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ITC ASSIGNMENT - 1

Q:1

Soln $H(x) = \sum P(x) \log \frac{1}{P(x)}$ and $H(X/Y) = \sum\sum P(x,y) \log \frac{1}{P(x/y)}$

or

$\sum P(x/y) \log \frac{1}{P(x/y)}$ — (1)

$\Rightarrow$ when x & y are independen

$P(x,y) = P(x) P(y)$

$P(x,y) \Rightarrow P(\cancel{y}) P(x/y) = P(x) \cancel{P(y)}$

$P(x/y) = P(x)$ $\rightarrow$ Put in Eq (1)

$$\boxed{H(x/y) = \sum P(x) \log \frac{1}{P(x)} = H(x)}$$

Q:2

Soln Generally or on an average we Can Say that Yes Conditioning Reduces Entropy,

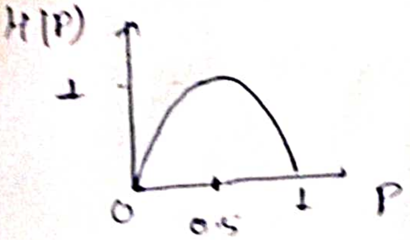
$I(X,Y) = H(X) - H(X/Y)$

$\because I(X,Y) \geq 0$

$H(X) - H(X/Y) \geq 0 \Rightarrow H(X) \geq H(X/Y)$ — (1)

$\rightarrow$ ~~But~~ knowing random variable Y can reduce the uncertainity in ~~x~~ that's why Eq (1) holds good, But $H(X/Y=y)$ may be greater. For Example in a court case, new evidence might increase uncertainity

Q 3 $\quad H(x) = \sum p(x) \log \frac{1}{p(x)}$



$\because \quad 0 \le P(x) \le 1 \Rightarrow \log \frac{1}{P(x)} \ge 0$

$H(x) \ge 0$

$\Rightarrow$ minimum value of $\boxed{H(x) = 0}$ when $P(x) = 0$

Q:4

$H(x) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8$

$= \frac{1}{2} + \frac{1}{2} + 2 \times \frac{3}{8} \Rightarrow 1 + \frac{3}{4} \Rightarrow \boxed{H(x) = \frac{7}{4} \text{ bit}}$

$H(y) = 4 \times \frac{1}{4}\log 4 = 4 \times \frac{1}{2} = 2$ bits

$H(x/y) = \sum_{i=1}^{4} P(Y=i)\, H(X/Y=i)$

$\Rightarrow \frac{1}{4} H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$

$+ \frac{1}{4}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4} H(1,0,0,0)$

$= \frac{1}{4} \times \frac{7}{4} + \frac{1}{4} \times \frac{7}{4} + \frac{1}{4} \times 2 + \frac{1}{4} \times 0$

$\boxed{H(x/y = \frac{11}{8} \text{ bits}}$

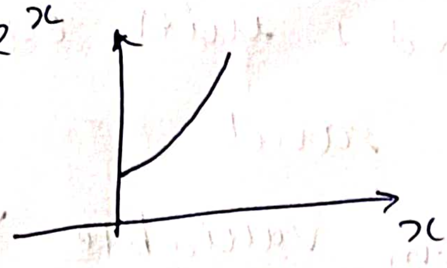
Similarly $\boxed{H(Y/\cancel{x}) = \frac{13}{8}}$

$H(x,y) = H(y) + H(x/y) = 2 + \frac{11}{8}$

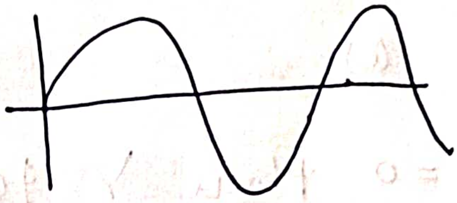
$\boxed{H(x,y) = \frac{27}{8} \text{ bits}}$

Q:5

(a) $Y = 2^x$ → always increasing function



(b) $Y = \cos x$ →



Let $y = g(x)$

$$p(y) = \sum_{y=g(x)} p(x)$$

Lets.
Consider any set of x's that map onto a single y

$$\sum p(x) \log p(x) \le \sum p(x) \log p(y)$$

$$= p(y) \log p(y)$$

Since log is monotonic increasing fxn and $p(x) \le \sum p(x) = p$

e Extending this argument to the entire range of x and y

$$H(X) = -\sum p(x) \log p(x)$$

$$= -\sum_y \sum_{x: y=g(x)} p(x) \log p(x)$$

$$= -\sum_y p(y) \log p(y)$$

$$= H(y) \quad \text{if } g \text{ is one to one with } p=1$$

(a) $Y = 2^X$ is one to one so $\boxed{H(X) = H(Y)}$

(b) $Y = \cos x$ not neccessarily one to one

$$\boxed{H(x) \ge H(Y)}$$

Q:6

Example : Let x be a random variable that takes values 0 and 1 with equal probabi (i.e a fair coin toss) and

Let another random variable Y which is defined as

(i) if $X=0$ then Y takes values 0 & 1 with equal probability

(ii) if $X=1$ then Y always 0

$$Y = \begin{cases} 0 & X=0 \\ 1 & X=0 \\ 0 & X=1 \end{cases}$$

→ So when $X=0$ Y has some uncertainity and when $X=1$ Y has no uncertainity, therefore knowing X reduces uncertainity

$$H(Y/x) < H(Y)$$

→ ~~also~~ Also knowing Y does not provide Compl information about X because when $Y=0$, X co $X=0$ or $X=1$. This increases the uncertainity

$$H(X/Y) > H(X)$$

Q:7 (a) $I(X:Y|Z) \not< I(X;Y)$

(b) $I(X;Y|Z) > I(X;Y)$

Solⁿ

from Data processing Inequality

if $X \rightarrow Y \rightarrow Z$ then

$I(X:Z) \leq I(X:Y)$ and

$I(X;Z) \leq I(Y,Z)$

$I(X:Y|Z) = H(X|Z) - H(X|YZ)$

and

$I(X:Y) = H(X) - H(X|Y)$

if $H(X|Y) = H(X|YZ) \Rightarrow I(X:Y|Z) = 0$

i.e $I(X:Z) = 0$ or X and Z are independent

$I(X:Y|Z) = H(X|Z) - H(X|YZ) = 0$

$I(X:Y) = H(X) - X(X|Y) = H(X) = 1$

$$\boxed{I(X;Y) > I(X;Y|Z)}$$

(b) Let X, Y be independent fair binary random variable

Let $Z = X + Y$ then

$$I(X:Y) = 0$$

and $I(X:Y/Z) = H(X/Z) = \frac{1}{2}$

So $\boxed{I(X:Y) < I(X;Y/Z)}$

Q:8 fanno's Lemma

$$H(P) + P\log(L-1) \geq H(U/\hat{U})$$

$L=3$ (given that x takes values from ternary set)

$$H(P) + P\log 2 \geq H(X/Y) \qquad X \sim U, \; Y \sim \hat{U}$$

$$H(P) + P \geq H(X/Y) \quad -① $$

$$I(X:Y) = H(X) - H(X|Y) \quad -②$$

from eq ① and ②

$$I(X;Y) \geq H(X) - H(P) - P$$

lower Bound

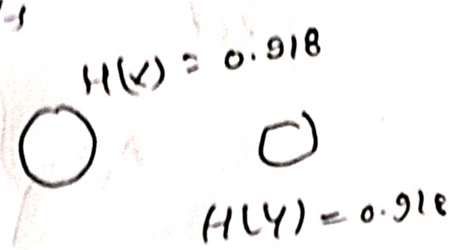
$$I(X;Y) = 3 \times \frac{1}{3}\log_2 3 - H(P) - P$$

$$= \log_2 3 - H(P) - P$$

Q: 9

Sol^n (a) $H(X) = \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 = 0.918$ bits

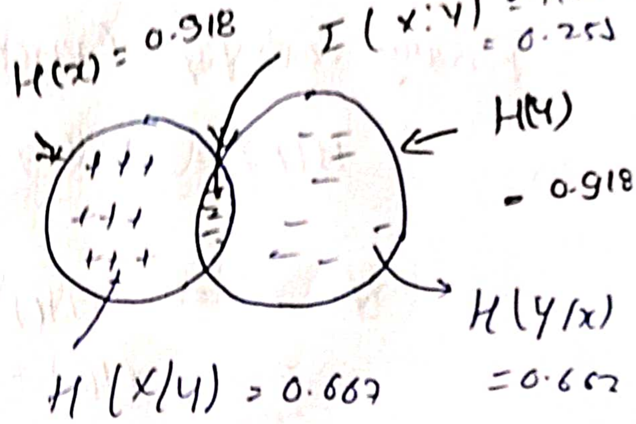
$H(Y) = 0.918$ bits



(b) $H(X/Y) = \frac{1}{3}H(X/Y=0) + \frac{2}{3}(X/Y=1)$

$= 0.667$ bits

$H(Y/X) = 0.667$ bits



(c) $H(X,Y) = H(X) + H(Y/X)$

$= 0.918 + 0.667$

$H(X,Y) = 1.585$ bits

(d) $H(Y) - H(Y/X) = 0.251$ bits

(e) $I(X:Y) = H(Y) - H(Y/X)$

$= 0.918 - 0.667$

$= 0.251$

Q: 10

Sol^n

$H(P) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 = 1.5$ bits

$H(q) = \frac{1}{3}\log 3 + \frac{1}{3}\log 3 + \frac{1}{3}\log 3 = 1.58491$ bits

$D[P||q] = \frac{1}{2}\log 3/2 + \frac{1}{4}\log 3/4 + \frac{1}{4}\log 3/4 = 0.08496$

$D(q||P) = \frac{1}{3}\log\frac{2}{3} + \frac{1}{3}\log\frac{4}{3} + \frac{1}{3}\log\frac{4}{3} = 0.08170$

Q:11

Let

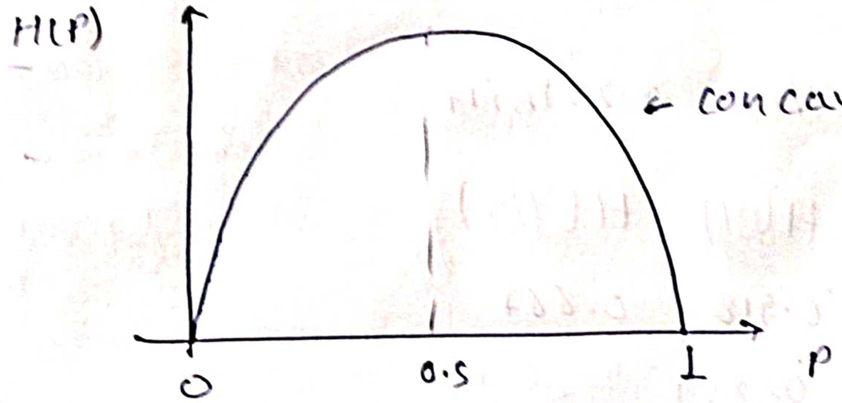
$$X = \begin{cases} 1 & \text{with } P(1) = P \\ 0 & \text{with } P(0) = 1-P \end{cases}$$

Binary Entropy function

$$H(P) = P \log \frac{1}{P} + (1-P) \log_2 \frac{1}{1-P}$$

if $P = 0$ $H(P) = 0$

$P = 1$ $H(P) = 0$



the function of H(P) is concave so ~~[illegible]~~ and symmetric around $P = \frac{1}{2}$ i.e $P = 1-P$ source are equiprobable

M-2 $\frac{d}{dP} H(P) = 0$ & $\frac{d^2}{dP^2} H(P) < 0$

after solving

$$P = \frac{1}{2}$$

Q: 12

$$H(x) = \sum p_i \log \frac{1}{p_i}$$

$X_i = 1, 2, 3 \ldots$ no of tosses to get first Head
$x_1, x_2 \ldots x_n$

$P(X = x_1) = \frac{1}{2}$ $\qquad P(X = x_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$P(X = x_2) = \frac{1}{4} \Rightarrow \frac{1}{2} \times \frac{1}{2}$

$$H(X) = 0 + \underbrace{1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8}} + \ldots$$

$$\Rightarrow \frac{a}{1-r} + \frac{d \times r}{(1-r)^2} \qquad a = 0 \quad d = \frac{1}{2} \quad r = \frac{1}{2}$$

$$H(x) = \frac{0}{1 - \frac{1}{2}} + \frac{\frac{1}{2}(1)}{\left(1 - \frac{1}{2}\right)^2} \qquad H(X) = 2$$

For biased / unfair coin

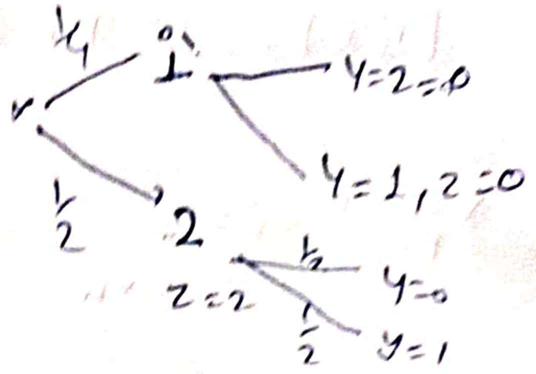
$P_i = P$

$$\Rightarrow 0 + 1 \cdot \frac{1}{P} + 2 \cdot \frac{1}{P^2} + \ldots$$

$$\Rightarrow \frac{\frac{1}{P}}{\left(1 - \frac{1}{P}\right)^2} = \frac{1/P}{(1 - 1/P)^2}$$

Q: 13

$$P(x) = \begin{cases} \frac{1}{4} & x = 0, 1 \\ 1/2 & x = 2 \end{cases}$$



$P(x=0) = \frac{1}{4}$ $P(x=1) = \frac{1}{4}$ $P(x=2) = \frac{1}{2}$

$H(x) = \sum p(x) \log \frac{1}{p(x)} = \frac{1}{4} \log 4 + \frac{1}{4} \log 4 + \frac{1}{2} \log 2$

$= 2 \times \frac{1}{4} \log 4 + \frac{1}{2} \log 2$

$\Rightarrow \log 2 + \frac{1}{2} \log_2 = 1.5$ bits

Now for Y

when $x=0$ $Y=0$ with probability $=1$

when $x=1$ $Y=1$ with $p=1$

when $x=2$ $Y \begin{cases} 0 \\ 1 \end{cases}$ with $p = 1/2$

$P(Y=0) = P(Y=0 / x=0) P(x=0) + P(Y=0/x=2) * P(x=2)$

$\Rightarrow \frac{1}{4} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

$P(Y=1) \Rightarrow P(Y=1/x=1) P(x=1) + P(Y=1/x=2) P(x=2)$

$\Rightarrow 1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$H(Y) = \frac{1}{2} \log 2 + \frac{1}{2} \log 2 = \boxed{H(Y) = 1 \text{ bit}}$

Q:13

Soln Now for Z

when $X=0$ $Z=0$ $P=1$

$X=1$ $Z=0$ $P=1$

$X=2$ $Z=1$ $P=1$

$P(Z=0) = P(Z=0/X=0)P(X=0) + P(Z=0/X=1)P(X=1)$

$= 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = \frac{1}{2}$

$P(Z=1) = P(Z=1/X=2)\,P(X=2) \Rightarrow 1 \cdot \frac{1}{2} = \frac{1}{2}$

$H(Z) = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 \Rightarrow \boxed{H(Z) = 1 \text{ bit}}$

Now

$H(Y/X) = \sum P(X=x_i)\, H(Y/X=x_i)$

$H(Y/X=0) = 0$, when $X=0$, $Y=0$

$H(Y/X=1) = 0$, when $X=1$ $Y=1$

$H(Y/X=2) = 1$ bit when $X=2$ Y is either 0 or 1

overall Entropy

$H(Y/X) = \sum P(X=x_i)\, H(Y/X=x_i)$

$= \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1$

$\boxed{H(Y/X) = \frac{1}{2}}$

⇒ Now $H(X,Y)$ Joint Entropy

$P(X=0, Y=0) = \frac{1}{4}$

$P(X=1, Y=1) = \frac{1}{4}$

$P(X=2, Y=0) = \frac{1}{4}$

$P(X=2, Y=1) = \frac{1}{4}$

$H(X,Y) = \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{4}$

$= \frac{4\log_2 4}{4} = 2$

$\boxed{H(X,Y) = 2\text{bits}}$

M-2

$H(X,Y) = H(X) + H(Y/X)$

$\Rightarrow 1.5\text{ bits} + 0.5\text{ bits} = 2\text{bits}$

⇒ $H(X/Z)$.

$P(X=0/Z=0) = 1/2$ $\quad P(X=2/Z=1) = 1$

$P(X=1/Z=1) = 1/2$ $\quad P(X=2/Z=0) = 1$

$H(X/Z) = \frac{1}{2}\log 2 + \frac{1}{2}\log 1 = 0$

$\boxed{H(X/Z) = 1/2}$

$\Rightarrow H(X,Z) = H(Z) + H(X/Z) \Rightarrow 1 + \frac{1}{2} = 1.5$

⇒ For $H(Y,Z)$

$P(Y=0, Z=0) = \frac{1}{4}$ $\quad P(Y=0, Z=1) = \frac{1}{4}$

$P(Y=1, Z=0) = \frac{1}{4}$ $\quad P(Y=1, Z=1) = \frac{1}{4}$

$$H(Y,Z) = \sum_{i} P(Y=y_i / Z=z_i) \log \frac{1}{P(Y=y_i, Z=z)}$$

$$\Rightarrow \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$$

$$\boxed{H(Y,Z) = 2\text{bits}}$$

Now $H(Z/Y)$

When $Y=0$ $\quad P(Z=0/Y=0)=1 \rightarrow H(Z/Y=0) = 0\text{ bits}$

no uncertainity

$Y=1$ $\quad P(Z=0/Y=0) = 0$ $\quad H(Z/Y=1) = 0$

and

$P(Z=1/Y=1) = 1$ $\quad$ no uncertainity

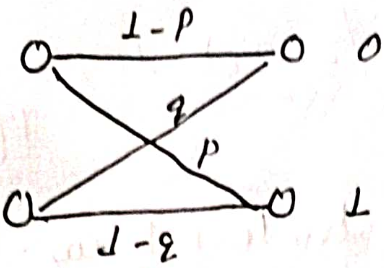
$$H(Z/Y) = P(Y=0)\,H(Z/Y=0) + P(Y=1)\,H(Z/Y=1)$$

$$\Rightarrow \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 = 0$$

$H(X)$	$H(Y)$	$H(Z)$	$H(Y/X)$	$H(X,Y)$	$H(X/Y)$	$H(X,Z)$	$H(Y,Z)$
1.5 bits	1 bit	1 bit	½ bit	2 bit	1 bit	2 bit	2 bit

Q: 14 Non Symmetric Binary Channel

$P(0) = \alpha$



$\Omega(P) = P \log \frac{1}{P} + 1-P \log \frac{1}{1-}$

$P(1) = 1-\alpha$

$$I(X,Y) = \Omega(q + (1-p-q)\alpha) - \alpha\,\Omega(q)$$

Solⁿ ~~P(X~~ $H(X) = a \log \frac{1}{a} + (1-a) \log \frac{1}{1-a}$

$P(Y=1 / X=0) = P$ $\quad P(X=0) = a$ $\quad P(Y=0 / X=0$

$= 1-$

$P(Y=0 / X=1) = q$ $\quad P(X=1) = 1-a$

Q:15

Soln $P\left(\frac{S_j}{S_i}\right) = P/2 \quad i \neq j$

$q = 3$

$M=1$ first order

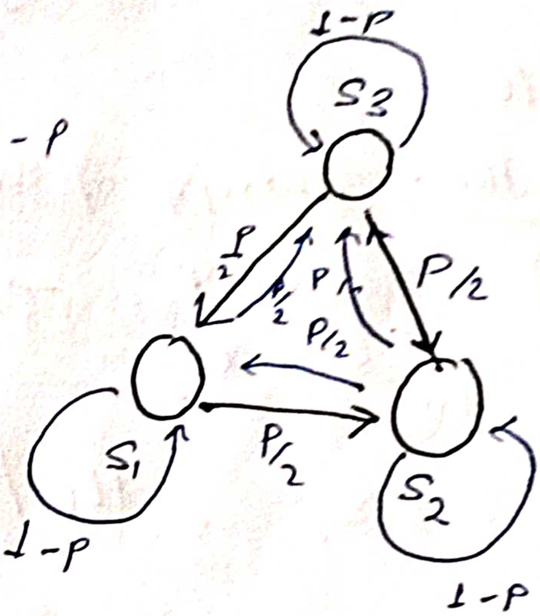
No of States $= 3^1$
$= 3$

for S_1

$P(S_2/S_1) = P/2$

$P(S_3/S_1) = P/2$

$P(S_1/S_1) = 1 - \left(P/2 + \frac{P}{2}\right) = 1 - P$



Similary for S_2, S_3

b) $P(S_i) = \sum_{j=1}^{3} P(S_i/S_j)\, P(S_j)$

$P(S_1) = P(S_1)(1-P) + P(S_2)\frac{P}{2} + P(S_3)\frac{P}{2}$

$P(S_2) = P(S_2)(1-P) + P(S_1)\frac{P}{2} + P(S_3)\frac{P}{2}$

$P(S_3) = P(S_3)(1-P) + P(S_1)\frac{P}{2} + P(S_2)\frac{P}{2}$

(c) $H(S_j/S_i) = \sum_{i=1}^{3} P(S_j/S_i) \log \frac{1}{P(S_j|S_i)}$

$H|S_j|S_i| = \frac{P}{2} \log \frac{1}{P/2} + \frac{P}{2} \log \frac{1}{P/2} + (1-P) \log \frac{1}{1-P}$

(d) $H(S_j/S_i) = P \log \frac{1}{P/2} + 1-P \log \frac{1}{1-P}$

using property of Binary Entropy i.e Symmetric and maximum when Equiprobable

$$\frac{p}{2} = 1 - p \Rightarrow p = \frac{2}{3}$$

e) maximum value of $H(S_2/S_1)$ when $p = 2/3$

$$H(S_j/S_i) = -\left[\frac{2}{3}\log\frac{1}{3} + 1 - \frac{2}{3}\log\frac{1}{3}\right]$$

$$= -\log\frac{1}{3} \Rightarrow \log 3 = 1.58 \text{ bits}$$

Q:16

Solution: Entropy of Sum

(a) $Z = X + Y$. $\rightarrow P(Z=z/X=x) = P(Y=z-x/X=x)$

$Y = z - x$

$$H(Z/X) = \Sigma P(x)\, H(Z/X=x)$$

$$= \sum_x P(x) \sum_y P(Z=z/X=x) \log \frac{1}{P(Z=z/X=x)}$$

$$= \sum_x P(x) \sum_y P(Y=z-x/X=x) \log \frac{1}{P(Y=z-x/X=x)}$$

$$\Rightarrow \sum_x P(x) \sum_y P(Y=z-x/X=x) \log \frac{1}{P(Y=z-x/X=x)}$$

$$= \Sigma P(x)\, H(Y/X=x)$$

$$= H(Y/X)$$

If X and Y are independent then

$$H(Y/X) = H(Y)$$

$\therefore$ $I(X;Z) \geq 0$

$$H(Z) \geq H(Z/X) = H(Y/X) = H(Y)$$

Similarly $H(Z) \geq H(X)$

16 b) Let joint distribution for X and Y

$$X = -Y = \begin{cases} 1 & p = 1/2 \\ 0 & p = 1/2 \end{cases}$$

then $H(X) = H(Y)$ and $Z = 0$ with $p = 1$

Hence $H(Z) = 0$

c)

$$H(Z) \leq H(X,Y) \leq H(X) + H(Y)$$

Because Z is a function of (X,Y) and $H(X,Y)$

$$H(X,Y) = H(X) + H(Y/X) \leq H(X) + H(Y)$$

we have inequality if X,Y is function of Z

and $H(Y) = H(Y/X)$ i.e X and Y are independent

):17

2]n

$$I(X:Z/Y) = H(X/Y) - H(X/Z,Y)$$
$$I(Z:Y/X) = H(Z/X) - H(Z/X,Y)$$
$$I(X:Z) = H(X) - H(X/Z)$$
$$I(Z:Y) = H(X) - H(X/Z)$$

Substituting in Right Hand Side

$$I(Z:Y/X) - I(Z:Y) + I(X:Z)$$

$$= \left(H(Z/X) - H(Z/X,Y)\right) - \left[H(Z) - H(Z/Y)\right]$$

$= H(X|Z) + H(Z|Y) - H(Z) - H(X|Y,Z)$

$\therefore \quad H(X,Z|Y) = H(X|Y) + H(Z|Y)$

$= -I(X:Z) + [H(Z|Y) - H(Z|X,Y)]$
$\quad + [H(X) - H(X|Z)]$

$= H(X|Y) - H(X|Z,Y$

$= I(X:Z|Y)$

Q: 18

$H(X) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + 2 \times \frac{1}{8}\log 8$

$= \frac{1}{2} + \frac{1}{4}\times 2 + \frac{1}{4}\times 3$

$= 1.75$ bits

Redundancy $R_{max} = 1 - \frac{H(X)}{\max H}$

$= 1 - \frac{H(X)}{\log_2 M}$

$= 1 - \frac{1.75}{\log_2 4}$

$= 1 - \frac{1.75}{2} = 0.125$ bits