Physics 370 Homework #4

7 problems

Solutions

▷ **1.**
According to an observer at Earth's equator, by how much would his clock and one on a satellite in geosynchronous orbit differ in one day?

Answer:
The time dilation due to gravity is given by Equation 2-31:

$$\frac{\Delta t_{Earth}}{\Delta t_{satellite}} = 1 - \frac{GM}{c^2}\left(\frac{1}{r_E} - \frac{1}{r_S}\right)$$

The radius of the Earth is $6.371 \times 10^6\,\mathrm{m}$. For a satellite orbiting the Earth we have

$$\begin{aligned}
\frac{mv^2}{r} &= G\frac{Mm}{r^2} && \text{centripetal force}\\
\implies r &= \frac{GM}{v^2} && \text{simplify}\\
\implies v &= \frac{2\pi r}{T} && \text{distance over time}\\
\implies r &= \frac{GMT^2}{4\pi^2 r^2}\\
\implies r^3 &= \frac{GMT^2}{4\pi^2}\\
\implies r &= \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67\times10^{-11})(5.972\times10^{24})(8.64\times10^4)^2}{4\pi^2}} = 4.22\times10^7\,\mathrm{m}
\end{aligned}$$

where $T = 1\,\mathrm{day} = 86400\,\mathrm{s}$. Thus the time dilation is

$$\begin{aligned}
\frac{\Delta t_E}{\Delta t_s} &= 1 - \frac{(6.67\times10^{-11})(5.972\times10^{24})}{(3\times10^8)^2}\left(\frac{1}{6.371\times10^6} - \frac{1}{4.22\times10^7}\right)\\
&= 1 - \left(4.43\times10^{-3}\right)\left(1.33\times10^{-7}\right) = 1 - 5.90\times10^{-10}
\end{aligned}$$

So a clock on earth will move slower by 1 part in 6×10^{-10}.

▷ **2.**

(a) Show, using the chain rule, that

$$\frac{dU}{d\lambda} = \frac{8\pi V hc}{e^{hc/\lambda k_B T} - 1}\frac{1}{\lambda^5}$$

(b) According to *Wien's Law*, the wavelength λ_{max} of maximum thermal emission of electromagnetic energy from a body of temperature T obeys

$$\lambda_{\text{max}} T = 2.898 \times 10^{-3}\,\text{m} \cdot \text{K}$$

Show this is true by obtaining an expression that, when solved, would yield the wavelength at which this function is maximized. Solving this equation is impossible, but show that the value of $\lambda_{\text{max}} T$ solves it to a reasonable degree.

Answer:

(a) We know from the text that

$$\frac{dU}{df} = \frac{hf^3}{e^{hf/kT} - 1}\frac{8\pi V}{c^3}$$

Now $c = f\lambda \implies f = c\lambda^{-1}$. Thus

$$\begin{aligned}
\frac{dU}{d\lambda} &= \frac{dU}{df}\frac{df}{d\lambda} \\
&= \left(\frac{hf^3}{e^{hf/kT} - 1}\frac{8\pi V}{c^3}\right)\left(-\frac{c}{\lambda^2}\right) \\
&= -\left(\frac{hc^3/\lambda^3}{e^{hc/\lambda kT} - 1}\frac{8\pi V}{c^3}\right)\left(\frac{c}{\lambda^2}\right) \\
&= -\frac{8\pi V hc}{e^{hc/\lambda kT} - 1}\frac{1}{\lambda^5}
\end{aligned}$$

The minus sign isn't really significant.

(b) The energy density $\dfrac{dU(\lambda)}{d\lambda}$ reaches its maximum value when $\dfrac{d}{d\lambda}\dfrac{dU}{d\lambda} = 0$. For simplicity, I'm going to write $f(\lambda) = (e^{hc/\lambda kT} - 1)^{-1}$, and note that

$$f'(\lambda) = \frac{hc}{\lambda^2 kT} e^{hc/\lambda kT}(e^{hc/\lambda kT} - 1)^{-2} = \frac{hc}{\lambda^2 kT} e^{hc/\lambda kT} f(\lambda)^2$$

Thus

$$
\begin{aligned}
0 &= \frac{d}{d\lambda}\frac{8\pi V hc}{f(\lambda)}\frac{1}{\lambda^5} \\
&= \frac{d}{d\lambda}\lambda^{-5} f(\lambda) \\
&= -5\lambda^{-6} f(\lambda) + \lambda^{-5} f'(\lambda) \\
\implies \ 5\lambda^{-6} f(\lambda) &= \lambda^{-5} f'(\lambda) \\
5 &= \lambda \frac{f'(\lambda)}{f(\lambda)} \\
5 &= \lambda \frac{hc}{\lambda^2 kT} e^{hc/\lambda kT} f(\lambda) \\
&= \frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT}-1} = \frac{hc}{\lambda kT}\frac{1}{1-e^{-hc/\lambda kT}}
\end{aligned}
$$

If $\lambda T = 2.898 \times 10^{-3}\,\mathrm{m\,K}$, then

$$
\frac{hc}{\lambda kT} = \frac{(6.636 \times 10^{-34})(3 \times 10^8)}{(1.38 \times 10^{-23})(2.898 \times 10^{-3})} = 5.0
$$

so that this is the solution. Note that

$$
\lambda_{\text{max}} = \frac{2.898 \times 10^{-3}\,\mathrm{m\,K}}{T}
$$

so that as the temperature increases, the wavelength of light decreases inversely, and the frequency increases linearly: hotter objects put out higher frequency light.

▷ **3.**

The electromagnetic intensity of all wavelengths thermally radiated by a body of temperature T is given by

$$
I = \sigma T^4 \quad \text{where} \quad \sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2/K^4}
$$

This is the *Stefan-Boltzmann law*. To derive it, show that the total energy of the radiation in a volume V at temperature T is

$$
U = \frac{8\pi^5 k_B^4 V T^4}{15 h^3 c^3}
$$

by integrating Planck's spectral energy density over all frequencies. Note that

$$
\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}
$$

Intensity, or power per unit area, is then the product of energy per unit volume and distance per unit time. But because intensity is a flow in a given direction away from the blackbody, c is not the correct speed. For radiation moving uniformly in all directions, the average *component* of velocity in a given direction is $\frac{1}{4}c$.

Answer:

The total energy represented by the spectrum dU/df is

$$\begin{aligned} U &= \int_0^\infty \frac{dU}{df}\,df \\ &= \int_0^\infty \frac{hf^3}{e^{hf/kT}-1}\frac{8\pi V}{c^3}df \end{aligned}$$

With an integral like this, it's best to get all of the quantities with units *out* of the integral. Some of the constants can be factored out right away, but to handle that exponential, we make a change of variables $x = hf/kT$, or $f = \frac{kT}{h}x$. This means that $df = \frac{kT}{h}dx$, and x will also range from 0 to ∞. Making the substitution,

$$\begin{aligned} U &= \int_0^\infty \frac{h(kT/h)^3x^3}{e^x-1}\frac{8\pi V}{c^3}\frac{kT}{h}\,dx \\ \implies U &= \frac{8\pi V(kT)^4}{h^3c^3}\int_0^\infty \frac{x^3}{e^x-1}\,dx \\ \implies \quad &= \frac{8\pi V(kT)^4}{h^3c^3}\frac{\pi^4}{15} = \frac{8\pi^5 V}{15h^3c^3}(kT)^4 \end{aligned}$$

Notice that the total energy is proportional to T^4: hotter objects put out a *lot* more energy than colder objects. The energy per unit volume is

$$\frac{U}{V} = \frac{8\pi^5k^4}{15h^3c^3}T^4$$

According to the problem, the intensity I is the energy per unit volume times the speed which is $\frac{1}{4}c$. Thus

$$I = \left(\frac{8\pi^5k^4}{15h^3c^3}T^4\right)\frac{c}{4} = \frac{2\pi^5k^4}{15h^3c^2}T^4 = \left(5.6\times10^{-8}\,\frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}^4}\right)T^4$$

▷ **4.**
Light of 300 nm wavelength strikes a metal plate, and photoelectrons are produced moving as fast as $0.002c$.
(a) What is the work function of the metal?
(b) What is the threshold wavelength for this metal?

Answer:

(a) The energy of each photon is

$$E = hf = h\frac{c}{\lambda} = \frac{(6.636 \times 10^{-34})(3 \times 10^{8})}{300 \times 10^{-9}} = 6.636 \times 10^{-19}\,\mathrm{J}$$

When a photon hits an electron, this energy goes into releasing the electron (ϕ) and the kinetic energy ($\frac{1}{2}mv^2$):

$$\begin{aligned} E &= \phi + \frac{1}{2}mv^2 \\ \Longrightarrow \quad \phi &= E - \frac{1}{2}mv^2 \\ &= 6.636 \times 10^{-19}\,\mathrm{J} - \frac{1}{2}(9.11 \times 10^{-31}\,\mathrm{kg})((0.002)(3 \times 10^{8}\,\mathrm{m/s}))^2 \\ &= 6.636 \times 10^{-19}\,\mathrm{J} - 1.640 \times 10^{-19}\,\mathrm{J} \\ &= \boxed{5.00 \times 10^{-19}\,\mathrm{J}} \end{aligned}$$

(b) The threshold wavelength for the metal is the wavelength of photon whose energy is just equal to the work function ϕ:

$$\begin{aligned} \phi &= \frac{hc}{\lambda_t} \\ \Longrightarrow \quad \lambda_t &= \frac{hc}{\phi} \\ &= \frac{(6.636 \times 10^{-34})(3 \times 10^{8})}{5.00 \times 10^{-19}} = 3.98 \times 10^{-7}\,\mathrm{m} \\ &= \boxed{398\,\mathrm{nm}} \end{aligned}$$

▷ **5.**

When a beam of monoenergetic electrons is directed at a tungsten target, X-rays are produced with wavelengths no shorter than 0.062 nm. How fast are the electrons in the beam moving?

Answer:

A minimum wavelength of $0.062\,\mathrm{nm}$ means a maximum energy of

$$E_{\max} = \frac{hc}{\lambda_{\min}} = \frac{(6.636 \times 10^{-34})(3 \times 10^{8})}{0.062 \times 10^{-9}} = 3.21 \times 10^{-15}\,\mathrm{J}$$

This must be the equal to the kinetic energy of a single electron. We don't know if the electrons are moving relativistically or not, so let's assume they aren't, and if the speed is greater than $0.1c$ we'll need to reconsider. In this case,

$$3.21 \times 10^{-15} = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2(3.21 \times 10^{-15})}{(9.11 \times 10^{-31})}} = 8.39 \times 10^{7} = 0.28c$$

Hmm, that's fast enough that maybe we should use the relativistic formula for the electron instead. Noting that $mc^2 = (9.11 \times 10^{-31})(3 \times 10^{8})^2 = 8.2 \times 10^{-14}\,\mathrm{J}$,

$$\begin{aligned}
KE &= (\gamma - 1)mc^2 \\
\gamma &= \frac{1}{\sqrt{1-(v/c)^2}} = \frac{KE}{mc^2} + 1 \\
\sqrt{1-\left(\frac{v}{c}\right)^2} &= \frac{mc^2}{KE + mc^2} \\
\left(\frac{v}{c}\right)^2 &= 1 - \frac{m^2c^4}{(KE+mc^2)^2} \\
v &= c\sqrt{1 - \frac{m^2c^4}{(KE+mc^2)^2}} \\
&= c\sqrt{1 - \frac{(8.2 \times 10^{-14})^2}{(3.21 \times 10^{-15} + 8.2 \times 10^{-14})^2}} = \boxed{0.27c}
\end{aligned}$$

▷ **6.**

A photon has the same momentum as an electron moving at $10^6\,\mathrm{m/s}$.

(a) Determine the photon's wavelength.

(b) What is the ratio of the kinetic energies of the two? (Note: a photon is *all* kinetic energy.)

Answer:____________________________________

(a) An electron moving at $10^6\,\mathrm{m/s}$ has momentum

$$p = \gamma m u = \frac{(9.11\times 10^{-31}\,\mathrm{J})(10^6\,\mathrm{m/s})}{\sqrt{1-\left(\frac{10^6}{3\times 10^8}\right)^2}} = 9.11\times 10^{-25}\,\frac{\mathrm{kg\,m}}{\mathrm{s}}$$

(The γ wasn't necessary in this case, but it was worth checking.) The photon, being massless, has energy $E = pc$, and thus wavelength

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \implies \lambda = \frac{h}{p} = \frac{6.636\times 10^{-34}}{9.11\times 10^{-25}} = \boxed{7.28\times 10^{-10}\,\mathrm{m}}$$

(b) Because γ for the electron is so small, we can approximate its kinetic energy as $\frac{1}{2}mu^2 = \frac{p^2}{2m}$, and so the ratio of the kinetic energies is

$$\frac{KE_p}{KE_e} = \frac{pc}{p^2/2m} = \frac{2mc}{p} = \frac{2(9.11\times 10^{-31})(3\times 10^8)}{9.11\times 10^{-25}} = 600$$

The photo has 600 times the kinetic energy as the electron does, because most of the electron's energy is in its rest energy.

▷ **7.**
Show that the laws of momentum and energy conservation forbid the complete *absorption* of a photon by a free electron. (This isn't the photoelectric effect, because electrons aren't entirely free in a metal.)

Answer:____________________________________
Suppose the photon's initial momentum is p_p and the electron's initial momentum is p_e. If the photon is absorbed, then the final momentum of the system $p_p + p_e$ is the final momentum p'_e of the electron.

Now let's consider energy. The energy of the photon is $E_p = p_p$ (in $c = 1$ units), and the energy of the electron initially is given by the formula $E_e^2 = p_e^2 + m_e^2$ (equation 2.28 in the book). The final energy of the system is the final energy of the electron, which is

$$(E'_e)^2 = (p'_e)^2 + m_e^2 = (p_p + p_e)^2 + m_e^2 = p_p^2 + p_e^2 + 2p_p p_e + m_e^2$$

The total final energy must be equal to the total initial energy $E_e + E_p$, and so

$$E'_e = E_p + E_e = p_p + \sqrt{p_e^2 + m_e^2}$$

$$(E'_e)^2 = p_p^2 + p_e^2 + m_e^2 + 2p_p\sqrt{p_e^2 + m_e^2}$$

Setting these two equal to each other:

$$
\begin{aligned}
p_p^2 + p_e^2 + 2p_p p_e + m_e^2 &= p_p^2 + p_e^2 + m_e^2 + 2p_p\sqrt{p_e^2 + m_e^2} \\
\implies 2p_p p_e &= 2p_p\sqrt{p_e^2 + m_e^2} \\
\implies p_e &= \sqrt{p_e^2 + m_e^2} \\
\implies p_e^2 &= p_e^2 + m_e^2 \\
\implies 0 &= m_e^2
\end{aligned}
$$

But the mass of the electron is *not* zero. Therefore the photon cannot be completely absorbed by a free electron.