Oct 7 Decision-making

Let's recap last time! What were the EVPI and EVIU?

Announcements and To-Dos

No min form

last time

Announcements:

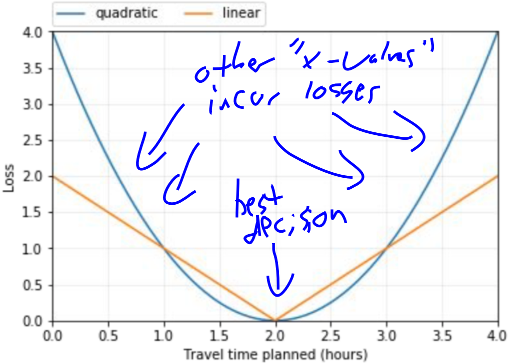
1. "Exam" should be posted tonight.

Last time we learned:

1. Started with Decision-making criteria

yes for today!

Full decision-making for an agent both requires us specify *what we value* but also uncertainty.



Linear and Quadratic Loss functions.

A *loss function* determines the falloff in utility as we move further and further away from the sweet spot. Suppose the proper value is d. Then the loss incurred when the true value is x might be:

1. **Linear:**

$$L_l(d,x) = \begin{cases} a(x-d) & x \geq d \\ b(d-x) & x < d \end{cases}$$

2. **Quadratic:**

$$L_q(d,x) = k(d-x)^2$$

3. Or something else! (Normal / Gaussian)

Probability Theory and Loss

if we took decision d:

$Var[X] = E[(X-\bar{x})^2] = \int (x-\bar{x})^2 f(x)\,dx \rightarrow f(x)$

computed $d - f(x) =$ error dist.

We also need a probability distribution to describe the risks associated with our actions.

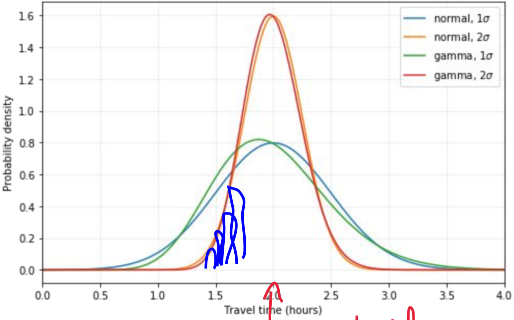
Properties of the population *mean* μ_X or $E[X]$:

1. For a random variable X with pdf x, $E[X] = \int_x x f(x)\,dx$. or $(\sum x\, P(X=x))$
2. We think of $E[\cdot]$ as the average value of an underlying population.
3. It's the unique minimizer c of $f(c) = \int_x (c-x)^2 f(x)\,dx$ ← pdf

↑ like variance!

Sample mean: minimizes $\sum_{data} (X_i - \boxed{\bar{X}})^2$

no better choice of #.



Uncertainly in travel time

centered or average around "best decision" true value.

EVPI and EVIU

We introduce two major measures for how and when to include uncertainly in a model:

1. The **Expected Value of Perfect Information** (EVPI) focuses on how much uncertainty is driving our loss: how much do we expect to lose *because* there is uncertainty.
2. The **Expected Value of Including Uncertainty** (EVIU) is a measure for our decisions: how much better do they get if we describe and use uncertainty at all.
3. For quantifying both, we need to set up an overall framework. Consider 3 types of decisions:

EVPI and EVIU

To understand EVPI and EVIU, we have to understand how our decisions might change when faced with uncertainty.

1. d_{Bayes} will be the "best" on-average decision, called the **Bayes' decision**. It minimizes *expected* loss. We will by trying to find this decision in some future algorithms.
2. We could also make other decisions: d_{pi} might be the decision we make with perfect information: *after* observing the "uncertain" elements.
3. We could also make other decisions: d_{iu} might be the decision we make ignoring all uncertainty.

EVIU

X, Y random

$$E[aX + bY] = aE[X] + bE[Y]$$

The **expected value of including uncertainty** is the difference in expected (loss) value between the decision we make with when we include uncertainty and the decision we make while ignoring it.

ignoring uncertainty — using distribution $f(x)$

$$EVPI = E[\text{Loss from } d_{iu} - \text{Loss from } d_{Bayes}]$$

$$=$$

$$=$$

$$=$$

Where x holds all of the possible states, and $f(x)$ is the uncertainty pdf for x: our probability (density) for each of those states.

EVIU

The **expected value of including uncertainty** is the difference in expected (loss) value between the decision we make with when we include uncertainty and the decision we make while ignoring it.

$$
\begin{aligned}
EVPI &= E[\text{Loss from } d_{iu} - \text{Loss from } d_{Bayes}] \\
&= \int_x \left(L(d_{iu}, x) - L(d_{Bayes}, x)\right) f(x)\, dx \\
&= \int_x L(d_{iu}, x) f(x)\, dx - \int_x L(d_{Bayes}, x) f(x)\, dx \\
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&= \int_x L(d_{iu}, x) f(x)\, dx - \int_x L(d_{Bayes}, x) f(x)\, dx \\
&= E_x[L(d_{iu}, x)] - E_x[L(d_{Bayes}, x)]
\end{aligned}
$$

Expected loss of:

d_{iu} vs. d_{Bayes}

Where x holds all of the possible states, and $f(x)$ is the uncertainty pdf for x: our probability (density) for each of those states.

EVPI

The **expected value of perfect information** is the difference in expected (loss) value between the decision we make with perfect information and the decision we make "just doing our best."

best guess using f(x)

best guess with perfect information "of the f(x)"

$$EVPI = E[\text{Loss from } \underline{d_{Bayes}} - \text{Loss from } d_{pi}]$$

$$=$$

$$=$$

$$=$$

Where again, x holds all of the possible states, and $f(x)$ is the uncertainty pdf for x: our probability (density) for each of those states.

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&= E_x[L(d_{Bayes}, x)] - E_x[L(d_{pi}, x)]
\end{aligned}
$$

Where again, x holds all of the possible states, and $f(x)$ is the uncertainty pdf for x: our probability (density) for each of those states.

EVUI

(all of)

→ translates "there is NO value to including f(x) if Loss is quadratic"!

Property: The EVUI for the quadratic loss function is zero!

Proof:

EVUI

Property: The EVUI for the quadratic loss function is zero!

Proof:

1) Recall the definition:

$$EVIU = E_x[L(d_{iu}, x)] - E_x[L(d_{Bayes}, x)]$$

2) What's the Bayes' decision? d_{Bayes} is the decision that *minimizes expected loss*. So we choose d_{bayes} to be the decision d such that $E_x[L(d, x)]$ is as small as possible. Then:

minimize $\underline{\text{Expected}}$ $[K(d-x)^2]$

minimize over "d" $\int K(d-x)^2 f(x)\,dx$

$= E[K(d-x)^2]$

EVUI

Property: The EVUI for the quadratic loss function is zero!

Proof:

1) Recall the definition:

$$EVIU = E_x[L(d_{iu}, x)] - E_x[L(d_{Bayes}, x)]$$

2) What's the Bayes' decision? d_{Bayes} is the decision that *minimizes expected loss*. So we choose d_{bayes} to be the decision d such that $E_x[L(d, x)]$ is as small as possible. Then:

FOIL

E[d] = d.

related to variance

$$E_x[L(d, x)] = \boxed{E_x[k(d-x)^2]}$$
$$= k \cdot E_x[d^2 - 2dx + x^2] = k\left(d^2 - 2dE_x[x] + \boxed{E_x[x^2]}\right)$$

EVUI

E[x] doesn't depend on d.

Property: The EVUI ($E_x[L(d_{iu}, x)] - E_x[L(d_{Bayes}, x)]$) for the quadratic loss function is zero!

Proof: We have that $E_x[L(d, x)] = k\left(d^2 - 2dE_x[x] - E_x[x^2]\right)$.

d/dd

depends on f(x) not on decision

Choose "d" to minimize.

$$\frac{d\, E_x[L(d,x)]}{d\, d} = k\left(2d - 2E_x[x] + 0\right)$$

set

$$0 = k\left(2d - 2E_x[x]\right)$$

$\Rightarrow \quad 2d = 2E_x[x]$ or $d = E_x[x]$

EVUI

Property: The EVUI $(E_x[L(d_{iu}, x)] - E_x[L(d_{Bayes}, x)])$ for the quadratic loss function is zero!

Proof: We have that $E_x[L(d, x)] = k\left(d^2 - 2dE_x[x] - E_x[x^2]\right)$. We're choosing d, so we differentiate with respect to d and set equal to zero.

$$\frac{dE_x[L(d,x)]}{dd} = \frac{d}{dd} k\left(d^2 - 2dE_x[x] + E_x[x^2]\right)$$
$$= 2kd - 2kE_x[x]$$
$$0 \stackrel{Set}{=} 2kd - 2kE_x[x]$$
$$\implies d = E_x[x]$$

So we have $d_{Bayes} = E_x[x]$: when the loss is quadratic, the Bayes' decision is the ~~once~~ one that returns the average value of the error of X itself!

EVUI

Property: The EVUI ($E_x[L(d_{iu}, x)] - E_x[L(d_{Bayes}, x)]$) for the quadratic loss function is zero!

Proof: We have that $d_{Bayes} = E_x[x]$.

By a symmetric argument, $d_{iu} = E_x[x]$. Here, we choose the minimizer of the loss function $k(d-x)^2$, or the point with the lowest squared-deviation from the rest of the loss function. In this way the loss and probability *expectation* functions are the same: they're both trying to minimize "squared error," where $E_x[x]$ does so of the *distribution* and d_{iu} does so of the loss function.
Since the two decisions are the same, the difference in loss between them is then 0!

We assumed that, neglecting uncertainty, the "best" estimate of x is $E_x[x]$. If we're risk-averse and/or asymmetric loss, might make a different assumption.

quadratic

EVPI

Property: Under quadratic loss, EVPI will likely not be zero.

This is intuitive, because once we've *observed* the randomness, it might change our decisions considerably.

Proof:

$$EVPI = E_x[L(d_{Bayes}, x)] - \underbrace{E_x[L(d_{pi}, x)]}_{0,\text{ since we won't have any loss if we make the perfect choice!}}$$

$d_{Bayes} = \mu_x$

so

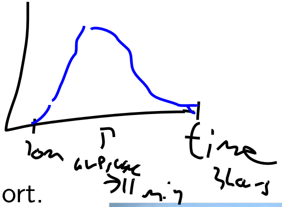
$$EVPI = E_x[E_x[x]\ x] = E_x[k(\bar{x} - x)^2]$$

or

$$EVPI = \int k(x - \mu_x)^2 f(x)\, dx$$

w/ that k: defn of Var[X].

Decision Theory in Practice.

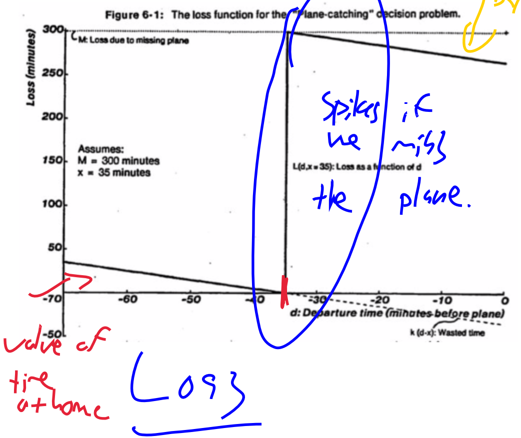


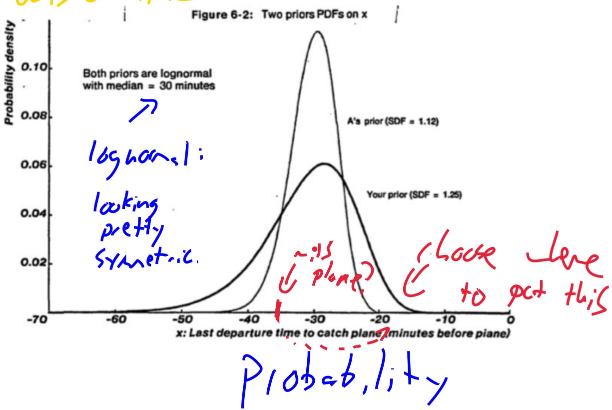
A common example for this is imagining driving to the airport.

1. The decision d represents how long you allow for the drive, security lines, check-in, and walking to the gate.
2. The state variable x represents how long it *will actually take* to get there.
3. What makes sense as a loss function?
4. What makes sense as a prior distribution on x?

Decision Theory in Practice

value of missing by lots of time



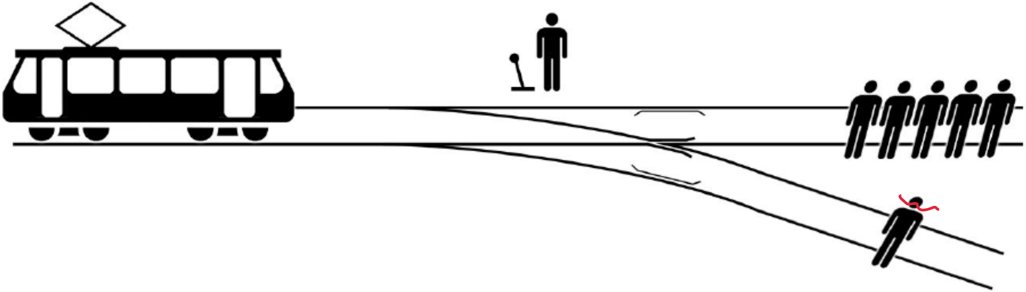


Figures from *Henrion, 1982: The Value of Knowing How Little You Know* 26

We revisit this problem with some simulations later.

Decision Theory in Practice

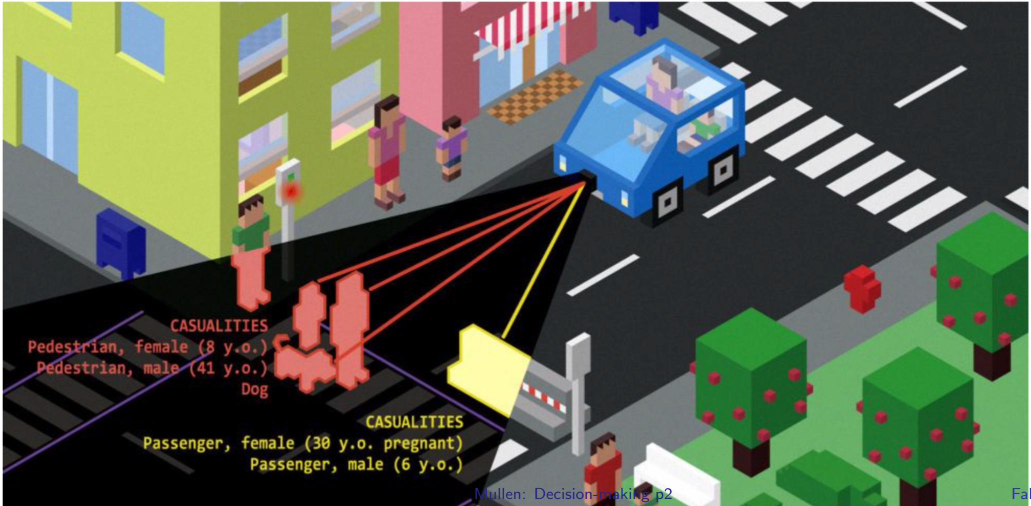
Decision-theory heavily overlap with Ethics in AI. Imagine the "trolley problem":



What do we do? What do we value?

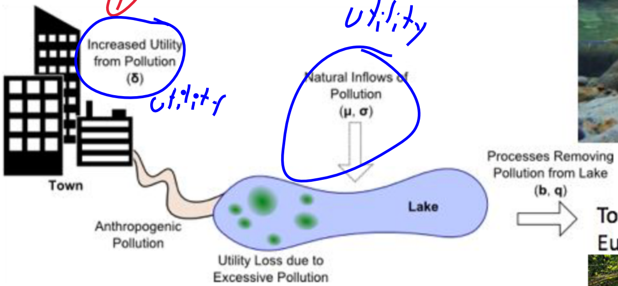
Decision Theory in Practice

The "trolley problem" is iconic in philosophy and ethics, but it's not that different than the decision that e.g. a self-driving autonomous car might have to make:



A Lake Problem

decisions !



More phosphorous discharge supports economic growth

Phosphorous concentration below threshold: Healthy lake

Too much phosphorous: Eutrophication

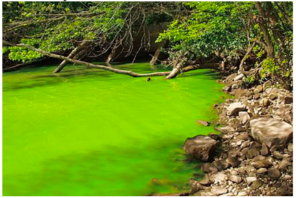
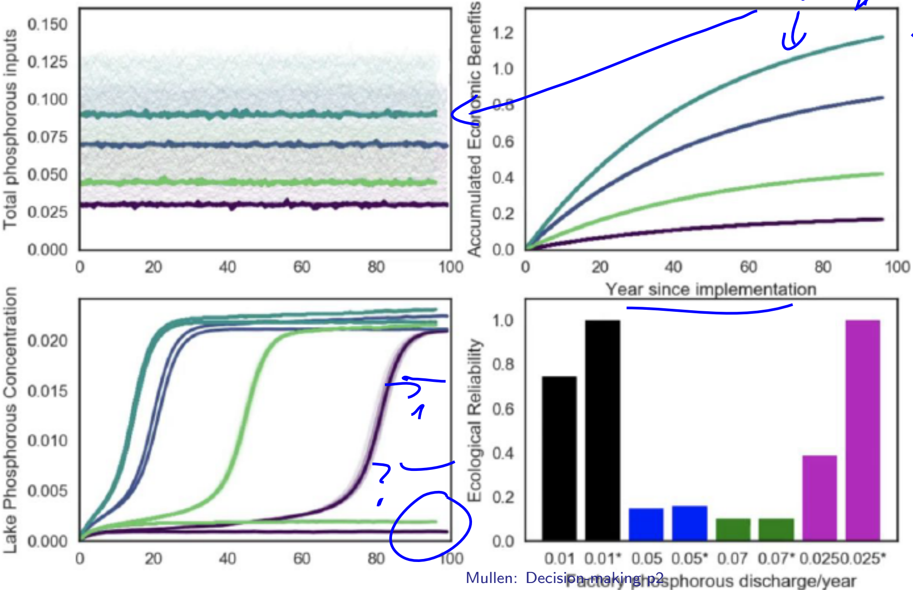


Figure adapted from Hadka et al., 2015

A Lake Problem: States

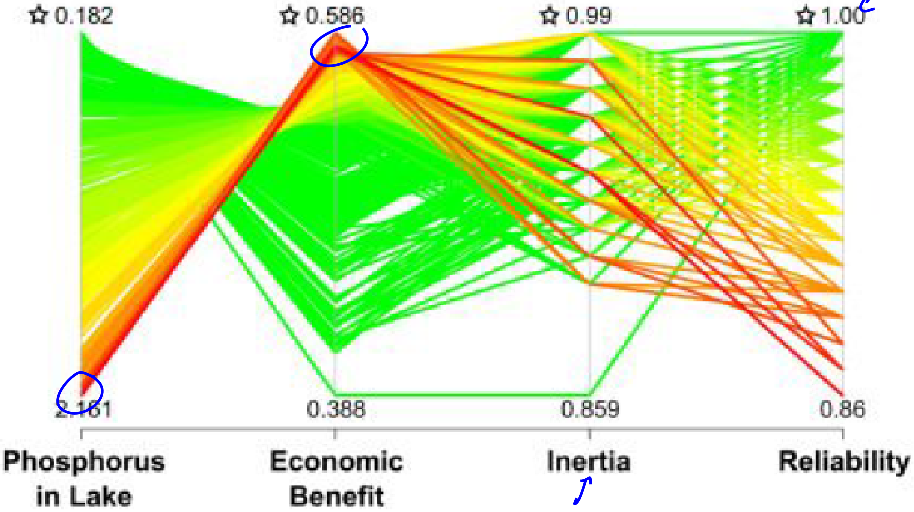


not $

different production regimes

?

A Lake Problem: Utilities



"mix"

each line is a state/set of choices

Mixing Utilities

Generally, if we want to mix some number of varying utility functions, we may use a set of weights $w_1, w_2, \ldots w_n$ for the n things we "care" about. Our eventual Bayes' decision , then is actually a *function* of those weights as well as the underlying probability distributions and loss functions (for each weight).

Consider for example breaking the lake problem down to just 2 "goals:" economic versus environmental utility and loss.

$$w_1 \cdot Econ + w_2 \cdot Envir$$

but since we're just finding a minimizer, what matters more is the *ratio* of the weights, so we often fix the all weights to sum to 1. Then we might instead write:

$$U = w \cdot U_{Econ} + (1 - w) \cdot U_{Envir}$$

Moving Forward

- This Week:
  1. nb day Friday
- Next time: Decision-Making!