353 Digital Communication (Trilochan Sir) | 4 credit | Repeat midSem (20)

↳ 4 Modules → 1] Converting Analog to Digital (Source Coding)
→ 2] Baseband communication
→ 3] Modulation technique & ~~En~~ Coding

Internal Evaluation
- Surprise Quiz
- minor
- Assignment
- Probject* (10%) Harware or Simulation for both lab & theory

4-5 slides project records

Text Book - 1. Communication System Symon Haykin
2. Digital Communication by J. Proakish (Random press)
3. Morden Digital & analog comm. Syst, BP Lathi
4. Digital Communication: Fundamentals and application by Vish B. Sklar (Exercise.)

Objectives:
1. knowledge about random signal and their properties
2. Generalise the concept of Digital source generation.
3. Analytic knowledge on baseband and paseband communication.
4. Understand the different digital modulation scheme.
5. Able to realise an efficient Digital commu system in terms of BER and SNR
6. Basic knowledge of Coding theorm

Purpose: to transmit a signal with information to a destination

8.

Source → Transmitter → Channel → Reciver → Destination

Tx Rx

Eg: fm radio

Source : microphone
Channel : free space
Receiver : speaker

TV

Sou Camera ← Image, Video
microphone
free Space : Channel

Channel : wired, wireless

Randomness in channel deterministi

noise ← Thermal Noise, flicker Noise

Distribution : probability.

AWGN - Additive white Gaussian Noise

Source $\xrightarrow{s(t)}$ Σ → $y(t) = s(t) + n(t)$

$n(t)$

Early day's digital Communication

Telegraph

A - Z

0 - 9

'.' , ',' , ')' , '?' ~~using~~ Morse code encoded using

(.) & (—)

short pulse long pulse.

most frequent - short code

rare occurance - long code Aim - average code length is to minimum

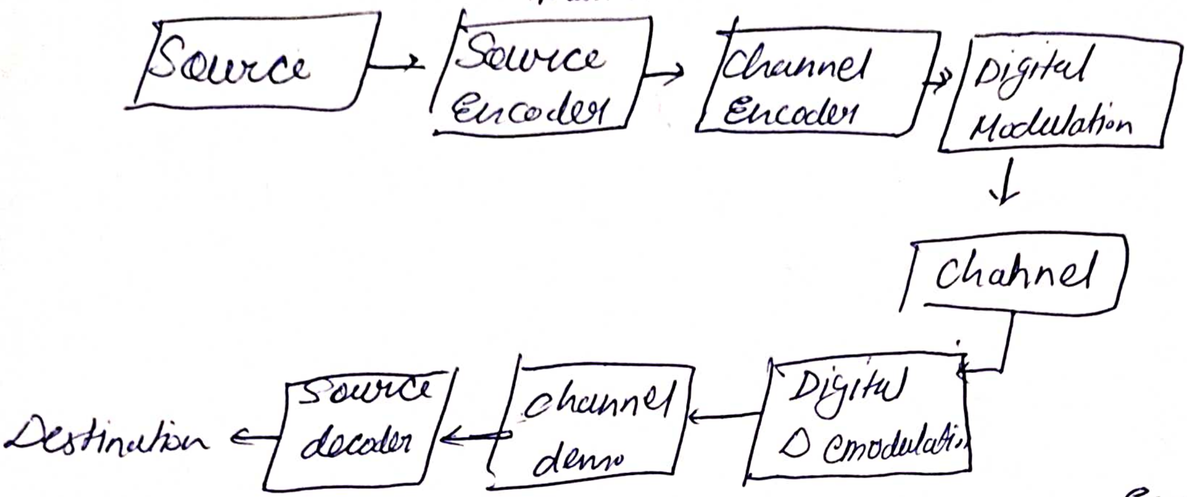
Redudance - Remove unused information

E →·

Z → - - -· ·

→ Remove the redundancy

→ compression



- We are adding redundancy to improve Error correction/ Error detection because of channel noise

0	→ 000	000	→ 0
1	→ 111	001	→ 0
		010	→ 0
		011	→ 1
		100	→ 0
		101	→ 1
		110	→ 1
		111	→ 1

Probability And Random Process

Experiment

↓

outcome

* All the outcome in a set is called Sample Space

eg Throwing a Dice

$S = \{1, 2, 3, 4, 5, 6\}$

Event - Subset of Sample Space

$S = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{2, 4, 6\}$ $A_2 = \{1, 2, 3\}$

$P(A_1) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$P(A_2) = \frac{1}{2}$

$$A = \bigcup_{i=1}^{N} A_i \qquad P(A) \leq \sum_{i=1}^{N} P(A)$$

<u>Exp 1</u> $\qquad$ <u>Exp 2</u>

$S = \{x_1, x_2, \cdots x_N\}$ $\qquad$ $S = \{y_1, y_2, \cdots y_N\}$

$P(x_i, y_i) \Rightarrow$ joint probability

$\rightarrow$ probability of x_i from Exp 1 & probability y_i from Exp 2

$$\sum_{i=1}^{N} \sum_{j=1}^{M} P(x_i, y_i) = 1 \qquad \int \int P(x, y)\, dx\, dy = 1$$

$$\underline{\sum_{j=1}^{M} P(x_i, y_j)} = P(x_i) \qquad \int P(x, y)\, dy = P(x)$$

$\rightarrow$ marginal probability

$$\sum_{i=1}^{N} (x_i, y_j) = P(y_j)$$

Conditional Probability

$P(A/B) \Rightarrow$ probability of A when B occured.

$$= \frac{P(A, B)}{P(B)}$$

$$P(A, B) = P(A)\, P(B/A)$$

$$= P(B)\, P(A/B)$$

$$P(A,B,C) = P(A)\,P(B/A)\,P(C/_{A,B})$$

$$P(A_1, A_2, \cdots A_n) = P(A_1) + P(A_2|A_1) + P(A_3/_{A_1A_2}) + \cdots P(A_n/_{A_1,A_2 \cdots A_1})$$

→ If A & B are Independent

$$\boxed{P(A|B) = P(A)} \quad \& \quad P(A,B) = P(A)P(B)$$

Random Variable: It is mapping from Sample space to Real no.

$$X : S \rightarrow \mathbb{R}$$

X = Random process
x - random variable

$\{H, T\}$ $\quad H \rightarrow -L \quad T \rightarrow L$ $\quad$ or $\quad H \rightarrow 0 \quad T \rightarrow L$

Continous Random Variable

when the image of sample space is Continous then the rand. varible is called Conti. rand varible

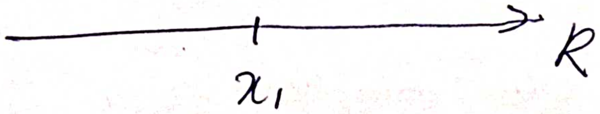
discrete random variable

when image is disconti. then discrete rand var.

Probability Distribution function/ Cummulative Distribution.

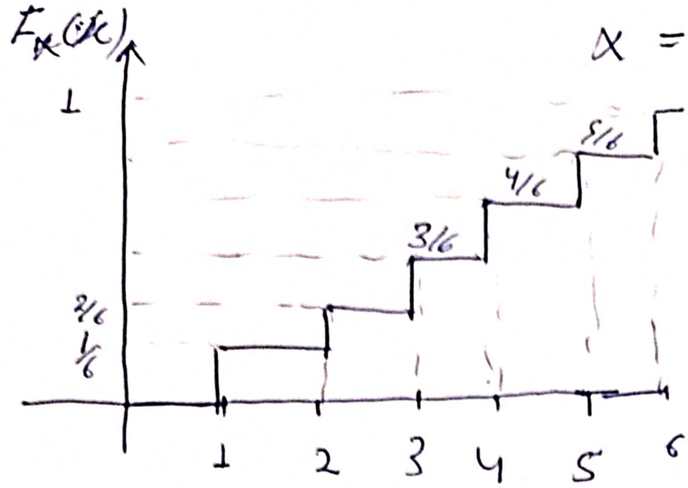
$$F_X(x) = P(X \leq x_1)$$

probability of random process from $-\infty$ to x_1

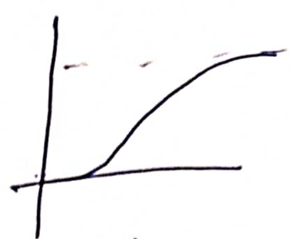


CDF → non decreasing function

$x = \{1, 2, 3, 4, 5, 6\}$

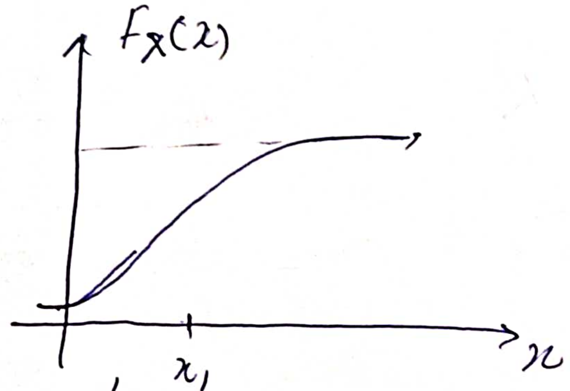


x DRV



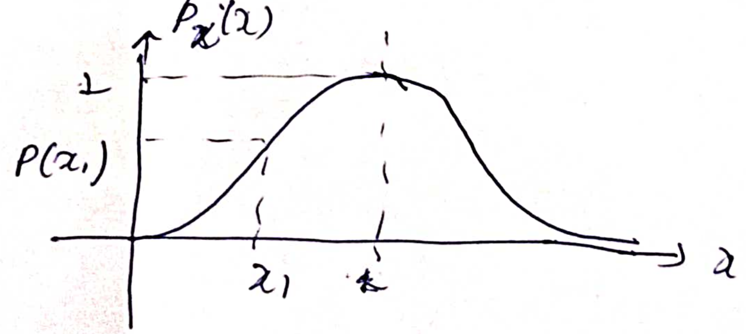
Cont^n R.V

<u>Pdf</u> : probability density function



$$p_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$

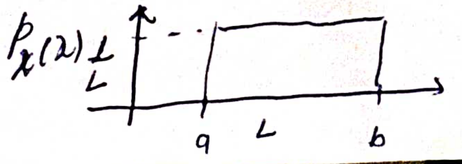
$$F_X = \int_{-\infty}^{x} p_X(x)\,dx$$

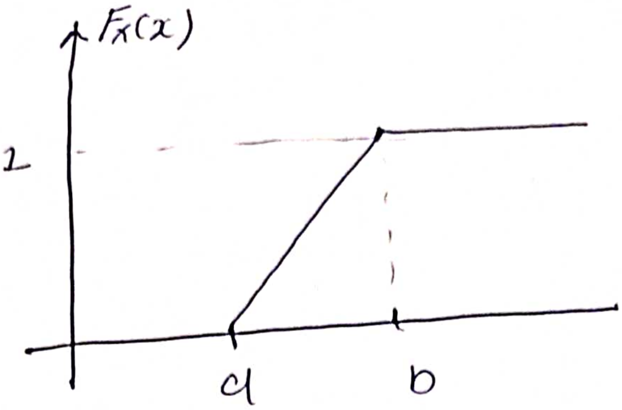


$$P(x_1 \le X \le x_2) = F_X(x_2) - F_X(x_1)$$

$$= \int_{x_1}^{x_2} p(x)\,dx$$

Ex Uniform Distribution $\int_a^b P(x)\,dx = 1$



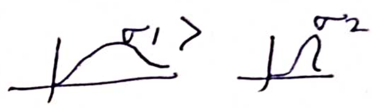


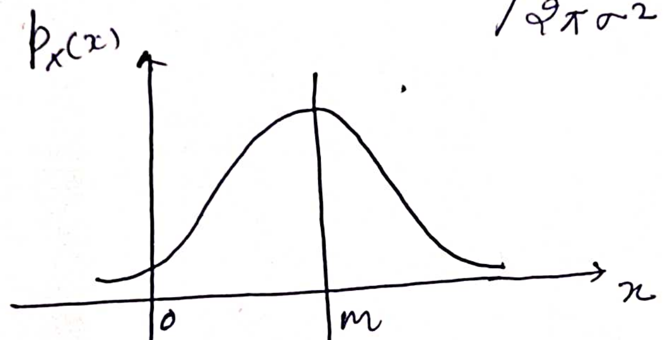
Gaussian distribution

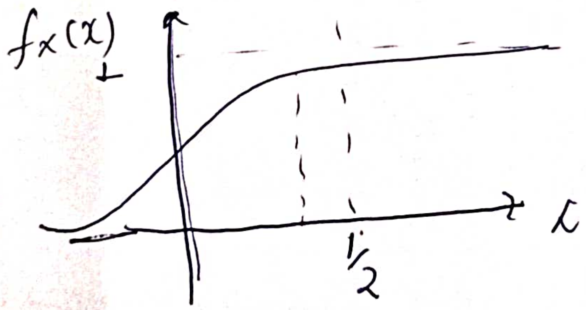
$$P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

m = mean

σ^2 = varian







multiple r.v.

x_1, x_2

$$F_{X_1,X_2}(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2)$$

$$p_{X_1,X_2} = \frac{\partial}{\partial x_1 \partial x_2} F_{X_1 X_2}(x_1, x_2)$$

$$F_{X_1 X_2} = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} p_{X_1,X_2}(y_1, y_2)\, dy_1\, dy_2$$

$$F_{X_1/X_2}(x_1 | X_2 = x_2) = \frac{\int_{-\infty}^{x_1} P_{X_1 X_2}(y, x_2)\, dy}{P_{X_2}(x_2)}$$

$$P_{X_1|X_2}(x_1 | X_2 = x_2) = \left.\frac{\partial F(x/x_2)}{\partial x}\right|_{x = x_1}$$

$$= \frac{P_{X_1 X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

<u>Independent r.v</u>

$$P_{X_1 X_2}(x_1, x_2) = P_{X_1}(x_1)\, P_{X_2}(x_2)$$

$$F_{X_1 X_2}(x_1, x_2) = F_{X_1}(x_1)\, F_{X_2}(x_2)$$

$\frac{2}{40}$ $\frac{5}{40}$

Mean / Expectation

$$\frac{0 \times 2 + 1 \times 5 + 2 \times 6 + \cdots 10 \times 5}{40}$$

$x, \ P_x(x)$

$$\mu_x = E(x) = \bar{x} = \sum_{i=1}^{n} x_i P_i \qquad \left| \ \mu_x = \int_{-\infty}^{\infty} x p(x) dx \right.$$

D. r.v — C. r.v

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 p(x)\, dx = E\left[(x - \mu_x)^2\right] = \sigma_x^2$$

$E(x^2) = \int x^2 . p(x) dx$ - Second moment of r.v x

$\int x^n p(x) dx$ - n^{th} moment of r.v

$$E[(x-\mu_x)^2] = E(x^2+\mu_x^2-2x\mu_x)$$

$$= E(x^2) + E(\mu_x^2) - 2\mu_x E(x)$$

$$= E(x^2) + \mu_x^2 - 2\mu_x^2$$

$$\boxed{\sigma_x^2 = \underset{\text{2nd mo}}{E(x^2)} - \mu_x^2}$$

Covariance - varies w.r.t mean

x y

$E[xy] = R_{xy}$ Correlⁿ

$$C_{xy} = E[(x-\mu_x)(y-\mu_y)^*]$$

$$= E(xy - x\mu_y - y\mu_x + \mu_x\mu_y)$$

$$= E(xy) - \mu_y \underset{\mu_x}{E(x)} - \mu_x \underset{\mu_y}{E(y)} + \mu_x xy$$

$$= E(xy) - 2\mu_x\mu_y + \mu_x\mu_y$$

$$\boxed{C_{xy} = E[xy^*] - \mu_x\mu_y^*}$$

if C_{xy} then x and y are uncorrelated

$E(xy) = R_{xy} = 0$ then orthogonal / Independent vector → underline

Covariance Matrix

$$x_1 \quad x_2 \quad \cdots \quad x_n$$

$$\downarrow \quad \downarrow \quad \quad \downarrow$$

$$\mu_1 \quad \mu_2 \quad \quad \mu_n$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

covariance b/w x_i, x_j

$$E[(x_i - \mu_i)(x_j - \mu_j)]$$

$$E[(\underline{x}_i - \underline{\mu})(x_j - \mu_j)^T]$$

$$\underbrace{\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}}_{n \times 1} \underbrace{\begin{bmatrix} (x_1 - \mu_1 & \cdots & x_n - \mu_n \end{bmatrix}}_{1 \times n}$$

$$= \begin{bmatrix} E[(x_1-\mu_1)(x_1-\mu_1)] & E[(x_1-\mu_1)(x_2-\mu_2)] & \cdots & E[(x_1-\mu_1)(x_n-\mu_n)] \\ E[(x_2-\mu_2)(x_1-\mu_1)] & \cdots & & E[(x_2-\mu_2)(x_n-\mu_n)] \\ \vdots & & & \\ E[(x_n-\mu_n)(x_1-\mu_1)] & \cdots & & E[(x_n-\mu_n)(x_n-\mu_n)] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{x_1}^2 & C_{x_1 x_2} & \cdots & C_{x_1 x_n} \\ C_{x_2 x_1} & \sigma_{x_2}^2 & \cdots & C_{x_2 x_n} \end{bmatrix}$$

$p \rightarrow$ probability distribution

$P \rightarrow$ probability

$x_1 \; x_2 \cdots x_n$

$\mu_1 \; \mu_2 \cdots \mu_n$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$$

$$C_{xx} = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & & & \vdots \\ \mu_{n1} & & & \mu_{nn} \end{bmatrix}$$

$$\mu_{ij} = E_x[(x_i - \mu_i)(x_j - \mu_j)]$$

$\mu_{ij} = \mu_{ji}$

$\mu_{ii} = \sigma_i^2$ Symmetric Matrix

$$F_y(y) = P(Y \le y)$$

$$y = ax + b$$

$$< P(ax + b \le y)$$

$$P\left(x \le \frac{y-b}{a}\right) = F_x\left(\frac{y-b}{a}\right)$$

$$F_y(y) = F_x\left(\frac{y-b}{a}\right)$$

$$p_y(y) = \frac{d}{dy} F_y(y)$$

$$= \frac{d}{dy}\left[F_x\left(\frac{y-b}{a}\right)\right]$$

$$= \frac{1}{a} p_x\left(\frac{y-b}{a}\right)$$

multiple r.v

$x = [x_1 \cdots x_n]$

$y_i = g_i(\underline{x})$

$g_i^{-1}(y_i) = x_i$

$x = f_i(y)$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \frac{\partial f_1}{\partial y_2} & - - - - & & \frac{\partial f_n}{\partial y_2} \\ \vdots & & & \\ \frac{\partial f_1}{\partial y_n} & - - - - & & \frac{\partial f_n}{\partial y_n} \end{bmatrix}_{n \times n}$$

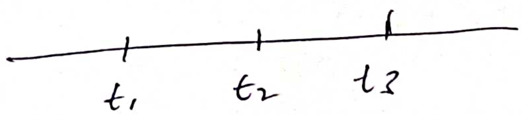
~~$P_y(y) = P_x(y) \times |J|$~~

$P_y(y) = P_x[f(y)]\,|J|$

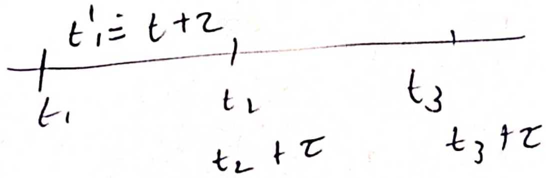
Jacobian Matrix

Discrete Random process

$X(t)$ $\quad t_1 < t_2 < t_3$

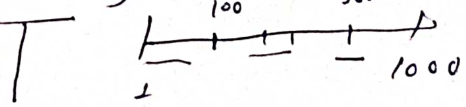


$P(x_{t_1}, x_{t_2}, x_{t_3})$



$P(x_{t_1'}, x_{t_2'}, x_{t_3'})$

if PDf are equal then we call random process strictly stationary



$E[x(n)] = \mu \rightarrow$ first order stationary

Correlation

$$r_{xy} = E[XY]$$

$$r_{xy}(k) = E[x(n)y(n-k)$$

1st & 2nd by kunt

discrete $\phi(t_1 t_2) = E(x_{t_1} y_{t_2})$

Auto correlation

$$R_x(k) = E[x(n)x(n-k)]$$

$$\phi_{xx}(t_1 t_2) = E(x(t_1)\,x(t_2))]$$

$$\phi_{xx}(t_1, t_1+\tau) = E[x(t)\,x(t-\tau)]$$

$$= \phi(\tau),\ R_{xx}(k)$$

~~stand.~~
1st order - μ const
2n - correl^n

$R_{xx}(k)$ corr l^n depend only on time gap or time delay τ in cont signal

R in discrete

$$\phi_{xx}(\tau) = \phi(\tau) \quad \text{- 2nd order stationary}$$

If any random process ~~delay~~ ~~obeys~~ obeys.

- 1st order stationary

$$E[x(\cancel{k})] = \mu_x$$

$$E[x(t)] = \mu_x$$

LPC

and 2^{nd} order

$$\phi(\tau) = E[x(t)x(t-\tau)]$$

$$R_{xx}(k) = E[x(n)x(n-k)]$$

if both satisfy a then we call process is called WSS - wide Sense Stationary (by default)

IID - Independent & Identical Distribution.

$x_1, x_2 \dots x_n \rightarrow x_1 <$ mean $= \mu$, variance $= \sigma^2$

$x_2 <$ mean μ, var σ^2

~~covariance~~

$$E[x[n]\,x(n-k)]$$

$$= N\,\delta(k)$$

$$K=0,\; E[x(n)^2] = \sigma^2$$

$$R_{xx}(k) = N\,\delta(k)$$

$$P(x_1, x_2 \dots x_n) = p(x_1)\,p(x_2) \dots p(x_n)$$

~~$x_1\; x_2 \dots$~~

Uncorrelated $\Rightarrow$ covariance $= 0$

Independence $=$ corr^n $= 0$

#

$x_1 \quad x_2 \quad \dots \quad x_n$

$x_i \sim N(0, \sigma^2)$

$N(\mu, \sigma^2)$

$$M = \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ 0 & & \cdots & \sigma^2 & 0 \\ 0 & 0 & \cdots & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 I_n$$

$$r_{11} = E[x_1 x_1]$$

$$= E[|x_1|^2]$$

$$= \sigma_x^2 = r$$

$$R_{xx}(k) = \frac{N}{2}\delta(x)$$

$$\sigma^2$$

$$P_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$P_{x_1 x_2 \cdots x_n}(x_1, x_2, \cdots x_n)$$

General

M = covariance matrix

$$P_{x_1 x_2 \cdots x_n}(x_1, x_2 \cdots x_n) = \frac{1}{[2\pi \det[M]]^{\frac{N}{2}}} e^{-\underset{1\times n}{(x-\mu)^t}\,\underset{n\times n}{M^{-1}}\,\underset{n\times 1}{(x-\mu)}}$$

IID

$$P_{x_1 x_2 \cdots x_n}(x_1, x_2 \cdots x_n) = P_{x_1}(x_1) P_{x_2}(x_2) \cdots P_{x_n}(x_n)$$

$$= \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdots \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right.$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{N}{2\sigma^2}(x-\mu)^2}$$

Power Spectral Density

$$\phi(\tau) = E[x(t)\, x(t-\tau)]$$

FT of $\phi(t)$

$$\phi(f) = \int_{-\infty}^{\infty} \phi(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$\phi(\tau) = \int_{-\infty}^{\infty} \phi(f)\, e^{j2\pi f t}\, df$$

Small $\rightarrow$ $$\phi(0) = \int_{-\infty}^{\infty} \phi(f)\, e^{0}\, df$$

$$= \int_{-\infty}^{\infty} \phi(f)\, df$$

$$\phi(0) = E[|x(t)|^2]$$

= power ~~spectral density~~ of signal

$\phi(f) \rightarrow$ is called PSD / FT of $\phi(\tau)$ / FT of Autocorrln

$$\phi(\tau) = E[x(t)\, x(t-\tau)]$$

$$\phi(-\tau) = E(x(t)\, x(t+\tau)]$$

$$= E[x(t-\tau)\, x(t)]$$

$$= \phi(\tau)$$

for Complex Signal

$$\phi(t_1, t_2) = \frac{1}{2} E[x(t_1)\, x^*(t_2)]$$

$$= \frac{1}{2} E[x(t)\, x^*(t-\tau)]$$

Q: What is PSD of IID distributed random process.

$$R_{xx}(k) = \frac{N_0}{2} \delta(k)$$

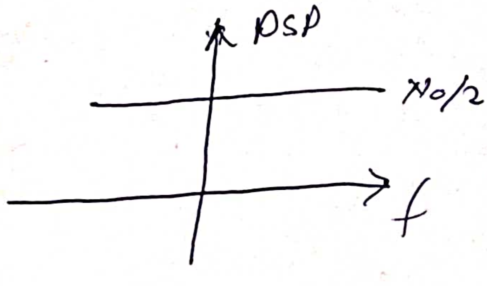
$$\phi(\tau) = \frac{N_0}{2} \delta(\tau)$$

$$\text{PSD } \phi(f) = \int_{-\infty}^{\infty} \phi(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{N_0}{2} \delta(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$= \frac{N_0}{2} \quad \text{costant}$$

$$= 10^{-5} \ \frac{\text{watt}}{Hz}$$



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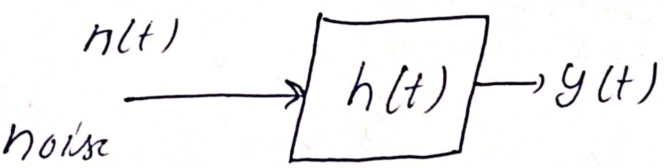
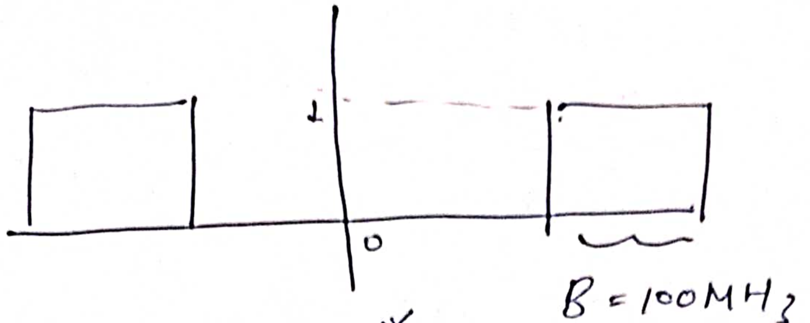


$$\phi_y(f) = |H(f)|^2 \phi_n(f)$$

Q: A Noise signal having PSD 10^{-12} watt/Hz given to a ideal BPF with 100 MHz BW find output power

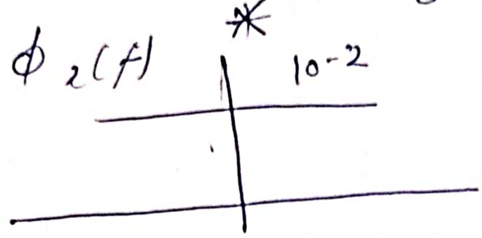


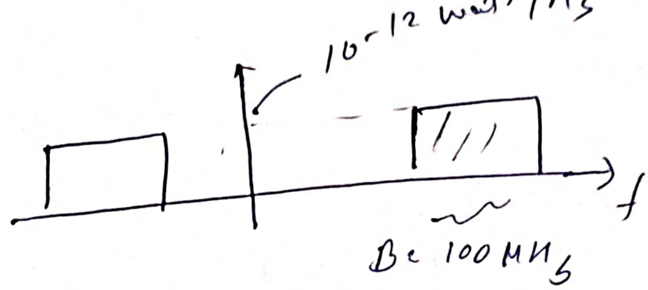
10^{-12}

100×10^{6}

10^{8}

$*$





$$P_o = \int_{-\infty}^{\infty} \phi_x(f)\,df$$

$$= 2 \times 10^{-12} \times 10^{8}$$

$$P = 2 \times 10^{-4} \text{ Watt}$$

$$P = 0.20 \text{ mW}$$

Noise is uncorrelate PSD

Question: Prove that PSD $\phi(e^{j\omega})$ of a signal $x(n)$ is real for (real valued signal)

$$\phi(e^{j\omega}) = \text{DTFT of } \cancel{x(n)}\ R_{xx}(k)$$

$$= \sum_{k=-\infty}^{\infty} R_{xx}(k)\, e^{-j\omega k}$$

$$= R_{xx}(0) + \sum_{k=1}^{\infty} R_{xx}(k)\, e^{-j\omega k} + \sum_{k=-1}^{-\infty} R_{xx}(k)\, e^{-j\omega k}$$

$$= \text{''} + \text{''} + \sum_{k=1}^{\infty} R_{xx}(-k)\, e^{j\omega k}$$

$$= \text{''} + \text{''} + \left(\sum_{k=1}^{\infty} R_{xx}(k)\, e^{-j\omega k}\right)^{*}$$

Hemissian Symmetry

$$R_{xx}(-k) = E[x(n)\, x(n+k)]$$

$$= E[x(n-k)\, x(k)] = R_{xx}(k)$$

$$R_{xx}(k) = R_{xx}^{*}(-k)$$

$$\phi(e^{j\omega}) = R_{xx}(0) + \underline{\sum_{k=1}^{\infty} R_{xx}(k) e^{-j\omega k}} + \left(\sum_{k=1}^{\infty} \{R_{xx}(k)\, e^{-j\omega k}\right)^{*}$$

$$= R_{xx}(0) + 2\sum_{k=1}^{\infty} Re\left[R_{xx}(k) e^{-j\omega k}\right]$$

Q: 2: A_i $i = 1, 2, 3, 4$ $B_j = 1, 2, 3$ → outcome

Event 1 E 2

$P(A_1, B_1) = 0.1$ $P(A_1, B_2) = 0.08$ $P(A_1, B_3) = 0.13$

$P(A_2, B_1) = 0.05$ $P(A_2, B_2) = 0.03$ $P(A_2, B_3) = 0.09$

$P(A_3, B_1) = 0.05$ $P(A_3, B_2) = 0.12$ $P(A_3, B_3) = 0.14$

$P(A_4, B_1) = 0.11$ $P(A_4, B_2) = 0.04$ $P(A_4, B_3) = 0.06$

Determine marginal probability $P(A_i)_{i=1,2,3,4}$ & $P(B_j)_{j=1,2,3}$

Sol^n

$$P(A_i) = \sum_{j=1}^{3} P(A_i, B_j) \quad i = 1, 2, 3, 4$$

$$P(A_1) = \sum_{j=1}^{3} P(A_1 B_j) = P(A_1, B_1) + P(A_2, B_2) + P(A_3, B_3)$$

$$= 0.31$$

$P(A_2) = 0.17$ $P(A_3) = 0.31$ $P(A_4) = 0.21$

$$P(B_j) = \sum_{i=1}^{4} P(A_i, B_j)$$

$$P(B_1) = \sum_{i=1}^{4} P(A_i, B_1) = 0.1 + 0.05 + 0.05 + 0.11 = 0.31$$

$P(B_2) = 0.27$ $\quad P(B_3) = 0.42$

$H(W \rightarrow P(A_i|B_j)$

Q: Consider a r.v Y defined as $Y = aX^3 + b$ $a>0$ (14)

where x is a Gaussian r.v with pdf $P_x(x)$.

Determine PDF of Y in terms of pdf of X

Solⁿ.

$$F_Y(y) = P(Y \le y)$$

$$= P(aX^3 + b \le y)$$

$$= P\left(X \le \left(\frac{y-b}{a}\right)^{1/3}\right)$$

$$= F_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

Now $\quad p_Y(y) = \frac{d}{dy} F_X(y) = \frac{d}{dy} F_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$

$$= P_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right] \frac{d}{dy}\left(\frac{y-b}{a}\right)^{1/3}$$

$$= \frac{1}{3a\left[\frac{y-b}{a}\right]^{2/3}} \times P_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

Q: X_R, X_i - satistical independent gaussian r.v (Real, Imag)

Rotational transformation of form

$$Y_R + jY_i = [X_R + jX_i)e^{j\phi}$$

results another pair (Y_R, Y_i) of gaussian r.v

Show that (Y_R, Y_i) pair has same joint pdf as (X_R, X_i)

Solⁿ: $\quad p(X_R, X_i) = P_X(X_R)p(X_i)$ $\quad$ ∵ Independent

$$p_x(x_R) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{x_R^2}{2\sigma^2}} \quad \because \text{ assume } \mu = 0$$

$$p_x(x_R, x_i) = \frac{1}{2\pi\sigma^2} \cdot e^{-\left(\frac{x_R^2 + x_i^2}{2\sigma^2}\right)}$$

$$y_R + j y_i = (x_R + j x_i)\, e^{j\phi}$$

$$x_R + j x_i = (y_R + j y_i)\, e^{-j\phi}$$

$$= (y_R + j y_i)(\cos\phi - j\sin\phi)$$

$$x_R = y_R \cos\phi + y_i \sin\phi$$

$$x_i = -y_R \sin\phi + y_i \cos\phi$$

$$J = \begin{vmatrix} \frac{\partial x_R}{\partial y_R} & \frac{\partial x_i}{\partial y_R} \\ \frac{\partial x_R}{\partial y_i} & \frac{\partial x_i}{\partial y_i} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} = 1$$

$$p_y(y_R, y_i) = \frac{1}{\cancel{|J|}\,1} \, p_x(y_R\cos\phi + y_i\sin\phi,\ -y_R\sin\phi + y_i\sin\phi)$$

$$\Rightarrow \frac{1}{2\pi\sigma^2} \, e^{-\left[(y_R\cos\phi + y_i\sin\phi)^2 + (-y_R\sin\phi + y_i\sin\phi)^2\right]}$$

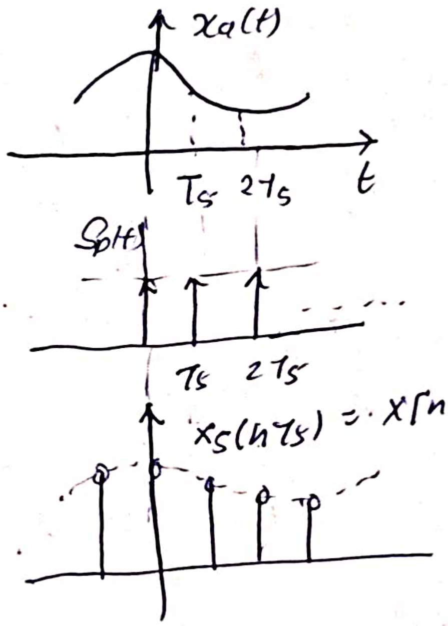
$$\Rightarrow \frac{1}{2\pi\sigma^2} \, e^{-\frac{(y_R^2 + y_i^2)}{2\sigma^2}}$$

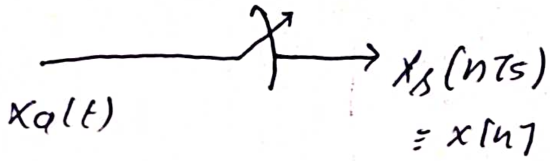
2nd part

J.G. Prokish → 2nd Chapter
Simon Hyking + Sampling

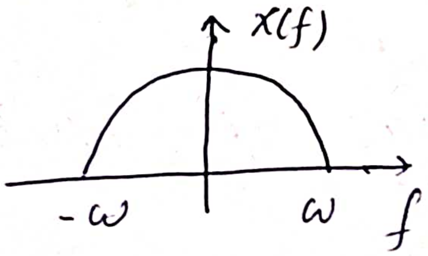
Sampling : Continous time signal $\xrightarrow{\text{Sampling}}$ Discrete Time Signal
↳ Discretizating time

uniform sampling
T_s : Sampling period
$\frac{1}{T_s} = F_s$: Sampling freq.

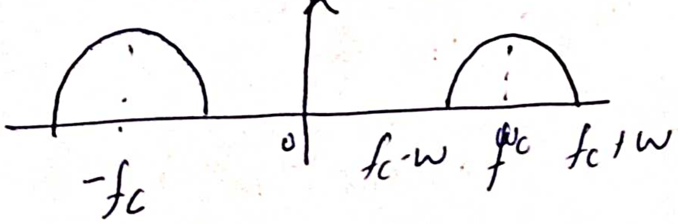




Question 1) Can we ~~rec~~ recover the original signal
2) Is their any loss of information.
loss can be minimize if signal is Bandlimited.



$|X(f)| = 0 \quad |f| \geq W$



Nyquiest Theorem / Rate

$f_s \geq 2W$ If sampling freq satisfy this condn then we can recover the original signal from sampled sign

$f_s = 2W$ [critical rate]

Q: The Covariance matrix of r.v is X_1, X_2, X_3 is

$$\begin{bmatrix} \mu_{11} & 0 & \mu_{13} \\ 0 & \mu_{22} & 0 \\ \mu_{31} & 0 & \mu_{33} \end{bmatrix}$$ The linear transformation $\underline{Y} = A\underline{X}$ $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

find covariance matrix of y.

<u>Sol^n</u> $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\mu_x = \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \vdots \\ \mu_{x_3} \end{bmatrix}$

Covarian matrix $M_X = E\left[(\underline{x} - \mu_x)(x - \mu_x)^T\right]$

$$M_y = E\left[(Y - \mu_y)(Y - \mu_y)^T\right]$$

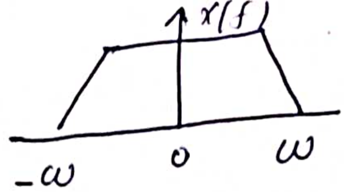
$$= E\left[(AX - A\mu_x)(AX - A\mu_x)^T\right]$$

$$\Rightarrow M_y = E\left[A(X - \mu_x)(X - \mu_x)A^T\right]$$

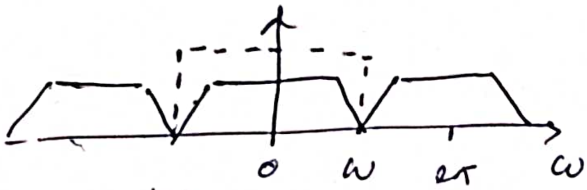
$$= A\, E\left[(X - \mu_x)(X - \mu_x)^T\right] A^T$$

$\boxed{\text{S-6 Eg Sum}}$

$x_a(t) \xleftarrow{F^{-1}} X(f)$



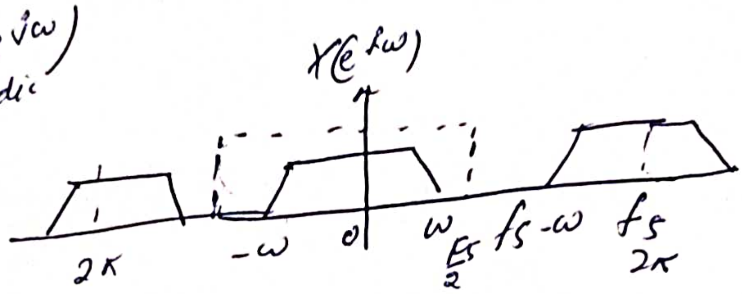
$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ periodic



Critical

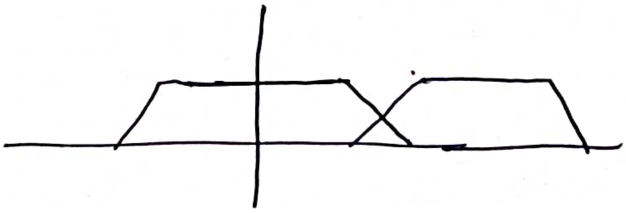
$f_s = 2W$

$\frac{f_s}{2} = W$



$f_s - W \geq \omega$

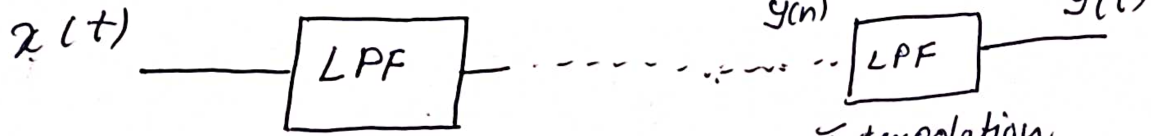
$\Rightarrow \boxed{f_s \geq 2W}$



$\Leftarrow f_s < 2W$

↳ aliasing

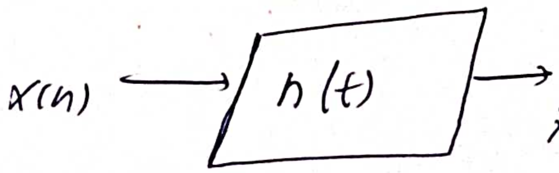
anti-aliasing filter



Interpolation

$X_s[n] \xleftrightarrow{LPF} \hat{x}(t)$ $\quad X_\delta(t) = \sum_{-\infty}^{\infty} \delta(t - nT_s)$

$$x[n] = \sum_{n=-\infty}^{\infty} x[nT_s]\,\delta(t - nT_s)$$ Ideal sampling

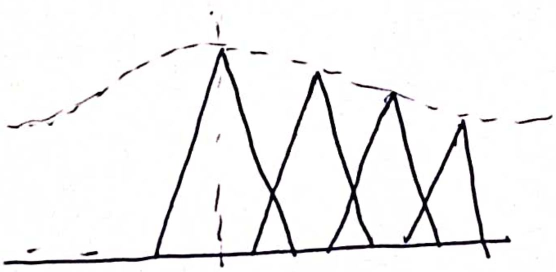


$\hat{x}(t) = \sum x(nT_s)\,h(t - nT_s)$

LTI

Let

$\longrightarrow$



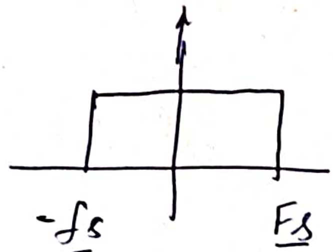
$\hat{x}(t) = \sum x(nT_s) h(t - T_s)$

$h(t) = ?$ ~~Satch~~ such that $\hat{x}(t) = x(t)$

find $\hat{x}(f) = X(f)$

$h(t) = \text{sinc}(f_s t) = \frac{\sin \pi f_s t}{\pi f_s t}$

$H(f) =$



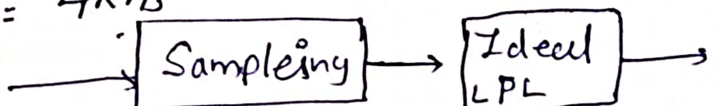
$\hat{x}(f) = \mathcal{F}\Big(\sum \underset{\text{Constan}}{x(nT_s)}\, h(t - T_s)\Big)$

$= \underbrace{\sum_{-\infty}^{\infty} x[n]\, e^{-j2\pi \left(\frac{f}{f_s}\right) n}}_{X(e^{j\omega})}$

$= X(f)$

Ex Undersampling

$f_m = 4kHz$

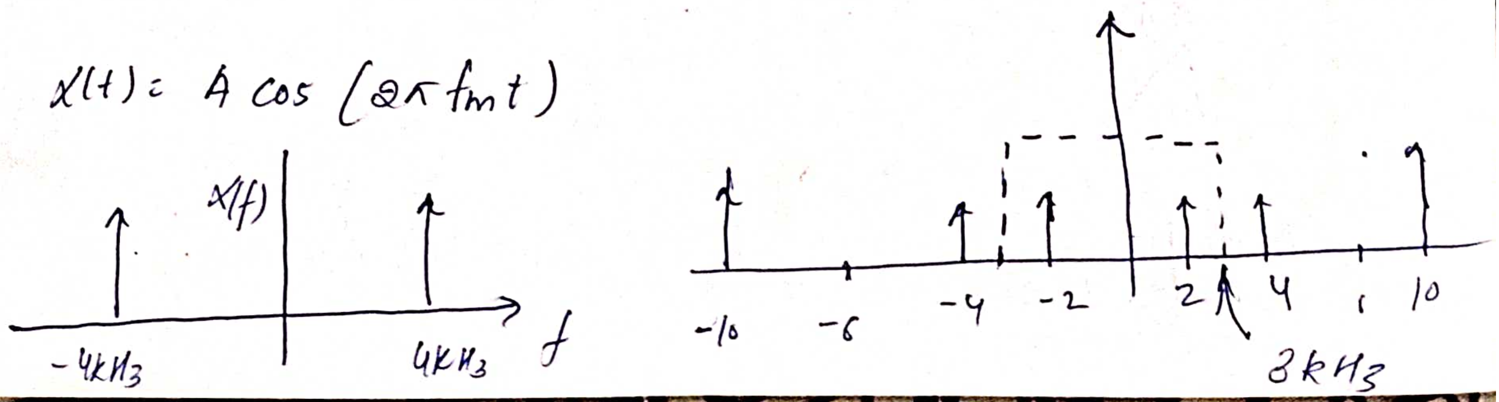


$f_o = ?$ 2kHz

Sinusoidal $f_s = 6kHz$ $f_c = \frac{f_s}{2} = \frac{6kHz}{2}$

Supposed to 8kHz

$x(t) = A\cos(2\pi f_m t)$



↳ Anti alising filter used to check sampling freq.

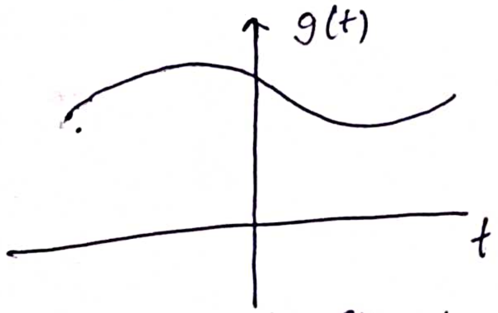
Practical Aspect of Sampling

↳ sampling is accomplished by means of high speed switching transition circuit

↳ practically the switching time can not be zero !!!.

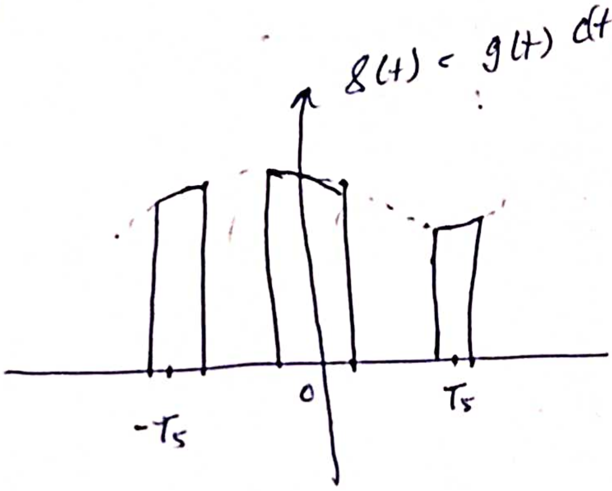
type of sampling

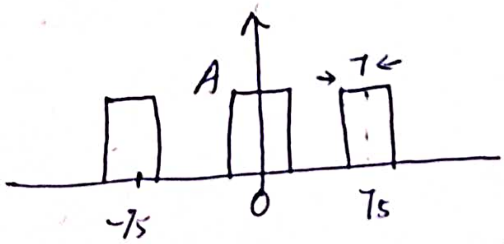
① Natural Sampling



Analog Signal

(a) Analog Signal





Sampling function

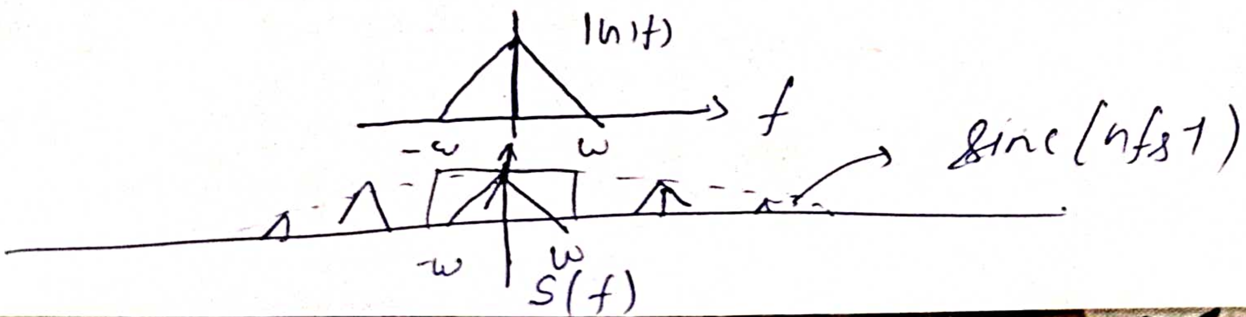
$c(t)$ = fourier Series

→ $f_s T A \cdot \sum C_n e^{j 2\pi n f_s t}$

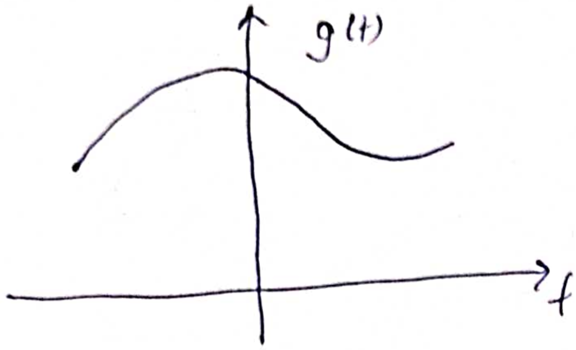
$C_n = \text{sinc}(n f_s t)$

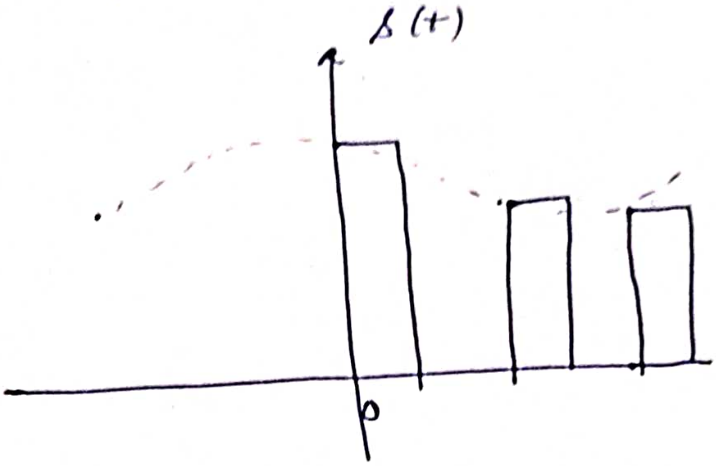
→ $s(t) = f_s T A \sum_{-\infty}^{\infty} \text{sin}(n f_s t) e^{j 2\pi n f_s t} g(t)$

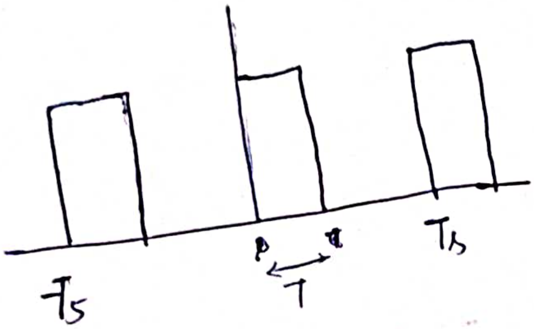
⇒ $S(f) = f_s T A \sum_{-\infty}^{\infty} \text{sinc}(n f_s T) \cdot G(f - n f_s)$ $TA = 1$

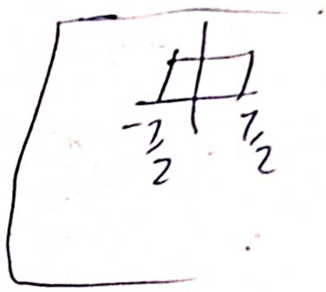


Q flat - Top Sampling / PLOSE Amplitude modulation / Sample & Hold (S/H)









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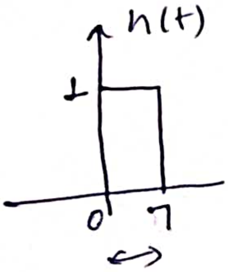


$$h(t) = rect\left(\frac{t - T/2}{T}\right)$$

$$= rect\left(\frac{t}{T} - \frac{1}{2}\right)$$

$$h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{other} \end{cases}$$

* Ideal sampling / Impulse sampling

$$g_s(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s)$$

$$G_s(f) = f_s \sum_{-\infty}^{\infty} G(f - nf_s)$$

$$g(t) = \sum_{-\infty}^{\infty} g_s(t) \circledast h(t) \quad \leftarrow \text{flat top sampling}$$

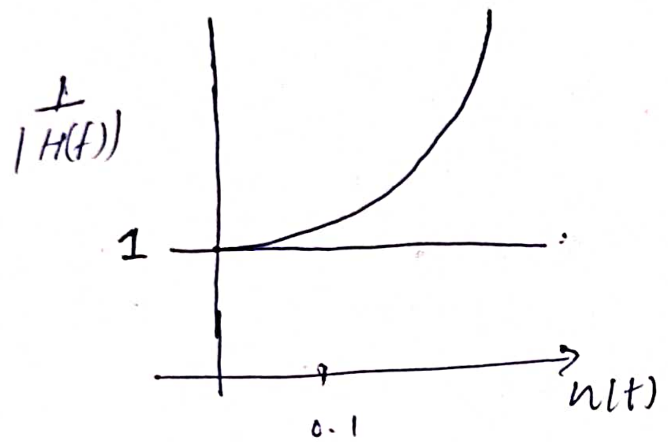
$$= \sum g(nT_s)\,\delta(t - nT_s) \circledast h(t)$$

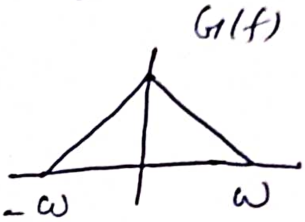
$$= \sum_{-\infty}^{\infty} g(nT_s)\,h(t - nT_s)$$

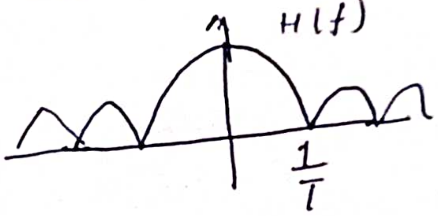
$$S(f) = G_s(f) H(f)$$

$$\Rightarrow f_s \sum H(f) G(f - nf_s)$$

$$H(f) = T \operatorname{sinc}(fT) e^{-2\pi \frac{fT}{2}}$$





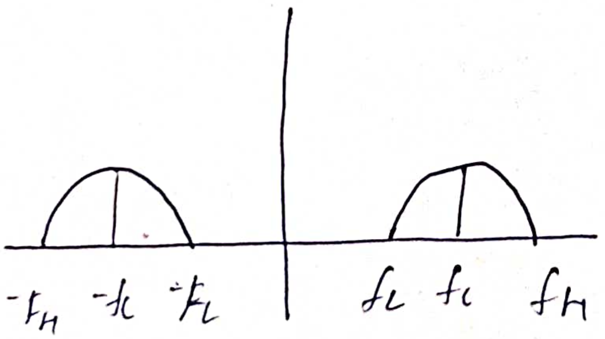


↳ allowed duty cycle 7.10% $= \frac{T}{T_s} = 0.1$

Sampling of Band pass Signal

Sampling Range : f_H = highest freq

f_L : Lowest freq $\quad B = f_H - f_L$



$f_s = 2B$

If f_H or f_L is one of the Harmony of f_s

$f_H = 100$ MHz

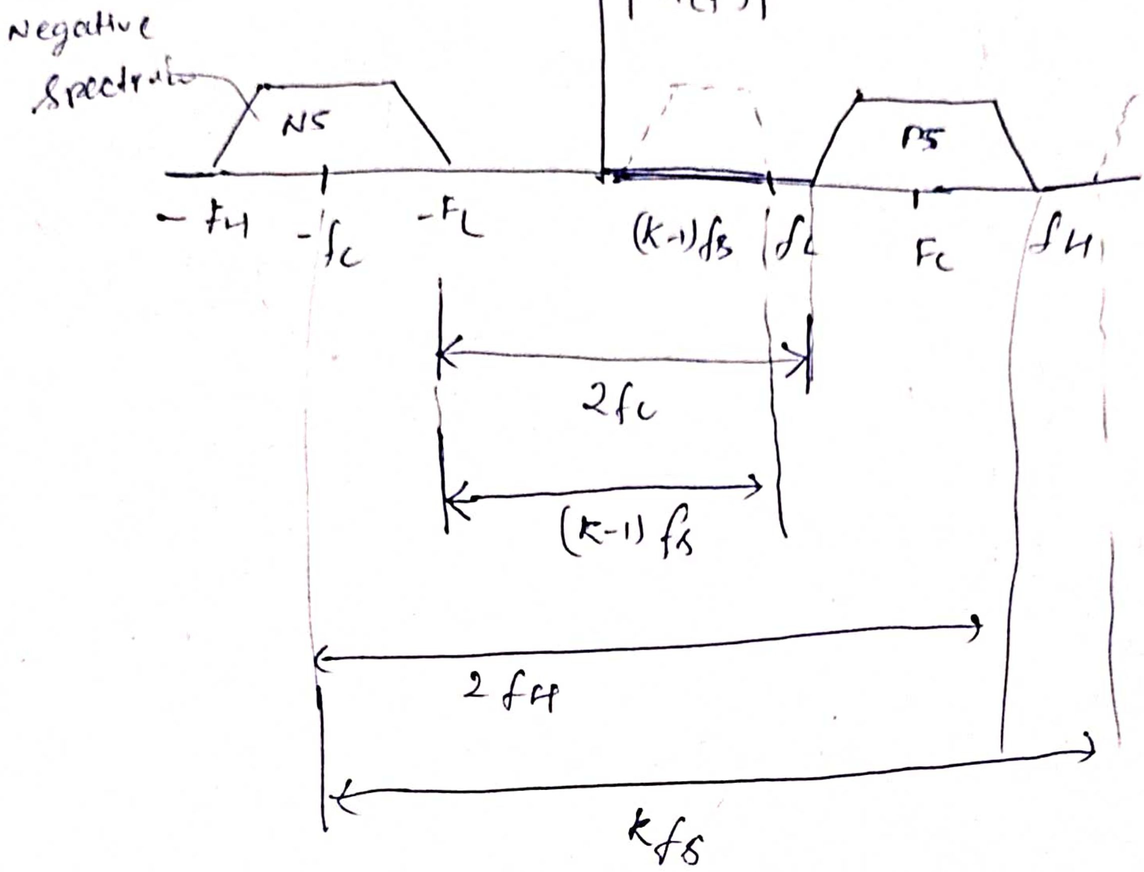
$f_L = 90$ MHz $\quad B = 10$ MHz

Minimum Sampling frequency allowed is

$$F_s = 2(f_H - f_L)$$

provided that either f_L or f_H is harmonic of f_s

$F_H \neq kB$ $\qquad\qquad\qquad$ $f_s \ll f_c$



$$2f_L \geq (k-1) f_s$$
$$2F_H \geq k f_s$$

$$\Rightarrow \boxed{\boxed{\frac{2f_H}{k} \leq f_s} \leq \frac{2f_L}{k-1}} \qquad \therefore f_L = f_H - B$$

$$\frac{1}{f_s} \leq \frac{k}{2f_H} \qquad \Rightarrow \quad (k-1) \leq \frac{k}{2f_H} 2(f_H - B)$$

$$(k-1) f_s \leq 2 f_L \Rightarrow (k-1) f_s \leq 2(f_H - B)$$

$$\boxed{k \leq \frac{f_H}{B}}, \quad \boxed{k_{max} = \frac{f_H}{B}}$$

$$\Rightarrow \frac{k-1}{k} \leq \frac{f_H - B}{f_H} \Rightarrow 1 - \frac{1}{k} \leq 1 - \frac{B}{f_H} \Rightarrow \frac{1}{k} \geq \frac{B}{f_H} \Rightarrow \boxed{k \leq \frac{f_H}{B}}$$

$$\boxed{k_{max} = \frac{f_H}{B}} \quad , \quad \boxed{f_s \geq \frac{2f_H}{k}}$$

Band pass Sampling Theorem

A Band pass signal with Higest freq f_H and Bandwidth B can be recovered from its samples through a BPF filtering by sampling it with frequency

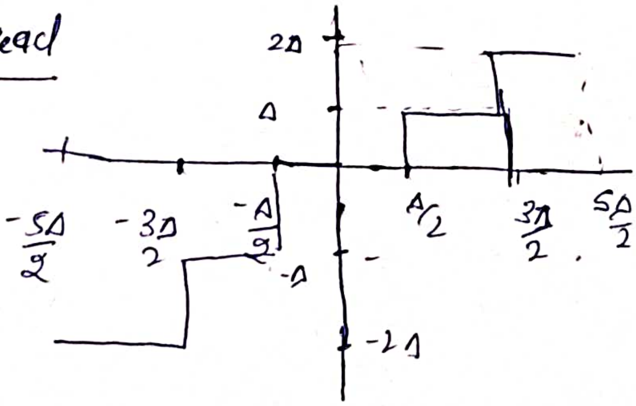
$$f_s = \frac{2f_H}{k}$$ where k is the largest integer Not exceeding f_H/B

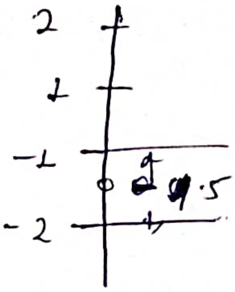
$\boxed{\text{Summery - 4 sampling Ideal, Normal, flat top, BP}}$

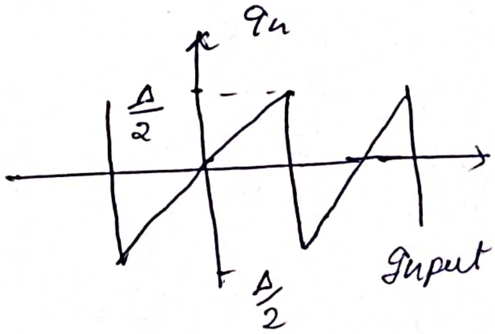
Quantization

↳ Descritizing Sample in Amplitude

Midtread

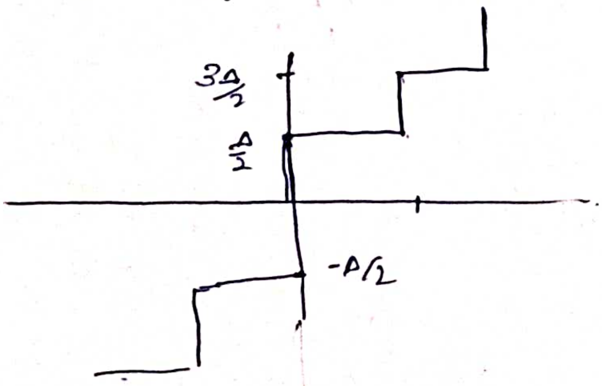




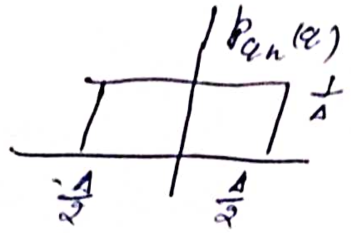


$$\tilde{x} = x_n + q_n$$

Quantization Error. $q_n = \tilde{x}_n - x_n$



Mid Rize
↳ origin mid b/w rize path



Max^m quantization Error = $\Delta/2$

Range $-\frac{\Delta}{2}$ to $\frac{\Delta}{2}$

$$p_{q_n}(q) = \text{pdf of } q_n = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} \le q_n \le \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

Mean of $q_n = 0$

i.e. MSE $= \sigma_{q_n}^2 = E(q_n^2) = \int_{-\Delta/2}^{\Delta/2} q^2\, p(q)\, dq$

mean sqr Error

$$= \int_{-\Delta/2}^{\Delta/2} q^2 \times \frac{1}{\Delta}\, dq = \frac{\Delta^2}{12}$$

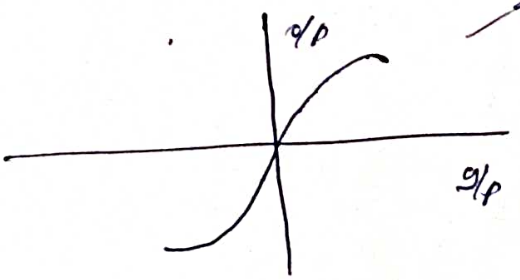
less Δ → Bits ↑↑

$$\boxed{\sigma_{q_n}^2 = \frac{\Delta^2}{12}}$$

Non-linear Quantization

Compander

$x(t) \rightarrow$ Compression — Uniform quantization — Expander

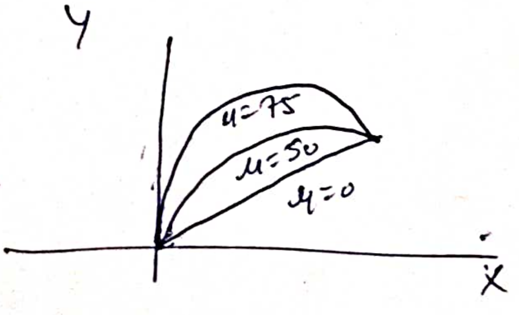


A - low - Europian

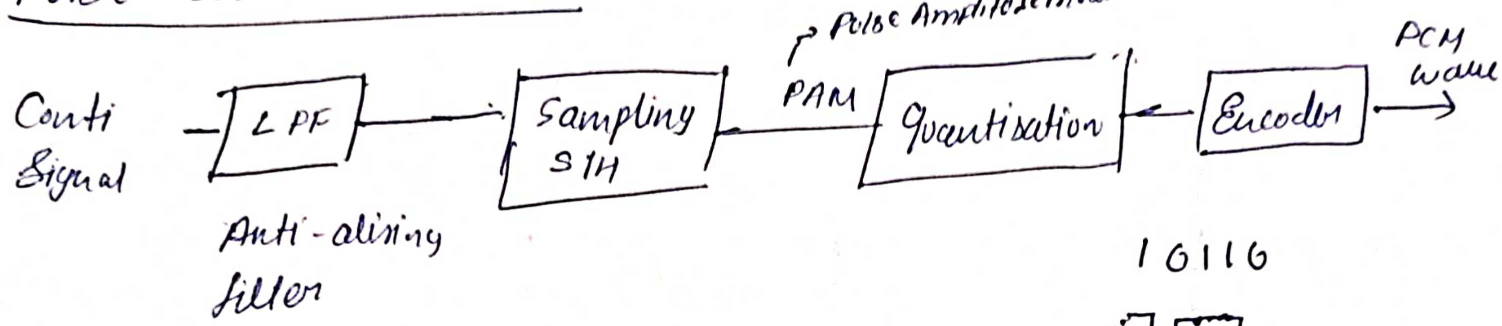
μ - low → American

$$|y| = \frac{\log(1 + \mu|x|)}{\log(1 + x)}\ \text{sin}(x)$$

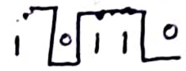
$0 \le \mu \le 255$

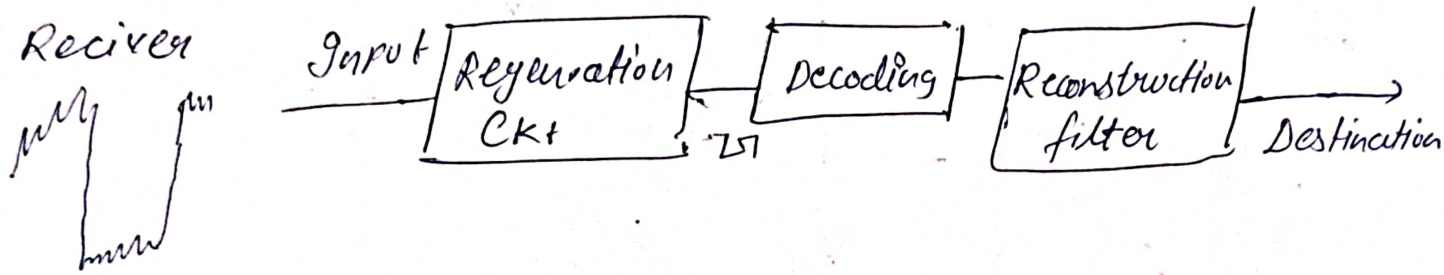


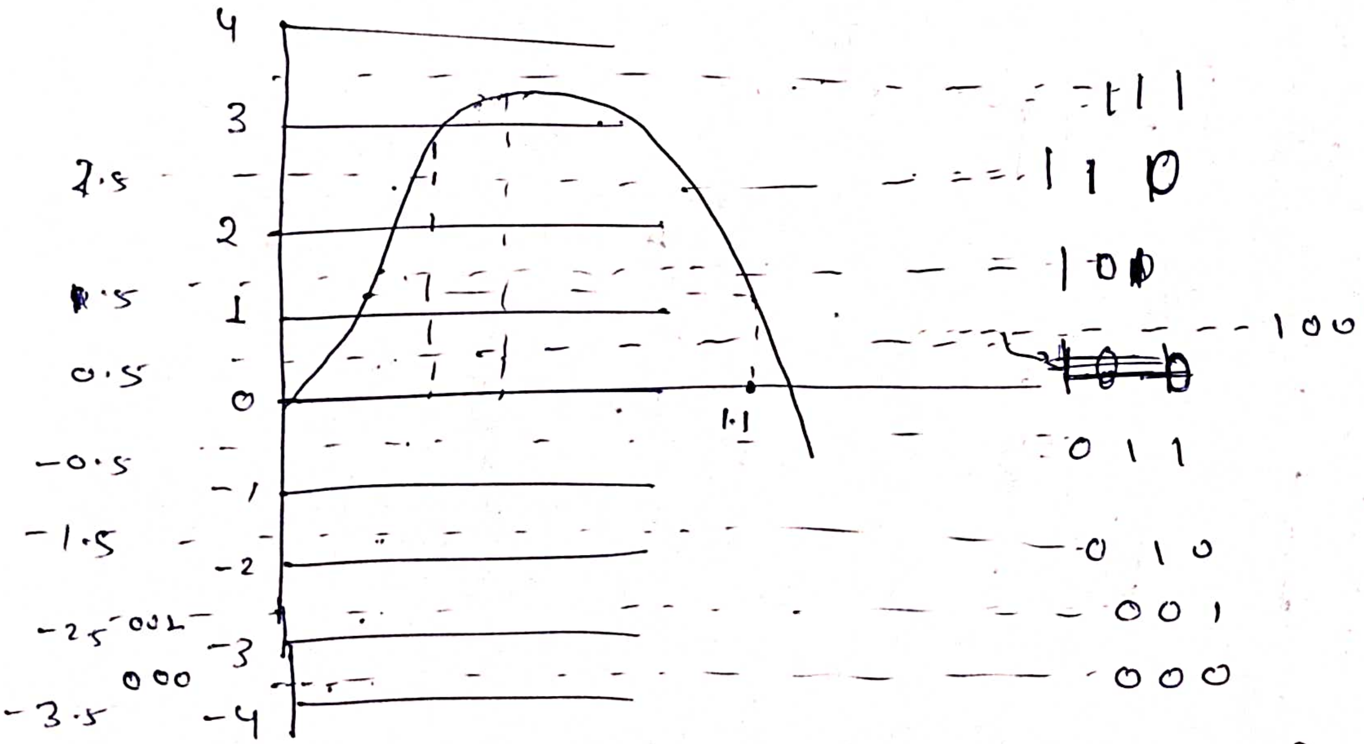
Pulse Code Modulation PCM



10110







$x_n = 0$	$x_n = 2.8V$	$x_n = 3.2V$	$x_n = 1.1V$
$\hat{x}_n = 0.5$	$\hat{x}_n = 2.5$	$\hat{x}_n = 3.5$	$\hat{x} = 1.5$
code = 100	code = 110	code = 111	code = 101

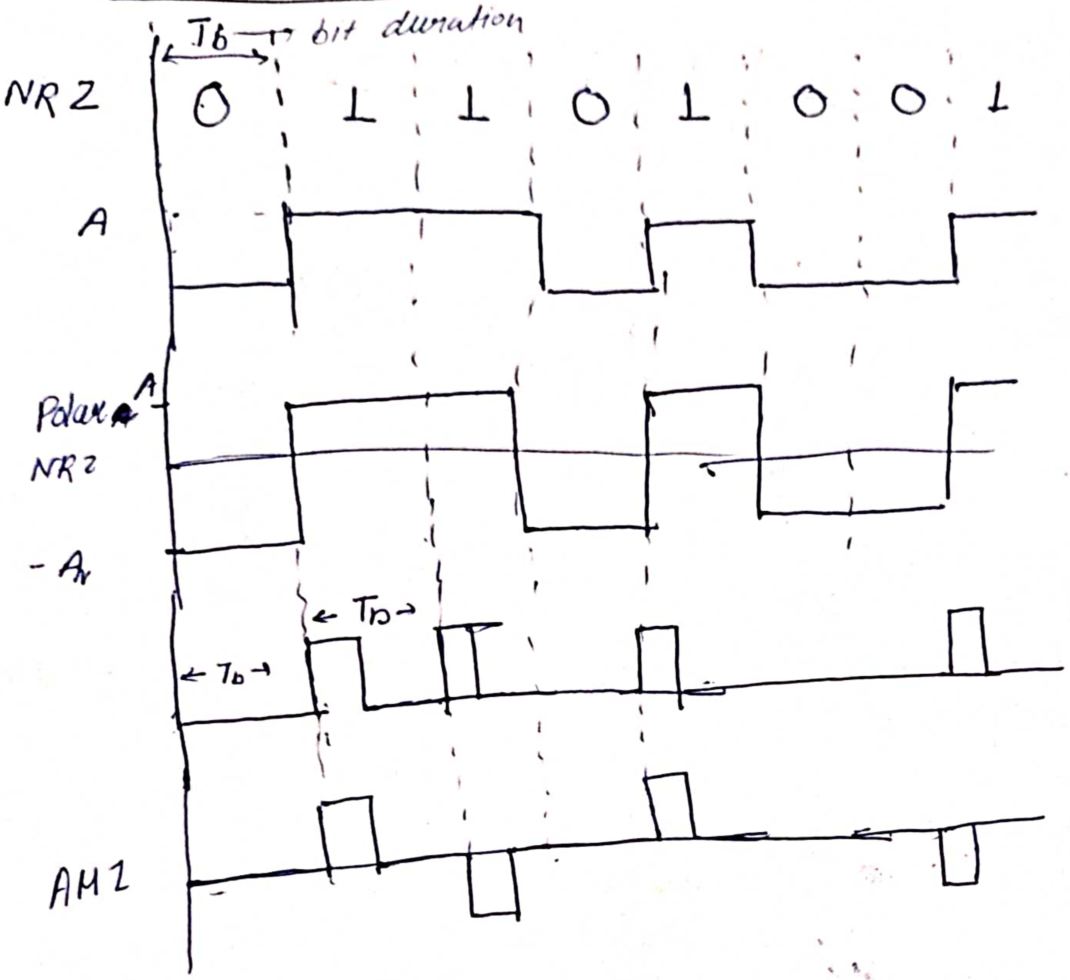
Line coding

Eg Audio Signal
Speech Signal
0.3 – 3.3 kHz
$f_s = 8\,kHz$ 8-bit PCM 256-levels

Output bit rates : $R = 8\,kHz \times 8\,bit$
$= 64\,k\,\frac{bit}{sec}$
$= 64\,kbps$

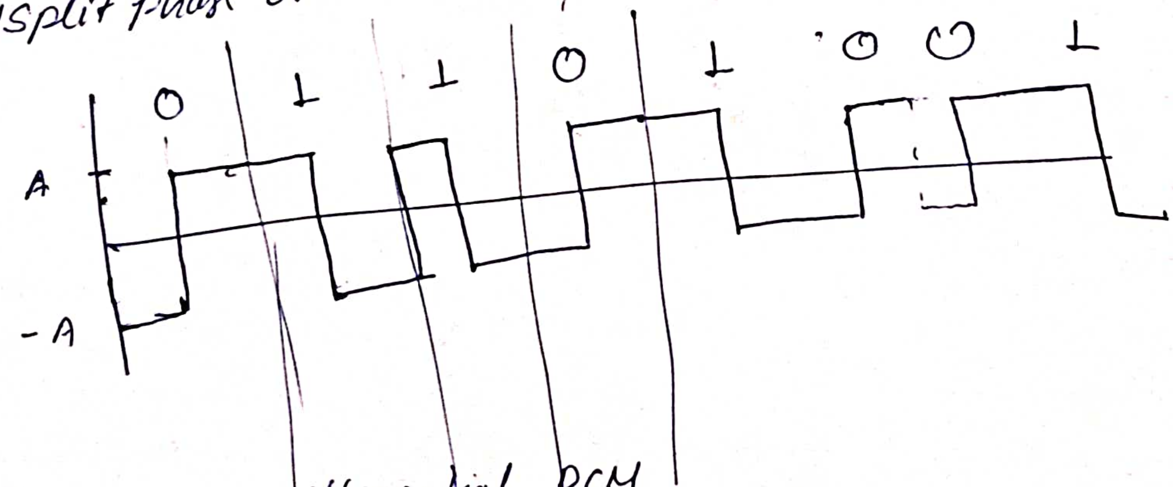
Line Coding

NRZ
Non Return to zero



Alternate mark Inversion
or
Bipolar return to Zero

Split phase or manchester Code



DPCM - Differential PCM

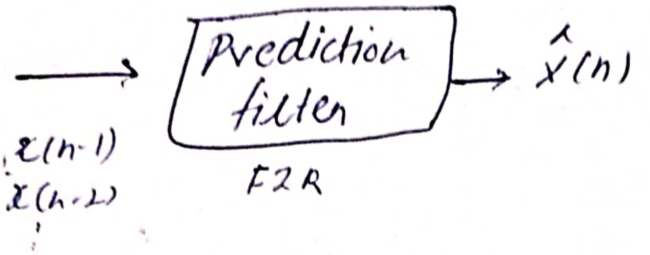
PCM - $R = nf_s$

n = no of bits used to Encode Each Sample

f_s = sampling freq.

$$e(n) = m(nT_s) - m[(n-1)T_s]$$

$m(n-1) + e(n) = m(n)$



prediction ~~predict~~ signal ~~symbol~~ from past

$$e(n) = m(n) - \hat{m}(n)$$ predicted.

$$e_q(n) = e(n) + q(n)$$

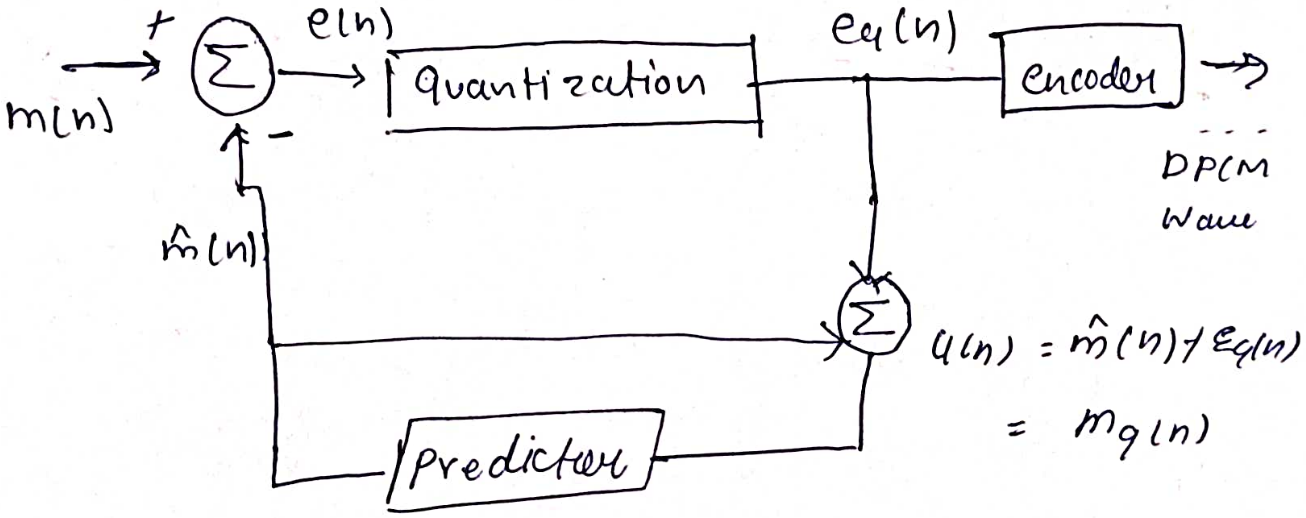
Quantization Error

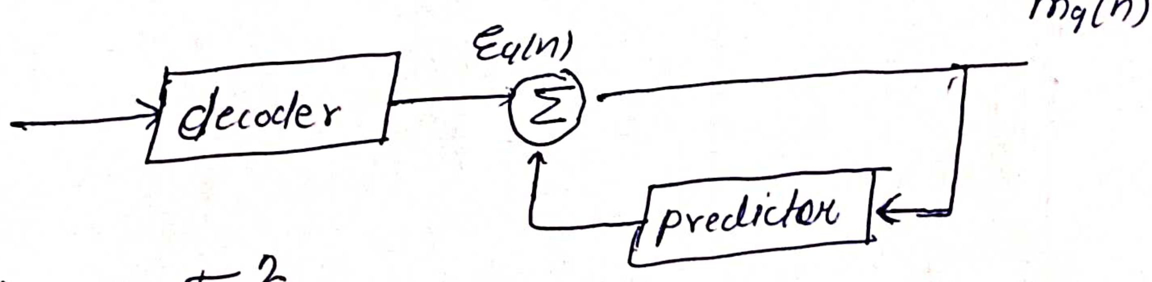
Quantized Signal

$$m_q(n) = \hat{m}(n) + e_q(n)$$

$$= \hat{m}(n) + e(n) + q(n)$$

$$m_q(n) = m(n) + q(n)$$





$$(SNR)_o = \frac{\sigma_m^2}{\sigma_Q^2}$$

Error of Prediction

$$= \left(\frac{\sigma_m^2}{\sigma_E^2}\right)\left(\frac{\sigma_E^2}{\sigma_q^2}\right)$$

$$= G_p \; G_q$$

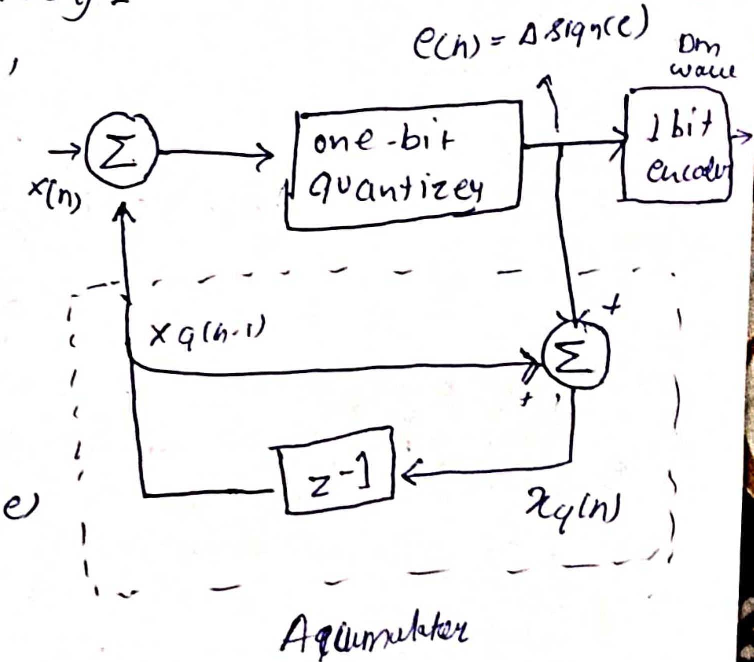
Processing gain

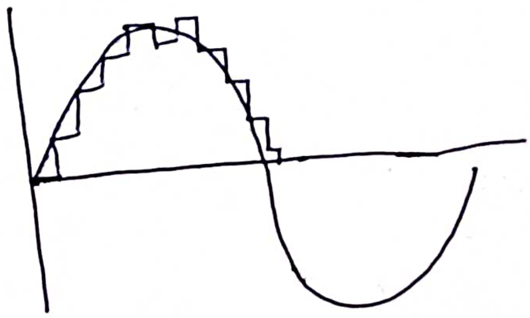
Delta Modulator (DM) → why we call delta- becus error is bounded by Δ

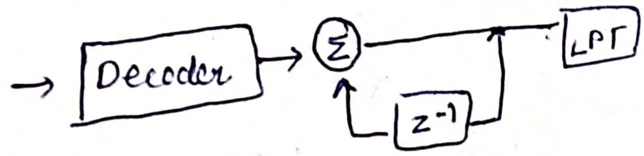
$x_n > x_{n+1}$ (quantized) ⇒ ~~binary~~ transmit binary '1'

$x_n < x_{n+1}$ → transmit binary '0'

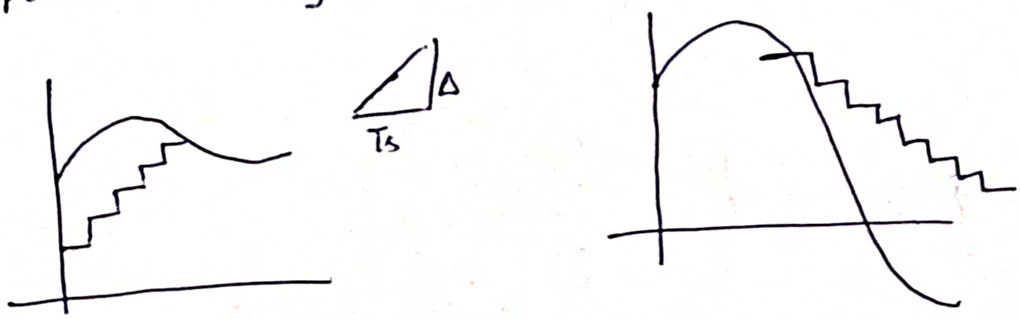
$$e(n) = x(n) - x_q(n-1)$$
$$= \Delta\, \text{sing}(e)$$
$$x_q(n) = x(n) + q(n)$$
$$= x_q(n-1) + \Delta\, \text{sing}(e)$$





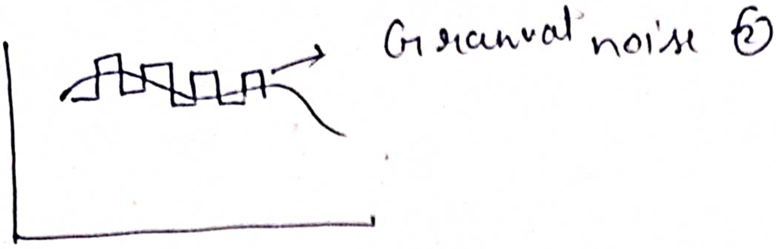


↳ if slope of signal is very less or very more then slope overloading



① Slope Overloading

A.
$$\frac{\Delta}{T_s} \geq \left| \frac{dx(t)}{dt} \right|_{max}$$



	Slope overloading	Granular Noise
$\Delta \uparrow$	$\downarrow$	$\uparrow$
$\Delta \downarrow$	$\uparrow$	$\downarrow$

→ So we have to change Δ value with slope → adaptive

Adaptive Delta modulation, ADM

$$\Delta_n = \Delta_{n-1}\, K^{e_n e_{n-1}}$$

$1\ 1 \rightarrow \uparrow$, $-1\ 1 \rightarrow \downarrow$, $1\ 0 \rightarrow \downarrow$

$\Delta_n < \Delta_{max}$

e_n = error

K = constant > 1

1.5 common choice.

$e_n = 1 \quad x_n > x_q(n-1)$

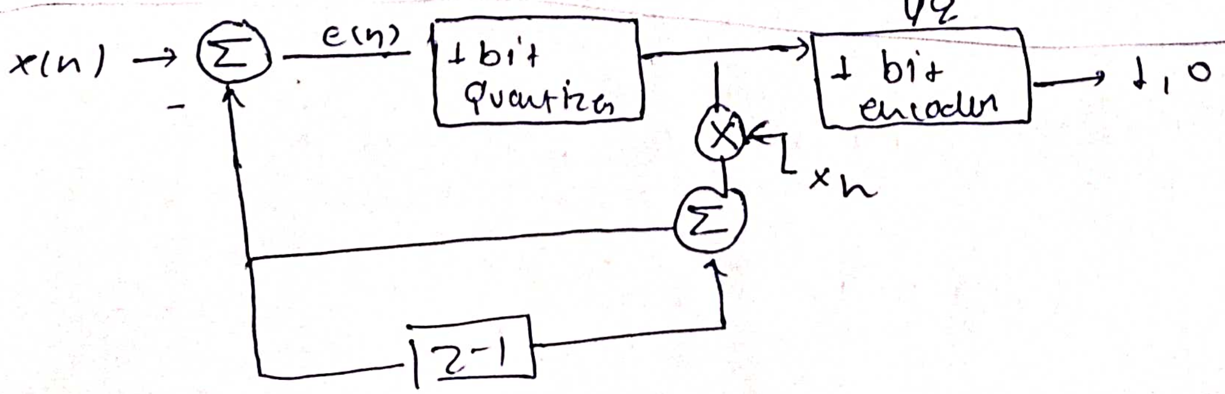
$e_n = -1 \quad x_n < x_q(n-1)$

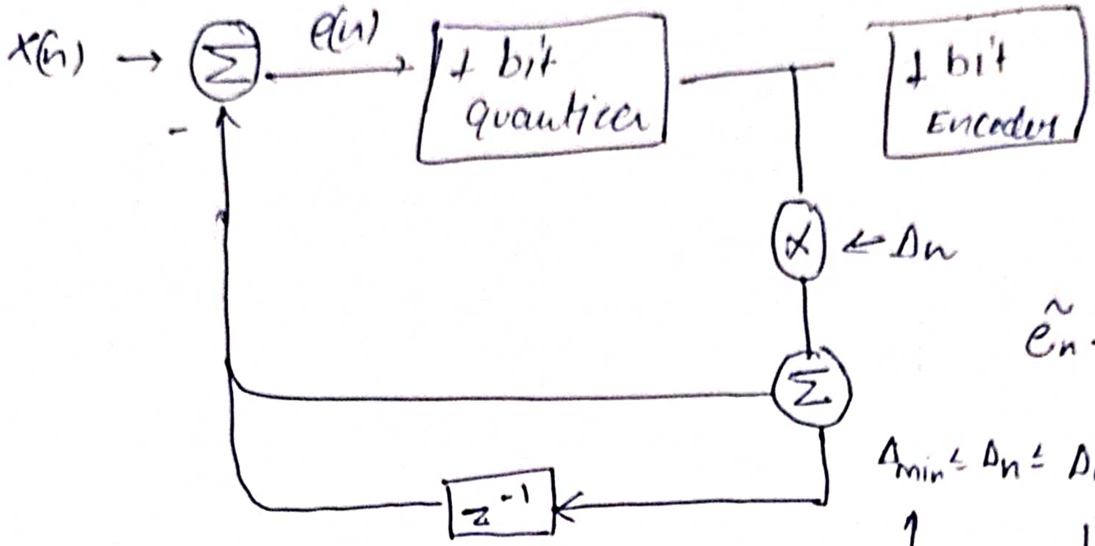
PCM, DM, DPCM, ADM

SQNR → Signal to Quantisation noise sqr

$$SQNR = \frac{\sigma_x^2}{\sigma_q^2}$$

$$SQNR_{dB} = 10 \log\left(\frac{\sigma_x^2}{\sigma_q^2}\right)$$



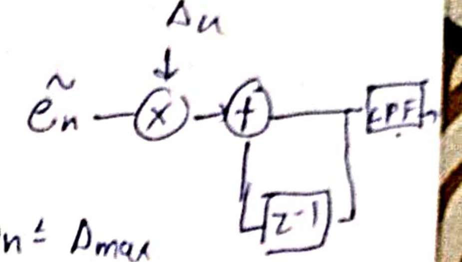


$\Delta_{min} \le \Delta_n \le \Delta_{max}$

↑ Granual Noise

↓ Slope overloading

ADM Tx



ADM Rx

$|x_{max}| = |x_{min}|$

$n \rightarrow$ no of bits

$$\Delta = \frac{x_{max} - x_{min}}{2^n}$$

$$= \frac{2x_{max}}{2^n}$$

SQNR for sinusoidal Signal

Signal Power $= \frac{A^2}{2}$

Quantisation noise $= \frac{\Delta^2}{12} = \left(\frac{2A}{2^n}\right)^2 / 12$

$$\Rightarrow SQNR = 10 \log \left(\frac{A^2 \times \cancel{12}}{2 \cdot \frac{4A^2}{2^{2n}}\ 12^2} \right) \quad \frac{2^{2n}}{8}\ 12^2$$

$$= 10 \log \left(\frac{A^2}{2} \times \frac{12}{\frac{4A^2}{2^{2n}}} \right)$$

$\tilde{e} \rightarrow$ Error difference

$$= \log \left(2^{2n} * \frac{3}{2} \right)$$

$$= 10 \log \left(\frac{3}{2} \right) + 2n (\log 2) 10$$

$$= 6n + 1.8 \text{ dB}$$

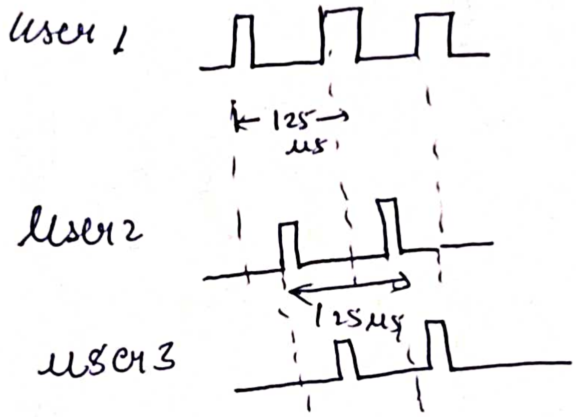
Time-Division Multiplexing

10 user → 1 signal — foundation

$f_s = 8$ kHz

$T_s = 125\ \mu s$

Sq. wave contain all freq



GSM = 8

transition smooth

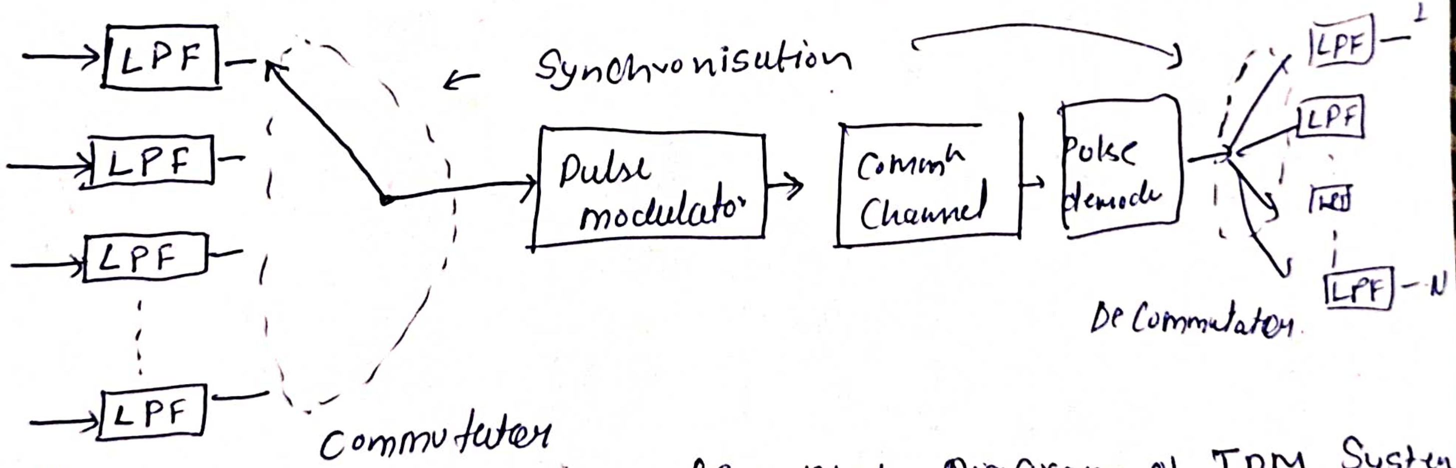
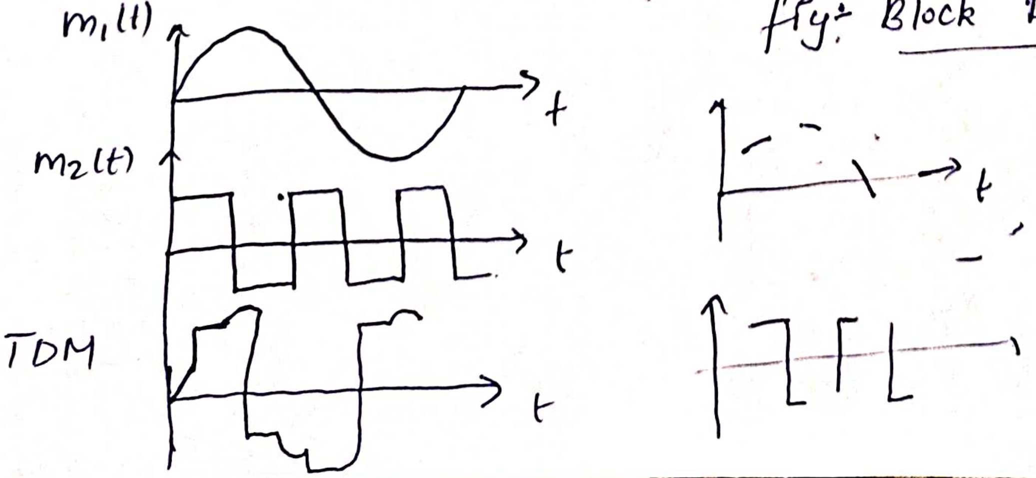


fig: Block Diagram of TDM System



Range

$$\Delta = \frac{2V}{m} \qquad M = \text{No of levels} = 2^n$$

$$\sigma_q^2 = \frac{\Delta^2}{12}$$

$$= \frac{4V^2}{M^2} \times \frac{1}{12}$$

$$= \frac{V^2}{3M^2} = \frac{V^2}{3(2^n)^2}$$

Quantisation Noise $\boxed{\sigma_q^2 = \frac{V^2}{3\cdot 4^n}}$

Signal is Random (Assume zero mean)

Signal Power $= \sigma_x^2$

$$SQNR = 10\log_{10}\frac{\sigma_x^2}{\sigma_q^2}$$

$$= 10\log\frac{\sigma_x^2}{V^2/3\cdot 4^n}$$

$$= \log\frac{\sigma_x^2}{V^2} + 10\log_{10}3 + 10\log 2^{2n}$$

$$= 10\log\frac{\sigma_x^2}{V^2} + 6n + 4.8$$

$V_{max} = 4\sigma_x \longrightarrow$ 99.99%

$$= 10\log\frac{\sigma_x^2}{4^2\sigma_x^2} + 6n + 4.8$$

$$\boxed{SQNR = 6n - 7.24}$$

$\Delta = \frac{2V}{m}$ (2V → Range) M = No of levels $= 2^n$

$\sigma_q^2 = \frac{\Delta^2}{12}$

$= \frac{4V^2}{M^2} \times \frac{1}{12}$

$= \frac{V^2}{3M^2} = \frac{V^2}{3(2^n)^2}$

Quantisation Noise $\boxed{\sigma_q^2 = \frac{V^2}{3 \cdot 4^n}}$

Signal is Random (Assume zero mean)

Signal Power $= \sigma_x^2$

$SQNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_q^2}$

$= 10 \log \frac{\sigma_x^2}{V^2/3 \cdot 4^n}$

$= \log \frac{\sigma_x^2}{V^2} + 10 \log_{10} 3 + 10 \log 2^{2n}$

$= 10 \log \frac{\sigma_x^2}{V^2} + 6n + 4.8$

$V_{max} = 4\sigma_x$ → 99.99%

$= 10 \log \frac{\sigma_x^2}{4^2 \sigma_x^2} + 6n + 4.8$

$\boxed{SQNR = 6n - 7.24}$

Q: A Speech Signal has a total duration of 10s It is Sampled at rate of 8 kHz. SQNR is required to be 40 dB, Calculate minimum Storage capacity to accomodated the digitized speech signal.

Sol^n Total time $= 10s$

$f_s = 8\,kHz$

$SQNR = 40\,dB = 6n + 1.8$

$n = 7$

Bits $= 128 = 2^7$

No of samples per sec = 8000

in 10s = 80,000

min memory req = 80,000 × 7

= 560000 = 560kb

Q: A PCM system uses a uniform quantizer followed by 7-bit encoder. The bit rate of the system is equal to 50×10^6 b/s (27)

a) what is maximum message BW for which the system operates satisfactory?

b) Determine output SQNR when a full-load sinusoidal signal of freq 1MHz is applied to the input.

Sol^n a) Let msg BW = W and $f_s = 2W \rightarrow T_s = \frac{1}{2W}$

T_b = bit duration $= \frac{T_s}{n} = \frac{1}{2W_n n} = T_b$

$$\text{Bit rate} = \frac{1}{T_b} = 2Wn = R_b \Rightarrow W = \frac{R_b}{2n}$$

$$W = \frac{50 \times 10^6 \text{ bit/s}}{2 \times 7} = 3.57 \text{ MHz}$$

b) $SQNR = 6n + 1.8 = 42 + 1.8 = 43.8\,dB$

Q: Consider a signal given $m(t) = A \tanh(\beta t)$ where A & β are constants. Determine the minimum step size Δ for a DM of this signal which ~~square~~ require to avoid slope overload

Sol^n $$\frac{\Delta}{T_s} \geq \left|\frac{dx(t)}{dt}\right|_{max} = \left.\frac{dm(t)}{dt}\right|_{max} = \frac{d}{dt} A \tanh(\beta t)$$

$$\Rightarrow A\beta\,\text{sech}(\beta t)\Big|_{max}$$

$$\Rightarrow A\beta$$

$$\Delta \geq A\beta T_s$$

$$\because \text{sech}(\beta t) = \frac{1}{\cos(\beta t)}$$

$$= \frac{2}{e^{\beta t} + e^{-\beta t}}$$

Q: A linear DM is designed to operate on speech signal limited to ~~operate on speech 3kHz~~ 3.4 kHz. This specification of modulator are follows.

↳ Sample rate = $10 f_{Nyq}$

↳ Step size = 100 mV

Find max^m amplitude of the input signal.

Sol^n

$$f_s = 10\ 2f_m = 20 f_m = 20 \times 3.4 \times 10^3 = 68\,kHz$$

$$T_s = \frac{1}{68\,k}\ \text{sec}$$

Let $m(t) = A\cos 2\pi f_m t$

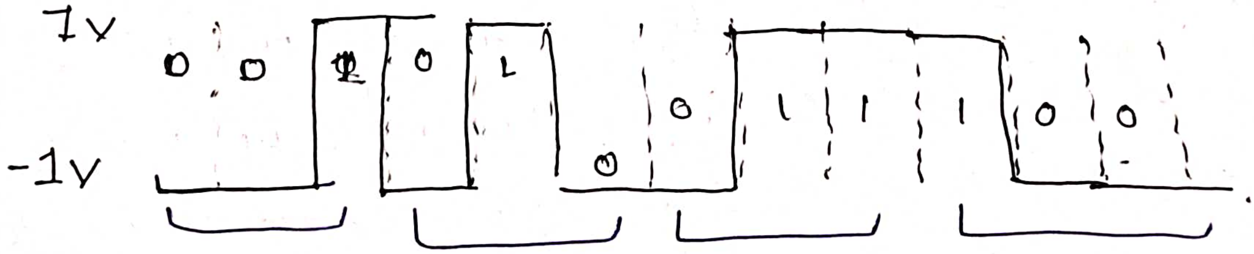
$$\frac{\Delta}{T_s} \geq \frac{d\,m(t)}{dt} = A\,2\pi f_m \Big|_{max}$$

$$\Delta \geq 2\pi f_m A\, T_s$$

$$A \leq \frac{\Delta}{2\pi f_m T_s} \leq \frac{\Delta f_s}{2\pi f_m} = \frac{20}{2\pi}\Delta = \frac{10\Delta}{\pi}$$

$$A \leq \frac{10\Delta}{\pi} = \frac{10 \times 100 \times 10^{-3}}{\pi} = 0.318$$

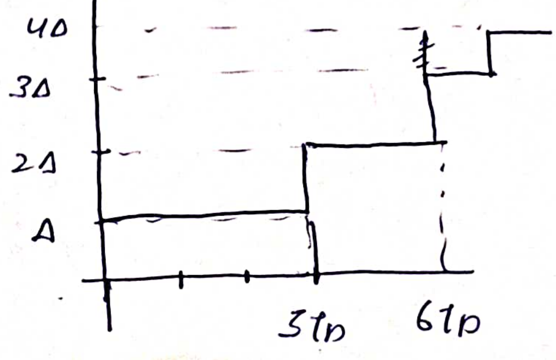
Q:



3 bit PCM

find the sampled version of analog signal for which the PCM wave is given

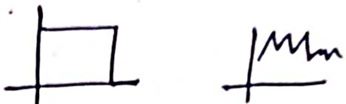
t/T_b	1	2	3	4
	001	010	011	100



$$T_b = \frac{T_s}{n} = \frac{T_s}{3}$$

Base band Pulse Communication

- Transmission of digital data over baseband channel
- Base band transmission of digital data requires the use of low-pass channel having a BW large enough to accomodate all the Essential freq contentent of the signal.
- channel is dispersive in nature



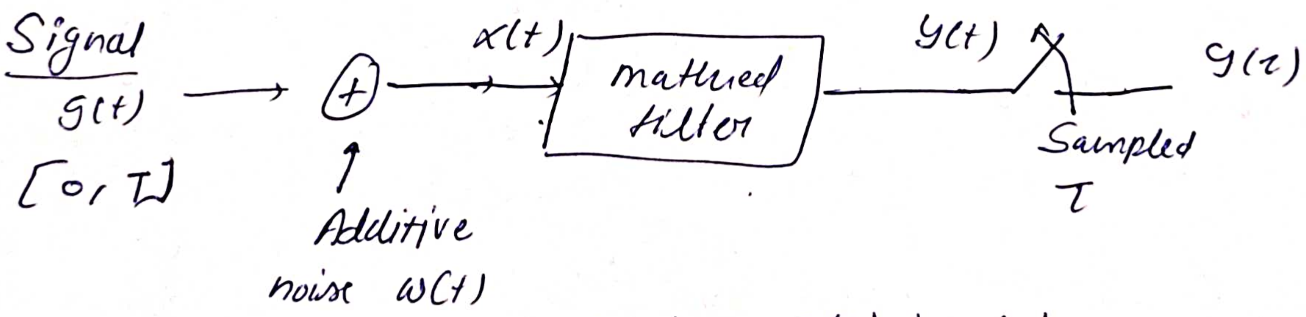
- Can't have infinite BW → (Constant gain over all frequency)
- Channel Adds noise
- 'Additive noise'

① → Each recived pulse may be affected by adjecent pulses theirby giving rise to Intersymbol Interfarance (ISI)

② The pulses are subject to channel Noise which is additive

Matched Filter

→ It is used for detection of pulse signal of known waveform that is ~~Inverse~~ in additive white noise
immersed



$$x(t) = g(t) + w(t) \qquad 0 \le t \le T$$

WSS

$$\text{PSD of noise} = \sigma_w^2$$

$$E[w(t)] = 0$$

$$E[w(t_1)\,w(t_2)] = \sigma_w^2\,\delta(t_1 - t_2)$$

Complex Signal

$$E[w(t_1)\,w(t_2)] = 0$$

$$E[w(t_1)\,w^*(t_2)] = \sigma_{w}^2\,\delta(t - t_1)$$

Matched filter - LTI System

LTI System having Impulse response $= h(t)$

$$y(t) = x(t) * h(t)$$

$$= [g(t) + w(t)] * h(t)$$

$$= g(t) * h(t) + w(t) * h(t)$$

$$= g_0(t) + n(t)$$

$$g_0(t) = g(t) * h(t)$$

$h(t) \xrightarrow{F.T} H(f)$

$g(t) \rightarrow G(f)$

↳ Pulse signal to Noise ratio

$$\eta = \frac{|g_0(T)|^2}{E[n^2(t)]}$$

$$g_0(T) = \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi fT} \, df$$

↳ (*)

$\rightarrow \boxed{h(t)} \rightarrow y(t)$

PSD σ_x^2 $\quad H(f)$ $\quad$ PSD $\sigma_x^2 |H(f)|^2$

PSD of $w(t)$

$$= S_w(f) = \sigma_w^2$$

PSD of $n(t)$ is given by

$$S_n(f) = \sigma_w^2 |H(f)|^2$$

Now $E[n^2(t)]$

$$= \int_{-\infty}^{\infty} S_n(f) \, df$$

$$= \sigma_w^2 \int_{-\infty}^{\infty} |H(f)|^2 \, df \quad \rightarrow (**)$$

$$\eta = |g_0(T)|^2 / E[n^2(t)]$$

$$= \frac{\left| \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi f T} df \right|^2}{\sigma_w^2 \int_{-\infty}^{\infty} |H(f)|^2 df} \quad - \; (**)$$

Schwartz's inequality

$\phi_1(x)$, $\phi_2(x)$ two complex fxn

$$\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx < \infty \quad \& \quad \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx < \infty$$

then

$$\left| \int_{-\infty}^{\infty} |\phi_1(x) \phi_2(x)|^2 dx \right| \leq \left[\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \right] \left[\int_{-\infty}^{\infty} |\phi_2(x)|^2 dx \right]$$

Equality holds when

$$\phi_1(x) = k^* \phi_2^*(x)$$

$$\phi_1(f) = H(f)$$

$$\phi_2(f) = G(f) e^{j2\pi f T}$$

Applying schwartz's inequality

$$\left| \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi f T} df \right|^2$$

$$\leq \int_{-\infty}^{\infty} |H(f)|^2 df \left[\left| \int_{-\infty}^{\infty} G(f) e^{j2\pi f T} df \right. \right.$$

$$\eta \leq \frac{\int_{-\infty}^{\infty} |G(f)|^2 df}{\sigma_w^2}$$

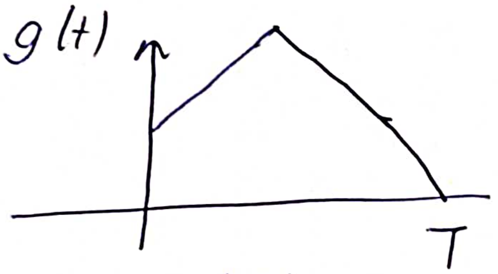
$$\eta_{max} = \int_{-\infty}^{\infty} \frac{|G(f)|^2}{\sigma_w^2} \, df$$

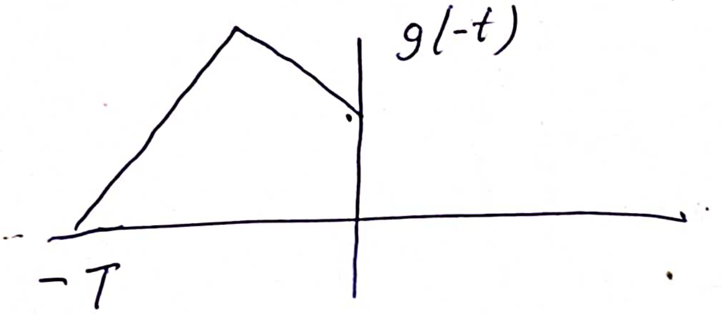
when $H_{opt}(f) = k\left(G(f)\, e^{j2\pi f T}\right)^*$

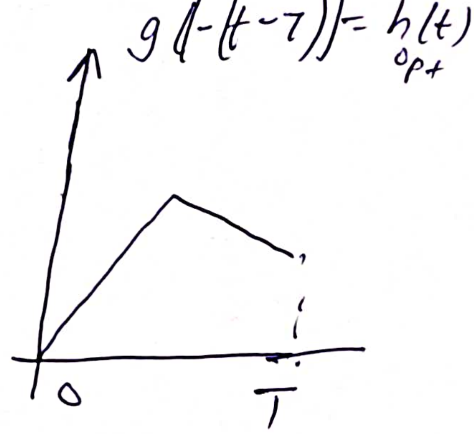
$$h_{opt}(t) = k\, g^*(-(t-T))$$

Real signal

$$h_{opt}(t) = k\, g[-(t-T)]$$

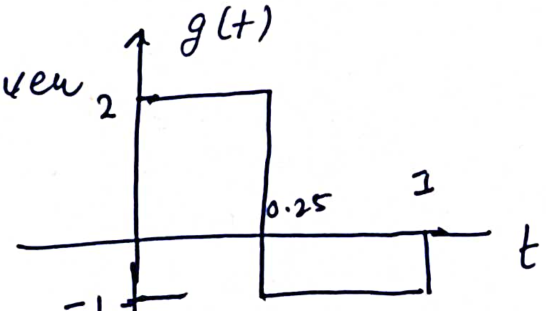






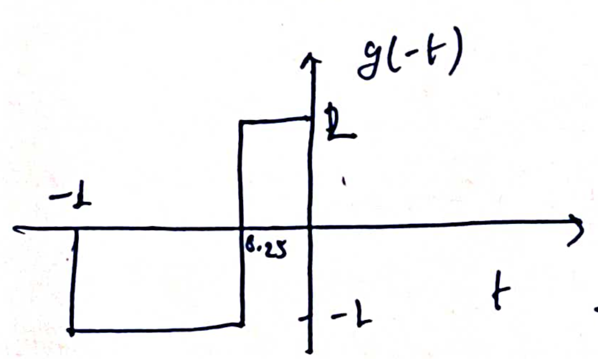
* ~~that~~ with this h(t) designed we can obtain h(t) optimum ~~Lower Bit Rate~~, + Better SNR, Lower Bit Error Rate

Q: find h optimum from g(t)? Given

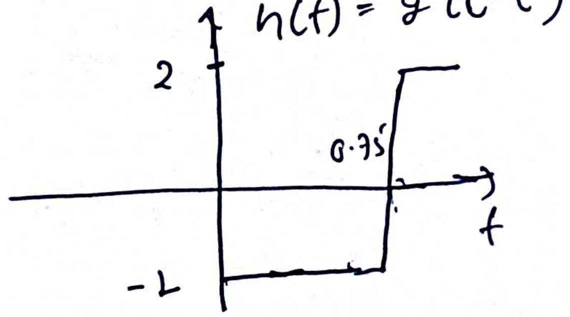


$$h_{opt} = k\, g^*(T-t)$$

Soln



$\Longrightarrow$



→ when $h(t) = h_{opt}(t) = K g^*(T-t)$ then n_{max}?

$$g_o(T) = \int_{-\infty}^{\infty} h_{opt}(f)\, G(f)\, e^{j2\pi f T}\, df$$

$$= \int_{-\infty}^{\infty} K\, G^*(f)\, e^{-j2\pi f T}\, G(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty} K\, |G(f)|^2\, df \qquad \because z z^* = |z|^2$$

$$g_o(T) = K \underbrace{\int_{-\infty}^{\infty} |G(f)|^2\, df}_{\text{Energy } E} \qquad \therefore E = \underbrace{\int_{-\infty}^{\infty} |g(t)|^2\, dt}_{\text{energy of } g(t)}$$

$$\boxed{g(T) = K E}$$

when $h(t) = h_{opt}(t)$

Rayleigh Energy theorem

$$E = \int_{-\infty}^{\infty} |g(t)|^2\, dt$$

$$E = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

*

$$E\left[(n(t))^2\right] = \sigma_w^2 \int_{-\infty}^{\infty} |h_{opt}(t)|^2\, df$$

$$= \sigma_w^2 \int_{-\infty}^{\infty} |K\, G^*(f)\, e^{-j2\pi f T}|\, df$$

$$= \sigma_w^2\, |K|^2 \int_{-\infty}^{\infty} |G^*(f)|\, df$$

$$E\left[n(t)\right]^2 = \sigma_w^2\, |K|^2\, E$$

Now, $\eta_{max} = \dfrac{|g_o(T)|^2}{E[n(t)^2]}$ when $h(t) = h_{opt}(t)$

$$\eta_{max} = \frac{|k|^2 E^2}{\sigma_w^2 |k|^2 \cdot E} = \frac{E}{\sigma_w^2}$$

$$\boxed{\eta_{max} = \frac{E}{\sigma_w^2}}$$

Signal Energy to noise spectral density

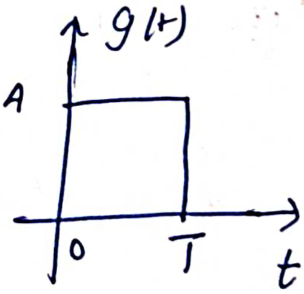
$E \longrightarrow J$

$\sigma_w^2 \longrightarrow$ watt/Hz

$\eta_{max} \rightarrow$ dimensionless.

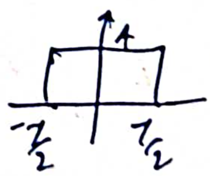
$SNR = \dfrac{\text{Signal Power}}{\text{Noise Power}} = \dfrac{E}{\sigma_w^2} \rightarrow$ No/2 PSD of Noise

Matched filter for rectangular pulse



Let $g(t) = A\,\text{rect}\left(\frac{t}{T} - \frac{T}{2}\right)$

$h_{opt}(t) = k\,g^*(t - \tau)$



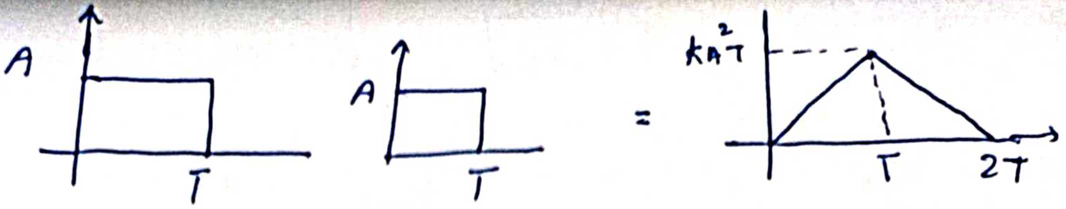
$= A\,\text{rect}\left(\frac{t}{\tau}\right)$

↳ Same wave ~~wave~~ when fold and shift we will get same signal

$h_{opt}(t) = k\,g(t)$

$g_o(t) = g(t) * h_{opt}(t)$

$g_o(t) = k\,g(t) * g(t)$



$$y(t) = \int_0^t x(t)\, h_{opt}(t-\tau)\, d\tau$$

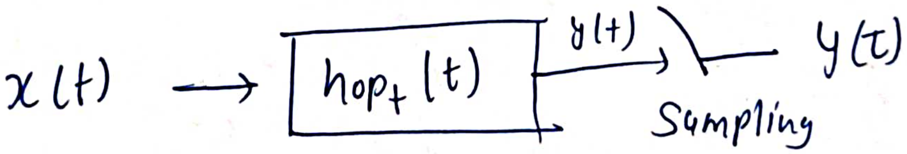
$$y(t) = \int_0^t x(t-\tau)\, h_{opt}\, d\tau$$

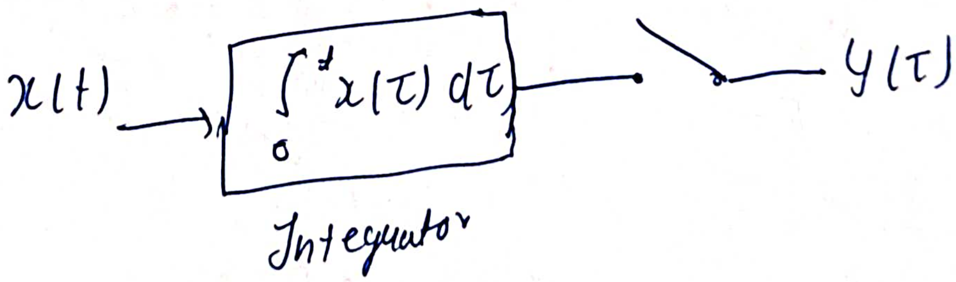
$$y(t) = KA \int_0^t x(\tau)\, d\tau \qquad h_{opt} = KA$$

$$y(t) = KA \int_0^t x(\tau)\, d\tau \qquad 0 \le t \le T$$

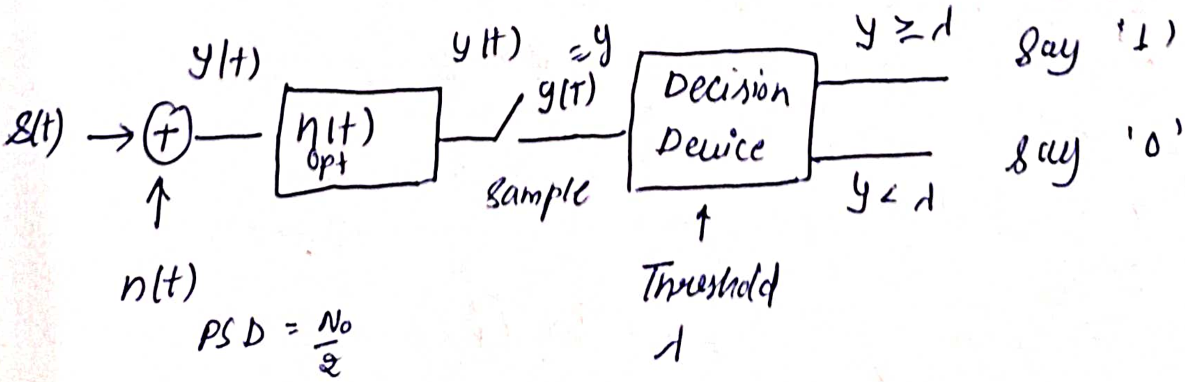
$$= KA \int_{t-T}^{t} x(\tau)\, d\tau \qquad T \le t \le 2T$$

$y(t)$ is interest

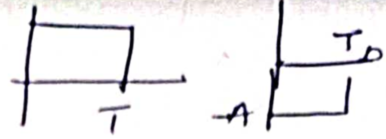




Error rate Due to Noise



$$s(t) = A\,\mathrm{rect}\left(\frac{t}{T_b} - \frac{T_b}{2}\right) \hookrightarrow \text{"1" is transmitted}$$



$$= -A\,\mathrm{rect}\left(\frac{t}{T_b} - \frac{T_b}{2}\right) \hookrightarrow \text{"0" is transmitted}$$

Recived Signal

Reciver should have knowledge of
- Starting and End time of the pulse
- pulse shape

$x(t) = A + n(t)$ when "1" was transmitted but "0"

$= -A + n(t)$ " "0" "

noise $\quad 0 \le t \le T_b$

Reciver supposed to determine the polarity of the transmitted pulse

Error – Two type of Error

↪ if "1" is decided but '0' was actually transmitted, the Error is called Error of first kind.

↳ if "0" is detected but "1" → Error of second kind

Tx	Rx	Error
0 →	1	first kind
1 →	0	second kind

Probability of Error of first kind $\quad 0 \to 1$

$$x(t) = -A + n(t) \qquad 0 \le t \le T_b$$

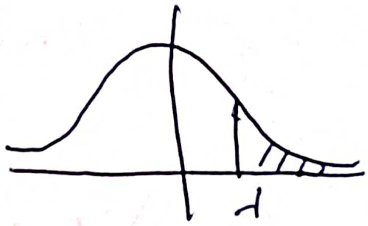
$$y(T) = KA \int_0^{T_b} x(t)\,dt$$

$$y = KA \int \left(-A + n(t)\right) dt$$

~~$y = KA^2 T_b +$~~

$w(t)$

$$y = kA \int_0^{T_b} (-A + n(t))\,dt$$



$$= -KA^2 T_b + KA \int_0^{T_b} n(t)\,dt$$

$$P_y(y/0) = \frac{1}{\sqrt{2\pi \sigma_{y_0}^2}}\, e^{-\frac{(y-m_{y_0})^2}{2\sigma_{y_0}^2}}$$

$$m_{y_0} = m_0 = E(y)$$

$$= E\Big[-\underbrace{KA^2}_{\text{Const}} T_b + \int_0^{T_b} \cancel{B}\, n(t)\,dt\Big]$$

$$= -KA^2 T_b + \int_0^{T_b} \underbrace{E[n(t)]}_{\text{zero mean}}\,dt$$

we assume

$$= -KA^2 T_b$$

$$\underline{m_{y_0} = -KA^2 T_b}$$

$$\sigma_{y_0}^2 = E\left[(y - m_{y_0})^2\right]$$

$n(t) = w(t)$

notation

$$= E\left[(KA)^2 \iint \cancel{A}(t)\, n(u)\,dt\,du\right]$$

$$= KA \int_0^{T_b}\int_0^{T_b} E[n(t)\,n(u)]\,dt\,du$$

covariance

$$\therefore E[w(t)w(t)] = \frac{N_0}{2}$$

$$= \frac{N_0}{2}\,\delta(t-u)$$

$$\sigma_{y_0}^2 = (KA)^2 \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\,\delta(t-u)\,dt\,du$$

$$\sigma_{y_0}^2 = (KA)^2 \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\,\delta(t-u)\,dt\,du$$

$$\therefore \int_0^{T_b} \delta(\underset{\underset{T_b}{\shortparallel}}{t}-u) = 1$$

$$(KA)^2 \int_0^{T_b} \frac{N_0}{2} \cdot 1\, du$$

$$\boxed{\sigma_{y_0}^2 = (KA)^2\, T_b\, \frac{N_0}{2}}$$

$\cancel{P(y/0)}$ $P(y/0) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-m_{y_0})^2}{2\sigma^2}}$

2000 trans →

$$m_{y_0} = -KA^2 T_b$$

$$\sigma_{y_0}^2 = (KA)^2\, T_b\, \frac{N_0}{2}$$

$$P_0\,(y > \lambda) = \int_\lambda^\infty P(y/0)\,dy$$

$$= \int_\lambda^\infty \frac{1}{\sqrt{2\pi\sigma_{y_0}^2}}\, e^{-\left(\frac{y-m_{y_0}}{2\sigma_{y_0}^2}\right)^2} dy$$

Let $V = \dfrac{y-m_{y_0}}{\sigma_{y_0}} \rightarrow dV = \dfrac{dy}{\sigma_{y_0}}$

$dy = \sigma_{y_0}\, dv$

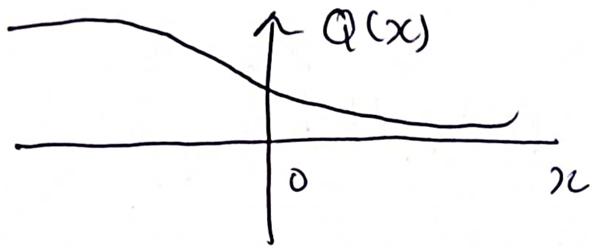
when $y = \lambda$

$$V = \frac{\lambda - m_{y_0}}{\sigma_{y_0}}$$

$$P_{e_0} = \int_{\frac{\lambda - m_{y_0}}{\sigma_{y_0}}}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-V^2/2}\, dV$$

$$P_{e0} = \int_{\frac{1-m_{y_0}}{\sigma_{y_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv$$

$$\therefore \; Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \qquad \mu = 0, \; \sigma = 1$$



$$P_{e0} = Q\left(\frac{1-m_{y_0}}{\sigma_{y_0}}\right)$$

<u>Error of second kind</u>

$1 \to 0$

$$x(t) = A + w(t) \qquad 0 \le t \le T_b$$

$$y(T) = KA \int_0^{T_b} (A + w(t))\, dt$$

$$= KA^2 T_b + KA \int_0^{T_b} w(t)\, dt$$

$$P(y|1) = \frac{1}{\sqrt{2\pi \sigma_{y_0}^2}} e^{-\frac{(y-m_{y_0})^2}{2\sigma_y^2}}$$

$$m_{y_0} = m_0 = E(y)$$

$$= E\left[kA^2 T_b + \int_0^{T_b} w(t)\,dt\right]$$

$$= kA^2 T_b + \int_0^{T_b} E[w(t)]\, dt$$

$\hookrightarrow$ zero mean we assume

$$m_{y_1} = KA^2 T_b$$

Now $\sigma_y^2 = E\left[(y - m_{y_0})^2\right]$

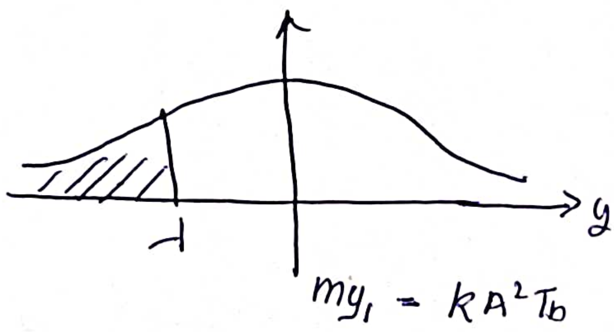
$$\therefore \quad y = KA^2 T_b + \int_0^{T_b} KA\, w(t)\, dt$$

$$= E\left[\left(\cancel{KA^2 T_b} + \int_0^{T_b} KA\, w(t)\, dt - \cancel{(KA^2 T_b)}\right)\right]$$

$$= E\left[\left(KA\int_0^{T_b} w(t)\, dt\right)^2\right]$$

σ

$$\sigma_{y_1} = (KA)^2\, T_b\, \frac{N_0}{2}$$



$$P_{e_1} = P(y < d)$$

$$= \int_{-\infty}^{d} P(y/1)\, dy$$

$$= 1 - Q\left(\frac{d - m_{y_1}}{\sigma_{y_1}}\right)$$

$\therefore$ property $= Q(-u)$
$= 1 - Q(u)$

$$P_{e_1} = 1 - Q\left(\frac{d - m_{y_1}}{\sigma_{y_1}}\right)$$

Some text book

$$\operatorname{erfc}(u) = \frac{1}{\sqrt{\pi}} \int_u^\infty e^{-x^2}\,dx$$

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-x^2/2}\,dx$$

$$\operatorname{erfc}(u) = 2Q(u\sqrt{2})$$

$$Q(u) = \frac{1}{2}\operatorname{erfc}(u/\sqrt{2})$$

P_0 = apriori (before) probability of 0

P_1 = " " 1

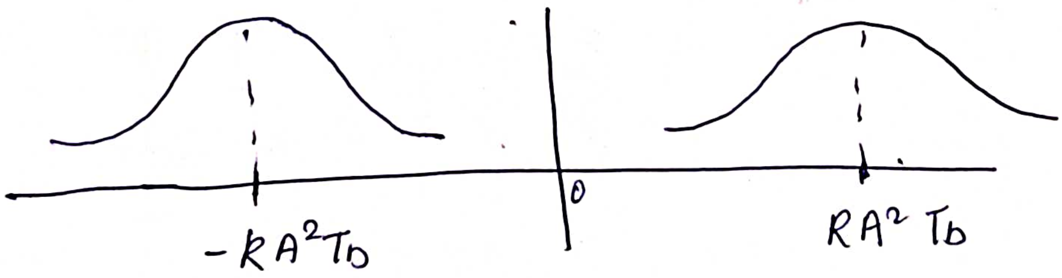
$0 < P_0, P_1 < 1$

$P_0 + P_1 = 1$

Avg probability Error

$$P_e = P_0 P_{e0} + P_1 P_{e0}$$

$$P_e = P_0\, Q\left(\frac{\lambda - m_{y_0}}{\sigma_{y_0}}\right) + P_1\left[1 - Q\left(\frac{\lambda - m_{y_1}}{\sigma_{y_1}}\right)\right]$$



threshold without noise $\lambda = 0$

$$P_e = P_0\, Q\left(-\frac{m_{y_0}}{\sigma_{y_0}}\right) + P_1\left(1 - Q\left(\frac{\lambda - m_{y_1}}{\sigma_{y_1}}\right)\right)$$

$$= P_0\, Q\left(-\frac{m_{y_0}}{\sigma_{y_0}}\right) + P_1\left[1 - Q\left(-\frac{m_{y_1}}{\sigma_{y_1}}\right)\right]$$

$$= P_0\, Q\left(\frac{m_{y_1}}{\sigma_{y_1}}\right) + P_1\left(1 - Q\left(-\frac{m_{y_1}}{\sigma_{y_1}}\right)\right) \quad \because Q(-u) = 1 - Q(u)$$

$$P_0\, Q\left(\frac{m_{y_1}}{\sigma_{y_1}}\right) + P_1\, Q\left(\frac{m_{y_1}}{\sigma_{y_1}}\right)$$

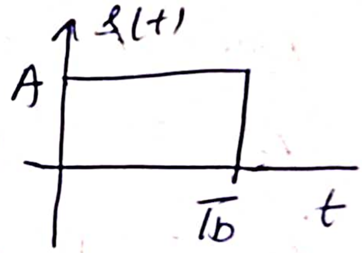
$$= \underbrace{(P_0 + P_1)}_{1}\, Q\left(\frac{m_{y_1}}{\sigma_{y_1}}\right)$$

$$P_e = Q\left(\frac{m_{y_1}}{\sigma_{y_1}}\right) = Q\left(\frac{KA^2 T_b}{\sqrt{(KA)^2 T_b N_0/2}}\right)$$

$$= Q\left(\frac{\cancel{K}A^2 T_b}{\cancel{KA}\sqrt{T_b N_0/2}}\right)$$

$$P_e = Q\left(\sqrt{\frac{2A^2 T_b}{N_0}}\right)$$

$$\boxed{P_e = Q\left(\sqrt{2\frac{E}{N_0}}\right)}$$



$$E = \int_0^{T_b} s^2\, dt$$

$$E = A^2 T_b$$