Physics 370 Homework #5

5 problems

Solutions

▷ **1.**

For the Compton effect as outlined in the notes, write the equations for conservation of energy and momentum in units where $h = c = 1$, and derive the equation

$$\lambda' - \lambda = \frac{1}{m}(1 - \cos\theta)$$

Answer:

The energy of the photon is $hf = hc/\lambda$ or just $1/\lambda$ where λ is its wavelength; the energy of the electron is $\gamma_u mc^2 = \gamma_u m$ when it moves with speed u. Thus conservation of energy gives us

$$\frac{1}{\lambda} + m = \frac{1}{\lambda'} + \gamma m$$

The horizontal component of the momentum is

$$\frac{1}{\lambda} = \frac{1}{\lambda'}\cos\theta + \gamma mu\cos\phi$$

while the vertical component of the momentum is

$$0 = \frac{1}{\lambda'}\sin\theta - \gamma mu\sin\phi$$

Notice that ϕ and γ do not appear in the final equation, so we would like to eliminate them. Let's solve the momenta equations for $\cos\phi$ and $\sin\phi$:

$$\gamma mu\cos\phi = \frac{1}{\lambda} - \frac{1}{\lambda'}\cos\theta$$

$$\gamma mu\sin\phi = \frac{1}{\lambda'}\sin\theta$$

Square both equations and add them together, using the fact that $\cos^2\phi + \sin^2\phi = 1$:

$$\begin{aligned}
\gamma^2 m^2 u^2 &= \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\cos\theta\right)^2 + \frac{1}{\lambda'^2}\sin^2\theta \\
&= \frac{1}{\lambda^2} - \frac{2}{\lambda\lambda'}\cos\theta + \frac{1}{\lambda'^2}\cos^2\theta + \frac{1}{\lambda'^2}\sin^2\theta \\
&= \frac{1}{\lambda^2} + \frac{1}{\lambda'^2} - \frac{2}{\lambda\lambda'}\cos\theta \\
&= \frac{1}{\lambda^2} + \frac{1}{\lambda'^2} - \frac{2}{\lambda\lambda'} + \frac{2}{\lambda\lambda'} - \frac{2}{\lambda\lambda'}\cos\theta \\
&= \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)^2 + \frac{2}{\lambda\lambda'}(1 - \cos\theta)
\end{aligned}$$

The left-hand side can be simplified using the energy equation, and by noting that

$$\gamma = \frac{1}{\sqrt{1-u^2}} \implies \sqrt{1-u^2} = \frac{1}{\gamma} \implies u^2 = 1 - \frac{1}{\gamma^2}$$

and so

$$\gamma^2 u^2 = \gamma^2\left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 - 1$$

$$\implies \gamma^2 m^2 u^2 = m^2(\gamma^2 - 1) = m^2\gamma^2 - m^2$$

Now, according to the energy equation,

$$\gamma m = \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + m$$

Therefore

$$\begin{aligned}
\implies \gamma^2 m^2 u^2 &= m^2\gamma^2 - m^2 \\
&= \left[\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + m\right]^2 - m^2 \\
&= \left[\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)^2 + 2m\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + m^2\right] - m^2 \\
&= \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)^2 + 2m\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)
\end{aligned}$$

Putting this all together,

$$\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)^2+2m\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)^2+\frac{2}{\lambda\lambda'}(1-\cos\theta)$$

$$2m\frac{\lambda'-\lambda}{\lambda\lambda'}=\frac{2}{\lambda\lambda'}(1-\cos\theta)$$

$$\implies \lambda'-\lambda=\frac{1}{m}(1-\cos\theta) \qquad \textbf{Q.E.D.}$$

▷ **2.**

(a) What are the SI units for h/c and h/c^2?
(b) In the units where $h=c=1$, you can write mass in terms of inverse meters or in terms of inverse seconds. How many kilograms are equivalent to $1\,/\mathrm{m}$?
(c) How many kilograms are equivalent to $1\,/\mathrm{s}$?

Answer:

(a) The units of h and c are $[h]=\mathrm{J}\cdot\mathrm{s}=\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}^2}\mathrm{s}=\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\mathrm{s}}$ and $[c]=\mathrm{m/s}$. Thus

$$\left[\frac{h}{c}\right]=\frac{\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}}{\mathrm{m/s}}=\mathrm{kg}\cdot\mathrm{m}$$

and

$$\left[\frac{h}{c^2}\right]=\frac{[h/c]}{[c]}=\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{m/s}}=\mathrm{kg}\cdot\mathrm{s}$$

(b) To convert $1\,/\mathrm{m}$ to kilograms, we can multiply it by something with the units $\mathrm{kg}\cdot\mathrm{m}$; that is, h/c. Thus

$$1\,/\mathrm{m}\frac{6.626\times10^{-34}}{3\times10^{8}}\,\mathrm{kg}\cdot\mathrm{m}=\boxed{2.2\times10^{-42}\,\mathrm{kg}}$$

which is much less massive than an electron.

(c) Similarly, we can multiplyby h/c^2,

$$1\,/\mathrm{s}\frac{6.626\times10^{-34}}{(3\times10^{8})^2}\,\mathrm{kg}\cdot\mathrm{s}=\boxed{7.4\times10^{-51}\,\mathrm{kg}}$$

▷ **3.**

A 0.057 nm X-ray photon "bounces off" an initially stationary electron and scatters with a wavelength of 0.061 nm. Find the directions of scatter of

(a) …the photon

(b) …the electron

(c) How fast is the electron moving after the collision?

Answer:

This is the Compton effect.

(a) The angle that the photon emerges is θ, which appears in the Compton formula we derived above:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

$$0.061 \times 10^{-9} - 0.057 \times 10^{-9} = \frac{6.626 \times 10^{-34}}{(9.11 \times 10^{-31})(3 \times 10^{8})}(1 - \cos\theta)$$

$$4 \times 10^{-12} = 2.42 \times 10^{-12}(1 - \cos\theta)$$

$$\Longrightarrow 1.65 = 1 - \cos\theta$$

$$\Longrightarrow \cos\theta = 1 - 1.65 = -0.65$$

$$\Longrightarrow \theta = \boxed{130.5^\circ}$$

(b) Let's rewrite the two momentum equations as so:

$$\gamma_u m_e u \sin\phi = \frac{h}{\lambda'}\sin\theta$$

$$\gamma_u m_e u \cos\phi = \frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta$$

The ratio of these equations is

$$\tan\phi = \frac{\frac{1}{\lambda'}\sin\theta}{\frac{1}{\lambda} - \frac{1}{\lambda'}\cos\theta} = \frac{\lambda\sin\theta}{\lambda' - \lambda\cos\theta}$$

$$= \frac{(0.057\,\text{nm})\sin 130.5^\circ}{0.061\,\text{nm} - 0.057\,\text{nm}\cos 130.5^\circ}$$

$$\tan\phi = 0.442$$

$$\implies \phi = \boxed{23.9^\circ}$$

(c) We can calculate the electron's speed by using the energy equation:

$$\frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + \gamma mc^2$$

$$\implies \frac{hc}{mc^2}\left[\frac{1}{\lambda} - \frac{1}{\lambda'}\right] + 1 = \gamma$$

$$\implies \gamma = \frac{h}{mc}\left[\frac{1}{\lambda} - \frac{1}{\lambda'}\right] + 1$$

$$= (2.424\times10^{-12})\left[\frac{1}{0.057\times10^{-9}} - \frac{1}{0.061\times10^{-9}}\right] + 1$$

$$= 1.00279$$

and the speed is

$$\gamma = \frac{1}{\sqrt{1-(u/c)^2}} = 1 - \left(\frac{u}{c}\right)^2 = \frac{1}{\gamma^2}$$

$$\implies u = c\sqrt{1 - \frac{1}{\gamma^2}}$$

$$\implies = (3\times10^8)\sqrt{1 - \frac{1}{(1.00279)^2}}$$

$$= \boxed{2.24\times10^7\,\text{m/s}}$$

▷ **4.**

A stationary muon μ^- annihilates with a stationary antimuon μ^+ (same mass, 1.88×10^{-28} kg, but opposite charge). The two disappear, replaced by electromagnetic radiation.
(a) Why is it not possible for a single photon to result?
(b) Suppose two photons result. Describe their possible directions of motion and wavelengths.

Answer:________________________________

(a) The initial momentum is zero; a single photon can't have zero momentum.
(b) To conserve momentum, the photons must move **in opposite directions**. The total energy of each photon must equal the mass of each muon.

$$\frac{hc}{\lambda} = m_\mu c^2 \implies \lambda = \frac{hc}{m_\mu c^2} = \frac{h}{m_\mu c} = \boxed{1.18 \times 10^{-14}\,\mathrm{m}}$$

▷ **5.**

How fast would you have to run to have a wavelength of 10^{-10} m? If the speed is relativistic, don't forget to use relativistic quantities!

Answer:________________________________
Massive objects actually need to move very slowly to have a relatively large wavelength, so let's ignore this relativistic red herring.

My wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{mu} \implies u = \frac{h}{m\lambda} = \frac{6.626 \times 10^{-34}\,\mathrm{J\,s}}{(100\,\mathrm{kg})(10^{-10}\,\mathrm{m})} = \boxed{6.626 \times 10^{-26}\,\mathrm{m/s}}$$