Introduction to Data Science With Probability and Statistics

Lecture 6: Independence, Total Probability Theorem, Bayes' Theorem.

CSCI 3022 - Summer 2020
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What will we learn today?

- Independence (Cont.)
- Multiplication Rule (Cont.)
- The law of total probability
- Bayesian Thinking/Bayes' theorem.
- Numerical Summary Stats notebook review.

Probability - Review

Conditional Probability: The probability that A occurs given that C has occurred is

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

Product Rule: $P(A \cap C) = P(A \mid C)\, P(C)$

Independence: Events A and B are independent if and only if

1) $P(A \mid B) = P(A)$
2) $P(B \mid A) = P(B)$
3) $P(A \cap B) = P(A)P(B)$

Law of Total Probability: If $C_1, C_2, \dots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \dots \cup C_m = \Omega$.
Then the probability of an arbitrary event A can be expressed as:

$$P(A) = P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \dots + P(A \mid C_m)P(C_m)$$

Independence - Die Rolls

Suppose you are going roll a standard, balanced 6-sided die and you are interested about 2 events.

$A = \{1, 2\}$ and $B = \{2, 3, 4, 5, 6\}$. Are these events independent?

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A = {1, 2} and B = {2, 3, 4, 5, 6}. Are these events independent?

$P(A) = 2/6$, $P(B) = 5/6$, $P(B|A) = 1/2$, $P(A|B) = 1/5$

$P(A \cap B) = 1/6$

$$\left.\begin{array}{l} - \ P(B|A) \neq P(B) \\ - \ P(A|B) \neq P(A) \\ - \ P(A \cap B) \neq P(A)P(B) \end{array}\right\} \text{Not Independent.}$$

Independence Properties

If A and B are two independent events, Then the following hold as well. The intuition we can have is that if the occurrence of an event has no effect on our belief about the other event, then the non-occurrence also does not have any effect.

1. ~A and B are also independent
2. A and ~B are also independent
3. ~A and ~B are also independent.

Exercise: Try to prove the above ones.

Multiplication Rule : Radar Detection

If an aircraft is present in a certain area, the radar correctly registers its presence with the probability 99%. If it is not present, it false registers an aircraft presence with 10% probability. We assume that an aircraft is present in that area with 5% probability. We are wondering about the following probabilities.

1. Of a False alarm, i.e falsely indicates the presence of an aircraft.
2. Of a missed detection, i.e nothing registers despite the presence of an aircraft.

Multiplication Rule : Radar Detection

Let, $A = \{\text{aircraft is present}\}$; $B = \{\text{radar registers aircraft presence}\}$

$\therefore$ $A^c = \{\text{aircraft not present}\}$; $B^c = \{\text{radar registers no aircraft presence}\}$

Given;

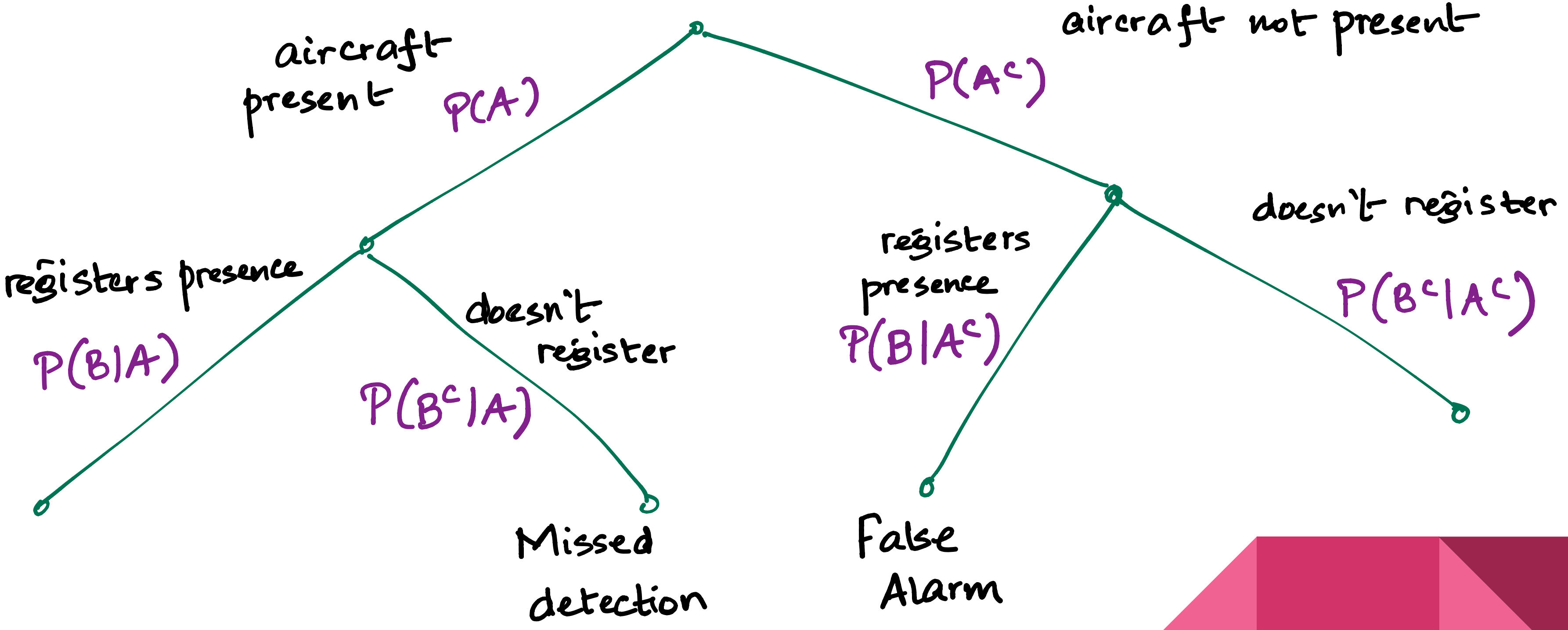
$P(B \mid A) = 0.99$; $P(B \mid A^c) = 0.10$, $P(A) = 0.05$

(i) False alarm : $P(A^c \cap B) = \underbrace{P(A^c)}_{1-P(A)} P(B \mid A^c) = 0.95 * 0.10 = \boxed{0.095}$

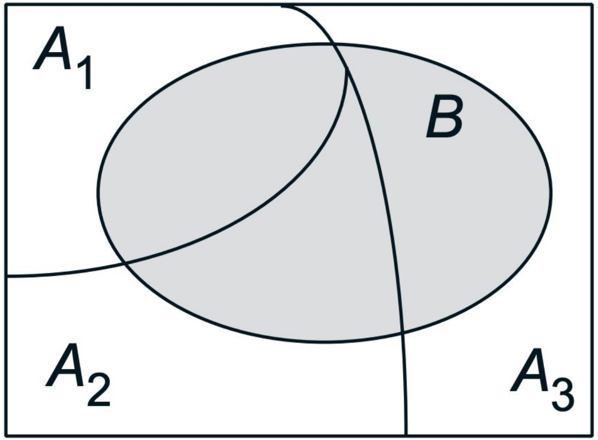
(ii) Missed detection : $P(A \cap B^c) = P(A) \underbrace{P(B^c \mid A)}_{1-P(B \mid A)}$

$= 0.05 * 0.01 = \boxed{0.0005}$

Multiplication Rule : Radar Detection



Total Probability theorem



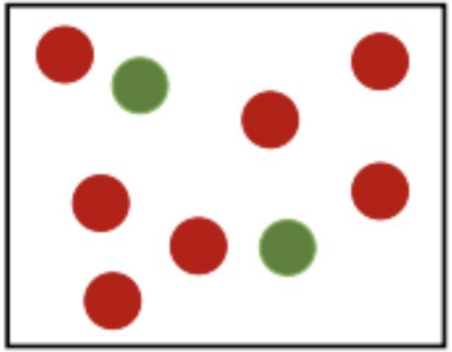
Let $A_1, A_2, A_3, ..., A_n$ be disjoint events that form a partition of the sample space and assume $P(A_i) > 0, \forall i = 1, .., n$ .Then any event B will have the probability as the following

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + ... + P(A_n \cap B)$$

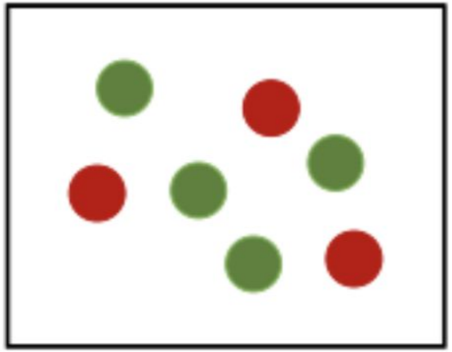
$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + ... + P(A_n)P(B|A_n)$$

Balls in A Box

You have two boxes with red and green balls. Box1 has 2 green and 7 red balls, Box2 has 4 greens and 3 reds. You randomly pick a box (Assume that all outcomes are equally likely) and then randomly pick a ball out of it (Assume that all outcomes are equally likely). What is the probability that the ball you picked was green?

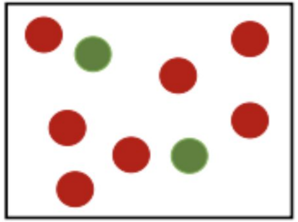


Box 1: 2 greens, 7 reds



Box 2: 4 greens, 3 reds

Balls in A Box



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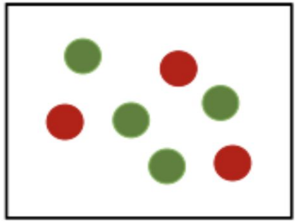
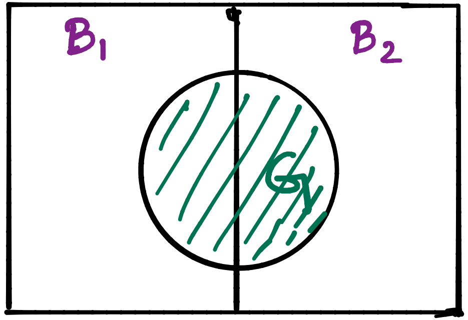


Box 2: 4 greens, 3 reds

Let $P(G)$ = prob of picking a green ball.

$P(B_1)$ = prob. of choosing Box 1.

$P(B_2)$ = prob. of choosing Box 2.



$$\therefore P(G) = P(B_1 \cap G) + P(B_2 \cap G)$$

$$= P(B_1)P(G|B_1) + P(B_2)P(G|B_2)$$

$$= \frac{1}{2}\cdot\frac{2}{9} + \frac{1}{2}\cdot\frac{4}{7} = \frac{1}{9} + \frac{2}{7}$$

$$= \frac{25}{63}$$

Chess Tournament

You enter a chess tournament where your probability of winning game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

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Let A_i be the event of playing with opponent A_i

B be the event of you winning.

Then; given;

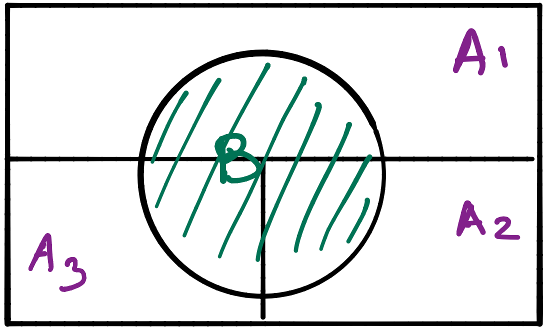
$$P(A_1) = 0.5, \quad P(A_2) = 0.25, \quad P(A_3) = 0.25$$

Chess Tournament

$P(A_1) = 0.5,\ P(A_2) = 0.25,\ P(A_3) = 0.25$

Also given;

$P(B|A_1) = 0.3;\quad P(B|A_2) = 0.4,\ P(B|A_3) = 0.5$



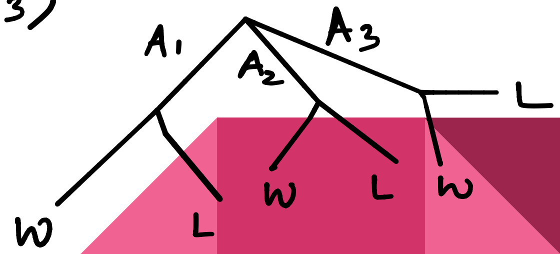
So,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(B|A_3)$$

$$= 0.5 * 0.3 + 0.25 * 0.4 + 0.25 * 0.5$$

$$= \boxed{0.375}$$



Bayes' Theorem

Let $A_1, A_2, A_3, ..., A_n$ be disjoint events that form a partition of the sample space and assume $P(A_i) > 0, \forall i = 1, .., n$.
Then any event B will have the probability as the following

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + ... + P(A_n)P(B|A_n)}$$

Bayesian Thinking

Bayes' Theorem has started an entire field of statistics: Bayesian Inference. There are various ways to interpret this celebrated theorem. But I will you two ways which I personally most relate to. In both cases, consider yourself a detective and you have to solve a murder case and assume that you have 3 suspects for the crime.

- One way to think of it is to go evidence first. You see the some evidence in the crime scene, and then you try to figure out which of the 3 suspect is most likely to be the killer.(Finding the *cause* by seeing the *effect*)
- Another direction is to have a prior belief regarding the real killer among the 3 suspects, but then a new evidence comes up in the investigation which leads you to update your beliefs accordingly.

Bayes' Theorem

Let $A_1, A_2, A_3, \ldots, A_n$ be your N suspects and $P(A_i) > 0, \forall i = 1, \ldots, n$ where P(i) is the probability that ith suspect has committed the murder. B is the evidence of that you found in the crime scene.

Prior probability of A_i being the cause

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)}$$

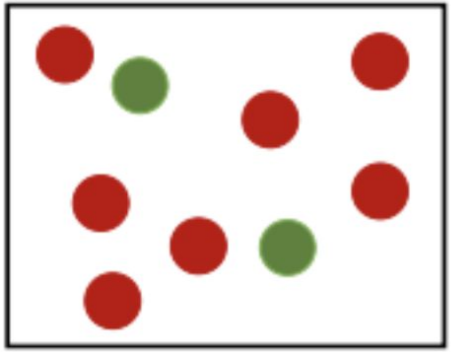
Total probability of the evidence / effect.

posterior probability of A_i after having evidence B

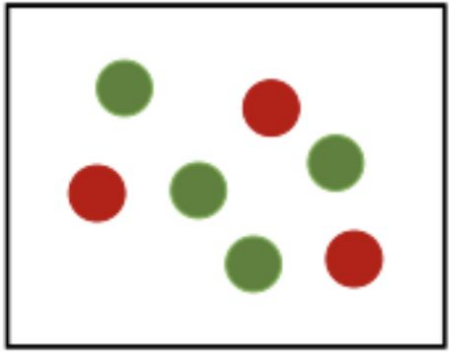
Balls Problem Revisited

You have two boxes with red and green balls. Box1 has 2 green and 7 red balls, Box2 has 4 greens and 3 reds. You randomly pick a box (Assume that all outcomes are equally likely) and then randomly pick a ball out of it (Assume that all outcomes are equally likely).

You see that the ball you picked was green. Now you want to know what is the probability that it came from Box 1?

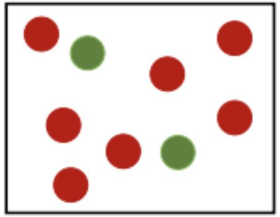


Box 1: 2 greens, 7 reds



Box 2: 4 greens, 3 reds

Balls in A Box Revisited



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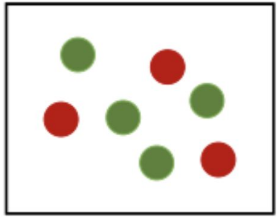


Box 2: 4 greens, 3 reds

Priors, $P(B_1) = 0.5$, $P(B_2) = 0.5$

Now;

$$P(G) = P(G \cap B_1) + P(G \cap B_2)$$
$$= P(B_1)P(G|B_1) + P(B_2)P(G|B_2) = 25/63$$

Probability of getting the ball from Box 1.

$$P(B_1|G) = \frac{P(B_1)P(G|B_1)}{P(G)} = \frac{1/9}{25/63} = \frac{1}{9} \times \frac{\cancel{63}\,7}{25} = \boxed{7/25}$$

posterior belief.

Radar Revisited

If an aircraft is present in a certain area, the radar correctly registers its presence with the probability 99%. If it is not present, it false registers an aircraft presence with 10% probability. We assume that an aircraft is present in that area with 5% probability.

You see that the radar has registered aircraft presence. You want to know what is the probability that an aircraft is actually present.

Radar Revisited

Let, $A = \{\text{aircraft is present}\}$; $B = \{\text{radar registers aircraft presence}\}$

$\therefore$ $A^c = \{\text{aircraft not present}\}$; $B^c = \{\text{radar registers no aircraft presence}\}$

Given;

$P(B|A) = 0.99$; $P(B|A^c) = 0.10$, $P(A) = 0.05$

Posterior Belief of aircraft presence;

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$= \frac{0.05 * 0.99}{0.05 * 0.99 + 0.95 * 0.1} = \boxed{0.34}$$

Chess Tournament Revisited

You enter a chess tournament where your probability of winning game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent.

Twist: Online tournament and you can’t see your opponent because of the anon option.

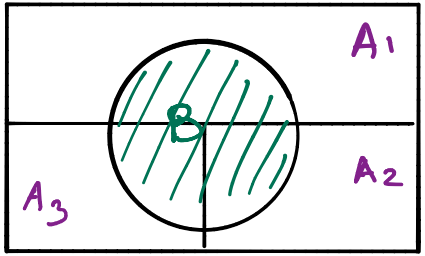
You have won! So, now you want to what is the probability that your opponent was of type 2.

Chess Tournament Revisited

$P(A_1) = 0.5, \quad P(A_2) = 0.25, \quad P(A_3) = 0.25$

Also given;

$P(B|A_1) = 0.3; \quad P(B|A_2) = 0.4, \quad P(B|A_3) = 0.5$



$$P(A_2|B) = \frac{P(A_2)\,P(B|A_2)}{P(B)}$$

$$= \frac{P(A_2)\,P(B|A_2)}{P(A_2)\,P(B|A_2) + P(A_1)\,P(B|A_1) + P(A_3)\,P(B|A_3)}$$

$$= \frac{0.25 * 0.4}{0.375} = \boxed{0.26}$$