PROBABILITY
&
QUEUEING THEORY

(As per SRM UNIVERSITY Syllabus)

Dr. P. GODHANDARAMAN

M.Sc., M.A., M.B.A., Ph.D.

Assistant Professor (Senior Grade)

Department of Mathematics

SRM IST, Kattankulathur – 603 203

Email : godhanda@gmail.com, Mobile : 9941740168

PROBABILITY AND QUEUEING THEORY

UNIT – I : RANDOM VARIABLES

Syllabus

- Review of Probability Concepts - Types of Events, Axioms
- Conditional Probability, Multiplication Theorem, Applications
- Discrete Random Variable
- Continuous Random Variable
- Expectation and Variance
- Higher Order Moments
- Moment Generating Function
- Function of Random Variable (One Dimensional Only)
- Chebychev's Inequality

PROBABILITY

Probability (or) Chance: Probably, Chances, Likely, Possible - The terms convey the same meaning.

Example:

1. ***Probably*** your method is correct
2. The ***chances*** of getting ranks Ram and Gothai are equal.
3. It is ***likely*** that Ram may not come for taking his classes today.
4. It is ***possible*** to reach the college by 8.30am.

Ordinary Language: The word probability means uncertainty about happening.

Mathematics or Statistics : A numerical measure of uncertainty is practiced by the important branch of statistics is called the **Theory of Probability.**

Day to Day Life:

- ***Certainty*** - Every day the sun rises in the east
- ***Impossibility*** - It is possible to live without water
- ***Uncertainty*** - Probably Raman gets that job.

In the theory of probability, we represent certainty by 1, impossibility by 0 and uncertainty by a positive fraction which lies between 0 and 1.

Applications : There is no area in **social, physical (or) natural sciences** where the probability theory is not used.

- It is the base of the fundamental laws of statistics.
- It gives solutions to betting of games.
- It is extensively used in business situations characterized by uncertainty.
- It is essential tool in statistical inference and forms the basis of the Decision Theory.

Random Experiment (or) Trial and Event (or) Cases:

Consider an experiment of throwing a **coin**. When tossing a coin, we may get a head or tail. Here tossing of a coin is a **trial** and getting a head or tail is an **event.**

Throwing of a **die** is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Favourable Events : The number of outcomes favourable to an event in an experiment is the number of outcomes which entail the happening of the event.

Example: In tossing 2 coins the cases favourable to the event of getting a head are HT, TH, and HH.

Exhaustive Events : The total number of possible outcomes in any **trial** is known as exhaustive events.

Example: In tossing a coin the possible outcomes are getting a head or tail. Hence we have 2 exhaustive events in throwing a coin.

Mutually Exclusive Event:

Two events are said to be mutually exclusive when the occurrence of one affects the occurrence of the other. In other words, if A & B are mutually exclusive events and if A happens then B will not happen and vice versa.

Example: In tossing a coin the events head or tail are mutually exclusive, since both tail & head cannot appear in the same time.

Equally Likely Events: Two events are said to be equally likely if one of them cannot be expected in preference to the other. ***Example:*** In tossing a coin, head or tail are equally likely events.

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

Independent Event : Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. ***Example :*** In tossing a coin, the event of getting a head in the 1st toss is independent of getting a head in the 2nd toss, 3rd toss, etc.

Mathematical Definition of Probability:

If P is the notation for probability of happening of the event, then $P(A) = \frac{Number\ of\ Favourable\ Cases}{Total\ Number\ of\ Exhaustive\ Cases} = \frac{m}{n}$

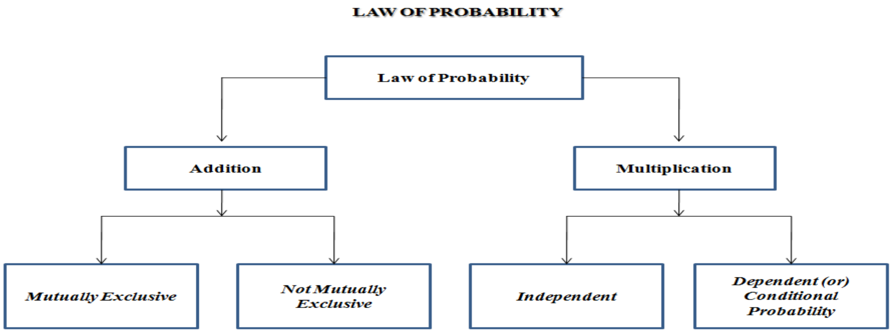
Statistical Definition of Probability:

If in n trials, an event E happens m times, then $P(E) = \lim_{n\to\infty} \frac{m}{n}$

Axiomatic Definition of Probability:

1. For any event A, $P(A) \geq 0$.
2. $P(S) = 1$
3. If $A_1, A_2, A_3, \dots, A_n$ are finite number of disjoint events of S, then
   $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \cdots = \sum P(A_i)$

LAW OF PROBABILITY



ADDITION LAW OF PROBABILITY

Case (i): When events are mutually exclusive

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Case (ii): When events are not mutually exclusive

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

THEOREM : ADDITION LAW OF PROBABILITY

If A and B are any two events and are not disjoint, then $\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

Proof : $A \cup B = A \cup (\bar{A} \cap B)$ Since A and $(\bar{A} \cap B)$ are disjoint,

$$P(A \cup B) = P[A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B) = P(A) + P(\bar{A} \cap B) + P(A \cap B) - P(A \cap B)$$
$$= P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B) = P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$$
$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

MULTIPLICATION LAW OF PROBABILITY

Case (i): When events are independent : The probability that both independent events, A and B will occur is equal to product of the probabilities of each event, then $P(A \cap B) = P(A)\,P(B)$.

Case (ii): When events are dependent (or) conditional probability: If the occurrence of an event A is affected by the occurrence of the another event B, then the events A and B are dependent. $P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)$

THEOREM : MULTIPLICATION LAW OF PROBABILITY

For two events A and B, $\boldsymbol{P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)}$, $\boldsymbol{P(A) > 0, P(B) > 0}$ ***where*** $\boldsymbol{P(B/A)}$ ***represents the conditional probability of occurrence of B when the event A has already happened and*** $\boldsymbol{P(A/B)}$ ***is the conditional probability of happening A, given that B has already happened.***

Proof: $P(A) = \frac{n(A)}{n(S)}$, $P(B) = \frac{n(B)}{n(S)}$, $P(A \cap B) = \frac{n(A\cap B)}{n(S)}$, For the conditional event A/B, the favorable outcomes must be one of the sample points of B, that is for the event A/B, the sample space is B and out of the $n(B)$of sample points, $n(A \cap B)$ pertain to the occurrence of the event A. $P(A \cap B) = \frac{n(B)}{n(S)} \frac{n(A\cap B)}{n(B)} = P(B)\,P(A/B)$.

Similarly we can prove, $P(A \cap B) = \frac{n(A)}{n(S)} \frac{n(A\cap B)}{n(A)} = P(A)\,P(B/A)$

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

PROBLEMS IN PROBABILITY

1. ***A box contains 4 red, 5 white & 6 black balls. What is the probability that 2 balls drawn are red & black?***

   Solution : $P(red\ \&\ black) = \frac{4C_1 \times 6C_1}{15C_2} = \frac{8}{35} = 0.2286$

2. ***From a group of 3 Indians, 4 Pakistanis and 5 Americans, a subcommittee of 4 people is selected by lots. Find the probability that the subcommittee will consist of (i) 2 Indians and 2 Pakistanis (ii) 1 Indian, 1 Pakistani and 2 Americans (iii) 4 Americans.***

   Solution : (i) $\text{P(2 Indians and 2 Pakistanis)} = \frac{3C_2 \times 4C_2}{12C_4}$

   (ii) $\text{P(1 Indian, 1 Pakistani and 2 Americans)} = \frac{3C_1 \times 4C_1 \times 5C_2}{12C_4}$ (iii) $\text{P(4 Americans)} = \frac{5C_4}{12C_4}$

3. ***A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) Both are good (ii) Both have major defects (iii) At least 1 is good (iv) At most 1 is good (v) Exactly 1 is good (vi) Neither has major defects (vii) Neither is good***

   Solution: (i) $\text{P(Both are good)} = \frac{10C_2}{16C_2} = \frac{3}{8}$ (ii) $\text{P(Both have major defects)} = \frac{2C_2}{16C_2} = \frac{1}{120}$

   (iii) $\text{P(At least 1 is good)} = \frac{(10C_1 \times 6C_1) + 10C_2}{16C_2} = \frac{7}{8}$ (iv) $\text{P(At most 1 is good)} = \frac{(10C_0 \times 6C_2) + (10C_1 \times 6C_1)}{16C_2} = \frac{5}{8}$

   (v) $\text{P(Exactly 1 is good)} = \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$

   (vi) $\text{P(Neither has major defects)} = \text{P(both are non major defectives)} = \frac{14C_2}{16C_2} = \frac{91}{120}$

   (vii) $\text{P(Neither is good)} = \text{P(both are defective)} = \frac{6C_2}{16C_2} = \frac{1}{8}$

4. ***Four cards are drawn at random from a well shuffled pack of cards. Find the probability that (i) All the 4 are queens (ii) There is one card from each suit (iii) 3 cards are diamonds and 1 spade (iv) All the 4 cards are hearts, and one of them is a jack.***

   Solution: (i) $P(All\ the\ 4\ are\ queens) = \frac{4C_4}{52C_4} = 0.0000037$

   (ii) $P(one\ card\ from\ each\ suit) = \frac{13C_1 \times 13C_1 \times 13C_1 \times 13C_1}{52C_4} = 0.1055$

   (iii) $P(3\ cards\ are\ diamonds\ and\ 1\ spade) = \frac{13C_3 \times 13C_1}{52C_4} = 0.0137$

   (iv) $P(All\ the\ four\ cards\ are\ hearts, and\ one\ of\ them\ is\ a\ jack.) = \frac{1C_1 \times 12C_3}{52C_4} = 0.00081$

5. ***Two dice are thrown. Find the probability that (i) The total of the numbers on the top faces is 9 (ii) The top face numbers are same (iii) The sum of the numbers on the top faces is less than 7.***

   Solution:
$$n(S) = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

   (i) $\text{P(total of the numbers on the top faces is 9)} = \frac{4}{36} = 0.111$

   (ii) $\text{P(top face numbers are same)} = \frac{6}{36} = 0.1667$

   (iii) $\text{P(sum of the numbers on the top faces is} < 7) = \frac{15}{36} = 0.4167$

PROBLEMS IN ADDITION LAW OF PROBABILITY

1. ***The probability that a company director will travel by train is $\frac{1}{5}$ and by bus is $\frac{2}{3}$. What is the probability of his travelling by train or bus?***

   Solution: Let A – Travelling by train, B – Travelling by Bus, $P(A) = \frac{1}{5}$, $P(B) = \frac{2}{3}$

   $P(A \cup B) = P(A) + P(B)$ ***(∵ A and B are mutually exclusive)***

   $P(train\ or\ bus) = P(A \cup B) = \frac{1}{5} + \frac{2}{3} = \frac{13}{15} = 0.867$

2. ***A is known to hit the target in 2 out of 5 shots. B is known to hit the target in 3 out of 4 shots. Find the probability of the target being hit when both try?***

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

Solution : $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{4}$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***($\because$ A & B are not mutually exclusive)***

$P(A \cap B) = P(A)\ P(B) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$ ***($\because$ A and B are independent)***

$P(A \cup B) = \frac{2}{5} + \frac{3}{4} - \frac{3}{10} = \frac{17}{20} = 0.85$

3. ***If the probability is 0.30 that a teaching job applicant has a P.G. degree, 0.70 for his work experience and 0.2 for both, out of 300 applicants, how many will have either a P.G. degree or work experience?***

Solution : $P(A) = 0.30$, $P(B) = 0.70, P(A \cap B) = 0.20$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***($\because$ A and B are not mutually exclusive)***

$P(A \cup B) = 0.3 + 0.7 - 0.2 = 0.8$

4. ***If A, B & C are any 3 events such that $\boldsymbol{P(A) = P(B) = P(C) = \frac{1}{4}, P(A \cap B) = P(B \cap C) = P(A \cap B \cap C) = 0}$, $\boldsymbol{P(C \cap A) = \frac{1}{8}}$. Find the probability that at least 1 of the events $A, B,$ and C occurs.***

Solution : $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8} = 0.625$

PROBLEMS IN MULTIPLICATION LAW OF PROBABILITY

1. ***Two persons A & B appear in an interview for two vacancies for the same post. The probability of A selection is 1/7 & that of B selection is 1/5. What is the probability that (i) Both of them (ii) None of them will be selected.***

Solution: $\mathrm{P(A)} = \frac{1}{7}$, $\mathrm{P(B)} = \frac{1}{5}$, $\mathrm{P(\overline{A})} = 1 - \frac{1}{7} = \frac{6}{7}$, $\mathrm{P(\overline{B})} = 1 - \frac{1}{5} = \frac{4}{5}$

(i) $P(A \cap B) = P(A)\ P(B) = \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$ ***($\because$ A and B are independent)***

(ii) $P(\overline{A} \cap \overline{B}) = P(\overline{A})\ P(\overline{B}) = \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$ ***($\because$ $\overline{A}$ and $\overline{B}$ are independent)***

2. ***If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and selectivity is 0.18. What is the prob. that a system with high fidelity will also have selectivity?***

Solution: Let A – selectivity and B – fidelity, $\mathrm{P(B) = 0.81, P(A \cap B) = 0.18}$

$\mathrm{P(A/B)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(B)}} = \frac{0.18}{0.81} = \frac{2}{9} = 0.222$

3. ***A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the prob. that the other one is also good?***

Solution : Let A − One of the tubes drawn is good & B − Other tube is good

$\mathrm{P(A)} = \frac{6C_1}{10C_1} = \frac{3}{5}$, $\mathrm{P(A \cap B)} = \frac{6C_2}{10C_2} = \frac{1}{3}$, Using Conditional Probability, $\mathrm{P(B/A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{3}{5}\right)} = \boldsymbol{\frac{5}{9}}$

RANDOM VARIABLE

The outcomes of many random experiments may be non-numerical. It is inconvenient to deal with these descriptive outcomes mathematically.

Example: When toss a coin we get two outcomes, namely head or tail. We can assign numerical values; say 1 to head and 0 to tail. This interpretation is easy and attractive from mathematical point of view and also practically meaningful.

Example: Three students sat for an examination & X denotes the number of students who passed. Describe the RV X.

Sample Space S	None	S_1	S_2	S_3	S_1S_2	S_2S_3	S_3S_1	$S_1S_2S_3$
No. of Students who passed X	0	1	1	1	2	2	2	3

$n(S) = 8$, $P(X = 0) = \frac{1}{8}$, $P(X = 1) = \frac{3}{8}$, $P(X = 2) = \frac{3}{8}$, $P(X = 3) = \frac{1}{8}$

TYPES OF RANDOM VARIABLE

Random Variable

Discrete Random Variable | **Continuous Random Variable**

DISCRETE RANDOM VARIABLE

A random variable X is discrete, if it assumes only finite number or countably infinite number of values.

Example : (i) *The mark obtained by a student in an examination. It's possible values are 0, 85 or 100.*

(ii) The number of students who are absent for a particular period.

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

1. Probability Mass Function (p.m.f.) $\sum_{i=1}^{\infty} P(x_i) = 1$ 2. Mean $E(X) = \sum_{i=1}^{\infty} x_i\, P(x_i)$, $E(X^2) = \sum_{i=1}^{\infty} {x_i}^2\, P(x_i)$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$

CONTINUOUS RANDOM VARIABLE

A RV X is continuous, if it takes all possible values between certain limits or in an interval which may be finite or infinite. ***E.g:***(i)The density of milk taken for testing at a farm.(ii)The operating time between two failures of a computer.

1. Probability Density Function (p.d.f.) $\int_{-\infty}^{\infty} f(x)dx = 1$ 2. Mean $\mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f(x)dx$, $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

PROPERTIES OF EXPECTATION

If X and Y are random variables and a, b are constants, then

1. $E(a) = a$ 2. $E(aX) = aE(X)$ 3. $E(aX + b) = aE(X) + b$ 4. $E(X - \bar{X}) = 0$ 5. $|E(X)| \le E(|X|)$
6. $E(X) \ge 0, if\ X \ge 0$ 7. $E(X + Y) = E(X) + E(Y)$ ***(Additive Theorem)***
8. $E(XY) = E(X)E(Y)$ ***(∵ A and B are independent)***
9. $E\big(a\, g(X)\big) = aE\big(g(X)\big)$ 10. $E(g(X) + a) = E\big(g(X)\big) + a$
11. $(E[g(X)\ ]) = g[E(X)]$ $[\boldsymbol{g(X)\ is\ linear\ in\ X}]$
12. $P(X \ge a) \le \frac{E(X)}{a}, a > 0$ ***(Markov Inequality)***
13. $P\{|X - E(X)| \ge k\} \ge \frac{\sigma_X^2}{\epsilon^2}$ ***(Chebyshev's Inequality)***

PROPERTIES OF VARIANCE

1. $Var(X) \ge 0$ 2. $E(X^2) \ge [E(X)]^2$ 3. $Var(b) = 0, b\ constant$
4. If X is a random variables, a is constants then $Var(aX) = a^2 Var(X)$
5. If a and b are constants, $Var(aX \pm b) = a^2 Var(X)$
6. If X and Y are two independent RV, a and b are constants then $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$

PROPERTIES OF CUMULATIVE DISTRIBUTION FUNCTION

1. If F is the distribution function of the RV X and if $a < b$, then
   $P(a < X \le b) = P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = F(b) - F(a)$
2. If F is the distribution function of one dimensional RV X, then (i) $0 \le F(X) \le 1$ (ii) $F(X) \le F(Y)$, if $x < y$
   In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.
3. If F is the distribution function of one dimensional random variable X, then
   $F(-\infty) = \lim_{x \to -\infty} F(X) = 0$ and $F(\infty) = \lim_{x \to \infty} F(X) = 1$ 4. $f(x) = \frac{d}{dx}\big(F(x)\big)$

MOMENTS

Definition: The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

Moments about Origin (Raw Moments)

Discrete : $\mu_n' = E(X^n) = \sum_i x_i^n p_i\, , n \ge 1.$ ***Continuous:*** $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n\, f(x)dx\, , n \ge 1$

Moment about Mean (Central Moments)

Discrete : $\mu_n = E[(X - \bar{X})^n] = \sum_i (x_i - \bar{X})^n\, p_i$, ***Continuous:*** $\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n\, f(x)dx\, ,\ n \ge 1$

Relationship between moments about origin and moment about mean

$$\mu_r = \mu_r' - rC_1\, \mu\, \mu_{r-1}^1 + rC_2\, \mu^2\, \mu_{r-2}' - \cdots$$

Hence, $\mu_1 = 0$, $\mu_2 = \mu_2' - (\mu_1')^2$, $\mu_3 = \mu_3' - 3\mu_2'\, \mu_1' + 2(\mu_1')^3$, $\mu_4 = \mu_4' - 4\mu_3'\, \mu_1' + 6\mu_2'(\, \mu_1')^2 - 3(\mu_1')^4$

MOMENT GENERATING FUNCTION

Definition : Moment generating function of a random variable about the origin is defined as

Discrete : $M_X(t) = E(e^{tX}) = \sum_x e^{tx}\, p(x)$, ***Continuous :*** $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\, f(x)dx$

Where the integration or summation is taken over the entire range of X, t being a real parameter, assuming that integration or summation is absolutely convergent.

$$M_X(t) = 1 + t\, \mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' \, , \quad \text{Where } \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

Note: 1. $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$ 2. $M_{CX}(t) = M_X(Ct)$, C being a constant. 3. $M_{X=a}(t) = e^{-at}\, M_X(t)$

4. If $X_1, X_2, \ldots X_n$ are n independent RVs, then $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t). M_{X_2}(t) \ldots M_{X_n}(t)$

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

PROBLEMS IN DISCRETE RANDOM VARIABLE

1. ***A discrete RV X has the following probability distribution***

$\boldsymbol{x}$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$\boldsymbol{p(x)}$	$\boldsymbol{a}$	$\boldsymbol{3a}$	$\boldsymbol{5a}$	$\boldsymbol{7a}$	$\boldsymbol{9a}$	$\boldsymbol{11a}$	$\boldsymbol{13a}$	$\boldsymbol{15a}$	$\boldsymbol{17a}$

(i) Find the value of $\boldsymbol{a}$ (ii) $\boldsymbol{P(X<3)}$ (iii) $\boldsymbol{P(X\geq 3)}$ (iv) $\boldsymbol{P(0<X<3)}$ (v) Find the distribution function of X.

Solution

(i) $\sum_{x=0}^{8} P(x) = 1 \Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)+P(8) = 1$

$a+3a+5a+7a+9a+11a+13a+15a+17a = 1 \Rightarrow 81a = 1 \Rightarrow \boldsymbol{a = \frac{1}{81}}$

(ii) $P(X<3) = P(X=0)+P(X=1)+P(X=2) = a+3a+5a = 9a = 9\times\frac{1}{81} = \frac{1}{9}$

(iii) $P(X\geq 3) = 1-P(X<3) = 1-\frac{1}{9} = \frac{8}{9}$

(iv) $P(0<X<3) = P(X=1)+P(X=2) = 3a+5a = 8a = \frac{8}{81}$

(v)

$\boldsymbol{x}$	0	1	2	3	4	5	6	7	8
$\boldsymbol{p(x)}$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$\boldsymbol{F(x)}$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

2. ***A discrete random variable X has the probability function given below:***

$\boldsymbol{x}$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$\boldsymbol{p(x)}$	*0*	$\boldsymbol{K}$	$\boldsymbol{2K}$	$\boldsymbol{2K}$	$\boldsymbol{3K}$	$\boldsymbol{K^2}$	$\boldsymbol{2K^2}$	$\boldsymbol{7K^2+K}$

Find (i) The value of K (ii) $\boldsymbol{P(1.5<X<4.5/X>2)}$ (iii) The smallest value of $\boldsymbol{\lambda}$ for which $\boldsymbol{P(X\leq\lambda)>1/2}$.

Solution:

(i) $\sum_{x=0}^{7} P(x) = 1 \Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7) = 1$

$0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1 \Rightarrow 10K^2+9K = 1$

$(10K-1)(K+1) = 0 \Rightarrow K = \frac{1}{10}, -1 \Rightarrow \boldsymbol{K = \frac{1}{10}}$ ***($\because \boldsymbol{K=-1}$, which is meaningless)***

(ii) $P(1.5<X<4.5/X>2) = \frac{P[(1.5<X<4.5)\cap(X>2)]}{P(X>2)}$ $\quad\because \boldsymbol{P(A/B) = \frac{P(A\cap B)}{P(B)}}$

$P(1.5<X<4.5/X>2) = \frac{P(3)+P(4)}{P(3)+P(4)+P(5)+P(6)+P(7)} = \frac{\left(\frac{5}{10}\right)}{\left(\frac{7}{10}\right)} = \boldsymbol{\frac{5}{7}}$

(iii) $P(X\leq\lambda) > \frac{1}{2}$, $\lambda = 0,\ P(X\leq 0) = 0 \ngtr \frac{1}{2}$; $\lambda = 1,\ P(X\leq 1) = \frac{1}{10} \ngtr \frac{1}{2}$;

$\lambda = 2,\ P(X\leq 2) = \frac{3}{10} \ngtr \frac{1}{2}$; $\lambda = 3,\ P(X\leq 3) = \frac{5}{10} \ngtr \frac{1}{2}$; $\lambda = 4,\ P(X\leq 4) = \frac{8}{10} > \frac{1}{2}$

The smallest value of $\boldsymbol{\lambda}$ for which $\mathbf{P(X\leq\lambda) > 1/2}$ is 4.

3. ***If the RV X takes the values 1, 2, 3 & 4 such that $\boldsymbol{2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)}$, find the probability distribution and cumulative distribution function of X.***

Solution: Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = 30K$

x	1	2	3	4
$p(x)$	$15K$	$10K$	$30K$	$6K$

$\sum_{x=1}^{4} P(x) = 1 \Rightarrow P(1)+P(2)+P(3)+P(4) = 1 \Rightarrow 15K+10K+30K+6K = 1 \Rightarrow 61K = 1 \Rightarrow \boldsymbol{K = \frac{1}{61}}$

Cumulative distribution function of X

x	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

4. ***A discrete RV X has the following probability distribution***

$\boldsymbol{x}$	*-2*	*-1*	*0*	*1*	*2*	*3*
$\boldsymbol{p(x)}$	***0.1***	$\boldsymbol{K}$	***0.2***	$\boldsymbol{2K}$	***0.3***	$\boldsymbol{3K}$

Find (i) K (ii) $\boldsymbol{P(X<2)}$ (iii) $\boldsymbol{P(-2<X<2)}$ (iv) the cdf of X (v) the mean of X.

Solution

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

(i) $\sum_{x=-2}^{3} P(x) = 1 \Rightarrow P(-2)+P(-1)+P(0)+P(1)+P(2)+P(3) = 1 \Rightarrow 6K + 0.6 = 1 \Rightarrow$ $\boldsymbol{K = \frac{1}{15}}$

	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$

(ii) $P(X<2) = P(-2)+P(-1)+P(0)+P(1) = \frac{1}{10}+\frac{1}{15}+\frac{2}{10}+\frac{2}{15} = \frac{1}{2}$

(iii) $P(-2<X<2) = P(-1)+P(0)+P(1) = \frac{1}{15}+\frac{2}{10}+\frac{2}{15} = \frac{2}{5}$

(iv)

x	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$
$F(X)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{4}{5}$	1

(v) ***Mean of X***

$$E(X) = \sum_{x=-2}^{3} x\,P(x) = (-2)P(-2)+(-1)P(-1)+0\,P(0)+1\,P(1)+2\,P(2)+3\,P(3)$$
$$= \left(-2\times\frac{1}{10}\right)+\left(-1\times\frac{1}{15}\right)+\left(0\times\frac{2}{10}\right)+\left(1\times\frac{2}{15}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{3}{15}\right) = \frac{16}{15}$$

5. ***If X is RV having the density function*** $\boldsymbol{f(x) = \begin{cases} \frac{x}{6} \; for\; x = 1,2,3 \\ 0, otherwise \end{cases}}$. ***Find*** $\boldsymbol{E(X^3+2X+7)}$ ***and*** $\boldsymbol{Var\,(4X+5)}$.

Solution

x	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$$E(X) = \sum_{x=1}^{3} x\,P(x) = \left(1\times\frac{1}{6}\right)+\left(2\times\frac{2}{6}\right)+\left(3\times\frac{3}{6}\right) = \frac{7}{3}$$
$$E(X^2) = \sum_{x=1}^{3} x^2\,P(x) = \left(1\times\frac{1}{6}\right)+\left(4\times\frac{2}{6}\right)+\left(9\times\frac{3}{6}\right) = 6$$
$$E(X^3) = \sum_{x=1}^{3} x^3\,P(x) = \left(1\times\frac{1}{6}\right)+\left(8\times\frac{2}{6}\right)+\left(27\times\frac{3}{6}\right) = \frac{49}{3}$$
$$E(X^3+2X+7) = E(X^3)+2E(X)+7 = \frac{84}{3}$$
$$Var(X) = E(X^2)-[E(X)]^2 = \frac{5}{9}, \qquad Var\,(4X+5) = 4^2 Var(X) = 16\times\frac{5}{9} = \frac{80}{9}$$

6. ***If X has the distribution function*** $\boldsymbol{F(x) = \begin{cases} 0, & x<1 \\ \frac{1}{3}, & 1\le x<4 \\ \frac{1}{2}, & 4\le x<6 \\ \frac{5}{6}, & 6\le x<10 \\ 1, & x\ge 10 \end{cases}}$

Find (i) The probability distribution of X (ii) $\boldsymbol{P(2<X<6)}$ ***(iii) Mean of X (iv) Variance of X.***

Solution

(i) For the given c.d.f., the probability distribution of X is

$P(X=1) = F(1)-F(0) = \frac{1}{3}-0 = \frac{1}{3}$, $\quad P(X=4) = F(4)-F(1) = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$,

$P(X=6) = F(6)-F(4) = \frac{5}{6}-\frac{1}{2} = \frac{1}{3}$, $\quad P(X=10) = F(10)-F(6) = 1-\frac{5}{6} = \frac{1}{6}$

x	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(ii) $P(2<X<6) = P(X=4) = \frac{1}{6}$

(iii) $E(X) = \sum_i x_i\,P(x_i) = \left(1\times\frac{1}{3}\right)+\left(4\times\frac{1}{6}\right)+\left(6\times\frac{2}{6}\right)+\left(10\times\frac{1}{6}\right) = \frac{14}{3}$

$E(X^2) = \sum_i x_i^2\,P(x_i) = \left(1\times\frac{1}{3}\right)+\left(16\times\frac{1}{6}\right)+\left(36\times\frac{2}{6}\right)+\left(100\times\frac{1}{6}\right) = \frac{95}{3}$

$Var(X) = E(X^2)-[E(X)]^2 = \frac{89}{3}$

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

7. ***When a die is thrown, X denotes the number that turns up. Find $E(X), E(X^2), Var(X)$ and standard deviation.***

Solution: $p = \frac{1}{6}$, X = 1, 2, 3, 4, 5, 6 Here X is a discrete RV

$$E(X) = \sum_i x_i P(x_i) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = 3.5$$

$$E(X^2) = \sum_i x_i^2 P(x_i) = \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(9 \times \frac{1}{6}\right) + \left(16 \times \frac{1}{6}\right) + \left(25 \times \frac{1}{6}\right) + \left(36 \times \frac{1}{6}\right) = \frac{91}{6} = 15.167$$

$Var(X) = E(X^2) - [E(X)]^2 = 2.9166$, $S.D. = \sigma_X = \sqrt{Var(X)} = 1.7078$

8. ***A coin is tossed until a head appears. What is the expectation of the number of tosses required?***
Solution: Let X – No. of tosses required to get the 1st head. The 1st head may appear in the 1st or 2nd … and so on. The events are H, TH, TTH, TTTH, … $p = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

x	1	2	3	4	5	…
$p(x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	…

$$E(X) = \sum_i x_i P(x_i) = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} = 2 \qquad [\because \mathbf{(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots}]$$

9. ***By throwing a fair dice, a player gains Rs. 20 if 2 turns up, gains Rs. 40 if 4 turns up and loses Rs. 30 if 6 turns up. He never loses or gains if any other number turns up. Find the expected value of money he gains.***
Solution: Let X – money won on an trial. x_i = Amount of money won, if the faces show $i = 1, 2, 3, 4, 5, 6$.

	1	2	3	4	5	6
x	0	20	0	40	0	-30
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_i x_i P(x_i) = \left(0 \times \frac{1}{6}\right) + \left(20 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(40 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(-30 \times \frac{1}{6}\right) = 5$$

10. **Find the first three moments of X if X has the following distribution**

$\boldsymbol{x}$	**-2**	**1**	**3**
$\boldsymbol{p(x)}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Solution : $\mu_n' = E(X^n) = \sum_i x_i^n p_i$, $n \geq 1$

$n = 1$, $\mu_1' = E(X) = \sum_i x_i P(x_i) = \left(-2 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(3 \times \frac{1}{4}\right) = 0$

$n = 2$, $\mu_2' = E(X^2) = \sum_i x_i^2 P(x_i) = \left(4 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(9 \times \frac{1}{4}\right) = \frac{9}{2}$

$n = 3$, $\mu_3' = E(X^3) = \sum_i x_i^3 P(x_i) = \left(-8 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(27 \times \frac{1}{4}\right) = 3$

11. ***A RV X has the probability function $f(x) = \frac{1}{2^x}, x = 1, 2, 3\ldots$ Find the (i) moment generating function (ii) Mean***

Solution :

(i) $M_X(t) = \sum_x e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \cdots\right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2 - e^t}$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{e^t}{2 - e^t}\right)\right]_{t=0} = \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0} = \frac{(2 - e^0)e^0 - e^0(-e^0)}{(2 - e^0)^2} = 2$

12. ***If a RV X has moment generating function $M_X(t) = \frac{3}{3-t}$, obtain the standard deviation of X.***

Solution : $M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1 - \frac{t}{3}\right)} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \left(\frac{t}{3}\right) + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{3}\right) + \frac{t^2}{2!}\left(\frac{2}{9}\right) + \frac{t^3}{3!}\left(\frac{6}{27}\right) + \cdots$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}$, $\mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{3}$, $\mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{2}{9}$

Variance $= \mu_2' - (\mu_1')^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$, Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

PROBLEMS IN CONTINUOUS RANDOM VARIABLE

1. ***If*** $p(x) = \begin{cases} x\,e^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ ***(i) Show that*** $p(x)$ ***is a p.d.f. (ii) Find its distribution function*** $P(x)$.

Solution

(i) $\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_{0}^{\infty} p(x)dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} x\, e^{-\frac{x^2}{2}}dx = \int_{0}^{\infty} x\, e^{-\frac{x^2}{2}}dx$

Put $x^2 = t$, $2x\, dx = dt \Rightarrow x\, dx = \frac{dt}{2}$, $x = 0,\ t = 0$ and $x = \infty,\ t = \infty$

$\int_{-\infty}^{\infty} p(x)dx = \int_{0}^{\infty} e^{-\frac{t}{2}}\frac{dt}{2} = \frac{1}{2}\int_{0}^{\infty} e^{-\frac{t}{2}}dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = -e^{-\infty} + e^{0} = 1$ $\quad (\because \boldsymbol{e^{-\infty} = 0},\ \boldsymbol{e^{0} = 1})$

$\therefore p(x)$ is a p.d.f. of a RV X.

(ii) $F(X) = P(X \leq x) = \int_{0}^{x} p(x)dx = \int_{0}^{x} x\, e^{-\frac{x^2}{2}}dx = 1 - e^{-\frac{x^2}{2}},\ x \geq 0$

2. ***If the density function of a continuous RV X is given by*** $f(x) = \begin{cases} ax, & 0 \leq x \leq 1 \\ a, & 1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0, & otherwise \end{cases}$***. Find (i) a (ii) c.d.f.***

Solution: (i) $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{0}^{1} ax\, dx + \int_{1}^{2} a\, dx + \int_{2}^{3} (3a - ax)dx = 1$

$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1 \Rightarrow \frac{a}{2} + a(2-1) + \left(9a - \frac{9a}{2}\right) - \left(6a - \frac{4a}{2}\right) = 1 \Rightarrow \boldsymbol{a = \frac{1}{2}}$

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ \frac{1}{2}, & 1 \leq x \leq 2 \\ \frac{3-x}{2}, & 2 \leq x \leq 3 \\ 0, & otherwise \end{cases}$$

(ii) ***c.d.f. of X:*** $F(X) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$

If $x < 0$, then $F(X) = 0$, since $f(x) = 0$ for $x < 0$

If $0 \leq x \leq 1$, then $F(X) = \int_0^x \frac{x}{2}\, dx = \left[\frac{x^2}{4}\right]_0^x = \frac{x^2}{4}$

If $1 \leq x \leq 2$, then $F(X) = \int_0^x f(x)\, dx = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^x \left(\frac{1}{2}\right) dx = \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^x = \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{1}{4}(2x-1)$

If $2 \leq x \leq 3$, then $F(X) = \int_0^x f(x)\, dx = \int_0^1 \left(\frac{x}{2}\right) dx + \int_1^2 \left(\frac{1}{2}\right) dx + \int_2^x (3a - ax)\, dx$

$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \left[\frac{3x}{2} - \frac{x^2}{4}\right]_2^x = \frac{1}{4} + \frac{2}{2} - \frac{1}{2} + \left(\frac{3x}{2} - \frac{x^2}{4}\right) - \left(\frac{6}{2} - \frac{4}{4}\right) = \frac{1}{4}(6x - x^2 - 5)$

If $x \geq 3$, then $F(X) = 1$

$$F(x) = \begin{cases} \frac{x^2}{4}, & 0 \leq x \leq 1 \\ \frac{1}{4}(2x-1), & 1 \leq x \leq 2 \\ \frac{1}{4}(6x - x^2 - 5), & 2 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$$

3. ***A continuous RV X has a pdf*** $\boldsymbol{f(x) = 3x^2, 0 \leq x \leq 1}$***. Find a and b such that***

(i) $\boldsymbol{P(X \leq a) = P(X > a)}$ ***(ii)*** $\boldsymbol{P(X > b) = 0.05}$

Solution:

(i) $P(X \leq a) = P(X > a) \Rightarrow \int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx \Rightarrow \int_0^a 3\,x^2 dx = \int_a^1 f(x)dx \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = 3\left[\frac{x^3}{3}\right]_a^1$

$a^3 = 1 - a^3 \Rightarrow 2a^3 = 1 \Rightarrow a^3 = \frac{1}{2} \Rightarrow \boldsymbol{a} = \left(\boldsymbol{\frac{1}{2}}\right)^{\frac{1}{3}} = \boldsymbol{0.7937}$

(ii) $P(X > b) = 0.05 \Rightarrow \int_b^1 3\,x^2 dx = 0.05 \Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05 \Rightarrow \boldsymbol{b = (0.95)^{\frac{1}{3}} = 0.9830}$

4. ***A Continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1+x)$. Find $P(X < 4)$.***

Solution : $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{2}^{5} k(1+x)dx = 1 \Rightarrow k\left[x+\frac{x^2}{2}\right]_2^5 = 1 \Rightarrow k\left[\left(5+\frac{25}{2}\right)-\left(2+\frac{4}{2}\right)\right] = 1 \Rightarrow \boldsymbol{k = \frac{2}{27}}$

$$P(X<4) = \frac{2}{27}\int_2^4 (1+x)dx = \frac{2}{27}\left[x+\frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left[\left(4+\frac{16}{2}\right)-\left(2+\frac{4}{2}\right)\right] = \frac{16}{27}$$

5. ***A RV X has a pdf $f(x) = kx^2e^{-x}, x \geq 0$. Find k, mean, variance and $E(3X^2-2X)$.***

Solution: $\int_{-\infty}^{\infty} f(x)dx = 1, \quad \int_0^{\infty} kx^2e^{-x}dx = 1$

Differentiation : $u = x^2, \quad u' = 2x, \quad u'' = 2, \quad u''' = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}$ $\quad (\because \int \boldsymbol{uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$k\left[x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 1 \Rightarrow k[(0-0+0)-(0-0+2)] = 1 \Rightarrow \boldsymbol{k = \frac{1}{2}}$ $\quad (\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$

Mean of *X* $E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_0^{\infty} x\left(\frac{1}{2}x^2e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^3e^{-x}dx$

Differentiation : $u = x^3, \quad u' = 3x^2, \quad u'' = 6x, \quad u''' = 6, \quad u'^v = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}, v_4 = \frac{e^{-x}}{(-1)^4}$ $(\because \int \boldsymbol{uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$E(X) = \frac{1}{2}\left[x^3\frac{e^{-x}}{(-1)} - 3x^2\frac{e^{-x}}{(-1)^2} + 6x\frac{e^{-x}}{(-1)^3} - 6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty} = 3$ $\quad (\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$

$E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)dx = \int_0^{\infty} x^2\left(\frac{1}{2}x^2e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^4e^{-x}dx$

Differentiation : $u = x^4, \quad u' = 4x^3, \quad u'' = 12x^2, \quad u''' = 24x, \quad u'^v = 24, \quad u^v = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, \quad v_2 = \frac{e^{-x}}{(-1)^2}, \quad v_3 = \frac{e^{-x}}{(-1)^3}, \ v_4 = \frac{e^{-x}}{(-1)^4}, v_5 = \frac{e^{-x}}{(-1)^5}, \ v_6 = \frac{e^{-x}}{(-1)^6}$

$$E(X^2) = \frac{1}{2}\left[x^4\frac{e^{-x}}{(-1)} - 4x^3\frac{e^{-x}}{(-1)^2} + 12x^2\frac{e^{-x}}{(-1)^3} - 24x\frac{e^{-x}}{(-1)^4} + 24x\frac{e^{-x}}{(-1)^5}\right]_0^{\infty} = 12$$

$V(X) = E(X^2) - [E(X)]^2 = 12 - 9 = 3\,, \quad E(3X^2-2X) = 3E(X^2) - 2E(X) = 3(12) - 2(3) = 30$

6. ***The prob. distribution function of a RV X is*** $\boldsymbol{f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}}$. ***Find the mean and variance.***

Solution

$E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_{-\infty}^{0} x\,f(x)dx + \int_0^1 x\,f(x)dx + \int_1^2 x\,f(x)dx + \int_2^{\infty} x\,f(x)dx$

$\quad = \int_{-\infty}^{0} 0\,dx + \int_0^1 x\,(x)dx + \int_1^2 x\,(2-x)dx + \int_2^{\infty} 0\,dx = \int_0^1 x^2dx + \int_1^2 (2x-x^2)dx$

$E(X) = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left(4-\frac{8}{3}\right) - \left(1-\frac{1}{3}\right) = 1$

$E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)dx = \int_{-\infty}^{0} x^2\,f(x)dx + \int_0^1 x^2\,f(x)dx + \int_1^2 x^2\,f(x)dx + \int_2^{\infty} x^2\,f(x)dx$

$\quad = \int_0^1 x^3x + \int_1^2 (2x^2 - x^3)dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{1}{4} + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{7}{6}$

$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6}$

7. ***The distribution function of a RV X is given by $F(x) = 1-(1+x)e^{-x}, \ x \geq 0$. Find the density function, mean and variance of X.***

Solution

$f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}[1-(1+x)e^{-x}] = [0-(1+x)(-e^{-x}) - e^{-x}] = e^{-x} + xe^{-x} - e^{-x} = xe^{-x}, x \geq 0$

$E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_0^{\infty} x^2e^{-x}dx = \left[x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 2$

$E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)dx = \int_0^{\infty} x^3e^{-x}dx = \left[x^3\frac{e^{-x}}{(-1)} - 3x^2\frac{e^{-x}}{(-1)^2} + 6x^2\frac{e^{-x}}{(-1)^3} - 6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty} = 6$

$V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

8. ***The cdf of a continuous RV X is given by*** $F(x) = \begin{cases} \mathbf{0}, & x < 0 \\ x^2, & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2, & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$

Find the p.d.f. of X and evaluate $P(|X| \le 1)$ ***and*** $P\left(\frac{1}{3} \le X < 4\right)$ ***using both the pdf and cdf.***

Solution: $f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x), & \frac{1}{2} \le x < 3 \\ 0, & x \ge 3 \end{cases}$$

pdf : $P(|X| \le 1) = P(-1 \le X \le 1) = \int_{-1}^{0} 0\, dx + \int_{0}^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3-x)dx = 2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = \frac{13}{25}$

cdf: $P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1) = \frac{13}{25}$

pdf : $P\left(\frac{1}{3} \le X < 4\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{3} \frac{6}{25}(3-x)dx + \int_{3}^{4} 0\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{3} = \frac{8}{9}$

cdf: $P\left(\frac{1}{3} \le X < 4\right) = F(4) - F\left(\frac{1}{3}\right) = 1 - \frac{1}{9} = \frac{8}{9}$

9. ***The first four moments of a distribution about*** $x = 4$ ***are 1, 4, 10, 45. Show that the mean is 5, variance is 3,*** $\mu_3 = 0, \mu_4 = 26$ ***.***

Solution: Let $\mu_1', \mu_2', \mu_3', \mu_4'$ be the first four moments about $X = 4$.

Given $\mu_1' = 1,\ \mu_2' = 4,\ \mu_3' = 10,\ \mu_4' = 45$ about $X = 4$.

$E(X-4) = 1 \Rightarrow E(X) - 4 = 1 \Rightarrow E(X) = 5,\qquad \mu_2 = \mu_2' - (\mu_1')^2 = 4 - 1 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 = 10 - 3(4)(1) + 2(1) = 0$

$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = 45 - 4(10)(1) + 6(4)(1) - 3(1) = 26$

10. ***The first three moments about the origin are 5, 26, 78. Show that the first three moments about the value*** $x = 3$ ***are*** $2, 5, -48$.

Solution: Let $\mu_1', \mu_2', \mu_3', \mu_4'$ be the first four moments about $x = 3$. Given $E(X) = 5, E(X^2) = 26, E(X^3) = 78$

$\mu_1' = E(X-3) = E(X) - 3 = 5 - 3 = 2,\qquad \mu_2' = E(X-3)^2 = E(X^2) - 6E(X) + 9 = 26 - 6(5) + 9 = 5,$

$\mu_3' = E(X-3)^3 = E(X^3) - 9E(X^2) + 27E(X) - 27 = 78 - 9(26) + 27(5) - 27 = -48,$

11. ***If X has probability density function given by*** $f(x) = \frac{x+1}{2}, -1 \le x \le 1.$ ***Find the 1st four central moments.***

Solution: $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$

$n = 1,\ \mu_1' = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^2 + x)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$

$n = 2,\ \mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^3 + x^2)dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$

$n = 3,\ \mu_3' = E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^4 + x^3)dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{5}$

$n = 4,\ \mu_4' = E(X^4) = \int_{-\infty}^{\infty} x^4 f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^5 + x^4)dx = \frac{1}{2}\left[\frac{x^6}{6} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5}$

Moment about Mean (Central Moments)

$\mu_r = \mu_r' - rC_1\,\mu\,\mu_{r-1}^1 + rC_2\,\mu^2\,\mu_{r-2}' - rC_3\,\mu^3\,\mu_{r-3}' + rC_4\,\mu^4\,\mu_{r-4}' - \cdots$

$r = 1,\ \mu_1 = 0$

$r = 2,\ \mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{9}$

$r = 3,\ \mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 = -\frac{8}{135}$

$r = 4,\ \mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{48}{405}$

12. ***A RV X has density function given by $f(x) = \begin{cases} 2\,e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$. Obtain the (i) moment generating function (ii) Four moments about the origin (iii) Mean (iv) Variance.***

Solution: $M_X(t) = \int_{x=-\infty}^{\infty} e^{tx} f(x)dx = \int_{x=0}^{\infty} e^{tx}\, 2\,e^{-2x}\, dx = \int_{x=0}^{\infty} 2\, e^{-(2-t)x}\, dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{2}{2-t}$

$M_X(t) = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{1}{2}\right) + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\left(\frac{3}{2}\right) + \cdots$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}$, $\quad r = 1,\ \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{2}$

$r = 2,\ \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{1}{2}$, $\quad r = 3,\ \mu_3' = coefficient\ of\ \frac{t^3}{3!} = \frac{3}{4}$

$r = 4,\ \mu_4' = coefficient\ of\ \frac{t^4}{4!} = \frac{3}{2}$, Mean $= \mu_1' = \frac{1}{2}$, Variance $= \mu_2' - (\mu_1')^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

13. ***Find the moment generating function of the RV whose moments are given by $\mu_r' = (r+1)!\,2^r$. Find also mean and variance.***

Solution: $\mu_1' = 2!\,2^1,\ \ \mu_2' = 3!\,2^2,\ \ \mu_3' = 4!\,2^3,\ \ M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\mu_3' + \cdots$

$M_X(t) = 1 + \frac{t}{1!}(2!\,2^1) + \frac{t^2}{2!}(3!\,2^2) + \frac{t^3}{3!}(4!\,2^3) + \cdots = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots = (1-2t)^{-2}$

Mean $= \mu_1' = 4$, $\mu_2' = 24$, Variance $= \mu_2' - (\mu_1')^2 = 24 - 16 = 8$

FUNCTION OF RANDOM VARIABLE

One to One Transformation of Random Variables:

Consider that a random variable X is linearly transformed into an another random variable Y. Let Y be $T(x)$.

A monotonically increasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 < x_2$. For example, $y = ax, a > 0$

A monotonically decreasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 > x_2$. For example, $y = ax, a < 0$

If the transformation is monotonically increasing $f_Y(y) = f_X(x)\,\frac{dx}{dy}$

If the transformation is monotonically decreasing $f_Y(y) = f_X(x)\,\left(-\frac{dx}{dy}\right)$

In general, for a linear transformation $\boldsymbol{f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|, where\ x = g^{-1}(y)}$

Non - One to One Transformation of Random Variables:

For a transformation which is non - one to one, the transformation will be broken up into transformations each of which one to one. $\quad \boldsymbol{f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_n)\left|\frac{dx_n}{dy}\right|}$

PROBLEMS IN FUNCTION OF RANDOM VARIABLE

1. ***Consider a RV X with p.d.f. $f(x) = e^{-x}, x \geq 0$ with transformation $y = e^{-x}$. Find the transformed density function.***

Solution: $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{e^{-x}}{|-e^{-x}|} = \frac{y}{y} = 1, 0 < y \leq 1$

2. ***Let X be a RV with p.d.f. $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. Find the p.d.f. of $Y = 8X^3$.***

Solution: $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$, *Let* $y = 8x^3 \Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \left(\frac{y}{8}\right)^{\frac{1}{3}}$, $\quad \frac{dx}{dy} = \frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1}\frac{1}{8} = \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}}$

$f_Y(y) = 2x\,\frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = 2\left(\frac{y}{8}\right)^{\frac{1}{3}}\frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = \frac{1}{12}\left(\frac{y}{8}\right)^{-\frac{1}{3}}$, ***Range:*** $0 < x < 1 \Rightarrow 0 < \left(\frac{y}{8}\right)^{\frac{1}{3}} < 1 \Rightarrow 0 < y < 8$

3. ***If X is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ find the pdf of $Y = \tan X$.***

Solution: Given $f_X(x) = \frac{1}{b-a} = \frac{1}{\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = \pi$, $\ f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$, *Let* $y = \tan x \Rightarrow x = \tan^{-1} y$, $\ \frac{dx}{dy} = \frac{1}{1+y^2}$

$f_Y(y) = \pi\,\frac{1}{1+y^2}$ $\qquad$ ***Range:*** $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\infty < y < \infty$

4. ***If X has an exponential distribution with parameter 1, find the pdf of $Y = \sqrt{X}$.***

Solution: *Given* $\lambda = 1$, $f_X(x) = \lambda\, e^{-\lambda x}, x > 0 = e^{-x}, x > 0$, $\quad f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$, $\quad$ *Let* $y = \sqrt{x} \Rightarrow x = y^2$,

$\frac{dx}{dy} = 2y$, $\quad f_Y(y) = 2y\ e^{-y^2}$ $\qquad$ ***Range:*** $x > 0 \Rightarrow y > 0$

Dr. P. Godhandaraman, Assistant Professor, Dept of Maths, SRM IST, Kattankulathur, Mobile: 9941740168, Email: godhanda@gmail.com

5. ***If the continuous RV X has p.d.f.*** $\boldsymbol{f(x)=\frac{2}{9}(x+1), -1<x<2}$. ***Find the p.d.f. of*** $\boldsymbol{Y=X^2}$.

Solution: The transformation function $Y=X^2$ is not monotonic in $(-1,2)$. So we divide the interval into two parts. i.e., $(-1,\ 1)$ and $(1,2)$. Since $(-1,1)$ is a symmetric interval, we have *Let* $y=x^2 \Rightarrow x=y^{\frac{1}{2}}$

Range: $-1<x<0 \Rightarrow -1<y^{\frac{1}{2}}<0 \Rightarrow 0<y<1$

$$f_Y(y)=\frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y})+f_X(-\sqrt{y})\right]=\frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y})+\frac{2}{9}(1-\sqrt{y})\right]=\frac{2}{9\sqrt{y}}, 0<y<1$$

Range: $1<x<2 \Rightarrow 1<y^{\frac{1}{2}}<2 \Rightarrow 1<y<4$ strictly increasing

$$f_Y(y)=\frac{2}{9}(x+1)\frac{1}{2}y^{-\frac{1}{2}}=\frac{2}{9}\left(y^{\frac{1}{2}}+1\right)\frac{1}{2}y^{-\frac{1}{2}}=\frac{1}{9}\left(y^{\frac{1}{2}}+1\right)y^{-\frac{1}{2}}=\frac{1}{9}\left(1+y^{-\frac{1}{2}}\right), 1<y<4$$

TCHEBYCHEFF INEQULAITY

If X is a RV with $E(X)=\mu$ and $V(X)=\sigma^2$, then $P\{|X-\mu|\geq c\}\leq\frac{\sigma^2}{c^2}$ or $P\{|X-\mu|<c\}\geq 1-\frac{\sigma^2}{c^2}$, $c>0$.

Alternative Form **: If we put $c=k\sigma$, where $k>0$ then Tchebycheff inequality takes the form**

$$P\left\{\left|\frac{X-\mu}{k}\right|\geq\sigma\right\}\leq\frac{1}{k^2} \text{ or } P\left\{\left|\frac{X-\mu}{k}\right|\leq\sigma\right\}\geq 1-\frac{1}{k^2}$$

***PROBLEMS IN* TCHEBYCHEFF INEQULAITY**

1. ***A RV X has mean*** $\boldsymbol{\mu=12}$ ***and variance*** $\boldsymbol{\sigma^2=9}$ ***and an unknown probability distribution. Find*** $\boldsymbol{P(6<X<18)}$.

Solution: Since the probability distribution of X is not known, we can not find the value of the required probability. We can find only a lower bound for the probability using Tchebycheff inequality.

$$P\{|X-\mu|\geq c\}\leq\frac{\sigma^2}{c^2}, c>0 \quad \text{(or)} \quad P\{|X-\mu|<c\}\geq 1-\frac{\sigma^2}{c^2}, c>0$$

$$P\{-c<(X-\mu)<c\}\geq 1-\frac{\sigma^2}{c^2} \quad \Rightarrow P\{\mu-c<X<\mu+c\}\geq 1-\frac{\sigma^2}{c^2}$$

$$\mu=12,\ \sigma^2=9,\ P\{12-c<X<12+c\}\geq 1-\frac{9}{c^2}$$

$$Put\ c=6,\ P\{12-6<X<12+6\}\geq 1-\frac{9}{6^2} \Rightarrow \boldsymbol{P\{6<X<18\}\geq\frac{3}{4}}$$

2. ***A fair die is tossed 720 times. Use Tchebycheff inequality to find a lower bound for the probability of getting 100 to 140 sixes.***

Solution: Let X – no. of sixes obtained when a fair die is tossed 720 times. $p=\frac{1}{6},\ q=\frac{5}{6},\ n=720$

X follows a binomial distribution with mean $np=120$ and variance $npq=100$, that is $\mu=120, \sigma=10$

By Tchebycheff inequality $P\{|X-\mu|\leq k\sigma\}\geq 1-\frac{1}{k^2} \Rightarrow P\{|X-120|\leq 10k\}\geq 1-\frac{1}{k^2}$

$$P\{-10k<(X-120)<10k\}\geq 1-\frac{1}{k^2} \Rightarrow P\{120-10k<X<120+10k\}\geq 1-\frac{1}{k^2}$$

$$Put\ k=2,\ P\{100<X<140\}\geq 1-\frac{1}{4} \Rightarrow \boldsymbol{P\{100<X<140\}\geq\frac{3}{4}}$$

3. ***A discrete RV X takes the values*** $\boldsymbol{-1,0,1}$ ***with probabilities*** $\boldsymbol{\frac{1}{8},\frac{3}{4},\frac{1}{8}}$ ***respectively. Evaluate*** $\boldsymbol{P\{|X-\mu|\geq 2\sigma\}}$***and compare it with the upper bound given by Tchebycheff inequality.***

Solution: $E(X)=\sum_{x=-1}^{1}x\,P(x)=\left(-1\times\frac{1}{8}\right)+\left(0\times\frac{3}{4}\right)+\left(1\times\frac{1}{8}\right)=0$

$E(X^2)=\sum_{x=-1}^{1}x^2\,P(x)=\left(1\times\frac{1}{8}\right)+\left(0\times\frac{3}{4}\right)+\left(1\times\frac{1}{8}\right)=\frac{1}{4}$, $\quad V(X)=E(X^2)-[E(X)]^2=\frac{1}{4}-0=\frac{1}{4}$

$P\{|X-\mu|\geq 2\sigma\}=P\{X\geq 1\}=P(X=-1\ or\ X=1)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$

By Tchebycheff inequality, $P\{|X-\mu|\geq k\sigma\}\leq\frac{1}{k^2} \Rightarrow \boldsymbol{P\{|X-\mu|\geq 2\sigma\}\leq\frac{1}{4}}$

All the Best

Dr. P. GodhandaRaman,

M.Sc., M.A., M.B.A., Ph.D.,

Assistant Professor (Senior Grade)

Department of Mathematics

SRM IST, Kattankulathur – 603 203

Email : godhanda@gmail.com, Mobile : 9941740168