Physics 370 Homework #10

5 problems

Solutions

▷ **1.**

A *paramagnet*, in its simplest formulation, is a collection of magnetic dipoles, each with magnetic dipole moment $\vec{\mu}$. When placed in a magnetic field $\vec{B} = B\hat{z}$, each dipole has energy $U = -\vec{\mu}\cdot\vec{B} = -\mu_z B$. Suppose the z–component of the dipole moment is *quantized*, so that it can only take one of two values: $\mu_z = +\mu_0$ (call this "up") and $\mu_z = -\mu_0$ ("down"). If there are $N_\uparrow$ dipoles that point up, and $N_\downarrow$ that point down, then the total energy is

$$E = -N_\uparrow \mu_0 B + N_\downarrow \mu_0 B = (N_\downarrow - N_\uparrow)\mu_0 B$$

(a) Rewrite the energy formula in terms of the total number of dipoles N and the number $N_\downarrow$ that point down, and then solve that equation for $N_\downarrow$. (Note that, if N is constant, $N_\downarrow$ parametrizes the energy.)
(b) Find the multiplicity Ω of the macrostate where there are $N_\downarrow$ dipoles pointing down. (This is also an energy macrostate.)
(c) Write an expression for the entropy as a function of $N_\downarrow$, and use Stirling's approximation $\ln x! \approx x\ln x - x$ $(x \gg 1)$ to eliminate the factorials from the expression assuming that the number of dipoles is large.
(d) Show that

$$kT = 2\mu_0 B \left(\ln \frac{N_\uparrow}{N_\downarrow}\right)^{-1}$$

(e) Find the temperature of the paramagnet when $N_\downarrow$ is $\frac{1}{4}N$ and $\frac{1}{2}N$.

Answer:

(a) $N = N_\uparrow + N_\downarrow$, so $N_\uparrow = N - N_\downarrow$. Thus

$$E = \left(N_\downarrow - (N - N_\downarrow)\right)\mu_0 B = \left(2N_\downarrow - N\right)\mu_0 B$$

$$\implies N_\downarrow = \frac{E}{2\mu_0 B} + \frac{1}{2}N$$

Note also that $N_\uparrow = N - N_\downarrow = \frac{1}{2}N - \frac{E}{2\mu_0 B}$.
(b) If there are N dipoles, there are $\Omega = \binom{N}{N_\downarrow}$ ways to have $N_\downarrow$ of them point down.
(c) The entropy is

$$S = k\ln\Omega = k\ln\frac{N!}{N_\downarrow!(N-N_\downarrow)!} = k\left[\ln N! - \ln N_\downarrow! - \ln N_\uparrow!\right]$$

Using Stirling's approximations,

$$
\begin{aligned}
S/k &\approx (N\ln N - N) - (N_\downarrow \ln N_\downarrow - N_\downarrow) - (N_\uparrow \ln N_\uparrow - N_\uparrow) \\
&= N\ln N - N_\downarrow \ln N_\downarrow - N_\uparrow \ln N_\uparrow - (N - N_\downarrow - N_\uparrow) \\
&= N\ln N - N_\downarrow \ln N_\downarrow - N_\uparrow \ln N_\uparrow
\end{aligned}
$$

(d) The temperature of the paramagnet is given by the expression

$$
\begin{aligned}
\frac{1}{T} = \frac{\partial S}{\partial E} &= k\frac{\partial}{\partial E}(N\ln N) - k\frac{\partial}{\partial E}(N_\downarrow \ln N_\downarrow) - k\frac{\partial}{\partial E}(N_\uparrow \ln N_\uparrow) \\
&= 0 - k\frac{\partial N_\downarrow}{\partial E}\frac{\partial}{\partial N_\downarrow}(N_\downarrow \ln N_\downarrow) - k\frac{\partial N_\downarrow}{\partial E}\frac{\partial N_\uparrow}{\partial N_\downarrow}\frac{\partial}{\partial N_\uparrow}(N_\uparrow \ln N_\uparrow)
\end{aligned}
$$

The derivatives are

$$
\begin{aligned}
\frac{\partial N_\downarrow}{\partial E} &= \frac{\partial}{\partial E}\left(\frac{E}{2\mu_0 B} + \frac{1}{2}N\right) = \frac{1}{2\mu_0 B} \\
\frac{\partial N_\uparrow}{\partial N_\downarrow} &= \frac{\partial N - N_\downarrow}{\partial N_\downarrow} = -1 \\
\frac{\partial}{\partial x}(x\ln x) &= \ln x + \frac{x}{x} = \ln x + 1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{T} &= -\frac{1}{2\mu_0 B}k(1+\ln N_\downarrow) + \frac{1}{2\mu_0 B}k(1+\ln N_\uparrow) \\
\implies \frac{1}{kT} &= \frac{1}{2\mu_0 B}\ln\frac{N_\uparrow}{N_\downarrow} \\
\implies kT &= 2\mu_0 B\left(\ln\frac{N - N_\downarrow}{N_\downarrow}\right)^{-1}
\end{aligned}
$$

(e) If $N_\downarrow = \gamma N$, then

$$
kT = 2\mu_0 B\left(\ln\frac{N - \gamma N}{\gamma N}\right)^{-1} = 2\mu_0 B\left[\ln\left(\frac{1}{\gamma} - 1\right)\right]^{-1}
$$

If $\gamma = \frac{1}{4}$, then

$$kT = 2\mu_0 B \left[\ln(4-1)\right]^{-1} = \frac{2\mu_0 B}{\ln 3}$$

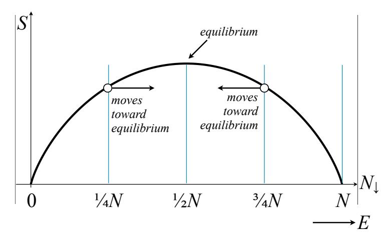
which is fine. However, if $\gamma = \frac{1}{2}$, then

$$kT = 2\mu_0 B \left[\ln(2-1)\right]^{-1} = \frac{2\mu_0 B}{\ln 1} = \infty$$

Wait, what? Infinite temperature, when half the dipoles point up? It becomes even more puzzling if you try $\gamma = \frac{3}{4}$:

$$kT = 2\mu_0 B \left[\ln\left(\frac{1}{3}\right)\right]^{-1} = -\frac{2\mu_0 B}{\ln 3}$$

or a negative temperature!



If we look at a graph of $S(E)$, it's clear that the slope $\frac{\partial S}{\partial E}$ is zero at the top and negative on the right-hand side, and so the temperature $T = \left(\frac{\partial S}{\partial E}\right)^{-1}$ is infinite at the top and negative on the right. What does this even mean? Notice that the entropy is maximum when exactly half the dipoles point up and half down, and so the system will try to reach this equilibrium point if it can. When $N_\downarrow = \frac{1}{4}N$, this means increasing the number of down-pointing spins, increasing the system's energy. (This is normal behavior: in most systems, when the energy increases its entropy increases as well.) However, when most of the spins point down already, the system actually wants to *lose* energy to gain entropy and reach equilibrium. A paramagnet in that state will give heat to *any* object it touches, and because heat flows from hot to cold, that means that the paramagnet with a negative temperature is hotter than any normal object! Negative temperatures are weird, and they come from the fact that the paramagnet can only store a limited amount of energy (all spins pointing down), so that the entropy has to curve down. In normal systems, the entropy always increases with energy, energy has no limit, and temperatures remain positive.

▷ **2.**

The entropy of an *ideal gas* (point particles that do not interact with one another) has the form

$$S = kN\left[\ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln\left(\frac{E}{N}\right) + C\right]$$

where C is a constant. Find an expression of the ideal gas's energy as a function of temperature $E(T)$.

Answer:____________________________________

We first need to find $\frac{1}{T(E)} = \frac{\partial S}{\partial E}$:

$$\begin{aligned}
\frac{1}{T(E)} &= \frac{\partial S}{\partial E} = \frac{\partial}{\partial E} kN \left[\ln V - \ln N + \frac{3}{2} \ln E - \frac{3}{2} \ln N + C \right] \\
&= kN \frac{3}{2} \frac{\partial}{\partial E} \ln E \\
&= \frac{3Nk}{2E}
\end{aligned}$$

Solving this for E gives us

$$E = \frac{3}{2} NkT$$

which matches up with the equipartition theorem: there are 3 degrees of freedom for each particle, and N particles, so $3N$ degrees of freedom in total, each with average energy $\frac{1}{2}kT$.

▷ **3.**

In a large number of distinguishable harmonic oscillators, how high does the temperature have to be for the probability of occupying the ground state to be less than $\frac{1}{2}$?

Answer:

We really need the partition function from the next question to do this one. The probability that the harmonic oscillator is in its ground state is

$$\begin{aligned}
P &= \frac{e^{-\hbar\omega_0/2kT}}{Z} \\
&= e^{-\hbar\omega_0/2kT} \frac{1 - e^{-\hbar\omega_0/kT}}{e^{-\hbar\omega_0/2kT}} \\
\frac{1}{2} &> P = 1 - e^{-\hbar\omega_0/kT} \\
\frac{1}{2} &< e^{-\hbar\omega_0/kT} \\
-\frac{\hbar\omega_0}{kT} &> -\ln 2 \\
\implies T &> \frac{\hbar\omega_0}{k \ln 2}
\end{aligned}$$

▷ **4.**
In class we showed that a system in contact with a thermal reservoir T has an average energy

$$\langle E\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{1}{kT^2}\frac{1}{Z}\frac{\partial Z}{\partial T}$$

where $\beta = \frac{1}{kT}$ and Z is the *partition function* of a system, defined as

$$Z = \sum_s e^{-E_s/kT}$$

summed over all possible microstates of the system.
(a) The possible microstates of a harmonic oscillator have energy $E_j = \left(j+\frac{1}{2}\right)\hbar\omega_0$, where $j = 0, 1, 2, \ldots$. Find the partition function of a harmonic oscillator, and do out the sum so you have a closed expression. (Hint: the sum is a power series.)
(b) Find the average energy of the harmonic oscillator.
(c) What is the energy approximately in the low-temperature limit $kT \ll \hbar\omega_0$ and the high-temperature limit $kT \gg \hbar\omega_0$?

Answer:____________________________________

(a) The partition function is

$$\begin{aligned} Z &= \sum_{j=0}^{\infty} e^{-(j+\frac{1}{2})\hbar\omega_0/kT} \\ &= \sum_{j=0}^{\infty} e^{-j\hbar\omega_0/kT} e^{-\hbar\omega_0/2kT} \\ &= e^{-\hbar\omega_0/2kT}\sum_{j=0}^{\infty}\left(e^{-\hbar\omega_0/kT}\right)^j \end{aligned}$$

This is a power series, and $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$; therefore

$$Z = \frac{e^{-\hbar\omega_0/2kT}}{1-e^{-\hbar\omega_0/kT}}$$

We can rewrite this in a couple of other forms if we like:

$$Z = \frac{1}{e^{\hbar\omega_0/2kT} - e^{\hbar\omega_0/2kT}} = \frac{1}{2}\,\mathrm{csch}\,\frac{\hbar\omega_0}{2kT}$$

(b) If we write $Z = \frac{1}{2}\,\mathrm{csch}\, a\beta$ where $a = \frac{1}{2}\hbar\omega_0$, then

$$\frac{\partial Z}{\partial \beta} = \frac{\partial}{\partial \beta}\frac{1}{2}\,\mathrm{csch}\, a\beta = -\frac{1}{2}a\coth a\beta\,\mathrm{csch}\, a\beta$$

and so

$$
\begin{aligned}
\langle E\rangle &= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \\
&= -\frac{1}{\frac{1}{2}\operatorname{csch} a\beta}\left[-\frac{1}{2}a\coth a\beta\operatorname{csch} a\beta\right] \\
&= a\coth a\beta \\
&= \frac{1}{2}\hbar\omega_0\coth\frac{\hbar\omega_0}{2kT}
\end{aligned}
$$

(c) It may be easier to think about this in terms of exponentials:

$$\langle E\rangle = \frac{1}{2}\hbar\omega_0\frac{e^{a\beta}+e^{-a\beta}}{e^{a\beta}-e^{-a\beta}}$$

If $kT \ll \hbar\omega_0$, then $a\beta = \frac{1}{2}\frac{\hbar\omega_0}{kT} \gg 1$. In this case, $e^{-a\beta} \approx 0$, and so

$$\langle E\rangle \approx \frac{1}{2}\hbar\omega_0\frac{e^{a\beta}}{e^{a\beta}} = \frac{1}{2}\hbar\omega_0$$

at low temperatures. (The harmonic oscillator is stuck in the ground state.)

At high temperatures, $a\beta \ll 1$, and we can use a Taylor expansion to write $e^{a\beta} \approx 1+a\beta$ and $e^{-a\beta} \approx 1-a\beta$, so

$$\langle E\rangle = \frac{1}{2}\hbar\omega_0\frac{(1+a\beta)+(1-a\beta)}{(1+a\beta)-(1-a\beta)} = \frac{1}{2}\hbar\omega_0\frac{2}{2a\beta} = \frac{1}{2}\hbar\omega_0\frac{1}{\frac{1}{2}\hbar\omega_0}kT = kT$$

▷ **5.**

Find the *most probable* speed of an ideal gas, according to the Maxwell Speed Distribution; that is, find the mode of the distribution.

Answer:

The Maxwell Speed Distribution is

$$f(v) = C4\pi v^2 e^{-mv^2/2kT}$$

where C is the normalization constant. The most probable speed is the one where $f'(v) = 0$:

$$
\begin{aligned}
f'(v) &= 4\pi C\left[2ve^{-mv^2/2kT} + v^2\left(-2mv/2kT\right)e^{-mv^2/2kT}\right] \\
&= 4\pi C\left[2v - \frac{mv^3}{kT}\right]e^{-mv^2/2kT}
\end{aligned}
$$

This is equal to zero if the value in brackets is zero:

$$0 = 2v - \frac{mv^3}{kT} \implies 2v = \frac{mv^3}{kT}$$

Either $v = 0$ (which isn't the maximum), or else

$$2 = \frac{mv^2}{kT} \implies v^2 = \frac{2kT}{m} \implies \boxed{v_{mp} = \sqrt{\frac{2kT}{m}}}$$