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CONTENT COVERED:

1. Theory Notes
2. Explanation
3. Derivation
4. Example
5. Shortcut & Formula Summary
6. Previous year Paper Q. Sol.

Noted-: Single Source Follow, Revise

Multiple Time Best key of Success

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Chapter 1 LINEAR ALGEBRA

There are two types of linear Equations

1) $AX = B$ —— ①

non homogenous linear eqn

$AX = 0$ —— ②

homogenous linear equation

Solving these two eqn by Matrix Method is known as linear Algebra.

Matrix :- It is a collection of numbers (real or complex) in fixed number of rows and columns.

Representation: $[A]_{R\times C} = \{a_{ij}\}_{(R\times C)}$ → no. of elements

no. of columns

no. of rows

where (R, C) → dimension or Space

i :- position of elements in Rows

j : position of element in Column

a : element itself

A : Name of Matrix

Types of Matrix according to Dimensions :-

Case I If $R = C$

eg. $R = C = 3$

$$[A]_{3\times3} = \begin{array}{cccc} C_1 & C_2 & C_3 & \\ \left[\begin{array}{c} a_{11} \\ a_{21} \\ a_{31} \end{array}\right. & \begin{array}{c} a_{12} \\ a_{22} \\ a_{32} \end{array} & \left.\begin{array}{c} a_{13} \\ a_{23} \\ a_{33} \end{array}\right] & \begin{array}{c} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \end{array}_{3\times3}$$

Square Matrix

Square Matrix is very common and useful matrix in linear Algebra because:

1) for the solution of linear eqn the least no. of variables should be equal to number of equations

ii) Rows Represents the number of eqn & column represent the no. of variable.

iii) Determinant, Cofactor, adjoint, Inverse, characteristic eqn are only valid for square matrix.

iv) In computer graphics and programming the main matrix for array is a square matrix.

v) diagonal is only Valid for Square Matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

$\rightarrow i<j$

$i>j$

Off Diagonal or Secondary Diag.

Leading Diagonal or primary Diag. or ON Diagonal or main Diagonal

$i=j$

vi) In the point of view of the dimensions all well as diagonal the square matrix is the Symmetric about its dimensions.

vii) Here the diagonal is as a mirror for the position of upper and lower diagonal elements.

Case II :- If $R \neq C$ shape is Rectangular

a) If $R<C$ Horizontal Matrix

$$A_{2\times3} = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \end{array} \\ \downarrow C_1 \quad \downarrow C_2 \quad \downarrow C_3 \end{array}$$

b) If $R>C$ Vertical Matrix

$$A_{3\times2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}_{3\times2}$$

Case III : If $R=1$, $C>1$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}_{1\times3}$$

{Row Vector}

Case IV : If $C=1$ $R>1$

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}_{3\times1}$$

Column Vector

Formation of Matrix according to position :-

$$A = \{a_{ij}\}_{3\times3} \quad \begin{cases} i+j & , i=j \\ i-j & , i<j \\ 0 & i>j \end{cases}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 6 \end{bmatrix}_{3\times3}$$

Q1. form $[A] = \{a_{ij}\}_{4\times4}$ $\begin{cases} \frac{i+j}{2} & i=j \\ i-j & i\neq j \end{cases}$. find the sum of all elements

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times4}$$

$$= \begin{bmatrix} 1 & -1 & -2 & -3 \\ 1 & 2 & -1 & -2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 1 & 4 \end{bmatrix}_{4\times4}$$

Sum of all element $= 0 + 10$ { Symmetric with -ve sign element upper & lower to diag }

$= 10$

$\Sigma \rightarrow$ Sigma = Sum

$\Pi \rightarrow$ Pi = product

Q2. Sum all elements of Matrices $[A]_{n\times n} = \{a_{ij}\}_{n\times n}$ which is defined as follows

i) $\{a_{ij}\} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

ii) $\{a_{ij}\} = \begin{cases} i & i=j \\ 0 & i\neq j \end{cases}$

iii) $\{a_{ij}\} = \{ i+j \quad \forall\ i,j \}$

iv) $\{a_{ij}\} = \{ i^2 - j^2 \ \forall\ i,j$

v) $\{a_{ij}\} = \begin{cases} i^2 & i=j \\ 0 & \text{otherwise} \end{cases}$

vi) $\{a_{ij}\} = \begin{cases} i+j & , i=j \\ 0 & i\neq j \end{cases}$

vii) $\{a_{ij}\} = \begin{cases} i^3 & i=j \\ 0 & i\neq j \end{cases}$

viii) $\{a_{ij}\} = \begin{cases} \frac{i+j}{2} & i=j \\ i-j & i\neq j \end{cases}$

Solution:

For Summation :-

(I) Sum of A.P $= \frac{n}{2}(a + l)$; n = no. of terms

first term (a), last term (l)

(II) Sum of all natural number upto n :-

$$1+2+3+ \cdots + n = \frac{n(n+1)}{2}$$

(III) Sum of all natural number's square

$$1^2+2^2+3^2+ \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(IV) Sum of cubes of all natural no. $1^3+2^3+\cdots + n^3 = \frac{n^2(n+1)^2}{4}$

Sol 1 $\{a_{ij}\} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & 0 & 0 & 1 & & 0 \\ & & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n}$$

Sum $= 1+1+\cdots n = n$ //

Sol 2: $\{a_{ij}\} : \begin{cases} i & i=j \\ 0 & \text{else} \end{cases}$

$$\begin{bmatrix} 1 & 0 & & & & 0 \\ \vdots & 2 & & & & 0 \\ & & 3 & & & 0 \\ & & & 4 & & \vdots \\ 0 & & & 0 & 5 & \cdots n \end{bmatrix}$$

Sum: $\frac{n}{2}(1+n)$ //

Sol 3:

$$\begin{bmatrix} 2 & 3 & 4 & 5 & \cdots & 1+n \\ 3 & 4 & 5 & 6 & \cdots & 2+n \\ \vdots & & & & & \\ n+1 & n+2 & & & & n+n \end{bmatrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \\ \rightarrow R_n \end{matrix}$$

Sum of Row

$$S_1 = 2+3+4+5 --- (n+1)$$
$$= 1+2+3+4 -- n$$
$$S_1 = \frac{n(n+1)}{2}$$

wrong concept

$$S_1 = 2+3+4+5 -+ (n+1)$$
$$= \frac{n}{2}\left[(n+1)+2\right] = \frac{n}{2}\left[(n+3)\right]$$

$$S_2 = 3+4+5+ -- (n+2)$$
$$\frac{n}{2}\left[3+(n+2)\right] = \frac{n}{2}\left[(n+5)\right]$$

$$S_3 = \frac{n}{2}\left[(n+7)\right]$$

$$\text{Sum of } R_n = \frac{n}{2}\left[(n+1)+(n+n)\right]$$
$$= \frac{n}{2}\left[n+\underbrace{(2n+1)}_{\text{odd}}\right]$$

Total Sum

$$R_T = \frac{n}{2}\left[(n+3)+(n+5)+(n+7)+ -- n+(2n+1)\right]$$
$$\Rightarrow \frac{n}{2}\left[\frac{n}{2}\left(n+3+n+(2n+1)\right)\right]$$
$$\Rightarrow \frac{n}{2}\left[\frac{n}{2}\left[4n+4\right]\right]$$
$$= \frac{n}{2}\times\frac{n}{2}\,4(n+1)$$
$$R_T = n^2(n+1) \;//$$

(iv) $\{a_{ij}\} = \begin{cases} i^2 - j^2 & \forall\, i,j \\ & \cancel{\text{else}} \end{cases}$

$$\begin{bmatrix} 1^2-1^2 & 1^2-2^2 & 1^2-3^2 & \cdots & 1^2-n^2 \\ 2^2-1^2 & 2^2-2^2 & 2^2-3^2 & \cdots & 2^2-n^2 \\ \vdots & & & & \vdots \\ n^2-1^2 & \cdots & & \cdots & n^2-n^2 \end{bmatrix}_{n\times n}$$

$\Rightarrow$ Sum of R_1 :- $\frac{n}{2}\left[0 + 1-n\right]$

$R_2 = \frac{n}{2}\left[4 + 4\right.$

wrong move

mirror Image to diagonal and negativ

Sum of All element $S = 0$

v) $\{a_{ij}\} = \begin{cases} i^2 & i=j \\ 0 & i\neq j \end{cases}$

$$\begin{bmatrix} 1^2 & 0 & 0 & \cdots & 0 \\ 0 & 2^2 & & & \\ 0 & & 3^2 & & \\ \vdots & & & 4^2 \quad 5^2 & \\ 0 & & & & n^2 \end{bmatrix}$$

$S = \frac{n(n+1)(2n+1)}{6}$

vi) $\{a_{ij}\} = \begin{cases} i+i & i=j \\ 0 & i\neq j \end{cases}$

$$\begin{bmatrix} 2 & & & & \\ & 4 & & & \\ & & 6 & & \\ & & & 8 & \\ & & & & 2n \end{bmatrix}$$

$S = \frac{n}{2}(2+2n)$

$= n(n+1)$ //.

vii) $\{a_{ij}\} : \begin{cases} i^3 & i=j \\ 0 & i\neq j \end{cases}$

$$S = \frac{n^2}{4}\left[n+1\right]^2$$

viii)

$$S = \frac{n}{2}\left(\frac{2}{2} + \frac{n+n}{2}\right)$$

$$\Rightarrow \frac{n}{2}\left(\frac{2n+2}{2}\right) = \frac{n}{2}(n+1)$$

Elementary Transformations: are used to Simplify the elements of matrix

There are three types of Elementary xformation

i) we can multiply any non zero number to any line (Row or Column)

ii) we can Interchange two parallel lines (Row to Row, column to column)

* iii) we can multiply any non zero number to any line and add or Substract with another line.

ex. $R_1 \longrightarrow R_1 - 3R_2$

$\begin{bmatrix} R_1 \longrightarrow 2+R_1 \\ R_1 \longrightarrow R_1 \cdot R_2 \end{bmatrix}$ Invalid

*****

Basic Operations of Matrix :-

Transpose: It is valid for Square and nonsquare matrix.

To find transpose Interchange Rows & Column.

If $[A] = \{a_{ij}\}_{R\times C}$

$A^T = \{a_{ji}\}_{C\times R}$

eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

Determinant: Determinant is the expansion or value of matrix according to element's position.

Position Coefficient $(-1)^{i+j}$

$$A = \begin{bmatrix} a_{11} & \overset{-ve}{a_{12}} & a_{13} \\ -ve\,a_{21} & a_{22} & a_{23}\,-ve \\ a_{31} & \underset{-ve}{a_{32}} & a_{33} \end{bmatrix}_{3\times 3}$$

• determinant of 2x2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times 2}$$

$$|A| = a_{11}a_{22} - a_{21}a_{12} \qquad = [\text{Leading diagonal} - \text{off diag.}]$$

• determinant of 3x3

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times 3}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Total 6 terms in expansion

<u>note</u> :– No. of terms in expansion of 2×2 determinant is $2!$
of 3×3 determinant is $3!$
of $n\times n$ determinant is $n!$

eg. $$|A| = \begin{vmatrix} 1 & 2 & 4 \\ 7 & 8 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 1(56-0) - 2(49-0) + 4(0)$$

$$= 56 - 98 + 0$$

$$= -42$$

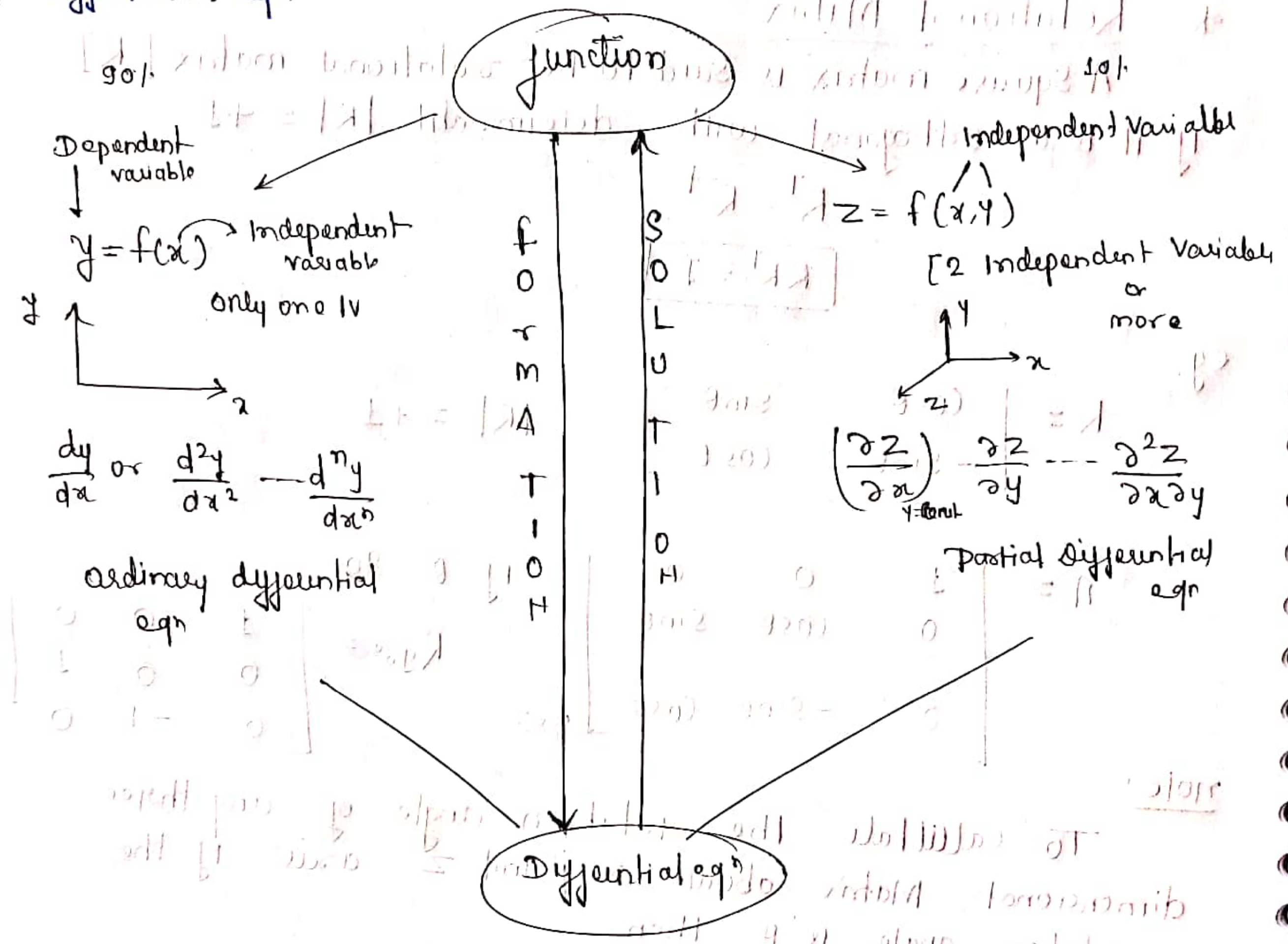
Cyrus Method for determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} \\ \\ a_{31} \end{matrix}$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}) - (a_{21}a_{12}a_{33} + a_{31}a_{22}a_{13} + a_{32}a_{23}a_{31})$$

Chapter: 2 Differential Eqⁿ

An Equation in which differential terms are involved is known as differential eqn.



* Representation of Differential eqn:

1. Order :- In D.E. order i.e. maximum no. of times of differentiation.
   Order is (are) always a positive Integer & degree

2. Degree :- The power of Higher order term in differential eqn after removing the fraction value & radical sign.
   - There is no relation betⁿ order & degree.
     A differential eqn can exist without degree but cannot exist without order.
   - By changing the representⁿ degree can be changed but order is fixed for every differential eqn
   - Order of D.E can only change by differentiating or Integrating the differential eqn.


Buy Now

Q1 find order & degree

① $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = y$ $\Rightarrow$ order = 2
degree = 1

② $\left(1 + \frac{d^2y}{dx^2}\right)^{3/2} = \left(\frac{d^2y}{dx^2}\right)^{2}$ $\Rightarrow$ o = 2
d = 4

③ $\left(\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^{2/3}$ $\Rightarrow \left(\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)^9 = \left(\frac{d^2y}{dx^2}\right)^2$ o = 2
d = 9

④ $y = e^{y'}$ $\Rightarrow$ o = 1

⑤ $\left(1 + \frac{1}{y'}\right)^{2/3} = (y'')^2$

⑥ $(y')^{2/3} + (y')^{3/2} = 1$

⑦ $y = \text{Log}\,(1 + y'')$

Sol 2 $\left(1 + \frac{d^2y}{dx^2}\right)^{3/2} = \left(\frac{d^2y}{dx^2}\right)^2$

$\Rightarrow \left(1 + \frac{d^2y}{dx^2}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^4$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^3 + \cdot \text{ — } = \left(\frac{d^2y}{dx^2}\right)^4$

Sol 3 $y = e^{y'}$

$y = 1 + y' + \frac{(y')^2}{2!} + \text{ - - } \infty$

o = 1
d = ∞

$\log y = y'$ ✗
donot change eqn

Sol 5 $\quad o = 2$

$d = 6$

Sol 6 $\quad (y')^{2/3} + (y')^{3/2} = 1$

Lcm of deno $\left(\frac{2}{3}, \frac{3}{2}\right)$

$= 6$

$(y')^{2/3 \times 6} + (y')^{3/2 \times 6} =$ any valu

short trick { only when same order

$(y')^4 + (y')^9$

order $= 1$

deg $= 9$

Sol 7 - $\quad y = \log(1 + y'')$

order $= 2$

deg $= \infty$

donot

$e^y = 1 + y''$

order $= 2$

deg $= 1$

~~X~~

* Formation of differential eqn

$$\text{function} \xrightarrow[\text{Removal of parameter}]{\text{Derivative}} \boxed{\text{Diff eqn of order } n}$$

n parameter
or
Independent const

eg (1) $y = c_1 x$

$$\frac{dy}{dx} = c_1$$

$\Rightarrow \quad y = y' x$

$\boxed{\text{no. of parameter} = \text{order of DE}}$

eg (2) $y = c_1 x + (c_1)^2$

$$\frac{dy}{dx} = c_1$$

so $\quad y = y'x + (y')^2$

eg (3) $y = ax + bx^2$

Sol $\quad \dfrac{dy}{dx} = a + 2bx$

$$\frac{d^2y}{dx^2} = 2b$$

So $\quad y = \dfrac{dy}{dx} - \dfrac{2}{2}\dfrac{d^2y}{dx^2} + \dfrac{d^2y}{dx^2}$

$$y = \frac{dy}{dx}$$

$$y = x(y' - y''x) + \frac{y''}{2}x^2$$

$$\Rightarrow \quad y''\left(\frac{x^2}{2} - x\right) + yx' - y = 0$$

eg(4) $y = Ae^{-x} + Be^{3x}$

DE $(D+1)(D-3)y = 0$

$\Rightarrow \left(\frac{dy}{dx} + y\right)\left(\frac{dy}{dx} - 3y\right) = 0$

$\Rightarrow \frac{d^2y}{dx^2} - \frac{2dy}{dx^2} - 3y = 0$

Q. form diff. eqn for function $y = Ax^2 + Bx^3$

Sol

$\frac{dy}{dx} =$

Standard form for $y = Ax^m + Bx^n$

$\Rightarrow x^4 y'' - xy'(m+n-1) + mny = 0$

* Solution of ordinary DEqn:

In general there are two type of soln

1) General Solution
11) particular Soln

$\boxed{\text{Differential eqn of order } n} \xrightarrow[\text{Add parameter}]{\text{Antiderivative}} \boxed{\text{function}}$ with n parameter.

- General Soln of n order differential eqn is n parameter family. i.e n number of Independent constants.
- General Soln is also known as Integral primitive or antiderivative
- Assigning some particular values to the parameter gives the particular Soln.
- General Soln is a Unique soln but particular Soln can be ∞.

eg (1) $\frac{dy}{dx} + \frac{x}{y} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow y\,dy + x\,dx = 0$

$\int$ both side

$\int y\,dy + \int x\,dx = c.$

$\frac{y^2}{2} + \frac{x^2}{2} = c$

Unique General soln i.e family is unique (Always get circle)

$\boxed{y^2 + x^2 = r}$ $\{ 2c = r$

family of circle

eg (2) $\frac{dy}{dx} - \frac{y}{x} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$

$\frac{dy}{y} = \frac{dx}{x}$

$\Rightarrow \cancel{-\frac{1}{y^2}} - \cancel{\frac{1}{x^2}} = \cancel{-c}$

$\Rightarrow \log y = \log x + c$

$\Rightarrow \boxed{y = x + e^c}$ straight line

eg (3) $\frac{dy}{dx} + \frac{y}{x} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow \frac{dy}{y} = -\frac{dx}{x}$

$\log y = -\log x + c$

~~log~~ $\log y + \log x = \log c$

$\boxed{xy = c}$ rectangular Hyperbola.

Chapter-3 Calculus

- Integral and differential Calculus (Series $\begin{cases} \text{Taylor} \\ \text{Mclaurin} \end{cases}$
- Vector Calculus — differential, Integral
- Maxima & Minima
- Mean Value theorem

Properties of Definite Integral :-

i) $$\int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$$

ii) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

iii) $$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

iv) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

v) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

vi) $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

vii) $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \text{ even} \\ 0 & ;\ f(-x) = -f(x) \text{ odd function} \end{cases}$$

viii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x) & : f(2a-x) = f(x) \\ 0 & : f(2a-x) = -f(x) \end{cases}$

eg ① $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$ is

a) 0
b) $a/2$
c) a
d) $2a$

eg ② $\int_0^{\pi/4} \frac{1-\tan x}{1+\tan}\,dx$ is

a) 0 b) 1 c) $\ln 2$
d) $2\ln 2$

eg ③ $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}\,dx$ is

a) 0
b) $-i$
c) i
d) $2i$

Sol) $I = \int_0^{a} \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$

$I = \int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

$\cancel{\int_0^{a} \sqrt{a-x} \, (\sqrt{a-x}}$

$I + I = \int_0^{a} \frac{\sqrt{a-x}+\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}\,dx$

$I = a/2$ //

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Sol2

$$I = \int_0^{\pi/4} \frac{1-\tan x}{1+\tan x}\, dx$$

$$I = \int_0^{\pi/4} \cancel{1-\tan} \quad \cancel{\cos} - \frac{\cos^2 x + \sin^2 x - 2\sin x \cos}{\cos 2x}$$

$$= \int_0^{\pi/4} \frac{1-\sin 2x}{\cos 2x}$$

$$\Rightarrow \int_0^{\pi/4} \sec 2x - \int_0^{\pi/4} \tan 2x$$

$$\left[\frac{1}{2}\log(\sec 2x)\right]_0^{\pi/4} = \frac{1}{2}\ln 2$$

Q3.

$$\int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}\, dx$$

$$\int_0^{\pi/2} e^{2ix}\, dx$$

$$\left(\frac{e^{2ix}}{2i}\right)_0^{\pi/2} = \frac{e^{i\pi}}{2i} - \frac{1}{2i}$$

$$\Rightarrow \frac{e^{i\pi}-1}{2i} = \frac{\cos\pi + i\sin\pi - 1}{2i}$$

$$= \frac{-2}{2i} = i$$

$$= c \;//$$

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Q. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ a) 0 B) $\pi/2$ c) $\pi/4$ d) NOT

Sol $\pi/4$ { half of upper limit for such problem }

Q 5, 15, 21, 27, 43, 44, 48, 49

Sol 5: $\int_0^{\pi} \sin^3\theta \, d\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow$ ~~$\sin \pi$~~ $\int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4}\,d\theta$

$\Rightarrow -\frac{3}{4}(\cos\theta)_0^{\pi} + \frac{1}{4}\left(\frac{\cos 3\theta}{3}\right)_0^{\pi}$

$\Rightarrow -\frac{3}{4}(-1-1) + \frac{1}{12}(\cos\pi - 1)$

$= +\frac{3}{2} - \frac{1}{6}$

$\frac{9-1}{6} = 8/6$

$= 4/3$

or

$\int_0^{\pi} (1-\cos^2\theta)\sin\theta \, d\theta$

$\Rightarrow$

Q 15

Sol $\int_0^{2} \frac{(x-1)^2 \sin(x-1)}{(x-1)^2 + \cos(x-1)}$

~~$f(2a-x)$~~ $f(2a-x)$

$=$ ~~$f(2$~~ $f(2-x)$

$(2-x-1)^2$

$(1-x)$

$-(\sin(1-x))$

$\Rightarrow$

0 // B)

Q21 $\left(\tan^{-1}x\right)_{-\infty}^{\infty}$

$\Rightarrow \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$ //

Q27. $\int_{-a}^{a} (\sin^6 x + \sin^7 x)\,dx$

$\Rightarrow 2\int_0^a \sin^6 x$ //.

Q43, 44. $I = \int_0^{\pi/4} \cos^2 x\,dx$

$\Rightarrow \frac{1 + \cos 2x}{2}$

$= \frac{1}{2}\left(\frac{\pi}{4}\right) + \left(\frac{\sin 2x}{4}\right)_0^{\pi/4}$

$\frac{\pi}{8} + \frac{1}{4}$ // A)

Q48. $\int \log x\,dx$

Sol ~~$\frac{1}{x}.x - \log x$~~ ~~log~~ ~~log x~~

$\int 1.\log x\,dx$

$\Rightarrow \log x \int dx - \int \frac{d}{dx}\log x \int dx$

$\Rightarrow x\log x - \int dx$

$\Rightarrow x\log x - x$

$= x(\log x - 1)$ //

* Special function for Single Integral.

1) Gamma function :-

$$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

must e^{-x}

for $n=1$ $\quad \Gamma(1) = \int_0^\infty e^{-x}\, dx = -\left[e^{-x}\right]_0^\infty$

$\Gamma(1) = 1$

- $\Gamma(1) = 1$ $\quad$ • $\Gamma(n+1) = n\Gamma(n)$ $\quad$ $n =$ fraction
- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(n+1) = n!$ ($n =$ Integer
- $\Gamma(-1/2) = -2\sqrt{\pi}$

$$\left[\begin{array}{l} \Gamma\left(-\frac{1}{2}+1\right) = \sqrt{\pi} \\ -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) = \sqrt{\pi} \end{array}\right.$$

$\Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi}$

eg ① $\quad I = \int_0^\infty e^{-x^2/2}\, dx$

$x^2/2 = t$

$dx = \dfrac{dt}{\sqrt{t}\sqrt{2}} =$ $\quad$ $x=0 \quad t=0$; $\quad x=\infty \quad t=\infty$

$$I = \int_0^\infty e^{-t} \frac{dt}{\sqrt{t}\sqrt{2}}$$

$$I = \int_0^\infty \frac{e^{-t}\, t^{-1/2}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\Gamma 1/2 = \sqrt{\pi/2} \,//.$$

Trigonometric form

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Q1. The value of - $\int_0^a \frac{x^7}{\sqrt{a^2-x^2}}\,dx$

A) $8/35\, a^7$
B) $16/35\, a^7$
C) $32/35\, a^7$
D) $64/35\, a^7$

Q2. $\int_{-\infty}^{0} e^{-x^2/20}\,dx$

a) $1/2$
b) $\sqrt{10}$
c) $\sqrt{5\pi}$
d) $\sqrt{\pi}$

Q3 $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$

a) $1/2$
b) 1
c) $\sqrt{\pi}$
d) 2π

Gaussian function → even function

Sol 1 $\cancel{a^2 - x^2} =$ $x = a\sin\theta$

$x^2 = a^2\sin^2\theta$

$$\int_0^{\cancel{a\sin a}} \frac{x^6 \cdot x}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a^2\sin^2\theta \cdot a\sin\theta}{a\sqrt{1-\sin^2\theta}}$$

$$\rightarrow a^{12}\int \frac{\sin^{13}\theta}{\cos\theta}$$

$$a^{12}\int_0^{\pi/2} \sin^{13}\theta \cos^{-1}\theta$$

$$\cancel{\frac{a^{12}\,\Gamma 7\;\Gamma 0}{2\,\Gamma 7}} \Rightarrow \frac{a^{12}}{2\,\Gamma 6+1}$$

$\Rightarrow \int \frac{x^6 . x}{\sqrt{a^2 - x^2}} dx$

Sol $\quad x = a\sin\theta \qquad a\cos\theta \frac{d\theta}{dx}$

$$\int_0^{\pi/2} \frac{a\sin\theta \cdot a^6\sin^6\theta}{a\sqrt{1-\sin^2\theta}} d\theta \cdot a\cos\theta$$

$$\Rightarrow \int_0^{\pi/2} a^7 \sin^7\theta$$

$$\Rightarrow \cancel{\frac{a^7 \,\Gamma 7+1}{2\,\Gamma \frac{8}{2}}} = \cancel{\frac{a^7}{2} \frac{\Gamma 6+1}{\Gamma 3+1}} \Rightarrow \cancel{\frac{a^7}{2} \frac{6\times5\times4\times3!}{3!}} \Rightarrow$$

$$\Rightarrow \frac{a^7\, \Gamma\frac{7+1}{2}\, \Gamma\frac{1}{2}}{2\,\Gamma\frac{7+0+2}{2}} = \frac{a^7\,\Gamma 4\,\Gamma 1/2}{2\,\Gamma\frac{9}{2}}$$

$$= \frac{a^7\,\Gamma 3+1\,\sqrt{\pi}}{2 \cdot \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}$$

$$= a^7 \times 16/35 \;//.$$

Q2.
Sol $\quad \int_{-\infty}^{0} e^{-x^2/20} dx \qquad$ even function

$$= \int_0^{\infty} e^{-x^2/20} dx = \int_0^{\infty} e^{-t} \frac{10\, dt}{\sqrt{20t}}$$

$x^2 = 20t \qquad \frac{x^2}{20} = t \qquad 2x dx = 20 dt \qquad = \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} \frac{10}{\sqrt{20}}$

$x = \sqrt{20t} \quad 1/\sqrt{20}$

$$\frac{10}{\sqrt{20}} \sqrt{+1/2} \qquad \frac{\sqrt{20}}{\cancel{10}} \sqrt{1/2}$$

$$\Rightarrow \frac{\cancel{10}}{\sqrt{20}} \cancel{\left(-2\sqrt{x}\right)} = \sqrt{5\pi} \; //$$

$$\Rightarrow \cancel{\frac{10}{\sqrt{5\times2\times2}} \left(-2\sqrt{x}\right)}$$

$$\Rightarrow \cancel{10\sqrt{\pi}}$$

* **Beta function :**

$$\beta(m,n) = \int_0^1 x^{m-1} \underbrace{(1-x)^{n-1}}_{\text{must convert to } (1-x)} dx \quad \text{--- } ①$$

let :

$1-x = y \quad \Rightarrow \quad -dx = dy$

$x \to 0 \Rightarrow y \to 1$

$x \to 1 \Rightarrow y \to 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} (y)^{n-1} \, -dy$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$$

$\boxed{\beta(m,n) = \beta(n,m)}$ duality of β function

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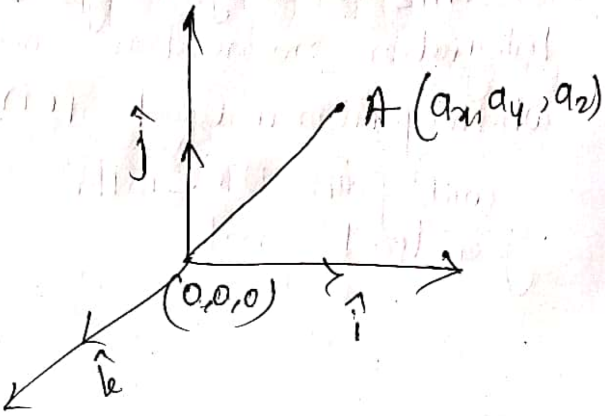
- Basics
- Vector differential Calculus (∇)
- Vector Integral Calculus

VECTOR CALCULUS

→ In general there are two types of quantities
i) Vector quantity = Magnitude (Direction)
ii) Scalar = only magnitude

① $\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$



$(a_x, a_y, a_z) = (1, 2, 3)$

$\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$

if $(a_x, a_y, a_z) = (x, xy, xyz)$

$\vec{A}(x,y,z) = x\hat{i} + xy\hat{j} + xyz\hat{k}$
⟶ Vector point function

② Magnitude :-

$$|A| = \sqrt{a_x^2 + a_y^2 + a_z^2} = +\text{value}$$

③ Unit Vector in direction of $\vec{A}$

$$\hat{A} = \frac{\vec{A}}{|A|}$$

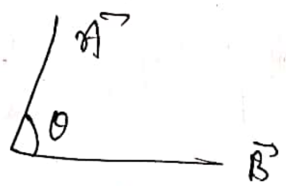
④ product of Vector :

a) General product : $\vec{A}(a)$

Vector applied on Scalar only mag. will change.

b) Dot product :- Inner product

Def: $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$



$\theta = 0$ $\quad \vec{A}\cdot\vec{B} = |A||B|$

$\theta = 90^\circ \Rightarrow \vec{A}\cdot\vec{B} = 0$

Math:

$\vec{A} = a_x\hat{i} + a_y\hat{j} + a_k\hat{k}$

$\vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\boxed{\vec{A}\cdot\vec{B} = a_x b_x + a_y b_y + a_z b_z}$$

Scalar quantity

c) Cross product

Def. $\boxed{\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n}}$

Vector quantity

$\hat{n}$ = normal unit Vector :

Resultant direction i.e $\perp$ of both $\vec{A}$ & $\vec{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Vector product

eg $\vec{A} = 2i - j + 3\hat{k}$

$\vec{B} = i - 4j + 5\hat{k}$

$|\vec{A}\cdot\vec{B}|$ & $|\vec{A}\times\vec{B}|$?

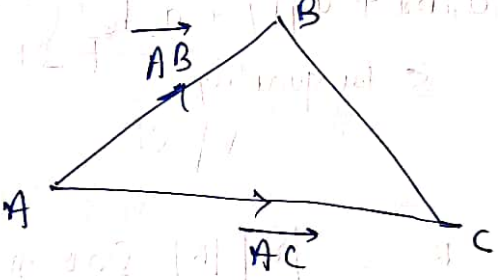
$|A\cdot B| = 2 + 4 + 15 = 21$ //

$$|A \times B| = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & -4 & 5 \end{vmatrix} = i(-5+12) - j(10-3) + k(-8+1)$$

$= 7i - 7j - 7k$ //

$$|A \times B| = \sqrt{49 + 49 + 49} = \sqrt{147}$$

⑤ Area of Δ:

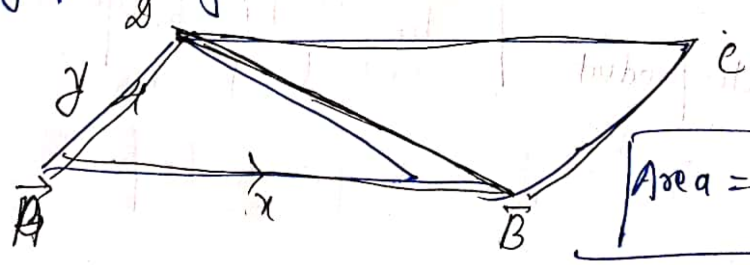


$\vec{AB} = \vec{B} - \vec{A}$

$\vec{AC} = \vec{C} - \vec{A}$

$$\boxed{\text{Area} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|}$$

⑥ Area of parallelogram



$$\boxed{\text{Area} = |\vec{x} \times \vec{y}|}$$

(7) Orthogonal Vectors:- Two Vectors are said to be Orthogonal if their dot product is equal to Zero. It is also known as perpendicular Vector ~~or Mutually Normal Vectors.~~

$$\boxed{\vec{A} \cdot \vec{B} = 0}$$

(8) Orthonormal Vector :- Two Vector are said to be orthonormal if their dot product is equal to Zero. $\boxed{\vec{A} \cdot \vec{B} = 0}$ as well as their magnitude must be equal to unity.

$$|\vec{A}| = |\vec{B}| = 1$$

note: Every Orthonormal Vector is orthogonal but reverse is not not true.

(9) parallel Vectors:- $k(\vec{A}) = \vec{B}$

Q38, Q33, Q47, Q49, Q50, Q51

Sol 38

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{|A||B|}\right)$$

$$= \cos^{-1}\left(\frac{0.866 \times 0.259 + 0.5 \times 0.966}{\sqrt{0.866 + 0.5}\ \sqrt{0.259 + 0.968}}\right)$$

1.168 1.10

$$= 45^0$$

Q33

A B

Q47.
Sol C

Q 49
Sol

particle

$x = t^3 + 2t$ $\qquad S = x\hat{i} + y\hat{j} + z\hat{k}$

$y = -3e^{-2t}$

$z = 2\sin 5t$

$$S = (t^3 + 2t)\hat{i} + -3e^{-2t}\hat{j} + 2\sin 5t\,\hat{k}$$

$$\frac{ds}{dt} = (3t^2 + 2)\hat{i} + 6e^{-2t}\hat{j} + 10\cos 5t\,\hat{k}$$

$$\frac{d^2s}{dt^2} = 6t\hat{i} + (-12e^{-2t})\hat{j} + (-10\times 5\sin 5t)\hat{k}$$

at $t = 0$

$= -12\hat{j}$

$|-12j| = 12$ //

Q 50

Sol

$\vec{OP} = a\hat{i} + b\hat{j}$

$\vec{OR} = c\hat{i} + d\hat{j}$

Area $= |\vec{OP} \times \vec{OR}|$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix}$$

$\Rightarrow \hat{k}\,(ad - bc)$

$= |(ad - bc)|$ // opt A

Q51

Sol $|a \times b|^2$

$|\vec{a} \times \vec{b}| = |a\, b \sin\theta|^2$

$= a^2 b^2 \sin^2\theta$

$= a^2 b^2 (1 - \cos^2\theta)$

$a^2 b^2 - a^2 b^2 \cos^2\theta$

$a^2 b^2 - (\vec{a} \cdot \vec{b})^2$ // opt(A)

Q70; Q75 Q17

Sol 70 $A = 4j + 3k$

$B = 0$

$C = 3i + 4k$

$BA = A - B = 4j + 3k - 3i - 4j$

$= -3i + 3k$

$BC = \underline{3i + 4k} - 0$

A.//

Q17 Sol B)

Q75 $\theta = \cos^{-1}\left(\dfrac{-\frac{3}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4} + \frac{1}{4}}\sqrt{\frac{3}{4} + \frac{1}{4}}}\right) = \cos^{-1}\left(\dfrac{+\frac{1}{2}}{\cancel{2}}\right)$

$\cos^{-1}\left(-\frac{1}{2}\right)$

$= 60^0$

Q. find Area of $\triangle$ ABC with Vertices A(1, 2, 1), B(3, 6, 2), (5, 1, 4)

$AB = (2, 4, 1)$
$AC = (4, -1, 3)$

$$\frac{1}{2}\left|\begin{matrix} i & j & k \\ 2 & 4 & 1 \\ 4 & -1 & 3 \end{matrix}\right|$$

$(12+1)i - j(6-4) + k(-2-16)k$

$\frac{1}{2}|13i - 2j - 18k|$

$\frac{1}{2}\sqrt{13^2+4+324}$

$\frac{1}{2}\sqrt{169+328}$

$\frac{1}{2}\sqrt{497} = 11.14$ //

$18 \times 18 = 324$

Q. $\vec{A} = (6, -1, 3)$ for what value of c
$\vec{b} = (4, c, -2)$ is $\perp$ to A

Ans $c = 18$

$\sqrt{\cancel{36+9+1}}$

3-D position Vector

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow (1)$$

Magnitude = $r = \sqrt{x^2+y^2+z^2}$

$$\vec{r} = P(x,y,z)$$

Del operator :

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$ nabla operator

dimension = Unit = $\frac{1}{m}$ or m^{-1}

∇ operation

- General product $\rightarrow$ Gradient
- Dot product $\rightarrow$ Divergence
- Cross product $\rightarrow$ Curl

I. Gradient :-

1. It is only valid for Scalar point function.
2. It is a general product of ∇ operator to any scalar point function at any point.

$$\phi(x,y,z) = \text{Scalar point function}$$

$$\text{grad}(\phi) = \nabla(\phi)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$\boxed{\text{grad}(\phi) = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}}$$ Vector Value

Result of gradient is a Vector quantity. which is known as <u>Normal Vector</u>.

* • If ϕ represents a plane then gradient of ϕ is a vector which represents the direction of that plane. which is perpendicular to the plane.

• physical Significance of Gradient: Gradient gives the Maximum rate of change of a scalar quantity at any point.

eg. $\phi = x^2 y^2 + y^2 z$

grad (1, 1, 1)

Sol grad(ϕ) : $2x\hat{i}\, y^2 + 0 + (2yx^2 + 2yz)\hat{j} + y^2\hat{k}$

at 1,1,1 $= 2\hat{i} + (2+2)\, j + 1\, k$

$= 2i + 4j + k$ // <u>Normal Vector</u>

Q1 - find the gradient of Scalar point fun $\phi = xy^2 + yz^2 + zx^2$

at point (1, -1, 1)

Sol $(y^2 + 2z\,)\hat{i} + (2yx + z^2)\,\hat{j} + (2zy + x^2)\,k$

$(1 + 2)\,i + (-2+1)\,\hat{j} + (-2+1)\,k$

$3i - \hat{j} - \hat{k}$ //

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Unit 4 Complex Variables

$\boxed{\sqrt{-3} \times \sqrt{-2} = -\sqrt{6}}$

$z = x + iy$ (1) is a Complex number where $i = \sqrt{-1}$

$i = \sqrt{-1}$ i (unit)

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Add. $i + i^2 + i^3 + i^4 = 0$

note :- ~~Σ four Consecutive~~ Σ of

Summation of i with four Consecutive power = 0

also, 4 Consecutive even power = 0

4 Consecutive odd power = 0

i^k = Real when k = even

Imaginary when k = odd

$(i^{59}) = (i^{58}) i = (i^2)^{29} \cdot i = -i$

$\bar{z} = x - iy$ — (2)

$z + \bar{z} = 2x$

$z - \bar{z} = 2iy$

$z\bar{z} = x^2 + y^2$ (Real)

$$\frac{z}{\bar z} = \frac{x+iy}{x-iy} = \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} + \frac{2ixy}{x^2+y^2}$$

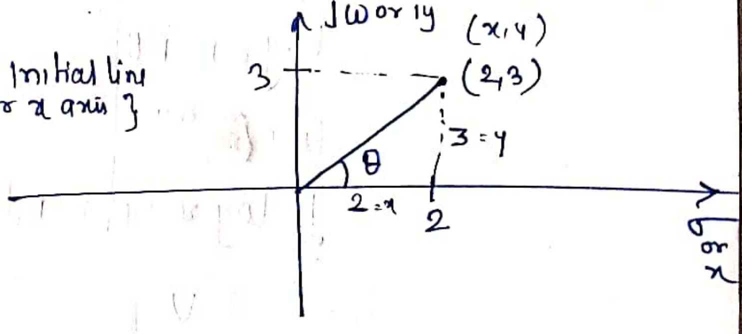
Real $\qquad \text{Im} = \frac{2xy}{x^2+y^2}$

$|z| = \sqrt{x^2+y^2}$

$$\boxed{|z|^2 = z.\bar z}$$

Argand plane

Z plane or S plane



• argument / angle {wrt to initial line or x axis}

ARG(z)

$z = x+iy$

$\theta = \tan^{-1} y/x$

$(x,y) = (2,3)$

$x+iy \quad 2+3i$

Complex variable

$U + iV = f(z) = f(x+iy)$

$U + iV = \underbrace{\sin x \cosh y}_{R \; u} + i \underbrace{\cos x \sinh y}_{V}$

$\cos i\theta = \cosh \theta$

$\sin i\theta = i \sinh \theta$

$u + iy = e^{x+iy}$

$= e^x . e^{iy}$

$\Rightarrow e^x (\cos y + i \sin y)$

$= \underbrace{e^x \cos y}_{U} + i \underbrace{e^x \sin y}_{V}$

Q $u+iv = \log(x+iy)$

$\Rightarrow e^{\cancel{u+iv}} = \cancel{x+iv}$

$u+iv = \log(r\cos\theta + r i\sin\theta)$

$\Rightarrow \log r(\cos\theta + i\sin\theta)$

$= \log r \cdot e^{i\theta}$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta$

$= \cancel{\log r e^{i\theta}}$

$= \underbrace{\frac{1}{2}\log x^2+y^2}_{u} + i\,\underbrace{\tan^{-1}(y/x)}_{v}$

Q. find argument of $z = \frac{1+i}{1-i}$ $\qquad \frac{1-1+2i}{1+1}$

$= 0 + i$

$\tan^{-1} 1/0 = \pi/2$

Q2. $z = x+iy$ find y

$\frac{3+5i}{2-i} = \frac{(3+5i)(2+i)}{4+1}$

$= \frac{6+13i-5}{5} = 13/5$

$= \frac{1}{3} + i$

Q3. $z = i^i$

$\cancel{\log z} = \cancel{i \log i}$ $\quad \log z = i \log i$

$= \cancel{i \log 1}$

$z = i^i$

$\Rightarrow$ ~~$\log z = i \log i$~~

$= \cancel{i(\log 1}$

$\Rightarrow$ ~~$a+ib = i^i$~~

~~$\log(a+ib)$~~

$z^2 = i^{2i}$

$z = i^i$

$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^i$

$\Rightarrow \left(e^{i\pi/2}\right)^i$

$= e^{-\pi/2}$ /

or

$i^i = e^{\log i^i}$

$= e^{i\log i}$

$= e^{i\left[\frac{1}{2}\log 1 + i\tan^{-1} 1/0\right]}$

$e^{i[\pi/2]}$ //

$\boxed{\log i = i\pi/2}$

Q4. $a+ib = i^{i^{i^{i \cdots \infty}}}$

$a+ib = i^{(a+ib)}$

$\Rightarrow$ ~~$\log(a+$~~ $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{a+ib}$

$= \left(e^{i\pi/2}\right)^{(a+ib)}$

$= e^{\pi/2[ia - b]}$

$\Rightarrow e^{ia\pi/2 - b\pi/2}$

~~cos~~

$\Rightarrow e^{ia\pi - b\pi}$

$e^{(ia-b)\pi}$

$=$

Sir Method

$a+ib = e^{\log i^{(a+ib)}}$

$= e^{(a+ib)\log i}$

$= e^{(a+ib)\, i\pi/2}$

$a+ib = e^{-\frac{\pi}{2}b} \cdot e^{\frac{ia\pi}{2}}$

$= e^{-\pi/2 b}\left[\cos\frac{a\pi}{2} + i\sin\left(\frac{a\pi}{2}\right)\right.$

$\Rightarrow e^{-\pi/2}$

Q4. , Q20, Q27, Q28, Q34, Q51 HW, Q50, 59, 65, 66

Sol 4 $z = x^{x}$

$z = i^{i}$ $\qquad e^{-\pi/2}$

Sol 20 $\varnothing$

Sol 27 $2 - 5i + 2i + 5$

$7 - 3i$

Q. 28. $\frac{(3+4i)(1+2i)}{5}$ $\qquad \frac{3+6i+4i-8}{5}$

$-\frac{5}{5} + \frac{10i}{5}$

$-1 + 2i$

$\sqrt{1+4} = \sqrt{5}$

34.

Sol $\frac{(2-3i)(-5-i)}{25+1}$ $\qquad \frac{-10-2i+15i-3}{26}$

$= \frac{-13}{26} + \frac{13}{26}i$

$= -0.5 + 0.5i$ //

50. $|e^{iz}|$

$|\cos z + i\sin z|$ $\qquad |e^{i(x+iy)}|$

$\cancel{\cos^2 + \sin^2}$

$e^{ix - y}$

$e^{ix} \cdot e^{-y}$

e^{-y} //

Q 66.

Sol $z = \frac{\sqrt{3}}{2} + i \sin\frac{1}{2}$ z^4.

$\cancel{(-\omega)^4}$ $\cancel{\omega^3 . \omega}$

$e^{i\pi/6}$

$= e^{2i\pi/3}$

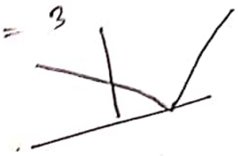
$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$\Rightarrow -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Random Question test

$x-3=0$

$x=3$



Q1. $f(x) = \begin{cases} |x-3| & x \geq 1 \\ \frac{x^2}{4} - \frac{3}{2}x + \frac{13}{4}, & x < 1 \end{cases}$

a) Continuous at $x = 1$

b) diff. at $x = 1$

c) Both a & b

d) None of these

Q2 $f(x) = \begin{cases} e^x & x < 1 \\ \ln x + ax^2 + bx & x \geq 1 \end{cases}$ $x \in R$

which statement is true

(1) $f(x)$ Not diff. at $x = 1$ for any value and of a & B

(2) $f(x)$ is differentiable for unique value of a and b

(3) $f(x)$ is diff at $x = 1$ $\forall$ a & b ; $a+b=e$

(4) $f(x)$ is diff. at $x=1$ $\forall$ a, b

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Q3. Max Value of f(x)

$$f(x) = \frac{1}{3}x(x^2-3) \quad \text{in} \quad -100 \le x \le 100$$

2/3

Q4. Find distance between the origin and the point at surface i.e nearest to its surface $z^2 = 1+xy$

a) 0 b) 1 c) 2 d) 3.

Q5. for Complex no. z

$$\lim_{z \to i} \frac{z^2+1}{z^3+2z-i(z^2+2)}$$ is

a) $-2i$

b) -1

c) i

d) $2i$

Sol 5
$$\lim_{z \to i} \frac{2z}{3z^2+2-i(2z)}$$

$-3 \quad +1 \quad +2$

$$\Rightarrow \frac{2i}{-3+2+2} = \frac{2i}{1} = 2i$$

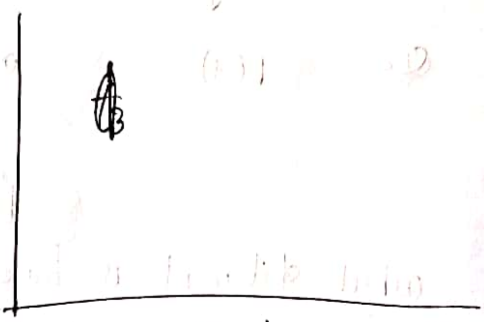
Sol 3
$$f(x) = \frac{1}{3}x(x^2-3)$$

$$= \frac{1}{3}\left[3x^2-3\right] = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1 \qquad 1, -1$$



$$f''(x) = \frac{1}{3}\left[6x\right]$$

$$= 2x$$

$$f(x) = -\frac{1}{3}(1-3) = \frac{2}{3}$$

$$-\frac{100}{3}(10000-3) \quad +ve$$

$$\frac{100}{3}(10000-3)$$

$$\Rightarrow \frac{9997 \times 100}{3}$$

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4 to 5 marks

Probablity & Statistics

- Basics (Definition) 5*
- Distribution 3*
  - Binomial
  - Poisson
  - Normal
- Random Variable 5*
  - Expected Value
  - RMS Value
  - Variance
  - Standard deviation
- Observation Data *
  - Mean
  - Mode
  - Median

  Central tendency

Random Space :- outcome is not always same.

Sample space(S):- Set of all possible outcomes of a Random experiment is called Sample space.

- It is also known as event space.
- The elements of Sample space are known as Sample points.
- Sample space can be finite or Infinite according to the random experiment.

1. Coin

a) Single Coin $S\{H, T\}$

$n(s) = 2$

b) Two Coin $2^2 = 4$

$n(s) = 4$

$S\{HT, TH, HH, TT\}$

c) Three Coin / toss : $2^3 = 8$

$n(s) = 8$

S(H, HT, HHH, HTH, HTT, THH, TTH
THT, TTT)

c) 4 Coin / 4 toss :

S = {

H H H H	H H H T
T H H H	T H H T
H T H H	H T H T
T T H H	T T H T
H H T H	H H T T
T H T H	T H T T
H T T H	H T T T
T T T H	T T T T

}

II. Dice

2. Single Dice / fair Dice / unbiased dice

$S = \{1, 2, 3, 4, 5, 6\}$

$n(S) = 6$

sum → no of element

2(1) 3(2) 4(3) 5(4) 6(5) 7(6) 8(5) 9(4) 10(3) 11(2) 12(1)

Two dice

$S = \{$

(1 1) (1 2) (1 3) (1 4) (1 5) (1 6)
(2 1) (2 2) (2 3) (2 4) (2 5) (2 6)
(3 1) (3 2) (3 3) (3 4) (3 5) (3 6)
(4 1) (4 2) (4 3) (4 4) (4 5) (4 6)
(5 1) (5 2) (5 3) (5 4) (5 5) (5 6)
(6 1) (6 2) (6 3) (6 4) (6 5) (6 6)

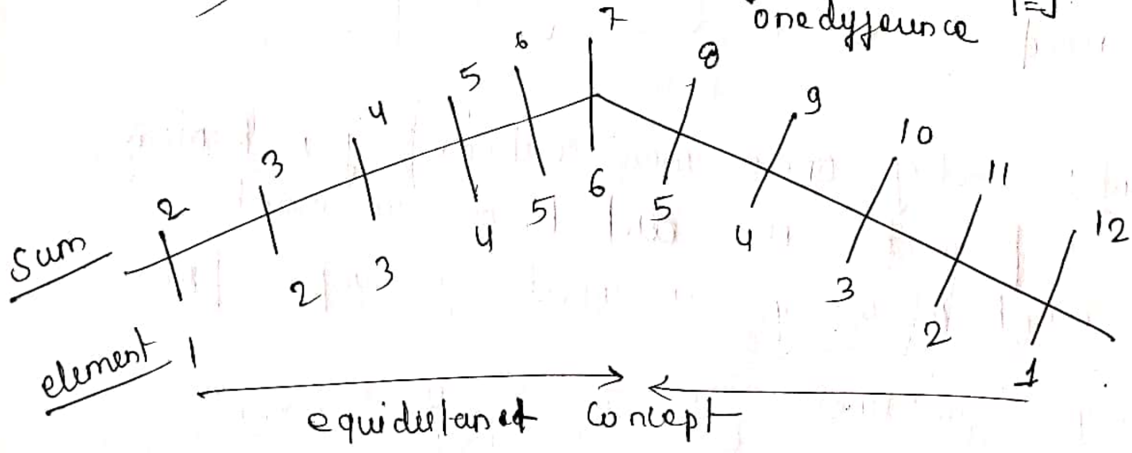
$\}$

$n(S) = 36$

6×6

one difference

i=j



III. Playing Card :

Suit :

13 Club

13 Spade

13 Heart

13 Diamond

$\boxed{\text{Total } 52}$

No. A (Ace) 1 2 3 4 5 6 7 8 9 10

Non face Card

	J (Jack)	Q (Queen)	K (King)
Club	✓	✓	✓
Spade	✓	✓	✓
Heart	✓	✓	✓
Diamond	✓	✓	✓

12 face Card

Event :- Set of one or more outcome of a Random experiment is said to be an event.
event is always a Subset of sample space.

eg. for 1 dice $\{1, 2, 3, 4, 5, 6\}$

event : more than 4
$= \{5, 6\}$

$\boxed{E \subseteq S}$ $\boxed{n(E) \leq n(S)}$

$$\boxed{\frac{n(E)}{n(S)} \leq 1}$$

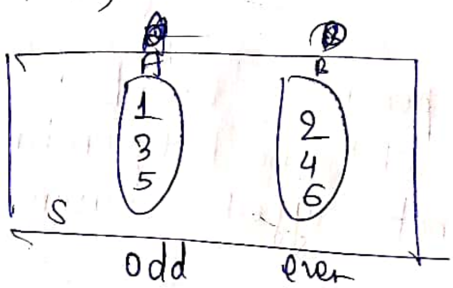
Types of Events:

1. Mutually Exclusive event or Disjoint Event.

- Two or more events are said to be mutually exclusive if they will not occur simultaneously in a single trial of same sample space.

eg. dice {1, 2, 3, 4, 5, 6}

for even & odd event



$P(A \cap B) = 0$

↳ A & B → (A & B simultaneous occur)

$n(A \cap B) = 0$

↳ No intersection of A & B

2. Mutually Exhaustive: A set of events is said to be mutually exhaustive if atleast one of them must occur or necessarily occur.

eg. In a coin

Let events A = Head

B = Tail

- It is mutually exclusive because occurance of any one prevents other.
- It is mutually exhaustive because in A and B atleast one must occur.

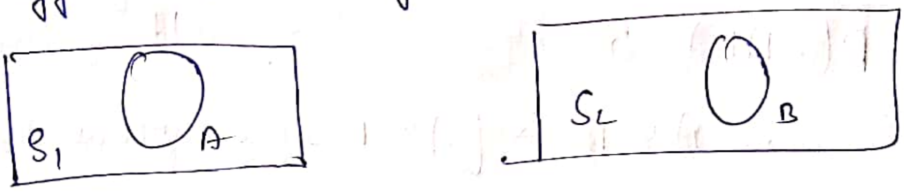
eg. Consider drawn of a card from pack of 52 cards

event A Card is a Spade
event B Card is a club
event C card is a Heat

- These three events are mutually ~~and~~ exclusive but not mutually exhaustive because may be the drawn card is a diamond.

3. Equally likely Events :- Set of two or more events are said to be equally likely if the chance of occurance are same.

*
4. Independent events :- Two or more events are said to be Independent if occurance of one doesnot affect occurance of other.



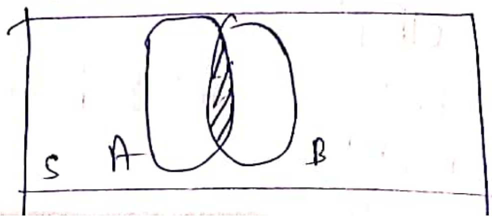
$$n(A \cap B) = 0$$

$$\underbrace{P(A \cap B)}_{\text{Simultaneous Occurance}} = P(A) \times P(B)$$

eg. Tossing two dice simultaneously

- event A No. 4 on first dice
- event B No. 3 on second dice

*
5. Dependent events :- Two or more events are said to be dependent if information about one give some information about another.



$n(A \cap B) \neq 0$

$P(A \cap B) \neq 0$

eg: Tossing two dice Simultaneously
event A no 4 on first Dice
event B Total Sum is 7.

* **Probablity** :- Probablity of an event is the ratio of favourable outcome to Total outcome

Let an Event E in Corresponding sample space S

then $\boxed{P(E) = \frac{n(E)}{n(S)}}$

$\boxed{P(E) \leq 1}$

1. For Mutually Exclusive
$P(A \cap B) = 0$

2. For Independent
$\boxed{P(A \cap B) = P(A) \cdot P(B)}$ Product Rule

3 $P(A)$ Probablity of Happening $P(A')$ or $P(\bar{A}) = 1 - P(A)$

4. If Two events A and B are not necessarily mutually exclusive then probablity of atleast one event

$\boxed{P(A \overset{or}{\cup} B) = P(A) + P(B) - P(A \cap B)}$ addition Rule

for mutually exclusive
$P(A \text{ or } B) = \boxed{P(A \cup B) = P(A) + P(B)}$ Pure addition Rule

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

for Independent $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A)\,P(B)$

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* Conditional probablity :-
(only applicable for dependent event)

$P(A|B)$ = Probablity of Happening (occurance) of A when B is already occured.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

{ for Independent
$$P(A|B) = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$
}

Terminology

Atleast 1	$(P(A) \geq 1)$	equal to or greater
At most 1	$P(A \leq 1)$	less or Equal
Union addition: OR	$P(A \cup B)$	Addition Rule
Product Intersection: AND	$P(A \cap B)$	Multiplicⁿ Rule

addition Rule can be used for at least One or, either or,
Multiplication Rule can be used for Simultaneously, one after another, as well as, alternatively, both

~~Q~~

Permutation: An arrangement of Set of r objects out of n objects given by permutation

$$^{n}p_{r} = \frac{n!}{(n-r)!}$$

Circular Permutation: Arrangement of n objects in the circle is given by $(\cancel{n-1})!$

$$(n-1)!$$

Combination: Number of ways of Selecting r objects out of n objects is given by combination.

$$\boxed{^{n}C_{r} = \frac{n!}{r!(n-r)!}} = \frac{^{n}p_{r}}{r!}$$

$$\boxed{^{n}p_{r} = r!\,^{n}C_{r}}$$

Q1. A Card is drawn from a pack of 52 Cards. What is the probablity either a club or face Card.

a) $21/51$ b) $22/52$ c) $23/52$ d) $24/54$

Q2. A Card is drawn from pack of 52 Card what probablity that Card is king or Ace

1) $1/13$ 2) $2/13$ 3) $3/13$ d) $4/13$

Sol: A = club = 13 $n(A) = 13$

B = face Card = 12 $\Rightarrow n(B) = 12$

$A \cap B$ = club & face = 3 $n(C) = 3$
(J Q K)

Chapter - 7 Numerical Method

Root | Area | Curve

I. Solution of Nonlinear Equation
1. Bisection Method 1
2. Regula-falsi or false position Method 1
3. Newton-Raphson Method 2

II. Numerical Integration
1. Trapezoidal Rule order 1
2. Simpson $1/3^{rd}$, $3/8^{th}$ Rule (2, 3)
3. Weddle's Rule

III Numerical Soln of Differential eqn
1. Euler's Method
2. Runge-kutta Method.

I. Numerical Integration:-

1. Trapezoidal Method:- The Trapezoidal rule is used to model the curve by using straight line. The order is one.

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]}$$

$$= \frac{h}{2}\left[(1^{st}+\text{Last}) + 2\times\text{Rest}\right]$$

note: $\int_{a=x_0}^{b=x_0+nh} y\,dx$ $\boxed{h = \frac{b-a}{n}}$ → no. of sub-interval

Step of subinterval

2. Simpson $1/3^{rd}$ Rule:- Simpson $1/3^{rd}$ rule is used to fit the curve by using parabola to approximate each point of the curve.

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+y_6\cdots)\right]}$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[\left(1^{st} + last\right) + 4 \times odd + 2 \times even\right]$$

* In Simpson $1/3^{rd}$ rule the given Interval must be divided into even number of equal Sub Interval.

3. **Simpson $\frac{3}{8}^{th}$ Rule:** For Simpsom 3/8 rule the given Interval must be divided into the number of Sub interval that should be Multiple of 3.

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3}{8}h\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \cdots) + 3(y_1 + y_2 + y_4 + y_5 + \cdots)\right]}$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3}{8}h\left[\left(1^{st} + last\right) + 2 \times \underset{\text{of } 3}{\text{multiple}} + 3 \times Rest\right]$$

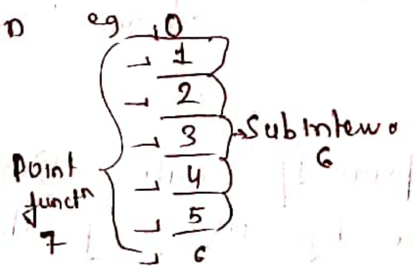
(trick 15-16-15-16)

4. **Weddle's Rule:** Sub interval Should be multiple of 6.

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + \cdots\right)}$$

- n Subinterval $\Rightarrow$ $(n+1)$ point function

eg



Q1. $y = \int_0^6 \frac{1}{1+x^2}\,dx$ ——— (i)

x	$y = \frac{1}{1+x^2}$	
$x_0 = 0$	$y_0 = 1$	
$= 1$	$= 0.5$	
2	$= 0.20$	$= 0.2$
3	$= 0.10$	$= 0.1$
4	$= 0.05$	$= 0.05$
5	$= 0.038$	$= 0.038$
6	$= 0.028$	

(i) By Trapezoidal $\quad h = \frac{b-a}{n} = 1 \quad$ $b = 6,\ a = 0$, $n = 6$

$$\int_0^6 \frac{1}{1+x^2}\,dx = 1 \times \frac{1}{2}\left[(1 + 0.028) + 2\times(0.5 + 0.2 + 0.1 + 0.05 + 0.038)\right]$$

$$= 1.4015$$

(ii) By Simpson $1/3^{rd}$ rule

(1.365)
3/8 - 1.355
w = 1.3725

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{1}{3}\left[(1+0.028) + 4\times(0.1 + 0.038) + 2\times(0.2 + 0.05)\right]$$

$$= 1.365$$

(iii) By Simpson 3/8 Rule

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3}{8}\left[(1+0.028) + 2\times(0.1) + 3\times(0.5+0.2+0.05+0.038)\right]$$

$$= 1.347$$

(iv) By Weddles Method:

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3}{16}\left(0.028 + 5\times0.038 + 0.05 + 6\times0.1 + 0.2 + 5\times0.5 + 1\right)$$

$$= 1.3704$$

Q2. A Second degree polynomial f(x) takes the following value. $f(x) = 1 - x + 4x^2$

x	0	1	2
$f(x)$	1	4	15

If $\int_0^2 f(x)\,dx$ is evaluated by following Rule the what is the error estimation.

i) Trapezoidal

ii) Simpson $1/3^{rd}$

iii) Simpson 3/8 → not applied ∵ subinterval is 2

Sol 1) Trapezoidal : $n = 2$ $h = \frac{2-0}{2} = 1$

$$\int_0^2 f(x)\,dx = \frac{1}{2}\left[(1+15) + 2\times 4\right]$$

$$= \frac{1}{2}\left[16 + 8\right]$$

$$= 12$$

Sol 2) Simpson $1/3^{rd}$

$$= \frac{1}{3}\left[16 + 4\times 4 + \cancel{2\times 15}\right]$$

$$= \frac{1}{3}\left[16 + 16 + \cancel{30}\right]$$

$$= \frac{32}{3} = 10.66$$

Trapezoidal Error Estimation = Exact − Approximate = −1.33

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note:- for the Calculation of Integral, trapezoidal rule is more exact if the function is polynomial of max deg 1 i.e (0,1).

By Simpson ~~1/3~~ Rule we get more exact Value of Integral if the polynomial is upto degree 3 ($1/3^{rd}$ is more exact upto degree 2 & 3/8 is more exact upto degree 3).

Book
Q15, Q13, Q28* Q64.

Sol 13

$a = -1$ $\quad n = 3 \quad h = \frac{b-a}{n} = \frac{2}{3}$

$b = 1$

x	-1	$-1/3$	$1/3$	1
y	1	$1/3$	$1/3$	1

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{3} \times \frac{1}{2}\left[(1+1) + 2(2/3)\right]$$

$$\Rightarrow \frac{1}{3}\left[2 + 4/3\right] = 1.1$$

Q15.

$h = \frac{b-a}{n}$; $n = \frac{3-1}{1} = 2$

Sol $\quad I =$

$$\frac{1}{2}\left[\frac{1}{1} + \cancel{\frac{1+2x}{3}}\right] 1/2 \Rightarrow \frac{1}{2}\left[1 + \frac{1}{3} + 1\right]$$

$$\cancel{\frac{1}{2}\left[\frac{3+1}{3}\right] = 2/3} \Rightarrow \frac{1}{2}\left[\frac{3+1+3}{3}\right] = 7/6$$

II. Numerical Sol^n of Nonlinear Equation :-

1. Bisection Method :-

- This Method is Gaurenteed to Convergence but rate of convergence is very low.
- The order of Convergence of bisection Method is linear i.e 1.

Drawback

→ By using Bisection Method we Cannot Locate Complex Root.

$$IF = \frac{x_i + x_{i+1}}{2}$$

$$\begin{Bmatrix} f(x_i) = -ve \\ f(x_{i+1}) = +ve \end{Bmatrix}$$

note : To check the Interval for the root of the function the function changes its sign in the given Interval.

Q1. find the Roots of the eq^n of by bisection Method upto 3rd approximation Interval (2,3)

$$f(x) = x^3 - 4x - 9 \quad \text{---} \quad (1)$$

$f(2) = -ve$

$f(3) = +ve$ i.e $x \in (2,3)$

1st Interation $x_1 = \frac{2+3}{2} = 2.5$

$$f(2.5) = \frac{125}{8} - \frac{20}{4} - 9$$

$$= -ve$$

$x \in (2.5, 3)$

2nd Iteration $x_2 = \frac{2.5+3}{2} = 2.75$

$f(2.75) = +ve$

$x \in (2.5, 2.75)$

$$x_3 = \frac{2.75 + 2.5}{2}$$

$$x_3 = 2.625 \quad //$$

Q2 The Root upto 3rd approx. for $f(x) = xe^x - 1$ $(0, 1)$

a) 0.525 (b) 0.625 c) 0.725 d) 0.825

$f(0) = 0 - 1 = -ve$

$f(1) = +ve$

$$x_1 = \frac{0+1}{2} = 0.5$$

$f(0.5) = -0.179$ ~~+ve~~ $-ve$

$x \in$ ~~(0, 0.5)~~ $(0.5, 1)$

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$f(0.75) = +ve$

$$x_3 = \frac{0.75 + 0.5}{2}$$

$$= 0.625$$

2. Regula falsi or (False position Method):

- This Method is also Gaureentied to Convergence.
- Convergence rate is faster than bisection method but slowler than Newton Raphson. the order is linear i.e first order.
- we cannot locate Complex Root.

Iterative formula = $\boxed{x_2 = x_0 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \times f(x_0)}$