect to S. Show that p' is equal to a multiple of p plus another term: what is the other term?

Answer:______________________________

The new momentum is

$$
\begin{aligned}
P' &= \sum_{i=1}^{N} \gamma_{u'_i} m_i u'_i \\
&= \sum_{i=1}^{N} (1 - u_i v)\gamma_{u_i}\gamma_v m_i \frac{u_i - v}{1 - u_i v} \\
&= \gamma_v \sum_{i=1}^{N} \gamma_{u_i} m_i (u_i - v) \\
&= \gamma_v \sum_{i=1}^{N} m_i u_i - \gamma_v v \sum_{i=1}^{N} \gamma_{u_i} m_i \\
&= \gamma_v P - \gamma_v v E
\end{aligned}
$$

where E is the total energy of the system. Thus if P and E are conserved, then P' will be conserved as well.

▷ **6.**

The light from galaxy NGC 221 consists of a recognizable spectrum of wavelengths. However, all are shifted toward the shorter-wavelength end of the spectrum. In particular, the calcium "line" ordinarily observed at 396.85 nm is observed at 396.58 nm. Is this galaxy moving toward or away from Earth? At what speed?

Answer:________________________________

This is the Doppler effect,

$$f' = f\sqrt{\frac{1-v}{1+v}}$$

where v is the speed of the particle moving away from the observer. We can solve this for v:

$$\begin{aligned}
\left(\frac{f'}{f}\right)^2 &= \frac{1-v}{1+v} \\
\left(\frac{f'}{f}\right)^2 (1+v) &= 1-v \\
v\left[\left(\frac{f'}{f}\right)^2 + 1\right] &= 1 - \left(\frac{f'}{f}\right)^2 \\
v &= \frac{1-(f'/f)^2}{1+(f'/f)^2}
\end{aligned}$$

We are given the wavelengths, not the frequencies. Because $\lambda = c/f$, $f'/f = \lambda/\lambda'$.

$$\begin{aligned}
&= \frac{1-(\lambda/\lambda')^2}{1+(\lambda/\lambda')^2} \\
&= \frac{\lambda'^2 - \lambda^2}{\lambda'^2 + \lambda^2} \\
&= \frac{(396.58)^2 - (396.85)^2}{(396.58)^2 + (396.85)^2} \\
&= \frac{-214.226}{314776} = \boxed{-6.8 \times 10^{-4}}
\end{aligned}$$

or $-2.0 \times 10^5\,\mathrm{m/s}$. This is negative, so the galaxy is moving towards us. (It should be blueshifted in that case, and we see that the wavelength is indeed observed to be slightly smaller.)