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18MAB302T- DISCRETE MATHEMATICS FOR ENGINEERS

Contents

Set Theory

Introduction:

Most of mathematics is based upon the theory of sets that was originated in 1895 by the German mathematician **G. Cantor** who introduced the concept of sets and defined a set as a collection of definite and distinguishable objects selected by means of certain rules or description.

Definition:

A **Set** is a well-defined collection of objects, called the elements or members of the set.

The adjective "well-defined" means that we should be able to determine if a given element is contained in the set under scrutiny.

Example of Sets:

- The set of all states in India.
- A pair of shoes.
- The set of all canadians.
- The collection of all self-financing colleges in a state.
- A collection of rocks.

Representation of a Set:

Capital letters A, B, C,… are generally used to denote sets and lower case letters a, b, c… to denote elements**.** If an element x is a member of any set A, it is denoted by $x \in A$ and if an element y is not a member of set A, it is denoted by $y \notin A$.

Sets can be represented in two ways:-

- Roster Notation
- Set Builder Notation

Roster Notation:

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Examples:

- The Set V of all vowels in English alphabet, V={a,e,i,o,u}.
- The set E of even positive integers less than or equal to 10: E={2,4,6,8,10}.
- The set P of positive integers less than100: P={1,2,3,…,99}.

Set Builder Notation:

The set is defined by specifying a property that elements of the set have in common. The set is described as A={x|p(x)}.

Examples:

- The set V={x|x is a vowel in English alphabet} is the same as V={a,e,i,o,u}.
- The set A={x|x=n^2, where n is a positive integer less than 6} is the same as A={1,4,9,16,25}.
- The set B={x|x is an even positive integer not exceeding 10} is the same as B={2,4,6,8,10}.

The following sets play an important role in discrete Mathematics:

N ={1,2,3,4,....}, the set of all natural numbers.

$Z = \{.....,-3,-2,-1,0,1,2,3,.....\}$,the set of all integers.

Z^+={1,2,3,....}, the set of all positive integers.

$Q = \{\frac{p}{q} \mid p \in z, q \in z, q \neq 0\}$, the set of all rational numbers.

R = the set of all real numbers.

Cardinality of a Set:

Cardinality of a set A, denoted by |A|, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Example:

- ❑ If A={1,4,9,16,25} then the cardinality of A, i.e. |A|=5 and
- ❑ If A={1,2,3,….} then |A|=∞.

Types of Sets:

Sets can be classified into many types. Some of which are universal, empty, singleton, finite, infinite, subset, proper set, improper set etc.

Universal Set:

The set which contains all the elements under consideration is called the Universal set and denoted as *U*.

Empty set :

The empty (or void, or null) set, symbolized by { } or Ø, contains no elements at all.

Example:

- A={x|x is an odd integer and 3<x<5}={ }.

Singleton Set or Unit Set:

Singleton set or unit set contains only one element. A singleton set is denoted by {x}.

Example:

- A={x|x∈*N* and 3<x<5} = {4}.

Finite Set:

A set which contains a definite number of elements is called a finite set.

Example:

- A=$\{ x^2 | x \in Z^+, x^2 < 100\} = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$ is a finite set.

Infinite Set:

A set which contains infinite number of elements is called an infinite set.

Example:

- A={x|x is an even positive integer}=A={2,4,6,8,…} is an infinite set.

Subset:

A set A is called a subset of a set B (symbolized by $A \subseteq B$) if every element of A is also an element of B.

Example:

- The set of all even positive integers between 1 and 100 is a **subset** of all positive integers between 1 and 100.

Note:

1. Any set is a subset of itself, i.e., $A \subseteq A$ and Ø is a subset of any set, i.e., $\emptyset \subseteq A$.
2. *If* $A \subseteq B$ and $B \subseteq C$ *then* $A \subseteq C$.
3. If $A \subseteq B$ then B is called the **superset** of A and is written as $B \supseteq A$.

Definition

Any subset A of the set B is called the **Proper Subset** of B, if there is at least one element of B which does not belong to A, i.e., if $A \subseteq B$ but $A \neq B$. It is denoted as A$\subset B$.

Two sets A and B are said to be **equal**, i.e., A=B, i.e., if $A \subseteq B$ and $B \subseteq A$.

Example:

- If A={a,b}, B ={a,b,c} and C ={b,c,a} then A and B are subsets of C, but A is a proper subset of C while B is not, Since B=C.

Power Set

Given a set *S*, the set of all subset of the set *S* is called the power Set of *S* and is denoted by P(*S*).

Example:

- If A={a,b,c} then
  P(A)=**{**{ },{a},{b}.{c},{a,b},{b,c}{a,c}{a,b,c}**}**.

Property:

If a Set *S* has n elements, then its power set has 2^n elements, i.e., if |*S*|=n then |P(*S*)|= 2^n.

Note:

In the above example, |A|=3 and so |P(A)|= 2^3=8.

Operations on Sets

Two or more sets can be combined to give rise to new sets. These operations that play an important role in many applications are discussed as follows.

If U is the Universal set and if A and B are any two sets , Then

1. $A \cup B$, read "A **union** B " contains all elements that are in A **or** B.

i.e., $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

2. $A \cap B$, read "A **intersection** B ", contains all elements that belong to both A **and** B.

i.e., $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

3. A^c *or* A' *or* $\bar{A}$, called the **complement** of A, contains all the elements which are **not** in A. i.e., $A' = \{x | x \in U \text{ and } x \notin A\}$.

4. Two sets A and B are called **Disjoint** sets if they do not have any element in common, then $A \cap B = \emptyset$.

Example:

- If $A=\{1,2,3\}$ and $B=\{4,5,6\}$ then $A \cap B = \emptyset$.Hence A and B are disjoint sets.

5. $A \setminus B$ or A-B, called the **Difference** of A and B or **Relative complement** of B with respect to A, contains the elements which are in A but not in B.

i.e., $A-B=\{x | x \in A \text{ and } x \notin B\}$.

Example:

- If $A=\{1,2,3\}$ and $B=\{1,3,5,7\}$ then $A-B=\{2\}$ and $B-A=\{5,7\}$.

6. $A \Delta B$ *or* $A \oplus B$ or A+B, called the **Symmetric difference** of A and B, contains the elements that are in exactly one of A and B but not both. It is obvious that $A \oplus B = (A-B) \cup (B-A)$.

Example:

- If $A=\{a,b,c,d\}$ and $B=\{c,d,e,f\}$ then $A \oplus B=\{a,b,e,f\}$.

The Algebraic laws of Set Theory

The binary operation of set “**Union**” and “**Intersection**” satisfy many identities. Several of these identities or "laws" have well established names. The pairs of laws are listed below:

For any subset *A* of universal set **U**, the following identities hold:

1. **Identity** laws:

a)$A \cup \emptyset = A$

b)$A \cap \mathbf{U} = A$

2. **Domination** laws:

a) $A \cup \mathbf{U} = \mathbf{U}$

b) $A \cap \emptyset = \emptyset$

3. **Idempotent** laws

a) $A \cup A = A$

b) $A \cap A = A$

4. **Inverse laws** or **Complement** laws:

a)$A \cup A' = \mathbf{U}$

b)$A \cap A' = \emptyset$

5. **Double complement** law or **Involution** law: $(A')' = A$

For any sets A, B, and C, the following identities hold:

6. **Commutative** laws:

$A \cup B = B \cup A$

$A \cap B = B \cap A$

7. **Associative** laws:

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

8. **Distributive** laws:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

9. **Absorption** laws:

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

10. **De Morgan's** laws

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

The Principle of Duality

In Each of the identities stated above, one of a pair of identities can be transformed into the other by using an important property of set algebra, namely, the **principal of duality** for sets, which asserts that for any true statement about sets, the **dual** statement obtained by interchanging Union(∪) as intersection(∩), and intersection (∩) as Union(∪), and interchanging **U** as ∅ and ∅ as **U** and reversing inclusions.

Proving set Identities using logical laws:

Let us establish some of the set identities.

(i) $\boldsymbol{A \cup A = A}$

(We recall that to prove that A=B we should establish that A$\subseteq B$ and B$\subseteq A$.)

Now $x \in A \cup A \Longrightarrow x \in A \; or \; x \in A$

$\Longrightarrow x \in$ A, $\quad \therefore A \cup A \subseteq$A

Also $x \in A \Longrightarrow x \in A \; or \; x \in A \quad \Longrightarrow x \in A \cup A$

$\therefore$ A $\subseteq A \cup A$

(ii) $\boldsymbol{A \cap B = B \cap A}$

Let us use the set builder notation to prove this identity.

$$
\begin{aligned}
A \cap B &= \{ x | x \in A \cap B \} \\
&= \{x | x \in A \text{ and } x \in B\} \\
&= \{x | x \in B \text{ and } x \in A\} \\
&= \{ x | x \in B \cap A\} \\
&= B \cap A
\end{aligned}
$$

(iii) $\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

$$
\begin{aligned}
A \cup (B \cap C) &= \{ x | x \in A \text{ or } x \in B \cap C\} \\
&= \{ x | x \in A \text{ or } (x \in B \text{ and } x \in C)\} \\
&= \{ x | (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\} \\
&= \{ x | (x \in A \cup B) \text{ and } (x \in A \cup C)\} \\
&= \{ x | x \in (A \cup B) \cap (A \cup C)\} \\
&= (A \cup B) \cap (A \cup C)
\end{aligned}
$$

[*****The Proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is left to the reader]

Examples

1. Prove that (A-C) ∩(C-B)=∅ analytically.

(A-C) ∩(C-B)={x| ($x \in A$ *and* x ∉ C) *and* (x ∈ C and x ∉ B)}

={ x| $x \in A$ *and*(x ∉ C *and* x ∈ C)and x ∉ B}

={ x| $x \in A$ *and*(x ∈ C and x ∈ $\bar{C}$)and x ∈ $\bar{B}$}

={ x| ($x \in A$ and x∈ ∅)and x ∈ $\bar{B}$}

={x|x∈ $A \cap \emptyset$and x ∈ $\bar{B}$}

={x|x∈ ∅ and x ∈ $\bar{B}$}

={ x|x∈ $\emptyset \cap \bar{B}$}

={x|x x∈ ∅} $= \emptyset$.

2.If A,B and C are sets , Prove that

$\overline{\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C})} = (\bar{\boldsymbol{C}} \cup \bar{\boldsymbol{B}}) \cap \bar{\boldsymbol{A}}$**, using set identities.**

$L.H.S = \overline{A \cup (B \cap C)}$

$=\bar{A} \cap \overline{B \cap C}$, by De Morgan's law

$=\bar{A} \cap (\bar{B} \cup \bar{C})$, by De Morgan's law

$=(\bar{B} \cup \bar{C}) \cap \bar{A}$, by commutative law

$=(\bar{C} \cup \bar{B}) \cap \bar{A}$, by commutative law

=*R.H.S.*

Partition of a Set

Definition:

If S is a non-empty set, A collection of disjoint non empty subsets of S whose union is S is called a **Partition** of S. In other words, the collection of subsets A_i is a Partition of S if and only if

1. $A_i \neq \phi$, for each i.
2. $A_i \cap A_j = \phi$ for i$\neq j$.
3. $\bigcup_i A_i = S$.

Note:

The sub sets in a Partition are also called ***blocks***, ***parts*** or ***cells*** of the partition.

Example: 1

Let the set be A={1, 2}Then the Partition of the set A= { {1}, {2} } { {1, 2} }.

Example: 2

Let the set be A ={1, 2, 3} Then the Partition of the set A are as follows:

- {{1}, {2}, {3}}
- {{1}, {2, 3}}
- {{2}, {1, 3}}
- {{3}, {1, 2}}
- {{1, 2, 3}}.

Note:

The following are not partitions of {1, 2, 3}:

- {{}, {1, 3}, {2}} is not a partition because one of its elements is the empty set.
- {{1, 2}, {2, 3}} is not a partition because the element 2 is contained in more than one block.
- {{1}, {2}} is not a partition of {1, 2, 3} because none of its blocks contains 3; however, it is a partition of {1, 2}.

(**Note:** The number of ways to Partition of a set is called a Bell number. In example:2 Bell number is 5.)

Cartesian Product of Sets

Definition:

If A and B are sets then the **Cartesian Product** of A and B *denoted as* $\boldsymbol{A \times B}$ is the set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. i.e., $A \times B = \{(a,b) | a \in A \text{ and } b \in B\}$.

Example:

If $A=\{a,b,c\}$ and $B=\{1,2\}$ then

- $A \times B = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$ and
  $B \times A = \{(1,a),(1,b)(1,c)(2,a)(2,b)(2,c)\}$.

Relations

Definition:

A **Relation** or a Binary relation R from set A to B is a subset of A×B which can be defined as aRb or (a,b) ∈ R or R(a,b).

A Binary relation R on a single set A is defined as a subset of A×A. For two distinct set, A and B with cardinalities m and n, the maximum cardinality of the relation R from A to B is mn.

Example:

Let R be a relation on A={1,2,3,4} defined by aRb if $a \leq b$.Then

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}.

Domain and Range of Relation:

If there are two sets A and B and a relation from A to B is R(a, b), then the **Domain** of R is defined as the set {a |(a, b) ∈ R for some b in B} and the **Range** of R is defined as the set {b |(a, b) ∈R for some a in A}.

Example:

Let A={2,3,5}, B={6,8,10} and a relation aRb if a divides b. Then

R={(2,6),(2,8),(2,10),(3,6),(5,10)}.

Domain of R = D(R) ={2,3,5}.

Range of R = R(R) ={6,8,10}.

Types of relations

1. A relation R on a set A is called a **Universal relation** if R= A×A.

Example:

- If A={1,2} then R ={(1,1),(1,2),(2,1)(2,2) =A×A.

- **2**. A relation R on a set A is called a Void relation if R=∅.

Example:

- If A={3,4,5} and R is defined as aRb if and only if a+b>10, then we get R=∅.

- A relation R on a set A is called an Identity relation if R = {(a,a)| a∈A}and is denoted by I_A.

Example:

- If A={1,2,3}then R ={(1,1),(2,2),(3,3)} is the identity relation on A.

4. If R is any relation from A to B, the **Inverse** of R denoted by R^{-1} is the relation from B to A which consists of those ordered pairs got by interchanging the elements of the ordered pairs in R.

Example:

- If A={2,3,5}, B={6,8,10} and a relation aRb if a divides b.

Then

$$R=\{(2,6),(2,8),(2,10),(3,6),(5,10)\}$$

Now

$$R^{-1}=\{(6,2),(8,2),(10,2),(6,3),(10,5)\}$$

Operations on Relations

1. If R and S denote two relations, the **Intersection** of R and S denoted by R$\cap S$ is defined by

$$a(R\cap S)b=aRb \wedge aSb, \text{ and}$$

the **Union** of R and S denoted by R$\cup S$ is defined by

$$a(R\cup S)b= aRb \vee aSb.$$

2. The **Difference** of R and S denoted by R-S is defined by

a(R-S)b=aRb$\wedge a\$b$.

3. The **Complement** of R denoted by R' or $\sim R$ is defined by

R' or $\sim R$=(A×B)-R.

Example

Let A={x,y,z}, B={1,2,3}, C={x,y} and D={2,3}.

Let R be a relation from A to B defined by R={(x,1), (x,2), (y,3)}.

And let S be a relation from C to D defined by S={(x,2), (y,3)}.

Then

1. $R\cap S$ ={(x,2), (y,3)} and $R\cup S$=R.

2. R-S={(x,1)}.

3. R'={(x,3),(y,1),(y,2),(z,1),(z,2),(z,3)}.

Composition of Relations:

If R is a relation from A to B, S is a Relation from B to C, then the **composition** of R and S denoted by R∘S is defined as R∘S={(a,c)|(a,b)∈ *R* and (b,c) ∈S}.

Example:

- Let R={(1,1),(1,3),(3,2),(3,4),(4,2)} and S={(2,1),(3,3),(3,4),(4,1)}.

  Then

1. R∘S={(1,3),(1,4),(3,1),(4,1)}

2. S∘R={(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)}

3. R∘R={(1,1),(1,3),(1,2),(1,4),(3,2)}

4. S∘S={(3,3),(3,4),(3,1)}

5. (R∘S) ∘R={(1,2),(1,4),(3,1),(3,3),(4,1),(4,3)}

6. R∘ (S∘R)={(1,2),(1,4),(3,1),(3,3),(4,1),(4,3)}

7. R^3= R∘ R∘R= (R∘ R) ∘R

   = R∘ (R∘R)={(1,1),(1,3),(1,2),(1,4)}

Properties of Relations

1. Reflexive Relation:

A relation R on a set A is called reflexive if (a,a) ∈ R holds for every element a ∈ A. i.e. if set A = {a, b} then R = {(a,a), (b,b)} is reflexive relation.

2. Irreflexive relation:

A relation R on a set A is called irreflexive if no (a,a) ∈ R holds for every element a ∈ A. i.e. if set A = {a,b} then R = {(a,b), (b,a)} is irreflexive relation.

3. Symmetric Relation:

A relation R on a set A is called symmetric if (b,a) ∈ R holds when (a,b) ∈R. i.e. The relation R={(4,5),(5,4),(6,5),(5,6)} on set A={4,5,6} is symmetric.

4. Asymmetric relation:

Asymmetric relation is opposite of symmetric relation. A relation R on a set A is called asymmetric if no (b,a) ∈ R when (a,b) ∈ R.

5. Transitive Relation:

A relation R on a set A is called transitive if (a,b) ∈ R and (b,c) ∈R then (a,c) ∈ R for all a,b,c ∈ A. i.e. Relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.

6. Antisymmetric Relation:

A relation R on a set A is called antisymmetric if (a,b) ∈ R and (b,a) ∈ R then a = b(otherwise it is called not antisymmetric i.e. if $a \neq b$).

Note:

Symmetric and anti-symmetric relations are not opposite because a relation R can contain both the properties or may not.

Example:

1. R={(1,3),(3,1)(2,3)}is neither symmetric nor antisymmetric, whereas the relation S={(1,1), (2,2)} is both symmetric and antisymmetric.

2. A relation is asymmetric if and only if it is both anti-symmetric and irreflexive.

Equivalence and Partial ordering Relation

Definition. 1

A relation is called an Equivalence Relation if it is reflexive, symmetric, and transitive.

Example:

Relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is an equivalence relation as it is reflexive, symmetric and transitive.

Definition. 2

A relation R is called **Partial ordering or Partial order relation** if R is reflexive, antisymmetric and transitive.

Example:

If R is a relation "greater than or equal to" (≥) on the set of integers Z then R is a Partial ordering relation since

(i) a$\geq a$ for every integer a, i.e. ≥ is reflexive.

(ii) a$\geq b$ and b$\geq a \Longrightarrow a = b,$ i.e. ≥ is antisymmetric.

(iii) a$\geq b$ and b$\geq c \Longrightarrow$ a$\geq c$, i..e ≥ is transitive.

Examples

1. If R is the relation on the set of ordered pairs of positive integers such that (a,b),(c,d)∈R whenever ad=bc, Show that R is an equivalence relation.

Solution:

(i) (a,b)R(a,b), since ab=ba

$\therefore$ R is reflexive.

(ii) when (a,b)R(c,d), ad=bc

$\Longrightarrow$ cb=da

This means that (c,d)R(a,b).

$\therefore$ R is symmetric

(iii) When (a,b)R(c,d) and (c,d)R(e,f), we have ad=bc and cf=de

$$\Longrightarrow \quad \frac{a}{b} = \frac{c}{d} \text{ and } \frac{c}{d} = \frac{e}{f}$$

$$\Longrightarrow \quad \frac{a}{b} = \frac{e}{f} \quad \Longrightarrow \quad af=be$$

This means that (a,b)R(e,f).

$\therefore$ R is transitive

2.If R is the relation on the set of positive integers such that (a,b) ∈R if and only if ab is a perfect square, Show that R is an equivalence relation.

Solution:

(i) Since $a.a = a^2$ is a perfect square, aRa

∴ R is reflexive.

(ii) When aRb, ab is a perfect square , say ab=x^2

$\Longrightarrow$ ba=x^2 , is a perfect square

That is., bRa.

∴ R is symmetric

(iii) When aRb and bRc, we write ab=x^2 and bc=y^2

$\Longrightarrow$ (ab)(bc)= $x^2.y^2$=$(xy)^2$

$\Longrightarrow$ab^2c=$(xy)^2$

$\Longrightarrow$ ac=$\left(\frac{xy}{b}\right)^2$, is a perfect square

i.e., aRc.

∴ R is transitive

3. If R is the relation on the set of positive integers such that (a,b) ∈R if and only if $a^2 + b$ is even, Prove that R is an equivalence relation.

Solution:

(i) $a^2 + a$ =a(a+1)=even, since a and (a+1) are consecutive integers.

i.e., aRa.

∴ R is reflexive.

(**ii**) When aRb, that is $a^2 + b$ is even, a and b must be both even or both odd. In either case, $b^2 + a$ is also even.

i.e., bRa.

∴ R is symmetric

(***iii***) When a,b,c are even, $a^2 + b$ and $b^2 + c$ are even, Also $a^2 + c$ is even when a,b,c are odd, $a^2 + b$ and $b^2 + c$ are even, Also $a^2 + c$ is even. Then aRb and bRc implies that aRc.

∴ R is transitive

4. If R is the relation on the set of integers such that $(a,b) \in R$ if and only if $3a+4b=7n$ for some integer n, Prove that R is an equivalence relation.

Solution:

(i) $3a+4a=7a$, when a is an integer

i.e., aRa .

$\therefore$ R is reflexive

(ii) When aRb, $3a+4b=7n$

Now,

$$3b+4a=7a+7b-(3a+4b)$$
$$=7a+7b-7n$$
$$=7(a+b-n)$$, where $(a+b-n)$ is an integer

i.e., bRa.

$\therefore$ R is symmetric

(iii) When aRb and bRc, we have $3a+4b=7m$ and $3b+4c=7n$

$$\Longrightarrow 3a+4b+3b+4c=7m+7n$$
$$\Longrightarrow 3a+7b+4c=7m+7n$$
$$\Longrightarrow 3a+4c=7m+7n-7b=7(m+n-b)$$

where $(m+n-b)$ is an integer. i.e., aRc

$\therefore$ R is transitive

Poset

Definition:

A set A together with a partial order relation R is called a **Partially ordered set** or **Poset.**

Example:

The set of integers under the relation "greater than or equal to"(≥) is a Partially ordered set. i.e.,(Z,≥) is a Poset.

Matrix Representation of Relation:

If R is the relation from the set A=$\{a_1 a_2, \dots, a_m\}$ to the set B=$\{b_1 b_2, \dots, b_n\}$, where the elements of A and B are assumed to be in a specific order, the relation R can be represented by the matrix

$$M_R = [m_{ij}], \text{ where}$$

$$m_{ij} = \begin{cases} 1, if\ (a_i, b_j) \in R \\ 0, if (a_i, b_j) \notin R \end{cases}$$

In other words, the zero-one matrix M_R has a '1' in its (i-j)th position when a_i is related to b_j and a '0' in this position when a_i is not related by b_j.

Example:

If A=[a_1, a_2, a_3] and B=[b_1, b_2, b_3, b_4] and R={$(a_1, b_2)(a_2, b_1)(a_2, b_3)(a_2, b_4)(a_3, b_2)(a_3, b_4)$

then the matrix of R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Conversely,

if R is the relation on A={1,3,4} represented by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then R={(1,1),(1,3),(3,3),(4,4)}, since $m_{ij} = 1$ means that the i^{th} element of A is related to the j^{th} element of A.

Operations

Definition:

If R and S are relations on a set A, represented by M_R and M_s respectively

then

1. $M_{R\cup s} = M_R \vee M_s$ (where the operation $'\vee'$ is called *join*).

2. $M_{R\cap S} = M_R \wedge M_s$(where the operation $'\wedge'$ is called *meet*).

3. $M_{R^{-1}} = {M_R}^T$(T-means the transpose of the matrix) and

4. $M_{R\circ S} = M_R \odot M_S$ (where $\odot$ is called the Boolean product),Boolean product of the matrices is similar to the ordinary product of matrices which is obtained by replacing the **multiplication** by the operation $\wedge$ and **addition** by the operation $\vee$.

Examples

If R and S are relations on a set A, represented by the matrices M_R and M_s as follows:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

1. $M_{R \cup s} = M_R \vee M_s = \begin{bmatrix} 1\vee1 & 0\vee0 & 1\vee1 \\ 0\vee1 & 1\vee0 & 1\vee0 \\ 1\vee0 & 0\vee1 & 0\vee0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. $M_{R \cap S} = M_R \wedge M_s = \begin{bmatrix} 1 \wedge 1 & 0 \wedge 0 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 1 \wedge 0 \\ 1 \wedge 0 & 0 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = {M_R}^T$

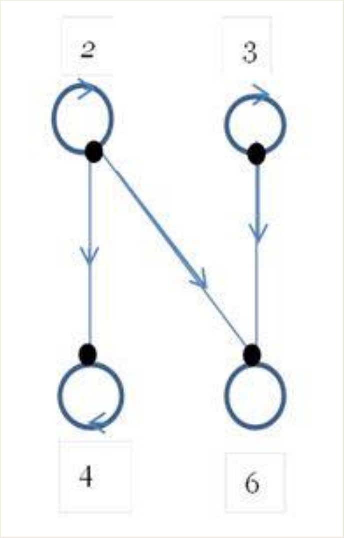
4. $M_{R \circ S} = M_R \odot M_S = \begin{bmatrix} 1\vee0\vee0 & 0\vee0\vee1 & 1\vee0\vee0 \\ 0\vee1\vee0 & 0\vee0\vee1 & 0\vee0\vee0 \\ 1\vee0\vee0 & 0\vee0\vee0 & 1\vee0\vee0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Representation of Relations By Graphs

Let R be a relation on a set A .To represent R graphically, each element of A is represented by a point. These points are called ***nodes*** or ***vertices***. Whenever the element a is related to b an arc is drawn from the point ‘a’ to the point ‘b’. These arcs are called ***arcs*** or ***edges***. The arcs start from the first element of the related pair and go to the second element. The direction is indicated by an arrow. The resulting diagram is called the **directed graph** or **digraph** of R. The edge of the form (a,a) represented by using an arc from the vertex a back to itself, is called a ***loop***.

Example:

If A={2,3,4,6} and R is defined by aRb if a divides b, then R={(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(6,6)}. The digraph representing the relation R is given in fig.



Figure

Hasse Diagrams For Partial Orderings

The simplified form of the digraph of a partial ordering on a finite set that contains sufficient information about the partial ordering is called a **Hasse diagram**, named after the Twentieth – century mathematician Helmut Hasse.

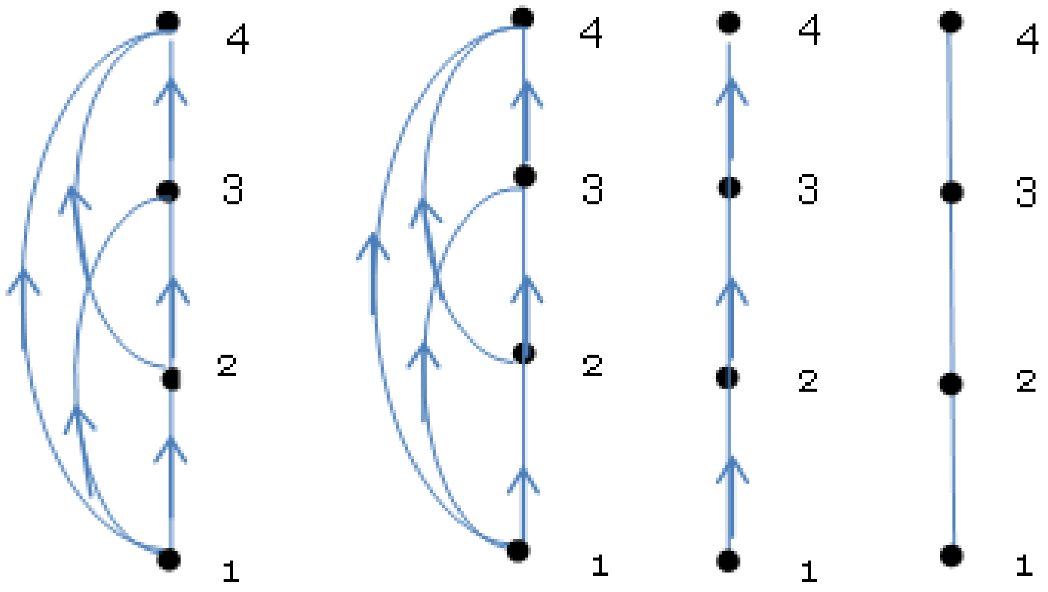
The simplification of the digraph as a Hasse diagram is achieved in three ways:

- Since the partial ordering is a reflexive relation, its digraph has loops at all vertices. We need not show these loops since they must be present.
- Since the partial ordering is transitive, we need not show those edges that must be present due to transitivity.
- If we assume that all edges are directed upward, we need not show the directions of the edges.

Thus the Hasse diagram representing a partial ordering can be obtained from its digraph, by removing all the loops, by removing all edges that are present due to transitivity and by drawing each edge without arrow so that its initial vertex is below its terminal vertex.

Example:

Let us Construct the Hasse diagram for the partial ordering $\{(a,b) \mid a \leq b\}$ on the set $\{1,2,3,4\}$ starting from its digraph.



<u>Terminology related to Posets</u>

<u>Definitions:</u>

When $\{P,\leq\}$ is a poset, an element $a \in P$ is called **maximal member** of P if there is no element b in P such that $a<b$.

Similarly, an element $a \in P$ is called **minimal member** of P if there is no element b in P such that $b<a$.

If there exists an element $a \in P$ such that $b \leq a$ for all $b \in P$,then ‘a’ is called the **greatest member** of P.

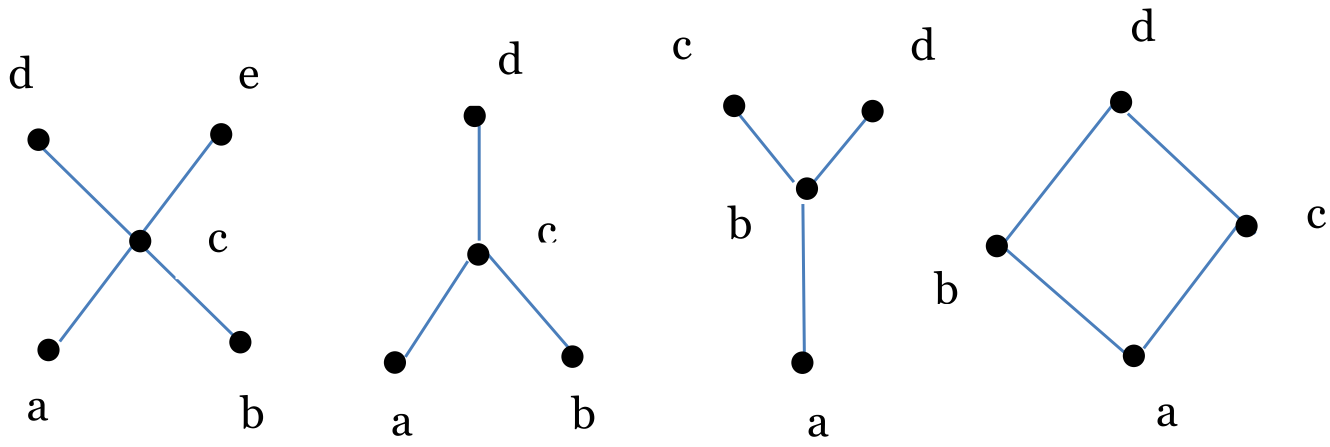
Similarly if there exists an element $a \in P$ such that $a \leq b$ for all $b \in P$, then ‘a’ is called the **least member** of P.

<u>Note :</u>

- The maximal and minimal, the greatest and least members of a poset can be easily identified using the Hasse diagram of the poset. They are the top and bottom elements in the diagram.
- A Poset can have more than one maximal member and more than one minimal member, whereas the greatest and least members, when they exist, are unique.

Example:

Hasse diagram of some posets are given below.



(1) (2) (3) (4)

From the above figure, in(1) a,b are minimal elements and d, e are maximal elements.

In(2), a,b are minimal elements and d is the greatest element(also the only maximal element).In (3), a is the least element and c,d are maximal elements. In(4) a is the least element and d is the greatest element.

Definitions:

When A is a subset of a poset $(P, \leq)$ and if u is an element of P such that $a \leq u$, for all $a \in A$, Then u is called an **upper bound** of A. Similarly an element $l \in P$ is called a **lower bound** of A, if for all $a \in A$, $l \leq a$.

- An element $x \in P$ is a **least upper bound**(LUB) (or) **supremum** of A, if x is an upper bound that is less than every other upper bound of A.
- An element $y \in P$ is the **greatest lower bound**(GLB) (or) **infimum** of A, if y is a lower bound that is greater than every other lower bound of A.

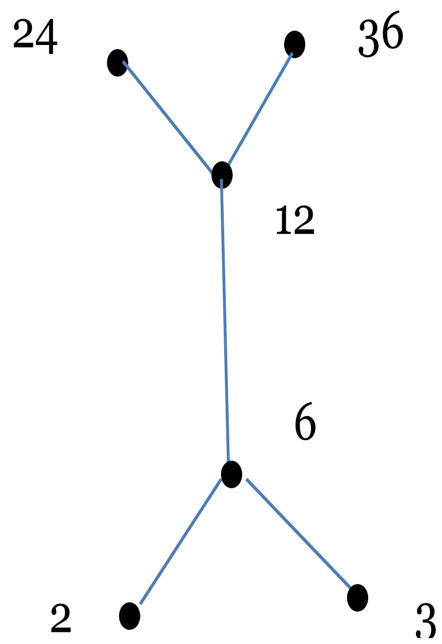
Note:

1. The upper and lower bounds of a subset of a Poset are not necessarily unique.

2. The LUB and GLB of a subset of a poset, if they exist are unique.

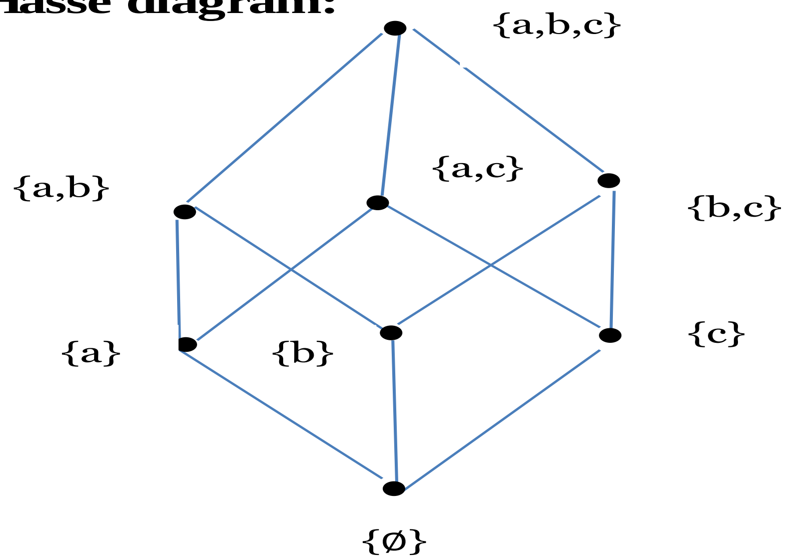
Examples:

1. Let x={2,3,6,12,24,36} and the relation '$\leq$' be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $(x, \leq)$.



2. Draw the Hasse diagram representing the partial ordering{(A,B)| A$\subseteq$ B} on the power set P(S), where S={a,b,c}.

Hasse diagram:



From the fig,{a,b,c} is the greatest element {Ø} is the least element of the poset.

The only upper bound of the subset ({a},{b},{c}) is {a,b,c} and hence the LUB of the subset.

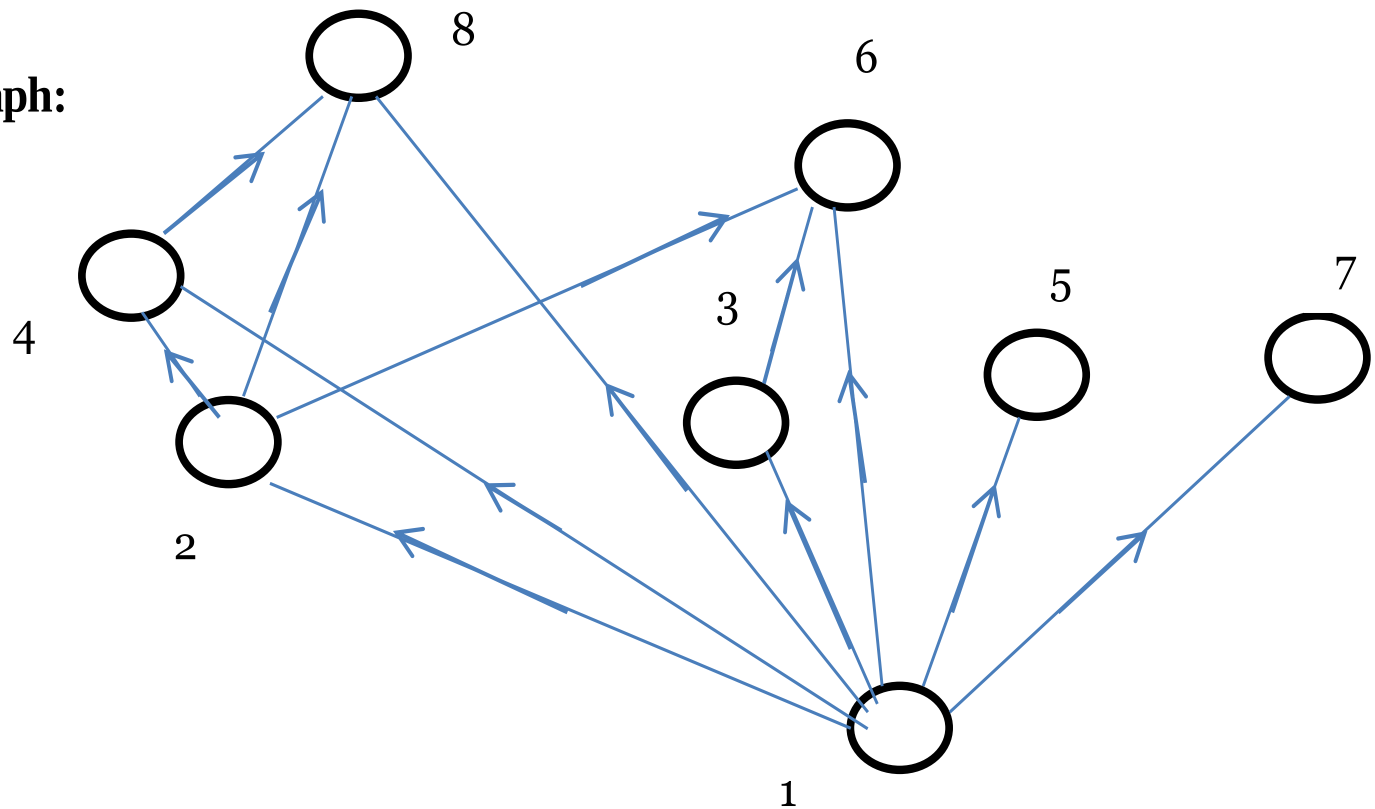
Note:

{a,b} is not an upper bound of the subset, as it is not related to {c},similarly{a,c} and {b,c} are not upper bounds of the given subset. The only lower bound of the subset({a,b},{a,c},{b,c}) is {Ø} and hence GLB of the given subset.

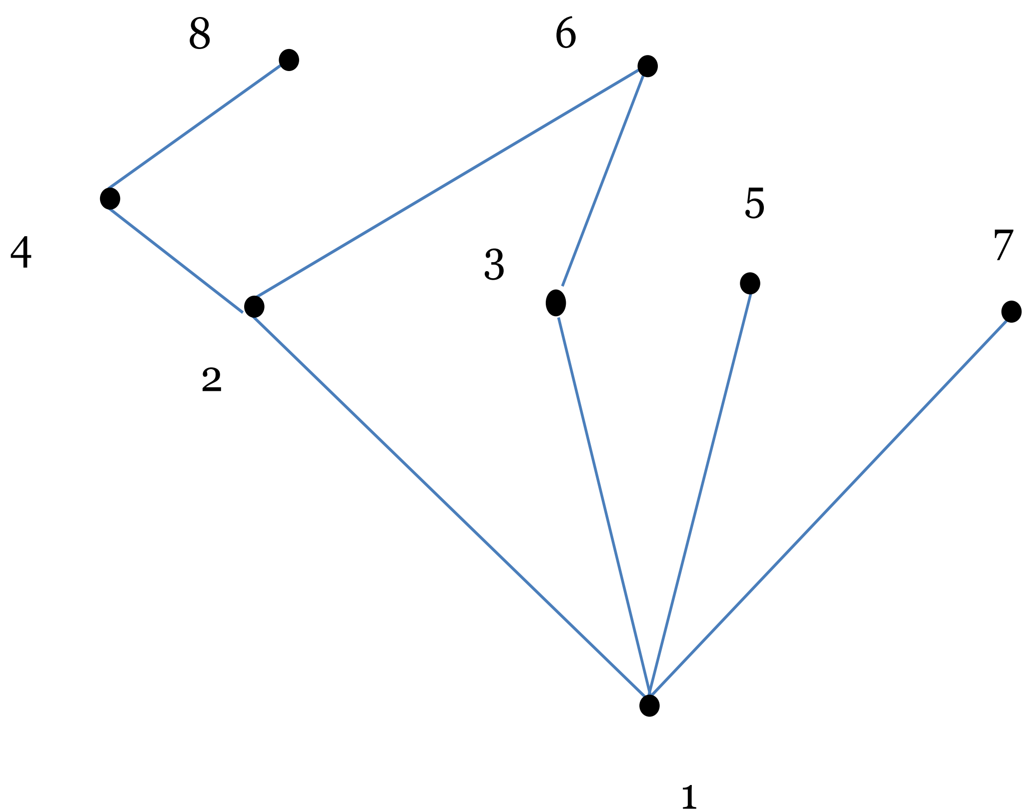
Note: {a},{b},{c} are not lower bounds of the given subset.

3. Draw the Digraph representing the partial ordering P={(a,b)| a divides b} on the set {1,2 3,4,5,6,7,8,}.Reduce it to the Hasse diagram.

Digraph:



Deleting all the loops at the vertices, deleting all the edges occurring due to transitivity, arranging all the edges to point upward and deleting all arrows, we get the corresponding Hasse diagram as given below.



Hasse Diagram

Problem Questions

1. Draw the Hasse diagram representing the Partial ordering P={(a,b)|a divides b} on the set {1,2,3,4,6,8,12} starting form the digraph of P

2. Draw the Hasse diagram for the divisibility relation on {2,4,5,10,12,20,25}starting from the digraph.

3. Draw the Hasse diagram for the less than or equal to relation on {0,2,5,10,11,15}starting from the digraph.

4. For the poset[{3,5,9,15,24,45}; divisor of], find (a) the maximal and minimal elements (b) the greatest and the least element (c) the upper bounds and LUB of {3,5} (d) the lower bounds and GLB of {15,45}.

Closures of Relations

Closures of Relations

Definition:

If R is a relation on a set A. If R does not possess a particular property, we can find the smallest relation R_1 on A so that R_1 possess the desired property and contains R. This R_1 is called the **Closure** of R.

(i) **Reflexive Closure**: The relation $R_1 = R \cup \Delta$ is the smallest reflexive relation containing R, where $\Delta = \{(a,a) / a \in A\}$. So Reflexive closure of $R = R \cup \Delta$

(ii) **Symmetric Closure**: The relation $R_1 = R \cup R^{-1}$ is the smallest symmetric relation containing R, where $R^{-1} = \{(y,x)/(x,y) \in R\}$. So symmetric closure of $R = R \cup R^{-1}$

(iii) **Transitive Closure:** If M_R is the matrix of R on A, then the matrix of the transitive closure of R is denoted by M_R^{∞} or $M_R^{+} = M_R \vee M_{R^2} \vee \ldots \vee M_{R^n}$ where $|A| = n$.

Problem 1:

Let A={1,2,3,4} and R_1 and R_2 be two relations on A given by R_1={(1,1), (1,2), (2,3),(3,4)} and R_2= {(1,2), (2,2), (2,3), (3,2), (4,1), (4,4)}. Find the reflexive and the symmetric closures of R_1 and R_2.

(i) Reflexive closure of $R_1 = R_1 \cup \Delta$

$$= R_1 \cup \{(1,1),(2,2),(3,3),(4,4)\}$$

$$=\{ (1,1), (1,2), (2,3),(3,4), (2,2),(3,3),(4,4)\}$$

(ii) Symmetric closure of $R_1 = R_1 \cup R_1^{-1}$

$$= R_1 \cup \{(1,1),(2,1),(3,2),(4,3)\}$$

$$= \{ (1,1),(1,2),(2,1),(2,3),(3,2),(3,4),(4,3)\}$$

Similarly try for R_2.

Transitive Closure and Warshall's Algorithm

Warshall's algorithm is used to compute the transitive closure of a relation.

Warshall's Algorithm

Let us assume $W_0 = M_R$

Step 1: First transfer to W_k all $1'$s in W_{k-1}.

Step 2: List the locations $p_1, p_2, p_{3,\dots\dots\dots}$ in cloumn K $of W_{k-1}$ where the entry is 1 and the locations of $q_1, q_2, q_{3,\dots\dots\dots}$ in row K of W_{k-1} where the entry is 1.

Step 3: Put 1 in all positions of (p_i, q_i) of W_k.

If $|A|$ =n compute till W_n

Problem: 1 Let A={1,2,3,4} and R= {(1,2),(2,3),(3,4),(2,1)}. Using Warshall's algorithm find the transitive closure of R.

Solution: Let us assume $W_0 = M_R$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As $|A| = 4$, we have to compute W_1, W_2, W_3, W_4(W_4 is the matrix of transitive closure).

K	In W_{k-1} Position of 1's in Col K (p_i)	In W_{k-1} Position of 1's in Row K (q_j)	W_K has 1's in (p_i, q_j)	W_K
1.	2	2	(2,2)	$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
2.	1,2	1,2,3	(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)	$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3.	1,2	4	(1,4),(2,4)	$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
4.	1,2,3	-	-	$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So the Transitive closure of R is given by

$R^\infty = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}.$

Problem 2: Find the transitive closure of $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: Let us assume $W_0 = M_R$.

As $|A| = 4$, we have to compute W_1, W_2, W_3, W_4 (W_4 is the matrix of transitive closure).

K	In W_{k-1} Position of 1's in Col K (p_i)	In W_{k-1} Position of 1's in Row K (q_j)	W_K has 1's in (p_i, q_j)	W_K
1.	1	1,4	(1,1),(1,4)	$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
2.	2	2,4	(2,2),(2,4)	$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3.	-	-	-	$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
4.	1,2,3,4	4	(1,4),(2,4),(3,4),(4,4)	$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Here we notice that $W_4 = W_0$. The transitive closure of R is given by R^{∞}={(1,1),(1,4),(2,2),(2,4),(3,4),(4,4)}.

Functions

Functions

Introduction:

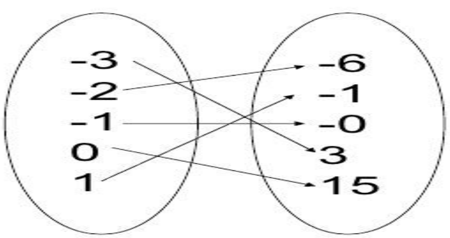
Function is a special kind of relation. If a relation f from X to Y is also said to be a function, the domain of f must be equal to X and if $(x, y) \in f$ and $(x, z) \in f$, then y must be equal to z.

Definition:

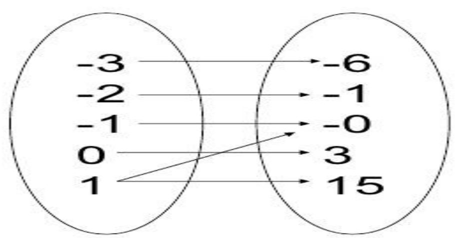
A relation f from a set X to another set Y is called a **function** if for every $x \in X$ there is a **unique** $y \in Y$ such that $(x, y) \in f$. It is represented as $f : X \to Y$ or $X \xrightarrow{f} Y$. The term "function" is also sometimes called as "transformation", "mapping" or "correspondence".

If $y = f(x)$, x is called an argument or **pre image** and y is called the **image** of x.

Function



Not a Function

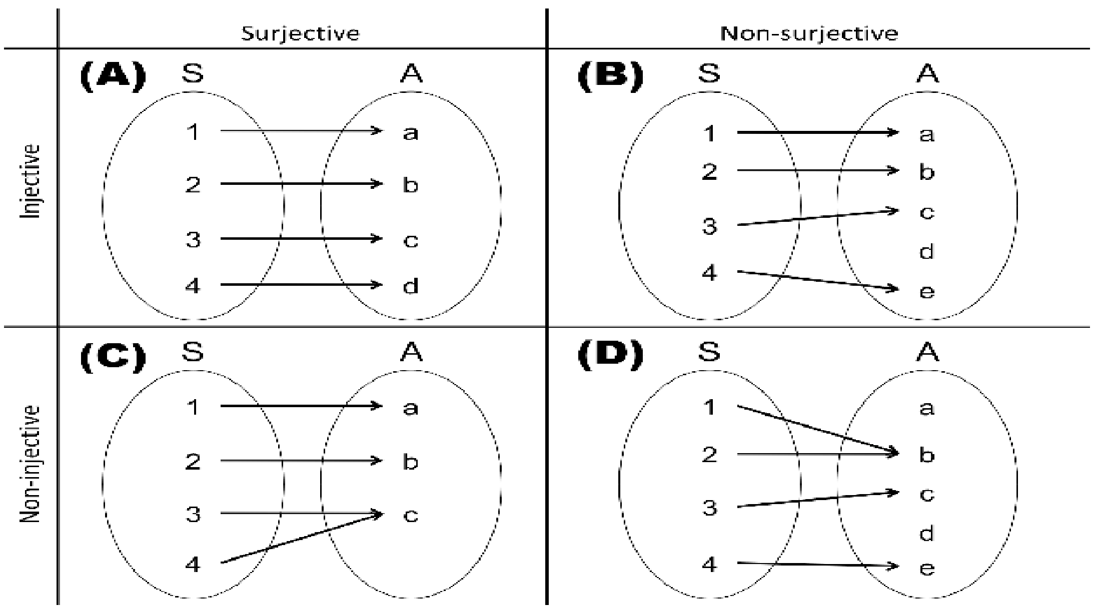


Types of functions:

(i) A function $f : X \to Y$ is called **one-to-one** (1-1) (or) injective (or) injection , if distinct elements of X are mapped into distinct elements of Y. In another words, f is 1-1 if and only if $f(x_1) = f(x_2)$ whenever $x_1 = x_2$ (or) $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

(ii) A function $f : X \to Y$ is **onto** (or) surjective (or) surjection if and only if for every element $y \in Y$ there is an element $x \in X$ such that $f(x) = y$.

(iii) A function f is called **one to one onto** (or) bijective (or) bijection (or) 1-1 correspondence if it is both 1-1 and onto.



Problem1:

If $f: Z^+ \to Z^+$ and is given by $f(x) = 3x$, verify if the function is bijective?

Solution:

(i) f is 1-1 if and only if $f(x_1) = f(x_2)$ whenever $x_1 = x_2$.

If $f(x) = f(y)$ then we have $3x = 3y$. So $x = y$. Hence the function is 1-1.

(ii) f is onto if and only if for every element $y \in Y$ there is an element $x \in X$ such that $f(x) = y$

Let $5 \in Z^+$. Now $3x = 5 \Rightarrow x = 5/3 \notin Z^+$. So f is not onto.

(iii) Since f is 1-1 but not onto it is **not bijective.**

Problem 2:

If $f: Z \to Z$ and is given by $f(x) = x + 5$, verify if the function is bijective?

Solution:

(i) f is 1-1 if and only if $f(x_1) = f(x_2)$ whenever $x_1 = x_2$.

If $f(x) = f(y)$ then we have $x + 5 = y + 5$. So $x = y$. Hence the function is 1-1.

(ii) f is onto if and only if for every element $y \in Y$ there is an element $x \in X$ such that $f(x) = y$

Let $y = f(x) = x + 5$. Now $x = y - 5 \in Z$. So f is onto.

(iii) Since f is 1-1 and onto it is **bijective.**

Composition, Identity and Inverse of function:

Definition:

(i) If $f : A \to B$ and $g : B \to C$, then the **composition** of f and g is a new function from A to C denoted by $g \circ f$ and is given by

$$(g \circ f)(x) = g(f(x)), \forall x \in A.$$

(ii) The function $f : A \to A$ where $f(x) = x$, where $x \in A$ is called the **identity function** on A, denoted by I_A.

(iii) If $f : A \to B$ and $g : B \to A$, then the function g is called the **inverse** of the function f, if $g \circ f = I_A$ and $f \circ g = I_B$.

Properties of function

Property: 1

Composition of functions is associative.(i.e) if $f : A \to B$, $g : B \to C$ and $h : C \to D$ then , $h \circ (g \circ f) = (h \circ g) \circ f$

Proof

Given $f : A \to B$ and $g : B \to C$. Let $x \in A, y \in B, z \in C$ so that $f : A \to B$ implies $y = f(x)$ and $g : B \to C$ implies $z = g(y)$.

$\{h \circ (g \circ f)\}(x) = h \circ (g(f(x)) = h \circ g(y) = h(g(y)) = h(z)$ and

$\{(h \circ g) \circ f\}(x) = (h \circ g)(f(x)) = (h \circ g)(y) = h(g(y)) = h(z)$.

Hence $h \circ (g \circ f) = (h \circ g) \circ f$.

Property: 2

When $f : A \to B$ and $g : B \to C$ are functions then $g \circ f : A \to C$ is an Injection, surjection or bijection according as f and g are injection, surjection or bijection.

Proof

(i) Let $a_1, a_2 \in A$. Then to prove $g \circ f$ is 1-1 .

$(g \circ f)(a_1) = (g \circ f)(a_2)$

$g(f(a_1)) = g(f(a_2))$ (as g is injective)

$f(a_1) = f(a_2)$ (as f is injective)

$\Rightarrow a_1 = a_2$. Hence 1-1.

(ii) To prove $g \circ f$ is surjective:

Let $c \in C$. Since g is onto, there is an element $b \in B$ such that $c = g(b)$.

Since f is onto , there is an element $a \in A$ such that $b = f(a)$.

Now $(g \circ f)(a) = (g(f(a)) = g(b) = c$. This implies for every element $c \in C$, there is an element $a \in A$ such that $(g \circ f)(a) = c$. So $g \circ f : A \to C$ is onto.

(iii)From (i) and (ii), it is clear that $g \circ f : A \to C$ is bijective if f and g are bijective.

Property: 3

The necessary and sufficient condition for the function $f : A \to B$ to be invertible is that f is 1-1 and onto.

Proof

case(i) if f is invertible , then to prove f is 1-1 and onto.

(a) To prove f is 1-1

Let $f : A \to B$ be invertible. Then there exists a unique function g:B → A

Such that $g \circ f = I_A$ and $f \circ g = I_B$ …… **(1)**

Now,

$$f(a_1) = f(a_2)$$

$$g(f(a_1)) = g(f(a_2))$$

$$g \circ f(a_1) = g \circ f(a_2)$$

$$\Rightarrow a_1 = a_2$$ (using (1))

So it is clear that f is 1-1.

(a) To prove f is onto:

Since g is a function, $g(b) \in A$ for $b \in B$.

Now

$b = I_B(b) = fog(b) = f(g(b))$

Therefore, for every $b \in B$, there exists an element $g(b) \in A$ such that

$f(g(b)) = b.$

So f is onto.

Case (ii): If f is 1-1 and onto, then to prove f is invertible.

For each $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Hence we define $g : B \to A$ by $g(b) = a$ ......(**2**) where $f(a) = b$.

If possible , let $g(b) = a_1$ and $g(b) = a_2$ where $a_1 \neq a_2$

$\Rightarrow f(a_1) = b, f(a_2) = b$

$\Rightarrow f(a_1) = f(a_2)$ where $a_1 \neq a_2$.

This implies that f is not 1-1, a contradiction to the assumption. Hence g is a unique function. Hence from (2) $g \circ f = I_A$, $f \circ g = I_B$.

So f is invertible.

Property: 4

The inverse of a function f, if it exists, is unique.

Proof

Let h and g be the inverses of f. (i.e) if $f : A \to B$ then $g : B \to A$ and also $h : B \to A$.

By definition $g \circ f = I_A$, $h \circ f = I_A$ and $f \circ g = I_B$, $f \circ h = I_B$

Now $h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$

So $h = g$.

Property: 5

If $f : A \to B$, $g : B \to C$ are invertible (inverse exists) functions, then $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof:

Given f and g are 1-1 and onto. So $g \circ f$ is also bijective. $\Rightarrow g \circ f$ is invertible.

Since $f : A \to B$ and $g : B \to C$, we have $f^{-1} : B \to A$ and $g^{-1} : C \to B$.

For any $a \in A$, let $b = f(a)$ and $c = g(b)$.

$\Rightarrow f^{-1}(b) = a, g^{-1}(c) = b$.

$(g \circ f)(a) = g(f(a)) = g(b) = c$

$\Rightarrow a = (g \circ f)^{-1}(c)$ .....**(1)**

And $(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$

$\Rightarrow a = (f^{-1} \circ g^{-1})(c)$ .....**(2)**

From (1) and (2) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Problem 1 :

If $f : Z \to N \cup \{0\}$ defined by $f(x) = \begin{cases} 2x - 1 & if \quad x > 0 \\ -2x & if \quad x \le 0 \end{cases}$,

(a) Prove that f is 1-1 and onto. (b) Determine f^{-1}.

Solution:

(a) Let $x_1, x_2 \in Z$ and $f(x_1) = f(x_2)$. Then either $f(x_1)$ and $f(x_2)$ are both odd or both even. (Because an odd number cannot be equal to an even number).

If they are both odd, then

$2x_1 - 1 = 2x_2 - 1 \Rightarrow x_1 = x_2$

If they are both even, then

$-2x_1 = -2x_2 \Rightarrow x_1 = x_2$.

Thus if $f(x_1) = f(x_2)$ then we get $x_1 = x_2$. So f is 1-1.

Let $y \in N$. If y is odd, then its pre image is $\frac{y+1}{2}$ since

$$f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y \quad \left(as \left(\frac{y+1}{2}\right) > 0\right).$$

If y is even, then its pre image is $-\dfrac{y}{2}$ since $f\left(\dfrac{-y}{2}\right) = -2\left(\dfrac{-y}{2}\right) = y \quad \left(as\left(-\dfrac{y}{2}\right) \leq 0\right)$.

Thus for any $y \in N$, the pre image is $\dfrac{y+1}{2} \in Z$ or $-\dfrac{y}{2} \in Z$. Hence f(x) is onto.

Consequently f is invertible.

(a) Let $y = f(x) = \begin{cases} 2x-1, x>0 \\ -2x, x \leq 0 \end{cases}$

Then $f^{-1}(y) = x = \begin{cases} \dfrac{y+1}{2}, y = 1,3,5\ldots \\ \dfrac{-y}{2}, y = 0,2,4,\ldots \end{cases}$

Or $f^{-1}(x) = \begin{cases} \dfrac{x+1}{2}, x = 1,3,5\ldots \\ \dfrac{-x}{2}, x = 0,2,4,\ldots \end{cases}$

Problem 2:

If A={1,2,3,4,5} , B={1,2,3,8,9} and the functions $f : A \to B$ and $g : A \to A$ are defined by

f ={(1,8), (3,9),(4,3),(2,1),(5,2)} and g={(1,2),(3,1),(2,2),(4,3),(5,2)}, find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ if they exist.

Solution:

(i) $f \circ g = f(g(x))$. Here $g : A \to A$ and $f : A \to B$, so $f \circ g : A \to B$

$(f \circ g)(1) = f(g(1)) = f(2) = 1$

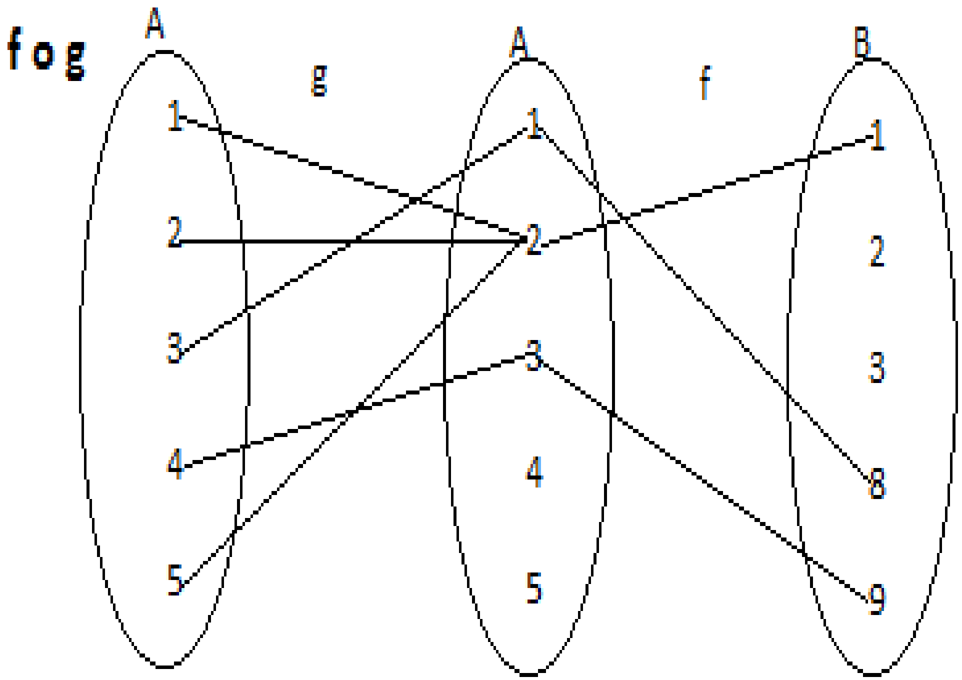
$(f \circ g)(2) = f(g(2)) = f(2) = 1$

$(f \circ g)(3) = f(g(3)) = f(1) = 8$

$(f \circ g)(4) = f(g(4)) = f(3) = 9$

$(f \circ g)(5) = f(g(5)) = f(2) = 1$

$\therefore f \circ g = \{(1,1), (2,1), (3,8), (4,9), (5,1)\}$



(i) $g \circ f = g(f(x))$. Here $f : A \to B$ and $g : A \to A$. So $g \circ f$ is not defined.

(ii) $f \circ f = f(f(x))$. Here $f : A \to B$ and $f : A \to B$. So $f \circ f$ is not defined.

(iii) $g \circ g = g(g(x))$. Here $g : A \to A$ and $g : A \to A$. So $g \circ g$ is defined.

$$(g \circ g)(1) = g(g(1)) = g(2) = 2$$

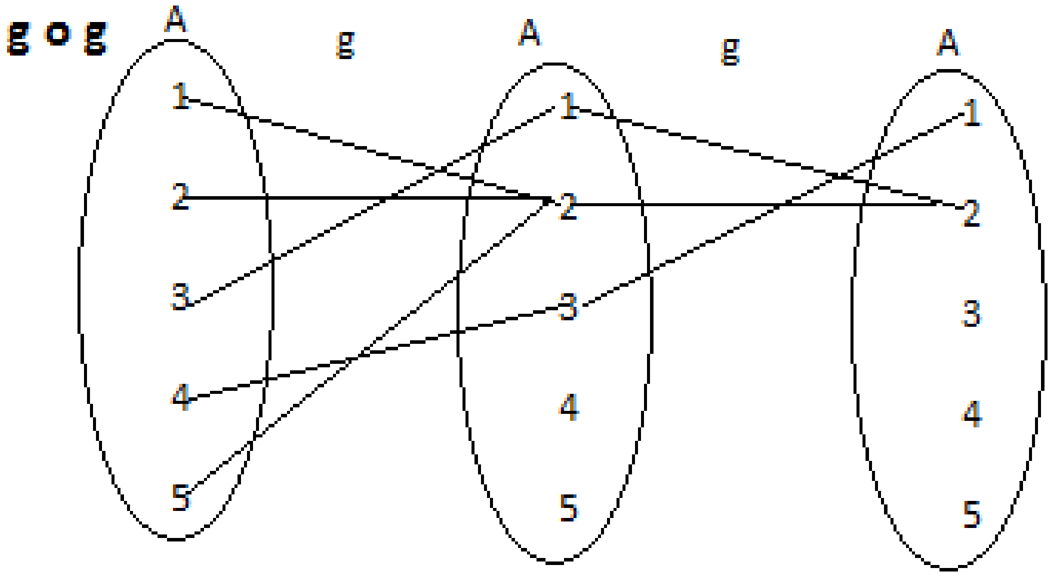
$$(g \circ g)(2) = g(g(2)) = g(2) = 2$$

$$(g \circ g)(3) = g(g(3)) = g(1) = 2$$

$$(g \circ g)(4) = g(g(4)) = g(3) = 1$$

$$(g \circ g)(5) = g(g(5)) = g(2) = 2$$

$$\therefore g \circ g = \{(1,2),(2,2),(3,2),(4,1),(5,2)\}.$$



Problem 3:

If S={1,2,3,4,5} and if the functions f ,g, and h are from $S \to S$ and are given by

f= { (1,2),(2,1),(3,4),(4,5),(5,3) }

g= { (1,3), (2,5),(3,1),(4,2),(5,4) }

h= { (1,2),(2,2,),(3,4),(4,3),(5,1) }

(a) Verify if $g \circ f = f \circ g$

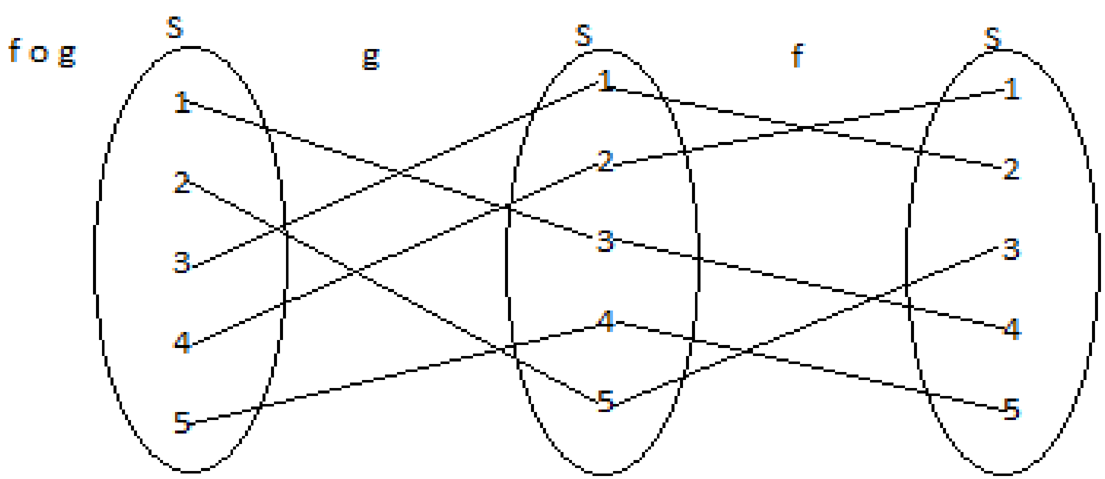
(b) Explain why f and g have inverses but h does not.

(c) Find f^{-1} and g^{-1}.

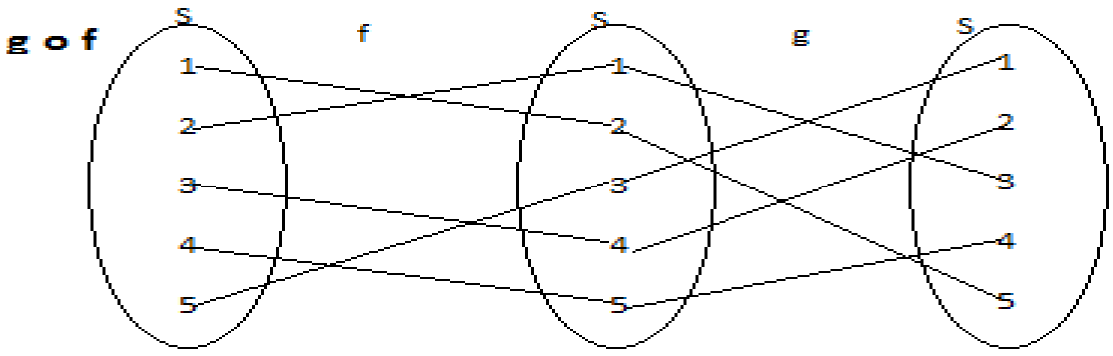
(d) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

Solution:

(a)



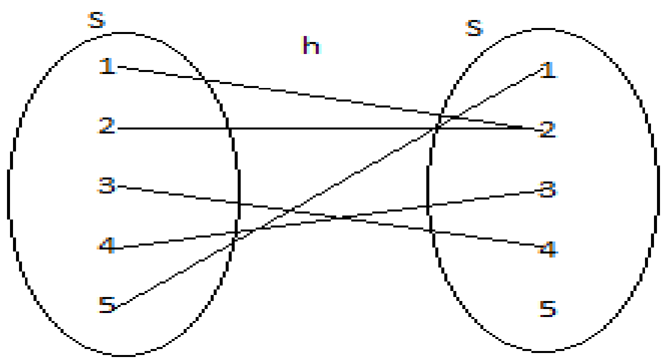
$\therefore\ f \circ g = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$



$\therefore\ g \circ f = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$

Therefore $g \circ f \neq f \circ g$

(a) Both f and g are 1-1 and onto. They are invertible. But h is not 1-1 and not onto



So h has no inverse.

(b) $f^{-1} = \{(2,1),(1,2),(4,3),(5,4),(3,5)\}$

$g^{-1} = \{(3,1),(5,2),(1,3),(2,4),(4,5)\}$

(c) Proof is left to the reader.

Problem 4:

If $f, g, h : R \to R$ defined by $f(x) = x^3 - 4x$, $g(x) = \dfrac{1}{x^2+1}$, $h(x) = x^4$, verify if $\{(f \circ g) \circ h\}(x) = \{f \circ (g \circ h)\}(x)$ after estimating their values.

Solution:

$$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = \left(\frac{1}{x^2+1}\right)^3 - 4\left(\frac{1}{x^2+1}\right)$$

$$\{(f \circ g) \circ h\}(x) = (f \circ g)h(x) = (f \circ g)(x^4) = \left(\frac{1}{x^8+1}\right)^3 - \frac{4}{x^8+1} \quad \ldots\ldots.(1)$$

$$(g \circ h)(x) = g(h(x)) = g(x^4) = \frac{1}{x^8+1}$$

$$\{f \circ (g \circ h)\}(x) = f(g \circ h(x)) = f\left(\frac{1}{x^8+1}\right) = \left(\frac{1}{x^8+1}\right)^3 - \frac{4}{x^8+1} \quad \ldots\ldots..(2)$$

From (1) and (2) we get $\{(f \circ g) \circ h\}(x) = \{f \circ (g \circ h)\}(x)$.


Thank You