18MAB204T-U1-Common-Lecture Notes

18MAB204T-Probability and Queueing Theory

Prepared by

Dr. S. ATHITHAN

Assistant Professor

Department of of Mathematics
Faculty of Engineering and Technology

SRM INSTITUTE OF SCIENCE AND TECHNOLOGY
Kattankulathur-603203, Chengalpattu District.

SRM INSTITUTE OF SCIENCE AND TECHNOLOGY
Kattankulathur-603203, Chengalpattu District.

Contents

1 Probability (Review) 2

2 Discrete and Continuous Distributions 5

3 Mathematical Expectation 7

4 Moments and Moment Generating Function 8

5 Examples 11

6 Function of a Random Variable 33

7 Exercise/Practice/Assignment Problems 36

1 Probability (Review)

1.1 Definition of Probability and Basic Theorems

1.1.1 Random Experiment

An Experiment whose outcome or result can be predicted with certainty is called a deterministic experiment.

An Experiment whose all possible outcomes may be known in advance, the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown cases is called a random experiment.

Definition 1.1.1 (Sample Space)**.** *Set of all possible outcomes which are assumed to be equally likely is called the sample space and is usually denoted by S. Any subset A of S containing favorable outcomes is called an event.*

1.1.2 The Approaches to Define the Probability

Definition 1.1.2 (Mathematical or Apriori Definition of Probability)**.** *Let S be a sample space and A be an event associated with a random experiment. Let n(S) and n(A) be the number of elements of S and A. Then the probability of event A occurring, denoted by P(A) and defined as*

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favorable to A}}{\text{Exhaustive number of cases in S(Set of all possible cases)}}$$

Definition 1.1.3 (Statistical or Aposteriori Definition of Probability)**.** *Let a random experiment be repeated n times and let an event A occur n_1 times out of n trials. The ratio $\frac{n_1}{n}$ is called the relative frequency of the event A. As n increases, $\frac{n_1}{n}$ shows a tendency to stabilize and to approach a constant value and is denoted by P(A) is called the probability of the event A.* i.e.,

$$P(A) = \lim_{n\to\infty} \frac{n_1}{n}$$

Definition 1.1.4 (Axiomatic Approach/Definition of Probability)**.** *Let S be a sample space and A be an event associated with a random experiment. Then the probability of the event A is denoted by P(A) is defined as a real number satisfying the following conditions/axioms:*

(i) $0 \le P(A) \le 1$

(ii) P(S)=1

(iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Axiom (iii) can be extended to arbitrary number of events. i.e., *If* $A_1, A_2, \ldots, A_n, \ldots$ *be mutually exclusive, then* $P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n \ldots) = P(A_1) + P(A_2) + P(A_2) + P(A_3) + \cdots + P(A_n) + \ldots$

Theorem 1.1. *The probability of the impossible event is zero.* i.e., *if ϕ is a subset of S containing no event, then* $P(\phi) = 0$.

Theorem 1.2. *If $\overline{A}$ be the complimentary event of A, then $P(\overline{A}) = 1 - P(A) \leq 1$.*

Theorem 1.3 (Addition Theorem of Probability)**.** *If A and B be any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$.*

The above Theorem can be extended for 3 events which is given by $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Theorem 1.4. *If $B \subset A$, then $P(B) \leq P(A)$.*

1.2 Conditional Probability and Independents Events

The conditional probability of an event B, assuming that the event A has happened, is denoted by $P(B/A)$ and is defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

Theorem 1.5 (Product Theorem of Probability)**.** $P(A \cap B) = P(A) \cdot P(B/A)$

The product theorem can be extended to three events such as

$$P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \text{ and } B)$$

There are few properties for the conditional distribution. Please go through it in the textbook.

When two events A and B are independent, we have $P(B/A) = P(B)$ and the Product Theorem takes the form $P(A \cap B) = P(A) \cdot P(B)$. The converse of this statement is also. *i.e.,* If $P(A \cap B) = P(A) \cdot P(B)$, then the events A and B are said to be independent.

This result can be extended to any number of events. *i.e.,* If $A_1, A_2, \ldots, A_n$ be n no. of events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \cdots \cdot P(A_n)$$

Theorem 1.6. *If the events A and B are independent, then the events $\overline{A}$ and $\overline{B}$ (and similarly A and $\overline{B}$, $\overline{A}$ and B) are also independent.*

1.3 Total Probability

Theorem 1.7 (Theorem of Total Probability). *If $B_1, B_2, \ldots, B_n$ be the set of exhaustive and mutually exclusive events and A is another event associated with(or caused by)B_i, then*

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

2 Discrete and Continuous Distributions

Definition 2.0.1. *A Random Variable (abbreviated as RV) is a function that assigns a real number $X(s)$ to every element $s \in S$, where S is the sample space corresponding to a random experiment E.*

Discrete Random Variable

If X is a random variable (RV) which can take a finite number or countably infinite number of values, X is called a discrete RV. When the RV is discrete, the possible values of X may be assumed as $x_1, x_2, \ldots, x_n, \ldots$. In the finite case, the list of values terminates and in the countably infinite case, the list goes upto infinity.

For example, the number shown when a die is thrown and the number of alpha particles emitted by a radioactive source are discrete RVs.

Probability Mass Function

If X is a discrete RV which can take the values $x_1, x_2, x_3, \ldots$ such that $P(X = x_i) = p_i$, then p_i is called the PROBABILITY FUNCTION or PROBABILITY MASS FUNCTION or POINT PROBABILITY FUNCTION, provided $p_i(i = 1, 2, 3, \ldots)$ satisfy the following conditions:

(a) $p_i \geq 0$, for all i, and

(b) $\sum_i p_i = 1$

The collection of pairs $\{x_i, p_i\}, i = 1, 2, 3, \ldots$ is called the probability distribution of the RV X, which is sometimes displayed in the form of a table as given below:

$X = x_i$	x_1	x_2	x_3	$\ldots$	x_n	$\ldots$
$P(X = x_i)$	p_1	p_2	p_3	$\ldots$	p_n	$\ldots$

Continuous Random Variable

If X is a RV which can take all values (i.e., infinite number of values) in an interval, then X is called a continuous RV.

For example, the length of time during which a vacuum tube installed in a circuit functions is a continuous RV.

Probability Density Function

If X is a continuous RV such that

$$P\left\{x - \frac{1}{2}dx \leq X \leq x - \frac{1}{2}dx\right\} = f(x)dx$$

then $f(x)$ is called the PROBABILITY DENSITY FUNCTION (PDF) of X, provided $f(x)$ satisfies the following conditions:

(a) $f(x) \geq 0$, for all $x \in R_X$ (Range space of) X, and

(b) $\displaystyle\int_{R_X} f(x)\,dx = 1$

Moreover, $P(a \le X \le b)$ or $P(a < X < b)$ is defined as $P(a \le X \le b) = \displaystyle\int_a^b f(x)\,dx$.

The curve $y = f(x)$ is called the probability curve of the RV X.

Note : When X is a continuous RV d $P(X = a) = P(a < X < a) = \displaystyle\int_a^a f(x)\,dx = 0$

This means that it is almost impossible that a continuous RV assumes a specific value. Hence $P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$.

Cumulative Distribution Function (CDF)

If X is an RV, discrete or continuous, then $P(X \le x)$ is called the CUMULATIVE DISTRIBUTION FUNCTION of X or DISTRIBUTION FUNCTION of X and denoted as $F(x)$. If X is discrete,

$$F(x) = \sum_j p_j$$

If X is continuous, $F(x)P(-\infty < X \le x) = \displaystyle\int_{-\infty}^{x} f(x)\,dx$

Properties of the cdf F(x)

(i) $F(x)$ is a non-decreasing function of x, *i.e.*, if $x_1 < x_2$, then $F(x_1) \le F(x_2)$

(ii) $F(-\infty) = 0$ and $F(\infty) = 1$.

(iii) If X is a discrete RV taking values $x_1, x_2, x_3, \ldots$ where $x_1 < x_2 < x_3 < \cdots < x_{i-1} < x_i < \ldots$ then $P(X = x_i) = F(x_i) - F(x_{i-1})$.

(iv) If X is a continuous RV, then $\dfrac{d}{dx}F(x) = f(x)$ at all points where $F(x)$ is differentiable.

Note : Although we may talk of probability distribution of a continuous RV, it cannot be represented by a table as in the case of a discrete RV. The probability distribution of a continuous RV is said to be known, if either its pdf or cdf is given.

For the problems on Discrete and Continuous Random Variables (RV's) we have to note down the following points from the table for understanding.

	Discrete ♯RV (Probability Distribution Function)	Continuous ♯RV (Probability Density Function)
Operator	Σ	$\int$
Takes Values of the form (Eg.)	$1, 2, 3, \ldots$	$0 < x < \infty$
Standard Notation	$P(X = x)$	$f(x)$
CDF*	$F(X = x) = P(X \leq x)$	$F(X = x) = P(X \leq x)$
Total Probability	$\sum_{-\infty}^{\infty} P(X = x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$

♯RV-Random Variable, *CDF-Cumulative Distribution Function or Distribution Function

3 Mathematical Expectation

Definition 3.0.1 (Expectation). *An average of a probability distribution of a random variable X is called the expectation or the expected value or mathematical expectation of X and is denoted by $E(X)$.*

Definition 3.0.2 (Expectation for Discrete Random Variable). *Let X be a discrete random variable taking values $x_1, x_2, x_3, \ldots$ with probabilities $P(X = x_1) = p(x_1), P(X = x_2) = p(x_2), P(X = x_3) = p(x_3), \ldots$, the expected value of X is defined as $E(X) = \sum_{i=-\infty}^{\infty} x_i p(x_i)$, if the right hand side sum exists.*

Definition 3.0.3 (Expectation for Continuous Random Variable). *Let X be a continuous random variable with pdf $f(x)$ defined in $(-\infty, \infty)$, then the expected value of X is defined as*

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

Note: $E(X)$ is called the mean of the distribution or mean of X and is denoted by $\overline{X}$ or μ.

3.1 Properties of Expectation

1. If c is any constant, then $E(c) = c$.
2. If a, b are constants, then $E(aX + b) = aE(X) + b$.
3. If (X, Y) is two dimensional random variable, then $E(X + Y) = E(X) + E(Y)$.
4. If (X, Y) is a two dimensional random variable, then $E(XY) = E(X)E(Y)$ only if X and Y are **independent** random variables.

3.2 Variance and its properties

Definition 3.2.1. *Let X be a random variable with mean $E(X)$, then the variance of X is defined as $\sigma^2 = E\{(X-\mu)^2\}$ and it is denoted by $Var(X)$ or σ_X^2.*

Note: $Var(X) = E(X^2) - [E(X)]^2$.

Properties of Variance

1. $Var(a) = 0$, where a is any constant.
2. $Var(aX) = a^2 Var(X)$
3. $Var(aX \pm b) = a^2 Var(X)$
4. $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$ only if X and Y are independent random variables.

4 Moments and Moment Generating Function

4.1 Moments

Definition 4.1.1 (Moments about origin)**.** *The r^{th} moment of a random variable X about the origin is defined as $E(X^r)$ and is denoted by μ_r'. Moments about origin are known as raw moments.*

$$\mu_r' = E(X^r) = \begin{cases} \sum\limits_{x=0}^{n} x^r P(X=x) & \text{if } X \text{ is discrete} \\ \int\limits_{x=0}^{\infty} x^r f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Note: By moments we mean the moments about origin or raw moments.

The first four moments about the origin are given by

1. $\mu_1' = E(X)$=Mean
2. $\mu_2' = E(X^2)$
3. $\mu_3' = E(X^3)$
4. $\mu_4' = E(X^4)$

Note: $Var(X) = E(X^2) - [E(X)]^2 = \mu_2' - {\mu_1'}^2$=Second moment - square of the first moment. Standard Deviation $SD = \sigma_X = \sqrt{Var(X)}$.

Definition 4.1.2 (Moments about mean or Central moments)**.** *The r^{th} moment of a random variable X about the mean μ is defined as $E[(X-\mu)^r]$ and is denoted by μ_r.*

The first four moments about the mean are given by

1. $\mu_1 = E(X-\mu) = E(X) - E(\mu) = \mu - \mu = 0$
2. $\mu_2 = E[(X-\mu)^2] = Var(X)$
3. $\mu_3 = E[(X-\mu)^3]$
4. $\mu_4 = E[(X-\mu)^4]$

Definition 4.1.3 (Moments about any point a)**.** *The r^{th} moment of a random variable X about any point a is defined as $E[(X-a)^r]$ and we denote it by m'_r.*

The first four moments about a point 'a' are given by

1. $m'_1 = E(X-a) = E(X) - a = \mu - a$
2. $m'_2 = E[(X-a)^2]$
3. $m'_3 = E[(X-a)^3]$
4. $m'_4 = E[(X-a)^4]$

Relation between moments about the mean and moments about any arbitrary point a

Let μ_r be the r^{th} moment about mean and m'_r be the r^{th} moment about any point a. Let μ be the mean of X.

$$\begin{aligned}
\therefore\ \mu_r &= E[(X-\mu)^r] \\
&= E[(X-a)-(\mu-a)]^r \\
&= E[(X-a)-m'_1]^r \\
&= E\left[(X-a)^r - {}^rC_1(X-a)^{r-1}m'_1 + {}^rC_2(X-a)^{r-2}(m'_1)^2 - \cdots + (-1)^r(m'_1)^r\right] \\
&= E(X-a)^r - {}^rC_1E(X-a)^{r-1}m'_1 + {}^rC_2E(X-a)^{r-2}(m'_1)^2 \\
&\quad - {}^rC_3E(X-a)^{r-3}(m'_1)^3 + {}^rC_4E(X-a)^{r-4}(m'_1)^4 - \cdots + (-1)^r(m'_1)^r \\
&= m'_r - {}^rC_1m'_{r-1}m'_1 + {}^rC_2m'_{r-2}(m'_1)^2 - {}^rC_3m'_{r-3}(m'_1)^3 + {}^rC_4m'_{r-4}(m'_1)^4 \\
&\quad - \cdots + (-1)^r(m'_1)^r
\end{aligned}$$

We define(fix) $m'_0 = 1$, then we have

$$\begin{aligned}
\mu_1 &= m'_1 - m'_0m'_1 = 0 \\
\mu_2 &= m'_2 - {}^2C_1m'_1\cdot m'_1 + (m'_1)^2 \\
&= m'_2 - (m'_1)^2 \\
\mu_3 &= m'_3 - {}^3C_1m'_2\cdot m'_1 + {}^3C_2m'_1\cdot(m'_1)^2 - (m'_1)^3\cdot \\
&= m'_3 - 3m'_2\cdot m'_1 + 2(m'_1)^3 \\
\mu_4 &= m'_4 - {}^4C_1m'_3\cdot m'_1 + {}^4C_2m'_2\cdot(m'_1)^2 - {}^4C_3m'_1\cdot(m'_1)^3 + (m'_1)^4\cdot \\
&= m'_4 - 4m'_3\cdot m'_1 + 6m'_2\cdot(m'_1)^2 - 3(m'_1)^4
\end{aligned}$$

4.2 Moment Generating Function (MGF)

Definition 4.2.1 (Moment Generating Function (MGF)). *The moment generating function of a random variable X is defined as $E(e^{tX})$ for all $t \in (-\infty, \infty)$. It is denoted by $M(t)$ or $M_X(t)$.*

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum\limits_{x=0}^{n} e^{tx} P(X = x) & \text{if } X \text{ is discrete} \\ \int\limits_{x=0}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

If X is a random variable. Its MGF $M_X(t)$ is given by

$$\begin{aligned} M_X(t) = E(e^{tX}) &= E\left(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots + \frac{(tX)^r}{r!} + \ldots\right) \\ &= 1 + \frac{t}{1!}E(X) + \frac{(t)^2}{2!}E(X^2) + \frac{(t)^3}{3!}E(X^3) + \cdots + \frac{(t)^r}{r!}E(X^r) + \ldots \\ &= 1 + \frac{t}{1!}\mu_1' + \frac{(t)^2}{2!}\mu_2' + \frac{(t)^3}{3!}\mu_3' + \cdots + \frac{(t)^r}{r!}\mu_r' + \ldots \end{aligned}$$

i.e.

$$\begin{aligned} \mu_1' &= \text{Coefft. of } t \text{ in the expansion of } M_X(t) \\ \mu_2' &= \text{Coefft. of } \frac{t^2}{2!} \text{ in the expansion of } M_X(t) \\ &\vdots \\ \mu_r' &= \text{Coefft. of } \frac{t^r}{r!} \text{ in the expansion of } M_X(t) \\ &\vdots \end{aligned}$$

$\therefore$ $M_X(t)$ generates all the moments about the origin. (That is why we call it as Moment Generating Function (MGF)).

Another phenomenon which we often use to find the moments is given below:

$$\mu_r' = M_X^{(r)}(0)$$

We have

$$M_X(t) = E(e^{tX}) = 1 + \frac{t}{1!}\mu_1' + \frac{(t)^2}{2!}\mu_2' + \frac{(t)^3}{3!}\mu_3' + \cdots + \frac{(t)^r}{r!}\mu_r' + \ldots$$

Differentiate with respect to t, we get

$$\begin{aligned}
M'_X(t) &= \mu'_1 + t\mu'_2 + \frac{(t)^2}{2!}\mu'_3 + \cdots + \frac{(t)^{r-1}}{(r-1)!}\mu'_r + \ldots \\
M''_X(t) &= \mu'_2 + t\mu'_3 + \frac{(t)^2}{2!}\mu'_4 + \cdots + \frac{(t)^{r-2}}{(r-2)!}\mu'_r + \ldots \\
&\vdots \\
M^{(r)}_X(t) &= \mu'_r + \text{terms of higher powers of } t \\
&\vdots
\end{aligned}$$

Putting $t = 0$, we get

$$\begin{aligned}
M'_X(0) &= \mu'_1 \\
M''_X(0) &= \mu'_2 \\
&\vdots \\
M^{(r)}_X(0) &= \mu'_r \\
&\vdots
\end{aligned}$$

The Maclaurin's series expansion given below will give all the moments.

$$M_X(t) = M_X(0) + \frac{t}{1!}M'_X(0) + \frac{t^2}{2!}M''_X(0) + \ldots$$

The MGF of X about its mean μ is $M_{X-\mu}(t) = E\left[e^{t(X-\mu)}\right]$.

Similarly, the MGF of X about any point a is $M_{X-a}(t) = E\left[e^{t(X-a)}\right]$.

Properties of MGF

1. $M_{cX}(t) = M_X(ct)$
2. $M_{X+c}(t) = e^{ct}M_X(t)$
3. $M_{aX+b}(t) = e^{bt}M_X(at)$
4. If X and Y are independent RV's then, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

5 Examples

Example: 5.1. *From 6 positive and 8 negative numbers, 4 are chosen at random(without replacement) and multiplied. What is the probability that the product is negative?*

Solution: If the product is negative(-ve), any one number is -ve (or) any 3 numbers are -ve.

No. of ways choosing 1 negative no. = ${}^6C_3 \cdot {}^8C_1 = 20 \cdot 8 = 160$.
No. of ways choosing 3 negative nos. = ${}^6C_1 \cdot {}^8C_3 = 6 \cdot 56 = 336$.

Total no. of ways choosing 4 nos. out of 14 nos. $= {}^{14}C_4 = 1001$.

$\therefore \quad P(\text{the product is -ve}) = \dfrac{160+336}{1001} = \dfrac{496}{1001}$.

Example: 5.2. *A box contains 6 bad and 4 good apples. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good.*

Solution: Let A be an event that the first drawn apple is good and let B be an event that the other one is also good.

$\therefore$ The required probability is $P(B/A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{{}^4C_2/{}^{10}C_2}{\frac{4}{10}} = \dfrac{6/45}{4/10} = \dfrac{1}{3}$

Aliter: Probability of the first drawn apple is good$=\dfrac{4}{10}$.
Once the first apple is drawn, the box contains only 9 apples in total and 3 good apples.

$\therefore$ Probability of the second drawn apple is also good$=\dfrac{3}{9} = \dfrac{1}{3}$.

Example: 5.3. *A random variable X has the following distribution*

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
P[X=x]	*0*	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k,(ii) the Cumulative Distribution Function (CDF) (iii) $P(1.5 < X < 4.5/X > 2)$ and (iv) the smallest value of α for which $P(X \le \alpha) > \dfrac{1}{2}$.

Solution:

(i) We know that $\sum_i P[X = x_i] = 1$.

Here,

$$
\begin{aligned}
P[X=0]+P[X=1]+P[X=2]+P[X=3]+&\\
P[X=4]+P[X=5]+P[X=6]+P[X=7] &= 1\\
\text{i.e. } 0+k+2k+2k+3k+k^2+2k^2+7k^2+k &= 1\\
\text{i.e. } 10k^2+9k-1 &= 0\\
\text{i.e. } (10k-1)(k+1) &= 0\\
\implies k = \frac{1}{10} \text{ or } k = -1 &
\end{aligned}
$$

Since the value of the probability is not negative, we take the value $k = \dfrac{1}{10}$.

$\therefore$ The probability distribution is given by

x	0	1	2	3	4	5	6	7
$P[X=x]$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$
$P[X=x]$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(ii) The Cumulative Distribution Function (CDF)

$$\begin{aligned}
F(x) &= P(X \le x) \\
F(0) &= P(X \le 0) = P(X=0) = 0 \\
F(1) &= P(X \le 1) = P(X=0) + P(X=1) = 0 + \frac{1}{10} = \frac{1}{10} \\
F(2) &= P(X \le 1) = P(X=0) + P(X=1) + P(X=2) = 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \\
&\vdots \\
F(7) &= P(X \le 7) = P(X=0) + P(X=1) + P(X=2) + P(X=2) \\
&\quad + P(X=4) + P(X=5) + P(X=6) + P(X=7) \\
&= \frac{100}{100} = 1
\end{aligned}$$

The detailed CDF $F(X=x)$ is given in the table.

x	0	1	2	3	4	5	6	7
$P[X=x]$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$
$P[X=x]$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
$F[X=x]$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	$\frac{100}{100}=1$

(iii)

$$\begin{aligned}
P(1.5 < X < 4.5/X > 2) &= \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X > 2)} \\
P(1.5 < X < 4.5) &= P(X=3) + P(X=4) = \frac{5}{10} \\
P(X > 2) &= 1 - P(X \le 2) \\
&= 1 - \{P(X=0) + P(X=1) + P(X=2)\} \\
&= 1 - \frac{3}{10} = \frac{7}{10} \\
P(1.5 < X < 4.5/X > 2) &= \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}
\end{aligned}$$

(iv) The smallest value of α for which $P(X \le \alpha) > \frac{1}{2} = 0.5$.

From the CDF table, we found that $P(X \leq 3) = \frac{5}{10} = 0.5$. But we need the probability which is more than 0.5, for this we have $P(X \leq 4) = \frac{8}{10} = 0.8 > 0.5$. $\therefore \quad \alpha = 4$ satisfies the given condition.

Example: 5.4. *A random variable X has the following distribution*

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
P[X=x]	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$

Find (i) the value of k,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 3/X > 2)$ and (iv) the smallest value of α for which $P(X \leq \alpha) > \frac{1}{2}$.

Solution:

x	0	1	2	3	4	5	6	7	8
$P[X = x]$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$
$P[X = x]$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

(i) The value of $k = \frac{1}{81}$

(ii) the Distribution Function (CDF)

x	0	1	2	3	4	5	6	7	8
$P[X = x]$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$
$P[X = x]$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$F[X = x]$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	$\frac{81}{81} = 1$

(iii) $P(0 < X < 3/X > 2)$

$$P(0 < X < 3/X > 2) \;=\; \frac{P[(0 < X < 3) \cap (X > 2)]}{P(X > 2)} = 0(\because \; there\ is\ no\ common\ elemen$$

(iv) The smallest value of α for which $P(X \leq \alpha) > \frac{1}{2} = 0.5$.

From the CDF table, we found that $P(X \leq 5) = \frac{36}{81} < 0.5$. But we need the probability which is more than 0.5, for this we have $P(X \leq 6) = \frac{49}{81} > 0.5$. $\therefore \quad \alpha = 6$ satisfies the given condition.

Example: 5.5. *Find the value of* k *for the pdf* $f(x) = \begin{cases} kx, & \textit{when } 0 \le x \le 1 \\ k, & \textit{when } 1 \le x \le 2 \\ 3k - kx, & \textit{when } 2 \le x \le 3 \\ 0, & \textit{otherwise} \end{cases}$. *Also find (a) the CDF of* X *(b)* $P(1.5 < X < 3.2/0.5 < X < 1.8)$

Solution:

$$\begin{aligned}
\text{We know that } \int_{-\infty}^{\infty} f(x)dx &= 1 \\
\textit{i.e. } \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx &= 1 \\
\textit{i.e. } \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} kx \cdot dx + \int_{1}^{2} k \cdot dx + \int_{2}^{3} (3k - kx)dx + \int_{3}^{\infty} 0 \cdot dx &= 1 \\
\textit{i.e. } 0 + k\left[\frac{x^2}{2}\right]_0^1 + k\,[x]_1^2 + \left[3k \cdot x - k \cdot \frac{x^2}{2}\right]_2^3 + 0 &= 1 \\
\textit{i.e. } 2k &= 1 \\
\implies \quad & \boxed{k = \frac{1}{2}}
\end{aligned}$$

(a) Since $k = \frac{1}{2}$, $\quad f(x) = \begin{cases} \frac{x}{2}, & \text{when } 0 \le x \le 1 \\ \frac{1}{2}, & \text{when } 1 \le x \le 2 \\ \frac{1}{2}(3 - x), & \text{when } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

The CDF of X is $F(x) = P(X \leq x) = \int\limits_{-\infty}^{x} f(x)dx$

$$
\begin{aligned}
\text{When } x < 0,\ F(x) &= 0, \text{ since } f(x) = 0 \text{ for } x < 0 \\
\text{When } 0 \leq x < 1,\ F(x) &= \int\limits_{-\infty}^{x} f(x)dx \\
&= \int\limits_{0}^{x} f(x)dx = \int\limits_{0}^{x} \frac{x}{2}\,dx = \boxed{\frac{x^2}{4}}
\end{aligned}
$$

$$
\begin{aligned}
\text{When } 1 \leq x < 2,\ F(x) &= \int\limits_{0}^{x} f(x)dx \\
&= \int\limits_{0}^{1} \frac{x}{2}\,dx + \int\limits_{0}^{x} \frac{1}{2}\,dx = \boxed{\frac{1}{4}(2x-1)}
\end{aligned}
$$

$$
\begin{aligned}
\text{When } 2 \leq x < 3,\ F(x) &= \int\limits_{0}^{x} f(x)dx \\
&= \int\limits_{0}^{1} \frac{x}{2}dx + \int\limits_{0}^{x} \frac{1}{2}\,dx + \int\limits_{0}^{x} \frac{1}{2}(3-x)\,dx = \boxed{\frac{1}{4}(-x^2+6x-5)}
\end{aligned}
$$

$$\text{When } x \geq 3,\ F(x) = 1$$

$$
\therefore \quad F(x) = \begin{cases} 0 & \text{when} x < 0 \\ \frac{x^2}{2}, & \text{when } 0 \leq x \leq 1 \\ \frac{1}{4}(2x-1), & \text{when } 1 \leq x \leq 2 \\ \frac{1}{4}(-x^2+6x-5), & \text{when } 2 \leq x \leq 3 \\ 1, & \text{when } x \geq 3 \end{cases}
$$

(b) $P(1.5 < X < 3.2/0.5 < X < 1.8) = \dfrac{P[(1.5 < X < 3.2) \cap (0.5 < X < 1.8)]}{P(0.5 < X < 1.8)}$

$$\begin{aligned}
P[(1.5 < X < 3.2) \cap (0.5 < X < 1.8)] &= P(1.5 < X < 1.8) \\
&= \int_{1.5}^{1.8} \frac{1}{2}\, dx = \boxed{\frac{3}{20}} \\
P(0.5 < X < 1.8) &= \int_{0.5}^{1} \frac{x}{2}\, dx + \int_{1}^{1.8} \frac{1}{2}\, dx = \boxed{\frac{3}{16}} \\
P(1.5 < X < 3.2/0.5 < X < 1.8) &= \frac{P[(1.5 < X < 3.2) \cap (0.5 < X < 1.8)]}{P(0.5 < X < 1.8)} \\
&= \boxed{\frac{16}{20}}
\end{aligned}$$

Example: 5.6. *Experience has shown that while walking in a certain park, the time X (in minutes) duration between seeing two people smoking has a density function*

$$f(x) = \begin{cases} kxe^{-x}, & \textit{when } x > 0 \\ 0, & \textit{otherwise} \end{cases}$$

Find the (a) value of k (b) distribution function of X (c) What is the probability that a person, who has just seen a person smoking will see another person smoking in 2 to 5 minutes, in at least 7 minutes?

Solution: Given the pdf of X is

$$f(x) = \begin{cases} kxe^{-x}, & \text{when } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \quad f(x) \geq 0 \ \forall \ x \quad \Longrightarrow \quad k > 0.$$

(a) We know that $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\begin{aligned}
\textit{i.e.} \int_{0}^{\infty} kxe^{-x}\, dx &= 1 \\
\textit{i.e.} \ k\left[x\frac{e^{-x}}{-1} - \frac{e^{-x}}{(-1)^2}\right]_0^{\infty} &= 1 \\
\Longrightarrow \boxed{k = 1}
\end{aligned}$$

$$\therefore \; f(x) = \begin{cases} xe^{-x}, & \text{when } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) The distribution function (CDF) is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

Here we have $x > 0$, $F(x) = \int_{0}^{x} xe^{-x}\,dx = 1 - (1+x)e^{-x}, \;\; x > 0$

(c)

$$\begin{aligned} P(2 < X < 5) &= \int_{2}^{5} f(x)\,dx \\ &= \int_{2}^{5} xe^{-x}\,dx \\ &= -6e^{-5} + 3e^{-2} \\ &= -0.04 + 0.406 = 0.366 \end{aligned}$$

and

$$\begin{aligned} P(X \geq 7) &= \int_{7}^{\infty} f(x)\,dx \\ &= \int_{7}^{\infty} xe^{-x}\,dx \\ &= 8e^{-7} \\ &= 0.007 \end{aligned}$$

Example: 5.7. *The sales of convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution function*

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ \dfrac{x^2}{2}, & \text{when } 0 \leq x < 1 \\ k(4x - x^2) - 1, & \text{when } 1 \leq x < 2 \\ 1, & \text{when } x \geq 2 \end{cases}$$

Suppose that this convenience store's total sales on any given day is less than 2000 units(Dollars or Pounds or Rupees). Find the (a) value of k (b) Let A and B be the events that tomorrow the store's total sales between 500 and 1500 units respectively. Find $P(A)$ and $P(B)$. (c) Are A and B are independent events?

Solution: The pdf of X is given by

$$f(x) = F'(x) = \frac{d}{dx}F(x) = \begin{cases} 0, & \text{when } x < 0 \\ x, & \text{when } 0 \leq x < 1 \\ k(4 - 2x), & \text{when } 1 \leq x < 2 \\ 0, & \text{when } x \geq 2 \end{cases}$$

Since $f(x)$ is a pdf,

$$\text{we have } \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\text{i.e. } \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$\text{i.e. } \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} x\, dx + \int_{1}^{2} k(4 - 2x)\, dx + \int_{2}^{\infty} 0 \cdot dx = 1$$

$$\text{i.e. } 0 + \left[\frac{x^2}{2}\right]_0^1 + k\left[4x - x^2\right]_1^2 + 0 = 1$$

$$\implies \boxed{k = \frac{1}{2}}$$

$$f(x) = \begin{cases} 0, & \text{when } x < 0 \\ x, & \text{when } 0 \leq x < 1 \\ (2 - x), & \text{when } 1 \leq x < 2 \\ 0, & \text{when } x \geq 2 \end{cases}$$

Since the total sales X is in thousands of units, the sales between 500 and 1500 units is the event A which stands for $\frac{1}{2} = 0.5 < X < \frac{3}{2} = 1.5$ and the sales over 1000 units is the event B which stands for $X > 1$. $\implies \quad A \cap B = 1 < X < 1.5$

Now

$$
\begin{aligned}
P(A) &= P(0.5 < X < 1.5) = \int_{0.5}^{1.5} f(x)dx \\
&= \int_{0.5}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx = \boxed{\frac{3}{4}}
\end{aligned}
$$

$$
\begin{aligned}
P(B) &= P(X > 1) = \int_{1}^{2} f(x)dx \\
&= \int_{1}^{2} (2-x)\,dx = \boxed{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
P(A \cap B) &= P(1 < X < 1.5) = \int_{1}^{1.5} f(x)dx \\
&= \int_{1}^{1.5} (2-x)\,dx = \boxed{\frac{3}{8}}
\end{aligned}
$$

The condition for independent events: $P(A) \cdot P(B) = P(A \cap B)$

Here, $P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$

$\therefore \quad A$ and B are independent events.

Example: 5.8. *If a random variable X has the following probability distribution, find* $E(X)$, $E(X^2)$, $Var(X)$, $E(2X+1)$, $Var(2X+1)$.

x	-1	0	1	2
$p(x)$	0.3	0.1	0.4	0.2

Solution: Here X is a discrete RV. $\therefore$

$$
\begin{aligned}
E(X) &= \sum_{i=-\infty}^{\infty} x_i p(x_i) \\
&= (-1) \times 0.3 + 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.2 \\
&= -0.3 + 0 + 0.4 + 0.4 = \boxed{0.5}
\end{aligned}
$$

$$\begin{aligned} E(X^2) &= \sum_{i=-\infty}^{\infty} x_i^2 p(x_i) \\ &= (-1)^2 \times 0.3 + 0^2 \times 0.1 + 1^2 \times 0.4 + 2^2 \times 0.2 \\ &= 0.3 + 0 + 0.4 + 0.8 = \boxed{1.5} \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= (1.5) - (0.5)^2 \\ &= 1.5 - 0.25 = \boxed{1.25} \end{aligned}$$

$$\begin{aligned} E(2X+1) &= 2E(X) + 1 \\ &= 2 \times (0.5) + 1 \\ &= 1 + 1 = \boxed{2} \end{aligned}$$

$$\begin{aligned} Var(2X+1) &= 2^2 Var(X) \\ &= 4 \times (1.25) = \boxed{5} \end{aligned}$$

Example: 5.9. *If a random variable* X *has the following probability distribution, find* $E(X)$, $E(X^2)$, $Var(X)$, $E(3X-4)$, $Var(3X-4)$.

x	0	1	2	3	4
$p(x)$	$\frac{1}{25}$	$\frac{3}{25}$	$\frac{5}{25}$	$\frac{7}{25}$	$\frac{9}{25}$

Solution: Here X is a discrete RV. $\therefore$

$$\begin{aligned} E(X) &= \sum_i x_i p(x_i) \\ &= \boxed{\frac{14}{5}} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum_{i=} x_i^2 p(x_i) \\ &= \boxed{\frac{46}{5}} \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{46}{5} - \left(\frac{14}{5}\right)^2 \\ &= \boxed{\frac{34}{25}} \end{aligned}$$

$$\begin{aligned} E(3X-4) &= 3E(X) - 4 \\ &= = \boxed{\frac{22}{5}} \end{aligned}$$

$$\begin{aligned} Var(3X-4) &= 3^2 Var(X) \\ &= \boxed{\frac{306}{25}} \end{aligned}$$

Example: 5.10. *A test engineer found that the distribution function of the lifetime X (in years) of an equipment follows a distribution function which is given by*

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ 1 - e^{-\frac{x}{5}}, & \text{when } x \geq 0 \end{cases}$$

Find the pdf, mean and variance of X.

Solution: The pdf of X is given by

$$f(x) = F'(x) = \frac{d}{dx}F = \begin{cases} 0, & \text{when } x < 0 \\ \frac{1}{5}e^{-\frac{x}{5}}, & \text{when } x \geq 0 \end{cases}$$

$$\begin{aligned} \text{The Mean } E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{0} x \frac{1}{5} e^{-\frac{x}{5}} dx \\ &= -\frac{1}{5}\left[(5x+25)e^{-\frac{x}{5}}\right]_0^{\infty} = \boxed{5} \end{aligned}$$

$$Var(X) = E(X^2) - [(X)]^2$$

$$\text{Now } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$
$$= \int_{-\infty}^{0} x^2 \frac{1}{5} e^{-\frac{x}{5}} dx$$
$$= -\frac{1}{5}\left[(5x^2 + 50x + 250)e^{-\frac{x}{5}}\right]_0^{\infty} = \boxed{50}$$

$$\therefore \quad Var(X) = 50 - [5]^2 = \boxed{25}$$

Example: 5.11. *Find the mean and standard deviation of the distribution*

$$f(x) = \begin{cases} kx(2-x), & \textit{when } 0 \le x \le 2 \\ 0, & \textit{otherwise} \end{cases}$$

Solution: Given that the continuous RV X whose pdf is given by

$$f(x) = \begin{cases} kx(2-x), & \text{when } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(x)$ is a pdf,

$$\text{we have } \int_{-\infty}^{\infty} f(x)dx = 1$$
$$i.e. \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$
$$i.e. \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{2} kx(2-x)\, dx + \int_{2}^{\infty} 0 \cdot dx = 1$$
$$i.e. \; 0 + k\left[x^2 - \frac{x^3}{3}\right]_0^2 + 0 = 1$$
$$\Longrightarrow \boxed{k = \frac{3}{4}}$$

$$\begin{aligned}\text{The Mean } E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{0} x \frac{3}{4} x(2-x) dx \\ &= -\frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \boxed{1}\end{aligned}$$

$$Var(X) = E(X^2) - [(X)]^2$$

$$\begin{aligned}\text{Now } E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_{-\infty}^{0} x^2 \frac{3}{4} x(2-x) dx \\ &= -\frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \boxed{\frac{6}{5}}\end{aligned}$$

$$\therefore \quad Var(X) = \frac{6}{5} - [1]^2 = \boxed{\frac{1}{5}} \implies S.D. = \sqrt{\frac{1}{5}}$$

Example: 5.12. *If X has the distribution function*

$$F(x) = \begin{cases} 0, & \text{when } x < 1 \\ \frac{1}{3}, & \text{when } 1 \le x < 4 \\ \frac{1}{2}, & \text{when } 4 \le x < 6 \\ \frac{5}{6}, & \text{when } 6 \le x < 10 \\ 1, & \text{when } x \ge 10 \end{cases}$$

Find (a) the probability distribution of X (b) Find the mean and variance of X.

Solution: Given that the CDF of X is

$$F(x) = \begin{cases} 0, & \text{when } x < 1 \\ \frac{1}{3}, & \text{when } 1 \leq x < 4 \\ \frac{1}{2}, & \text{when } 4 \leq x < 6 \\ \frac{5}{6}, & \text{when } 6 \leq x < 10 \\ 1, & \text{when } x \geq 10 \end{cases}$$

We know that $P(X = x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \ldots$, where F is constant in $x_{i-1} \leq x \leq x_i$.

The CDF changes its values at $x = 1, 4, 6, 10$. $\therefore$ The probability distribution takes its values as follows:

$$\begin{aligned} P(X = 1) &= F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3} \\ P(X = 4) &= F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \\ P(X = 6) &= F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3} \\ P(X = 10) &= F(10) - F(6) = 1 - \frac{5}{6} = \frac{1}{6} \end{aligned}$$

$\therefore$ The probability distribution is given by

x	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

Mean$= E(X) = \Sigma x p(x) = \frac{14}{3}$.

$E(X^2) = \Sigma x^2 p(x) = \frac{95}{3}$.

$Var(X) = E(X^2) - [E(X)]^2 = \frac{285}{9} - \frac{196}{9} = \frac{89}{9}$.

Example: 5.13. *If X has the distribution function*

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ \frac{1}{6}, & \text{when } 0 \le x < 2 \\ \frac{1}{2}, & \text{when } 2 \le x < 4 \\ \frac{5}{8}, & \text{when } 4 \le x < 6 \\ 1, & \text{when } x \ge 6 \end{cases}$$

Find (a) the probability distribution of X (b) Find the mean and variance of X.

Solution: Given that the CDF of X is

$$F(x) = \begin{cases} 0, & \text{when } x < 0 \\ \frac{1}{6}, & \text{when } 0 \le x < 2 \\ \frac{1}{2}, & \text{when } 2 \le x < 4 \\ \frac{5}{8}, & \text{when } 4 \le x < 6 \\ 1, & \text{when } x \ge 6 \end{cases}$$

We know that $P(X = x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \dots$, where F is constant in $x_{i-1} \le x \le x_i$.

The CDF changes its values at $x = 1, 4, 6, 10$. $\therefore$ The probability distribution takes its values as follows:

$$\begin{aligned} P(X = 0) &= \frac{1}{6} \\ P(X = 2) &= F(2) - F(0) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \\ P(X = 4) &= F(4) - F(2) = \frac{5}{8} - \frac{1}{2} = \frac{1}{8} \\ P(X = 6) &= F(6) - F(4) = 1 - \frac{5}{8} = \frac{3}{8} \end{aligned}$$

$\therefore$ The probability distribution is given by

x	0	2	4	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{3}{8}$

Mean=$E(X) = \Sigma x p(x) = \frac{37}{12}$.

$E(X^2) = \Sigma x^2 p(x) = \frac{101}{6}$.

$Var(X) = E(X^2) - [E(X)]^2 = \frac{101}{6} - \left[\frac{37}{12}\right]^2 = 16.83 - 9.51 = 7.32.$

Example: 5.14. *The first four moments about $X = 5$ are 2,5,12 and 48. Find the first four central moments.*

Solution: Let m'_1, m'_2, m'_3, m'_4 be the first four moments about $X = 5$. Given $m'_1 = 1, m'_2 = 5, m'_3 = 12, m'_4 = 48$ be the first four moments about $X = 5$ $m'_1 = E(X - 5) = 1 \implies E(X) - 5 = 1 \implies E(X) = \mu_1 = 6$

$$\begin{aligned}
\mu_2 &= m'_2 - (m'_1)^2 = 5 - 1 = 4 \\
\mu_3 &= m'_3 - 3m'_2 \cdot m'_1 + 2(m'_1)^3 \\
&= 12 - 3 \times 5 \times 1 + 2(1) = -1 \\
\mu_4 &= m'_4 - 4m'_3 \cdot m'_1 + 6m'_2 \cdot (m'_1)^2 - 3(m'_1)^4 \\
&= 48 - 4(12)(1) + 6(5)(1) - 3(1) = 27
\end{aligned}$$

Example: 5.15. *The first four moments about $x = 4$ are 1,4,10 and 45. Find the first four moments about the mean.*

Solution:

Let m'_1, m'_2, m'_3, m'_4 be the first four moments about $X = 4$. Given $m'_1 = 1, m'_2 = 4, m'_3 = 10, m'_4 = 45$ be the first four moments about $X = 4$ $m'_1 = E(X - 4) = 1 \implies E(X) - 4 = 1 \implies E(X) = \mu_1 = 5$

$$\begin{aligned}
\mu_2 &= m'_2 - (m'_1)^2 = 4 - 1 = 3 \\
\mu_3 &= m'_3 - 3m'_2 \cdot m'_1 + 2(m'_1)^3 \\
&= 10 - 3 \times 4 \times 1 + 2(1) = 0 \\
\mu_4 &= m'_4 - 4m'_3 \cdot m'_1 + 6m'_2 \cdot (m'_1)^2 - 3(m'_1)^4 \\
&= 45 - 4(10)(1) + 6(4)(1) - 3(1) = 26
\end{aligned}$$

Example: 5.16. *A random variable X has the pdf $f(x) = kx^2e^{-x}, \quad x \geq 0$. Find the r^{th} moment and hence find the first four moments.*

Solution: We know that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1 \\
\text{i.e., } \int_{0}^{\infty} kx^2e^{-x}\,dx &= 1 \\
\text{i.e., } k\int_{0}^{\infty} e^{-x}x^{3-1}\,dx &= 1 \\
\text{i.e., } k\Gamma(3) &= 1 \quad \left[\because\ \Gamma(n) = \int_{0}^{\infty} e^{-x}x^{n-1}\,dx\right] \\
\text{i.e., } k\cdot 2! &= 1 \implies k = \frac{1}{2}. \qquad [\because\ \Gamma(n) = (n-1)!]
\end{aligned}$$

Now, The r^{th} moment is given by

$$\begin{aligned}
\mu_r' &= E(X^r) \\
&= \int_{-\infty}^{\infty} x^r f(x)\,dx \\
&= \int_{0}^{\infty} x^r kx^2e^{-x}\,dx \\
&= k\int_{0}^{\infty} e^{-x}x^{r+3-1}\,dx \\
&= \frac{1}{2}\Gamma(r+3) = \frac{1}{2}(r+2)!
\end{aligned}$$

$$\begin{aligned}
\text{Now, } \quad \text{First Moment } \mu_1' &= \frac{1}{2}(3)! = 3 \\
\text{Second Moment } \mu_2' &= \frac{1}{2}(4)! = 12 \\
\text{Third Moment } \mu_3' &= \frac{1}{2}(5)! = 60 \\
\text{Fourth Moment } \mu_4' &= \frac{1}{2}(6)! = 360
\end{aligned}$$

Example: 5.17. *A random variable X has the pdf $f(x) = \frac{1}{2}e^{-\frac{x}{2}}, \quad x \geq 0$. Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Solution: The MGF of X is given by

$$
\begin{aligned}
M_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\
&= \frac{1}{2}\int_{0}^{\infty} e^{tx} e^{-\frac{x}{2}}\, dx \\
&= \frac{1}{2}\int_{0}^{\infty} e^{-\frac{1}{2}(1-2t)x}\, dx \\
&= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}(1-2t)x}}{-\frac{1}{2}(1-2t)}\right]_0^{\infty} \\
\therefore \quad M_X(t) &= \frac{1}{1-2t} \text{ if } t < \frac{1}{2}. \\
&= (1-2t)^{-1} = 1 + 2t + 4t^2 + 8t^3 + \ldots
\end{aligned}
$$

Now, Differentiating w.r.to t, we get

$$
\begin{aligned}
M'_X(t) &= 2 + 8t + 24t^2 + \ldots \\
M''_X(t) &= 8 + 48t + \ldots \\
\text{Now, First Moment=Mean= } E(X) = \mu'_1 &= M'_X(0) = 2 \\
\text{Second Moment= } E(X^2) = \mu'_2 &= M''_X(0) = 8 \\
\therefore \quad Var(X) &= E(X^2) - [E(X)]^2 = 8 - 4 = 4.
\end{aligned}
$$

Example: 5.18. *A random variable X has the pmf $p(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \ldots$. Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Solution: The MGF of X is given by

$$
\begin{aligned}
M_X(t) = E(e^{tX}) &= \sum_{x=-\infty}^{\infty} e^{tx} p(x) \\
&= \sum_{x=1}^{\infty} e^{tx}\frac{1}{2^x} = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x \\
&= \frac{e^t}{2}\left[1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\dots\right] \\
&= \frac{e^t}{2}\left[1-\left(\frac{e^t}{2}\right)\right]^{-1} \\
\therefore\ M_X(t) &= \frac{e^t}{2-e^t} \text{ if } e^t \neq 2.
\end{aligned}
$$

Now, Differentiating w.r.to t, we get

$$
\begin{aligned}
M'_X(t) &= \frac{2e^t}{(2-e^t)^2} \\
M''_X(t) &= \frac{2[(2-e^t)e^t+2e^{2t}]}{(2-e^t)^3}
\end{aligned}
$$

$$
\begin{aligned}
\text{Now,}\quad \text{First Moment=Mean=}\ E(X) = \mu'_1 &= M'_X(0) = 2 \\
\text{Second Moment=}\ E(X^2) = \mu'_2 &= M''_X(0) = 6 \\
\therefore\ Var(X) &= E(X^2) - [E(X)]^2 = 6 - 4 = 2.
\end{aligned}
$$

Example: 5.19. *A random variable X has the r^{th} moment of the form $\mu'_r = (r+1)!2^r$. Find the MGF(Moment Generating Function) and hence find its mean and variance.*

Solution: The MGF of X is given by

$$\begin{aligned} \text{Given } \mu'_r &= (r+1)!2^r \\ \therefore \quad \mu'_1 &= 2!2 \\ \mu'_2 &= 3!2^2 \\ \mu'_3 &= 4!2^3 \\ &\vdots \end{aligned}$$

$$\begin{aligned} \therefore \quad M_X(t) = E(e^{tX}) &= 1 + \frac{t}{1!}\mu'_1 + \frac{t^2}{2!}\mu'_2 + \frac{t^3}{3!}\mu'_3 + \cdots + \frac{t^r}{r!}\mu'_r + \ldots \\ &= 1 + \frac{t}{1!}2 + \frac{t^2}{2!}3!2^2 + \frac{t^3}{3!}4!2^3 + \ldots \\ &= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \ldots \\ \therefore \quad M_X(t) &= (1-2t)^{-2}. \end{aligned}$$

Now, Differentiating w.r.to t, we get

$$\begin{aligned} M'_X(t) &= -2(1-2t)^{-3}(-2) \\ M''_X(t) &= 6(1-2t)^{-4}(-2)^2 \\ \text{Now, First Moment=Mean= } E(X) = \mu'_1 &= M'_X(0) = 4 \\ \text{Second Moment= } E(X^2) = \mu'_2 &= M''_X(0) = 24 \\ \therefore \quad Var(X) &= E(X^2) - [E(X)]^2 = 24 - 16 = 8. \end{aligned}$$

Example: 5.20. *A random variable X takes the values -1,0,1 with probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P\{|X - \mu| \geq 2\sigma\}$ and compare it with the upper bound given by Tchebycheff's(Chebychev's) inequality.*

Solution:

$$\begin{aligned}
E(X) &= 0 \\
E(X^2) &= \frac{1}{4} \\
Var(X) &= \frac{1}{4} \\
\therefore \ P\{|X-\mu| \geq 2\sigma\} &= P\{|X| \geq 1\} \\
&= P\{X = -1 \text{ or } X = 1\} = \frac{1}{4} \\
\text{By Chebychev's inequality} & \\
P\{|X-\mu| \geq c\} &\leq \frac{\sigma^2}{c^2} \\
\text{Choosing } c = 2\sigma & \\
P\{|X-\mu| \geq 2\sigma\} &\leq \frac{1}{4}
\end{aligned}$$

In this problem both the values coincide.

Note: Alternative form of Chebychev's inequality: If we put $c = k\sigma$, where $k > 0$, then it takes the form

$$P\left\{\frac{|X-\mu|}{k} \geq \sigma\right\} \leq \frac{1}{k^2}$$

$$P\left\{\frac{|X-\mu|}{k} < \sigma\right\} \geq 1 - \frac{1}{k^2}$$

Example: 5.21. *A fair dice is tossed 720 times. Use Tchebycheff's(Chebychev's) inequality to find a lower bound for the probability of getting 100 to 140 sixes.*

Solution:

$$\begin{aligned}
p &= P\{\text{getting '6' in a single toss}\} = \frac{1}{6} \\
q &= 1 - \frac{1}{6} = \frac{5}{6} \text{ and} \\
n &= 720
\end{aligned}$$

X follows binomial distribution with mean $np = 120$ and variance $npq = 100$. *i.e.* $\mu = 120$ and $\sigma = 100$.

By Alternate form of Chebychev's inequality

$$\begin{aligned} P\{|X-\mu| \leq k\sigma\} &\geq 1-\frac{1}{k^2} \\ P\{|X-120| \leq 10k\} &\geq 1-\frac{1}{k^2} \\ P\{120-10k \leq X \leq 120+10k\} &\geq 1-\frac{1}{k^2} \\ \text{Choosing } k=2, &\quad \text{we get,} \\ P\{100 \leq X \leq 140\} &\geq \frac{3}{4} \end{aligned}$$

$\therefore$ Required lower bound is $\frac{3}{4}$.

---

6 Function of a Random Variable

In the analysis of electrical systems, we will be often interested in finding the properties of a signal after it has been subjected to certain processing operations by the system, such as integration, weighted averaging, etc. These signal processing operations may be viewed as transformations of a set of input variables to a set of output variables. If the input is a set of random variables (RVs), then the output will also be a set of RVs. In this chapter, we deal with techniques for obtaining the probability law (distribution) for the set of output RVs when the probability law for the set of input RVs and the nature of transformation are known.

Finding $f_Y(y)$, when $f_X(x)$ is given or known

Let us now derive a procedure to find $f_Y(y)$, the pdf of Y, when $Y = g(X)$, where X is a continuous RV with pdf $f_X(x)$ and $g(x)$ is a strictly monotonic function of x.

Case (i): $g(x)$ is a strictly increasing function of x.

$$\begin{aligned} f_Y(y) &= P[Y \leq y], \quad \text{where } f_Y(y) \text{ is the cdf of } Y \\ &= P[g(X) \leq y] \\ &= P[X \leq g^{-1}(y)] \\ &= f_X[g^{-1}(y)] \end{aligned}$$

Differentiating both sides with respect to y,

$$f_Y(y) = f_X(x)\frac{dx}{dy}, \quad \text{where } x = g^{-1}(y) \tag{6.1}$$

Case (ii): $g(x)$ is a strictly decreasing function of x.

$$\begin{aligned} F_Y(y) &= P[Y \le y], \text{ where } F_Y(y) \text{ is the cdf of } Y \\ &= P[g(X) \le y] \\ &= P[X \ge g^{-1}(y)] \\ &= 1 - P[X \le g^{-1}(y)] \\ &= 1 - F_X[g^{-1}(y)] \end{aligned}$$

Differentiating both sides with respect to y,

$$f_Y(y) = -f_X(x)\frac{dx}{dy}, \text{ where } x = g^{-1}(y) \tag{6.2}$$

Combining (6.1) and (6.2), we get

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right| \\ f_Y(y) &= f_X(x)\frac{dx}{|g'(x)|} \end{aligned}$$

Note : The above formula for $f_Y(y)$ can be used only when $x = g^{-1}(y)$ is single valued.

When $x = g^{-1}(y)$ takes finitely many values $x_1, x_2, x_3, \ldots, x_n$, *i.e.*, when the roots of the equation $y = g(x)$ are $x_1, x_2, x_3, \ldots, x_n$, the following extended formula should be used to find $f_Y(y)$:

$$\begin{aligned} f_Y(y) &= f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_1)\left|\frac{dx_n}{dy}\right| + \cdots + f_X(x_n)\left|\frac{dx_n}{dy}\right| \quad \text{or} \\ f_Y(y) &= f_X(x_1)\frac{dx_1}{|g'(x_1)|} + f_X(x_2)\frac{dx_2}{|g'(x_2)|} + \cdots + f_X(x_n)\frac{dx_n}{|g'(x_n)|} \end{aligned}$$

Example: 6.1. *Let X be a random variable with pdf $f(x) = \frac{2}{9}(x+1)$, $-1 < x < 2$. Find the pdf of the random variable $Y = X^2$.*

Solution:
Divide the interval into two parts (-1,1) and (1,2).

$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$

When $-1 < x < 1$, $f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y}) + \frac{2}{9}(1-\sqrt{y})\right] = \frac{2}{9\sqrt{y}}$, $0 < y < 1$.

When $1 < x < 2$, $f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y})\right]$, $1 < y < 4$.

Example: 6.2. *Let X be a random variable with pdf $f(x) = \dfrac{x}{12}$, $1 < x < 5$. Find the pdf of the random variable $Y = 2X - 3$.*

Solution:

$f_Y(y) = \left|\dfrac{dx}{dy}\right| f_X(x) = \dfrac{1}{2} \cdot \dfrac{x}{12}$

When $1 < x < 5$, $f_Y(y) = \dfrac{y+3}{48}$, $-1 < y < 7$.

Example: 6.3.
If X is uniformly distributed in $(-\pi/2, \pi/2)$, find the pdf of $Y = \tan X$.

Solution:

Given that $f_X(x)$ is uniformly distributed. $\therefore f_X(x) = \dfrac{1}{\pi/2 - (-\pi/2)} = \dfrac{1}{\pi}$

Now, $f_Y(y) = \left|\dfrac{dx}{dy}\right| f_X(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{(1+y^2)}$

When $x \in (-\pi/2, \pi/2)$, $f_Y(y) = \dfrac{1}{\pi(1+y^2)}$, $-\infty < y < \infty$.

7 Exercise/Practice/Assignment Problems

1. Find the probability of drawing two red balls in succession from a bag containing 3 red and 6 black balls when (i) the ball that is drawn first is replaced, (ii) it is not replaced.

2. A bag contains 3 red and 4 white balls. Two draws are made without replacement; what is the probability that both the balls are red.

3. A random variable X has the following distribution

X	-2	-1	0	1	2	3
P[X=x]	0.1	k	0.2	$2k$	0.3	$3k$

   Find (i) the value of k,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 3/X < 2)$ and (iv) the smallest value of α for which $P(X \leq \alpha) > \dfrac{1}{2}$.

   Ans: $k = \dfrac{1}{15}$

4. A random variable X has the following distribution

X	0	1	2	3	4
P[X=x]	k	$2k$	$5k$	$7k$	$9k$

   Find (i) the value of k,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 3/X < 2)$ and (iv) the smallest value of α for which $P(X \leq \alpha) > \dfrac{1}{3}$.

   Ans: $k = \dfrac{1}{24}$

5. A random variable X has a pdf

$$f(x) = \begin{cases} 3x^2, & \text{when } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   Find the (i) Find a, b such that $P(X \leq a) = P(X > a)$ and $P(X > b) = 0.05$,(ii) the Distribution Function (CDF) (iii) $P(0 < X < 0.5/X < 0.8)$. Ans: $a = \sqrt[3]{\dfrac{1}{2}}$, $b = \sqrt[3]{\dfrac{19}{20}}$

6. The amount of bread (in hundred kgs) that a certain bakery is to sell in a day is a random variable X with a pdf

$$f(x) = \begin{cases} Ax, & \text{when } 0 \leq x < 5 \\ A(10 - x), & \text{when } 5 \leq x < 10 \\ 0, & \text{otherwise} \end{cases}$$

   Find the (i) the value of A,(ii) the Distribution Function (CDF) (iii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$. (iv) the probability that in a day the sales is (a) more than 500 kgs (b) less than 500 kgs (c) between 250 and 750 kgs. Ans: $A = \dfrac{1}{25}$

7. The Cumulative Distribution Function (CDF) of a random variable X is given by

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2}, & \text{when } x > 2 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) the pdf of X,(ii) $P(X > 5/X < 5)$, $P(X > 5/2.5 < X < 7.5)$ (iii) $P(X < 3)$, $P(3 < X < 5)$.

8. A coin is tossed until a head appears. What is the expected value of the number of tosses?. Also find its variance.

9. The pdf of a random variable X is given by

$$f(x) = \begin{cases} a + bx, & \text{when } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the value of a, b if the mean is 1/2,(ii) the variance of X (iii) $P(X > 0.5/X < 0.5)$

10. The first three moments about the origin are 5,26,78. Find the first three moments about the value x=3. Ans: 2,5,-48

11. The first two moments about x=3 are 1 and 8. Find the mean and variance. Ans: 4,7

12. The pdf of a random variable X is given by

$$f(x) = \begin{cases} k(1 - x), & \text{when } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the (i) the the value of k,(ii) the r^{th} moment about origin (iii) mean and variance. Ans: $k = 2$

13. An unbiased coin is tossed three times. If X denotes the number of heads appear, find the MGF of X and hence find the mean and variance.

14. Find the MGF of the distribution whose pdf is $f(x) = ke^{-x}$, $x > 0$ and hence find its mean and variance.

15. The pdf of a random variable X is given by

$$f(x) = \begin{cases} x, & \text{when } 0 \leq x \leq 1 \\ 2 - x, & \text{when } 1 < x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

For this find the MGF and prove that mean and variance cannot be find using this MGF and then find its mean and variance using expectation.

16. The pdf of a random variable X is given by $f(x) = ke^{-|x|}$, $-\infty < x < \infty$ Find the (i) the the value of k,(ii) the r^{th} moment about origin (iii) the MGF and hence mean and variance. Ans: $k = \dfrac{1}{2}$

17. Find the MGF of the RV whose moments are given by $\mu_r' = (2r)!$. Find also its mean and variance.

18. Use Tchebycheff's(Chebychev's) inequality to find how many times a fair coin must be tossed in order to the probability that the ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55.

19. If X denotes the sum of the numbers obtained when 2 dice are thrown, obtain an upper bound for $P\{|X - 7| \geq 4\}$ (Use Chebychev's inequality). Compare with the exact probability.
20. A RV X has mean $\mu = 12$ and variance $\sigma^2 = 9$ and unknown probability distribution. Find $P[6 < X < 8]$.
21. If a RV X is uniformly distributed over $(-\sqrt{3}, \sqrt{3})$, compute $P\left[|X - \mu| \geq \dfrac{3\sigma}{2}\right]$.
22. A RV X is exponentially distributed with parameter 1. Use Chebychev's inequality to show that $P(-1 < X < 3) \geq 3/4$. Find the actual probability to compare.
23. Given a RV X with density function

$$f(x) = \begin{cases} 2x, & \text{when } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

find the pdf of $Y = 8X^3$.
24. If X has an exponential distribution with parameter a, find the pdf of $Y = \log X$.
25. If X has an exponential distribution with parameter 1, find the pdf of $Y = \sqrt{X}$.
26. If X is uniformly distributed in $(1, 2)$, find the pdf of $Y = 1/X$.
27. If X is uniformly distributed in $(-2\pi, 2\pi)$, find the pdf of (i) $Y = X^3$ and (ii) $Y = X^4$.

SOLVE/PRACTICE MORE PROBLEMS FROM THE TEXTBOOK AND REFERENCE BOOKS

VALUES OF $e^{-\lambda}$

λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$e^{-\lambda}$	1.000	.905	.819	.741	.670	.607	.549	.497	.449	.407
λ	1	2	3	4	5	6	7	8	9	10
$e^{-\lambda}$	.368	.135	.0498	.0183	.00674	.00248	.000912	.000335	.000123	.000045

Figure 7.1: Values of $e^{-\lambda}$.

Contact: (+91) 979 111 666 3 (or) athithansingaram@gmail.com
Visit: https://sites.google.com/site/lecturenotesofathithans/home