# 18MAB302T-Discrete Mathematics for Engineers
# UNIT 3

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- Proposition and logical operators
- Truth tables
- Tautology , Contradiction and Contingency
- Equivalences
- Implications
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# Proposition or Statement

- A declarative sentence which is true or false but not both is called a proposition
- Sentences which are exclamatory, interrogative or imperative in nature are not propositions
- Lower case letters are used to denote propositions

Examples of propositions are

- Chennai is a city
- 2+2=3

Examples of not a proposition are

- What a beautiful day!
- Shut the door
- What time is it?

- If a proposition is true , we say that the truth value of the proposition is true, denoted as T
- If a proposition is false , we say that the truth value of the proposition is false, denoted as F
- Atomic/ Primary/Primitive statement is a statement which does not contain any connectives

- Compound or Molecular statement is a statement which is constructed by combining one or more statements using connectives

- There are 5 connectives or logical operators
- Conjunction, disjunction, negation, conditional and biconditional

# Conjunction ∧

- When p and q are any two propositions, the proposition "p and q" is denoted by $p \wedge q$ and is called conjunction of p and q which is a compound statement that is true when p and q are true and is false otherwise. Truth Table is

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Disjunction v

- When p and q are any two propositions, the proposition “p or q” is denoted by p v q and is called disjunction of p and q which is a compound statement  that is false when p and q is false and is true otherwise. Truth Table is

| p | q | pvq |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Negation ¬

- Given a proposition p, negation of this proposition is ¬p. ¬ is read as “It is not the case that” or “It is false that.
- Example: p Chennai is a city

  ¬p Chennai is not a city
- Truth table is

| p | ¬p |
|---|---|
| T | F |
| F | T |

# Conditional→

- If p and q are any two propositions, the compound proposition “if p then q” is denoted by $p \to q$ is called a conditional proposition, which is false when p is true and q is false and is true otherwise, p is hypothesis and q is conclusion. Truth table is

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Example of conditional

- If I get up at 5 AM, I will go for a walk
- If I get up at 5 AM, I will not go for a walk
- If I have not got up by 5 AM, I may or may not not go for a walk
- ( the contract is not violated , so p→q is true)

# biconditional↔

- If p and q are any two propositions, the compound proposition "p if and only if q" is denoted by p↔qis called a biconditionalproposition, which is true when p and q have the same truth value and is false otherwise, Eg. $n$ is even if and only if $n^2$is even

| p | q | p↔q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Tautology

- A compound statement P=P($p_{1,}p_2,\ldots,p_n$) where $p_{1,}p_2,\ldots,p_n$ are variables is called a tautology , if it is true for every truth value assignment for $p_{1,}p_2,\ldots,p_n$.
- Example

| p | ¬p | pv¬p |
|---|---|---|
| T | F | T |
| F | T | T |

# 2. Show that ((p →q) ^ (q →r)) → (p →r) is a tautology.

| p | q | r | $p \to q$ | $q \to r$ | $((p \to q) \wedge (q \to r) \equiv a$ | $(p \to r) \equiv b$ | $a \to b$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | F | T | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

3. Show that the following compound proposition is a tautology.

$$\neg(p \vee (q \wedge r)) \rightarrow ((p \vee q) \wedge (p \rightarrow r))$$

| p | q | r | $q \wedge r$ | $p \vee (q \wedge r)$ | $\neg(p \vee (q \wedge r))$ (a) | $p \vee q$ | $(p \rightarrow r)$ | $((p \vee q) \vee (p \rightarrow r))$ (b) | (a)→(b) |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | T | T | T | T |
| T | T | F | F | T | F | T | F | T | T |
| T | F | T | F | T | F | T | T | T | T |
| T | F | F | F | T | F | T | F | T | T |
| F | T | T | T | T | F | T | T | T | T |
| F | T | F | F | F | T | T | T | T | T |
| F | F | T | F | F | T | F | T | T | T |
| F | F | F | F | F | T | F | T | T | T |

# Contradiction

- A compound statement P=P($p_{1,}p_2,\ldots,p_n$) where $p_{1,}p_2,\ldots,p_n$ are variables is called a contradiction , if it is false for every truth value assignment for $p_{1,}p_2,\ldots,p_n$.
- Example

| p | ¬p | p∧¬p |
|---|---|---|
| T | F | F |
| F | T | F |

# 2. Show that the following Compound Proposition is a contradiction.

$$p \wedge (\neg q \wedge (p \rightarrow q))$$

| $p$ | $q$ | $\neg q$ | $p \rightarrow q$ | $\neg q \wedge (p \rightarrow q)$ | $p \wedge (\neg q \wedge (p \rightarrow q))$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | T | F | F | F |
| F | T | F | T | F | F |
| F | F | T | T | T | F |

# Example 3

The truth table for: $(p \vee q) \wedge (\neg p \wedge \neg q)$, is a contradiction, as shown below.

| p | q | $(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ | $(p \vee q) \wedge (\neg p \wedge \neg q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | T | F | F |
| F | T | T | T | F | F | F |
| F | F | F | T | T | T | F |

# Equivalence of propositions A≡B

- Two component propositions A($p_{1,}p_2$,…,$p_n$) and B($p_{1,}p_2$,…,$p_n$) are said to be logically equivalent or equivalent , if they have the identical truth tables.
- Eg.

| p | q | pvq | ¬(pvq) | ¬p | ¬q | ¬pΛ¬q |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

- Note: Already p↔q is true whenever A and B have same true vales. So p↔q is a tautology when A and B are equivalent. Conversely A≡B when p↔q is a tautology. Example is

| p | q | p→q | ¬p | ¬pvq | (p↔q) ↔(¬pvq) |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

- **Example**: show that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \rightarrow q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

# Duality Law

- The dual of a compound proposition that contains only the logical operators ∨, ∧ and ¬is the proposition obtained by replacing ∨ by ∧, each∧ by ∨, each T by F

  and each F by T, where T and F are special variables
representing tautology and contradiction respectively

Dual of proposition $A$ is denoted by $A^*$

- Duality theorem: If A($p_{1,}p_2$,…,$p_n$)= B($p_{1,}p_2$,…,$p_n$) , where A and B are compound propositions, then

$A^*$ ($p_{1,}p_2$,…,$p_n$) = $B^*$ ($p_{1,}p_2$,…,$p_n$)

# Examples

- 1. write the dual of $p \vee \neg q$

  Ans: $p \wedge \neg q$
- 2. write the dual of$p \wedge (q \vee (r \wedge T))$

  Ans: $p \vee (q \wedge (r \vee F))$
- 3. write the dual of $(p \wedge \neg q) \vee (q \wedge F)$

  Ans:$(p \vee \neg q) \wedge (q \vee T)$
- 4. write the dual of $p \wedge q) \vee (\neg p \vee q) \vee F$

  Ans: $(p \vee q) \wedge (\neg p \wedge q) \wedge T$

# Table1:Laws of Algebra of propositions

| S.No | Name of law | Primal form | Dual form |
|---|---|---|---|
| 1. | Idempotent law | $p \vee p \equiv p$ | $p \wedge p \equiv p$ |
| 2. | Identity law | $p \vee F \equiv p$ | $p \wedge T \equiv p$ |
| 3 | Dominant law | $p \vee T \equiv T$ | $p \wedge F \equiv F$ |
| 4. | Complement law | $p \vee \neg p \equiv T$ | $p \wedge \neg p \equiv F$ |
| 5. | Commutative law | $p \vee q \equiv q \vee p$ | $p \wedge q \equiv q \wedge p$ |
| 6 | Associative law | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ |
| 7. | Distributive law | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ |
| 8. | Absorption law | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 9. | De Morgan's law | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ | $\neg(p \wedge q) \equiv \neg p \wedge \neg q$ |

# Table 2: Equivalences involving Conditionals

| S.No. | Equivalence |
|---|---|
| 1.. | $p \rightarrow q \equiv \neg p \vee q$ |
| 2. | $p \rightarrow q \equiv \neg q \rightarrow \neg p$ |
| 3. | $p \vee q \equiv \neg p \rightarrow q$ |
| 4. | $p \wedge q \equiv \neg(p \rightarrow \neg q)$ |
| 5. | $\neg(p \rightarrow q) \equiv p \wedge \neg q$ |
| 6. | $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ |
| 7. | $(p \rightarrow r) \wedge (q \wedge r) \equiv (p \vee q) \rightarrow r$ |
| 8. | $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ |
| 9. | $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ |

# Table 3: Equivalences involving biconditionals

| S.No. | Equivalence |
|---|---|
| 1. | $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ |
| 2. | $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ |
| 3. | $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ |
| 4. | $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$ |

# Tautological Implication

- A compound proposition A($p_{1,}p_2$,…,$p_n$)is said to tautologically imply or imply the compound proposition B($p_{1,}p_2$,…,$p_n$), if B is true whenever A is true or equivalently if and only if A→B is a tautology. This is denoted by $A \Rightarrow B$,read as “A implies B”.
- Note: ⇒ is not a connective and A ⇒ B is not a proposition.

# Example problems of implication

Show that (i) $p \Rightarrow (p \vee q)$ (ii) $(p \rightarrow q) \Rightarrow (\neg q \rightarrow \neg p)$

(i)

| p | q | p v q | p → (p v q) |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | T | T |
| F | F | F | T |

(ii)

| p | q | ¬p | ¬q | p→q | ¬q→¬p | (p→q)→(¬q→¬p) |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

# Table 4: Implications

| S.No. | Implication |
|---|---|
| 1 | $p \wedge q \Rightarrow p$ |
| 2 | $p \wedge q \Rightarrow q$ |
| 3 | $p \Rightarrow p \vee q$ |
| 4 | $\neg p \Rightarrow p \rightarrow q$ |
| 5 | $q \Rightarrow p \rightarrow q$ |
| 6 | $\neg(p \rightarrow q) \Rightarrow p$ |
| 7 | $\neg(p \rightarrow q) \Rightarrow \neg q$ |
| 8 | $p \wedge (p \rightarrow q) \Rightarrow q$ |
| 9 | $\neg q \wedge (p \rightarrow q) \Rightarrow \neg p$ |
| 10 | $\neg p \wedge (p \vee q) \Rightarrow q$ |
| 11 | $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$ |
| 12 | $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$ |

# Problem 1

Without using truth tables, prove that

$p \to (q \to p) \equiv \neg p \to (p \to q)$

Solution:

$p \to (q \to p) \equiv \neg p \vee (q \to p)$

$\equiv \neg p \vee (\neg q \vee p)$

$\equiv \neg q \vee (p \vee \neg p)$, by commutative and associative laws

$\equiv \neg p \vee T$, by complement law

$\equiv T$ ,by dominant law (1)

$\neg p \to (p \to q) \equiv p \vee (p \to q)$, by (1) of Table 1.10

$\equiv p \vee (\neg p \vee q)$, by (1) of Table 1.10

$\equiv (p \vee \neg p) \vee q$, by associative law

$\equiv T \vee q$, by complement law

$\equiv T$, by dominant law, (2)

From (1) and (2), the result follows.

# Problem 2

Prove the following implications without using truth tables:

$((p \vee \neg p) \rightarrow q) \rightarrow ((p \vee \neg p) \rightarrow r) \Rightarrow q \rightarrow r$

*Solution:*

$[((p \vee \neg p) \rightarrow q) \rightarrow ((p \vee \neg p) \rightarrow r)] \rightarrow (q \rightarrow r)$

$\equiv [(T \rightarrow q) \rightarrow (T \rightarrow r)] \rightarrow (q \rightarrow r)$

$\equiv [(F \vee q) \rightarrow (F \vee r)] \rightarrow (q \rightarrow r)$

$\equiv (q \rightarrow r) \rightarrow (q \rightarrow r)$

$\equiv T$

# Unit – 3
# Theory of Inference

**OUTLINE**

- Introduction
- Truth table Technique
- Examples
- Rules of Inference
- Direct Method
- Indirect Method
- Inconsistency
- Conditional Proof

# Introduction

- Inference Theory is concerned with the inferring of a conclusion from certain assumptions called premises by applying certain principles of reasoning called **Rules of Inference.**
- When a conclusion derived from a set of premises by using rules of inference, the process of such derivation is called **Formal Proof**.

# Truth table Technique

- A Conclusion C is said to follow from a set of premises $H_1$, $H_2$, …..$H_n$ if $H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n \Rightarrow C$
- **Problem 1** Consider $H_1 : \neg P$ , $H_2 : P \vee Q \Rightarrow$ **C :** Q

| P | Q | $H_1 : \neg P$ | $H_2 : P \vee Q$ | C : Q |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | F |

**$H_1$ and $H_2$ are true only in the 3rd row of the above table, in which case Conclusion C is also true .Hence the conclusion is valid**

**Problem 2** If two sides of a triangle are equal, then two opposite angles are equal. Two sides of a triangle are not equal. Therefore the opposite angles are not equal.

P : Two sides of a triangle are equal.

Q: The two opposite angles are equal

$\mathbf{H_1}: P \rightarrow Q$, $\mathbf{H_2}: \neg P \Rightarrow \mathbf{C}: \neg Q$

| P | Q | $H_1 : P \rightarrow Q$ | $H_2 : \neg P$ | $C : \neg Q$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | F | F | T |
| F | T | T | T | F |
| F | F | T | T | T |

# Conclusion:

$H_1$ and $H_2$ are true in the $3^{rd}$ and $4^{th}$ row of the above table, but Conclusion C is different for the $3^{rd}$ and $4^{th}$ row . Hence the conclusion is not a valid conclusion.

## Note:

The truth table technique becomes tedious if the premises contain a large number of variables

# Rules of Inference

- **Rule P :** A premise may be introduced at any step in the derivation.

- **Rule T:** A formula S may be introduced in the derivation, if S is tautologically implied by one or more preceding formula in the derivation.



**contd…….**

# Rules of Inference

| S. No. | Rule in tautological form | Name of the rule |
|---|---|---|
| 1. | $(P \wedge Q) \rightarrow P$<br>$(P \wedge Q) \rightarrow Q$ | simplification |
| 2. | $P \rightarrow (P \vee Q)$<br>$Q \rightarrow (P \vee Q)$ | Addition |
| 3. | $P, Q \rightarrow (P \wedge Q)$ | Conjunction |
| 4. | $(P, (P \rightarrow Q)) \rightarrow Q$ | Modus Ponens |
| 5. | $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ | Modus Tollens |
| 6. | $((P \rightarrow Q), (Q \rightarrow R)) \rightarrow (P \rightarrow R)$ | Hypothetical Syllagism |
| 7. | $((P \vee Q), \neg P) \rightarrow Q$ | Disjunctive Syllagism |
| 8. | $((P \vee Q), (P \rightarrow R), (Q \rightarrow R)) \rightarrow R$ | Dilemma |

# Direct Method

- From the given statement, identify the premises and conclusion. Conclusion is obtained with the help of premises by using the rules of inference .

**Problem 1** If it rains heavily, then travelling will be difficult. If students arrive on time, then travelling was not difficult. They arrived on time. Therefore, it did not rain heavily.

**Solution :**

P : It rains heavily

Q: Travelling is difficult

R: Students arrived on time.

**Premises:** $P \rightarrow Q$, $R \rightarrow \neg Q$ and $R \Rightarrow$ Conclusion C: $\neg P$

| Stage | Premises | Rules |
| --- | --- | --- |
| (1) | $P \rightarrow Q$ | Rule P |
| (2) | $\neg Q \rightarrow \neg P$ | Rule T, (1), Contra positive |
| (3) | $R \rightarrow \neg Q$ | Rule P |
| (4) | $R \rightarrow \neg P$ | Rule T, (3),(2), Hypothetical Syllagism |
| (5) | R | Rule P |
| (6) | $\neg P$ | Rule T, (4),(5) and Modus Ponens |

**Problem 2** Construct an argument using rules of inference to show that the hypothesis:

Radha Works hard. If Radha works hard, then she is a dull girl and if Radha is a dull girl, then she will not get the job imply the conclusion Radha will not get the job.

**Solution** /* Step1: Name the Sentences in the question by using capital letters */

**R:** Radha works hard

**D:** Radha is a dull girl

**J:** She will not get the job

**Step 2** **/ * construct the premises . Identify the conclusion */**

| Sentence | Premise |
| --- | --- |
| Radha Works hard | $H_1$ : R |
| If Radha works hard, then she is a dull girl | $H_2$ : $R \rightarrow D$ |
| If Radha is a dull girl, then she will not get the job | $H_3$ : $D \rightarrow J$ |

| | |
| --- | --- |
| **Imply the conclusion, Radha will not get the job** | **C : J** |

| Stage | Premises | Rules |
|---|---|---|
| (1) | R | Rule P |
| (2) | $R \rightarrow D$ | Rule P |
| (3) | D | Rule T, (1),(2) and Modus Ponens |
| (4) | $D \rightarrow J$ | Rule P |
| (5) | J | Rule T, (3),(4) and Modus Ponens |

Thus we have derived the conclusion **C** using the given premises $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H_3}$

**Problem 3** It is not sunny this afternoon and it is colder than yesterday. We will go to the playground only if it is sunny. If we do not go to the ground then we will go to a movie. If we go to a movie then we will return by sunset lead to the conclusion: we will return home by sunset.

**Solution**

**P**: It is sunny this afternoon.
**Q**: It is colder than yesterday.
**R**: We will go to the playground.
**S**: We will go to a movie.
**T**: We return home by sunset.

The premises are $\neg P \wedge Q$, $R \rightarrow P$, $\neg R \rightarrow S$, $S \rightarrow T$

**Conclusion C**: T

| Stage | Premises | Rules |
|---|---|---|
| **1** | $\neg R \rightarrow S$ | Rule P |
| **2** | $S \rightarrow T$ | Rule P |
| **3** | $\neg R \rightarrow T$ | Rule T , (1), (2) Hypothetical syllogism |
| **4** | $\neg P \wedge Q$ | Rule P |
| **5** | $\neg P$ | Rule T , (4), simplification |
| **6** | $R \rightarrow P$ | Rule P |
| **7** | $\neg S \rightarrow R$ | Rule T ,(1), Contra positive |
| **8** | $\neg S \rightarrow P$ | Rule T , (6), (7), Hypothetical syllagism |
| **9** | $\neg P \rightarrow S$ | Rule T ,(8) ,Contrapositive |
| **10** | S | Rule T (5), (9), Modus Ponens |
| **11** | T | Rule T (2), (10), Modus Ponens. |

**Problem 4:** Prove the following using direct method:

$P \vee Q, P \rightarrow R, Q \rightarrow S \Rightarrow S \vee R$

| Stage | Premises | Rules |
|---|---|---|
| **1** | $P \vee Q$ | Rule P |
| **2** | $\sim P \rightarrow Q$ | Rule T, (1), Equivalence |
| **3** | $Q \rightarrow S$ | Rule P |
| **4** | $\sim P \rightarrow S$ | Rule T, (2), (3) Hypothetical syllogism |
| **5** | $P \rightarrow R$ | Rule P |
| **6** | $\sim S \rightarrow P$ | Rule T, (4), Contra positive |
| **7** | $\sim S \rightarrow R$ | Rule T ,(5), (6), Hypothetical syllogism |
| **8** | $S \vee R$ | Rule T , (7), Equivalence |

# Indirect Method

- To prove a conclusion C
- Assume the conclusion is false
- Include $\neg$C( negation of the conclusion) as an additional premise .
- Additional and along with the given premises arrive at a contradiction .

**Problem 1** Using indirect method of proof to derive $P \to \neg S$ from $P \to (Q \vee R)$, $Q \to \neg P$, $S \to \neg R$, $P$.

| Stage | Premises | Rule |
|---|---|---|
| 1 | $P \to (Q \vee R)$ | Rule P |
| 2 | $P$ | Rule P |
| 3 | $(Q \vee R)$ | Rule T ,(1), (2) Modus Ponens |
| 4 | $S \to \neg R$ | Rule P |
| 5. | $\neg(P \to \neg S)$ | Additional Premise |
| 6. | $\neg(\neg P \vee \neg S)$ | Rule T, (5), Equivalence |
| 7 | $P \wedge S$ | Rule T, (6), Equivalence |
| 8 | $S$ | Rule T ,(7), Simplification |

# Contd…

| Stage | Premises | Rules |
|---|---|---|
| 9 | $\neg R$ | Rule T, (4) ,(8) , Modus Ponens |
| 10 | $\neg Q \rightarrow R$ | Rule T ,(3), $a \rightarrow b \equiv \sim a \vee b$ |
| 11 | $\neg R \rightarrow Q$ | Rule T ,(10),Contra positive |
| 12 | Q | Rule T ,(9),(11) Modus Ponens |
| 13 | $Q \rightarrow \neg P$ | Rule P |
| 14 | $\neg P$ | Rule T ,(12), (13) Modus Ponens |
| 15 | $P \wedge \neg P \equiv F$<br>A contradiction | Rule T, (2),(12) Negation law |

**Problem 2**: Prove by Indirect method $\neg Q$, $P \rightarrow Q$, $(P \vee R) \Rightarrow R$

| Stage | Premises | Rules |
| --- | --- | --- |
| 1 | $P \rightarrow Q$ | Rule P |
| 2 | $\neg Q$ | Rule P |
| 3 | $\neg P$ | Rule T, (1),(2),Modus Tollens |
| 4 | $(P \vee R)$ | Rule P |
| 5. | $\neg R$ | Additional Premise |
| 6. | $\neg P \wedge \neg R$ | Rule T, (3),(5),Addition |
| 7 | $\neg(P \vee R)$ | Rule T, (6), Demorgan's Law |
| 8 | $(P \vee R) \wedge \neg(P \vee R) \equiv F$<br>A contradiction | Rule T, (4),(7), Negation law |

# Inconsistency

- A set of premises $H_1, H_2, \ldots\ldots H_n$ is said to be inconsistent if their conjunction implies a contradiction (i.e.,) $(H_1 \wedge H_2 \wedge \ldots\ldots \wedge H_n) \Rightarrow F$

**Problem 1** Show that the following premises are inconsistent $P \rightarrow Q$, $Q \rightarrow R$, $S \rightarrow \neg R$ and $P \wedge S$

| Stage | Premises | Rules |
|---|---|---|
| (1) | $P \rightarrow Q$ | Rule P |
| (2) | $Q \rightarrow R$ | Rule P |
| (3) | $P \rightarrow R$ | Rule T, (1),(2),Hypothetical Syllagism |
| (4) | $S \rightarrow \neg R$ | Rule P |
| (5) | $R \rightarrow \neg S$ | Rule T, (4),Contra positive |
| (6) | $P \rightarrow \neg S$ | Rule T, (3),(5), Hypothetical Syllagism |
| (7) | $\neg P \vee \neg S$ | Rule T, (6), $a \rightarrow b \equiv \sim a \vee b$ |
| (8) | $\neg(P \wedge S)$ | Rule T, (6), Demorgan’s Law |
| (9) | $(P \wedge S)$ | Rule P |
| (10) | $(P \wedge S) \wedge \neg(P \wedge S)\ ) \equiv F$ A contradiction | Rule T, (8),(9), Negation law |

**Problem 2** Show that the following premises are inconsistent

- If Raja misses many classes, then he fails in the final examination.
- If Raja fails in the final examination, then he is uneducated.
- If Raja reads a lots of books, then he is uneducated .
- If Raja misses many classes and reads a lot of books.

## Solution

- M: Raja misses many classes.
- F: He fails in the final examination.
- E: He is educated.
- B: Raja reads a lots of books

$$M \rightarrow F,\ F \rightarrow \neg E,\ B \rightarrow E,\ M \wedge B$$

We have to prove $M \rightarrow F, F \rightarrow \neg E, B \rightarrow E, (M \wedge B) \Rightarrow F$

| Stage | Premises | Rules |
|---|---|---|
| (1) | $M \rightarrow F$ | Rule P |
| (2) | $F \rightarrow \neg E$ | Rule P |
| (3) | $M \rightarrow \neg E$ | Rule T , (1), (2) Hypothetical syllagism |
| (4) | $B \rightarrow E$ | Rule P |
| (5) | $\neg E \rightarrow \neg B$ | Rule T, (4), Contra positive |
| (6) | $M \rightarrow B$ | Rule T , (3), (5) Hypothetical syllagism |
| (7) | $M \wedge B$ | Rule P |
| (8) | M | Rule T, (7),Simplification |
| (9) | $\neg B$ | Rule T, (6),(9), Modus Tollens |
| (10) | B | Rule T, (7),Simplification |
| (11) | $B \wedge \neg B \equiv F$<br>A contradiction | Rule T, (8),(9), Negation law |

# Conditional Proof

Given a set of premises ($H_1$, $H_2$, …..$H_n$)
whose conclusion is of the form ($P \rightarrow Q$).

**The rule CP as follows:**

Along with the given set of premises ($H_1$, $H_2$, …..$H_n$) , we take the antecedent part P of the conclusion ($P \rightarrow Q$) as an additional premise and we prove only the consequent part Q of ($P \rightarrow Q$ ) as the conclusion .

**Problem 1** If A works hard, then B or C will enjoy themselves . If B enjoys himself, then A will not work hard. If D enjoys himself, then C will not. Therefore, if A works hard, D will not enjoy himself.

Translate the above into statements and prove the conclusion by using the CP -Rule

## Solution

**P:** A works hard

**Q:** B will enjoy themself

**R:** C will enjoy themself

**S:** D enjoys himself

$$P \rightarrow (Q \vee R), Q \rightarrow \neg P, S \rightarrow \neg R \Rightarrow P \rightarrow \neg S$$

# Contd…

| Stage | Premises | Rules |
|---|---|---|
| (1) | $P \rightarrow (Q \vee R)$ | Rule P |
| (2) | $P$ | Additional Premise |
| (3) | $(Q \vee R)$ | Rule T,(1),(2), Modus Ponens |
| (4) | $Q \rightarrow \neg P$ | Rule P |
| (5) | $\neg Q$ | Rule T,(1),(2), Modus Tollens |
| (6) | $R$ | Rule T , (3), (5), Disjunctive syllagism |
| (7) | $S \rightarrow \neg R$ | Rule P |
| (8) | $\neg S$ | Rule T,(6),(7), Modus Tollens and $P \rightarrow \neg S$ by CP- Rule |

**Problem 2** Prove the following using CP Rule

$P \rightarrow (Q \rightarrow S), \neg R \vee P, Q \Rightarrow R \rightarrow S$

| Stage | Premises | Rules |
|---|---|---|
| (1) | $\neg R \vee P$ | Rule P |
| (2) | $R$ | Additional Premise |
| (3) | $P$ | Rule T,(1),(2), Disjunctive Syllagism |
| (4) | $P \rightarrow (Q \rightarrow S)$ | Rule P |
| (5) | $(Q \rightarrow S)$ | Rule T,(1),(2), Modus Ponens |
| (6) | $Q$ | Rule P |
| (7) | $S$ | Rule T,(5),(6), Modus Ponens and<br>$(R \rightarrow S)$ by CP -Rule |

# Principle of Mathematical Induction

- Let $P(n)$ be the statement involving a natural number. If $P(1)$ is true and $P(k+1)$ is true, then all the assumptions of $P(k)$ is true.
- We conclude that a statement $P(n)$ is true for all natural numbers.

# Procedure

- To prove P(n) is true for all natural numbers n.
- **First Step**: We must prove that P(1) is true.
- **Second Step**: Assume P(k) is true, P(k+1)is true
- The first step is called the **Basis Step of the proof.**
- The second step is called the **Induction step of the proof.**

# Problems

1) Prove by induction,

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots..+n^2 = \frac{n(n+1)(2n+1)}{6}$$

**Solution:**

**Let** P(n)= $\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots..+n^2 = \frac{n(n+1)(2n+1)}{6}$

**Basis step:**

To Prove: P(1) is true.

LHS: $1^2 = 1$

RHS: $\frac{(1)(2)(3)}{6} = \frac{6}{6} = 1.$

LHS=RHS

$\therefore$ P(1) is true

**Induction Step:** Assume P(n) is true.

(i.e).. $1^2 + 2^2 + \ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

To Prove: P(n+1) is true

$$1^2 + 2^2 + \ldots\ldots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+1)}{6}$$

LHS: $1^2 + 2^2 + \ldots\ldots + n^2 + (n+1)^2$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)+6(n+1)^2}{6}$$

$$= \frac{(n+1)[2n^2+7n+6]}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

=RHS

$\therefore$ P(n+1) is true.

If P(n) is true, then P(n+1) is also true.

$\therefore$ By the principle of Mathematical Induction,

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots..+n^2 = \frac{n(n+1)(2n+1)}{6}$$

2) Show that $a^n - b^n$ is divisible by (a-b) for all $n \in N$.

Solution: Let P(n): $a^n - b^n$ is divisible by (a-b) for all $n \in N$.

Basis Step: Let n=1

$a^1 - b^1 = a - b$ is divisible by a-b

$\therefore$ P(1) is true.

Induction step: Assume P(n) is true

$a^n - b^n$ is divisible by (a-b)

To Prove: $a^{n+1} - b^{n+1}$ is divisible by (a-b)

$$
\begin{aligned}
a^{n+1} - b^{n+1} &= a^n.a - b^n.b \\
&= [c(a-b)+b^n]a - b^{n+1} \\
&= ac(a-b) + a b^n - b^{n+1} \\
&= ac(a-b) + b^n(a-b)
\end{aligned}
$$

$$a^{n+1} - b^{n+1} = (a-b)[ac+b^n]$$

$a^{n+1} - b^{n+1}$ is divisible by (a-b)

(i.e)., if P(n) is true, then P(n+1) is also true.

$\therefore$ By the principle of Mathematical Induction,

$a^n - b^n$ is divisible by (a-b) for all $n \in N$.

3) Prove by Mathematical Induction $2^n < n!$ for every integer $n \geq 4$

Proof: Let $P(n): 2^n < n!$

**Basis Step: Let $n = 4$**

(i.e) $2^4 = 16$ and $4! = 24$

$$\Rightarrow 2^4 < 4!$$

$\therefore 2^n < n!$ for every integer $n \geq 4$

(i.e)., $P(4)$ is true

**Induction Step:** Assume $P(n) = 2^n - n! < 0$

$\Rightarrow 2^n - n! = m$ where m is negative integer

To Prove: $P(n+1) = 2^{n+1} - (n+1)! < 0$

$2^{n+1} - (n+1)! = 2^n\, 2 - (n+1)!$

$= (m+n!)2 - (n+1)!$

$= 2m + n![2-n-1]$

$2^{n+1} - (n+1)! = 2m + n![1-n]$

2m is a negative number, (1-n) is also an negative number since $n \geq 4$.

$\therefore 2^{n+1} - (n+1)! = 2m + n![1-n]$ is a negative number.

(i.e)., if P(n) is true, then P(n+1) is also true.

$\therefore$ By the principle of Mathematical Induction,

$2^n < n!$ for every integer $n \geq 4$

4) Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n},\ n \geq 2$ .

Proof

Let P(n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ .--------(1)

Basis Step: Let n = 2

LHS of (1) = $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.707$

RHS of (1) = $\sqrt{2} = 1.414$

LHS > RHS

$\therefore$ P(2) is true

Induction Step:

Let us assume that P(n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ is true.

To Prove: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}$

LHS: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}}$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \frac{\sqrt{n}(\sqrt{n+1})+1}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \frac{(\sqrt{n^2+n})+1}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \frac{n+1}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}\,.$$

(i.e)., if P(n) is true, then P(n+1) is also true.

$\therefore$ By the principle of Mathematical Induction,

P(n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ is true.

# THANK YOU