Question 4

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Part 1

Statement: If a bivariate Gaussian random variable $Y := (Y1, Y2)$ has a diagonal covariance matrix, then the random variables $Y1, Y2$ are correlated.
This statement is false and we disprove it using the very definition of the covariance matrix. We denote the mean of this bivariate gaussian by the two-dimensional vector $\mu = \begin{bmatrix} E(Y_1) \\ E(Y_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$. By definition, the covariance matrix is :

$$Covariance\ Matrix = E((Y - \mu)(Y - \mu)^T)$$

Therefore, the second entry in the first row (and the first entry in the second column) is given by:

$$Covariance\ Matrix_{1,2} = Covariance\ Matrix_{2,1} = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = 0$$

since the covariance matrix is given to be diagonal. By definition, uncorrelated variables are those that satisfy the property $E(XY) = E(X)E(Y)$ We have

$$E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = 0$$

$$E(Y_1Y_2 + \mu_1\mu_2 - Y_1\mu_2 - \mu_1Y_2) = 0$$

$$E(Y_1Y_2) + \mu_1\mu_2 - \mu_2E(Y_1) - \mu_1E(Y_2) = 0$$

$$E(Y_1Y_2) + \mu_1\mu_2 - \mu_1\mu_2 - \mu_1\mu_2 = 0$$

$$E(Y_1Y_2) = \mu_1\mu_2 = E(Y_1)E(Y_2)$$

Here we have used the fact that expectation of a constant is the constant and $E(Y_1) = \mu_1$ & $E(Y_2) = \mu_2$
Hence, Y_1 and Y_2 are uncorrelated.

Part 2

Statement: If a bivariate Gaussian random variable $Z := (Z1, Z2)$ is such that the random variables $Z1, Z2$ are uncorrelated, then the random variables $Z1, Z2$ are independent.
This statement is true and can be proved as follows: We are given that random variables Z_1, Z_2 are uncorrelated and hence by definition,

$$E(Z_1 Z_2) = E(Z_1)E(Z_2)$$

$$E(Z_1 Z_2 - E(Z_1)E(Z_2)) = 0$$

$$E(Z_1 Z_2 + E(Z_1)E(Z_2) - E(Z_1)E(Z_2) - E(Z_1)E(Z_2)) = 0$$

$$E(Z_1 Z_2 + E(Z_1)E(Z_2) - Z_1 E(Z_2) - Z_2 E(Z_1)) = 0$$

$$E((Z_1 - E(Z_1))(Z_2 - E(Z_2))) = 0$$

This implied that in the covariance matrix, the non-diagonal entries will be 0 since both the non-diagonal entires are $E((Z_1 - E(Z_1))(Z_2 - E(Z_2))) = 0$.
Therefore, the covariance matrix is diagonal and since both the diagonal entries (representing variances) should be positive, we denote the covariance matrix as:

$$C = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

The inverse of the covariance matrix will hence be:

$$C^{-1} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}$$

We denote the joint pdf of Z_1, Z_2 as f, and their marginal pdfs as f_{Z_1} and f_{Z_2}
The pdf of the bivartiate gaussian is hence given by:

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{det(C)}} exp(-0.5(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} C^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^T))$$

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{det(C)}} exp(-0.5(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^T))$$

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{a^2 b^2}} exp(-0.5(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^T) = \frac{1}{2\pi ab} exp(\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2})$$

$$f(z_1, z_2) = \frac{1}{\sqrt{2\pi} a} exp(\frac{z_1^2}{2a^2}) \frac{1}{\sqrt{2\pi} b} exp(\frac{z_2^2}{2b^2})$$

$$f(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2)$$

Since the joint pdf of two random variables can be written as the product of the marginal pdf's for all values of z_1, z_2, the two random variables must be independent.
Hence, $Z - 1, Z_2$ are independent.
Hence proved.