Maths

Unit-4

Snedecor's F-distribution :-

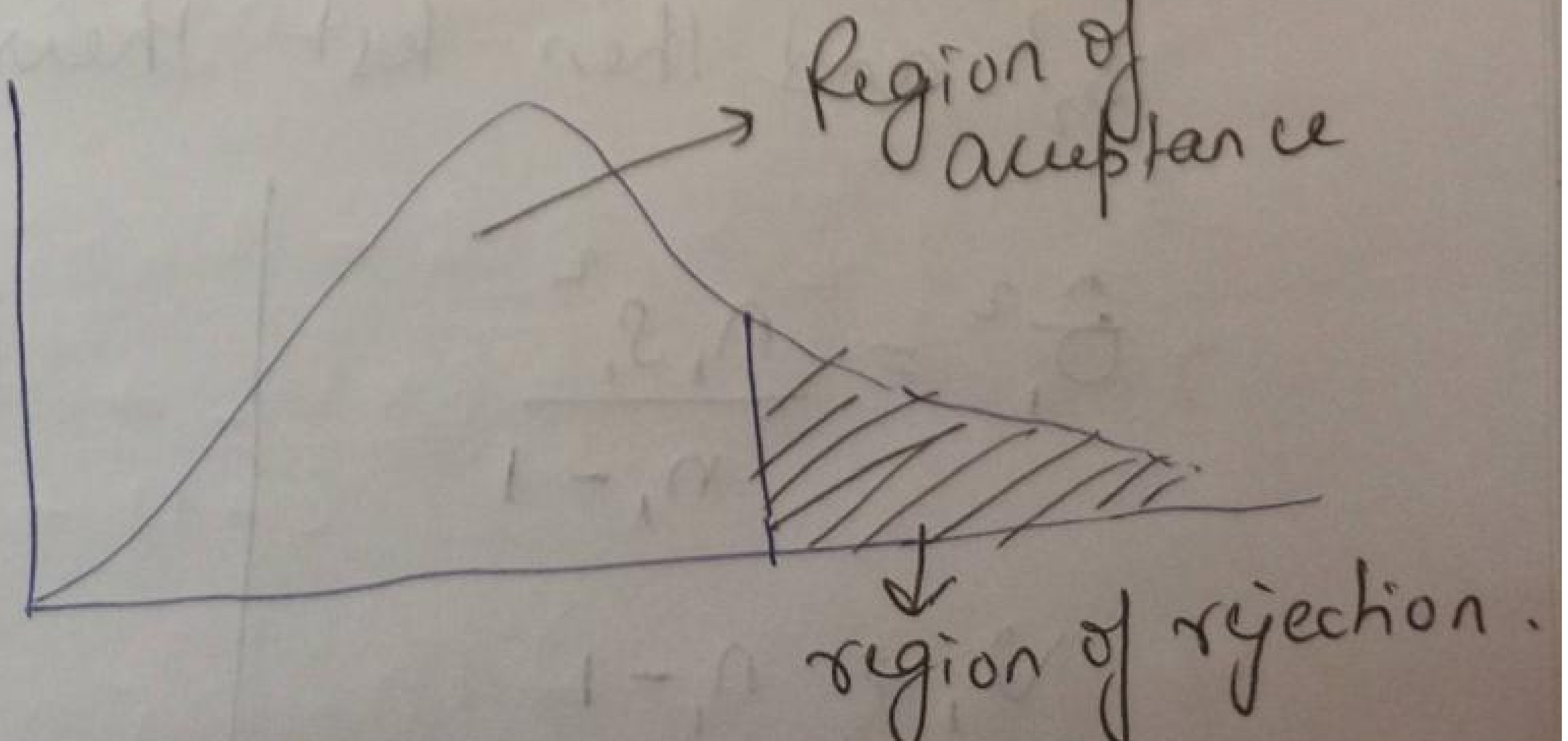
A random variable F is said to follow Snedecor's F-distribution or simply F-distribution if its p.d.f is given by

$$f(F) = \frac{(v_1/v_2)^{v_1/2}}{\beta\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \frac{F^{v_1/2 - 1}}{\left(1+\frac{v_1}{v_2}F\right)^{\left(\frac{v_1+v_2}{2}\right)}}, \quad F>0$$

Note :-
The mathematical random variable corresponding to the random variable F is also taken as F. v_1 and v_2 in $f(F)$ are the degree of freedom associated with F-distribution.

Properties of F-distribution :-

1) The Probability curve of the F-distribution is roughly sketched.



2) The square of the t-variate with n degree of freedom follows a F-distribution with 1 and n degree of freedom.

3) The mean of F distribution is $\frac{v_2}{v_2 - 2}$ $(v_2 > 2)$

4) The variance of F-distribution is

$$\frac{2v_2^2 (v_1 + v_2 - 2)}{v_1 (v_2 - 2)^2 (v_2 - 4)} \quad (v_2 > 4)$$

Use of F-distribution :-

F-distribution is used to test the quality of the variance of the population from which two small sample have been drawn.

F-test of significance of difference between Population variance and F-table :-

To ~~find~~ test the significance of difference b/w Population variance, we shall first find their estimated $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on sample variance S_1^2 and S_2^2 and then test their quality.

$$\hat{\sigma}_1^2 = \frac{n_1 S_1^2}{n_1 - 1}$$

$$v_1 = n_1 - 1$$

$$\hat{\sigma}_2^2 = \frac{n_2 S_2^2}{n_2 - 1}$$

$$v_2 = n_2 - 1$$

$$\boxed{F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}} \quad \text{with } \nu_1 = n_1 - 1 \text{ , } \nu_2 = n_2 - 1$$

→ when $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$

$$\boxed{F = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2}} \quad \text{with } \nu_1 = n_2 - 1 \text{ , } \nu_2 = n_1 - 1$$

→ when $\hat{\sigma}_2^2 > \hat{\sigma}_1^2$

Note :- 1) F test is not a two tailed test and is always a right tailed test as F can not be negative. i.e. $F > 0$.

2) We should always make $F > 1$ by taking larger of $\hat{\sigma}_2^2$, $\hat{\sigma}_1^2$ as numerator.

Q Two random sample give following data.

Sample No	Size	Mean	Variance
1	8	9.6	1.2
2	11	16.5	2.5

Can we conclude that the two sample have been drawn from same normal population.

Sol → Here, $n_1 = 8$, $\bar{x}_1 = 9.6$, $S_1^2 = 1.2$

$n_2 = 11$, $\bar{x}_2 = 16.5$, $S_2^2 = 2.5$

$$\hat{\sigma}_1^2 = \frac{n_1 S_1^2}{n_1 - 1} = \frac{8 \times 1.2}{8 - 1} = \frac{9.6}{7} = 1.37$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{11 \times 2.5}{11-1} = \frac{27.5}{10} = 2.75$$

$$F = \frac{\sigma_2^2}{\sigma_1^2} \text{ with } v_1 = n_2 - 1, \quad v_2 = n_1 - 1$$

$$\Rightarrow F = \frac{2.75}{1.37} \quad (v_1 = 10, \; v_2 = 7)$$

$$\Rightarrow F = 2.007 \quad \text{with } (v_1 = 10, \; v_2 = 7)$$

Let LOS be 5%

Now, $F_{0.05}(v_1 = 10, v_2 = 7) = 3.64$.

$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2$

$H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2$

$\because F < F_{0.05}(10, 7)$

$\Rightarrow H_0$ is accepted.

So, variance of both sample is same.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$t = \frac{9.6 - 16.5}{\sqrt{\frac{8 \times 1.2 + 11 \times 2.5}{8 + 11 - 2}\left(\frac{1}{8} + \frac{1}{11}\right)}}$$

$$t = -10.05$$

$$v = n_1 + n_2 - 2 = 8 + 11 - 2$$
$$= 17$$

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$t_{0.05}(v=17) = 2.11$

$H_0 : \bar{x}_1 = \bar{x}_2$

$H_1 : \bar{x}_1 \neq \bar{x}_2$

$|t| > t_{0.05}(v=17)$.

$\Rightarrow$ H_0 is rejected & difference is significant

$\Rightarrow$ It is not from same population.

Chi square distribution (χ^2).

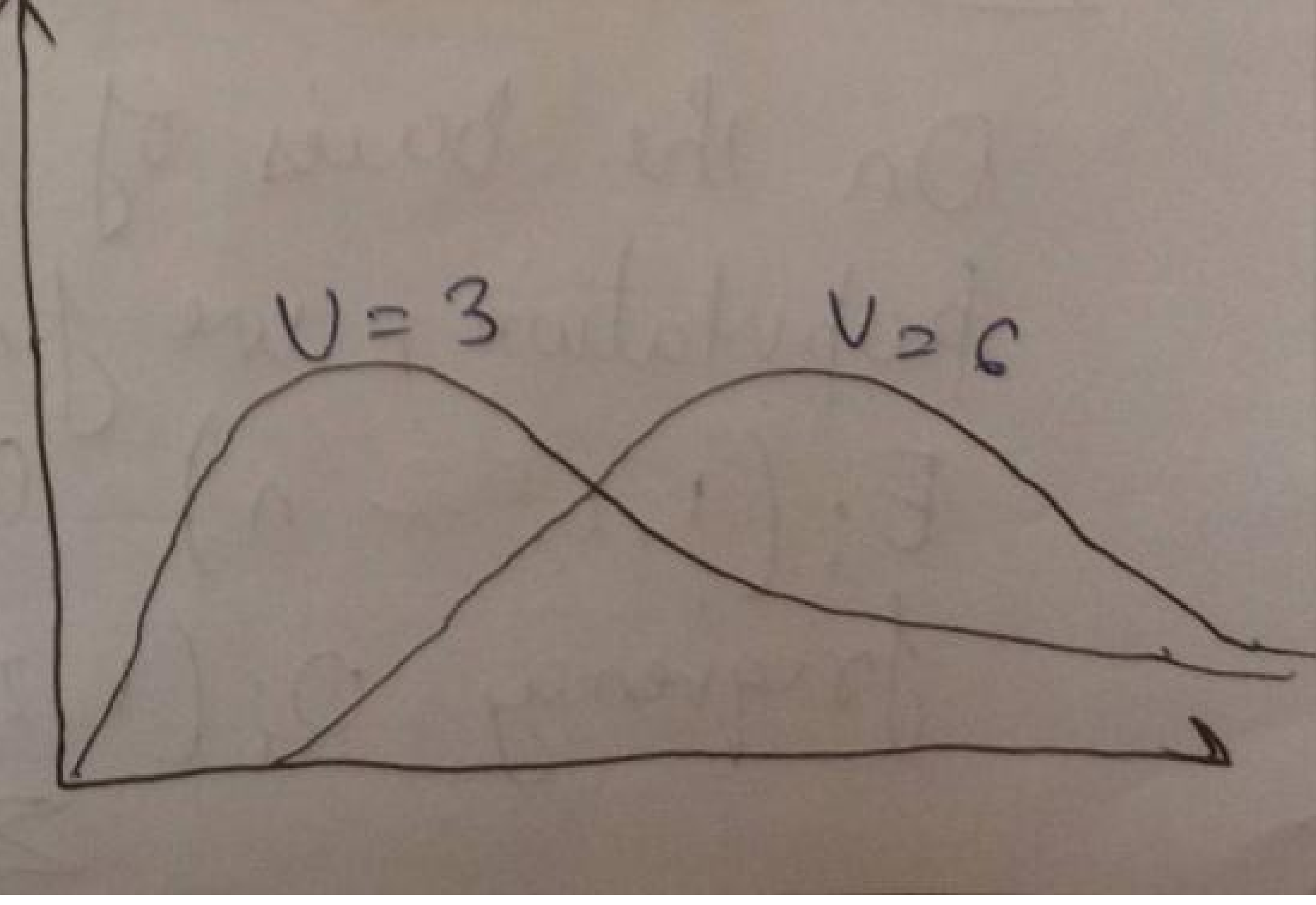
If $x_1, x_2 - - - x_n$ are normally distributed Independent random variable, then it is known that $(x_1^2 + x_2^2 + - - x_n^2)$ follows probability distribution called chi square distribution with n degree of freedom. The pdf of chi-square distribution is given by

$$f(\chi^2) = \frac{1}{2^{v/2}\sqrt{\frac{v}{2}}}(\chi^2)^{v/2-1} e^{-\chi^2/2}$$

$$0 < \chi^2 < \infty$$

Properties :-

1) A rough sketch of the probability ~~cu~~ curve of the χ^2 distribution with $v=3$ & $v=6$ is given in graph.



2) As ν become smaller and smaller the curve is skewed more and more to the right.
As ν increases, the curve become more and more symmetric.

3) The mean and variance of the χ^2 distribution are ν & 2ν respectively.

4) As ν tends to ∞, the χ^2 distribution becomes normal distribution.

χ^2 test

Goodness of fit
(Judge whether a given sample may be reasonable regarded as a sample from a certain hypothetical population)

Chi-square test of attributes
(test the independence of attributes).

χ^2 test of goodness of fit :-

On the basis of hypothesis assumed about the population, we find the expected frequencies $E_i (1, 2, \ldots n)$ corresponding to observed frequency $O_i (1, 2, \ldots n)$ such that

$$\Sigma O_i = \Sigma E_i$$

Then,

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$ with degree of freedom equal to no. of independent frequency.

* The Critical value of χ^2 for ν degree of freedom at α LOS denoted by $\chi^2_\nu(\alpha)$ and available in χ^2-table.

* If calculated $\chi^2 < \chi^2_\nu(\alpha)$ then H_0 is accepted else H_0 is rejected

Q. The following data gives the number of aircraft accident that occured during the ~~variance~~ various days of week.

Day	Mon	Tues	Wed	Thus	fri	Sat
No. of accident	15	19	13	12	16	15

Test whether the accident are uniformly distributed over the week.

Sol → H_0: accident is uniformly distributed over week.

Total no. of accident = 90

Expected No. of accident per day $= \frac{90}{6} = 15$

O_i	15	19	13	12	16	15
E_i	15	15	15	15	15	15

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1}{15}(0+16+4+9+1+0)$$

$$= \frac{30}{15} = 2$$

$v = 6 - 1$ → for $\sum E_i = \sum O_i$

$\chi_{0.05}(v=5) = 11.07$

$\Rightarrow \chi^2 < \chi^2 (v=5)$

So, H_0 is accepted

$\Rightarrow$ Accident are uniformly distributed over the week.

χ^2 test for Independent of attribute :-

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \quad \text{when } \sum_{i=1}^{n} O_i = \sum_{i=1}^{n} E_i$$

B →

A ↓

	B_1	B_2	Total
A_1	a	b	a+b
A_2	c	d	c+d
Total →	a+c	b+d	a+b+c+d = N

$$E[a] = \frac{(a+b)(a+c)}{N}$$

$$E[b] = \frac{(a+b)(b+d)}{N}$$

$$E[c] = \frac{(a+c)(c+d)}{N}$$

$$E[d] = \frac{(b+d)(c+d)}{N}$$

$$\nu = (r-1)(s-1)$$ $r \to$ No. of rows
$s \to$ No. of column

Not :- $\chi^2 < \chi^2_\nu(\alpha) \Rightarrow H_0$ is accepted & A & B are independent

else, H_0 is rejected & A & B are not independent.

Q. The following data is collected on two constraint. Based on this, can you say that there is no relation between smoking and literacy.

	Smokers	Non-smokers
Literates	83	57
Non-Literates	45	68

Sol → H_0 : Literacy & Smoking are independent

	Smokers	Non-smokers	Total
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

O	E	E [rounded]	$\frac{(O-E)^2}{E}$
83	$\frac{128 \times 140}{253} = 70.83$	71	$\frac{12^2}{71} = 2.03$
57	$\frac{140 \times 125}{253} = 69.17$	69	$\frac{12^2}{69} = 2.09$
45	$\frac{113 \times 128}{253} = 57.17$	57	$\frac{12^2}{57} = 2.53$
68	$\frac{113 \times 125}{253} = 55.83$	56	$\frac{12^2}{56} = 2.57$
253		253	$\chi^2 = 9.22$

$$v = (r-1)(s-1) = (2-1)(2-1) = 1$$

$\Rightarrow \chi^2_{0.05}(v=1) = 3.84.$

$\Rightarrow \chi > \chi_{0.05}(v=1)$

$\Rightarrow H_0$ is rejected

So, There is some relation between Smoker & Literacy.

Q. From the following table test the hypothesis that flower color is independent of flatness of leaf.

	flat leaves	Curved leaves	Total
white flower	99	36	135
Red flower	20	5	25
	119	41	160

Sol → H_0 : Flower color is independent of flatness.

$$E[99] = \frac{135 \times 119}{160} = 100.406$$

similarly

$E[36] = 34.594$, $E[20] = 18.594$

$E[5] = 6.406$

O_i	E_i [Rounded]	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E}$
99	100	1	0.01
36	35	1	0.0285
20	19	1	0.0588
5	6	1	0.166
160	160		$\chi^2 = 0.2633$

$\nu = 1$.

$\chi^2_{0.05}(\nu = 1) = 3.84$

$\Rightarrow \chi^2 < \chi^2_{0.05}(\nu = 1) \Rightarrow H_0$ is accepted

So, there is no relation b/w color & flatness of flower.

Queueing Theory

Input Pattern :-

We will deal with only those Queueing system in which the no. of arrivals per unit time has a poisson distribution with mean λ.

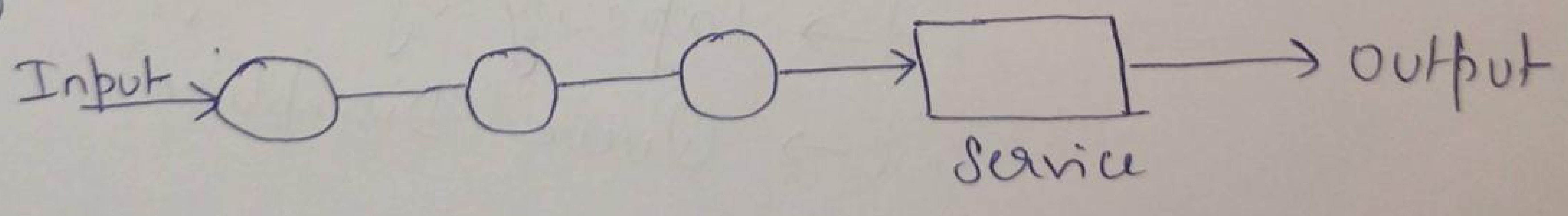
The time between consecutive arrivals has an exponential distribution with mean $\frac{1}{\lambda}$.

Service Pattern :-

We will deal with only those Queueing system in which the no. of customers serviced per unit time has a Poisson distribution with mean μ. or equivalently the inter-service time has an exponential distribution with mean $\frac{1}{\mu}$.

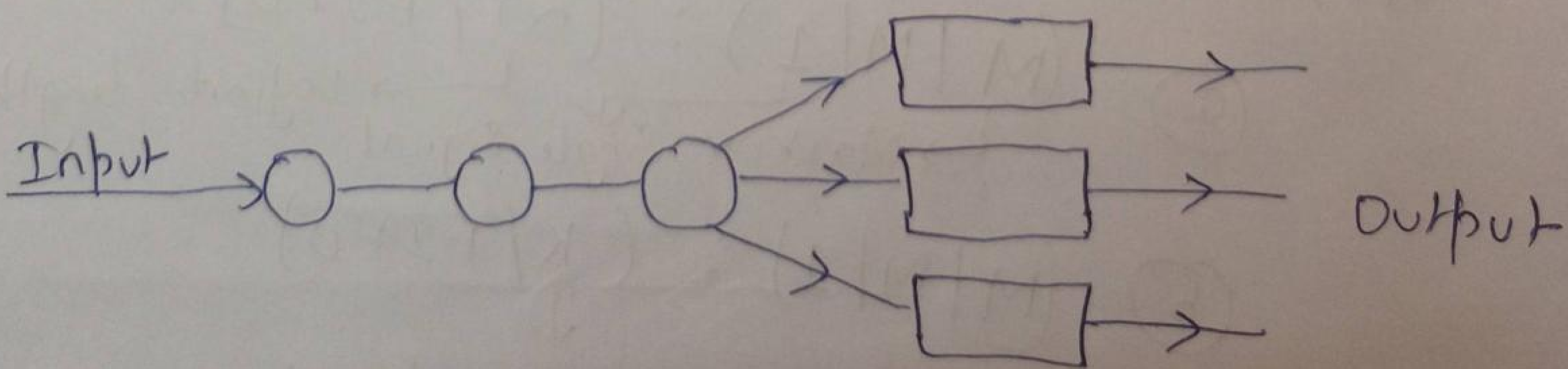
Types of Queueing System :-

(1)



Single Server Queueing System

(2)



Multiple Server Queueing System.

Queue discipline :-

1) FCFS | FIFO → First Come First Served / First In First Out

2) LCFS | LIFO → Last Come First Served / Last In First Out.

3) SIRO → Service in Random Order.

Symbolic Representation of Queueing Model :-

$(a/b/c) : (d/e)$

where, a → Type of distribution of number of arrival per unit time

b → Type of distribution of Service time

c → No. of Service

d → Capacity of the System

e → Queue Discipline.

* We have to study 2 Model in our Syllabus.

(a) $(M/M/1) : (\infty / FIFO)$

→ Markov Single Signal Infinite length

(b) $(M/M/1) : (K/FIFO)$.

finite lenght

Model I :- (M/M/1) : (∞/FIFO)

1. Average No. of Customer in the system

$$L_s = E[N] = \frac{\lambda}{\mu - \lambda}$$

2. Average No. of Customer in the Queue :-

$$L_q = E[N-1] = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

3. Average No. of customer in an Non-Empty Queue :-

$$L_w = \frac{\mu}{\mu - \lambda}.$$

4. Probability that the number of customers in the system exceeds k.

$$P(N > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

5. Probability Density function (pdf) of waiting time in the system :-

$$f(w) = (\mu - \lambda) e^{-(\mu - \lambda) w}$$

6. Average waiting time of customer in system

$$E[W_s] = \frac{1}{\mu - \lambda}$$

7. Probability that the waiting time of a customer in the system exceeds t.

$$P(W_s > t) = e^{-(\mu - \lambda)t}$$

8. Probability density function (pdf) of waiting time in the queue :-

$$g(w) = \begin{cases} \frac{\lambda}{\mu}(\mu - \lambda)e^{-(\mu-\lambda)w} & \text{for } w > 0 \\ \frac{\lambda}{\mu}(\mu - \lambda) & \text{for } w = 0 \end{cases}$$

9. Average waiting time of a customer in the queue :-

$$E[W_q] = \frac{\lambda}{\mu(\mu - \lambda)}$$

10. Average waiting time of a customer in the queue if ~~is~~ he has to wait :-

$$E[W_q / W_q > 0] = \frac{1}{\mu - \lambda}$$

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Relation Among $E[N_s]$, $E[N_q]$, $E[W_s]$ & $E[W_q]$

1) $E[N_s] = \frac{\lambda}{\mu - \lambda} = \lambda E[W_s]$.

2) $E[N_q] = \frac{\lambda^2}{\mu(\mu - \lambda)} = \lambda \cdot E[W_q]$.

3) $E[W_s] = E[W_q] + \frac{1}{\mu}$.

4) $E[N_s] = E[N_q] + \frac{\lambda}{\mu}$.

Q. Arrival of a telephone booth are considered to be Poisson with an average time of 12 min between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.

(a) Find the average number of person ~~arriving at~~ waiting in the system.

(b) what is the probability that a person arriving at booth have to wait in a queue?

(c) what is the probability that it will take more than 10 min altogether to wait for phone and complete his call?

(d) Estimate the fraction of the day when the phone will be in use.

(e) The telephone department will install a second booth, when convinced that an arrival has to wait on the average for atleast 3 min for phone. By how much the flow of arrival should increase in order to justify the second booth.

(f) What is average length of queue that forms from time to time.

Sol → Mean inter-arrival time = 12 min

$$\Rightarrow \frac{1}{\lambda} = 12 \text{ min}$$

$$\Rightarrow \lambda = \frac{1}{12} \text{ per minute}$$

arrival rate

Mean service time = 4 min

$$\Rightarrow \frac{1}{\mu} = 4 \text{ min}$$

$$\Rightarrow \mu = \frac{1}{4}/\text{min}$$

service rate

(a) Average No. of person waiting in ~~Queue~~ System

$$= E[N_S] = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} \Rightarrow \frac{\frac{1}{12}}{\frac{2}{12}} = \frac{1}{2} = 0.5$$

(b) Probability of a person arriving has to wait :-

$$\Rightarrow P(w>0) = 1 - P(w=0)$$

$$= 1 - P(\text{no customer in system})$$

$$\boxed{P_0 = 1 - \frac{\lambda}{\mu}}$$

$$= 1 - P_0$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right).$$

$$= \frac{\lambda}{\mu} = \frac{1/12}{1/4} = \frac{1}{3} \text{ Ans}$$

(c) Probability of person waiting atleast 10 min.

$$\Rightarrow P(w>10) = e^{-(\mu-\lambda)\times 10}$$

$$= e^{-\left(\frac{1}{4}-\frac{1}{12}\right)\times 10}$$

$$= e^{-\frac{2}{12}\times 10}$$

$$= e^{-\frac{5}{3}} = 0.1889 \text{ Ans}$$

(d) Probabity of phone is in use

$$= 1 - P(\text{not in use})$$

$$= 1 - P_0$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \frac{\lambda}{\mu} = \frac{1/12}{1/4} = \frac{1}{3} \text{ Ans}$$

(e) Second phone will be installed if $E[W_q] > 3$.

$$\Rightarrow \frac{\lambda}{\mu(\mu-\lambda)} > 3$$

- Let λ_B be required arrival rate to justify second booth

$$\Rightarrow \frac{\lambda_B}{\mu(\mu-\lambda_B)} > 3$$

$$\Rightarrow \frac{\lambda_B}{\frac{1}{4}\left(\frac{1}{4}-\lambda_B\right)} > 3.$$

$$\Rightarrow \lambda_B > \frac{3}{4}\left(\frac{1}{4}-\lambda_B\right)$$

$$\Rightarrow \lambda_B + \frac{3}{4}\lambda_B > \frac{3}{16}$$

$$\Rightarrow \frac{7\lambda_B}{4} > \frac{3}{16}$$

$$\Rightarrow \lambda_B > \frac{12}{16 \times 7}$$

$$\Rightarrow \lambda_B > \frac{3}{28}.$$

$\therefore$ arrival rate should be increase by $\frac{3}{28} - \frac{1}{12}$ to justify second phone

$$\Rightarrow \frac{3}{28} - \frac{1}{12} = \frac{1}{42} \text{ per } \underline{\text{min}}.$$

(f) Average lenght of queue:

$E[N_q \mid \text{queue is always there}]$

$= E[N_q \mid N_q > 0)$

$= E[N_q / N > 1)$

$$= \frac{E[N_q]}{P(N>1)} = \frac{E[N_q]}{1 - P_0 - P_1}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \times \frac{1}{1 - \left(1+\frac{\lambda}{\mu}\right)P_0}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \times \frac{1}{1 - \left(1+\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{\mu^2}{\lambda^2}$$

$$= \frac{\mu}{\mu-\lambda} = \frac{1/4}{1/4 - \frac{1}{12}} = 1.5$$ Ans

Q. Customer arrive at a one-man barber shop according to a Poisson distribution with mean interarrival time of 12 min. Customer spend an average of 10 min in barber's chair.

(a) What is expected number of customer in barber shop and in the queue?

(b) Calculate the percentage of time an arrival can walk straight into bair chair without having to wait.

(c) How much time can a customer expect to ~~wait~~ spend in barber shop.

(d) Management will provide another chair and hire another barber when customer waiting time in the shop exceeds 1.25 h. How much must the average rate of arrival increase to warrant a second barber.

(e) What is average time customer spend in the queue?

(f) What is the probability that waiting time in system exceeds 30 min.

(g) Calculate the percentage of customer who have to wait prior to getting ~~et~~ into barber chair.

(h) What is probability that more than 3 customer are in the system.

Sol $\longrightarrow$ Mean interarrival time = 12 min

$$\Rightarrow \frac{1}{\lambda} = 12 \text{ min} \Rightarrow \lambda = \frac{1}{12} \text{ per min.}$$

Mean service time $\Rightarrow \frac{1}{\mu} = 10$ min

$$\Rightarrow \mu = \frac{1}{10} \text{ per min.}$$

(a) $E[N_s] = \frac{\lambda}{\mu - \lambda}$

$= \frac{1/12}{\frac{1}{10} - \frac{1}{12}} \Rightarrow \frac{\frac{1}{12}}{\frac{2}{120}} \Rightarrow 5$

$E[N_q] = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\left(\frac{1}{12}\right)^2}{\frac{1}{10}\left(\frac{1}{10} - \frac{1}{12}\right)}$

$= \frac{1}{144} \times \frac{120 \times 10}{2}$

$= 4.17$ customers.

(b) P(customer walk straight to chair)

$= P(\text{No customer}) = P(N=0)$

$= P_0 = 1 - \frac{\lambda}{\mu}$

$= 1 - \frac{1/12}{1/10} \Rightarrow 1 - \frac{10}{12}$

$= \frac{2}{12} = \frac{1}{6}$ Ans

$\therefore$ percentage of time arrival doesn't need to wait $= \frac{1}{6} \times 100\% = 16.7\%$ Ans

(c) time customer has to spend

$\Rightarrow E[W] = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{10} - \frac{1}{12}}$

$= 60$ min

or

1 hr Ans

(d) Second chair will be installed if $E(W) > 1.25 \text{ hr}$

$$\Rightarrow E[W] > 75 \text{ min}$$

$$\Rightarrow \frac{1}{\mu - \lambda_r} > 75 \text{ min}$$

$$\Rightarrow \frac{1}{75} > \mu - \lambda_r$$

$$\Rightarrow \frac{1}{75} - \mu > -\lambda_r$$

$$\Rightarrow \lambda_r > \mu - \frac{1}{75}$$

$$\Rightarrow \lambda_r > \frac{1}{10} - \frac{1}{75}$$

$$\Rightarrow \lambda_r > \frac{13}{150}$$

$\therefore$ to warrant second barber, average arrival rate must increase by $\frac{13}{150} - \frac{1}{10} = \frac{1}{300}$ per minute

(e) Average time customer has to spend in Queue

$$E[W_q] = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{\frac{1}{12}}{\frac{1}{10}\left(\frac{1}{10} - \frac{1}{12}\right)}$$

$$= \frac{1}{12} \times \frac{10 \times 120}{2}$$

$$= 50 \text{ min}$$

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(f) P(waiting time exceeds 30)

$$= P(W>30) = e^{-(\mu-\lambda)30}$$

$$= e^{-\left(\frac{1}{10}-\frac{1}{12}\right)\times 30}$$

$$= e^{-\frac{2}{120}\times 30}$$

$$= e^{-0.5}$$

$$= 0.6065 \text{ Ans}$$

(g) P(a customer has to wait) = $P(W>0)$

$$= 1 - P(W=0)$$

$$= 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \frac{\lambda}{\mu} = \frac{1/12}{1/10}$$

$$= \frac{5}{6} \text{ Ans}$$

(h) $P(N>3) = \left(\frac{\lambda}{\mu}\right)^{3+1}$

$$= \left(\frac{\frac{1}{12}}{\frac{1}{10}}\right)^4 = \left(\frac{5}{6}\right)^4$$

$$= 0.4823 \text{ Ans}$$

Q. At what average rat must a clerk in a supermarket work in order to ensure a probability of 0.90 that a customer will not want longer than 12 min? It is assummed that there is only one counter at which customer arrive in a poisson fashion at an average rat of 15 per hr and that the lenght of service by clerk has an exponential distribution.

Sol → Here, $\lambda = 15$ per hr.

Let $\mu = \mu_B$ / hr.

P (customer not wait longer than 12 min i.e $\frac{1}{5}$ hr)

$= P\left(W_q \le \frac{1}{5}\right) = 0.90$

$\Rightarrow P\left(W_q > \frac{1}{5}\right) = 1 - 0.90$

$\Rightarrow P\left(W_q > \frac{1}{5}\right) = 0.10$

$\Rightarrow \int_{0.2}^{\infty} g(w)\,dw = 0.10$

$\Rightarrow \int_{0.2}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)w}\,dw = 0.10$

$= \frac{\lambda}{\mu}(\mu-\lambda) \times \left[\frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)}\right]_{0.2}^{\infty} = 0.10$

$$= \frac{\lambda}{\mu}\left[-e^{-(\mu-\lambda)\omega}\right]_{0.2}^{\infty} = 0.10$$

$$= \frac{\lambda}{\mu}\left[-e^{-\infty} + e^{-(\mu_B - 15)\times 0.2}\right] = 0.10$$

$$= \frac{15}{\mu_B}\left[0 + e^{-(\mu_B - 15)\times 0.2}\right] = 0.10$$

$$= e^{-(\mu_B - 15)\times 0.2} = \frac{0.10}{15} \times \mu_B$$

On taking log both the side.

$$-[\mu_B - 15]\times 0.2 = \log(0.10) + \log(\mu_B) - \log 15$$

$$= 0.2\times 15 - 0.2\mu_B = \log 1 + \log \mu_B - \log 150$$

$$= \log \mu_B + 0.2\mu_B = 3 - \log 1 + \log 150$$

$$= 0.2\mu_B + \log \mu_B = 8$$

On solving. $\mu_B = 24$.

$\therefore$ clerk must serve at rat of 24 customer per hour.

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Q. If people arrive to purchase cinema ticket at an average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 min before the picture starts and if it takes exactly 1.5 min to reach the correct seat after purchasing the ticket.

(a) Can he expect to be seated for the start of picture

(b) What is the probability that he will be seated for the start of the picture?

(c) How early he must arrive in order to be 99% sure of being seated for the start of picture?

Sol → Average arrival rate ⇒ $\lambda = 6$ per min.

Average service rate ⇒ $\mu = 7.5 + (2-1.5)$

$\mu = 8$ per minut

(a) Average waiting time to get the ticket

$$E[W] = \frac{1}{\mu-\lambda} = \frac{1}{8-6} = \frac{1}{2}.$$

E[total time to buy ticket & go to seat]

$$= \frac{1}{2} + 1.5 = 2 \text{ min}$$

∴ He can just be seated for start of movie.

(b) P(he will be seated at start of picture)

$= P(\text{total time} < 2 \text{ min})$

$= P(\text{Waiting time} + 1.5 < 2 \text{ min})$

$= P(W < 2 - 1.5) = P(W < 0.5)$

$= P(W < \frac{1}{2})$.

$= 1 - P(W > \frac{1}{2})$

$= 1 - e^{-\mu\left(1 - \frac{\lambda}{\mu}\right) \times \frac{1}{2}}$

$= 1 - e^{-8\left(1 - \frac{6}{8}\right) \times \frac{1}{2}}$

$= 1 - e^{-8 \times \frac{2}{8} \times \frac{1}{2}}$

$= 1 - e^{-1} = 0.63$ Ans

(c) $P(W < t) = 0.99$

$\Rightarrow P(W > t) = 1 - 0.99$

$\Rightarrow P(W > t) = 0.01$

$\Rightarrow e^{-\cancel{t}(\mu - \lambda)} = 0.01$

$\Rightarrow e^{-t(8-6)} = 0.01$

$= e^{-2t} = 0.01$

$-2t = \log 0.01$

$t = -\frac{\log 0.01}{2} = -\frac{-4.6}{2} = -2.3$

$\therefore t = 2.3$ min

i.e. P (ticket purchasing time < 2.3) = 0.99

$\therefore$ P (total time to get ticket & go to seat < 2.3 + 1.5) = 0.99

$\therefore$ He must arrive atleast 3.8 min early to be 99% sure of seeing start of picture.

Q A duplicating machine manufactured for office use is operated by an office assistant who earns Rs 5 per hour. The time to complete each Job varies according to an exponential distribution with mean 6 min. Assume a poisson input with an average rate of 5 Jobs per hour. If an 8-h day is used as a base determine.

(a) the percentage idle time of machine

(b) the average time a job is in system

(c) the average earning per day of the assistant.

Sol $\rightarrow$ Here, $\lambda = 5/hr.$

$$\frac{1}{\mu} = 6 \text{ min} = \frac{1}{\mu} = \frac{6}{60} \text{ hr.}$$

$$\Rightarrow \mu = 10/hr.$$

(a) $P(\text{machine idle}) = P(N=0) = P_0$

$$= \left(1 - \frac{\lambda}{\mu}\right) = 1 - \frac{5}{10} = \frac{1}{2}$$

% time it is idle $= \frac{1}{2} \times 100 = 50\%$ Ans

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(b) Average time a job in system

$= E[W] = \frac{1}{\mu - \lambda}$

$= \frac{1}{10-5} = \frac{1}{5} = 0.2$ hr

$= 12$ min

(c) Average Earning per day

$= E$[No. of Job/day] $\times$ earning per Job.

$= E$[No. of Job/day] $\times$ E[time in hour/Job] $\times$ earning/hr.

$= (8 \times 5) \times \frac{1}{5} \times$ Rs 5

$=$ Rs 40 Ans

Module III : (M/M/1) : (K/FIFO)

(1) Value of P_0 & P_n :-

$$P_0 = \begin{cases} \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{if } \lambda \neq \mu \\[2ex] \dfrac{1}{K+1} & \text{if } \lambda = \mu \end{cases}$$

[We can get this by finding $\lim_{\lambda/\mu \to 1} P_0$ (at $\lambda \neq \mu$).]

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$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right] & \text{if } \lambda \neq \mu \\ \frac{\lambda}{K+1} & \text{if } \mu = \lambda . \end{cases}$$

(2) Average no. of customer in the system :-

$$E[N] = \begin{cases} \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{if } \lambda \neq \mu \quad \text{--- (i)} \\ \frac{K}{2} & \text{if } \lambda = \mu . \end{cases}$$

(3) Average No. of customer in Queue :-

$$E[N_q] = E[N] - \frac{\lambda}{\mu}$$

or

$$E[N] - (1 - P_0)$$

which is true when the average arrival rate is λ throughout. Now, in eq. (i), $1 - P_0 \neq \frac{\lambda}{\mu}$ because the average arrival rate is λ as long as there is a vacancy in queue and it is zero when system is full.

Hence, we define the overall effective arrival rate denoted by λ' or λ_{eff} such that

$$\frac{\lambda'}{\mu} = 1 - P_0$$

$$\Rightarrow \lambda' = \mu(1 - P_0)$$

$$E[N_q] = E[N] - \frac{\lambda'}{\mu}.$$

(4) Average waiting time in System & in the Queue :-

$$E[W_s] = \frac{1}{\lambda'} \cdot E[N]$$

$$E[W_q] = \frac{1}{\lambda'} E[N_q].$$

Q. Patients arrive at a clinic according to poisson distribution at a rate of 30 patient per ~~floor~~ hour. The waiting room does not accomodate more than 14 patient. Examination time per patient is exponential with mean rate of 20 per hour

(a) Find effective arrival rate at the clinic

(b) What is probability that arriving patient will not wait?

(c) What is expected waiting time until a patient is discharged from the clinic.

Sol → Mean arrival rat $\lambda = 30$ per hour

Mean service rat $\mu = 20$ per hour

$k = 14 + 1 = 15$

(a) Effective arrival rate

$$\lambda' = \mu(1 - P_0)$$

$$\because \lambda \neq \mu \Rightarrow P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{1 - \frac{30}{20}}{1 - \left(\frac{30}{20}\right)^{k+1}}$$

$$= 0.00076$$

$$\lambda' = \mu(1 - P_0) = 20(1 - 0.00076)$$

$$= 19.98 \text{ per hr. Ans}$$

(b) P(Patient do not wait) = $P(N=0) = P_0$

$= 0.00076$ Ans

(c) $E[W_s] = \frac{1}{\lambda'} \cdot E[N]$

$$E[N] = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\cdot\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

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$$\Rightarrow E[N] = -3 - \frac{(15+1)\cdot\left(\frac{30}{20}\right)^{15+1}}{1-\left(\frac{30}{20}\right)^{15+1}}$$

$$= -3 - \frac{16\cdot\left(\frac{3}{2}\right)^{16}}{1-\left(\frac{3}{2}\right)^{16}}$$

$= 13$ patient approx

$$E[W_s] = \frac{1}{\lambda'} E[N]$$

$$= \frac{1}{19.98} \times 13 = 0.65 \text{ hr.}$$

or

$= 39$ min Ans

Q. The local one-person barber shop can accomodate a maximum of 5 person at a time (4 waiting & 1 getting hair cut). Customer arrive according to a poisson distribution with mean 5 per hr. The barber cut hair at an average rat of 4 per hour (Exponential service time).

(a) What is percentage of time is the barber idle?

(b) What fraction of the potential customers are turned away?

(c) What is expected number of customer waiting for hair cut?

(d) How much time can a customer expect to spend in barber shop?

Sol → Here. $\lambda = 5$
$\mu = 4$
$k = 5$

(a) P(barber is idle) = $P(N=0) = P_0$

$$\Rightarrow P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$\Rightarrow P_0 = \frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^6} = 0.0888$$

% of idle time = $0.8888 \times 100 = 9\%$ approx.

(b) P(a customer turned away)

$= P(N>5) = P(N=5) = P_5$

$$P_5 = \left(\frac{\lambda}{\mu}\right)^5 \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^6}\right]$$

$$= \left(\frac{5}{4}\right)^5 \left[\frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^6}\right]$$

$= 0.2711$

(c) $E[N_q] = E[N] - (1 - P_0)$

$$= \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} - (1 - P_0)$$

$$= \frac{5}{4-5} - \frac{6 \times \left(\frac{5}{4}\right)^6}{1 - \left(\frac{5}{4}\right)^6} - 1 + 0.0888$$

$$= (-5 + 8.1317) - 1 + 0.0888$$

$= 2.2$ Customer.

(d) $E[W] = \frac{1}{\lambda'} \cdot E[N]$

$$= \frac{1}{\mu(1 - P_0)} \cdot 3.1317$$

$$= \frac{1}{4(1 - 0.0888)} \cdot 3.1317$$

$= 0.8592$ hr

or

$= 51.5$ min Ans

Q. In a single server queueing system with poisson input and exponential service time, if mean arrival rate is 3 calling unit per hr. the expected service ~~xo~~ time is 0.25 hr. and maximum possible number of calling units in system is 2, find $P_n (n \geq 0)$, average no. of calling unit in system and in the queue. and average waiting time in the system and in the queue.

Sol → $\lambda = 3$

$\frac{1}{\lambda} = \frac{1}{4} \Rightarrow \lambda = 4.$

$K = 2$

P_0 :- $P_0 = \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}$

$= \dfrac{1 - \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^{3}} = 0.4324.$ Ans

$P_n = \left(\frac{\lambda}{\mu}\right)^n \left[\dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}\right]$

$= \left(\frac{\lambda}{\mu}\right)^n \cdot P_0$

$= \left(\frac{3}{4}\right)^n \cdot 0.4324$

$= 0.4324 \times (0.75)^n$ Ans

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$$E[N] = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{3}{4-3} - \frac{3\cdot\left(\frac{3}{4}\right)^{3}}{1-\left(\frac{3}{4}\right)^{3}}$$

$$= 3 - \frac{81}{37} = 0.8 \text{ approx}$$ Ans

$$E[N_q] = E[N] - (1-P_0)$$

$$= 0.8 - (1-0.4324)$$

$$= 0.24 \text{ (approx)}.$$ Ans

$$E[W_s] = \frac{1}{\lambda'} \cdot E[N]$$

$$= \frac{1}{\mu(1-P_0)} \cdot E[N]$$

$$= \frac{1}{4(1-0.4324)} \cdot 0.8$$

$$= \frac{5}{14} \text{ hr}$$

or

21.4 min

Ans

$$E[\cancel{N} W_q] = \frac{1}{\lambda'} \cdot E[N_q]$$

$$= \frac{1}{4(1-0.4324)} \cdot 0.24$$

$$= 0.1071 \text{ hr}$$

or

6.4 min   Ans