Introduction to Data Science With Probability and Statistics
Lecture 8 & 9: Random Variables

CSCI 3022 - Summer 2020
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What will we learn today?

- Refresher & Random Variable & PMF, CDF

- [ ] Counting – Permutations & Combinations
- [ ] Bernoulli Distribution
- [ ] Binomial Distribution
- [ ] Uniform Distribution

} Lec 9

- [ ] *A Modern Introduction to Probability and Statistics, Chapter 4*

Refresher

Example: Suppose that 1% of men over the age of 40 have prostate cancer. Also suppose that a test for prostate cancer exists with the following properties:

- 90% of people who have cancer will test positive
- 8% of people who do not have cancer will also test positive

What is the probability that a person who tests positive for cancer actually has cancer?

Refresher

Example: Suppose that 1% of men over the age of 40 have prostate cancer. Also suppose that a test for prostate cancer exists with the following properties:

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- 8% of people who do not have cancer will also test positive

What is the probability that a person who tests positive for cancer actually has cancer?

Let;

C : {event of having cancer}

C^c : {event of not having cancer}

P : {event of testing positive}

P^c : {event of testing negative}

Refresher

Given; Prior : $\mathbb{P}(c) = 0.01$; $\mathbb{P}(P|C) = 0.90$; $\mathbb{P}(P|C^c) = 0.08$

Evidence : positive test.

Posterior / updated believe : $\mathbb{P}(c|P) = \dfrac{\mathbb{P}(c \cap P)}{\mathbb{P}(P)}$

$$\therefore \mathbb{P}(c|P) = \frac{\mathbb{P}(c)\,\mathbb{P}(P|c)}{\mathbb{P}(c)\,\mathbb{P}(P|c) + \mathbb{P}(c^c)\,\mathbb{P}(P|c^c)}$$

$$= \frac{(0.01)(0.90)}{(0.01)(0.90) + (0.99)(0.08)} \approx 0.102 \approx \boxed{10\%}$$

Random Variables

A **discrete random variable** is a function that maps elements of the sample space Ω to a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$

<u>**Examples**</u>:

- Sum of the dice → Discrete R.V which can take values $\{2, 3, 4, \ldots, 12\}$
- Difference of the dice → Discrete R.V with values $\{0, 1, 2 \ldots, 5\}$
- Maximum of the dice → Discrete R.V with values $\{1, 2, 3, \ldots, 6\}$
- Number of coin flips until we get a Heads Discrete R.V with values $\{0, 1, 2 \ldots \infty\}$
- Number of Heads in n flips

Discrete R.V with values $\{1, 2, 3, \ldots, n\}$

Probability Mass Function

A **probability mass function** (pmf) is the map between the random variable's values and the probabilities of those values. $f(a) = P(X = a)$

Each of the random variables' values has some probability mass (or weight) associated with it.

Because the pmf is a probability function, the sum of all the masses must be 1!

For eg: A R.V 'X' is the result of a die roll

$\therefore$ X can take $\{1, 2, \ldots, 6\}$

$$\sum_{i=1}^{n} f(a_i) = 1$$

PMF: $P(X=1) = P(X=2) = \ldots P(X=6) = 1/6$

Probability Mass Function

Example: What is the pmf for the number of coin flips until a biased coin, with $P(H) = p$, comes up heads?

Probability Mass Function

Example: What is the pmf for the number of coin flips until a biased coin, with $P(H) = p$, comes up heads?

Let X = # flips till you get a Heads.

$X = \{1, 2, 3, \ldots\}$

$\Omega = \{H, TH, TTH, \ldots\}$

Coin flips are independent

So; $P(H) = p$

$P(TH) = P(T)P(H)$

$P(TTH) = P(T)P(T)P(H)$

$\vdots$

PMF

x	$P(X=x)$
1	p
2	$(1-p)p$
3	$(1-p)^2 p$
$\vdots$	$\vdots$

Cumulative Distribution Function

A **cumulative distribution function** (cdf) is a function whose value at a point *a* is the cumulative sum of probability masses up until that point *a*: $F(a) = P(X \leq a)$

Example: If I roll a single fair die, what is the cdf?

Question: What is the relationship between the pmf and the cdf?

Cumulative Distribution Function

A **cumulative distribution function** (cdf) is a function whose value at a point *a* is the cumulative sum of probability masses up until that point *a*: $F(a) = P(X \leq a)$

Example: If I roll a single fair die, what is the cdf? $X = \{1, 2, 3, 4, 5, 6\}$

- $P(X \leq 1) = P(X=1) = 1/6$ $F(1)$
- $P(X \leq 2) = P(X=1) + P(X=2) = 2/6$ $F(2)$

$\vdots$

$P(X \leq 6) = P(X=1) + \ldots + P(X=6) = 1$ $F(6)$

Question: What is the relationship between the pmf and the cdf?

$$F(a) = \sum_{x \leq a} P(X=a)$$

Cumulative Distribution Function

Example: What is the probability that I roll two dice and they add up to at least 9?

Cumulative Distribution Function

Example: What is the probability that I roll two dice and they add up to at least 9?

Let X : sum of dice results.

$$P(X \geq 9) = \underset{4/36}{P(X=9)} + \underset{3/36}{P(X=10)} + \underset{2/36}{P(X=11)} + \underset{1/36}{P(X=12)} = \boxed{\frac{10}{36}}$$

OR;

$$P(X \geq 9) = 1 - P(X < 9)$$
$$= 1 - P(X \leq 8)$$
$$= 1 - F(8)$$
$$= 1 - \frac{26}{36} = \boxed{\frac{10}{36}}$$

R.V Problem

Example: Suppose that you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice.

- What are the possible values that X can take?
- Which elements of the sample space map to which values of X?
- What is the pmf of the random variable X?

R.V Problem

Example: Suppose that you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice.

- What are the possible values that X can take? $X = \{1, 2, 3, 4, 5, 6\}$
- Which elements of the sample space map to which values of X? $\Omega = \{(1,1), (1,2) \ldots (6,6)\}$
- What is the pmf of the random variable X?

$P(x=1) = 1/36$; $P(x=2) = \cancel{3}/36$, $P(x=3) = 5/36$ ....

(1,1) (2,1) (2,2) (1,1) (1,3) (3,1) (2,3) (3,2) (3,3)

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

R.V Problem

<u>Example</u> (continued): Suppose that you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice.

- What is the probability that X is an even number?
- What is the probability that X is 3 or smaller?
- What is the complete cdf of X?

R.V Problem

Example (continued): Suppose that you roll two fair, six-sided dice. Let X be a random variable representing the maximum of the two dice.

- What is the probability that X is an even number? $\longrightarrow P(X=2)+P(X=4)+P(X=6)$ from table $= 21/36$
- What is the probability that X is 3 or smaller? $F(3) = \frac{9}{36}$
- What is the complete cdf of X?

x	1	2	3	4	5	6
$p(x)$	1/36	3/36	5/36	7/36	9/36	11/36
$F(x)$	1/36	4/36	9/36	16/36	25/36	36/36

Permutations

How many ways are there to order a set of 1 object?

How many ways are there to order a set of 2 objects?

How many ways are there to order a set of 3 objects?

What is a formula for the number of ways to order n objects?

Permutations

How many ways are there to order a set of 1 object? $\underline{\underline{1}}$ way

How many ways are there to order a set of 2 objects? AB, BA = $2! = \underline{\underline{2}}$ ways

How many ways are there to order a set of 3 objects? $3! = 1 \times 2 \times 3 = \underline{\underline{6}}$ ways

ABC, ACB, BCA, BAC, CAB, CBA

What is a formula for the number of ways to order *n* objects? $n!$

Permutations

- Counting **permutations** means counting the number of ways that a set of objects can be ordered.

Example: Parker, Nocona, Tatum, and Madeline line up at the ice cream truck. How many different orders could they stand in?

Permutations

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Example: Parker, Nocona, Tatum, and Madeline line up at the ice cream truck. How many different orders could they stand in?

There are 4 persons; Ans = $4! = 1 \times 2 \times 3 \times 4 = \boxed{24}$

$\underline{4}\ ,\ \underline{3}\ ,\ \underline{2}\ ,\ \underline{1}\ .$

all 4 are possible

all 3 possible except the first one

Permutations

What is the general formula for *r*-permutations of *n* objects?

<u>Example</u>: How many 3-character strings can we make if each character is a distinct letter from the English alphabet?

Permutations

What is the general formula for *r*-permutations of *n* objects? $P(n,r) = \frac{n!}{(n-r)!}$

Example: How many 3-character strings can we make if each character is a distinct letter from the English alphabet?

$\underline{26}, \underline{25}, \underline{24}$ $\therefore$ $26 \times 25 \times 24$ ways

$$P(26,3) = \frac{26 \times 25 \times 24 \times 23 \times 22 \times \cdots \times 1}{23 \times 22 \times \cdots \times 1}$$

$$= \frac{26!}{(26-3)!}$$

Combinations

- Counting **combinations** means counting the number of ways that a set of objects can be combined into subsets.

Example: How many 3-character combinations can we make if each character is a distinct letter from the English alphabet?

don't care about order.

So, CAB, ABC, BAC are indistinguishable.

Permutations → $\frac{26!}{23!}$ for combination divide it by all permutation size of 3 = 3!

$$= \frac{26!}{23! \times 3!}$$

Combinations

- Counting **combinations** means counting the number of ways that a set of objects can be combined into subsets.

Example: How many 3-character combinations can we make if each character is a distinct letter from the English alphabet?

General : $\boxed{C(n,r) = \dfrac{n!}{r!\,(n-r)!} = \dfrac{P(n,r)}{r!}}$

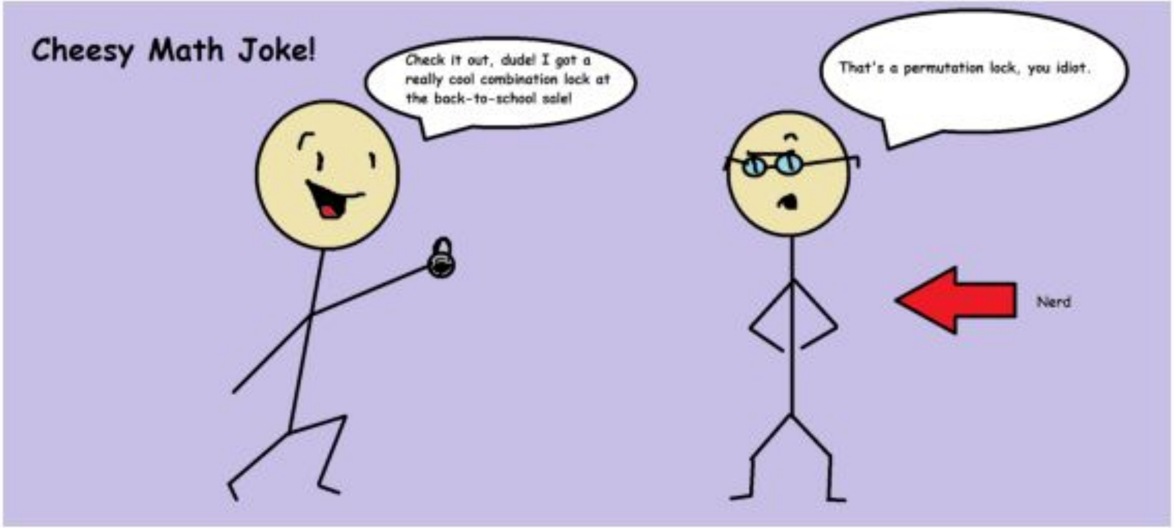
Combinations & Permutations

Combinations: Order does not matter.

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Permutations: Order does matter.

$$P(n,r) = \frac{n!}{(n-r)!}$$



Combinations

Example: If there are 10 problems on an exam, and you need at least 7 correct to pass, how many different ways are there to pass?

Combinations

Example: If there are 10 problems on an exam, and you need at least 7 correct to pass, how many different ways are there to pass?

To pass, no. of correct answers need to be 7, 8, 9 or 10

But there are multiple ways to select 7 correct problems. $= C(10, 7)$

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Final ans;

$$C(10,7) + C(10,8) + C(10,9) + C(10,10)$$

Combinations

Example: A coin is flipped 10 times. How many possible outcomes have exactly 2 Heads?

Example: A coin is flipped 10 times. How many possible outcomes have 2 Heads or fewer?

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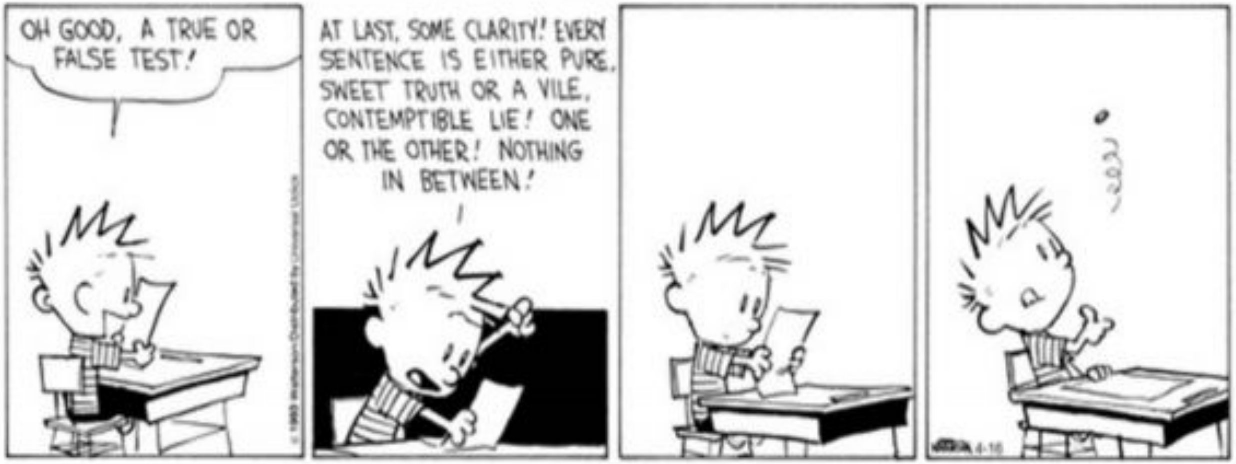
The Bernoulli Distribution

- The Bernoulli distribution is used to model experiments with only two possible outcomes; often referred to as “success” and “failure”, and encoded as 1 and 0, respectively.



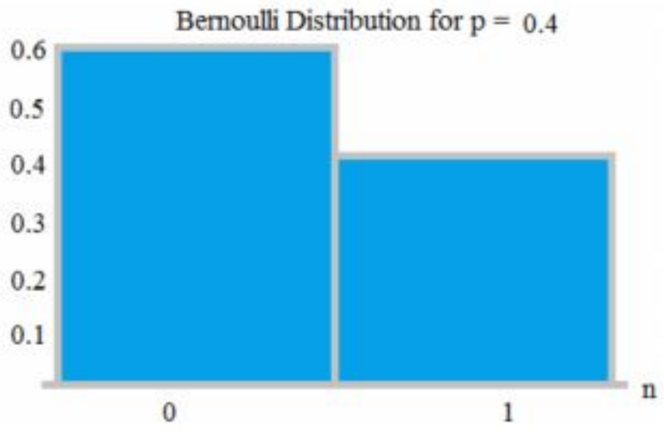
The Bernoulli Distribution

A discrete random variable X has a **Bernoulli distribution** with parameter p, where $0 \leq p \leq 1$, if its probability mass function is given by

$$f(1) = p_X(1) = P(X = 1) = p \quad \text{and} \quad p_X(0) = P(X = 0) = 1 - p$$

We denote this distribution by Ber(p).

Example:



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Describing the pmf with a single equation:

$$p_X(1) = p$$
$$p_X(0) = 1 - p$$

$$p_X(x) = \ ?$$

Sum of Bernoulli Random Variables

Example: Suppose that you show up to a quiz completely unprepared. The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way. What is the probability that you get 3 questions correct?

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Sum of Bernoulli Random Variables

<u>Example</u> (continued): What is the probability that you get 0 questions correct?

What is the probability that you get 1 problem correct?

Sum of Bernoulli Random Variables

<u>Example</u> (continued): What is the probability that you get 0 questions correct?

What is the probability that you get 1 problem correct?

Sum of Bernoulli Random Variables

Example (continued): What is the probability that you get 3 questions correct?

Sum of Bernoulli Random Variables

<u>**Example**</u> **(continued):** What is the probability that you get 3 questions correct?

Sum of Bernoulli Random Variables → A Binomial Distribution!

Example : What is the probability that you get *k* questions correct out of *n* problems total?

Sum of Bernoulli Random Variables → A Binomial Distribution!

Example : What is the probability that you get k questions correct out of n problems total?

Binomial Distribution

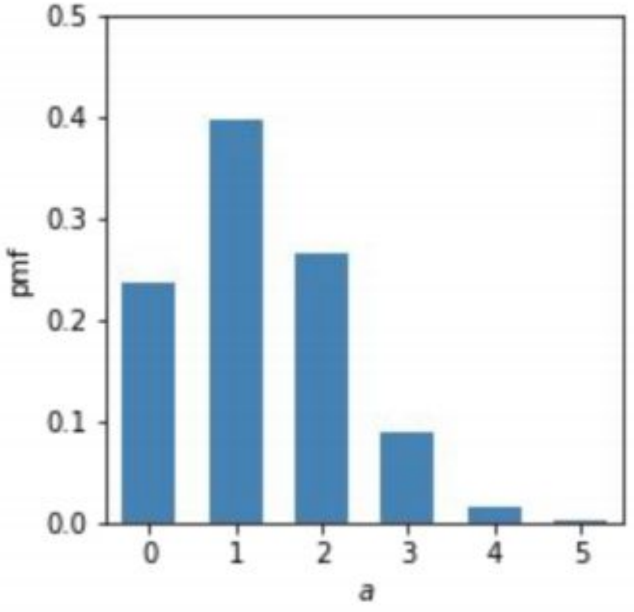
A discrete random variable X has a **Binomial Distribution** with parameters *n* and *p*, where $n = 1, 2, \dots$ and $0 \leq p \leq 1$, if its probability mass function is given by

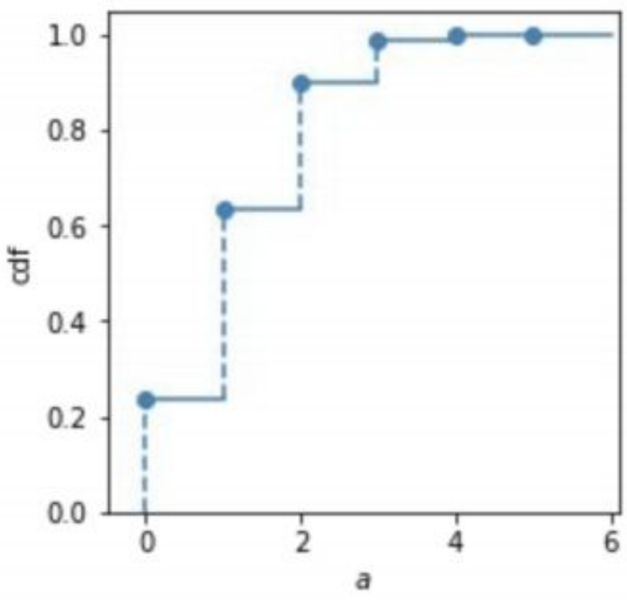
$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \dots, n$$

We denote this distribution by Bin(n, p).

Binomial Distribution

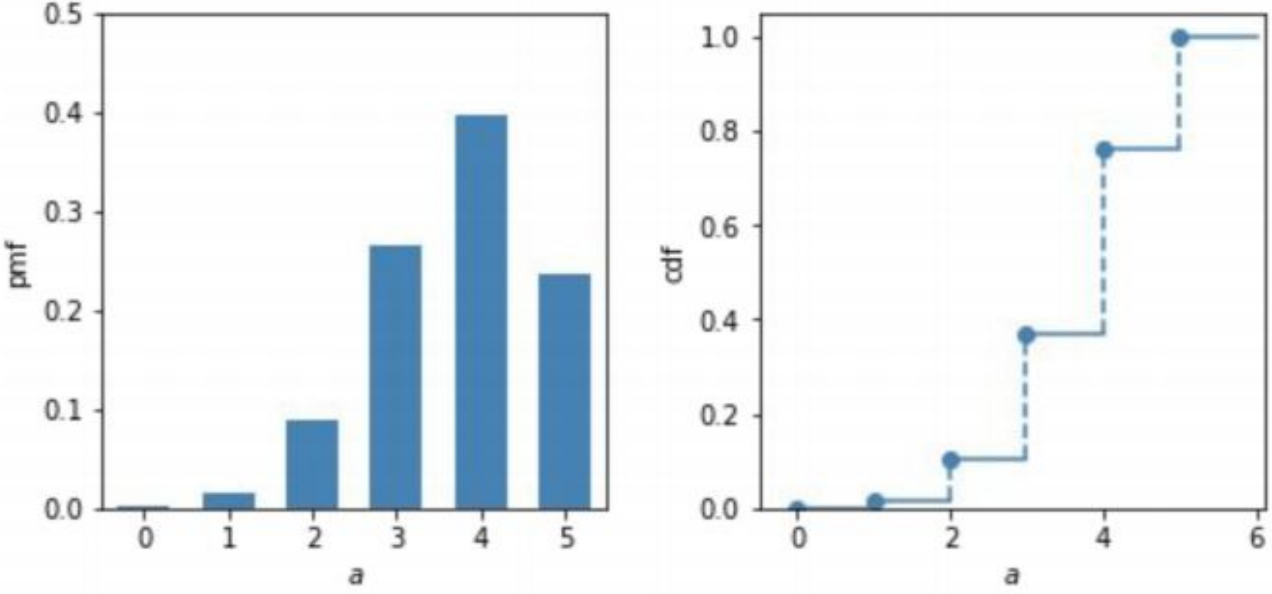
Example: n=5, p=0.25





Binomial Distribution

Example: n=5, p=0.75



Discrete Uniform Distribution

A discrete random variable X has a **Discrete Uniform Distribution** with parameters *a* and *b*, where $n = b - a + 1$ if:

$$p_X(k) = \frac{1}{n} \quad \text{for } k = a, a+1, a+2, \dots, b$$

Example : What is the distribution of a fair die?

Discrete Uniform Distribution

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Example : What is the distribution of a fair die?