Rotary Inverted Pendulum

Experiment No. 1

EE615: Control and Computational Laboratory

August 17, 2025

Group 10

Full Name	Roll No.
Sahil Patil	24M1087
Shivangi Sharma	24M0027

Contents

1 | Aim

Design a swing up and stabilization controller for the inverted pendulum.

2 | Objectives

The experiment can be performed in two steps which requires the following objectives to be achieved.

- Swing-Up Control : The pendulum begins in a stable downward position. The controller torques it to an unstable upright position
- Balancing (Stabilization) Control : Once the pendulum reaches the upright (inverted) position, it must be stabilized and balanced.

3 | Block Diagram

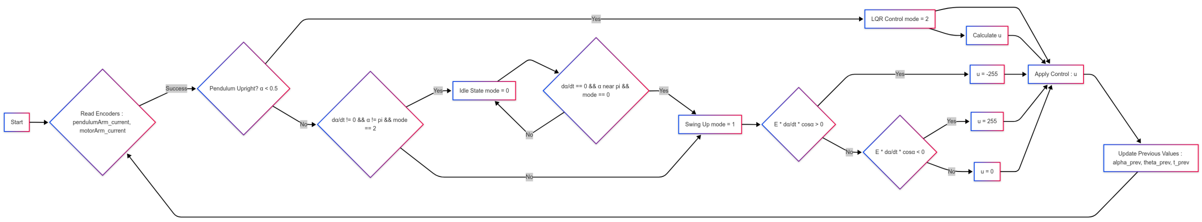

Figure 3.1: Swing-Up/Balance Closed-Loop System

4 | Procedure

4.1 | Balance Control Design

4.1.1 | Open Loop Modeling

The pendulum setup can move freely in two rotary directions. Thus, it is a two degree of freedom, or 2 DOF, system. As described in Figure 4.1, the arm rotates about the YO axis and its angle is denoted by the symbol θ and the pendulum hanging under the arm rotates about its pivot and its angle is called α. The arm pivot is attached to the shaft of the DC motor and the motor input voltage is the control variable.In the inverted pendulum experiment, the pendulum angle, α, is taken to be positive when the pendulum rotates counter-clockwise. That is, when the arm moves in the positive clockwise direction the inverted pendulum moves clockwise (i.e. the hanging pendulum moves counter-clockwise) and that is taken to be α¿0. When the arm rotates in a positive clockwise direction the pendulum moves in the clockwise direction, which in turn is taken to be positive. The pendulum model parameters are defined in Table A.1. The linear equations of motion of the system are found by linearizing the non-linear equations of motions about point $\alpha = \pi$. For the state

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \alpha \\ \frac{\partial}{\partial t}\theta \\ \frac{\partial}{\partial t}\alpha \end{bmatrix} \tag{4.1}$$

the linear state-space representation of pendulum setup is

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + B\mathbf{u}(x) \tag{4.2}$$

$$y(t) = C\mathbf{x}(t) + D\mathbf{u}(x) \tag{4.3}$$

where, A, B, C, D matrices are defined as in Appendix A

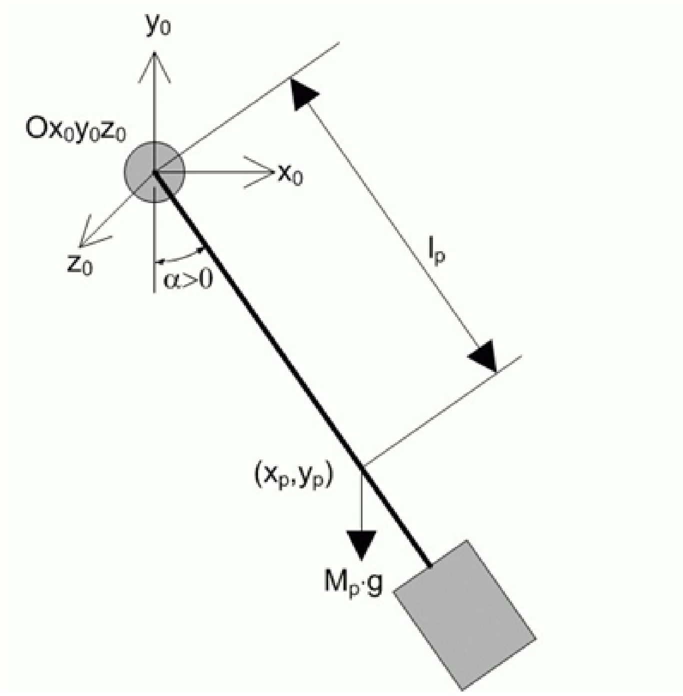

Figure 4.1: Free Body Diagram of Pendulum

4.1.2 | LQR Control Design

The *linear quadratic regulator* problem is: given a plant model

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{4.4}$$

find a control input $\mathbf{u}$ that minimizes the cost function

$$J = \int_0^\infty \mathbf{x}(t)^T Q\mathbf{x}(t) + \mathbf{u}(t)^T R\mathbf{u}(t)\, dt \tag{4.5}$$

where Q is an $n \times n$ positive semi-definite weighing matrix and R is an $r \times r$ positive definite symmetric matrix. That is, find a control gain K in the state feedback control law

$$\mathbf{u} = K\mathbf{x} \tag{4.6}$$

such that the quadratic cost function J is minimized.

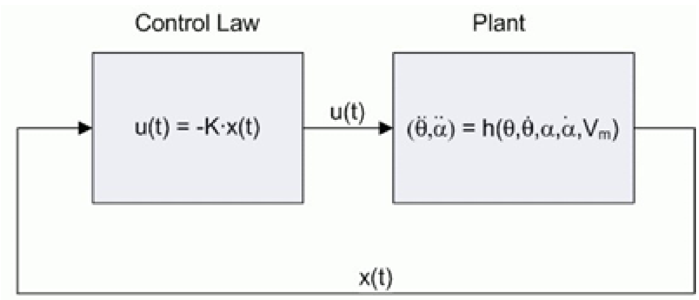

Figure 4.2: Closed Loop Control System

4.2 | Swing Up Control Design

Consider a single pendulum. Let its mass be m and let the moment of inertia with respect to the pivot point be J. Furthermore, let l be the distance from the pivot to the center of mass. The angle between the vertical and the pendulum is θ, where θ is positive in the clockwise direction. The acceleration of gravity is g and the acceleration of the pivot is u. The acceleration u is positive if it is in the direction of the positive x-axis. The equation of motion for the pendulum is

$$J\ddot{\theta} - mgl\sin\theta + mul\cos\theta = 0. \tag{4.7}$$

The system has two state variables, the angle θ and the rate of change of the angle $\dot{\theta}$.
The model given by Equation (4.7) is based on several assumptions: friction has been neglected and it has been assumed that the pendulum is a rigid body. It has also been assumed that there is no limitation on the velocity of the pivot. The energy of the uncontrolled pendulum ($u = 0$) is

$$E = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos\theta - 1). \tag{4.8}$$

Computing the derivative of E with respect to time we find

$$\frac{dE}{dt} = J\dot{\theta}\ddot{\theta} - mgl\dot{\theta}\sin\theta = -ml\dot{\theta}\cos\theta, \tag{4.9}$$

The system is simply an integrator with varying gain. Controllability is lost when the coefficient of u in the right-hand side of Equation (4.9) vanishes. This occurs for $\dot{\theta} = 0$ or $\theta = \pm\pi/2$, i.e., when the pendulum is horizontal or when it reverses its velocity. Control action is most effective when the angle θ is 0 or π and the velocity is large. To increase energy the acceleration of the pivot u should be positive when the quantity $\dot{\theta}\cos\theta$ is negative. A control strategy is easily obtained by the Lyapunov method. With the Lyapunov function $V = (E - E_0)^2/2$, and the control law

$$u = k(E - E_0)\dot{\theta}\cos\theta, \tag{4.10}$$

we find that

$$\frac{dV}{dt} = -mlk((E - E_0)\dot{\theta}\cos\theta)^2. \tag{4.11}$$

But we have given the maximum control input in either case.
Code:

```
// Swing Up
E = 0.5*Jp*alpha_dot*alpha_dot + m*g*L*(cos(alpha) - 1);

if((E)*alpha_dot*cos(alpha) > 0) {
  control_input = -255;
}
else if((E)*alpha_dot*cos(alpha) < 0) {
  control_input = 255;
}
else {
  control_input = 0;
}
```

4.3 | Switching Logic

This code snippet sets different modes for a pendulum system based on its angle (α) and angular velocity ($\dot{\alpha}$). The modes represent different states of the pendulum:

Mode 0 : (Idle)
Mode 1 : (Swing Up)
Mode 2 : (Balancing)

1. **Balancing Mode (mode = 2)**
   If the absolute value of α is less than 0.50 radians (i.e., the pendulum is near the upright position),Set mode = 2 (Balancing Mode).

```
// Setting the modes of pendulum
// 0 ==> idle
// 1 ==> Swing UP
// 2 ==> Balancing
if((abs(alpha) < 0.50)) {
  mode = 2;
}
else if((alpha_dot != 0) && ((alpha != 3.14) || (alpha != -3.14)) && (mode == 2)) {
  mode = 0;
}
else if((alpha_dot == 0) && (((alpha <= 3.14) && (alpha >= 3.13)) || ((alpha >= -3.14) && (alpha <= -3.13))) && (mode == 0)) {
  mode = 1;
}
```

2. **Transition to Idle Mode (mode = 0)**
   If the pendulum is moving ($\dot{\alpha} \neq 0$), and it is not exactly at π or $-\pi$ radians (not at the downward position), and it was previously in Balancing Mode (mode == 2), then set mode = 0 (Idle Mode).

3. **Swing-Up Mode (mode = 1)**
   If the pendulum is not moving ($\dot{\alpha} = 0$), and its angle is very close to π or $-\pi$ radians (near the downward position), and it was previously in Idle Mode (mode == 0), then set mode = 1 (Swing Up Mode).

5 | Calculations and Analysis

The linearized system is given as :

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 123.979167598469 & -1.57720932796986 & 0 \\ 0 & 111.623391037245 & -0.725262974268975 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 56.3893217007458 \\ 25.9300312573820 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D = 0$$

The Q and R matrices obtained are:

$$Q = \begin{pmatrix} 1258.925 & 0 & 0 & 0 \\ 0 & 1318256.738 & 0 & 0 \\ 0 & 0 & 15.848 & 0 \\ 0 & 0 & 0 & 3981.071 \end{pmatrix}$$

$$R = 0$$

6 | Results

For balancing the pendulum in upright position the the optimal gain values obtained are,

$$k_\theta = -35.48 \quad k_\alpha = 1415.60 \quad k_{\dot{\theta}} = -29.27 \quad k_{\dot{\alpha}} = 114.57$$

for the above Q and R matrices.

7 | Challenges Faced and Solutions

Few of the challenges encountered while performing the experiment are :

1. **Nonlinearity of Dynamics**
   Challenge: The system follows nonlinear differential equations, making traditional linear control methods insufficient in some cases.
   Solution: Use Linearization around the upright position for small deviations. Apply nonlinear control methods like feedback linearization or sliding mode control.

2. **Actuator Constraints and Saturation**
   Challenge: The motor driving the rotary arm has limited torque and speed, which can lead to saturation.
   Solution: Optimize control effort using LQR weighting to keep control actions within actuator limits.

3. **Swing-up Control (Bringing the Pendulum to Upright Position)**
   Challenge: The pendulum starts in the downward position and must be swung up before stabilization. **Solution:** Use Energy-based control to pump energy into the system until it reaches the upright position.

4. **Hardware Wear and Tear**
   Challenge: Repeated motion can lead to mechanical wear in motors and joints.

8 | Conclusion

The control of a Rotary Inverted Pendulum requires solving two key problems: swing-up control and balancing control. Due to the system's inherent nonlinearity and instability, achieving these objectives demands a well-designed combination of energy-based control, state feedback, and optimal control strategies.
For swing-up control, energy-based methods such as the energy shaping approach or trajectory planning are effective in gradually increasing the pendulum's energy to reach the upright position. Once the pendulum is near the top, a switching mechanism transitions the system to a stabilizing controller.
For balancing, advanced control strategies like Linear Quadratic Regulator (LQR) to ensure that the pendulum remains upright and responds to disturbances robustly.

A | Appendix

Symbol	Description	Value	Unit
M_p	Mass of the pendulum assembly (weight and link combined).	0.027	kg
l_p	Length of pendulum center of mass from pivot.	0.153	m
L_p	Total length of pendulum.	0.191	m
r	Length of arm pivot to pendulum pivot.	0.08260	m
J_m	Motor shaft moment of inertia.	$3.00E{-}005$	$\mathrm{kg \cdot m^2}$
M_{arm}	Mass of arm.	0.028	kg
g	Gravitational acceleration constant.	9.810	$\mathrm{m/s^2}$
J_{eq}	Equivalent moment of inertia about motor shaft pivot axis.	$1.23E{-}004$	$\mathrm{kg \cdot m^2}$
J_p	Pendulum moment of inertia about its pivot axis.	$1.10E{-}004$	$\mathrm{kg \cdot m^2}$
B_{eq}	Arm viscous damping.	0.000	$\mathrm{N \cdot m/(rad/s)}$
B_p	Pendulum viscous damping.	0.000	$\mathrm{N \cdot m/(rad/s)}$
R_m	Motor armature resistance.	3.30	Ω
K_t	Motor torque constant.	0.02797	$\mathrm{N \cdot m}$
K_m	Motor back-electromotive force constant.	0.02797	$\mathrm{V/(rad/s)}$

Table A.1: Table of Parameters for the Pendulum System

State-Space Matrix	***Expression***
A	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{rM_p^2 l_p^2 g}{J_{eq}^p + M_p l_p^2 J_{eq}^p + J_p M_p r^2} & 0 & -\frac{K_t K_p (J_p + M_p l_p^2)}{(J J_{eq}^p + M_p l_p^2 J_{eq}^p + J_p M_p r^2) R_m} & 0 \\ \frac{M_p l_p g (J_{eq}^p + M_p r^2)}{J J_{eq}^p + M_p l_p^2 J_{eq}^p + J_p M_p r^2} & 0 & \frac{M_p l_p K_r K_p}{(J J_{eq}^p + M_p l_p^2 J_{eq}^p + J_p M_p r^2) R_m} & 0 \end{bmatrix}$
B	$\begin{bmatrix} 0 \\ 0 \\ \frac{K_t (J_p + M_p l_p^2)}{(J J_{eq}^p + M_p l_p^2 J_{eq}^p + J_p M_p r^2) R_m} \\ \frac{M_p l_p K_r r}{(J J_{eq}^p + M_p l_p^2 J_{eq}^p + J_p M_p r^2) R_m} \end{bmatrix}$
C	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
D	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Table A.2: Linear State-Space Matrices