# Introduction to Data Science With Probability and Statistics

## Lecture 12: Expectation of Discrete and Continuous Random Variables

CSCI 3022 - Summer 2020

Sourav Chakraborty
Dept. of Computer Science
University of Colorado Boulder

# What will we learn today?

- ❑ Expected Value
- ❑ Expectation of Random Variables
- ❑ Expectation of Functions of Random Variables
- ❑ Linearity of Expectation
- ❑ Expected Values and their relationship to the pmf/pdf

- ❑ *A Modern Introduction to Probability and Statistics, Chapter 7*

# Review from Last Time

A **probability mass function** (pmf) is the map between the random variable's values and the probabilities of those values.

$$f(a) \;=\; P(X = a)$$

A random variable X is **continuous** if for some function $f: \mathbb{R} \to \mathbb{R}$ and for any numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

The **probability density function** (pdf) must satisfy:

1) $f(x) \geq 0$ for all x
2) $\int_{-\infty}^{\infty} f(x)\, dx \;= 1$

# Expected Value: Discrete Random Variables

**Intuition**: Think of masses of weight $p(a_i)$ placed at the points $a_i$, $E[X]$ is the balance point.

Center of Mass (from Calculus/Physics) is an analogous concept.

Example: Suppose that I write the homework questions as either: easy (takes 10 minutes), medium (60 minutes), or hard (120 minutes). The probability that each question is easy/medium/hard is: 0.2, 0.3, 0.5 respectively.

If a homework consists of 5 questions, what's the average time it takes to do the homework?

# Expected Value: Discrete Random Variables

The **expectation** or **expected value** of a discrete random variable X that takes the values $a_1, a_2, \ldots$ and with pmf $p$ is given by

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

**Example**: Suppose that I write the homework questions as either: easy (takes 10 minutes), medium (60 minutes), or hard (120 minutes). The probability that each question is easy/medium/hard is: 0.2, 0.3, 0.5 respectively.

If a homework consists of 5 questions, what's the average time it takes to do the homework?

# Expected Value: Discrete Random Variables

The **expectation** or **expected value** of a discrete random variable X that takes the values $a_1, a_2, \ldots$ and with pmf $p$ is given by

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

**Example**: Suppose that I write the homework questions as either: easy (takes 10 minutes), medium (60 minutes), or hard (120 minutes). The probability that each question is easy/medium/hard is: 0.2, 0.3, 0.5 respectively.

If a homework consists of 5 questions, what's the average time it takes to do the homework?

Let $X$ = time taken to solve one HW question.

$\therefore$ we want $= 5E[X] = 5\left((0.2)(10) + (0.3)(60) + (0.5)(120)\right)$

$= \underline{\underline{400 \text{ mins}}}$.

# Expected Value: Discrete Random Variables

**Example**: Let X be a Bernoulli random variable with parameter $p$. What is $E[X]$?

# Expected Value: Discrete Random Variables

**Example**: Let X be a Bernoulli random variable with parameter $p$. What is $E[X]$?

$$X \sim Ber(p) = \begin{cases} 1, & p. \\ 0, & (1-p) \end{cases}$$

$$E[X] = (1)(p) + (1-p)(0)$$

$$= p.$$

# Geometric Random Variable

$$x \sim Geo(p) = (1-p)^{k-1} \cdot p.$$

$$E[x] = P(x=1)\cdot 1 + P(x=2)\cdot 2 + P(x=3)\cdot 3 + \dots \infty$$

$$= p\cdot 1 + (1-p)\cdot p\cdot 2 + (1-p)^2 p\cdot 3 + \dots$$

$$= p\sum_{i=0}^{\infty} (1-p)^i \cdot i \qquad \{\text{ Looks like a geometric series }\}$$

# Geometric Random Variable

(Cont.)

$$E[x] = 1p + 2p(1-p) + 3p(1-p)^2 + 4p(1-p)^3 + \dots \quad \text{--- Eq. } 1$$

$$(1-p)\,E[x] = 1p(1-p) + 2p(1-p)^2 + 3p(1-p)^3 + \dots \quad \text{--- Eq. } 2$$

(Multiply $(1-p)$ to Eq. 1)

Now; Eq. 1 − Eq. 2. we get:

$$p\,E[x] = p + p(1-p) + p(1-p)^2 + p(1-p)^3 + \dots \quad \text{--- Eq. } 3$$

$$\Rightarrow E[x] = 1 + (1-p) + (1-p)^2 + (1-p)^3$$

(Dividing $p$ from Eq. 3)

$$= \frac{1}{1-(1-p)}$$

$$\therefore \boxed{E[x] = \frac{1}{p}}$$

# Expected Value: Discrete Random Variables

**Example**: Suppose you and a friend are avoiding studying by each rolling a fair die. You decide that the first time that you roll the same number, you'll go back to work.

What is the expected number of times you'll roll the dice before getting a match?

# Expected Value: Discrete Random Variables

**Example**: Suppose you and a friend are avoiding studying by each rolling a fair die. You decide that the first time that you roll the same number, you'll go back to work.

What is the expected number of times you'll roll the dice before getting a match?

Success = getting same roll results.

$P(\text{success}) = 6/36 = 1/6$ ((1,1), (2,2), (3,3) ... (6,6) out of 36 out.)

X : the trial number when you get your first success.

$\therefore X \sim Geo(1/6)$

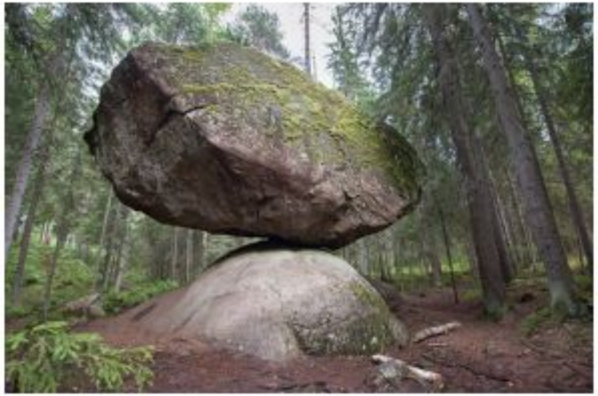
$\therefore E[X] = 1/p = 1/(1/6) = 6$ //

# Expected Value: Continuous Random Variables

The **expectation** or **expected value** of a continuous random variable X with probability density function f is:

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

**Intuition**: Think of a single big rock balancing on a fulcrum.



Kummakivi rock in Finland

# Exponential Random Variable

$$X \sim Exp(\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \underbrace{\int_{-\infty}^{0} x f(x)\,dx}_{0} + \int_{0}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$E[X] = \lambda \int_{0}^{\infty} \underbrace{x}_{u} \cdot \underbrace{e^{-\lambda x}\,dx}_{dv}$$

Integration by parts.

$$du = dx \quad ; \quad v = \int dv = \int e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x}$$

# Exponential Random Variable

(cont.) $E[x] = \lambda\left[x \cdot \frac{e^{-\lambda x}}{-\lambda} - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda}\,dx\right]$ $\left\{\because \left[uv\Big|_0^\infty - \int_0^\infty v\,du\right]\right\}$

$= \frac{x}{e^{-\lambda x}}\Big|_0^\infty + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty$

L'Hopital's rule

$\lim_{x\to\infty} \frac{1}{-\lambda e^{-\lambda x}} = 0$

$\left(0 - \left(-\frac{1}{\lambda}e^{\lambda \cdot 0}\right)\right)$

$= 1/\lambda$

$$\boxed{E[x] = \frac{1}{\lambda}}$$

# Expected Value: Continuous Random Variables

The lifetime, in years of a certain brand of battery is Exponentially distributed with parameter $\lambda = 0.25$

How long, on average will this battery last?

# Expected Value: Continuous Random Variables

The lifetime, in years of a certain brand of battery is Exponentially distributed with parameter $\lambda = 0.25$

How long, on average will this battery last?

$X \sim Exp(0.25)$ { lifetime of the battery }

$\therefore E[X] = \frac{1}{\lambda} = 4$ years.

# Expectation of Functions of Random Variables

**Change-of-Variables Formula**: Let X be a random variable and let $g\colon\ \mathbb{R} \to\ \mathbb{R}$ be a function.

If X is discrete and take the values $a_1, a_2, \ldots$ then

$$E[g(x)] = \sum_i g(a_i)P(X = a_i)$$

If X is continuous, with probability density function f, then

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

# Expectation of Functions of Random Variables

Suppose an architect is designing a community and wants to maximize the diversity in size of his square buildings that are of width X and depth X. X is uniformly distributed by 0 and 10 meters. What is the expected value of the area of the building?

# Expectation of Functions of Random Variables

Suppose an architect is designing a community and wants to maximize the diversity in size of his square buildings that are of width X and depth X. X is uniformly distributed by 0 and 10 meters. What is the expected value of the area of the building?

$X \sim U(0,10)$, $\therefore$ pdf $X$ : $p(x) = \frac{1}{10-0} = \frac{1}{10}$ $\left\{ f(x) = \frac{1}{b-a} \right\}$

$$\therefore \quad E[X] = \int_{-\infty}^{\infty} g(x) \cdot p(x)\, dx = \int_{0}^{10} x^2 \cdot \frac{1}{10}\, dx$$

$$= \frac{x^3}{3} \Big|_0^{10} \cdot \frac{1}{10} = 33.\overline{33}\, m^2$$

# Expectation of Functions of Random Variables

**Expectation is a Linear Function.**

$$E[aX + b] = aE[X] + b$$

Proof:

$$E[aX+b] = \sum_x (ax+b)\, p_X(x)$$

$$= a\sum x\, p_X(x) + b\sum_x p_X(x)$$

$$= a\, E[X] + b \qquad \left\{ \because \sum_x p_X(x) = 1 \right\}$$

# Next Time:

- Variance