PROBABILITY

&

RANDOM PROCESSES

(As per SRM UNIVERSITY Syllabus)

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15MA209 – PROBABILITY AND RANDOM PROCESSES
UNIT – I : Probability Distributions

Syllabus:

- Random Variables
- Moments
- Moment Generating Function
- Binomial Distribution
- Poisson Distribution
- Geometric Distribution
- Exponential Distribution
- Normal Distribution
- Function of Random Variables

RANDOM VARIABLE

The outcomes of many random experiments may be non-numerical. It is inconvenient to deal with these descriptive outcomes mathematically.

Example: When toss a coin we get two outcomes, namely head or tail. We can assign numerical values; say 1 to head and 0 to tail. This interpretation is easy and attractive from mathematical point of view and also practically meaningful.

Example: Three students sat for an examination & X denotes the number of students who passed. Describe the RV X.

Sample Space S	None	S_1	S_2	S_3	S_1S_2	S_2S_3	S_3S_1	$S_1S_2S_3$
No. of Students who passed X	0	1	1	1	2	2	2	3

$$n(S) = 8, \quad P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}$$

TYPES OF RANDOM VARIABLE

Random Variable

Discrete Random Variable | **Continuous Random Variable**

DISCRETE RANDOM VARIABLE

A random variable X is discrete, if it assumes only finite number or countably infinite number of values.

Example **:** (i) *The mark obtained by a student in an examination. It's possible values are 0, 85 or 100.*

(ii) The number of students who are absent for a particular period.

1. Probability Mass Function (p.m.f.) $\sum_{i=1}^{\infty} P(x_i) = 1$ 2. Mean $E(X) = \sum_{i=1}^{\infty} x_i\, P(x_i)$, $E(X^2) = \sum_{i=1}^{\infty} {x_i}^2\, P(x_i)$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$

CONTINUOUS RANDOM VARIABLE

A RV X is continuous, if it takes all possible values between certain limits or in an interval which may be finite or infinite. ***E.g:***(i)The density of milk taken for testing at a farm.(ii)The operating time between two failures of a computer.

1. Probability Density Function (p.d.f.) $\int_{-\infty}^{\infty} f(x)dx = 1$ 2. Mean $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$, $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$
3. Variance $V(X) = E(X^2) - [E(X)]^2$ 4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

PROPERTIES OF EXPECTATION

If X and Y are random variables and a, b are constants, then

1. $E(a) = a$ 2. $E(aX) = aE(X)$ 3. $E(aX + b) = aE(X) + b$ 4. $E(X - \bar{X}) = 0$ 5. $|E(X)| \le E(|X|)$
6. $E(X) \ge 0, if\ X \ge 0$ 7. $E(X + Y) = E(X) + E(Y)$ ***(Additive Theorem)***
7. $E(XY) = E(X)E(Y)$ ***(∵ A and B are independent)***
8. $E\big(a\, g(X)\big) = aE\big(g(X)\big)$ 9. $E(g(X) + a) = E\big(g(X)\big) + a$
10. $(E[g(X)\ ]) = g[E(X)]$ $[\boldsymbol{g(X)\ is\ linear\ in\ X}]$
11. $P(X \ge a) \le \frac{E(X)}{a}, a > 0$ ***(Markov Inequality)***
12. $P\{|X - E(X)| \ge k\} \ge \frac{\sigma_X^2}{\epsilon^2}$ ***(Chebyshev's Inequality)***

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PROPERTIES OF VARIANCE

1. $Var(X) \geq 0$ 2. $E(X^2) \geq [E(X)]^2$ 3. $Var(b) = 0, b\ constant$
2. If X is a random variables, a is constants then $Var(aX) = a^2 Var(X)$
4. If a and b are constants, $Var(aX \pm b) = a^2 Var(X)$
5. If X and Y are two independent RV, a and b are constants then $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$

PROPERTIES OF CUMULATIVE DISTRIBUTION FUNCTION

1. If F is the distribution function of the RV X and if $a < b$, then
   $P(a < X \leq b) = P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = F(b) - F(a)$
2. If F is the distribution function of one dimensional RV X, then (i) $0 \leq F(X) \leq 1$ (ii) $F(X) \leq F(Y)$, if $x < y$
   In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.
3. If F is the distribution function of one dimensional random variable X, then
   $F(-\infty) = \lim_{x\to-\infty} F(X) = 0$ and $F(\infty) = \lim_{x\to\infty} F(X) = 1$ 4. $f(x) = \frac{d}{dx}\left(F(x)\right)$

MOMENTS

Definition: The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

Moments about Origin (Raw Moments)

Discrete : $\mu_n' = E(X^n) = \sum_i x_i^n p_i\ , n \geq 1.$ ***Continuous:*** $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx\ , n \geq 1$

Moment about Mean (Central Moments)

Discrete : $\mu_n = E[(X - \bar{X})^n] = \sum_i (x_i - \bar{X})^n p_i$, ***Continuous:*** $\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f(x)dx\ ,\ n \geq 1$

Relationship between moments about origin and moment about mean

$$\mu_r = \mu_r' - rC_1\, \mu\, \mu_{r-1}^1 + rC_2\, \mu^2\, \mu_{r-2}' - \cdots$$

Hence, $\mu_1 = 0$, $\mu_2 = \mu_2' - (\mu_1')^2$, $\mu_3 = \mu_3' - 3\mu_2'\, \mu_1' + 2(\mu_1')^3$, $\mu_4 = \mu_4' - 4\mu_3'\, \mu_1' + 6\mu_2'(\, \mu_1')^2 - 3(\mu_1')^4$

MOMENT GENERATING FUNCTION

Definition : Moment generating function of a random variable about the origin is defined as

Discrete : $M_X(t) = E(e^{tX}) = \sum_x e^{tx}\, p(x)$, ***Continuous :*** $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\, f(x)dx$

Where the integration or summation is taken over the entire range of X, t being a real parameter, assuming that integration or summation is absolutely convergent.

$$M_X(t) = 1 + t\, \mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' \ ,\quad \text{Where } \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

Note: 1. $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$ 2. $M_{CX}(t) = M_X(Ct)$, C being a constant. 3. $M_{X=a}(t) = e^{-at}\, M_X(t)$

1. If $X_1, X_2, \ldots X_n$ are n independent RVs, then $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t).M_{X_2}(t) \ldots M_{X_n}(t)$

PROBLEMS IN DISCRETE RANDOM VARIABLE

1. ***A discrete RV X has the following probability distribution***

x	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a (ii) $P(X < 3)$ ***(iii)*** $P(X \geq 3)$ ***(iv)*** $P(0 < X < 3)$ ***(v) Find the distribution function of X.***

Solution

(i) $\sum_{x=0}^{8} P(x) = 1 \Rightarrow P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) = 1$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1 \Rightarrow 81a = 1 \Rightarrow \boldsymbol{a = \frac{1}{81}}$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = a + 3a + 5a = 9a = 9 \times \frac{1}{81} = \frac{1}{9}$

(iii) $P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{1}{9} = \frac{8}{9}$

(iv) $P(0 < X < 3) = P(X = 1) + P(X = 2) = 3a + 5a = 8a = \frac{8}{81}$

(v)

x	0	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

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2. ***A discrete random variable X has the probability function given below:***

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) The value of K (ii) $P(1.5 < X < 4.5/X > 2)$ (iii) The smallest value of λ for which $P(X \le \lambda) > 1/2$.

Solution:

(i) $\sum_{x=0}^{7} P(x) = 1 \Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7) = 1$

$$0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1 \Rightarrow 10K^2+9K = 1$$

$$(10K-1)(K+1) = 0 \Rightarrow K = \frac{1}{10}, -1 \Rightarrow \mathbf{K = \frac{1}{10}} \quad (\because K = -1, \text{ which is meaningless})$$

(ii) $P(1.5 < X < 4.5/X > 2) = \frac{P[(1.5<X<4.5)\cap(X>2)]}{P(X>2)}$ $\quad \because P(A/B) = \frac{P(A\cap B)}{P(B)}$

$$P(1.5 < X < 4.5/X > 2) = \frac{P(3)+P(4)}{P(3)+P(4)+P(5)+P(6)+P(7)} = \frac{\left(\frac{5}{10}\right)}{\left(\frac{7}{10}\right)} = \frac{5}{7}$$

(iii) $P(X \le \lambda) > \frac{1}{2}$, $\lambda = 0$, $P(X \le 0) = 0 \ngtr \frac{1}{2}$; $\lambda = 1$, $P(X \le 1) = \frac{1}{10} \ngtr \frac{1}{2}$;

$\lambda = 2$, $P(X \le 2) = \frac{3}{10} \ngtr \frac{1}{2}$; $\lambda = 3$, $P(X \le 3) = \frac{5}{10} \ngtr \frac{1}{2}$; $\lambda = 4$, $P(X \le 4) = \frac{8}{10} > \frac{1}{2}$

The smallest value of λ for which $P(X \le \lambda) > 1/2$ is 4.

3. ***If the RV X takes the values 1, 2, 3 & 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability distribution and cumulative distribution function of X.***

Solution: Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = 30K$

x	1	2	3	4
$p(x)$	$15K$	$10K$	$30K$	$6K$

$\sum_{x=1}^{4} P(x) = 1 \Rightarrow P(1)+P(2)+P(3)+P(4) = 1 \Rightarrow 15K+10K+30K+6K = 1 \Rightarrow 61K = 1 \Rightarrow \mathbf{K = \frac{1}{61}}$

Cumulative distribution function of X

x	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

4. ***A discrete RV X has the following probability distribution***

x	-2	-1	0	1	2	3
$p(x)$	0.1	K	0.2	$2K$	0.3	$3K$

Find (i) K (ii) $P(X < 2)$ (iii) $P(-2 < X < 2)$ (iv) the cdf of X (v) the mean of X.

Solution

(i) $\sum_{x=-2}^{3} P(x) = 1 \Rightarrow P(-2)+P(-1)+P(0)+P(1)+P(2)+P(3) = 1 \Rightarrow 6K+0.6 = 1 \Rightarrow \mathbf{K = \frac{1}{15}}$

	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$

(ii) $P(X < 2) = P(-2)+P(-1)+P(0)+P(1) = \frac{1}{10}+\frac{1}{15}+\frac{2}{10}+\frac{2}{15} = \frac{1}{2}$

(iii) $P(-2 < X < 2) = P(-1)+P(0)+P(1) = \frac{1}{15}+\frac{2}{10}+\frac{2}{15} = \frac{2}{5}$

(iv)

x	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$
$F(X)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{4}{5}$	1

(v) ***Mean of X***

$$E(X) = \sum_{x=-2}^{3} x\,P(x) = (-2)P(-2)+(-1)P(-1)+0\,P(0)+1\,P(1)+2\,P(2)+3\,P(3)$$

$$= \left(-2\times\frac{1}{10}\right)+\left(-1\times\frac{1}{15}\right)+\left(0\times\frac{2}{10}\right)+\left(1\times\frac{2}{15}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{3}{15}\right) = \frac{16}{15}$$

5. ***If X is RV having the density function*** $f(x) = \begin{cases} \frac{x}{6} \text{ for } x = 1, 2, 3 \\ 0, \text{otherwise} \end{cases}$. ***Find*** $E(X^3 + 2X + 7)$ ***and*** $Var\ (4X + 5)$.

Solution

x	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$$E(X) = \sum_{x=1}^{3} x\, P(x) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{2}{6}\right) + \left(3 \times \frac{3}{6}\right) = \frac{7}{3}$$

$$E(X^2) = \sum_{x=1}^{3} x^2\, P(x) = \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{2}{6}\right) + \left(9 \times \frac{3}{6}\right) = 6$$

$$E(X^3) = \sum_{x=1}^{3} x^3\, P(x) = \left(1 \times \frac{1}{6}\right) + \left(8 \times \frac{2}{6}\right) + \left(27 \times \frac{3}{6}\right) = \frac{49}{3}$$

$$E(X^3 + 2X + 7) = E(X^3) + 2E(X) + 7 = \frac{84}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{9}, \qquad Var\ (4X + 5) = 4^2 Var(X) = 16 \times \frac{5}{9} = \frac{80}{9}$$

6. ***If X has the distribution function*** $F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{5}{6}, & 6 \le x < 10 \\ 1, & x \ge 10 \end{cases}$

Find (i) The probability distribution of X (ii) $P(2 < X < 6)$ ***(iii) Mean of X (iv) Variance of X.***

Solution

(i) For the given c.d.f., the probability distribution of X is

$$P(X = 1) = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}, \qquad P(X = 4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

$$P(X = 6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}, \qquad P(X = 10) = F(10) - F(6) = 1 - \frac{5}{6} = \frac{1}{6}$$

x	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(ii) $P(2 < X < 6) = P(X = 4) = \frac{1}{6}$

(iii) $E(X) = \sum_i x_i\, P(x_i) = \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{2}{6}\right) + \left(10 \times \frac{1}{6}\right) = \frac{14}{3}$

$$E(X^2) = \sum_i x_i^2\, P(x_i) = \left(1 \times \frac{1}{3}\right) + \left(16 \times \frac{1}{6}\right) + \left(36 \times \frac{2}{6}\right) + \left(100 \times \frac{1}{6}\right) = \frac{95}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{89}{3}$$

7. ***When a die is thrown, X denotes the number that turns up. Find*** $E(X), E(X^2), Var(X)$ ***and standard deviation.***

Solution: $p = \frac{1}{6}$, X = 1, 2, 3, 4, 5, 6 Here X is a discrete RV

$$E(X) = \sum_i x_i\, P(x_i) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = 3.5$$

$$E(X^2) = \sum_i x_i^2\, P(x_i) = \left(1 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(9 \times \frac{1}{6}\right) + \left(16 \times \frac{1}{6}\right) + \left(25 \times \frac{1}{6}\right) + \left(36 \times \frac{1}{6}\right) = \frac{91}{6} = 15.167$$

$$Var(X) = E(X^2) - [E(X)]^2 = 2.9166, \qquad S.D. = \sigma_X = \sqrt{Var(X)} = 1.7078$$

8. ***A coin is tossed until a head appears. What is the expectation of the number of tosses required?***

Solution: Let X – No. of tosses required to get the 1st head. The 1st head may appear in the 1st or 2nd … and so on. The events are H, TH, TTH, TTTH, ... $p = \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

x	1	2	3	4	5	…
$p(x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	…

$$E(X) = \sum_i x_i\, P(x_i) = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} = 2 \qquad [\because (1 - x)^{-2} = 1 + 2x + 3x^2 + \cdots]$$

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9. ***By throwing a fair dice, a player gains Rs. 20 if 2 turns up, gains Rs. 40 if 4 turns up and loses Rs. 30 if 6 turns up. He never loses or gains if any other number turns up. Find the expected value of money he gains.***

Solution: Let X – money won on an trial. x_i = Amount of money won, if the faces show $i = 1, 2,3,4,5,6$.

	1	2	3	4	5	6
x	0	20	0	40	0	-30
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_i \ x_i \, P(x_i) = \left(0 \times \frac{1}{6}\right) + \left(20 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(40 \times \frac{1}{6}\right) + \left(0 \times \frac{1}{6}\right) + \left(-30 \times \frac{1}{6}\right) = 5$$

10. **Find the first three moments of X if X has the following distribution**

x	-2	1	3
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Solution : $\mu_n' = E(X^n) = \sum_i x_i^n p_i \, , \quad n \geq 1$

$$n = 1, \ \mu_1' = E(X) = \sum_i \ x_i \, P(x_i) = \left(-2 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(3 \times \frac{1}{4}\right) = 0$$

$$n = 2, \ \mu_2' = E(X^2) = \sum_i \ x_i^2 P(x_i) = \left(4 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(9 \times \frac{1}{4}\right) = \frac{9}{2}$$

$$n = 3, \ \mu_3' = E(X^3) = \sum_i \ x_i^3 P(x_i) = \left(-8 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(27 \times \frac{1}{4}\right) = 3$$

11. ***A RV X has the probability function $f(x) = \frac{1}{2^x}, x = 1, 2, 3\ldots$ Find the (i) moment generating function (ii) Mean***

Solution :

(i) $M_X(t) = \sum_x e^{tx} \, p(x) = \sum_{x=1}^{\infty} e^{tx} \, \frac{1}{2^x} = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \cdots \right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2-e^t}$

(ii) $E(X) = \left[\frac{d}{dt} \, M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0} = \frac{(2-e^0)e^0 - e^0(-e^0)}{(2-e^0)^2} = 2$

12. ***Find the moment generating function for the following function given by***

x	0	1	2	3	4	5	6
$p(x)$	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Solution :

$$M_X(t) = \sum_{x=0}^{6} e^{tx} \, p(x) = e^{0t} \, p(0) + e^t \, p(1) + e^{2t} \, p(2) + e^{3t} \, p(3) + e^{4t} \, p(4) + e^{5t} \, p(5) + e^{6t} \, p(6)$$

$$= \frac{1}{49}[1 + 3e^t + 5e^{2t} + 7e^{3t} + 9e^{4t} + 11e^{5t} \ + 13e^{6t}]$$

13. ***If a RV X has moment generating function $M_X(t) = \frac{3}{3-t}$, obtain the standard deviation of X.***

Solution : $M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1-\frac{t}{3}\right)} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \left(\frac{t}{3}\right) + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{3}\right) + \frac{t^2}{2!}\left(\frac{2}{9}\right) + \frac{t^3}{3!}\left(\frac{6}{27}\right) + \cdots$

$$\mu_r' = coefficient \ of \ \frac{t^r}{r!}, \quad \mu_1' = coefficient \ of \ \frac{t^1}{1!} = \frac{1}{3}, \quad \mu_2' = coefficient \ of \ \frac{t^2}{2!} = \frac{2}{9}$$

Variance $= \mu_2' - (\mu_1')^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$, Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

PROBLEMS IN CONTINUOUS RANDOM VARIABLE

1. ***If $p(x) = \begin{cases} x \, e^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (i) Show that $p(x)$ is a p.d.f. (ii) Find its distribution function $P(x)$.***

Solution

(i) $\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_{0}^{\infty} p(x)dx = \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\infty} x \, e^{-\frac{x^2}{2}} dx = \int_{0}^{\infty} x \, e^{-\frac{x^2}{2}} dx$

Put $x^2 = t$, $2x \, dx = dt \Rightarrow x \, dx = \frac{dt}{2}$, $\quad x = 0, \ t = 0$ and $x = \infty, \ t = \infty$

$$\int_{-\infty}^{\infty} p(x)dx = \int_0^{\infty} e^{-\frac{t}{2}} \frac{dt}{2} = \frac{1}{2}\int_0^{\infty} e^{-\frac{t}{2}} \, dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = -e^{-\infty} + e^0 = 1 \qquad (\because \boldsymbol{e^{-\infty} = 0, \ e^0 = 1})$$

$\therefore p(x)$ is a p.d.f. of a RV X.

(ii) $F(X) = P(X \leq x) = \int_0^x p(x)dx = \int_0^x x \, e^{-\frac{x^2}{2}} dx = 1 - e^{-\frac{x^2}{2}}, \ \ x \geq 0$

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2. ***A continuous RV X has a pdf $f(x) = 3x^2, 0 \le x \le 1$. Find a and b such that***
(i) $P(X \le a) = P(X > a)$ (ii) $P(X > b) = 0.05$
Solution:

(i) $P(X \le a) = P(X > a) \Rightarrow \int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx \Rightarrow \int_{0}^{a} 3\,x^2 dx = \int_{a}^{1} f(x)dx \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = 3\left[\frac{x^3}{3}\right]_a^1$

$a^3 = 1 - a^3 \Rightarrow 2a^3 = 1 \Rightarrow a^3 = \frac{1}{2} \Rightarrow \boldsymbol{a} = \left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\frac{1}{3}} = \mathbf{0.7937}$

(ii) $P(X > b) = 0.05 \Rightarrow \int_{b}^{1} 3\,x^2 dx = 0.05 \Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05 \Rightarrow \boldsymbol{b} = (\mathbf{0.95})^{\frac{1}{3}} = \mathbf{0.9830}$

3. ***A Continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$.***

Solution : $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{2}^{5} k(1+x)dx = 1 \Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1 \Rightarrow k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1 \Rightarrow \boldsymbol{k} = \frac{\mathbf{2}}{\mathbf{27}}$

$$P(X < 4) = \frac{2}{27}\int_{2}^{4}(1+x)dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = \frac{16}{27}$$

4. ***A RV X has a pdf $f(x) = kx^2 e^{-x}, x \ge 0$. Find k, mean, variance and $E(3X^2 - 2X)$.***

Solution: $\int_{-\infty}^{\infty} f(x)dx = 1, \quad \int_{0}^{\infty} kx^2 e^{-x} dx = 1$

Differentiation : $u = x^2, \quad u' = 2x, \quad u'' = 2, \quad u''' = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}$ $\quad (\because \int \boldsymbol{uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$k\left[x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^\infty = 1 \Rightarrow k[(0 - 0 + 0) - (0 - 0 + 2)] = 1 \Rightarrow \boldsymbol{k} = \frac{\mathbf{1}}{\mathbf{2}}$ $\quad (\because \boldsymbol{e^{-\infty} = 0, \; e^0 = 1})$

Mean of X $E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_{0}^{\infty} x\left(\frac{1}{2}x^2 e^{-x}\right)dx = \frac{1}{2}\int_{0}^{\infty} x^3 e^{-x} dx$

Differentiation : $u = x^3, \quad u' = 3x^2, \quad u'' = 6x, \quad u''' = 6, \quad u'^v = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}, v_4 = \frac{e^{-x}}{(-1)^4}$ $(\because \int \boldsymbol{uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots})$

$E(X) = \frac{1}{2}\left[x^3\frac{e^{-x}}{(-1)} - 3x^2\frac{e^{-x}}{(-1)^2} + 6x\frac{e^{-x}}{(-1)^3} - 6\frac{e^{-x}}{(-1)^4}\right]_0^\infty = 3$ $\quad (\because \boldsymbol{e^{-\infty} = 0, \; e^0 = 1})$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{\infty} x^2\left(\frac{1}{2}x^2 e^{-x}\right)dx = \frac{1}{2}\int_{0}^{\infty} x^4 e^{-x} dx$

Differentiation : $u = x^4, \quad u' = 4x^3, \quad u'' = 12x^2, \quad u''' = 24x, \quad u'^v = 24, \quad u^v = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, \quad v_2 = \frac{e^{-x}}{(-1)^2}, \quad v_3 = \frac{e^{-x}}{(-1)^3}, \; v_4 = \frac{e^{-x}}{(-1)^4}, v_5 = \frac{e^{-x}}{(-1)^5}, \; v_6 = \frac{e^{-x}}{(-1)^6}$

$$E(X^2) = \frac{1}{2}\left[x^4\frac{e^{-x}}{(-1)} - 4x^3\frac{e^{-x}}{(-1)^2} + 12x^2\frac{e^{-x}}{(-1)^3} - 24x\frac{e^{-x}}{(-1)^4} + 24x\frac{e^{-x}}{(-1)^5}\right]_0^\infty = 12$$

$$V(X) = E(X^2) - [E(X)]^2 = 12 - 9 = 3, \quad E(3X^2 - 2X) = 3E(X^2) - 2E(X) = 3(12) - 2(3) = 30$$

5. ***The prob. distribution function of a RV X is $f(x) = \begin{cases} \boldsymbol{x}, & \mathbf{0} < \boldsymbol{x} < 1 \\ \mathbf{2} - \boldsymbol{x}, & \mathbf{1} < \boldsymbol{x} < 2 \\ \mathbf{0}, & \boldsymbol{x} > 2 \end{cases}$. Find the mean and variance.***

Solution

$E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_{-\infty}^{0} x\,f(x)dx + \int_{0}^{1} x\,f(x)dx + \int_{1}^{2} x\,f(x)dx + \int_{2}^{\infty} x\,f(x)dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,(x)dx + \int_{1}^{2} x\,(2-x)dx + \int_{2}^{\infty} 0\,dx = \int_{0}^{1} x^2 dx + \int_{1}^{2}(2x - x^2)dx$

$E(X) = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 1$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{0} x^2 f(x)dx + \int_{0}^{1} x^2 f(x)dx + \int_{1}^{2} x^2 f(x)dx + \int_{2}^{\infty} x^2 f(x)dx$

$= \int_{0}^{1} x^3 x + \int_{1}^{2}(2x^2 - x^3)dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{1}{4} + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{7}{6}$

$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6}$

6. ***The distribution function of a RV X is given by $F(x) = 1 - (1 + x)e^{-x}, \; x \ge 0$. Find the density function, mean and variance of X.***
Solution:

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$f(x)=\frac{d}{dx}[F(x)]=\frac{d}{dx}[1-(1+x)e^{-x}]=[0-(1+x)(-e^{-x})-e^{-x}]=e^{-x}+xe^{-x}-e^{-x}=xe^{-x}, x\geq 0$

$E(X)=\int_{-\infty}^{\infty}x\,f(x)dx=\int_0^{\infty}x^2e^{-x}dx=\left[x^2\frac{e^{-x}}{(-1)}-2x\frac{e^{-x}}{(-1)^2}+2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty}=2$

$E(X^2)=\int_{-\infty}^{\infty}x^2\,f(x)dx=\int_0^{\infty}x^3e^{-x}dx=\left[x^3\frac{e^{-x}}{(-1)}-3x^2\frac{e^{-x}}{(-1)^2}+6x^2\frac{e^{-x}}{(-1)^3}-6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty}=6$

$V(X)=E(X^2)-[E(X)]^2=6-4=2$

7. ***The cdf of a continuous RV X is given by*** $\boldsymbol{F(x)=\begin{cases}0, & x<0\\ x^2, & 0\leq x<\frac{1}{2}\\ 1-\frac{3}{25}(3-x)^2, & \frac{1}{2}\leq x<3\\ 1, & x\geq 3\end{cases}}$

Find the p.d.f. of X and evaluate $\boldsymbol{P(|X|\leq 1)}$ ***and*** $\boldsymbol{P\left(\frac{1}{3}\leq X<4\right)}$ ***using both the pdf and cdf.***

Solution: $f(x)=\frac{d}{dx}[F(x)]$

$$f(x)=\begin{cases}0, & x<0\\ 2x, & 0\leq x<\frac{1}{2}\\ \frac{6}{25}(3-x), & \frac{1}{2}\leq x<3\\ 0, & x\geq 3\end{cases}$$

pdf : $P(|X|\leq 1)=P(-1\leq X\leq 1)=\int_{-1}^{0}0\,dx+\int_0^{\frac{1}{2}}2x\,dx+\int_{\frac{1}{2}}^{1}\frac{6}{25}(3-x)dx=2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}+\frac{6}{25}\left[3x-\frac{x^2}{2}\right]_{\frac{1}{2}}^{1}=\frac{13}{25}$

cdf: $P(|X|\leq 1)=P(-1\leq X\leq 1)=F(1)-F(-1)=\frac{13}{25}$

pdf : $P\left(\frac{1}{3}\leq X<4\right)=\int_{\frac{1}{3}}^{\frac{1}{2}}2x\,dx+\int_{\frac{1}{2}}^{3}\frac{6}{25}(3-x)dx+\int_3^4 0\,dx=2\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{1}{2}}+\frac{6}{25}\left[3x-\frac{x^2}{2}\right]_{\frac{1}{2}}^{3}=\frac{8}{9}$

cdf: $P\left(\frac{1}{3}\leq X<4\right)=F(4)-F\left(\frac{1}{3}\right)=1-\frac{1}{9}=\frac{8}{9}$

8. ***The first four moments of a distribution about*** $\boldsymbol{x=4}$ ***are 1, 4, 10, 45. Show that the mean is 5, variance is 3,*** $\boldsymbol{\mu_3=0,\mu_4=26}$ **.**

Solution: Let $\mu_1',\ \mu_2',\ \mu_3',\ \mu_4'$ be the first four moments about $X=4$.

Given $\mu_1'=1,\ \mu_2'=4,\ \mu_3'=10,\ \mu_4'=45$ about $X=4$.

$E(X-4)=1\Rightarrow E(X)-4=1\Rightarrow E(X)=5,\qquad \mu_2=\mu_2'-(\mu_1')^2=4-1=3$

$\mu_3=\mu_3'-3\mu_2'\,\mu_1'+2(\mu_1')^3=10-3(4)(1)+2(1)=0$

$\mu_4=\mu_4'-4\mu_3'\,\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4=45-4(10)(1)+6(4)(1)-3(1)=26$

9. ***The first three moments about the origin are 5, 26, 78. Show that the first three moments about the value*** $\boldsymbol{x=3}$ ***are*** $\boldsymbol{2,5,-48}$**.**

Solution: Let $\mu_1',\ \mu_2',\ \mu_3',\ \mu_4'$ be the first four moments about $x=3$. Given $E(X)=5, E(X^2)=26, E(X^3)=78$

$\mu_1'=E(X-3)=E(X)-3=5-3=2,\qquad \mu_2'=E(X-3)^2=E(X^2)-6E(X)+9=26-6(5)+9=5,$

$\mu_3'=E(X-3)^3=E(X^3)-9E(X^2)+27E(X)-27=78-9(26)+27(5)-27=-48,$

10. ***If X has probability density function given by*** $\boldsymbol{f(x)=\frac{x+1}{2},-1\leq x\leq 1}$. ***Find the 1st four central moments.***

Solution: $\mu_n'=E(X^n)=\int_{-\infty}^{\infty}x^n\,f(x)dx$

$n=1,\ \mu_1'=E(X)=\int_{-\infty}^{\infty}x\,f(x)dx=\frac{1}{2}\int_{-1}^{1}(x^2+x)dx=\frac{1}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1}=\frac{1}{3}$

$n=2,\ \mu_2'=E(X^2)=\int_{-\infty}^{\infty}x^2\,f(x)dx=\frac{1}{2}\int_{-1}^{1}(x^3+x^2)dx=\frac{1}{2}\left[\frac{x^4}{4}-\frac{x^3}{3}\right]_{-1}^{1}=\frac{1}{3}$

$n=3,\ \mu_3'=E(X^3)=\int_{-\infty}^{\infty}x^3\,f(x)dx=\frac{1}{2}\int_{-1}^{1}(x^4+x^3)dx=\frac{1}{2}\left[\frac{x^5}{5}-\frac{x^4}{4}\right]_{-1}^{1}=\frac{1}{5}$

$n=4,\ \mu_4'=E(X^4)=\int_{-\infty}^{\infty}x^4\,f(x)dx=\frac{1}{2}\int_{-1}^{1}(x^5+x^4)dx=\frac{1}{2}\left[\frac{x^6}{6}-\frac{x^5}{5}\right]_{-1}^{1}=\frac{1}{5}$

Moment about Mean (Central Moments)

$\mu_r=\mu_r'-rC_1\,\mu\,\mu_{r-1}^1+rC_2\,\mu^2\,\mu_{r-2}'-rC_3\,\mu^3\,\mu_{r-3}'+rC_4\,\mu^4\,\mu_{r-4}'-\cdots$

$r=1,\ \mu_1=0$

$r = 2, \quad \mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{9}$

$r = 3, \quad \mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 = -\frac{8}{135}$

$r = 4, \quad \mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'(\,\mu_1')^2 - 3(\mu_1')^4 = \frac{48}{405}$

11. ***A RV X has density function given by*** $f(x) = \begin{cases} 2\,e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$. ***Obtain the (i) moment generating function (ii) Four moments about the origin (iii) Mean (iv) Variance.***

Solution: $M_X(t) = \int_{x=-\infty}^{\infty} e^{tx} f(x)dx = \int_{x=0}^{\infty} e^{tx}\, 2\,e^{-2x}\,dx = \int_{x=0}^{\infty} 2\,e^{-(2-t)x}\,dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{2}{2-t}$

$M_X(t) = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{1}{2}\right) + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\left(\frac{3}{2}\right) + \cdots$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}, \qquad r = 1,\ \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{2}$

$r = 2,\ \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{1}{2}, \qquad r = 3,\ \mu_3' = coefficient\ of\ \frac{t^3}{3!} = \frac{3}{4}$

$r = 4,\ \mu_4' = coefficient\ of\ \frac{t^4}{4!} = \frac{3}{2},\ \ \text{Mean} = \mu_1' = \frac{1}{2},\ \ \text{Variance} = \mu_2' - (\mu_1')^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

12. ***Find the moment generating function of the RV whose moments are given by*** $\mu_r' = (r+1)!\,2^r$. ***Find also mean and variance.***

Solution: $\mu_1' = 2!\,2^1,\ \ \mu_2' = 3!\,2^2,\ \ \mu_3' = 4!\,2^3,\ \ M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\mu_3' + \cdots$

$M_X(t) = 1 + \frac{t}{1!}(2!\,2^1) + \frac{t^2}{2!}(3!\,2^2) + \frac{t^3}{3!}(4!\,2^3) + \cdots = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots = (1-2t)^{-2}$

Mean $= \mu_1' = 4$, $\mu_2' = 24$, Variance $= \mu_2' - (\mu_1')^2 = 24 - 16 = 8$

FUNCTION OF ONE DIMENSITIONAL RANDOM VARIABLE

One to One Transformation of Random Variables:

Consider that a random variable X is linearly transformed into an another random variable Y. Let Y be $T(x)$.

A monotonically increasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 < x_2$. For example, $y = ax, a > 0$

A monotonically decreasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 > x_2$. For example, $y = ax, a < 0$

If the transformation is monotonically increasing $f_Y(y) = f_X(x)\ \frac{dx}{dy}$

If the transformation is monotonically decreasing $f_Y(y) = f_X(x)\ \left(-\frac{dx}{dy}\right)$

In general, for a linear transformation $\boldsymbol{f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|, where\ x = g^{-1}(y)}$

Non - One to One Transformation of Random Variables:

For a transformation which is non - one to one, the transformation will be broken up into transformations each of which one to one. $\boldsymbol{f_Y(y) = f_X(x_1)\ \left|\frac{dx_1}{dy}\right| + f_X(x_2)\ \left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_n)\ \left|\frac{dx_n}{dy}\right|}$

PROBLEMS IN FUNCTION OF RANDOM VARIABLE

1. ***Consider a RV X with p.d.f.*** $\boldsymbol{f(x) = e^{-x}, x \ge 0}$ ***with transformation*** $\boldsymbol{y = e^{-x}}$. ***Find the transformed density function.***

Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right| = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{e^{-x}}{|-e^{-x}|} = \frac{y}{y} = 1, 0 < y \le 1$

2. ***Let X be a RV with p.d.f.*** $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. ***Find the p.d.f. of*** $\boldsymbol{Y = 8X^3}$.

Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = 8x^3 \Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \left(\frac{y}{8}\right)^{\frac{1}{3}}, \quad \frac{dx}{dy} = \frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1} \frac{1}{8} = \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}}$

$f_Y(y) = 2x\ \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = 2\left(\frac{y}{8}\right)^{\frac{1}{3}} \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = \frac{1}{12}\left(\frac{y}{8}\right)^{-\frac{1}{3}}$, ***Range:*** $0 < x < 1 \Rightarrow 0 < \left(\frac{y}{8}\right)^{\frac{1}{3}} < 1 \Rightarrow 0 < y < 8$

3. ***If X is uniformly distributed in*** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ***find the pdf of*** $\boldsymbol{Y = tan\,X}$.

Solution: Given $f_X(x) = \frac{1}{b-a} = \frac{1}{\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = \pi,\ \ f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = \tan x \Rightarrow x = \tan^{-1} y,\ \ \frac{dx}{dy} = \frac{1}{1+y^2}$

$f_Y(y) = \pi\,\frac{1}{1+y^2}$ ***Range:*** $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow -\infty < y < \infty$

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4. *If X has an exponential distribution with parameter 1, find the pdf of $Y = \sqrt{X}$.*

Solution: *Given* $\lambda = 1$, $f_X(x) = \lambda\, e^{-\lambda x}, x > 0 = e^{-x}, x > 0$, $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$, *Let* $y = \sqrt{x} \Rightarrow x = y^2$,

$\frac{dx}{dy} = 2y$, $f_Y(y) = 2y\ e^{-y^2}$ ***Range:*** $x > 0 \Rightarrow y > 0$

Theoretical Distributions

- ***Discrete Distribution***
  - ***Binomial***
  - ***Poisson***
  - ***Geometric***
- ***Continuous Distribution***
  - ***Exponential***
  - ***Normal***

DISCRETE DISTRIBUTION				
Discrete Probability Distribution	**Probability Mass Function (p.m.f.)**	**Moment Generating Function (m.g.f.) $M_X(t)$**	**Mean $E(X)$**	**Variance $V(X)$**
Binomial	$P(X = x) = nC_x p^x q^{n-x}, \ x = 0,1, \dots n$	$(q + pe^t)^n$	np	npq
Poisson	$P(X = x) = \frac{e^{-\lambda}\,\lambda^x}{x!}, \ x = 0,1, \dots \infty$	$e^{\lambda(e^t - 1)}$	λ	λ
Geometric	$P(X = x) = p\, q^{x-1}, \ x = 1, \dots \infty$	$\frac{p\, e^t}{1 - q\, e^t}$	$\frac{1}{p}$	$\frac{q}{p^2}$
CONTINOUS DISTRIBUTION				
Continuous Probability Distribution	**Probability Density Function (p.d.f.)**	**Moment Generating Function (m.g.f.) $M_X(t)$**	**Mean $E(X)$**	**Variance $V(X)$**
Exponential	$f(x) = \lambda\, e^{-\lambda x}, \quad x \geq 0$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$	$e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)}$	μ	σ^2

BINOMIAL DISTRIBUTION

Bernoulli Distribution : A Bernoulli distribution is one having the following properties

(i) The experiment consists of n repeated trials.

(ii) Each trial results in an outcome that may be classified under two mutually exclusive categories as a success or as a failure.

(iii) The probability of success denoted by p, remains constant from trial to trial.

(iv) The repeated trials are independent.

Binomial Distribution:

A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1 - p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials

$$\boldsymbol{P(X = x) = nC_x p^x q^{n-x}, \ \ x = 0, 1, \dots n}$$

The quantities n & p are called the parameters of binomial distribution.

Areas of Application:

1. Quality control measures and Sampling processes in industries to classify items as defective or non defective.
2. Medical applications as success or failure of a surgery, cure or no cure of a patient.

Moment Generating Function (m.g.f.) in Binomial Distribution

$$M_X(t) = \sum_{x=0}^{n} e^{tx}\, p(x) = \sum_{x=0}^{n} e^{tx}\, nC_x p^x q^{n-x} = \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= nC_0 (pe^t)^0 q^{n-0} + nC_1 (pe^t)^1 q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \cdots + nC_n (pe^t)^n q^{n-n}$$

$$= q^n + nC_1 (pe^t)^1 q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \cdots + (pe^t)^n$$

$$\boldsymbol{M_X(t) = (q + pe^t)^n}$$

Mean and Variance using Moment Generating Function in Binomial Distribution

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}(q+pe^t)^n\right]_{t=0} = [n(q+pe^t)^{n-1}\, pe^t]_{t=0} = n(q+pe^0)^{n-1}\, pe^0 = \boldsymbol{np}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt} np\,(q+pe^t)^{n-1}\, e^t\right]_{t=0} = np[(n-1)(q+pe^t)^{n-2}\, pe^t + (q+pe^t)^{n-1}e^t]_{t=0}$$

$$= np[(n-1)(q+pe^0)^{n-2}\, pe^0 + (q+pe^0)^{n-1}e^0] = np[(n-1)p+1] = \boldsymbol{n^2p^2 - np^2 + np}$$

$$V(X) = E(X^2) - [E(X)]^2 = n^2p^2 - np^2 + np - n^2p^2 = -np^2 + np = np(1-q) = \boldsymbol{npq}$$

Problems in Binomial Distribution

1. ***Four coins are tossed simultaneously. What is the probability of getting 2 heads and at least 2 heads?***

Solution : $n = 4,\ p = \frac{1}{2},\quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}, P(X = x) = nC_x p^x q^{n-x},\ \ x = 0,1,\dots n.$

(i) $P(2\ heads) = P(X = 2) = 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} = \frac{3}{8}$

(ii) $P(atleast\ 2\ heads) = P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} + 4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{4-3} + 4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4-4} = \frac{3}{8} + \frac{1}{4} + \frac{1}{16} = \frac{11}{16}$$

2. ***The probability that a patient recovers from a disease is 0.3. If 18 people are affected from this disease, What is the probability that (i) At least 10 survive (ii) Exactly 6 survive (iii) 4 to 7 survive***

Solution: $n = 18,\ p = 0.3,\quad q = 1 - p = 1 - 0.3 = 0.7,\ P(X = x) = 18C_x(0.3)^x(0.7)^{18-x}$

(i) $P(X \geq 10) = 1 - P(X < 10) = 1 - [P(0) + P(1) + \cdots + P(9)] = 1 - 0.9790 = 0.021$

(ii) $P(X = 6) = 18C_6(0.3)^6(0.7)^{18-6} = 0.1873$

(iii) $P(4\ to\ 7\ survice) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$

$$= 18C_4(0.3)^4(0.7)^{18-4} + 18C_5(0.3)^5(0.7)^{18-5} + 18C_6(0.3)^6(0.7)^{18-6} + 18C_7(0.3)^7(0.7)^{18-7} = 0.6947$$

3. ***In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good (ii) At most there are 3 defective (iii) Exactly there are 3 defective bulbs.***

Solution : $n = 20,\ p = 10\% = \frac{10}{100} = 0.1,\quad q = 1 - p = 1 - 0.1 = 0.9,\ P(X = x) = 20C_x(0.1)^x(0.9)^{20-x}$

(i) $P(X = 0) = 20C_0(0.1)^0(0.9)^{20-0} = 0.1216$

(ii) $P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= 20C_0(0.1)^0(0.9)^{20-0} + 20C_1(0.1)^1(0.9)^{20-1} + 20C_2(0.1)^2(0.9)^{20-2} + 20C_3(0.1)^3(0.9)^{20-3}$$

$$= 0.8671$$

(iii) $P(X = 3) = 20C_3(0.1)^3(0.9)^{20-3} = 0.1901$

4. ***It is known that the probability of an item produced by a certain machine will be defective is 5%. If the produced items are sent to the market in packets of 20, find the number of packets containing (i) At least 2 defective items (ii) At most 2 defective items in a consignment of 1000 packets using Binomial distribution.***

Solution : $n = 20,\ p = 5\% = \frac{5}{100} = 0.05,\quad q = 1 - p = 0.95\ ,\ P(X = x) = 20C_x(0.05)^x(0.95)^{20-x}$

(i) $P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - 20C_0(0.05)^0(0.95)^{20-0} - 20C_1(0.05)^1(0.95)^{20-1} = 1 - 0.3585 - 0.3774 = 0.2641$$

$N\ P(X \geq 2) = 1000 \times 0.2641 = 264$

(ii) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= 20C_0(0.05)^0(0.95)^{20-0} + 20C_1(0.05)^1(0.95)^{20-1} + 20C_2(0.05)^2(0.95)^{20-2} = 0.9246$$

$N\ P(X \leq 2) = 1000 \times 0.9246\ = 925$

5. ***Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) At least 1 boy (iii) At most 2 girls (iv) Children of both genders. Assume equal prob. for boys and girls.***

Solution : Considering each child as a trial $n = 4$. Assuming that birth of a boy is a success, $p = \frac{1}{2},\quad q = \frac{1}{2}$.

Let X denote the no. of successes (boys). $P(X = x) = nC_x p^x q^{n-x},\ \ x = 0,1,\dots n.\ P(X = x) = 4C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$

(i) $P(2\ boys\ and\ 2\ girls) = P(X = 2) = 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} = \frac{3}{8}$

Number of families having 2 boys and 2 girls $= N\ P(X = 2) = 800\left(\frac{3}{8}\right) = 300$

(ii) $P(\text{At least 1 boy}) = P(X \geq 1) = 1 - P(X = 0) = 1 - 4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{4-0} = 1 - \frac{1}{16} = \frac{15}{16}$

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Number of families having At least 1 boy $= N\,P(X \geq 1) = 800\left(\frac{15}{16}\right) = 750$

(iii) $P(\text{At most 2 girls}) = P(\text{exactly } 0, 1 \text{ or } 2 \text{ girls}) = P(4) + P(3) + P(2) = 1 - [P(0) + P(1)]$

$$= 1 - 4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{4-0} - 4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{4-1} = \frac{11}{16}$$

Number of families having At most 2 girls $= 800\left(\frac{11}{16}\right) = 550$

(iv) $P(\text{Children of both sexes}) = 1 - P(\text{children of the same sex}) = 1 - [P(\text{all are boys}) + P(\text{all are girls})]$

$$= 1 - P(4) - P(0) = 1 - 4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4-4} - 4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{4-0} = \frac{7}{8}$$

Number of families having Children of both sexes $= 800\left(\frac{7}{8}\right) = 700$

6. ***For a binomial distribution the mean is 6 and variance is 2. Find the distribution and find $P(X = 1)$.***

Solution : Mean $= np = 6$, Variance $= npq = 2$, $\frac{npq}{np} = \frac{2}{6} \Rightarrow \boldsymbol{q = \frac{1}{3}}$, $\quad p = 1 - q = 1 - \frac{1}{3} = \boldsymbol{\frac{2}{3}}$

$np = 6 \Rightarrow n = \frac{6}{p} \Rightarrow n = \frac{6}{\left(\frac{2}{3}\right)} = \frac{18}{2} = 9$, $\boldsymbol{n = 9}$, $\boldsymbol{p = \frac{2}{3}}$, $\boldsymbol{q = \frac{1}{3}}$, $P(X = 1) = 9C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^{9-1} = 0.0012$

7. ***A Binomial variable X satisfies the relation $9P(X = 4) = P(X = 2)$ when $n = 6$. Find the parameter p.***

Solution : $\quad n = 6, \quad P(X = x) = nC_x p^x q^{n-x}, \;\; x = 0, 1, \dots n$

$$9P(X = 4) = P(X = 2) \;\Rightarrow\; 9 \times 6C_4 p^4 q^{6-4} = 6C_2 p^2 q^{6-2} \;\Rightarrow\; 9 \times 6C_4 p^4 q^2 = 6C_2 p^2 q^4$$

$$9 \times 6C_4 p^2 = 6C_2 q^2 \;\Rightarrow\; 135p^2 = 15q^2 \;\Rightarrow\; 9p^2 - q^2 = 0 \;\Rightarrow\; 9p^2 - (1-p)^2 = 0 \;\Rightarrow 8p^2 + 2p - 1 = 0$$

$$p = -\frac{1}{2} \;(or)\; \frac{1}{4}, \qquad \boldsymbol{p = \frac{1}{4}} \;(\because \boldsymbol{p\ cannot\ be\ negative})$$

8. ***A discrete RV X has moment generating function $M_X(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$. Find $E(X)$, $Var(X)$ and $P(X = 2)$.***

Solution : $M_X(t) = (q + pe^t)^n$, $p = \frac{3}{4}$, $q = \frac{1}{4}$, $n = 5$, $E(X) = np = \frac{15}{4}$, $\quad V(X) = npq = \frac{15}{16}$

$$P(X = 2) = 5C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{5-2} = \frac{45}{512}$$

POISSON DISTRIBUTION

Definition : The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region represented as t, is $\boldsymbol{P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}}$, $\boldsymbol{x = 0, 1, \dots \infty}$. where λ is the average number of outcomes per unit time or region.

Poisson distribution as Limiting Form of Binomial Distribution

Poisson distribution is a limiting case of binomial distribution under the following conditions

(i) n, the number of trials is indefinitely large, i.e., $n \to \infty$.

(ii) p, the constant probability of success in each trial is very small, i.e., $p \to 0$.

(iii) $\lambda = np$ is finite or $p = \frac{\lambda}{n}$ and $q = 1 - \frac{\lambda}{n}$, where λ is a positive real number.

Areas of Application:

1. The number of misprints on a page of a book.
2. The number of deaths due to accidents in a month on national highway 47.
3. The number of break downs of a printing machine in a day.
4. The number of vacancies occurring during a year in a particular department.

Moment Generating Function (m.g.f.) in Poisson Distribution

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\,p(x) = \sum_{x=0}^{\infty} e^{tx}\,\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty} e^{tx}\,\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} \qquad \left(\because \boldsymbol{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots}\right)$$

$$= e^{-\lambda}\left[\frac{(\lambda e^t)^0}{0!} + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right] = e^{-\lambda}\left[1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right] = e^{-\lambda}\,e^{\lambda e^t} = \boldsymbol{e^{\lambda(e^t - 1)}}$$

Mean and Variance Using Moment Generating Function in Poisson Distribution

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}e^{\lambda(e^t-1)}\right]_{t=0} = \left[e^{\lambda(e^t-1)}\,\lambda e^t\right]_{t=0} = e^{\lambda(e^0-1)}\,\lambda e^0 = \boldsymbol{\lambda} \qquad (\because \boldsymbol{e^0 = 1})$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\,\lambda e^t\,e^{\lambda(e^t-1)}\right]_{t=0} = \lambda\left[e^t\,e^{\lambda(e^t-1)}\,\lambda e^t + e^t\,e^{\lambda(e^t-1)}\right]_{t=0} = \boldsymbol{\lambda(\lambda + 1)}$$

$$V(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \boldsymbol{\lambda}$$

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Problems in Poisson distribution

1. ***On an average a typist makes 2 mistakes per page. What is the prob. that she will make (i) No errors on a page (ii) 4 or more errors on a particular page?***

Solution : $\lambda = 2$, Let X represent the number of errors on a page. $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0,1, \dots \infty$

(i) $P(X = 0) = \frac{e^{-2}\, 2^0}{0!} = e^{-2} = 0.1353$

(ii) $P(X \geq 4) = 1 - P(X < 4) = 1 - [P(0) + P(1) + P(2) + P(3)] = 1 - \left(\frac{e^{-2}\, 2^0}{0!} + \frac{e^{-2}\, 2^1}{1!} + \frac{e^{-2}\, 2^2}{2!} + \frac{e^{-2}\, 2^3}{3!}\right)$

$= 1 - e^{-2}\left(1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = 0.1434$

2. ***The number of monthly breakdown of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (i) Without a breakdown (ii) With only 1 breakdown (iii) With at least 1 breakdown***

Solution : $\lambda = 1.8$, Let X denote the number of breakdown of the computer in a month.

(i) $P(X = 0) = \frac{e^{-1.8}\,(1.8)^0}{0!} = e^{-1.8} = 0.1653$

(ii) $P(X = 1) = \frac{e^{-1.8}\,(1.8)^1}{1!} = e^{-1.8}(1.8) = 0.2975$

(iii) $P(X \geq 1) = 1 - P(X = 0) = 0.8347$

3. ***It is known that the probability of an item produced by a certain machine will be defective is 5%. If the produced items are sent to the market in packets of 20, find the number of packets containing (i) At least 2 defective items (ii) At most 2 defective items in a consignment of 1000 packets using Poisson distribution***

Solution : $p = 5\% = 0.05$, $q = 0.95$, $n = 20$, $\lambda = np = 20 \times \frac{5}{100} = 1$, $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0,1, \dots \infty$

(i) $P(X \geq 2) = 1 - P(X < 2) = 1 - \left[\frac{e^{-1}\,(1)^0}{0!} + \frac{e^{-1}\,(1)^1}{1!}\right] = 1 - [e^{-1} + e^{-1}] = 0.2642$

$NP(X \geq 2) = 1000 \times 0.2642 = 264$

(ii) $P(X \leq 2) = P(0) + P(1) + P(2) = \frac{e^{-1}\,(1)^0}{0!} + \frac{e^{-1}\,(1)^1}{1!} + \frac{e^{-1}\,(1)^2}{2!} = e^{-1} + e^{-1} + \frac{e^{-1}}{2} = 0.9197$

$N\,P(X \leq 2) = 1000 \times 0.9197 = 920$

4. ***A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective. What is approximate probability that a box will fail to meet the guaranteed quality?***

Solution: $p = 5\% = 0.05$, $q = 0.95$, $n = 100$, $\lambda = np = 100 \times \frac{5}{100} = 5$, $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0,1, \dots \infty$

$P(X > 10) = 1 - P(X \leq 10)$

$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10)]$

$= 1 - e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!} + \frac{5^8}{8!} + \frac{5^9}{9!} + \frac{5^{10}}{10!}\right] = 0.014$

5. ***In a book of 520 pages, 390 typographical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.***

Solution : $\lambda = \frac{390}{520} = 0.75$, $P(1\ page\ contains\ no\ error) = P(X = 0) = \frac{e^{-0.75}\,(0.75)^0}{0!} = e^{-0.75}$

$P(5\ pages\ contains\ no\ error) = (e^{-0.75})^5 = 0.0235$

6. ***Let X be a RV following Poisson distribution such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$. Find the mean and standard deviation of X.***

Solution : $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0,1, \dots \infty$

$P(X = 2) = 9P(X = 4) + 90P(X = 6) \Rightarrow \frac{e^{-\lambda} \lambda^2}{2!} = 9\frac{e^{-\lambda} \lambda^4}{4!} + 90\frac{e^{-\lambda} \lambda^6}{6!}$, $Dividing\ by\ e^{-\lambda}\ \lambda^2$

$\frac{1}{2!} = \frac{9\,\lambda^2}{4!} + \frac{90\,\lambda^4}{6!} \Rightarrow \frac{1}{2} = \frac{3\,\lambda^2}{8} + \frac{\lambda^4}{8} \Rightarrow \frac{\lambda^4}{8} + \frac{3\,\lambda^2}{8} - \frac{1}{2} = 0$

$\lambda^4 + 3\,\lambda^2 - 4 = 0 \Rightarrow \lambda = 1, -4$,

Mean = $\lambda = 1$, Variance = $\lambda = 1$, S.D. = $\sqrt{Variance} = 1$

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GEOMETRIC DISTRIBUTION

Definition: If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of trials on which the first success occurs, is $\boldsymbol{P(X = x) = p\, q^{x-1},\ \ x = 1, 2, \ldots \infty}$ **.**

Application: Geometric distribution has important application in queueing theory, related to the number of units which are being served or waiting to be served at any given time.

Moment Generating Function (M.G.F.) in Geometric Distribution

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\, p(x) = \sum_{x=1}^{\infty} e^{tx}\, p\, q^{x-1} = \frac{p}{q}\sum_{x=1}^{\infty}(q\,e^t)^x = \frac{p}{q}[(q\,e^t)^1 + (q\,e^t)^2 + (q\,e^t)^3 + \cdots]$$

$$= \frac{p}{q}(q\,e^t)[1 + (q\,e^t)^1 + (q\,e^t)^2 + \cdots] = pe^t[1 - q\,e^t]^{-1} = \boldsymbol{\frac{pe^t}{1-q\,e^t}} \qquad [\because \boldsymbol{(1-x)^{-1} = 1 + x + x^2 + \cdots}]$$

Mean and Variance using Moment Generating Function in Geometric Distribution

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{pe^t}{1-q\,e^t}\right)\right]_{t=0} = \left[\frac{(1-q\,e^t)pe^t - pe^t(-q\,e^t)}{(1-q\,e^t)^2}\right]_{t=0} \qquad \left[\because \boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}}\right]$$

$$= \left[\frac{pe^t - p\,q\,e^{2t} + p\,q\,e^{2t}}{(1-q\,e^t)^2}\right]_{t=0} = \left[\frac{pe^t}{(1-q\,e^t)^2}\right]_{t=0} = \frac{pe^0}{(1-q\,e^0)^2} = \frac{p}{p^2} = \boldsymbol{\frac{1}{p}} \qquad (\because \boldsymbol{p + q = 1})$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\ \frac{pe^t}{(1-q\,e^t)^2}\right]_{t=0} = \left[\frac{(1-q\,e^t)^2 pe^t - pe^t 2(1-q\,e^t)^1(-q\,e^t)}{(1-q\,e^t)^4}\right]_{t=0} \qquad \left[\because \boldsymbol{\frac{d}{dx}(u\,v) = u\,v' + v\,u'}\right]$$

$$= \left[\frac{(1-q\,e^0)^2 pe^0 - pe^0 2(1-q\,e^0)^1(-q\,e^0)}{(1-q\,e^0)^4}\right] = \frac{p^3 + 2p^2q}{p^4} = p^2\left(\frac{p+2q}{p^4}\right) = \boldsymbol{\frac{1+q}{p^2}}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{1+q-1}{p^2} = \boldsymbol{\frac{q}{p^2}}$$

ANOTHER FORM OF GEOMETRIC DISTRIBUTION

Definition: If X denotes the number of failure before the first success, then $\boldsymbol{P(X = x) = p\, q^x,\ \ x = 0, 1, 2, \ldots \infty}$ **.**

Moment Generating Function (M.G.F.) in Geometric Distribution

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\, p(x) = \sum_{x=0}^{\infty} e^{tx}\, p\, q^x = p\sum_{x=0}^{\infty}(q\,e^t)^x = p[1 + (q\,e^t)^1 + (q\,e^t)^2 + (q\,e^t)^3 + \cdots]$$

$$= p[1 - q\,e^t]^{-1} = \boldsymbol{\frac{p}{1-q\,e^t}} \qquad [\because \boldsymbol{(1-x)^{-1} = 1 + x + x^2 + \cdots}]$$

Mean and Variance using Moment Generating Function in Geometric Distribution

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{p}{1-q\,e^t}\right)\right]_{t=0} = \left[p\frac{d}{dt}(1 - q\,e^t)^{-1}\right]_{t=0} = [p(-1)(1 - q\,e^t)^{-1-1}(-q\,e^t)]_{t=0}$$

$$= [pqe^t\ (1 - q\,e^t)^{-2}]_{t=0} = pqe^0\ (1 - q\,e^0)^{-2} = \boldsymbol{\frac{q}{p}} \qquad (\because \boldsymbol{p + q = 1})$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\ \frac{p\,q\,e^t}{(1-q\,e^t)^2}\right]_{t=0} = \left[\frac{(1-q\,e^t)^2 p\,q\,e^t - p\,q\,e^t 2(1-q\,e^t)^1(-q\,e^t)}{(1-q\,e^t)^4}\right]_{t=0} \qquad \left[\because \boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}}\right]$$

$$E(X^2) = \left[\frac{(1-q\,e^0)^2 p\,q\,e^0 - p\,q\,e^0 2(1-q\,e^0)^1(-q\,e^0)}{(1-q\,e^0)^4}\right] = \frac{p^3q + 2p^2q^2}{p^4} = p^2q\left(\frac{p+2q}{p^4}\right) = \boldsymbol{\frac{q^2+q}{p^2}}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{q^2+q}{p^2} - \frac{q^2}{p^2} = \frac{q^2+q-q^2}{p^2} = \boldsymbol{\frac{q}{p^2}}$$

Memory less Property of Geometric Distribution

If X is a RV with geormetic distribution, then X lacks memory, in the sense that $\boldsymbol{P(X > s + t/X > s) = P(X > t)}$**.**

Proof : $\quad P(X = x) = p\, q^{x-1},\ \ x = 1, \ldots \infty$

$$P(X > s + t/X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)}$$

$$P(X > k) = \sum_{x=k+1}^{\infty} p\, q^{x-1} = p\, q^{k+1-1} + p\, q^{k+2-1} + p\, q^{k+3-1} + \cdots = p\, q^k + p\, q^{k+1} + p\, q^{k+2} + \cdots$$

$$= p\, q^k(1 + q^1 + q^2 + \cdots) = p\, q^k(1 - q)^{-1} = \frac{p\, q^k}{p} = \boldsymbol{q^k}$$

Hence $P(X > s + t) = q^{s+t}$ *and* $P(X > s) = q^s$

$$P(X > s + t/X > s) = \frac{q^{s+t}}{q^s} = \frac{q^s\, q^t}{q^s} = q^t = \boldsymbol{P(X > t)}$$

Problems in Geometric Distribution

1. ***If the probability that an applicant for a drivers license will pass the road test on any given trial is 0.8. What is the probability that he will finally pass the test (i) On the fourth trial (ii) In less than 4 trials?***

   Solution : $p = 0.8,\ q = 1 - p = 0.2,\ P(X = x) = p\, q^{x-1},\ x = 1, \ldots \infty$

   (i) $\quad P(X = 4) = (0.8)\,(0.2)^{4-1} = 0.0064$

   (ii) $\quad P(X < 4) = P(1) + P(2) + P(3) = (0.8)\,[(0.2)^{1-1} + (0.2)^{2-1} + (0.2)^{3-1}] = 0.992$

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2. ***A typist types 2 letters erroneously for every 100 letters. What is the probability that the 10^{th} letter typed is the 1^{st} erroneous letter?***

Solution : $p = \frac{2}{100} = 0.02,\ q = 1 - p = 0.98,\ P(X = 10) = (0.02)\,(0.98)^{10-1} = 0.0167$

3. ***A die is tossed until 6 appears. What is the probability that it must be tossed more than 5 times?***

Solution : $p = \frac{1}{6},\ q = 1 - p = \frac{5}{6},\ P(X = x) = p\, q^{x-1},\ x = 1, \dots \infty$

$$P(X > 5) = 1 - P(X \le 5) = 1 - [P(1) + P(2) + P(3) + P(4) + P(5)]$$
$$= 1 - \frac{1}{6}\left[\left(\frac{5}{6}\right)^{1-1} + \left(\frac{5}{6}\right)^{2-1} + \left(\frac{5}{6}\right)^{3-1} + \left(\frac{5}{6}\right)^{4-1} + \left(\frac{5}{6}\right)^{5-1}\right] = 0.4019$$

4. ***A trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8. (i) What is the probability that the target would be first hit at the 6^{th} attempt? (ii) What is the probability that it takes less than 5 shots?***

Solution : $p = 0.8,\ q = 1 - p = 0.2,\ P(X = x) = p\, q^{x-1},\ x = 1, \dots \infty$

(i) $P(X = 6) = (0.8)\,(0.2)^{6-1} = 0.00026$

(ii) $P(X < 5) = P(1) + P(2) + P(3) + P(4) = (0.8)\,[(0.2)^{1-1} + (0.2)^{2-1} + (0.2)^{3-1} + (0.2)^{4-1}] = 0.9984$

5. ***The probability that a candidate can pass in an exam is 0.6. (i) What is the probability that he pass in the 3^{rd} trial (ii) What is the probability that he pass before the 3^{rd} trial?***

Solution : : $p = 0.6,\ q = 1 - p = 0.4,\ P(X = x) = p\, q^{x-1},\ x = 1, \dots \infty$

(i) $P(X = 3) = (0.6)\,(0.4)^{3-1} = 0.096$

(ii) $P(X < 3) = P(1) + P(2) = (0.6)\,[(0.4)^{1-1} + (0.4)^{2-1}] = 0.84$

6. ***A discrete RV X has moment generating function $\boldsymbol{M_X(t) = (5 - 4\, e^t)^{-1}}$ find $\boldsymbol{P(X = 5\ or\ 6)}$.***

Solution : $P(X = x) = p\, q^x,\ x = 0, 1, \dots \infty,\qquad M_X(t) = p(1 - q\, e^t)^{-1}$

$$M_X(t) = (5 - 4\, e^t)^{-1} = \frac{1}{5}\left(1 - \frac{4}{5}\, e^t\right)^{-1},\ p = \frac{1}{5},\ q = \frac{4}{5}$$

$$P(X = 5\ or\ 6) = P(x = 5) + P(X = 6) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5 + \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^6 = 0.118$$

7. ***A coin is tossed until the first head occurs. Assuming that the tosses are independent and the probability of a head occurring is p, find the value of p so that the probability that an odd number of tosses is required is equal to 0.6. Can you find a value of p so that the probability is 0.5 that an odd number of tosses is required?***

Solution : $P(X = x) = p\, q^{x-1},\ x = 1, \dots \infty$

$$P(X = odd\ number) = P(1) + P(3) + P(5) + \cdots = p\,(q^{1-1} + q^{3-1} + q^{5-1} + \cdots)$$
$$= p(1 + q^2 + q^4 + \dots) = p[1 + (q^2) + (q^2)^2 + \dots]$$
$$= p(1 - q^2)^{-1} = \frac{p}{1 - q^2} = \frac{p}{(1-q)(1+q)} = \frac{p}{p(1+q)} = \frac{1}{1+q}$$

$Now\ \ \frac{1}{1+q} = 0.6 \Rightarrow \frac{1}{2-p} = 0.6 \Rightarrow 0.6(2 - p) = 1 \Rightarrow 0.6p = 0.2 \Rightarrow p = \frac{1}{3}$

$Now\ \ \frac{1}{1+q} = 0.5 \Rightarrow \frac{1}{2-p} = 0.5 \Rightarrow 0.5(2 - p) = 1 \Rightarrow 0.5p = 0 \Rightarrow p = 0$

Though we get $p = 0$ it is meaningless. Hence the value of p cannot be found out.

EXPONENTIAL DISTRIBUTION

Definition: A continuous RV X defined in $(0, \infty)$ is said to follow an exponential distribution if the probability density function is $\boldsymbol{f(x) = \lambda\, e^{-\lambda x},\ x \ge 0}$.

Application: Exponential distribution is useful in queueing theory and reliability theory. Time to failure of a component and time between arrivals can be modeled using exponential distribution.

Moment Generating Function (M.G.F.) in Exponential Distribution

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} e^{tx}\, \lambda\, e^{-\lambda x}\, dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx = \lambda\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty} = \boldsymbol{\frac{\lambda}{\lambda - t}}$$

Mean and Variance using Moment Generating Function in Exponential Distribution

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right)\right]_{t=0} = \lambda\left[\frac{d}{dt}(\lambda - t)^{-1}\right]_{t=0} = \lambda[(-1)(\lambda - t)^{-2}(-1)]_{t=0} = \boldsymbol{\frac{1}{\lambda}}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\, \lambda\, (\lambda - t)^{-2}\right]_{t=0} = \lambda\, [(-2)(\lambda - t)^{-3}(-1)]_{t=0} = \boldsymbol{\frac{2}{\lambda^2}}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boldsymbol{\frac{1}{\lambda^2}}$$

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Memory less Property of Exponential Distribution

Statement: ***If X is exponential distributed with parameter λ, then for any 2 positive integers s and t*** $\boldsymbol{P(X > s + t/X > s) = P(X > t)}$.

Proof : $P(X > s + t/X > s) = \frac{P(X>s+t \cap X>s)}{P(X>s)} = \frac{P(X>s+t)}{P(X>s)}$, $\quad f(x) = \lambda\, e^{-\lambda x},\ x \geq 0$

$$P(X > k) = \int_k^{\infty} f(x)\, dx = \int_k^{\infty} \lambda\, e^{-\lambda x}\, dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^{\infty} = \boldsymbol{e^{-\lambda k}}$$

Hence $P(X > s + t) = e^{-\lambda(s+t)}$ *and* $P(X > s) = e^{-\lambda s}$

$$P(X > s + t/X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \boldsymbol{P(X > t)}$$

Problems in Exponential Distribution

1. ***The time (in hours) required to repairs a machine is exponential, distributed with parameter*** $\lambda = \frac{1}{2}$. ***(i) What is the probability that the repair time exceeds 2 hours? (ii) What is the conditional prob. that a repair takes at least 10h given that its duration exceeds 9h?***

   Solution: $f(x) = \lambda\, e^{-\lambda x},\ x > 0, \quad f(x) = \frac{1}{2}\, e^{-\frac{x}{2}},\ x > 0$

   (i) $P(X > 2) = \int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{2}\, e^{-\frac{x}{2}}dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^{\infty} = \frac{e^{-\infty} - e^{-1}}{-1} = e^{-1} = 0.3679$

   (ii) $P(X \geq 10/X > 9) = P(X > 1) = \int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{2}\, e^{-\frac{x}{2}}dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_1^{\infty} = \frac{e^{-\infty} - e^{-\frac{1}{2}}}{-1} = e^{-\frac{1}{2}} = 0.6065$

2. ***The mileage which car owners get with a certain kind of radial tire is a RV having an exponential distribution with mean 40,000 km. Find the prob. that one of these tires will last (i) Atleast 20,000 km (ii) At most 30,000 km***

   Solution: $Mean = \frac{1}{\lambda} = 40{,}000\ km,\ \lambda = \frac{1}{40{,}000},\ \ f(x) = \lambda\, e^{-\lambda x},\ x > 0, \quad f(x) = \frac{1}{40{,}000}\, e^{-\frac{x}{40{,}000}},\ x > 0$

   (i) $P(X \geq 20{,}000) = \int_{20{,}000}^{\infty} f(x)dx = \int_{20{,}000}^{\infty} \frac{1}{40{,}000}\, e^{-\frac{x}{40{,}000}}dx = \frac{1}{40{,}000}\left[\frac{e^{-\frac{x}{40{,}000}}}{-\frac{1}{40{,}000}}\right]_{20{,}000}^{\infty} = e^{-\frac{1}{2}} = 0.6065$

   (ii) $P(X \leq 30{,}000) = \int_0^{30{,}000} f(x)dx = \int_0^{30{,}000} \frac{1}{40{,}000}\, e^{-\frac{x}{40{,}000}}dx = \frac{1}{40{,}000}\left[\frac{e^{-\frac{x}{40{,}000}}}{-\frac{1}{40{,}000}}\right]_0^{30{,}000} = 1 - e^{-\frac{3}{4}} = 0.527$

3. ***The length of time a person speaks over phone follows exponential distribution with mean 6. What is the probability that the person will talk for (i) More than 8 min (ii) Less than 4 min (iii) Between 4 and 8 min***

   Solution: $Mean = \frac{1}{\lambda} = 6,\ \lambda = \frac{1}{6},\ f(x) = \lambda\, e^{-\lambda x},\ x > 0, \quad f(x) = \frac{1}{6}\, e^{-\frac{x}{6}},\ x > 0$

   (i) $P(X > 8) = \int_8^{\infty} f(x)dx = \int_8^{\infty} \frac{1}{6}\, e^{-\frac{x}{6}}\, dx = \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}}\right]_8^{\infty} = e^{-\frac{4}{3}} = 0.2635$

   (ii) $P(X < 4) = \int_0^4 f(x)dx = \int_0^4 \frac{1}{6}\, e^{-\frac{x}{6}}\, dx = \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}}\right]_0^4 = 1 - e^{-\frac{2}{3}} = 0.4865$

   (iii) $P(4 \leq X \leq 8) = \int_4^8 \frac{1}{6}\, e^{-\frac{x}{6}}\, dx = \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}}\right]_4^8 = e^{-\frac{2}{3}} - e^{-\frac{4}{3}} = 0.5134 - 0.2635 = 0.2499$

4. ***The amount of time that a watch can run without having to be reset is a random variable having exponential distribution, with mean 120 days. Find the prob. that such a watch will have to be reset in less than 24 days.***

   Solution:, $Mean = \frac{1}{\lambda} = 120\ days,\ \lambda = \frac{1}{120},\ \ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = \frac{1}{120}\, e^{-\frac{x}{120}},\ x > 0$

   $$P(X < 24) = \int_0^{24} f(x)dx = \int_0^{24} \frac{1}{120}\, e^{-\frac{x}{120}}dx = \frac{1}{120}\left[\frac{e^{-\frac{x}{120}}}{-\frac{1}{120}}\right]_0^{24} = 1 - e^{-\frac{1}{5}} = 0.1813$$

5. ***The number of kilo meters that a car can run before its battery has to be replaced is exponentially distributed with an average of 10,000 kms. If the owner desires to take a tour consisting of 8000 kms, what is the probability that he will be able to complete is his tour with our replacing the battery?***

   Solution:, $Mean = \frac{1}{\lambda} = 10{,}000,\ \lambda = \frac{1}{10{,}000},\ \ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = \frac{1}{10{,}000}\, e^{-\frac{x}{10{,}000}},\ x > 0$

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$$P(X > 8000) = \int_{8000}^{\infty} f(x)dx = \int_{8000}^{\infty} \frac{1}{10{,}000}\, e^{-\frac{x}{10{,}000}} dx = \frac{1}{10{,}000}\left[\frac{e^{-\frac{x}{10{,}000}}}{-\frac{1}{10{,}000}}\right]_{8000}^{\infty} = e^{-\frac{4}{5}} = 0.4493$$

6. ***In a construction site, 3 lorries unload materials per hour, on an average. What is the probability that the time between arrival of successive lorries will be (i) at least 30 minutes (ii) less than 10 minutes?***

Solution: $\lambda = 3,\ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = 3\, e^{-3x},\ x > 0$

(i) Probability that the time between arrival of successive lorries equal to 30 minutes or $\frac{1}{2}$ hour

$$P\left(X \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^{\infty} f(x)dx = \int_{\frac{1}{2}}^{\infty} 3\, e^{-3x}dx = 3\left[\frac{e^{-3x}}{-3}\right]_{\frac{1}{2}}^{\infty} = e^{-\frac{3}{2}} = 0.223$$

(ii) Probability that the time between arrival of successive lorries equal to 10 minutes or $\frac{1}{6}$ hour

$$P\left(X < \frac{1}{6}\right) = \int_{0}^{\frac{1}{6}} f(x)dx = \int_{0}^{\frac{1}{6}} 3\, e^{-3x}dx = 3\left[\frac{e^{-3x}}{-3}\right]_{0}^{\frac{1}{6}} = 1 - e^{-\frac{1}{2}} = 0.393$$

7. ***The life length X of an electronic component follows an exponential distribution. There are 2 processes by which the component may be manufactured. The expected life length of the component is 100h. if process I is used to manufacture, while it is 150 h if process II is used. The cost of manufacturing a single component by process I is Rs. 10, while it is Rs. 20 for process II. Moreover if the component lasts less than the guaranteed life of 200 h, a loss of Rs. 50 is to be borne by the manufacturer. Which process is advantageous to the manufacturer?***

Solution: If process I is used, the density function of X is given by

$$Mean = \frac{1}{\lambda} = 100,\ \lambda = \frac{1}{100},\ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = \frac{1}{100}\, e^{-\frac{x}{100}},\ x > 0$$

$$P(X \geq 200) = \int_{200}^{\infty} \frac{1}{100}\, e^{-\frac{x}{100}}dx = \frac{1}{100}\left[\frac{e^{-\frac{x}{100}}}{-\frac{1}{100}}\right]_{200}^{\infty} = e^{-2}$$

$$P(X < 200) = \int_{0}^{200} \frac{1}{100}\, e^{-\frac{x}{100}}dx = \frac{1}{100}\left[\frac{e^{-\frac{x}{100}}}{-\frac{1}{100}}\right]_{0}^{200} = 1 - e^{-2}$$

Similarly, if process II is used, the density function of X is given by

$$Mean = \frac{1}{\lambda} = 150,\ \lambda = \frac{1}{150},\ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = \frac{1}{150}\, e^{-\frac{x}{150}}\ x > 0$$

$$P(X \geq 200) = \int_{200}^{\infty} \frac{1}{150}\, e^{-\frac{x}{150}}dx = \frac{1}{150}\left[\frac{e^{-\frac{x}{150}}}{-\frac{1}{150}}\right]_{200}^{\infty} = e^{-\frac{4}{3}}$$

$$P(X < 200) = \int_{0}^{200} \frac{1}{150}\, e^{-\frac{x}{150}}dx = \frac{1}{150}\left[\frac{e^{-\frac{x}{150}}}{-\frac{1}{150}}\right]_{0}^{200} = 1 - e^{-\frac{4}{3}}$$

Let C_1 and C_2 be the costs per component corresponding to the processes I and II respectively.

Then $C_1 = \begin{cases} 10, & X \geq 200 \\ 60, & X < 200 \end{cases}$

$$E(C_1) = 10 \times P(X \geq 200) + 60 \times P(X < 200) = 10 \times e^{-2} + 60 \times (1 - e^{-2}) = 60 - 50e^{-2} = 53.235$$

Now $C_2 = \begin{cases} 20, & X \geq 200 \\ 70, & X < 200 \end{cases}$

$$E(C_2) = 20 \times P(X \geq 200) + 70 \times P(X < 200) = 20 \times e^{-\frac{4}{3}} + 70 \times \left(1 - e^{-\frac{4}{3}}\right) = 70 - 50e^{-\frac{4}{3}} = 56.765$$

Since $E(C_1) < E(C_2)$, process I is advantageous to the manufacturer.

NORMAL DISTRIBUTION

The Normal distribution was first described by De Moive in 1933 as the limiting form of Binomial distribution as the number of trials becomes infinite. This discovery came into limelight after its discovery by both Laplace and Gauss half a century later. So this distribution is also called Gaussion distribution.

Definition: A continuous RV X, with parameters μ and σ^2 is normal if it has a probability density function

$$\boldsymbol{f(x) = \frac{1}{\sigma\sqrt{2\,\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}},\ -\infty < x < \infty$$

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Standard Normal distribution:

If X is a RV following normal distribution with parameter μ and σ, then $z = \frac{X-\mu}{\sigma}$ is called a Standard Normal variate and the p.d.f. of the standard variate Z is given by $\boldsymbol{\varphi(z)} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}\,\boldsymbol{\pi}}}\, \boldsymbol{e}^{-\frac{z^2}{2}}, -\infty < z < \infty$

Application: (i) The most important continuous probability distribution in the statistics field is Normal distribution. (ii) In nature like rainfall and meteorological studies (iii) In industry (iv) In error calculation of experiments (v) Statistical quality control (vi) Radar applications and in research.

Problems in Normal Distribution

1. ***If X is normally distributed and the mean X is 12 and the SD is 4. Find out the probability of the following (i) $X \geq 20$ (ii) $X \leq 20$ (iii) $0 \leq X \leq 12$***

   Solution: $\mu = 12,\ \sigma = 4$

(i) $P(X \geq 20) = P\left(\frac{X-\mu}{\sigma} \geq \frac{20-\mu}{\sigma}\right) = P\left(Z \geq \frac{20-12}{4}\right) = P(Z \geq 2) = 0.5 - P(0 \leq Z \leq 2) = 0.5 - 0.4772 = 0.0228$

(ii) $P(X \leq 20) = P\left(\frac{X-\mu}{\sigma} \leq \frac{20-\mu}{\sigma}\right) = P\left(Z \leq \frac{20-12}{4}\right) = P(Z \leq 2) = 0.5 + P(0 \leq Z \leq 2) = 0.5 + 0.4772 = 0.9772$

(iii) $P(0 \leq X \leq 12) = P\left(\frac{0-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{12-\mu}{\sigma}\right) = P\left(\frac{-12}{4} \leq Z \leq \frac{12-12}{4}\right) = P(-3 \leq Z \leq 0) = P(0 \leq Z \leq 3) = 0.4987$

2. ***In an examination the marks obtained by the students in Maths, Physics and Chemistry are normally distributed about mean 50, 52, 48 and S.D. 15, 12, 16 respectively. Find the prob. of securing a total mark of 180 or above.***

   Solution: Let X, Y, Z be the marks of respective subjects. The total marks $T = X + Y + Z$

   $\mu = E(T) = E(X+Y+Z) = E(X) + E(Y) + E(Z) = 50 + 52 + 48 = 150$

   $\sigma^2 = V(T) = V(X+Y+Z) = V(X) + V(Y) + V(Z) = 15^2 + 12^2 + 16^2 = 225 + 144 + 256 = 625$, $\sigma = 25$

   $P(T \geq 180) = P\left(Z \geq \frac{180-150}{25}\right) = P(Z \geq 1.2) = 0.5 - P(0 \leq Z \leq 1.2) = 0.5 - 0.3849 = 0.1151$

3. ***If the actual amount of instant coffee which a filling machine puts into '6 – ounce' jars is a RV having a normal distribution with S.D. is 0.05 ounce and if only 3% of the jars are to contain less than 6 ounce of coffee, what must be the mean fill of these jars?***

   Solution: Let X be the actual amount of coffee put into the jars. Then X follows $N(\mu, \sigma)$, $\sigma = 0.05$

   $P(X < 6) = 3\% = \frac{3}{100} = 0.03 \Rightarrow P\left(-\infty < Z < \frac{6-\mu}{0.05}\right) = 0.03 \Rightarrow P\left(0 < Z < \frac{\mu-6}{0.05}\right) = 0.5 - 0.03 = 0.47$

   From the table, $\frac{\mu-6}{0.05} = 1.808$, $\boldsymbol{\mu = 6.0904\ ounces}$

4. ***In an engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%, between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of the students failed in the examination and 5% of the students got distinction. Find the percentages of students who have got first class and second class.***

   Solution: Let X follow the distribution $N(\mu, \sigma)$.

Given: $P(X < 45) = 0.10$ *and* $P(X > 75) = 0.05$

$P\left(-\infty < \frac{X-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right) = 0.1$ *and* $P\left(\frac{75-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \infty\right) = 0.05$

$P\left(-\infty < Z < \frac{45-\mu}{\sigma}\right) = 0.1$ *and* $P\left(\frac{75-\mu}{\sigma} < Z < \infty\right) = 0.05$

$P\left(0 < Z < \frac{\mu-45}{\sigma}\right) = 0.4$ *and* $P\left(0 < Z < \frac{75-\mu}{\sigma}\right) = 0.45$

From the table, $\frac{\mu-45}{\sigma} = 1.28$ *and* $\frac{75-\mu}{\sigma} = 1.64$

$\mu - 1.28\sigma = 45$ (1) *and* $\mu + 1.64\sigma = 75$ (2)

Solving equations (1) and (2), $\mu = 58.15$ *and* $\sigma = 10.28$

$P(Students\ gets\ first\ class) = P(60 < X < 75) = P\left(\frac{60-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{75-\mu}{\sigma}\right) = P\left(\frac{60-58.15}{10.28} < Z < \frac{75-58.15}{10.28}\right)$

$= P(0.18 < Z < 1.64) = P(0 < Z < 1.64) - P(0 < Z < 0.18)$

$= 0.4495 - 0.0714 = 0.3781$

Percentage of students getting first class = 38

Now percentage of students getting second class = 100 – (students who have failed, got 1^{st} class and got distinction)

Percentage of students getting second class $= 100 - (10 + 38 + 5) = 47.$

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5. ***The savings bank account of a customer showed an average balance of Rs. 150 and a S.D. of Rs. 50. Assuming that the account balances are normally distributed (i) What percentage of account is over Rs. 200? (ii) What percentage of account is between Rs. 120 & Rs. 170? (iii) What % of account is less than Rs. 75?***

Solution: $\mu = 150,\ \sigma = 50$

(i) $P(X \geq 200) = P\left(Z \geq \frac{200-150}{50}\right) = P(Z \geq 1) = 0.5 - P(0 \leq Z \leq 1) = 0.5 - 0.3413 = 0.1587$

Percentage of account is over Rs. 200 is 15.87%

(ii) $P(120 < X < 170) = P\left(\frac{120-150}{50} < Z < \frac{170-150}{50}\right) = P(-0.6 < Z < 0.4)$

$= P(0 < Z < 0.6) + P(0 < Z < 0.4) = 0.2257 + 0.1554 = 0.3811$

Percentage of account is between Rs. 120 & Rs. 170 is 38.11%

(iii) $P(X < 75) = P\left(Z < \frac{75-150}{50}\right) = P(Z < -1.5) = 0.5 - P(0 < Z < 1.5) = 0.5 - 0.4332 = 0.0668$

Percentage of account is less than Rs. 75 is 6.68%

6. ***In a newly constructed township, 2000 electric lamps are installed with an average life of 1000 burning hours and standard deviation of 200hours. Assuming the life of the lamps follows normal distribution, find***

(i) The number of lamps expected to fail during the first 700 hours.

(ii) In what period of burning hours 10% of the lamps fail.

Solution: $\mu = 1000,\ \sigma = 200$

(i) $P(X \leq 700) = P\left(\frac{X-\mu}{\sigma} < \frac{700-1000}{200}\right) = P(Z < -1.5) = P(Z > 1.5) = 0.5 - P(0 < Z < 1.5)$

$= 0.5 - 0.4332 = 0.0668$ ***(∵From Normal Table)***

The no. of lamps that fail to burn in the first 700 hours $= 2000 \times 0.0668 = 133.6 \approx 134$

(ii) Let t be the period at which 10% of lamps fail.

$$P(X \leq t) = 0.1 \Rightarrow P\left(\frac{X-\mu}{\sigma} \leq \frac{t-1000}{200}\right) = 0.1 \Rightarrow P\left(Z \geq \frac{1000-t}{200}\right) = 0.1$$

$P\left(0 \leq Z \leq \frac{1000-t}{200}\right) = 0.5 - 0.1 = 0.4 \Rightarrow \frac{1000-t}{200} = 1.28 \Rightarrow \boldsymbol{t = 744}$ ***(∵From Normal Table)***

7. ***The marks obtained by a number of students in a certain subject are approximately normally distributed with mean 65 and standard deviation 5. If 3 students are selected at random from this group, what is the probability that at least 1 of them would have scored above 75?***

Solution: $\mu = 65,\ \sigma = 5$

$P(X > 75) = P\left(\frac{X-\mu}{\sigma} > \frac{75-\mu}{\sigma}\right) = P\left(Z > \frac{75-65}{5}\right) = P(Z > 2) = 0.5 - P(0 < Z < 2) = 0.5 - 0.4772 = 0.0228$

$p = P(a\ student\ scores\ above\ 75) = 0.0228$, $q = 1 - p = 0.9772\ and\ n = 3$

$P(y) = nC_y p^y q^{n-y},\ \ y = 0,1,\dots n$, $P(y) = 3C_y(0.0228)^y(0.9772)^{3-y},\ \ y = 0,1,\dots n$

$P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(0) = 1 - 3C_0(0.0228)^0(0.9772)^{3-0} = 0.0667$

8. ***In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and S.D.***

Solution: Let mean by μ and standard deviation σ

31% of the items are under 45	***and***	***8% are over 64***
$P(X < 45) = 31\%$	***and***	$P(X > 64) = 8\%$
$P\left(\frac{X-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right) = 0.31$	***and***	$P\left(\frac{X-\mu}{\sigma} > \frac{64-\mu}{\sigma}\right) = 0.08$
$P\left(Z < \frac{45-\mu}{\sigma}\right) = 0.31$	***and***	$P\left(Z > \frac{64-\mu}{\sigma}\right) = 0.08$
$P\left(0 < Z < \frac{\mu-45}{\sigma}\right) = 0.5 - 0.31 = 0.19$	***and***	$P\left(0 < Z < \frac{64-\mu}{\sigma}\right) = 0.5 - 0.08 = 0.42$
$\frac{\mu-45}{\sigma} = 0.5$	***and***	$\frac{64-\mu}{\sigma} = 1.4$
$\mu - 0.5\sigma = 45$	***and***	$\mu + 1.4\sigma = 64$

Solving for μ and σ, we get $\boldsymbol{\mu = 50}$ and $\boldsymbol{\sigma = 10}$

9. ***In a distribution exactly normal 7% of the items are under 35 and 89% are under 63. What are the mean and S.D. of the distribution?***

Solution: Let mean by μ and standard deviation σ

7% of the items are under 35	***and***	***89% are over 63***
$P(X < 35) = 7\%$	***and***	$P(X < 63) = 89\%$

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$$P\left(\frac{X-\mu}{\sigma} < \frac{35-\mu}{\sigma}\right) = 0.07 \quad \textit{and} \quad P\left(\frac{X-\mu}{\sigma} < \frac{63-\mu}{\sigma}\right) = 0.89$$

$$P\left(Z < \frac{35-\mu}{\sigma}\right) = 0.07 \quad \textit{and} \quad P\left(Z < \frac{63-\mu}{\sigma}\right) = 0.89$$

$$P\left(0 < Z < \frac{\mu-35}{\sigma}\right) = 0.5 - 0.07 = 0.43 \quad \textit{and} \quad P\left(0 < Z < \frac{63-\mu}{\sigma}\right) = 0.89 - 0.5 = 0.39$$

$$\frac{\mu-35}{\sigma} = 1.48 \quad \textit{and} \quad \frac{63-\mu}{\sigma} = 1.23$$

$$\mu - 1.48\sigma = 35 \quad \textit{and} \quad \mu + 1.23\sigma = 63$$

Solving for μ and σ, we get $\boldsymbol{\mu = 50.3}$ and $\boldsymbol{\sigma = 10.33}$

10. In a normal distribution of a large group of men 5% are under 60 in height and 40% are between 60 and 65. Find the mean height and S.D.

Solution: Let mean by μ and standard deviation σ

5% of the items are under 60 **_and_** ***40% are between 60 and 65***

$$P(X < 60) = 5\% \quad \textit{and} \quad P(60 < X < 65) = 40\% = 0.4$$

$$P\left(\frac{X-\mu}{\sigma} < \frac{60-\mu}{\sigma}\right) = 0.05 \quad \textit{and} \quad P(X < 65) = P(X < 60) + P(60 < X < 65)$$

$$P\left(Z < \frac{60-\mu}{\sigma}\right) = 0.05 \quad \textit{and} \quad P\left(\frac{X-\mu}{\sigma} < \frac{65-\mu}{\sigma}\right) = 0.05 + 0.4 = 0.45$$

$$P\left(0 < Z < \frac{\mu-60}{\sigma}\right) = 0.5 - 0.05 = 0.45 \quad \textit{and} \quad \left(0 < Z < \frac{\mu-65}{\sigma}\right) = 0.5 - 0.45 = 0.05$$

$$\frac{\mu-60}{\sigma} = 1.645 \quad \textit{and} \quad \frac{\mu-65}{\sigma} = 0.13$$

$$\mu - 1.645\sigma = 60 \quad \textit{and} \quad \mu + 0.13\sigma = 65$$

Solving for μ and σ, we get $\boldsymbol{\mu = 65.42}$ and $\boldsymbol{\sigma = 3.29}$

11. If X is $N(3,4)$. Find k so that $P(|X-3| > k) = 0.05$.

Solution: Let mean by μ and standard deviation σ. $\mu = 3,\ \sigma^2 = 4,\ \sigma = 2,\ Z = \frac{X-\mu}{\sigma} = \frac{X-3}{2}$

$$P(|X-3| > k) = 0.05 \Rightarrow P\left(\left|\frac{X-3}{2}\right| > \frac{k}{2}\right) = 0.05 \Rightarrow P\left(|Z| > \frac{k}{2}\right) = 0.05 \Rightarrow 2\,P\left(Z > \frac{k}{2}\right) = 0.05$$

$$P\left(Z > \frac{k}{2}\right) = 0.025 \Rightarrow 0.5 - P\left(0 < Z < \frac{k}{2}\right) = 0.025 \Rightarrow P\left(0 < Z < \frac{k}{2}\right) = 0.475 \Rightarrow \frac{k}{2} = 1.96 \Rightarrow \boldsymbol{k = 3.92}$$

Unit – II : Two Dimensional Random Variables

Syllabus

- Two Dimensional Discrete Random Variables
- Two Dimensional Continuous Random Variables
- Marginal Distribution Function
- Conditional Distribution Function
- Independent Random Variables
- Cumulative Distribution Function
- Transformation of Random Variables
- Central Limit Theorem

TWO DIMENSIONAL RANDOM VARIABLE

Definitions: Let S be the sample space associated with a random experiment E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes $s \in S$. Then (X, Y) is called a two dimensional RV.

Example: Throwing a die and tossing a coin simultaneously. We get a two dimensional sample space. The outcome of throwing a die is represented as random variable Y and the outcome of tossing a coin is represented as random variable X. When both these random variables occur simultaneously or jointly, the probabilities of joint occurrences can be written in a matrix form called joint probability matrix $P(X, Y)$.

Two Dimensional Random Variable

Two Dimensional Discrete Random Variable | **Two Dimensional Continuous Random Variable**

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DISCRETE **Two Dimensional Random Variable**	**CONTINUOUS** **Two Dimensional Random Variable**
Joint Probability Mass Function (j.p.m.f.) $\sum_i \sum_j P(x_i, y_j) = 1$	***Joint Probability Density Function (j.p.d.f.)*** $\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x,y)dx\,dy = 1$
Marginal Probability Function of X $P(X = x_i) = \sum_j P(x_i, y_j)$	***Marginal Probability Function of X*** $f(x) = \int_{y=-\infty}^{\infty} f(x,y)dy$
Marginal Probability Function of Y $P(Y = y_i) = \sum_i P(x_i, y_j)$	***Marginal Probability Function of Y*** $f(y) = \int_{x=-\infty}^{\infty} f(x,y)dx$
Conditional Probability X given Y $P(x/y) = \frac{P(x,y)}{P(y)}$	***Conditional Probability X given Y*** $f(x/y) = \frac{f(x,y)}{f(y)}$
Conditional Probability Y given X $P(y/x) = \frac{P(x,y)}{P(x)}$	***Conditional Probability Y given X*** $f(y/x) = \frac{f(x,y)}{f(x)}$
Independent Event $P(x,y) = P(x)\,P(y)$	***Independent Event*** $f(x,y) = f(x)\,f(y)$

TWO DIMENSIONAL DISCRETE RANDOM VARIABLE

Definitions: If the possible values of (X,Y) are finite or countably infinite, (X,Y) is called a two dimensional discrete random variable. The values of (X,Y) can be represented as $(x_i,\ y_j), i = 1, 2, \ldots,\ j = 1, 2, \ldots$.

PROBLEMS IN TWO DIMENSIONAL DISCRETE RANDOM VARIABLE

1. ***From the following distribution of (X,Y) find (i) $P(X \le 1)$ (ii) $P(Y \le 3)$ (iii) $P(X \le 1, Y \le 3)$ (iv) $P(X \le 1/Y \le 3)$ (v) $P(Y \le 3 / X \le 1)$ (vi) $P(X + Y \le 4)$.***

X \ Y	1	2	3	4	5	6
0	*0*	*0*	*1/32*	*2/32*	*2/32*	*3/32*
1	*1/16*	*1/16*	*1/8*	*1/8*	*1/8*	*1/8*
2	*1/32*	*1/32*	*1/64*	*1/64*	*0*	*2/64*

Solution:

X \ Y	1	2	3	4	5	6	$P(X = x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$P(X = 0) = \frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$P(X = 1) = \frac{20}{32}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$P(X = 2) = \frac{4}{32}$
$P(Y = y)$	$P(Y = 1) = \frac{3}{32}$	$P(Y = 2) = \frac{3}{32}$	$P(Y = 3) = \frac{11}{64}$	$P(Y = 4) = \frac{13}{64}$	$P(Y = 5) = \frac{6}{32}$	$P(Y = 6) = \frac{16}{64}$	*1*

(i) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{20}{32} = \frac{28}{32} = \mathbf{\frac{7}{8}}$

(ii) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \mathbf{\frac{23}{64}}$

(iii) $P(X \le 1, Y \le 3) = P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3) = 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \mathbf{\frac{9}{32}}$

(iv) $P(X \le 1/Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{\left(\frac{9}{32}\right)}{\left(\frac{23}{64}\right)} = \mathbf{\frac{18}{23}}$

(v) $P(Y \leq 3 / X \leq 1) = \frac{P(X\leq1, Y\leq3)}{P(X\leq1)} = \frac{\left(\frac{9}{32}\right)}{\left(\frac{7}{8}\right)} = \frac{\mathbf{9}}{\mathbf{28}}$

(vi) $P(X+Y \leq 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4) + P(1,1) + P(1,2) + P(1,3) + P(2,1) + P(2,2) = \frac{\mathbf{13}}{\mathbf{32}}$

2. ***The joint probability function (X, Y) is given by $P(x, y) = k(2x + 3y), x = 0, 1, 2\,; y = 1, 2, 3$***

(i) Find the marginal distributions (ii) Find the probability distribution of $(X+Y)$. (iii) Find all conditional probability distributions.

Solution:

X \ Y	1	2	3	$P(X=x)$
0	$3k$	$6k$	$9k$	$P(X=0) = 18k$
1	$5k$	$8k$	$11k$	$P(X=1) = 24k$
2	$7k$	$10k$	$13k$	$P(X=2) = 30k$
$P(Y=y)$	$P(Y=1) = 15k$	$P(Y=2) = 24k$	$P(Y=3) = 33k$	$72k$

$\sum_{i=0,1,2} \sum_{j=1,2,3} P(x_i, y_j) = 1 \Rightarrow 72k = 1 \Rightarrow \boldsymbol{k = \frac{1}{72}}$

X \ Y	1	2	3	$P(X=x)$
0	$\frac{3}{72}$	$\frac{6}{72}$	$\frac{9}{72}$	$P(X=0) = \frac{18}{72}$
1	$\frac{5}{72}$	$\frac{8}{72}$	$\frac{11}{72}$	$P(X=1) = \frac{24}{72}$
2	$\frac{7}{72}$	$\frac{10}{72}$	$\frac{13}{72}$	$P(X=2) = \frac{30}{72}$
$P(Y=y)$	$P(Y=1) = \frac{15}{72}$	$P(Y=2) = \frac{24}{72}$	$P(Y=3) = \frac{33}{72}$	1

(i) The marginal probability distribution of X

$P(X=x)$	$P(X=0) = \frac{18}{72}$	$P(X=1) = \frac{24}{72}$	$P(X=2) = \frac{30}{72}$

The marginal probability distribution of Y

$P(Y=y)$	$P(Y=1) = \frac{15}{72}$	$P(Y=2) = \frac{24}{72}$	$P(Y=3) = \frac{33}{72}$

(ii) Find the probability distribution of $(X+Y)$

$X+Y$	$P(X=x, Y=y)$
1	$P(0,1) = \frac{3}{72}$
2	$P(0,2) + P(1,1) = \frac{5}{72} + \frac{6}{72} = \frac{11}{72}$
3	$P(0,3) + P(1,2) + P(2,1) = \frac{7}{72} + \frac{8}{72} + \frac{9}{72} = \frac{24}{72}$
4	$P(1,3) + P(2,2) = \frac{11}{72} + \frac{10}{72} = \frac{21}{72}$
5	$P(2,3) = \frac{13}{72}$
Total	*1*

(iii) The conditional probability distribution of X given Y : $P(X=x/Y=y) = \frac{P(X=x,\ Y=y)}{P(Y=y)}$

$P(X=0/Y=1) = \frac{P(X=0,\ Y=1)}{P(Y=1)} = \frac{\left(\frac{3}{72}\right)}{\left(\frac{15}{72}\right)} = \frac{1}{5}, \quad P(X=1/Y=1) = \frac{P(X=1,\ Y=1)}{P(Y=1)} = \frac{\left(\frac{5}{72}\right)}{\left(\frac{15}{72}\right)} = \frac{1}{3}$

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$$P(X=2/Y=1)=\frac{P(X=2,\ Y=1)}{P(Y=1)}=\frac{\left(\frac{7}{72}\right)}{\left(\frac{15}{72}\right)}=\frac{7}{15},\quad P(X=0/Y=2)=\frac{P(X=0,\ Y=2)}{P(Y=2)}=\frac{\left(\frac{6}{72}\right)}{\left(\frac{24}{72}\right)}=\frac{1}{4}$$

$$P(X=1/Y=2)=\frac{P(X=2,\ Y=2)}{P(Y=2)}=\frac{\left(\frac{8}{72}\right)}{\left(\frac{24}{72}\right)}=\frac{1}{3},\quad P(X=2/Y=2)=\frac{P(X=2,\ Y=2)}{P(Y=2)}=\frac{\left(\frac{10}{72}\right)}{\left(\frac{24}{72}\right)}=\frac{5}{12}$$

$$P(X=0/Y=3)=\frac{P(X=0,\ Y=3)}{P(Y=3)}=\frac{\left(\frac{9}{72}\right)}{\left(\frac{33}{72}\right)}=\frac{3}{11},\quad P(X=1/Y=3)=\frac{P(X=1,\ Y=3)}{P(Y=3)}=\frac{\left(\frac{11}{72}\right)}{\left(\frac{33}{72}\right)}=\frac{1}{3}$$

$$P(X=2/Y=3)=\frac{P(X=2,\ Y=3)}{P(Y=3)}=\frac{\left(\frac{13}{72}\right)}{\left(\frac{33}{72}\right)}=\frac{13}{33}$$

The conditional probability distribution of Y given X : $\boldsymbol{P(Y=y/X=x)=\frac{P(X=x,\ Y=y)}{P(X=x)}}$

$$P(Y=1/X=0)=\frac{P(X=0,\ Y=1)}{P(X=0)}=\frac{\left(\frac{3}{72}\right)}{\left(\frac{18}{72}\right)}=\frac{1}{6},\quad P(Y=2/X=0)=\frac{P(X=0,\ Y=2)}{P(X=0)}=\frac{\left(\frac{6}{72}\right)}{\left(\frac{18}{72}\right)}=\frac{1}{3}$$

$$P(Y=3/X=0)=\frac{P(X=0,\ Y=3)}{P(X=0)}=\frac{\left(\frac{9}{72}\right)}{\left(\frac{18}{72}\right)}=\frac{1}{2},\quad P(Y=1/X=1)=\frac{P(X=1,\ Y=1)}{P(X=1)}=\frac{\left(\frac{5}{72}\right)}{\left(\frac{24}{72}\right)}=\frac{5}{24}$$

$$P(Y=2/X=1)=\frac{P(X=1,\ Y=2)}{P(X=1)}=\frac{\left(\frac{8}{72}\right)}{\left(\frac{24}{72}\right)}=\frac{1}{3},\quad P(Y=3/X=1)=\frac{P(X=1,\ Y=3)}{P(X=1)}=\frac{\left(\frac{11}{72}\right)}{\left(\frac{24}{72}\right)}=\frac{11}{24}$$

$$P(Y=1/X=2)=\frac{P(X=2,\ Y=1)}{P(X=2)}=\frac{\left(\frac{7}{72}\right)}{\left(\frac{30}{72}\right)}=\frac{7}{30},\quad P(Y=2/X=2)=\frac{P(X=2,\ Y=2)}{P(X=2)}=\frac{\left(\frac{10}{72}\right)}{\left(\frac{30}{72}\right)}=\frac{1}{3}$$

$$P(Y=3/X=2)=\frac{P(X=2,\ Y=3)}{P(X=2)}=\frac{\left(\frac{13}{72}\right)}{\left(\frac{30}{72}\right)}=\frac{13}{30}$$

3. ***The joint p.d.f. of (X,Y) where X & Y are discrete is given in the following table. Are X and Y are independent.***

X \ Y	**0**	**1**	**2**
0	**0.1**	**0.04**	**0.06**
1	**0.2**	**0.08**	**0.12**
2	**0.2**	**0.08**	**0.12**

Solution:

X \ Y	***0***	***1***	***2***	$\boldsymbol{P(X=x)}$
0	0.1	0.04	0.06	$\boldsymbol{P(X=0)=0.2}$
1	0.2	0.08	0.12	$\boldsymbol{P(X=1)=0.4}$
2	0.2	0.08	0.12	$\boldsymbol{P(X=2)=0.4}$
$\boldsymbol{P(Y=y)}$	$\boldsymbol{P(Y=0)=0.5}$	$\boldsymbol{P(Y=0)=0.2}$	$\boldsymbol{P(Y=0)=0.3}$	***1***

If X and Y are independent $\boldsymbol{P(X=x,Y=y)=P(x=x)\,P(y=y)}$

$P(x=0)\,P(y=0)=0.2\times 0.5=0.1=P(X=0,Y=0)$

$P(x=0)\,P(y=1)=0.2\times 0.2=0.04=P(X=0,Y=1)$

$P(x=0)\,P(y=2)=0.2\times 0.3=0.06=P(X=0,Y=2)$

$P(x=1)\,P(y=0)=0.4\times 0.5=0.2=P(X=1,Y=0)$

$P(x=1)\,P(y=1)=0.4\times 0.2=0.08=P(X=1,Y=1)$

$P(x=1)\,P(y=2)=0.4\times 0.3=0.12=P(X=1,Y=2)$

$P(x=2)\,P(y=0)=0.4\times 0.5=0.2=P(X=2,Y=0)$

$P(x=2)\,P(y=1)=0.4\times 0.2=0.08=P(X=2,Y=1)$

$P(x=2)\,P(y=2)=0.4\times 0.3=0.12=P(X=2,Y=2)$ ***∴ X and Y are independent.***

4. ***Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X,Y).***

Solution: As there are only 2 white balls in the box, X can take the values 0, 1, 2 & Y can take the values 0, 1, 2, 3.

$P(X=0, Y=0) = P(\text{drawing 3 balls none of which is white or red}) = P(\text{all the 3 balls drawn are black})$

$P(X=0, Y=0) = \frac{4C_3}{9C_3} = \frac{1}{21}$

$P(X=0, Y=1) = P(\text{drawing 1 red and 2 black balls}) = \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$

$P(X=0, Y=2) = P(\text{drawing 2 red and 1 black balls}) = \frac{3C_2 \times 4C_1}{9C_3} = \frac{1}{7}$

$P(X=0, Y=3) = P(\text{drawing 3 red balls}) = \frac{3C_3}{9C_3} = \frac{1}{84}$

$P(X=1, Y=0) = P(\text{drawing 1 white and 2 black balls}) = \frac{2C_1 \times 4C_2}{9C_3} = \frac{1}{7}$

$P(X=1, Y=1) = P(\text{drawing 1white , 1 red and 1 black balls}) = \frac{2C_1 \times 3C_1 \times 4C_1}{9C_3} = \frac{2}{7}$

$P(X=1, Y=2) = P(\text{drawing 1 white and 2 red balls}) = \frac{2C_1 \times 3C_2}{9C_3} = \frac{1}{14}$

$P(X=1, Y=3) = P(\text{drawing 1 white and 3 red balls}) = 0$ (Since only 3 balls are drawn)

$P(X=2, Y=0) = P(\text{drawing 2 white and 1 black balls}) = \frac{2C_2 \times 4C_1}{9C_3} = \frac{1}{21}$

$P(X=2, Y=1) = P(\text{drawing 2 white and 1 red balls}) = \frac{2C_2 \times 3C_1}{9C_3} = \frac{1}{28}$

$P(X=2, Y=2) = P(\text{drawing 2 white and 2 red balls}) = 0$ (Since only 3 balls are drawn)

$P(X=2, Y=3) = P(\text{drawing 2 white and 3 red balls}) = 0$ (Since only 3 balls are drawn)

The joint probability distribution of (X, Y) may be represented in the form of a table as given below

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

5. ***The joint probability mass function of two RVs X and Y is given by*** $p(x, y) = \frac{1}{27}(2x + y), x = 0, 1, 2, y = 0, 1, 2.$
(i) Find the conditional distribution of Y given $X = 2$ ***(ii) Find the conditional distribution of X given*** $Y = 1$
Solution:

X \ Y	0	1	2	$P(X = x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$P(X=0) = \frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$P(X=1) = \frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$P(X=2) = \frac{15}{27}$
$P(Y = y)$	$P(Y=0) = \frac{6}{27}$	$P(Y=0) = \frac{9}{27}$	$P(Y=0) = \frac{12}{27}$	*1*

(i) ***The conditional probability distribution of Y given*** $X = 2$

$P(Y = y/X = 2) = \frac{P(X=2,\ Y=y)}{P(X=2)}$, $\quad P(Y = 0/X = 2) = \frac{P(X=2,\ Y=0)}{P(X=2)} = \frac{\left(\frac{4}{27}\right)}{\left(\frac{15}{27}\right)} = \frac{4}{15}$

$P(Y = 1/X = 2) = \frac{P(X=2,\ Y=1)}{P(X=2)} = \frac{\left(\frac{5}{27}\right)}{\left(\frac{15}{27}\right)} = \frac{1}{3}$, $\quad P(Y = 2/X = 2) = \frac{P(X=2,\ Y=2)}{P(X=2)} = \frac{\left(\frac{6}{27}\right)}{\left(\frac{15}{27}\right)} = \frac{2}{5}$

(ii) ***The conditional probability distribution of X given*** $Y = 1$ ***:***

$P(X = x/Y = 1) = \frac{P(X=x,\ Y=1)}{P(Y=1)}$, $\quad P(X = 0/Y = 1) = \frac{P(X=0,\ Y=1)}{P(Y=1)} = \frac{\left(\frac{1}{27}\right)}{\left(\frac{9}{27}\right)} = \frac{1}{9}$

$P(X = 1/Y = 1) = \frac{P(X=1,\ Y=1)}{P(Y=1)} = \frac{\left(\frac{3}{27}\right)}{\left(\frac{9}{27}\right)} = \frac{1}{3}$, $\quad P(X = 2/Y = 1) = \frac{P(X=2,\ Y=1)}{P(Y=1)} = \frac{\left(\frac{5}{27}\right)}{\left(\frac{9}{27}\right)} = \frac{5}{9}$

TWO DIMENSIONAL CONTINUOUS RANDOM VARIABLE

Definitions : If (X, Y) can assume all values in a specified region R in xy - plane, (X, Y) is called a two dimensional continuous random variable.

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PROBLEMS IN TWO DIMENSIONAL DISCRETE RANDOM VARIABLE

1. ***The joint pdf of a two dimensional RV (X,Y) is given by $f(x,y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$. Compute (i) $P(X>1)$ (ii) $P\left(Y<\frac{1}{2}\right)$(iii) $P\left(X>1, Y<\frac{1}{2}\right)$ (iv) $P\left(X>1/Y<\frac{1}{2}\right)$ (v) $P\left(Y<\frac{1}{2}/X>1\right)$ (vi) $P(X<Y)$ (vi) $P(X+Y\le 1)$.***

Solution: ***Method I***

(i) $P(X>1) = \int_{y=0}^{1}\int_{x=1}^{2}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \int_{y=0}^{1}\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_1^2 dy = \int_{y=0}^{1}\left[\left(\frac{4y^2}{2}+\frac{8}{24}\right)-\left(\frac{y^2}{2}+\frac{1}{24}\right)\right]dy$

$= \int_{y=0}^{1}\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy = \left[\frac{3y^3}{6}+\frac{7y}{24}\right]_0^1 = \frac{3}{6}+\frac{7}{24} = \frac{19}{24}$

Method II - Using Marginal Density Function of X

$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)dy = \int_{y=0}^{1}\left(xy^2+\frac{x^2}{8}\right)dy = \left[\frac{xy^3}{3}+\frac{x^2y}{8}\right]_0^1 = \left(\frac{x}{3}+\frac{x^2}{8}\right), 0\le x\le 2$

(i) $P(X>1) = \int_{x=1}^{2} f_X(x)dx = \int_{x=1}^{2}\left(\frac{x}{3}+\frac{x^2}{8}\right)dx = \left[\frac{x^2}{6}+\frac{x^3}{24}\right]_1^2 = \left(\frac{4}{6}+\frac{8}{24}\right)-\left(\frac{1}{6}+\frac{1}{24}\right) = \frac{3}{6}+\frac{7}{24} = \frac{19}{24}$

Method I

(ii)$P\left(Y<\frac{1}{2}\right) = \int_{y=0}^{\frac{1}{2}}\int_{x=0}^{2}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \int_{y=0}^{\frac{1}{2}}\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^2 dy = \int_{y=0}^{\frac{1}{2}}\left(\frac{4y^2}{2}+\frac{8}{24}\right)dy = \left[\frac{2y^3}{3}+\frac{8y}{24}\right]_0^{\frac{1}{2}} = \frac{1}{4}$

Method II - Using Marginal Density Function of Y

$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)dx = \int_{x=0}^{2}\left(xy^2+\frac{x^2}{8}\right)dx = \left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^2 = \left(2y^2+\frac{1}{3}\right), 0\le y\le 1$

(ii)$P\left(Y<\frac{1}{2}\right) = \int_{y=0}^{\frac{1}{2}} f_Y(y)\,dy = \int_{y=0}^{\frac{1}{2}}\left(2y^2+\frac{1}{3}\right)dy = \left[\frac{2y^3}{3}+\frac{y}{3}\right]_0^{\frac{1}{2}} = \frac{2}{24}+\frac{1}{6} = \frac{6}{24} = \frac{1}{4}$

(iii) $P\left(X>1, Y<\frac{1}{2}\right) = \int_{y=0}^{\frac{1}{2}}\int_{x=1}^{2}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \int_{y=0}^{\frac{1}{2}}\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_1^2 dy = \int_{y=0}^{\frac{1}{2}}\left[\left(\frac{4y^2}{2}+\frac{8}{24}\right)-\left(\frac{y^2}{2}+\frac{1}{24}\right)\right]dy$

$= \int_{y=0}^{\frac{1}{2}}\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy = \left[\frac{3y^3}{6}+\frac{7y}{24}\right]_0^{\frac{1}{2}} = \frac{1}{16}+\frac{7}{48} = \frac{10}{48} = \frac{5}{24}$

(iv) $P\left(X>1/Y<\frac{1}{2}\right) = \frac{P\left(X>1,Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)} = \frac{\left(\frac{5}{24}\right)}{\left(\frac{1}{4}\right)} = \frac{5}{6}$

(v) $P\left(Y<\frac{1}{2}/X>1\right) = \frac{P\left(X>1,Y<\frac{1}{2}\right)}{P(X>1)} = \frac{\left(\frac{5}{24}\right)}{\left(\frac{19}{24}\right)} = \frac{5}{19}$

(iv) $P(X<Y) = \int_{y=0}^{1}\int_{x=0}^{y}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \int_{y=0}^{1}\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^y dy = \int_{y=0}^{1}\left(\frac{y^4}{2}+\frac{y^3}{24}\right)dy = \left[\frac{y^5}{10}+\frac{y^4}{96}\right]_0^1 = \frac{53}{480}$

(v) $P(X+Y\le 1) = \int_{y=0}^{1}\int_{x=0}^{1-y}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \int_{y=0}^{1}\left[\frac{x^2\,y^2}{2}+\frac{x^3}{24}\right]_0^{1-y} dy = \int_{y=0}^{1}\left(\frac{(1-y)^2\,y^2}{2}+\frac{(1-y)^3}{24}\right)dy$

$= \int_{y=0}^{1}\left[\frac{(y^2-2y^3+y^4)}{2}+\frac{(1-y)^3}{24}\right]dy = \left[\frac{y^3}{6}-\frac{y^4}{8}+\frac{y^5}{10}+\frac{(1-y)^4}{-96}\right]_0^1 = \frac{13}{480}$

2. ***If the joint pdf of a two dimensional RV (X,Y) is given by $f(x,y) = x^2+\frac{xy}{3}$, $0<x<1, 0<y<2$. Find (i) $P\left(X>\frac{1}{2}\right)$ (ii) $P(Y<X)$(iii) $P\left(X<\frac{1}{2}, Y<\frac{1}{2}\right)$ (iv) $P\left(Y<\frac{1}{2}/X<\frac{1}{2}\right)$.***

Solution: ***Marginal Density Function of X and Y***

$f_X(x) = \int_{y=0}^{2}\left(x^2+\frac{xy}{3}\right)dy = \left[x^2y+\frac{xy^2}{6}\right]_0^2 = \left(2x^2+\frac{2x}{3}\right), 0<x<1$

$f_Y(y) = \int_{x=0}^{1}\left(x^2+\frac{xy}{3}\right)dx = \left[\frac{x^3}{3}+\frac{x^2y}{6}\right]_0^1 = \left(\frac{1}{3}+\frac{y}{6}\right), 0<y<2$

(i) $P\left(X>\frac{1}{2}\right) = \int_{x=\frac{1}{2}}^{1} f_X(x)dx = \int_{x=\frac{1}{2}}^{1}\left(2x^2+\frac{2x}{3}\right)dx = \left[\frac{2\,x^3}{3}+\frac{x^2}{3}\right]_{\frac{1}{2}}^1 = \left(\frac{2}{3}+\frac{1}{3}\right)-\left(\frac{1}{12}+\frac{1}{12}\right) = 1-\frac{1}{6} = \frac{5}{6}$

(ii) $P(Y<X) = \int_{x=0}^{1}\int_{y=0}^{x}\left(x^2+\frac{xy}{3}\right)dy\,dx = \int_{x=0}^{1}\left[x^2y+\frac{xy^2}{6}\right]_0^x dx = \int_{x=0}^{1}\left(x^3+\frac{x^3}{6}\right)dx = \left[\frac{7x^4}{24}\right]_0^1 = \frac{7}{24}$

(iii) $P\left(X<\frac{1}{2}, Y<\frac{1}{2}\right) = \int_{y=0}^{\frac{1}{2}}\int_{x=0}^{\frac{1}{2}}\left(x^2+\frac{xy}{3}\right)dx\,dy = \int_{y=0}^{\frac{1}{2}}\left[\frac{x^3}{3}+\frac{x^2y}{6}\right]_0^{\frac{1}{2}} dy = \int_{y=0}^{\frac{1}{2}}\left(\frac{1+y}{24}\right)dy = \frac{1}{24}\left[y+\frac{y^2}{2}\right]_0^{\frac{1}{2}} = \frac{5}{192}$

(iv) $P\left(Y<\frac{1}{2}/X<\frac{1}{2}\right) = \frac{P\left(X<\frac{1}{2}, Y<\frac{1}{2}\right)}{P\left(X<\frac{1}{2}\right)} = \frac{P\left(X<\frac{1}{2}, Y<\frac{1}{2}\right)}{1-P\left(X>\frac{1}{2}\right)} = \frac{\left(\frac{5}{192}\right)}{1-\frac{5}{6}} = \frac{5}{192}\times 6 = \frac{5}{32}$

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3. ***The joint pdf of a two dimensional RV (X,Y) is given by $f(x,y) = kxy\, e^{-(x^2+y^2)},\ x > 0, y > 0$. Find the value of k and prove also that X and Y are independent.***

Solution: ***By the property of the joint pdf***

$\int_{y=0}^{\infty}\int_{x=0}^{\infty} kxy\, e^{-(x^2+y^2)}dx\, dy = 1 \Rightarrow k\left[\int_{y=0}^{\infty} y\, e^{-y^2}dy\right]\left[\int_{x=0}^{\infty} x\, e^{-x^2}dx\right] = 1$

$u = x^2 \Rightarrow du = 2x\, dx \Rightarrow x\, dx = \frac{du}{2}$, **Range**: $x = 0 \Rightarrow u = 0, x = \infty \Rightarrow u = \infty$

$v = y^2 \Rightarrow dv = 2y\, dy \Rightarrow y\, dy = \frac{dv}{2}$, **Range**: $y = 0 \Rightarrow v = 0, y = \infty \Rightarrow v = \infty$

$k\left[\int_{v=0}^{\infty}\frac{e^{-v}dv}{2}\right]\left[\int_{u=0}^{\infty}\frac{e^{-u}du}{2}\right] = 1 \Rightarrow \frac{k}{4}\left[\int_{v=0}^{\infty} e^{-v}dv\right]\left[\int_{u=0}^{\infty} e^{-u}du\right] = 1 \Rightarrow \frac{k}{4}\left[\frac{e^{-v}}{-1}\right]_0^{\infty}\left[\frac{e^{-u}}{-1}\right]_0^{\infty} = 1 \Rightarrow \boldsymbol{k = 4}$

Marginal Density Function of X and Y

$f_X(x) = \int_{y=0}^{\infty} 4xy\, e^{-(x^2+y^2)}dy = 4xe^{-x^2}\int_{y=0}^{\infty} y\, e^{-y^2}dy = 4xe^{-x^2}\int_{v=0}^{\infty}\frac{e^{-v}dv}{2} = 2xe^{-x^2}\left[\frac{e^{-v}}{-1}\right]_0^{\infty} = 2xe^{-x^2}, x > 0$

$f_Y(y) = \int_{x=0}^{\infty} 4xy\, e^{-(x^2+y^2)}dx = 4ye^{-y^2}\int_{y=0}^{\infty} xe^{-x^2}dx = 4y\, e^{-y^2}\int_{u=0}^{\infty}\frac{e^{-u}du}{2} = 2y\, e^{-y^2}\left[\frac{e^{-u}}{-1}\right]_0^{\infty} = 2y\, e^{-y^2}, y > 0$

$f_X(x)\, f_Y(y) = 2xe^{-x^2}2y\, e^{-y^2} = 4xy\, e^{-(x^2+y^2)} = f_{XY}(x,y)$ $\quad\therefore$ **The RVs x and y are independent.**

4. ***The joint pdf of a two dimensional RV (X,Y) is given by $f(x,y) = \begin{cases} 2 - x - y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & otherwise \end{cases}$.***

(i) Find the marginal density functions of X and Y. (ii) Conditional density functions.

Solution: (i) The Marginal Density Function of X and Y

$f_X(x) = \int_{y=0}^{1}(2 - x - y)dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1 = 2 - x - \frac{1}{2} = \frac{3-2x}{2},\ 0 \le x \le 1$

$f_Y(y) = \int_{x=0}^{1}(2 - x - y)dx = \left[2x - \frac{x^2}{2} - yx\right]_0^1 = 2 - \frac{1}{2} - y = \frac{3-2y}{2},\ 0 \le y \le 1$

(ii) The Conditional Density Function of X and Y

$f(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{(2-x-y)}{\left(\frac{3-2y}{2}\right)} = \frac{2(2-x-y)}{3-2y}, \qquad f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{(2-x-y)}{\left(\frac{3-2x}{2}\right)} = \frac{2(2-x-y)}{3-2x}$

5. ***If X and Y have joint pdf $f(x,y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & otherwise \end{cases}$.***

(i) Find the marginal density functions of X and Y. (ii) Check whether X and Y are independent.

Solution: The Marginal Density Function of X and Y

$f_X(x) = \int_{y=0}^{1}(x + y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2},\ 0 < x < 1$

$f_Y(y) = \int_{x=0}^{1}(x + y)dx = \left[\frac{x^2}{2} + yx\right]_0^1 = \frac{1}{2} + y,\ 0 < y < 1$

$f_X(x)\, f_Y(y) = \left(x + \frac{1}{2}\right)\left(\frac{1}{2} + y\right) \ne f(x,y)$ $\qquad\therefore$ ***X* and Y are not independent.**

TRANSFORMATION OF TWO DIMENSIONAL RANDOM VARIABLE

Let (X,Y) be a continuous random variable with joint probability density function $f(x,y)$. Let U and V be transformations such that $U = u(x,y), V = v(x,y)$. Then the joint probability density function (U,V) is $\boldsymbol{f(u,v) = f(x,y)|J|}$, where J is the jacobian of the transformation. $\boldsymbol{J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$ and $|J|$ is the modulus value of the Jacobian of the transformation.

1. ***If X and Y each follow an exponential distribution with parameter 1 and are independent, find the pdf of $U = X - Y$.***

Solution: Exponential distribution $f(x) = \lambda\, e^{-\lambda x}, x > 0$, Given: Parameter is 1, (i.e.) $\lambda = 1$,

$f(x) = e^{-x}, x > 0$ and $f(y) = e^{-y}, y > 0$. Since X and Y are independent.

$f(x,y) = f(x)f(y) = e^{-x}\, e^{-y} = e^{-(x+y)}, x > 0, y > 0$.; Let $u = x - y$ and $v = y \Rightarrow x = u + v$ and $y = v$;

$\frac{\partial x}{\partial u} = 1;\ \frac{\partial x}{\partial v} = 1;\ \frac{\partial y}{\partial u} = 0;\ \frac{\partial y}{\partial v} = 1; \quad |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$f(u,v) = f(x,y)|J| \Rightarrow f(u,v) = e^{-(x+y)}\, 1 = e^{-(u+v+v)} = e^{-(u+2v)}$

Range: $x > 0 \Rightarrow u + v > 0 \Rightarrow u > -v \Rightarrow -u < v$ and $y > 0 \Rightarrow v > 0$

$\boldsymbol{f(u,v) = e^{-(u+2v)},\ -u < v \text{ when } u < 0 \text{ and } v > 0 \text{ when } u > 0}$

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The marginal density function of U:

$$f(u)=\int_{v=-\infty}^{\infty} f(u,v)dv=\int_{v=-u}^{\infty} e^{-(u+2v)}dv=e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty}=e^{-u}\left[\frac{e^{-\infty}}{-2}-\frac{e^{2u}}{-2}\right]=\frac{e^{u}}{2},\ \boldsymbol{f(u)}=\begin{cases}\frac{e^{u}}{2}, u<0\\ \frac{e^{-u}}{2}, u>0\end{cases}.$$

2. ***If the joint pdf of (X,Y) is given by $f(x,y)=x+y, 0\le x\le 1; 0\le y\le 1$, find the pdf of $U=XY$.***

Solution: Let $u=xy$ and $v=y\Rightarrow x=\frac{u}{v}$ and $y=v$; $\frac{\partial x}{\partial u}=\frac{1}{v}$; $\frac{\partial x}{\partial v}=-\frac{u}{v^2}$; $\frac{\partial y}{\partial u}=0$; $\frac{\partial y}{\partial v}=1$

$$|J|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{v} & -\frac{u}{v^2}\\ 0 & 1\end{vmatrix}=\frac{1}{v};\ f(u,v)=f(x,y)|J|\Rightarrow f(u,v)=(x+y)\left(\frac{1}{v}\right)=\left(\frac{u}{v}+v\right)\left(\frac{1}{v}\right)$$

Range: $0\le x\le 1\Rightarrow 0\le\frac{u}{v}\le 1\Rightarrow 0\le u\le v$ and $0\le y\le 1\Rightarrow 0\le v\le 1$

$$\boldsymbol{f(u,v)=\left(\frac{u}{v}+v\right)\left(\frac{1}{v}\right),\ 0\le u\le v;\ \ 0\le v\le 1}$$

The marginal density function of U:

$$f(u)=\int_{v=-\infty}^{\infty} f(u,v)dv=\int_{v=u}^{1}\left(\frac{u}{v}+v\right)\left(\frac{1}{v}\right)dv=\int_{v=u}^{1}\left(\frac{u}{v^2}+1\right)dv=\left[\frac{u\,v^{-2+1}}{-1}+v\right]_u^1=2(1-u), 0<u<1.$$

3. ***If X and Y are independent RVs with $f(x)=e^{-x}U(x)$ and $f(y)=3e^{-3y}U(y)$, find $f(z)$, if $Z=\frac{X}{Y}$***

Solution: Since X and Y are independent. $f(x,y)=f(x)f(y)=e^{-x}\,3e^{-3y}=3e^{-(x+3y)}, x\ge 0, y\ge 0$

Let $z=\frac{x}{y}$ and $w=y\Rightarrow x=zw$ and $y=w$; $\frac{\partial x}{\partial z}=w$; $\frac{\partial x}{\partial w}=z$; $\frac{\partial y}{\partial z}=0$; $\frac{\partial y}{\partial w}=1$

$$|J|=\begin{vmatrix}\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}\end{vmatrix}=\begin{vmatrix}w & z\\ 0 & 1\end{vmatrix}=w;\ f(u,v)=f(x,y)|J|\Rightarrow f(u,v)=3e^{-(x+3y)}\,w=3w\,e^{-(zw+3w)}=3w\,e^{-(z+3)w}$$

Range: $x\ge 0\Rightarrow zw\ge 0\Rightarrow z\ge 0, w\ge 0$ and $y\ge 0\ \Rightarrow w\ge 0$; $\boldsymbol{f(u,v)=3w\,e^{-(z+3)w},\ z\ge 0, w\ge 0}$

The marginal density function of Z:

$$f(z)=\int_{w=-\infty}^{\infty} f(z,w)dw=\int_{w=0}^{\infty}3w\,e^{-(z+3)w}dw=3\left[(w)\frac{e^{-(z+3)w}}{-(z+3)}-(1)\frac{e^{-(z+3)w}}{[-(z+3)]^2}\right]_0^{\infty}=\frac{3}{(z+3)^2},\ z\ge 0.$$

4. ***If X and Y are independent RVs each normally distributed with mean zero and variance σ^2, find the density functions of $R=\sqrt{X^2+Y^2}$ and $\varphi=tan^{-1}\left(\frac{Y}{X}\right)$.***

Solution: Normal distribution $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty<x<\infty$, Given: $\mu=0$, variance $=\sigma^2$

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{x^2}{2\sigma^2}}, -\infty<x<\infty$ and $f(y)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{y^2}{2\sigma^2}}, -\infty<y<\infty$. Since X and Y are independent.

$$f(x,y)=f(x)f(y)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{x^2}{2\sigma^2}}\times\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{y^2}{2\sigma^2}}=\boldsymbol{\frac{1}{2\pi\sigma^2}\,e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}}, -\infty<x<\infty; -\infty<y<\infty.$$

Let $r=\sqrt{x^2+y^2}$ & $\varphi=tan^{-1}\left(\frac{y}{x}\right)\Rightarrow x=r\cos\varphi$ & $y=r\sin\varphi$;

$\frac{\partial x}{\partial r}=\cos\varphi$; $\frac{\partial x}{\partial\varphi}=-r\sin\varphi$; $\frac{\partial y}{\partial r}=\sin\varphi$; $\frac{\partial y}{\partial\varphi}=r\cos\varphi$

$$|J|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r(\cos^2\varphi+\sin^2\varphi)=r$$

$f(r,\varphi)=f(x,y)|J|\Rightarrow f(r,\varphi)=\frac{1}{2\pi\sigma^2}\,e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}\,r=\frac{r}{2\pi\sigma^2}\,e^{-\left(\frac{r^2}{2\sigma^2}\right)}$; ***Range:*** $r\ge 0; 0\le\varphi\le 2\pi$;

The marginal density function of R and φ :

$$f(r)=\int_{\varphi=0}^{2\pi} f(r,\varphi)d\varphi=\int_{\varphi=0}^{2\pi}\frac{r}{2\pi\sigma^2}\,e^{-\left(\frac{r^2}{2\sigma^2}\right)}d\varphi=\frac{r}{2\pi\sigma^2}\,e^{-\left(\frac{r^2}{2\sigma^2}\right)}[\varphi]_0^{2\pi}=\frac{r}{2\pi\sigma^2}\,e^{-\left(\frac{r^2}{2\sigma^2}\right)}\times 2\pi=\frac{r}{\sigma^2}\,e^{-\left(\frac{r^2}{2\sigma^2}\right)}, r\ge 0$$

$f(\varphi)=\int_{r=0}^{\infty} f(r,\varphi)dr=\int_{r=0}^{\infty}\frac{r}{2\pi\sigma^2}\,e^{-\left(\frac{r^2}{2\sigma^2}\right)}dr$; Put $t=\frac{r^2}{2\sigma^2}\Rightarrow dt=\frac{2r}{2\sigma^2}dr\Rightarrow r\,dr=\sigma^2dt$

$$f(\varphi)=\int_{t=0}^{\infty}\frac{\sigma^2}{2\pi\sigma^2}\,e^{-t}dt=\frac{1}{2\pi}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}=\frac{1}{2\pi}, 0\le\varphi\le 2\pi$$

All the Best

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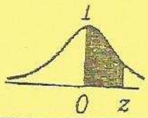
Normal Distribution Table

Area under the Normal curve
from 0 to z



z	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993

All the Best

Regards!

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