Introduction to Data Science With Probability and Statistics
Lecture 4: An Introduction to Probability

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What will we learn today?

- Probability definitions
- Brief review of Set Notation
- DeMorgans Laws
- Probability of Disjoint Events
- Probability of Non-Disjoint Events
- Probability of Complementary Events
- Conditional Probability

- *A Modern Introduction to Probability and Statistics, Chapter 2*

Probability

Aspects of the world **seem** random and unpredictable.

- Are we short or tall?
- Do we have Mom's eyes or Dad's?
- How long will it take to drive to the airport?
- Which grocery store line should I get in?
- Is the eye of the hurricane going to pass over New Orleans?

❖ Probability is a way of thinking about unpredictable phenomena as if they were each generated from some random process.

Probability

- There are more than one school of thoughts regarding the interpretation of probabilities of events.
- When we say that the probability of getting a *heads* after tossing a coin is 0.5, we interpret that as: if we toss the same coin numerous times, roughly half of the times, we are going to see *heads.* In this case, the probability measure can be thought of as a way of representing the *frequency of occurrence* of the particular event.

Probability

- Consider another case: One of your friend is admitted to the hospital and is diagnosed a rare disease. The doctors are using a brand new drug for the treatment. You being worried, ask the nurse about it and the nurse maintains that there is no reason to worry, because there is a 90% chance that he will be cured using the drug.
- What does that mean exactly?
- This is an one-time event, unlike the coin toss experiment and the sense of frequency doesn’t fit.
- In this case, it mainly represents the *subjective belief* of the nurse.

Probabilistic Models

A probabilistic model is the mathematical description of an uncertain event. There are two main ingredients

1. A Sample Space (Ω), which is the set of all possible outcomes of that random experiment. An event is a collection of outcomes.
2. A probability law, which assigns a non-negative number $P(A)$ for all the possible outcomes and the events of that random experiment. The value $P(A)$ is called the probability of the event/outcome A.

Probability – Basic Definitions

The **sample space Ω** is the set of all possible outcomes of the experiment.

- ❖ Think of a random process as a trial or experiment.

<u>Example:</u> If we flip a fair coin a single time, what is the sample space?

<u>Example:</u> If we are doing a poll, and ask each person their birth month, what is the sample space?

- ➢ Observation: These are **discrete** sample spaces because there are a countably finite number of outcomes.

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Example: If we are doing a poll, and ask each person their birth month, what is the sample space?

$$\Omega = \{Jan, Feb, \ldots, Dec\}$$

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Probability – Discrete vs. Continuous

There is continuous math, like derivatives and integrals. Or the flow of water out of a faucet.

Then there is discrete math, like counting, sorting, and enumeration. Or an individual droplet of water.

Discrete Probability – Basic Definitions

The **cardinality** of a set A is the number of elements in that set; denoted |A|.

For each event A in Ω the **probability** is calculated by P(A) = $\frac{|A|}{|\Omega|}$

For each event in Ω the **probability** is a measure between 0 and 1 of how likely it is for the event to occur.

Observation: The sum of the probability of each distinct outcome in Ω is 1. Why?

Example

Experiment : Rolling of a fair die.

$\Omega = \{1,2,3,4,5,6\}$

$P(1) = 1/6$

$P(2) = 1/6$

$\vdots$

$P(6) = 1/6$

observation.

$$\sum_{i \in \Omega} P(i) = 1$$

Set Notation – Quick Review

What is the difference between the following:

{0, 1}

[0, 1]

(0, 1)

Set Notation – Quick Review

What is the difference between the following:

{0, 1} ← A set containing elem 0&1.

All real numbers in the range 0 to 1, inclusive. → [0, 1]

(0, 1) ← All real numbers in the range 0 to 1, exclusive

Set Operations

The **intersection** of two events is the subset of outcomes in **both** events.

Intersection = “and”

Question: What is $P(A \cap B)$?

The **union** of two events is the subset of outcomes in **one or both** events.

union = “or”

Question: What is $P(A \cup B)$?

Set Operations

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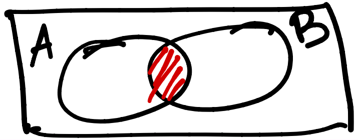

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$$= \frac{|A \cap B|}{|\Omega|}$$

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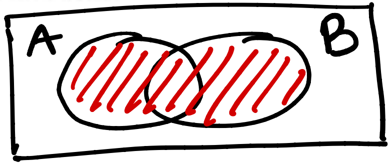

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Set Operations

The **complement** of an event A is the set of all outcomes in Ω that are **not in** A

Question: What is $P(A^c)$?

When the intersection of two events is empty, we call those two events **disjoint** or **mutually exclusive**.

Set Operations

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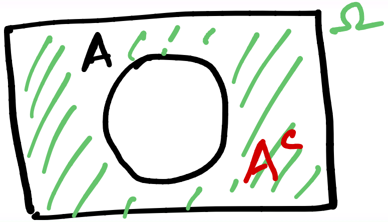
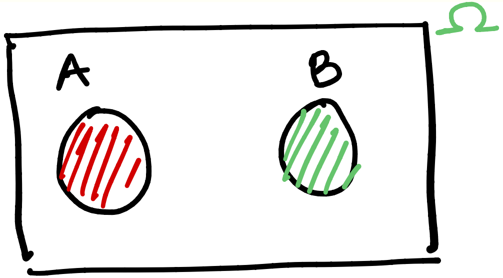


Question: What is $P(A^c)$?

$P(A) + P(A^c) = P(\Omega) = 1.$

$\Rightarrow P(A^c) = 1 - P(A)$ //

When the intersection of two events is empty, we call those two events **disjoint** or **mutually exclusive**.



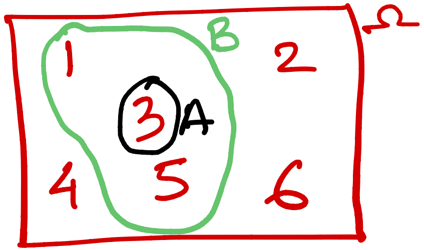
Set Operations

If all outcomes of event A are also outcomes of event B, we say that A is a **subset** of B.

Example :

B : A die roll gives odd number as a result.

A : A dieroll gives 3 as the result.



Notation

Complement: A^c or $\bar{A}$ or $\sim A$

Intersection: $A \cap B$

Union: $A \cup B$

Disjoint: $A \cap B = \phi$

Subset: $A \subseteq B$, $A \subset B$

Set Operations

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Notation

Complement:

Intersection:

Union:

Disjoint:

Subset:

De Morgan's Law for Sets

Complement of a Union: $(A \cup B)^c = A^c \cap B^c$

Complement of an Intersection: $(A \cap B)^c = A^c \cup B^c$

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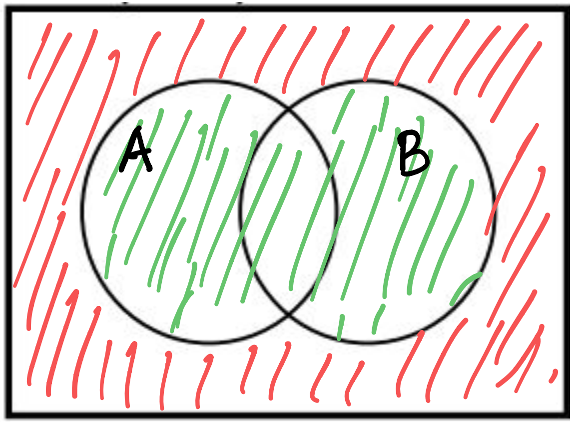
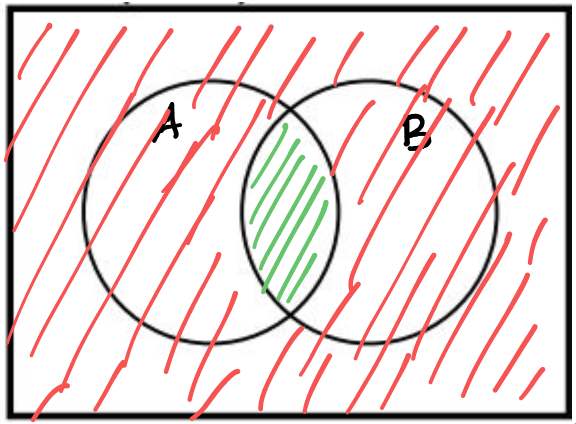


$(A \cap B)^c = A^c \cup B^c$



Probability

Example: A biased coin is a coin with a modified probability function. Instead of $P(\{H,T\}) = \{½, ½\}$, a biased coin's probability function is $P(\{H,T\}) = \{p, q\}$. What can we say about q?

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$\Omega = \{H,T\}$, $P(H) = p$, $P(T) = q$.

we know that

$P(H) + P(T) = 1.$

$\Rightarrow p + q = 1.$

$\Rightarrow \underline{q = 1 - p}$

$P(\{H,T\}) = \{p, 1-p\}$

Probability Functions

A random process with two outcomes with fixed probabilities assigned to each outcome is called a **Bernoulli Trial.**

A **probability function** P assigns to each event A a number P(A) in [0, 1] such that

1) $P(\Omega) = 1$
2) $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint events

Recall: $\Omega = \{1,2,3,4,5,6\}$

$P(1 \cup 3) = 1/6 + 1/6 = \underline{\underline{1/3}}$

Note that a probability function has two key properties:

1) The probability of the entire sample space is 1.
2) The probability of the union of disjoint events is the sum of the probability of each event.

Probability of Non-Disjoint Events

Question: What is the probability of the union of events A and B if they are not disjoint?

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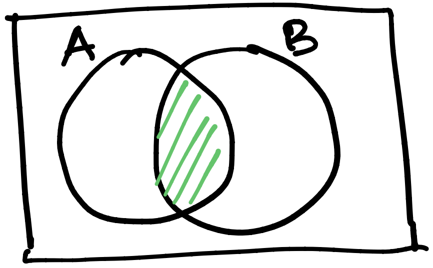

from set theory.

$$|A \cup B| = |A| + |B| - |A \cap B| \quad \text{—— } ①$$

Dividing by $|\Omega|$, ① becomes

$$\frac{|A \cup B|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability of Non-Disjoint Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: Consider rolling a fair die. Let Ω be the sample space. Let $A = \{2, 4, 6\}$ (the event of rolling an even number) and $B = \{1, 3, 6\}$ (the event of rolling a 1, 3, or 6). What is $P(A \cup B)$?

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$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 3/6 + 3/6 - 1/6$$

$$= \underline{\underline{5/6}}.$$

Probability of N~~on~~-Disjoint Events

Question: What is the probability of the complement of an event A?

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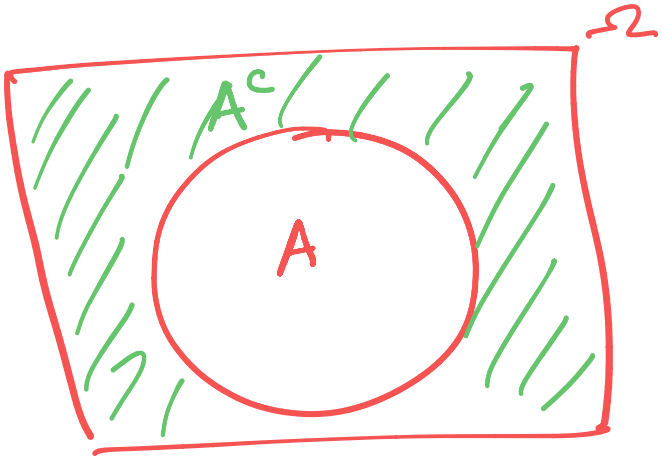

$P(\Omega) = 1.$

$\Rightarrow P(A) + P(A^c) = 1.$

$\Rightarrow P(A^c) = 1 - P(A)$

Probability of the Complement

$$P(A^c) = 1 - P(A)$$

Example: $A = \{2, 4\}$, $\Omega = \{1, 2, 3, 4, 5, 6\}$. Find $P(A^c)$.

Probability of the Complement

$$P(A^c) = 1 - P(A)$$

Example: $A = \{2,4\}$, $\Omega = \{1,2,3,4,5,6\}$. Find $P(A^c)$.

$A^c = \{1,3,5,6\} \Rightarrow P(A^c) = \underline{\underline{4/6}}$.

$P(A) = 2/6$.

$\Rightarrow P(A^c) = 1 - P(A) = 1 - 2/6 = \underline{\underline{4/6}}$

Conditional Probability

Conditional Probability provides us with a way to reason about the outcome of an experiment, based on *partial information*. Here are some examples:

- In an experiment involving 2 successive die rolls, we are told that the sum of the 2 rolls is 9. How likely it is that the first roll was 6?
- How likely it is that the person has a disease given that the medical test was negative?
- In an experiment involving die roll, we are told that the value is even. How likely it is that the actual value is 4?

Conditional Probability

In precise terms, given an experiment, a corresponding sample space, and a probability law, suppose that we know that the outcome is within some even B. We wish to quantify the likelihood that the outcome also belongs to another event A.

That is represented by the expression P(A | B). Which says that conditional probability of the event A, given that B has occurred.

Conditional Probability

<u>Example:</u> You are rolling a die once, and we are told that the outcome was an even number, what is the probability that the value was actually 4?

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Let $B = \{2, 4, 6\}$ & $A = \{4\}$.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

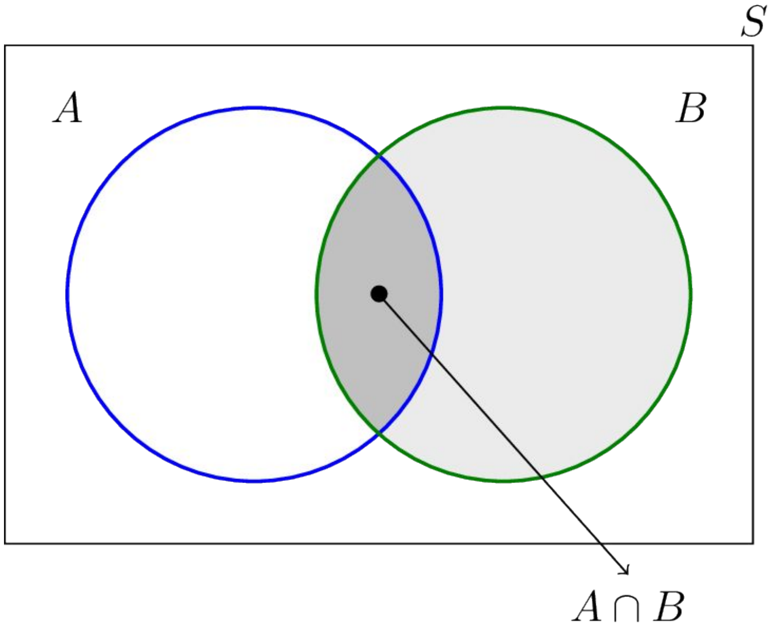
$P(A \mid B) = \frac{1}{3}$

Conditional Probability

$P(B) = 3/6 = 1/2$

$P(A \cap B) = 1/6$

$P(A|B) = \dfrac{1/6}{1/2} = \underline{\underline{1/3}}.$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability - Problem

Problem: You toss a coin thrice successively. Find that value of P(A | B) where A is the event where more heads than tails come up, and B being the event where the first toss outcome was heads.

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A = { more heads come up } , B = { 1st toss is a head }

$$\Omega = \{\underbrace{HHH, HHT, HTH, HTT}, THH, THT, TTH, TTT\}$$

$$P(B) = 4/8 = 1/2$$

$$P(A \cap B) = 3/8$$

$$\therefore \quad P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = \boxed{3/4}$$

Next Time:

More Probability!