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ON-LINE COVERING THE UNIT SQUARE BY SQUARES AND THE THREE-DIMENSIONAL UNIT CUBE BY CUBES

We say that a sequence $C_1, C_2, \ldots$ of sets of Euclidean d-space E^d *permits a covering* of a set $C \subset E^d$ if there are rigid motions σ_i such that C is contained in the union of sets $\sigma_i C_i$ for $i = 1, 2, \ldots$ A survey of results on the covering of convex bodies by sequences of convex bodies can be found in the paper [2] of Groemer. We consider *on-line* covering methods in which we are given C_i, where $i > 1$, only after the motion σ_{i-1} has been provided; at the beginning we are given C_1. The on-line restriction is introduced here by analogy to the on-line restriction for packing problems (see [1] and [5]).

The first solution of an on-line covering problem was given by Kuperberg [4] who proved that the unit cube I^d of E^d can be on-line covered by every sequence of cubes of the total volume at least 4^d. The paper [3] presents two on-line methods of a covering of I^d by cubes. The first method permits an on-line covering of I^d by every sequence of cubes of the total volume at least $3 \cdot 2^d - 4$. The second method is much more efficient. For $d > 3$ it enables an on-line covering of the unit cube by arbitary sequence of cubes whose sum of volumes is at least $2^d + (\frac{4}{3}\sqrt{2} + \frac{2}{3})(\frac{3}{2})^d$. In the present paper we consider the cases $d = 2$ and $d = 3$. We apply an improvement of the second method. The unit square can be covered by this improved method if the total area of a sequence of squares used for the covering is at least $\frac{7}{4} \cdot \sqrt[3]{9} + \frac{13}{8} \cong 5.265$. The three-dimensional cube can be covered by this method by every sequence of cubes of the total volume at least $\frac{143}{32} + \frac{3}{8}(\frac{9}{2})^{3/4} + \frac{63}{8}(\frac{2}{9})^{1/4} \cong 11.034$.

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1. Description of the method

Let p and q be real numbers such that $pq = 2$ and that $\frac{4}{3} \le p < 2$. Clearly, $1 < q \le \frac{3}{2}$.

We will on-line cover the unit cube

$$I^d = \{(x_1, \dots, x_d);\ 0 \le x_1 \le 1, \dots, 0 \le x_d \le 1\}$$

by a sequence $Q_1, Q_2, \dots$ of cubes. We call a cube of our sequence *very big* if its sides are of length at least 1, and we call it *big* if its sides are at least $\frac{1}{2}p$ but smaller than 1.

There is a cube N_i in Q_i whose side is of the greatest possible length of the form 2^{-r} or $p \cdot 2^{-r-2}$, where r is a non-negative integer. If Q_i is not big and if the length of the side of N_i is of the form 2^{-r}, where r is a non-negative integer, then we put $B_i = N_i$ and $T_i = \emptyset$. If Q_i is big, then let us denote by h_i the length of the side of Q_i and let us find in Q_i a rectangular parallelotope M_i containing N_i and having $d-1$ perpendicular sides of length $\frac{1}{2}$ and the side of length h_i perpendicular to them; we put $B_i = M_i$ and $T_i = \emptyset$. If the side of N_i is of the form $p \cdot 2^{-r-2}$, where r is a non-negative integer, we put $B_i = \emptyset$ and $T_i = N_i$.

For every non-empty B_i we find the greatest number $b_i \le 1$ such that every point of I^d whose last coordinate is smaller than b_i has been covered by cubes Q_j such that $j < i$. The set of all points of the cube I^d having the last coordinate b_i is called *the i-th bottom* of the cube. Of course, if $b_i > 0$, then the i-th bottom is covered by cubes Q_j, where $j < i$. A point of the *i-th* bottom is called *surface* if there is a point in I^d with the last coordinate greater and with unchanged the other coordinates which has not been covered by cubes Q_j, where $j < i$. We find a rigid motion σ_i such that $\sigma_i B_i$ contains a surface point of the i-th bottom and that it has the form

$$\{(x_1, \dots, x_d);\ a_k 2^{-r} \le x_k \le (a_k + 1)2^{-r} \text{ for } k = 1, \dots, d-1 \\ \text{and } b_i \le x_d \le b_i + 2^{-r}\},$$

where 2^{-r} denotes the length of the side of the cube N_i, and where the numbers $a_1, \dots, a_{d-1}$ belong to the set $\{0, \dots, 2^r - 1\}$. Let us add that if Q_i is big, we find a rigid motion σ_i such that $\sigma_i Q_i$ is between the hyperplanes $x_k = 0$ and $x_k = 1$ for $k = 1, \dots, d-1$, and that $\sigma_i B_i$ contains a surface point of the i-th bottom and that it has the form

$$\{(x_1, \dots, x_d);\ a_k 2^{-1} \le x_k \le (a_k + 1)2^{-1} \text{ for } k = 1, \dots, d-1 \\ \text{and } b_i \le x_d \le b_i + h_i\},$$

where $a_1, \dots, a_{d-1} \in \{0, 1\}$. As long as I^d is not covered, every i-th bottom contains surface points. We say that a big cube Q_i is *early* if $\sigma_i B_i \subset I^d$ and if $b_i + h \le b_m$ for an $m > i$, and we say that it is *late* in the opposite case.

The sequence $T_1, T_2, \dots$ is used for covering I^d by a similar method. For every T_i we find the greatest number $t_i \le 1$ such that all points of I^d with the last co-ordinate greater than $1 - t_i$ are covered by cubes Q_j, where $j < i$. The set of all points of I^d with the last coordinate equal to $1 - t_i$ is called *the i-th top*. A point of the i-th top is called *surface* if there is a point in I^d with the last coordinate smaller and with unchanged the other coordinates which has not been covered by cubes Q_j, where $j < i$. If $T_i \neq \emptyset$, we find a rigid motion σ_i such that $\sigma_i T_i$ contains a surface point of the i-th top and that it is of the form

$$\{(x_1, \dots, x_d);\ a_k p 2^{-r-2} \le x_k \le (a_k + 1) p 2^{-r-2} \text{ for } k = 1, \dots, d-1 \\ \text{and } 1 - t_i - p2^{-r-2} \le x_d \le 1 - t_i\},$$

where $p \cdot 2^{-r-2}$ denotes the length of the side of the cube T_i and where $a_1, \dots, a_k \in \{0, 1, \dots, 3 \cdot 2^r - 1\}$. As long as I^d is not covered, every i-th top contains surface points.

So for every index i a rigid motion σ_i is found such that the non-empty set from the sets B_i or T_i is moved in order to cover our unit cube. Together with the set we move the corresponding cube Q_i. If $b_n + t_n \ge 1$, then we can stop the covering process because I^d has been covered by cubes $\sigma_1 Q_1, \dots, \sigma_{n-1} Q_{n-1}$.

Remark. Our method in the case when $p = q = \sqrt{2}$ is an improvement of the method of the current bottom and top considered in [3]. We change here the definitions of the i-th bottom, of the i-th top, and of the surface points. Moreover, we precisely determine where to put big cubes. The introduced improvements eliminate putting a cube in a place totally covered by previous cubes.

From the comparison of the presented method with the method of the current bottom and top we conclude that Lemmas 1 and 2 of [3] can be generalized here as follows

LEMMA 1. *The set*

$$\{(x_1, \dots, x_d);\ 0 \le x_1 \le 1, \dots, 0 \le x_d \le a\}$$

can be covered by our method by every sequence of cubes whose sides are of the form 2^{-r}, where $r \in \{1, 2, \dots\}$, if the total volume of the cubes is at least

$$b(a) = \frac{2^d - 1}{2^{d-1}} \left[a + \frac{2^{d-1} - 1}{2^d - 1} \right].$$

LEMMA 2. *The set*

$$\{(x_1, \dots, x_d);\ 0 \le x_1 \le 1, \dots, a \le x_d \le 1\}$$

can be covered by our method by every sequence of cubes whose sides are of the form $p \cdot 2^{-r-1}$*, where* $r \in \{1, 2, \ldots\}$*, if the total volume of the cubes is at least*

$$t(a) = \frac{2^d - 1}{2^{d-1}} \left[(1-a) \left(\frac{3p}{4} \right)^{d-1} + \left(3^{d-1} - \frac{2^{d-1}}{2^d - 1} \right) \left(\frac{p}{4} \right)^d \right].$$

2. Covering the unit square by a sequence of squares

THEOREM 1. *The unit square can be on-line covered by every sequence of squares of the total area at least*

$$\frac{7}{4}\sqrt[3]{9} + \frac{13}{8} \cong 5.265. \tag{1}$$

Proof. Applying the method described in the preceding part, where $p = \sqrt[3]{3}$, we on-line cover the unit square I^2 by arbitrary sequence $Q_1, Q_2, \ldots$ of squares. If in the sequence there is a very big square or if there are at least four big squares, then the sequence covers the whole I^2. So let us assume further that there is no very big square and that the sequence contains at most three big squares.

Let us explain that the value $\sqrt[3]{3}$ for p has been chosen in order to minimize the maximum of the numbers $p^2 b(0) + q^2 t(0)$ and $p^2 t(1) + q^2 t(1)$. It is easy to check that for $p = \sqrt[3]{3}$ the value of $p^2 b(a) + q^2 t(a)$ does not depend on a and that

$$p^2 b(a) + q^2 t(a) = 2\sqrt[3]{9} + \frac{7}{8}. \tag{2}$$

Observe that every big square Q_i contains a square Q_i' of sides of length $\frac{1}{2}p$ such that

$$|Q_i| - |Q_i'| < 1 - \frac{1}{4}p^2 \quad \text{and} \quad |Q_i'| = p^2 |N_i|. \tag{3}$$

Case 1, *in which the sequence* $Q_1, Q_2, \ldots$ *does not contain big squares.*

From Lemmas 1 and 2, and from (2) we see that I^2 can be covered by every sequence $Q_1, Q_2, \ldots$ of the total area at least $2\sqrt[3]{9} + \frac{7}{8} \simeq 5.035$.

Case 2, *in which the sequence* $Q_1, Q_2, \ldots$ *contains exactly one big square.*

Denote by Q_m the unique big square in our sequence.

For $d = 2$ the value of $t(a)$ defined in Lemma 2 is

$$\frac{3}{2}\left[(1-a) \cdot \frac{3}{4}p + \frac{7}{48}p^2 \right]. \tag{4}$$

Particularly, the value $\frac{7}{48}p^2$ in (4) originates from

$$\left(\frac{p}{4}\right)^2+\left(\frac{p}{4}\right)^2+\left(\frac{p}{8}\right)^2+\left(\frac{p}{16}\right)^2+\ldots=\frac{7}{48}\cdot p^2 \tag{5}$$

It estimates from above the area "under" the i-th top which is covered by squares $\sigma_j T_j$ for $j < i$.

Below we consider two subcases; when the difference $t_m - b_m$ is smaller than $\frac{1}{4}p$, and when it is at least $\frac{1}{4}p$.

Assume that $t_m - b_m < \frac{1}{4}p$. Clearly, at most two of the three rectangles

$$\left\{(x,y);\ \frac{1}{4}pk \le x \le \frac{1}{4}p(k+1),\ b_m \le y \le t_m\right\}, \quad \text{where } k=0,1,2,$$

has been covered by squares $\sigma_j Q_j$ for $j < m$. If two of the above three rectangles has been covered by squares $\sigma_j Q_j$ for $j < m$, then the third of them is covered by $\sigma_m Q_m$. So since now we assume that at most one of the three rectangles has been covered by squares $\sigma_j Q_j$ for $j < m$. Observe that for $d = 2$ the total area of squares $\sigma_1 B_1, \sigma_2 B_2, \ldots$ in Lemma 1 is smaller than $\frac{3}{2}(a+\frac{1}{3})$, and the total area of squares $\sigma_1 T_1, \sigma_2 T_2, \ldots$ in Lemma 2 is smaller than $\frac{3}{2}[(1-a)\cdot\frac{3}{4}p+\frac{7}{48}p^2]$. So from the inequality $t_m - b_m < \frac{1}{4}p$ we conclude that the total area of the squares $\sigma_1 N_1, \sigma_2 N_2, \ldots$ is less than $\frac{3}{2}[a+\frac{1}{3}+(1-a)\frac{3}{4}p+\frac{7}{48}p^2-\frac{1}{2}(t_m-b_m)]$. Moreover, instead of one of the components $(\frac{1}{4}p)^2$ in (5) we have now another component; $\frac{1}{4}p(t_m-b_m)$ or $\frac{1}{3}(\frac{1}{4}p)^2$. This and (3) imply that the difference between the loss and the profit (in comparison to the situation when we would not have this square) in the total area of the sequence of squares which permits a covering of the square I^2 is not smaller than $1-(\frac{1}{2}p)^2-\frac{3}{2}[\frac{1}{2}(t_m-b_m)+(\frac{1}{4}p)^2-\frac{1}{4}p(t_m-b_m)]\cdot q^2$ for the component $\frac{1}{4}p(t_m-b_m)$, or at least $1-(\frac{1}{2}p)^2-\frac{3}{2}[\frac{1}{2}(t_m-b_m)+(\frac{1}{4}p)-\frac{1}{3}(\frac{1}{4}p)^2]\cdot q^2$ for the component $\frac{1}{3}(\frac{1}{4}p)^2$. Observe that both the numbers are not greater than $1-\frac{1}{4}\sqrt[3]{9}-\frac{1}{4}=\frac{3}{4}-\frac{1}{4}\sqrt[3]{9}$.

Now assume that $t_m - b_m \ge \frac{1}{4}p$. Similarly like in the preceding paragraph we get that the total area of squares $\sigma_1 N_1, \sigma_2 N_2, \ldots$ is smaller than $\frac{3}{2}[a+\frac{1}{3}+(1-a)\frac{3}{4}p+\frac{7}{48}p^2-\frac{1}{2}(t_m-b_m)]$ if $t_m-b_m \le \frac{1}{2}p$, and that it is smaller than $\frac{3}{2}[a+\frac{1}{3}+(1-a)\frac{3}{4}p+\frac{7}{48}p^2-\frac{1}{2}(\frac{1}{2}p-\frac{1}{2})]$ if $t_m-b_m > \frac{1}{2}p$. Similarly we observe that in both situations the diference between the loss and the profit (in comparison to the situation when we would not have this square) in the total area of the sequence of cubes which can cover I^2 is smaller than $\frac{3}{4}-\frac{1}{3}\sqrt[3]{9}$.

Since in each of the above subcases the difference between the loss and the profit is not greater than $\frac{3}{4}-\frac{1}{4}\sqrt[3]{9}$, then I^2 can be covered by our method provided the total area of the squares in the sequence is at least $\frac{3}{4}-\frac{1}{4}\sqrt[3]{9}+2\sqrt[3]{9}+\frac{7}{8}=\frac{13}{8}+\frac{7}{4}\sqrt[3]{9}\cong 5.265$.

Case 3, *in which there are exactly two big squares in the sequence* $Q_1, Q_2, \ldots$

At least one of the two big squares must be an early square. Denote it by Q_m. Put $S_m = M_m \setminus N_m$. We have $|S_m| = \frac{1}{2}(h_m - \frac{1}{2})$. Since the rectangle $\sigma_m S_m$ is a subset of I^2 such that $b_m + h_m \leq b_w$ for an index $w > m$, and since it covers only points not covered earlier in the covering process, it replaces a number of squares of the form Q_j counted in (2) whose sum of volumes is $\frac{1}{2}(h_m - \frac{1}{2}) \cdot \frac{3}{2}p^2$. So when we have an early big square in the sequence, the diference between the loss and the profit (in comparison to the situation when we would not have this square) in the total area of the sequence of cubes which is able to cover I^2 is

$$h_m^2 - \left(\frac{1}{2}p\right)^2 - \frac{1}{2}\left(h_m - \frac{1}{2}\right) \cdot \frac{3}{2}p^2,$$

which is not greater than the negative number $1 - \frac{5}{8}\sqrt[3]{9}$.

We see that I^2 permits a covering by our method if the total area of the squares of the sequence $Q_1, Q_2, \ldots$ is at least $1 - (\frac{1}{2}p)^2 + 1 - \frac{5}{8}\sqrt[3]{9} + 2\sqrt[3]{9} + \frac{7}{8} \cong 5.215$.

Case 4, *in which there are exactly three big squares in the sequence* $Q_1, Q_2, \ldots$

Since two of the three squares are early, I^2 permits a covering by our method if the total area of the squares of the sequence $Q_1, Q_2, \ldots$ is at least $1 - (\frac{1}{2}p)^2 + 2(1 - \frac{5}{8}\sqrt[3]{9}) + 2\sqrt[3]{9} + \frac{7}{8} \simeq 4.915$. ■

3. Covering the three-dimensional unit cube by a sequence of cubes

THEOREM 2. *The cube* I^3 *permits an on-line covering by every sequence of three-dimensional cubes of the total volume at least*

$$\frac{143}{32} + \frac{3}{8}\left(\frac{9}{2}\right)^{3/4} + \frac{63}{8}\left(\frac{2}{9}\right)^{1/4} \cong 11.034.$$

Proof. Applying our method for $p = (\frac{9}{2})^{1/4}$ we on-line cover I^3 by arbitrary sequence $Q_1, Q_2, \ldots$ of cubes. Let us assume that there is no very big cube and that the sequence contains at most seven big cubes; in the opposite case the thesis is trivially true. It is easy to check that for $p = (\frac{9}{2})^{1/4}$ the value of $p^3 b(a) + q^3 t(a)$ does not depend on a and that $p^3 b(a) + q^3 t(a) = \frac{3}{4}p^3 + \frac{63}{8}p^{-1} + \frac{59}{32}$.

Case 1, *in which the sequence* $Q_1, Q_2, \ldots$ *does not contain big cubes.*

Of course, the cube I^3 permits a covering by our method if the total volume of cubes of the sequence $Q_1, Q_2, \ldots$ is at least $\frac{3}{4}p^3 + \frac{63}{8}p^{-1} + \frac{59}{32} \cong 9.568$.

Case 2, *in which the sequence $Q_1, Q_2, \ldots$ contains at least one big cube.* Observe that for every early big cube Q_m the difference between the lose and the profit (in comparison to the situation when we would not have this cube) is equal to

$$h_m^3 - \left(\frac{1}{2}p\right)^3 - \left(\frac{1}{2}\right)^2\left(h_m - \frac{1}{2}\right)p^2 \cdot \frac{7}{4}.$$

This number is not greater than $1 - \frac{11}{32}p^3$, and thus it is negative.

Of course, our sequence contains at most three late big cubes. Analogically like in Case 2 of Theorem 2 we can show that for every late big cube the difference between the lose and the profit is equal to $1 - \frac{1}{8}p^3 - \frac{1}{8}$.

We conclude that I^3 can be covered by our method by every sequence of cubes of the total volume at least

$$3\left(1-\frac{1}{8}p^3-\frac{1}{8}\right)+\frac{3}{4}p^3+\frac{63}{8}p^{-1}+\frac{59}{32}=\frac{143}{32}+\frac{3}{8}\left(\frac{9}{2}\right)^{\frac{3}{4}}+\frac{63}{8}\left(\frac{2}{9}\right)^{\frac{1}{4}} \cong 11.034. \blacksquare$$

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