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On-line packing and covering a disk with disks

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Abstract A circular disk D of area $|D|$ can be on-line covered with any sequence of circular disks of total area not smaller than $6.488|D|$. Furthermore, any sequence of circular disks whose total area does not exceed $0.197|D|$ can be on-line packed into D.

Keywords On-line covering · On-line packing · Disk

Mathematics Subject Classification (2000) 52C15 · 05B40

1 Introduction

Let $C, C_1, C_2, \dots$ be planar convex bodies. We say that the sequence (C_i) *permits a covering* of C if there exist rigid motions σ_i such that $C \subseteq \bigcup \sigma_i C_i$. We say that (C_i) can be *packed* into C if there exist rigid motions σ_i such that $\bigcup \sigma_i C_i \subseteq C$ and that $\sigma_i C_i$ have mutually disjoint interiors. The *on-line* restriction means that at the beginning we are given only the first set C_1; then we learn each successive set C_i only after the motion σ_{i-1} has been provided. The placement of each set $\sigma_i C_i$ cannot be changed afterwards.

The area of C is denoted by $|C|$.

Let D be a circular disk. Denote by $c(D)$ the least number such that any (finite or infinite) sequence of circular disks with total area greater than $c(D)|D|$ permits an on-line covering of D. Furthermore, denote by $p(D)$ the greatest number such that

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any (finite or infinite) sequence of circular disks with total area smaller than $p(D)|D|$ can be on-line packed into D.

In Dumitrescu and Jiang (2010) it is proved that $c(D) < 9.763$. In this paper we show that $c(D) < 6.488$. On the other hand, $c(D) \geq 2.25$; three disks of diameters smaller than the side length of an equilateral triangle inscribed in D do not permit a covering of D.

Any sequence of squares with total area not greater than $1/3$ can be on-line packed into the unit square (see Han et al. 2008). Obviously, any disk D of diameter d can be inscribed into a square S of side length d as well as D contains an inscribed square R of side length $d/\sqrt{2}$. Since $|S|/|D| = 4/\pi$ and $|D|/|R| = \pi/2$ it follows that $p(D) \geq \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \approx 0.167$. On the other hand, $p(D) \leq 0.5$; two disks of diameters greater than a half of the diameter of D cannot be packed into D. The aim of this paper is to show that $p(D) > 0.197$.

Other results concerning on-line packings and coverings are given in Kuperberg (1994), Lassak (1997) and Lassak and Zhang (1991).

By $[a, b] \times [c, d]$ we mean $\{(x, y);\ a \leq x \leq b,\ c \leq y \leq d\}$.

2 The method of the first free subbrick

In Sect. 3 we will use the *method of the first free subbrick* introduced in Januszewski and Lassak (1997). Let

$$\lambda = \left[\frac{1}{17}(7 - 4 \cdot 2^{1/2})\right]^{1/2} \approx 0.281.$$

By a *brick of size* p, where $p \in \{0, 1, 2, \ldots\}$, we mean a rectangle of side lengths $\lambda 2^{(3-p)/2}$, $\lambda 2^{(2-p)/2}$.

We can dissect any brick of size p into two congruent bricks, called *subbrick*, of size $p+1$. Consequently, any brick of size p is dissected into 2^q subbricks of size $p+q$. A brick of size 0 is also called a subbrick.

Let

$$\begin{aligned}
K_1 &= [-\lambda/2, \lambda/2] \times [\lambda, \lambda + \lambda\sqrt{2}/2],\\
K_2 &= [-\lambda/2, \lambda/2] \times [-\lambda - \lambda\sqrt{2}/2, -\lambda],\\
K_3 &= [-\lambda\sqrt{2}, \lambda\sqrt{2}] \times [-\lambda, \lambda].
\end{aligned}$$

Obviously, both K_1 and K_2 are bricks of size 3 and K_3 is a brick of size 0.

Assume that $B_1, B_2, \ldots$ is a sequence of bricks. We will pack these bricks into $K_1 \cup K_2 \cup K_3$.

Let q be an arbitrary non-negative integer. We enumerate all 2^q subbricks of K_1 of size $3+q$ by integers from 1 to 2^q. In particular, K_1 is the first brick of size 3. We enumerate all 2^q subbricks of K_2 of size $3+q$ by integers from 1 to 2^q. Finally, we enumerate all 2^q subbricks of K_3 of size q by integers from 1 to 2^q. The only requirement is that the integers $2m-1$ and $2m$ are given to the subbricks of size $s+1$ of the subbrick of size s whose number is m.

By a *free* subbrick we mean a subbrick whose interior has an empty intersection with any brick packed before. For each B_i from the sequence we find the smallest integer $j \in \{1, 2, 3\}$ such that there is a free subbrick of K_j congruent to B_i. We pack B_i into the congruent free subbrick of K_j with the smallest possible number.

We stop the packing process if a subbrick B_z from the sequence cannot be packed into $K_1 \cup K_2 \cup K_3$.

3 On-line disks packing

Theorem 1 *Any sequence of circular disks whose total area does not exceed*

$$\frac{35 - 20\sqrt{2}}{34}|D| \approx 0.1975|D|$$

can be on-line packed into a circular disk D.

Proof Without loss of generality we can assume that D is a disk with radius $1/2$ centered at the origin.

Let $D_1, D_2, \ldots$ be a sequence of disks and let

$$\sum |D_i| \le \frac{35 - 20\sqrt{2}}{34}|D| = \frac{5}{2}\lambda^2|D|.$$

Denote by d_i the diameter of D_i.

It is easy to see that any D_i is contained in a brick B_i with the longest side shorter than $2d_i$. Consequently, by $|D_i| = \frac{1}{4}\pi d_i^2$ and $\sqrt{2}d_i \cdot 2d_i > |B_i|$ we have

$$|D_i| > \frac{\pi\sqrt{2}}{16}|B_i|.$$

Since

$$\left(\frac{1}{2}\lambda\right)^2 + \left(\lambda + \frac{\sqrt{2}}{2}\lambda\right)^2 = \frac{1}{4}$$

and

$$(\lambda\sqrt{2})^2 + \lambda^2 < \frac{1}{4},$$

it follows that D contains the bricks K_1, K_2, K_3.

We pack $B_i \supset D_i$ into $K_1 \cup K_2 \cup K_3 \subset D$ by the method of the first free subbrick.

Contrary to the statement assume that it is impossible to pack $D_1, D_2, \ldots$ into D by this method. Let D_z be the disk which stops the packing process. We show that this leads to the false inequality

$$\sum_{i=1}^{z} |D_i| > \frac{5}{2}\lambda^2 |D|.$$

Obviously, $|D_i| = \frac{1}{4}\pi d_i^2 \leq \frac{5}{2}\lambda^2 |D| = \frac{5}{2}\lambda^2 \cdot \frac{1}{4}\pi$ and, consequently, $d_i < 2\lambda$, for each $i \in \{1, 2, \dots, z\}$. Hence at least one disk is packed into K_3. Consider two cases.

Case 1. Either $d_z \leq \lambda\sqrt{2}/2$ *or there is a disk of diameter smaller than or equal to* $\lambda\sqrt{2}/2$ *packed into* K_3. By the description of the packing method we deduce that at least one brick has been packed into K_1 and at least one brick has been packed into K_2 as well.

By the Proposition of Januszewski and Lassak (1997) we conclude that

$$\sum_{i=1}^{z} |B_i| > |K_1| + |K_2| + |K_3| = 5\sqrt{2}\lambda^2.$$

Consequently,

$$\sum_{i=1}^{z} |D_i| > \frac{\pi\sqrt{2}}{16} \sum_{i=1}^{z} |B_i| = \frac{\pi\sqrt{2}}{16} \cdot 5\sqrt{2}\lambda^2 = \frac{5}{8}\lambda^2\pi = \frac{5}{2}\lambda^2 |D|.$$

Case 2. There is no disk of diameter smaller than or equal to $\lambda\sqrt{2}/2$ *packed into* K_3 *and* $d_z > \lambda\sqrt{2}/2$.

Subcase 2A: $d_z > \sqrt{2}\lambda$.

There is at least one disk (of diameter greater than $\lambda\sqrt{2}/2$) packed into K_3. Consequently,

$$\sum_{i=1}^{z} |D_i| > \pi\left(\frac{\sqrt{2}\lambda}{4}\right)^2 + |D_z| > \pi\left(\frac{\sqrt{2}\lambda}{4}\right)^2 + \pi\left(\frac{\sqrt{2}\lambda}{2}\right)^2 = \frac{5}{2}\lambda^2 |D|.$$

Subcase 2B: $\lambda < d_z \leq \sqrt{2}\lambda$.

At most one subbrick of K_3 of size 2 is free. Consequently, at least three subbricks of K_3 of size 2 are not free and

$$\sum_{i=1}^{z} |D_i| \geq \frac{\pi\sqrt{2}}{16} \cdot 3\sqrt{2}\lambda^2 + |D_z| > \frac{\pi\sqrt{2}}{16} \cdot 3\sqrt{2}\lambda^2 + \pi\left(\frac{\lambda}{2}\right)^2 = \frac{5}{2}\lambda^2 |D|.$$

Subcase 2C: $\lambda\sqrt{2}/2 < d_z \leq \lambda$. There is no free subbrick of K_3. Consequently,

$$\sum_{i=1}^{z} |D_i| \geq \frac{\pi\sqrt{2}}{16} \cdot 4\sqrt{2}\lambda^2 + |D_z| > \frac{\pi\sqrt{2}}{16} \cdot 4\sqrt{2}\lambda^2 + \pi\left(\frac{\sqrt{2}\lambda}{4}\right)^2 = \frac{5}{2}\lambda^2 |D|.$$

□

4 A method of covering with squares

Let S_i be a square of side length s_i smaller than 1, for $i = 1, 2, \ldots$. Let $R_i \subset S_i$ be a rectangle of width w_i and of height s_i, where $w_i \le s_i < 2w_i$ and where $w_i \in \{2^{-1}, 2^{-2}, 2^{-3}, \ldots\}$.

Denote by P_l and P_r the following sets:

$$P_l = ([-0.5, 0] \times [-0.5, 0.5]) \setminus ([-0.5, -0.25] \times [-0.5, -0.25\sqrt{3}]),$$
$$P_r = ([0, 0.5] \times [-0.5, 0.5]) \setminus ([0.25, 0.5] \times [-0.5, -0.25\sqrt{3}]).$$

We describe an on-line method of covering of $P_l \cup P_r$ with $S_1, S_2, \ldots$. This method is a modification of the method presented in Januszewski (2009).

The squares $S_i \supset R_i$ will be divided into two types depending on the width of R_i.

If $w_i \in \{2^{-1}, 2^{-3}, 2^{-5}, \ldots\}$, then S_i is an *l-square*.

If $w_i \in \{2^{-2}, 2^{-4}, 2^{-6}, \ldots\}$, then S_i is an *r-square*.

The description of the covering method is inductive.

Let $i \ge 1$. Assume that if $i > 1$, then the motions $\sigma_1, \ldots, \sigma_{i-1}$ have been provided and that the sets $Q_1, \ldots, Q_{i-1}$ have been defined.

Let $q \in \{l, r\}$. Denote by b_i^q the greatest number not greater than 0.5 such that each point of P_q whose y-coordinate is smaller than b_i^q belongs to $\bigcup_{j=1}^{i-1} Q_j$ (if $i = 1$, then the union of Q_j is taken as an empty set and $b_1^q = -0.5$). The set of points of P_q with y-coordinate b_i^q is called the ith *q-bottom*. A point (x_1, y_1) of the ith q-bottom is an *iq-surface* point if $x_1 \ne 0.25$ and $x_1 \ne -0.25$ and if no point $(x_1, y_1 + \delta)$ belongs to $\bigcup_{j=1}^{i-1} Q_j$, for $\delta > 0$. By an *il-place* we mean a set containing an il-surface point which has the form

$$\{(x, y);\ -0.5 + kw_i \le x \le -0.5 + (k + 1)w_i,\ b_i^l \le y \le b_i^l + s_i\},$$

where $k \in \{0, 1, \ldots, (2w_i)^{-1} - 1\}$. By an *$ir$-place* we mean a set containing an ir-surface point which has the form

$$\{(x, y);\ kw_i \le x \le (k + 1)w_i,\ b_i^r \le y \le b_i^r + s_i\},$$

where $k \in \{0, 1, \ldots, (2w_i)^{-1} - 1\}$.

S_i is a q-square, where either $q = l$ or $q = r$. If P_q is not covered yet, then we take $v = q$. Otherwise, we take $v = l$ provided $q = r$ and we take $v = r$ provided $q = l$. We say that S_i is *used for the covering of P_v*.

(1) If $b_i^v < 0.5 - s_i$, then S_i is a *basic square*. Furthermore, R_i is a *basic rectangle*. We move $S_i \supset R_i$ so that $\sigma_i R_i$ is located in an iv-place.
The *part of S_i used for the covering* is defined as

$$Q_i = (P_v \cap \sigma_i R_i) \setminus \bigcup_{j=1}^{i-1} Q_j$$

(if $i = 1$, then the union of sets Q_j is taken as an empty set). Obviously, if either $w_i \leq 0.25$ or $w_i = 0.5$ and, at the same time, $b_i^v \geq -0.25\sqrt{3}$, then $P_v \cap \sigma_i R_i = \sigma_i R_i$.

(2) If $b_i^v \geq 0.5 - s_i$, then S_i is a *special square*.
Take $k_l = -0.5$ provided the point $(-0.5, 0.5)$ is not covered by any special square preceding S_i. Otherwise, denote by k_l the greatest number such that any point of the segment $[-0.5, k_l] \times [0.5, 0.5]$ is covered by a special square preceding S_i.
Take $k_r = 0.5$ provided the point $(0.5, 0.5)$ is not covered by any special square preceding S_i. Otherwise, denote by k_r the smallest number such that any point of the segment $[0.5 - k_r, 0.5] \times [0.5, 0.5]$ is covered by a special square preceding S_i.
If $v = l$, then $\sigma_i S_i = [k_l, k_l + s_i] \times [0.5, 0.5 - s_i]$.
If $v = r$, then $\sigma_i S_i = [0.5 - k_r - s_i, 0.5 - k_r] \times [0.5, 0.5 - s_i]$.
Furthermore, we take $Q_i = \emptyset$.

5 On-line covering with disks

Let $q \in \{l, r\}$. Moreover, let $q^* = l$ if $q = r$ and $q^* = r$ if $q = l$. Assume that S_i is a basic q-square used for the covering of P_{q^*}. By Lemma 1 of Januszewski (2009) we deduce that the area of the part of S_i used for the covering exceeds $\frac{1}{3}|S_i|$, i.e.,

$$|S_i| < 3|Q_i|.$$

Lemma *Assume that S_i is a basic q-square used for the covering of P_q and that P_{q^*} is not covered by the squares preceding S_i. The area of the part of S_i used for the covering exceeds $\frac{2}{5}|S_i|$, i.e., $|S_i| < \frac{5}{2}|Q_i|$.*

Proof If $\sigma_j R_j \subset P_q$ for $j < i$, then S_j is a q-square. Hence no basic rectangle of width $\frac{1}{2}w_i$ has a non-empty intersection with $Int\ \sigma_i R_i$. At most three basic rectangles of width $\frac{1}{4}w_i$ (and of height smaller than $\frac{1}{2}w_i$) have a non-empty intersection with $Int\ \sigma_i R_i$. The part of $\sigma_i R_i$ covered with basic rectangles preceding R_i is of area smaller than

$$3 \cdot \frac{1}{4}w_i \cdot \frac{1}{2}w_i + 3 \cdot \frac{1}{16}w_i \cdot \frac{1}{8}w_i + \cdots = \frac{3w_i^2}{8}\left(1 + \frac{1}{16} + \frac{1}{64} \cdots\right) = \frac{3w_i^2}{8} \cdot \frac{16}{15} = \frac{2}{5}w_i^2.$$

It is easy to see that

$$|Q_i| > |R_i| - \frac{2}{5}w_i^2 = w_i s_i - \frac{2}{5}w_i^2.$$

Since

$$\frac{2}{5}s_i^2 - w_i s_i + \frac{2}{5}w_i^2 < 0$$

for $w_i \leq s_i < 2w_i$, it follows that $|Q_i| > \frac{2}{5}s_i^2 = \frac{2}{5}|S_i|$. □

Theorem 2 *A circular disk D can be on-line covered with any sequence of circular disks whose total area is not smaller than*

$$(8.125+0.6875\sqrt{3}-2\sqrt{2})|D| \approx 6.487|D|.$$

Proof Let D be a circular disk and let (D_i) be a sequence of circular disks with total area greater than or equal to

$$(8.125+0.6875\sqrt{3}-2\sqrt{2})|D|.$$

We lose no generality in assuming that D is a disk with radius 0.5 centered at the origin.

It is easy to see that D is contained in $P_l \cup P_r$. Furthermore, any D_i contains an inscribed square S_i of side length s_i such that $|D_i|/|S_i| = \pi/2$. Hence

$$\sum |S_i| = \frac{2}{\pi}\sum |D_i| \geq 0.5(8.125+0.6875\sqrt{3}-2\sqrt{2}) \approx 3.24.$$

We can assume that the diameter d_i of each D_i is smaller than 1; otherwise D can be covered by D_i. Consequently, we can assume that the side length of each S_i is smaller than $0.5\sqrt{2}$.

We cover $P_l \cup P_r$ by $S_1, S_2, \ldots$ by the method described in Sect. 4. Obviously, if $P_l \cup P_r \subseteq \bigcup \sigma_i S_i$, then $D \subset \bigcup \sigma_i D_i$.

We prove Theorem 2 indirectly. Assume that we apply the covering method and that $P_l \cup P_r$ is not contained in $\bigcup \sigma_i S_i$. We show that this leads to the false inequality

$$\sum |S_i| < 0.5(8.125+0.6875\sqrt{3}-2\sqrt{2}).$$

Consider three cases. Take $a = 0.5 - 0.25\sqrt{3}$.

Case 1. Neither P_l nor P_r is covered.

The parts of the basic squares used for the covering are pairwise disjoint and they are contained in $P_l \cup P_r$. By the Lemma we deduce that the total area of the basic squares is not greater than $2.5|P_l| + 2.5|P_r|$. The total area of the special squares is smaller than $0.5^2 + 0.5^2$. Hence the total area of the squares is smaller than

$$2.5(0.5-0.25a)+2.5(0.5-0.25a)+0.5 = 3-1.25a < 3,$$

which is a contradiction.

Case 2. P_l is covered.

Let S_j be the last special square used for the covering of P_l. The total area of the squares used for the covering of P_l is smaller than

$$2.5|P_l| + 0.5^2 + s_j^2.$$

Subcase 2A. The point $(0.5, 0.5)$ belongs to $\sigma_j S_j$. Obviously, $s_j > 0.5$.

Since P_r is not covered, it follows that $b_n^r < 0.5 - s_j$ for each positive integer n such that S_n has been used for the covering of P_r. The total area of parts of basic rectangles lying above the nth r-bottom (for any positive integer n) is smaller than

$$\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{8}\cdot\frac{1}{4}+\cdots=\frac{1}{8}\left(1+\frac{1}{4}+\cdots\right)=\frac{1}{8}\cdot\frac{4}{3}=\frac{1}{6}.$$

Thus the total area of the squares used for the covering of P_r is smaller than

$$3\left[0.5(1-s_j)+\frac{1}{6}\right]+0.5^2.$$

Consequently, the total area of the squares in the sequence is smaller than

$$2.5(0.5-0.25a)+0.25+s_j^2+1.5-1.5s_j+0.5+0.25.$$

It is easy to check that this upper bound is smaller than 3.21 provided $0.5 < s_j < 0.5\sqrt{2}$, which is a contradiction.

Subcase 2B. The point $(0.5, 0.5)$ does not belong to $\sigma_j S_j$.

First assume that the special squares have pairwise disjoint interiors.

The total area of the bottom squares is not greater than

$$2.5|P_l|+3|P_r|=5.5(0.5-0.25a).$$

The special squares have pairwise disjoint interiors and they have side lengths smaller than $0.5\sqrt{2}$. Moreover, the sum of s_m, taken over all indexes m such that S_m is a special square, is smaller than 1. It is easy to check that the total area of the special squares is smaller than $(0.5\sqrt{2})^2+(1-0.5\sqrt{2})^2$. Hence the total area of the squares is smaller than

$$5.5(0.5-0.25a)+(0.5\sqrt{2})^2+(1-0.5\sqrt{2})^2=0.5(8.125+0.6875\sqrt{3}-2\sqrt{2}),$$

which is a contradiction.

Assume that there is a special square S_m used for the covering of P_r such that $Int\ S_j \cap Int\ S_m \neq \emptyset$. Obviously, $s_j < s_m < 0.5$, otherwise $P_l \cup P_r$ is covered. Furthermore, since P_r is not covered, it follows that $b_n^r < 0.5 - s_j$ for any integer n such that S_n has been used for the covering of P_r.

At this point we show that the total area of the squares used for the covering of P_r is smaller than $1.75 - s_j^2$. Denote by k the integer such that $2^{-k} \le s_j < 2^{-k+1}$. Let

$$U_1=[0,2^{-k}]\times[0.5-s_j,s_j],\quad U_2=[2^{-k},2\cdot 2^{-k}]\times[0.5-s_j,s_j].$$

There is $t \in \{1, 2\}$ such that each basic rectangle of width not smaller than 2^{-k} has an empty intersection with $Int\ U_t$, otherwise P_r is covered. Consequently, the area of the part of U_t covered with basic rectangles does not exceed

$$2^{-k-1} \cdot 2^{-k} + 2^{-k-2} \cdot 2^{-k-1} + \cdots = \frac{2}{3} \cdot 2^{-2k}.$$

This implies that the total area of the basic rectangles used for the covering of P_r does not exceed

$$\frac{1}{2} - |U_t| + \frac{2}{3} \cdot 2^{-2k}.$$

Hence the sum of areas of all basic squares used for the covering of P_r is not greater than

$$3\left(\frac{1}{2} - |U_t| + \frac{2}{3} \cdot 2^{-2k}\right) = \frac{3}{2} + 2 \cdot 2^{-2k} - 3 \cdot 2^{-k} s_j.$$

Since $2^{-k} \leq s_j < 2^{-k+1}$, it follows that this upper bound is not greater than $1.5 - s_j^2$. Obviously, the total area of the special squares used for the covering of P_r is smaller than 0.5^2. Hence, the total area of the squares used for the covering of P_r is smaller than $1.75 - s_j^2$.

Consequently,

$$\sum |S_i| < 2.5(0.5 - 0.25a) + 0.5^2 + s_j^2 + 1.75 - s_j^2 < 3.21,$$

which is a contradiction.

Case 3: P_r is covered.

We argue in a way similar to the argument of Case 2. □

The method described in Sect. 4 can be used also for the covering of

$$I = [-0.5, 0.5] \times [-0.5, 0.5]$$

with squares $S_1, S_2, \ldots$. In that case, if I is not covered, then the total area of the squares in the sequence is smaller than $2.5 \cdot 0.5 + 3 \cdot 0.5 + 1 = 3.75$.

Corollary *A unit square can be on-line covered with any sequence of squares whose total area is not smaller than* 3.75.

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