Unit - 5

Markov Process

A model is said to be markov model if the value of a random process depend only on the just previous value and is independent of all values in distant past.

It is often said that a Markov Process is one in which the future value is ~~*~~ independent of past values, given the present value.

Example :-

Consider the experiment of tossing a fair coin a number of times.

Consider R.V that represent the total no. of heads in first n trials and is given by $S_n = X_1 + X_2 -- X_n$.

The possible value of S_n are $0, 1, 2, -- n$. If $S_n = k$ $(k = 0, 1, 2. - n)$, then the Random variable S_{n+1} assume only two possible values namely $(k+1)$ if $(n+1)^{th}$ trial result in head. & k if $(n+1)^{th}$ trial result in tail.

Thus $P\left[S_{n+1} = k+1 / S_n = k\right] = \frac{1}{2}$.

The Conditional Probability of S_{n+1} depend on value of S_n and not on manner in which S_n was reached.

This is simple example of Markov Chain.

Markov chain :- If for all n,

$$P[x_n = a_n, x_{n-1} = a_{n-1}, x_{n-2} = a_{n-2} \ldots x_0 = a_0]$$
$$= P[x_n = a_n / x_{n-1} = a_{n-1}],$$

Then X_n is called the Markov Chain and $a_1, a_2 \ldots a_n$ are states of markov chain.

One - step transistion Probability :-

The conditional probability $P[x_n = a_j / x_{n-1} = a_i]$ is called one step transistion probability from state a_i to a_j. and it is denoted by

$$P_{ij}(n-1, n)$$

Homogenous ~~Probability~~ ~~Matrix~~ & Markov Chain :-

If one step transistion probability does not depend on the step that is

$$P_{ij}(n-1, n) = P_{ij}(m-1, m)$$

Then, X_n is called homogenous markov chain.

Transistional Probability Matrix :-

When the markov chain is Homogenous, the one-step transition probability is denoted by P_{ij}. The matrix $P = \{P_{ij}\}$ is called transistional Probability Matrix, ~~shortly~~ (tpm).

Note :- The tpm of Markov chain is a stochastic Matrix. Since $P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ for all i.

n-step transition Probability :-

The conditional Probability that the process is in state a_j at step n, given that it was in state a_i at step 0, i.e. $P[X_n = a_j / X_0 = a_i]$ is called the n-step transition probability and is denoted by $P_{ij}^{(n)}$.

Example :- Assume that a man is at an integral point of the x-axis b/w the origin and the point x=3. He takes a unit step either to right with prob. 0.7 or to left with probability 0.3. Unless he is at origin when he takes a step to right to reach ~~n~~ n=1 or he is at point x=3, when he takes a step to the left to reach n=2.

Sol→ tpm is given below.

States of X_n

$$\text{States of } X_{n-1} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0.3 & 0 & 0.7 & 0 \\ 2 & 0 & 0.3 & 0 & 0.7 \\ 3 & 0 & 0 & 1 & 0 \end{array}$$

Probability Distribution :-

If the probability that the process is in stat a_i is p_i ($i = 1, 2, 3 \ldots k$), at any arbitrary step, then the row vector $P = (p_1, p_2 \ldots p_k)$ is called the probability distribution of the process at that time.

In Particular.
$p^{(0)} = (p_1^{(0)}, p_2^{(0)}, p_3^{(0)} \ldots p_k^{(0)})$ is called initial probability distribution.

Chapman - Kolmogorov Theorem :-

If P is the tpm of a homogenous Markov chain then the n-step tpm $p^{(n)}$ is equal to p^n.

i.e. $[P_{ij}^{(n)}] = [P_{ij}]^n$.

Q The tpm of a Markov chain $\{X_n\}$, n = 1, 2, 3. having three states 1, 2, 3. is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

and the initial distribution is $p^{(0)} = (0.7, 0.2, 0.1)$

find (i) $P[X_2 = 3]$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Sol →

NOTE :-

$$\underset{\substack{\text{n step}\\ \text{probability}\\ \text{distribution}}}{P^{(n)}} = \underset{\substack{(n-1)^{th}\text{ step}\\ \text{probability}\\ \text{distribution.}}}{P^{(n-1)}} \underset{\text{tpm}}{P}$$

Sol → $P^{(1)} = P^{(0)} \cdot P$

$$= [0.7, 0.2, 0.1] \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= [0.22, 0.43, 0.35]$$

$$P^{(2)} = P^{(1)} \cdot P$$

$$= [0.22, 0.43, 0.35] \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= [0.385, 0.336, \underbrace{0.279}_{3^{rd}\text{ stat}}] \quad 2^{nd}\text{ step}$$

(i) $P[\underset{2^{nd}\text{ step}}{X_2} = \underset{3^{rd}\text{ stat}}{3}] = 0.279$

(ii) $P[\underbrace{X_3 = 2}_{A}, \underbrace{X_2 = 3, X_1 = 3, X_0 = 2}_{B}]$

$= P(A \cap B) = P(A/B) \times P(B)$

$= P(X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2) \times P[X_2 = 3, X_1 = 3, X_0 = 2]$

$= P[X_3=2/X_2=3] \cdot P[X_2=3/X_1=3, X_0=2] \cdot P(X_1=3, X_0=2)$

$= P[X_3=2/X_2=3] \cdot P[X_2=3/X_1=3] \cdot P[X_1=3/X_0=2] \cdot P(X_0=2)$

$= 0.4 \times 0.3 \times 0.2 \times 0.2$

$= 0.0048$ Ans

Q A fair dice is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the tpm P of the markov chain X_n. Also find P^2 and $P[X_2=6]$.

Sol → State space → $[1, 2, 3, 4, 5, 6]$.

Let X_n = the maximum no. in first n trials $= 3$ (say).

Then, $X_{n+1} = 3$ if $(n+1)^{th}$ trial result in 1, 2, or 3.

$= 4$ if $(n+1)^{th}$ trial result in 4

$= 5$ if $(n+1)^{th}$ trial result in 5

$= 6$ if $(n+1)^{th}$ trial result in 6.

$\therefore P[X_{n+1}=3/X_n=3] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$

$P[X_{n+1}=i/X_n=3] = \frac{1}{6} \quad (i = 4, 5, 6)$.

Similarly, we can find this for all maximum assumption of 1, 2, 4, 5, 6

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(i) $\therefore$ tpm

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(ii) $$P^2 = P \cdot P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{36} \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$ Ans

(iii) Initial stat probability distribution is

$$p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

$$p^{(1)} = p^{(0)} \cdot P$$

$$p^{(1)} = \left[\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\right] \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p^{(1)} = \left[\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36}\right].$$

$$p^{(2)} = p^{(1)} \cdot P \Rightarrow \left[\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36}\right] \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p^{(2)} = \left[\frac{1}{216}, \frac{7}{216}, \frac{19}{216}, \frac{37}{216}, \frac{61}{216}, \frac{91}{216}\right]$$

$$P[X_2 = 6] = \frac{91}{216}$$ Ans

Regular Matrix :- A stochastic Matrix P is said to be regular matrix, if all the entries of P^m (for some ~~+ve~~ ~~inge~~ integer m) are +ve.

A homogenous markov chain is said to be regular if its tpm is tpm.

Stationary distribution or Steady stat distribution

If a homogenous markov chain is regular, then every sequence of state Probability distribution approaches a unique fixed distribution called the steady-state distribution of the Markov chain.

i.e. $\lim_{n\to\infty} \{P^n\} = \pi$, where the steady state probability distribution at step n, $P^{(n)} = \{P_1^{(n)} \cdots P_k^{(n)}\}$ and the stationary distribution $\pi = (\pi_1, \pi_2, \cdots \pi_k)$ are row vectors.

Note :- If P is the tpm of regular chain, then
$\pi P = \pi$ (π is a row vector).

Q. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train. if he drives one day, then the next day he is just as likely to drive again as he ~~is~~ to travel by train. Now suppose that on the first day of week, the man tossed a fair dice and drove to work if and only if 6 appeared.
Find (i) The probability that he takes a train on the third day.
(ii) The Probability that he drives to work in the long run.

Sol → State space = (train, car):

The tpm is

$$P = \begin{array}{c} \\ T \\ C \end{array}\begin{array}{cc} T & C \\ \end{array}$$

$$P = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} \end{array}$$

Initial state Probability distribution is

$$p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right).$$

Because of 1st day

$\frac{1}{6}$ ⟶ P(going by car) = P(getting 6) $= \frac{1}{6}$

$\frac{5}{6}$ ⟶ $1 - \frac{1}{6} = \frac{5}{6}$.

$$p^{(2)} = p^{(1)} \cdot P \Rightarrow \left[\frac{5}{6}, \frac{1}{6}\right]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$p^{(2)} = \left[\frac{1}{12}, \frac{11}{12}\right]$$

$$p^{(3)} = p^{(2)} \cdot P \Rightarrow \left[\frac{1}{12}, \frac{11}{12}\right]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$p^{(3)} = \left[\frac{11}{24}, \frac{13}{24}\right]$$

$\therefore$ P[man going by train on 3rd day] $= \frac{13}{24}$

(ii) Let $\pi = (\pi_1, \pi_2)$ be the distribution form of the state probability distribution or steady state distribution of Markov chain.

By property of π.

$$\pi P = \pi$$

$$\Rightarrow [\pi_1, \pi_2] \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_1, \pi_2]$$

$$= \left[\frac{\pi_2}{2}, \ \pi_1 + \frac{\pi_2}{2}\right] = [\pi_1, \pi_2]$$

$$\Rightarrow \frac{\pi_2}{2} = \pi_1 \quad --- \ (1)$$

$$\pi_1 + \frac{\pi_2}{2} = \pi_2 \quad --- \ (2)$$

Also, $\pi_1 + \pi_2 = 1$

Putting value of (1) in this

$$\frac{\pi_2}{2} + \pi_2 = 1$$

$$\Rightarrow \frac{3\pi_2}{2} = 1$$

$$\Rightarrow \pi_2 = \frac{2}{3}$$

$$\Rightarrow \pi_1 = 1 - \frac{2}{3}$$

$$\Rightarrow \pi_1 = \frac{1}{3}$$

$\Rightarrow P(\text{man travels by car on long run}) = \frac{2}{3}$ Ans

Q A gambler has Rs 2, He bets Rs 1 at a time and wins Rs 1 with Probability $\frac{1}{2}$. He stops playing if he loses Rs 2 or wins Rs 4. (a) What is the tpm of related Markov chain.
(b) What is the probability that he has lost his money at end of 5 plays.
(c) What is the probability that the game last more than 7 plays.

Solution :- Let X_n represent the amount with the player at the end of n^{th} round of the play.
State space of $X_n = (0, 1, 2, 3, 4, 5, 6)$ as
tpm of Markov chain is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The initial probability distribution of X_n is

$$p^{(0)} = (0, 0, 1, 0, 0, 0, 0).$$

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$$p^{(1)} = p^{(0)} \cdot P$$

$$= (0,0,1,0,0,0,0) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p^{(1)} = \left(0, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0, 0, 0\right).$$

similarly,

$$p^{(2)} = \left(\tfrac{1}{4}, 0, \tfrac{1}{2}, 0, \tfrac{1}{4}, 0, 0\right)$$

$$p^{(3)} = p^2(P) = \left(\tfrac{1}{4}, \tfrac{1}{4}, 0, \tfrac{3}{8}, 0, \tfrac{1}{8}, 0\right)$$

$$p^{(4)} = p^{(3)} P = \left(\tfrac{3}{8}, 0, \tfrac{5}{16}, 0, \tfrac{1}{4}, 0, \tfrac{1}{10}\right)$$

$$p^{(5)} = p^{(4)} P = \left(\tfrac{3}{8}, \tfrac{5}{32}, 0, \tfrac{9}{32}, 0, \tfrac{1}{8}, \tfrac{1}{16}\right)$$

$$p^{(6)} = p^{(5)} P = \left(\tfrac{29}{64}, 0, \tfrac{7}{32}, 0, \tfrac{13}{64}, 0, \tfrac{1}{8}\right)$$

$$p^{(7)} = p^{(6)} P = \left(\tfrac{29}{64}, \tfrac{7}{64}, 0, \tfrac{27}{128}, 0, \tfrac{13}{128}, \tfrac{1}{8}\right).$$

(i) P(man lost all his money at end of 5^{th} play)

$$= P(X_5 = 0) = \frac{3}{8} \quad \text{Ans}$$

(ii) P(game last more than 7 play)

$$= P(X_7 = 1, 2, 3, 4, \text{ or } 5)$$

$$= \frac{7}{64} + 0 + \frac{27}{128} + 0 + \frac{13}{128} = \frac{27}{64} \quad \text{Ans}$$

Classification of State of Markov Chain

* If $P_{ij}^{(n)} > 0$ for some n and for all i and j. Then every state can be reached from every other state, when this condition is satisfied the Markov chain is said to be Irreducible.
Otherwise, the chain is said to be reducible.

* State i of a markov chain is called a return state if $P_{ii}^{(n)} > 0$, for some $n > 1$.
The period d_i of a return state i is defined as $d_i = \gcd\{m : P_{ii}^{(n)} > 0\}$.
State i is said to be periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$.

Note :-

1) If a markov chain is finite and irreducible all its state are non-null persistent.
2) A non-null persistent & aperiodic state is called argodic.

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Q. Find the nature of Markov chain with the tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Sol. →

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = P$$

$$P^4 = P^2$$

In general, $P^{2n} = P^2$

$$P^{2n+1} = P.$$

We note :- $P_{00}^{(2)} > 0$, $P_{01}^{(1)} > 0$, $P_{02}^{(2)} > 0$

$P_{10}^{(1)} > 0$, $P_{11}^{(2)} > 0$, $P_{12}^{(1)} > 0$

$P_{20}^{(2)} > 0$, $P_{21}^{(1)} > 0$, $P_{22}^{(2)} > 0$

∴ Markov chain is irreducible.

Also, $P_{ii}^{(2)} = P_{ii}^{(4)} = P_{ii}^{(6)} > 0 \ \forall i$.
all the states of the chain is periodic w. finite and irreducible, all its states are non-null persistant

Also, $P_{ii}^{(2)} = P_{ii}^{(4)} = P_{ii}^{(6)} > 0 \ \forall i$, all the stat of chain is periodic with period
$= \gcd(2, 4, 6) = 2$

Since, the chain is finite and irreducible, all its states are non-null peristant.
Since the stat is ~~non~~ periodic, So, all states are not ergodic.

Q. Three boys A, B, C, are throwing a ball to each other. A always throw to B. and B always through to C. but C is just as likely to throw it B as to A. Show that the Process is Markovian. find the transition Matrix and classify the stat.

Sol → tpm:-

$$P = \text{Stat of } X_{n+1} \begin{array}{c} \\ A \\ B \\ C \end{array} \overset{\text{Stat of } X_n}{\begin{array}{ccc} A & B & C \\ \end{array}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Stat of X_n is dependent only on X_{n-1} but not on any other stat of X_{n-2} . . . .

$\therefore$ X_n is a Markovian chain.

Now, $$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

$$P^6 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} & \frac{1}{2} \\ 1/8 & 3/8 & 3/8 \end{bmatrix}$$

$P_{11}^{(3)} > 0$ $P_{12}^{(1)} > 0$ $P_{13}^{(2)} > 0$

$P_{21}^{(3)} > 0$ $P_{22}^{(2)} > 0$ $P_{23}^{(1)} > 0$

$P_{31}^{(1)} > 0$ $P_{32}^{(2)} > 0$ $P_{33}^{(2)} > 0$

$\therefore$ Markov chain is irreducible.

$P_{ii}^{(2)}, P_{ii}^{(3)}, P_{ii}^{(5)}, P_{ii}^{(6)} > 0$ for $i = 2, 3$.

$\Rightarrow$ Stat 2 & 3 are periodic with period $= \gcd(3, 5, 6, 2) = 1$.

$\Rightarrow$ aperiodic

$P_{ii}^{(3)}, P_{ii}^{(5)}, P_{ii}^{(6)} > 0$ for $i = 1$

$\Rightarrow$ Stat 1 are periodic with period $= \gcd(3, 5, 6) = 1$

$\Rightarrow$ aperiodic.

Therefore all states is also aperiodic

Since all stat is irreducible & chain is finite,

$\therefore$ All stat is non-persistent.

Also, since it is non-persistent & aperiodic

All its states are ergodic.