# 18MAB302T - DISCRETE MATHEMATICS

E. Suresh
Dept. of Mathematics
SRMIST

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- How beautiful is Rose?
- $2 + 2 = 3$
- What time is it?
- $X + Y = Z$
- Take a cup of coffee

In the above statements (2), (4) and (6) are obviously not proposition as they are not declarative in nature whereas (1) and (3) are propositions. Also (5) is not Proposition, since it is neither true nor false, as the value of X, Y and Z are not assigned

**Truth Value**

If a proposition is true we say that truth value of the proposition is true and denoted by T

If a proposition is false then the truth value issaid to be false and denoted by F

Note : Propositions are represented by P, Q, R, S,....

## Atomic proposition

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**Compound Proposition**

A proposition obtained from the combinations of two or more propositions by means of logical operators or connectivesis referred as molecular or compound proposition.

# Connectives

## Connectives

The word and phrases used to form compound propositions are called connectives.

The following symbols are used to represent connectives

| Symbol | Connective Word | Name |
|---|---|---|
| $\neg$ | not | Negation |
| $\wedge$ | and | Conjunction |
| $\vee$ | or | Disjunction |
| $\rightarrow$ | if...then... | Implication or Conditional |
| $\leftrightarrow$ | if and only if | Bi-conditional |

# Negation

**Negation**

If P is any proposition then negation of P, denoted by $\neg P$ which is read as not P, is a proposition whose truth value is false when P is true and true when P is false.

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Let P : Paris is in France

Then the negation of P is the statement.

$\neg P$ : It is not the case that Paris is in France

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**Example** : Consider the statement

Let P : Paris is in France

Then the negation of P is the statement.

$\neg P$ : It is not the case that Paris is in France

Normally it is written as

$\neg P$ : Paris is not in France **Truth Table**:

| P | $\neg P$ |
|---|---|
| T | F |
| F | T |

1. Find the negation of the proposition "All students are intelligent"

   Solution : Let P : All students are intelligent .

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   Solution : Let P : All students are intelligent .

   Then negation of P is given as

   $\neg$P : It is not the case that all students are intelligent

   [OR]

   $\neg$P : Some students are not intelligent

   [OR]

   $\neg$P : There exist a student who is not intelligent

1. Find the negation of the proposition "No student is intelligent"
   Solution : Let Q : No student is intelligent
   Then the negation of Q is given as
   $\neg Q$ : Some students are intelligent

# Conjunction

**Conjunction**

If P and Q are two propositions then conjunction of P and Q is the compound statement, denoted by $P \wedge Q$ and read as "P and Q ".

The truth value of $P \wedge Q$ is true when both P and Q are true, otherwise it is false.

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The truth value of $P \wedge Q$ is true when both P and Q are true, otherwise it is false.

Truth table for conjunction of P and Q is given as

| P | Q | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

## Example:

**Example:**

Form the conjunction of P and Q for each of the following

(i) P :Ram is healthy ; Q : He has Blue Eyes

(ii) P : It is Cold; Q : It is raining

(iii) P : 5 x + 6 = 26; Q : $x > 3$

**Example:**

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(i) P :Ram is healthy ; Q : He has Blue Eyes

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(iii) P : 5 x + 6 = 26; Q : $x > 3$

**Solution :** Conjunction of P and Q is P$\wedge$Q

(i) P$\wedge$Q : Ram is healthy and he has blue eyes

(ii) P$\wedge$Q : It is cold and raining

(iii) P$\wedge$Q : 5 x + 6 = 26 and $x > 3$

# Disjunction

## Disjunction

If P and Q are two propositions then disjunction of P and Q is the compound statement, denoted by $P \vee Q$and read as "P or Q"

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If P and Q are two propositions then disjunction of P and Q is the compound statement, denoted by $P \vee Q$and read as "P or Q"

Truth table for disjunction of P and Q is given as

| P | Q | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

## Example:

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Form the disjunction of P and Q for each of the following

(i) P :Arjun will go to Delhi; Q : Arjun will go to Chennai

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Form the disjunction of P and Q for each of the following

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**Solution :** Disjunction of P and Q is $P \vee Q$

(i) $P \vee Q$ : Arjun will go to Delhi or Chennai

(ii) $P \vee Q$ : It is cold or raining

# Problems

If P : It is cold and Q : It is raining

Write simple verbal sentence which describes each of the following statements

(i) $\neg P$ (ii) $P \wedge Q$ (iii) $P \vee Q$ (iv) $P \vee \neg Q$

# Problems

If P : It is cold and Q : It is raining

Write simple verbal sentence which describes each of the following statements

(i) $\neg P$ (ii) $P \wedge Q$ (iii) $P \vee Q$ (iv) $P \vee \neg Q$

**Solution :**

(i) $\neg P$ : It is not cold

(ii) $P \wedge Q$ : It is cold and raining

(iii) $P \vee Q$ : It is cold or raining

(iv) $P \vee \neg Q$ : It is cold or it is not raining

# Conditional Proposition

**Conditional Proposition**

If P and Q are two propositions, the compound proposition ”if P then Q”, denoted by $P \to Q$ , is called a conditional proposition or implication.

The proposition P is called antecedent or hypothesis and the proposition Q is called the consequent or conclusion.

**Conditional Proposition**

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**Truth table :**

| P | Q | $P \to Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The alternative terminologies used to express $P \to Q$ are the following

(i) P implies Q

(ii) P only if Q

(iii) Q if P  or Q when P

(iv) Q follows from P

(v) P is sufficient for Q

**Example :**

Form the conditional statement of P and Q for each of the following

(i) P : Tomorrow is Sunday ; Q : Today is Saturday

(ii) P : It rains; Q : I will carry an umbrella

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Form the conditional statement of P and Q for each of the following

(i) P : Tomorrow is Sunday ; Q : Today is Saturday

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**Solution :** Conditional statement of P and Q is $P \to Q$

(i) $P \to Q$ : If tomorrow is Sunday then today is Saturday

(ii) $P \to Q$ : If it rains then I will carry an umbrella

## Bi-Conditional Proposition

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If P and Q are two propositions, the compound proposition "P if and only if Q" denoted by $P \leftrightarrow Q$ is called a bi-conditional proposition.

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**Truth table :**

| P | Q | $P \leftrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Problem 1

Construct truth table for each of the following compound proposition

(i) $P \wedge (\neg Q \vee Q)$

(ii) $\neg(P \vee Q) \vee (\neg P \wedge \neg Q)$

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Construct truth table for each of the following compound proposition

(i) $P \wedge (\neg Q \vee Q)$

(ii) $\neg(P \vee Q) \vee (\neg P \wedge \neg Q)$

**Solution :** (i) Truth table for $P \wedge (\neg Q \vee Q)$

| P | Q | ¬Q | ¬Q ∨ Q | $P \wedge (\neg Q \vee Q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | T | T |
| F | T | F | T | F |
| F | F | T | T | F |

**Solution : (ii)** Truth table for $\neg(P \vee Q) \vee (\neg P \wedge \neg Q)$

Let A $\equiv \neg(P \vee Q) \vee (\neg P \wedge \neg Q)$

**Solution : (ii)** Truth table for $\neg(P \vee Q) \vee (\neg P \wedge \neg Q)$

Let A $\equiv \neg(P \vee Q) \vee (\neg P \wedge \neg Q)$

| P | Q | ¬P | ¬Q | P∨ Q | ¬ ( P∨ Q) | (¬ P ∧¬ Q) | A |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | F |
| T | F | F | T | T | F | F | F |
| F | T | T | F | T | F | F | F |
| F | F | T | T | F | T | T | T |

# Problem 2

Construct truth table for

(i) $(P \vee \neg Q) \rightarrow P$

(ii) $(\neg(P \wedge Q) \vee R) \rightarrow \neg P$

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# Problem 2

Construct truth table for

(i) $(P \vee \neg Q) \to P$

(ii) $(\neg(P \wedge Q) \vee R) \to \neg P$

Solution : (i) Truth table for $(P \vee \neg Q) \to P$

| P | Q | ¬Q | $P \vee \neg Q$ | $(P \vee \neg Q) \to P$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | T | T |
| F | T | F | F | T |
| F | F | T | T | F |

Solution : (ii) Truth table for $(\neg(P \wedge Q) \vee R) \rightarrow \neg P$ Let A $\equiv (\neg(P \wedge Q) \vee R) \rightarrow \neg P$

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| P | Q | R | $\neg P$ | $P \wedge Q$ | $\neg(P \wedge Q)$ | $\neg(P \wedge Q) \vee R$ | A |
|---|---|---|---|---|---|---|---|
| T | T | T | F | T | F | T | F |
| T | T | F | F | T | F | F | T |
| T | F | T | F | F | T | T | F |
| T | F | F | F | F | T | T | F |
| F | T | T | T | F | T | T | T |
| F | T | F | T | F | T | T | T |
| F | F | T | T | F | T | T | T |
| F | F | F | T | F | T | T | T |

# Problem 3

Construct truth table for

$$(\neg(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \rightarrow R))$$

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Construct truth table for

$(\neg(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \rightarrow R))$

Solution : Let A $\equiv (P \vee (Q \wedge R))$ ; B $\equiv (\neg(P \vee (Q \wedge R))$

and C $\equiv ((P \vee Q) \wedge (P \rightarrow R))$

Truth table is

# Problem 3

Construct truth table for

$(\neg(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \rightarrow R))$

Solution : Let A $\equiv (P \vee (Q \wedge R))$ ; B $\equiv (\neg(P \vee (Q \wedge R))$

and C $\equiv ((P \vee Q) \wedge (P \rightarrow R))$

Truth table is

| P | Q | R | $Q \wedge R$ | A | B | $P \vee Q$ | $P \rightarrow R$ | C | $B \leftrightarrow C$ |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | T | T | T | F |
| T | T | F | F | T | F | T | F | F | T |
| T | F | T | F | T | F | T | T | T | F |
| T | F | F | F | T | F | T | F | F | T |
| F | T | T | T | T | F | T | T | T | F |
| F | T | F | F | F | T | T | T | T | T |

# Problem 4

Problem-4 : Construct truth table for

(i) $(P \vee Q) \rightarrow R$

(ii) $(P \rightarrow R) \wedge (Q \rightarrow R)$

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Problem-4 : Construct truth table for

(i) $(P \vee Q) \rightarrow R$

(ii) $(P \rightarrow R) \wedge (Q \rightarrow R)$

Solution :(i) Truth table for $(P \vee Q) \rightarrow R$ is

| P | Q | R | $(P \vee Q)$ | $(P \vee Q) \rightarrow R$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | T | F |
| T | F | T | T | T |
| T | F | F | T | F |
| F | T | T | T | T |
| F | T | F | T | F |

Solution :(ii) Truth table for $(P \to R) \wedge (Q \to R)$ is

Solution :(ii) Truth table for $(P \to R) \wedge (Q \to R)$ is

| P | Q | R | $(P \to Q)$ | $(Q \to R)$ | $(P \to R) \wedge (Q \to R)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | F | F | F |
| T | F | T | T | T | T |
| T | F | F | F | T | F |
| F | T | T | T | T | T |
| F | T | F | T | F | F |
| F | F | T | T | T | T |
| F | F | F | T | T | T |

# Converse, Contrapositive and Inverse

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Let $P \to Q$ be conditional statement, then

(i) Converse of $P \to Q$ is $Q \to P$

(ii) Contrapositive of $P \to Q$ is $\neg Q \to \neg P$

# Converse, Contrapositive and Inverse

Let $P \to Q$ be conditional statement, then

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(ii) Contrapositive of $P \to Q$ is $\neg Q \to \neg P$

(iii) Inverse of $P \to Q$ is $\neg P \to \neg Q$

# Example

Find Converse, Contrapositive and Inverse for the following conditional statement

(i) If it rains then the crops will grow

(ii) If Raja is a poet, then he is poor

(iii) Only if Ram studies well, he pass the test

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(iii) Only if Ram studies well, he pass the test

Solution : (i) Let P : It rains and Q : The crops will grow

So, $P \to Q$ : If it rains then the crops will grow

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So, $P \to Q$ : If it rains then the crops will grow

Now converse $Q \to P$ : If the crops grow then there is rain

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(i) If it rains then the crops will grow

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Solution : (i) Let P : It rains and Q : The crops will grow

So, $P \rightarrow Q$ : If it rains then the crops will grow

Now converse $Q \rightarrow P$ : If the crops grow then there is rain

Contrapositive $\neg Q \rightarrow \neg P$ : If the crops do not grow then there is no rain

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Find Converse, Contrapositive and Inverse for the following conditional statement

(i) If it rains then the crops will grow

(ii) If Raja is a poet, then he is poor

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Solution : (i) Let P : It rains and Q : The crops will grow

So, $P \to Q$ : If it rains then the crops will grow

Now converse $Q \to P$ : If the crops grow then there is rain

Contrapositive $\neg Q \to \neg P$ : If the crops do not grow then there is no rain

Inverse $\neg P \to \neg Q$ : If it does not rain then the crops will not grow

Solution : (ii) Let P : Raja is a poet and Q : Raja is poor

So, $P \to Q$ : If Raja is a poet then he is poor

Now converse $Q \to P$ : If Raja is poor then he is a poet

Contrapositive $\neg Q \to \neg P$ : If Raja is not poor then he is not a poet

Inverse $\neg P \to \neg Q$ : If Raja is not a poet then he is not poor

Solution : (ii) Let P : Raja is a poet and Q : Raja is poor

So, $P \to Q$ : If Raja is a poet then he is poor

Now converse $Q \to P$ : If Raja is poor then he is a poet

Contrapositive $\neg Q \to \neg P$ : If Raja is not poor then he is not a poet

Inverse $\neg P \to \neg Q$ : If Raja is not a poet then he is not poor

Solution : (iii) Let P : Ram pass the test and Q : Ram studies well

So, $P \to Q$ : If Ram pass the test then he studies well

Now converse $Q \to P$ : If Ram studies well then he pass the test

Contrapositive $\neg Q \to \neg P$ : If Ram does not study well then he will not pass the test

Inverse $\neg P \to \neg Q$ : If Ram does not pass the test then he

# Tautology

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A statement that is always false is called a contradiction or absurdity

Note : If a proposition is neither a tautology nor a contradiction, it is called a contingency

# Equivalences of Propositions

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Two compound propositions A and B are said to be logically equivalent or simply equivalent, if they have identical truth values

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## Equivalences of Propositions

Two compound propositions A and B are said to be logically equivalent or simply equivalent, if they have identical truth values
i.e., if the truth value of A is equal to the truth value of B for every one of the $2^n$ possible sets of truth values
It is denoted as $A \equiv B$

Note : $A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology

## Tautological Implications

**Tautological Implications**

A compound proposition A is said to tautologically imply or simply imply the compound proposition B, if and only if, $A \to B$ is a tautology.
This is denoted by $A \Rightarrow B$, read as A implies B

# Problems

1. Determine which of the following compound proposition are tautologies and which of them are contradiction, using truth tables

(i) $(\neg Q \wedge (P \to Q)) \to \neg P$

(ii) $((P \to Q) \wedge (Q \to R)) \to (P \to R)$

(iii) $\neg(Q \to R) \wedge R \wedge (P \to Q)$

Solution :(i) Truth table for $(\neg Q \wedge (P \to Q)) \to \neg P$

Let A $\equiv (\neg Q \wedge (P \to Q)) \to \neg P$

| P | Q | $\neg P$ | $\neg Q$ | $P \to Q$ | $\neg Q \wedge (P \to Q)$ | A |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | T | T |

$\therefore (\neg Q \wedge (P \to Q)) \to \neg P$ is a tautology

Solution : (ii) Truth table for

$((P \to Q) \wedge (Q \to R)) \to (P \to R)$

Let A $\equiv ((P \to Q) \wedge (Q \to R))$ and

$B \equiv ((P \to Q) \wedge (Q \to R)) \to (P \to R) \equiv A \to (P \to R)$

| P | Q | R | $P \to Q$ | $Q \to R$ | A | $P \to R$ | B |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | F | T | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

Solution : (iii) Truth table for $\neg(Q \to R) \wedge R \wedge (P \to Q)$

Let A $\equiv \neg(Q \to R) \wedge R$

| P | Q | R | $P \to Q$ | $Q \to R$ | $\neg(Q \to R)$ | A | $A \wedge (P \to Q)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | F |
| T | T | F | T | F | T | F | F |
| T | F | T | F | T | F | F | F |
| T | F | F | F | T | F | F | F |
| F | T | T | T | T | F | F | F |
| F | T | F | T | F | T | F | F |
| F | F | T | T | T | F | F | F |
| F | F | F | T | T | F | F | F |

$\therefore \neg(Q \to R) \wedge R \wedge (P \to Q)$ is a contradiction

1. Show that the propositions $P \to Q$ and $\neg P \vee Q$ are logically equivalent

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Solution : To prove that : $P \to Q$ and $\neg P \vee Q$ are logically equivalent

It is enough to prove that $(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology

| P | Q | $\neg P$ | $(P \to Q)$ | $\neg P \vee Q$ | $(P \to Q) \leftrightarrow (\neg P \vee Q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

$(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology

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It is enough to prove that $(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology

| P | Q | $\neg P$ | $(P \to Q)$ | $\neg P \vee Q$ | $(P \to Q) \leftrightarrow (\neg P \vee Q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

$(P \to Q) \leftrightarrow (\neg P \vee Q)$ is a tautology

$\Rightarrow P \to Q$ and $\neg P \vee Q$ are logically equivalent

# IMPORTANT LOGICAL EQUIVALENCE

| S.No | Law | | |
|---|---|---|---|
| 1 | Idempotent | $P \vee P \equiv P$ | $P \wedge P \equiv P$ |
| 2 | Identity | $P \vee F \equiv P$ | $P \wedge T \equiv P$ |
| 3 | Dominant | $P \vee T \equiv T$ | $P \wedge F \equiv F$ |
| 4 | Complement | $P \vee \neg P \equiv T$ | $P \wedge \neg P \equiv F$ |
| 5 | Commutative | $P \vee Q \equiv Q \vee P$ | $P \wedge Q \equiv Q \wedge P$ |
| 6 | Associative | $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ | $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ |
| 7 | Distributive | $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ | $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ |
| 8 | Absorption | $P \vee (P \wedge Q) \equiv P$ | $P \wedge (P \vee Q) \equiv P$ |
| 9 | De Morgans | $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ | $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ |

# EQUIVALENCES INVOLVING CONDITIONAL

| S.No | Formula |
|---|---|
| 1 | $P \to Q \equiv \neg P \vee Q$ |
| 2 | $P \to Q \equiv \neg Q \to \neg P$ |
| 3 | $P \vee Q \equiv P \to Q$ |
| 4 | $P \wedge Q \equiv \neg(P \to \neg Q)$ |
| 5 | $\neg(P \to Q) \equiv P \wedge \neg Q$ |
| 6 | $(P \to Q) \wedge (P \to R) \equiv P \to (Q \wedge R)$ |
| 7 | $(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$ |
| 8 | $(P \to R) \wedge (Q \to R) \equiv (P \vee Q) \to R$ |
| 9 | $(P \to R) \vee (Q \to R) \equiv (P \wedge Q) \to R$ |

EQUIVALENCES INVOLVING BI-CONDITIONAL

| S.No | Formula |
|---|---|
| 1 | $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$ |
| 2 | $P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$ |
| 3 | $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$ |
| 4 | $\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$ |

# Problems

1. Without using truth tables, prove the following

   (i) $(\neg P \vee Q) \wedge (P \wedge (P \wedge Q)) \equiv P \wedge Q$

   (ii) $P \rightarrow (Q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow Q)$

   (iii) $\neg(P \leftrightarrow Q) \equiv (P \vee Q) \wedge \neg(P \vee Q) \equiv$
   $(P \wedge \neg Q) \vee (\neg P \wedge Q)$

Solution : (i)

$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

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$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

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$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$ [By idempotent]

Solution : (i)

$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$ [By idempotent]

$\equiv (P \wedge Q) \wedge (\neg P \vee Q)$ [By commutative]

Solution : (i)

$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$ [By idempotent]

$\equiv (P \wedge Q) \wedge (\neg P \vee Q)$ [By commutative]

$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q)$ [By distributive]

Solution : (i)

$$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$$

$$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q) \qquad \text{[By Associative]}$$

$$\equiv (\neg P \vee Q) \wedge (P \wedge Q) \qquad \text{[By idempotent]}$$

$$\equiv (P \wedge Q) \wedge (\neg P \vee Q) \qquad \text{[By commutative]}$$

$$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q) \qquad \text{[By distributive]}$$

$$\equiv (P \wedge Q \wedge \neg P) \vee (P \wedge Q \wedge Q) \qquad \text{[By commutative]}$$

Solution : (i)

$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$ [By idempotent]

$\equiv (P \wedge Q) \wedge (\neg P \vee Q)$ [By commutative]

$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q)$ [By distributive]

$\equiv (P \wedge Q \wedge \neg P) \vee (P \wedge Q \wedge Q)$ [By commutative]

$\equiv ((P \wedge \neg P) \wedge Q) \vee (P \wedge (Q \wedge Q))$ [By associative]

Solution : (i)

$$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$$

$$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q) \quad \text{[By Associative]}$$

$$\equiv (\neg P \vee Q) \wedge (P \wedge Q) \quad \text{[By idempotent]}$$

$$\equiv (P \wedge Q) \wedge (\neg P \vee Q) \quad \text{[By commutative]}$$

$$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q) \quad \text{[By distributive]}$$

$$\equiv (P \wedge Q \wedge \neg P) \vee (P \wedge Q \wedge Q) \quad \text{[By commutative]}$$

$$\equiv ((P \wedge \neg P) \wedge Q) \vee (P \wedge (Q \wedge Q)) \quad \text{[By associative]}$$

$$\equiv (F \wedge Q) \vee (P \wedge Q) \quad \text{[By complement \& idempotent]}$$

Solution : (i)

$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$ [By idempotent]

$\equiv (P \wedge Q) \wedge (\neg P \vee Q)$ [By commutative]

$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q)$ [By distributive]

$\equiv (P \wedge Q \wedge \neg P) \vee (P \wedge Q \wedge Q)$ [By commutative]

$\equiv ((P \wedge \neg P) \wedge Q) \vee (P \wedge (Q \wedge Q))$ [By associative]

$\equiv (F \wedge Q) \vee (P \wedge Q)$ [By complement & idempotent]

$\equiv F \vee (P \wedge Q)$ [By dominant]

Solution : (i)

$$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$$

$$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q) \quad \text{[By Associative]}$$

$$\equiv (\neg P \vee Q) \wedge (P \wedge Q) \quad \text{[By idempotent]}$$

$$\equiv (P \wedge Q) \wedge (\neg P \vee Q) \quad \text{[By commutative]}$$

$$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q) \quad \text{[By distributive]}$$

$$\equiv (P \wedge Q \wedge \neg P) \vee (P \wedge Q \wedge Q) \quad \text{[By commutative]}$$

$$\equiv ((P \wedge \neg P) \wedge Q) \vee (P \wedge (Q \wedge Q)) \quad \text{[By associative]}$$

$$\equiv (F \wedge Q) \vee (P \wedge Q) \quad \text{[By complement \& idempotent]}$$

$$\equiv F \vee (P \wedge Q) \quad \text{[By dominant]}$$

$$\equiv P \wedge Q \quad \text{[By dominant]}$$

Solution : (i)

$LHS \equiv (\neg P \vee Q) \wedge (P \wedge (P \wedge Q))$

$\equiv (\neg P \vee Q) \wedge ((P \wedge P) \vee Q)$ [By Associative]

$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$ [By idempotent]

$\equiv (P \wedge Q) \wedge (\neg P \vee Q)$ [By commutative]

$\equiv ((P \wedge Q) \wedge \neg P) \vee ((P \wedge Q) \wedge Q)$ [By distributive]

$\equiv (P \wedge Q \wedge \neg P) \vee (P \wedge Q \wedge Q)$ [By commutative]

$\equiv ((P \wedge \neg P) \wedge Q) \vee (P \wedge (Q \wedge Q))$ [By associative]

$\equiv (F \wedge Q) \vee (P \wedge Q)$ [By complement & idempotent]

$\equiv F \vee (P \wedge Q)$ [By dominant]

$\equiv P \wedge Q$ [By dominant]

$\equiv RHS$

Therefore $(\neg P \vee Q) \wedge (P \wedge (P \wedge Q)) \equiv P \wedge Q$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

LHS $\equiv P \to (Q \to P)$

Solution :(ii) To prove : $P \rightarrow (Q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow Q)$

LHS $\equiv P \rightarrow (Q \rightarrow P)$

$\equiv \neg P \vee (Q \rightarrow P)$ $\quad [P \rightarrow Q \equiv \neg P \vee Q]$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

LHS $\equiv P \to (Q \to P)$

$\equiv \neg P \vee (Q \to P)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee P)$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

LHS $\equiv P \to (Q \to P)$

$\equiv \neg P \vee (Q \to P)$ $[P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee P)$

$\equiv \neg P \vee (P \vee \neg Q)$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

LHS $\equiv P \to (Q \to P)$

$\equiv \neg P \vee (Q \to P)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee P)$

$\equiv \neg P \vee (P \vee \neg Q)$

$\equiv (\neg P \vee P) \vee \neg Q$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

LHS $\equiv P \to (Q \to P)$

$\equiv \neg P \vee (Q \to P)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee P)$

$\equiv \neg P \vee (P \vee \neg Q)$

$\equiv (\neg P \vee P) \vee \neg Q$

$\equiv (T) \vee \neg Q$

Solution :(ii) To prove : $P \to (Q \to P) \equiv \neg P \to (P \to Q)$

LHS $\equiv P \to (Q \to P)$

$\equiv \neg P \vee (Q \to P)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee P)$

$\equiv \neg P \vee (P \vee \neg Q)$

$\equiv (\neg P \vee P) \vee \neg Q$

$\equiv (T) \vee \neg Q$

$\equiv T$ $\quad$ —–¿ (1)

$$\text{RHS} \equiv \neg P \to (P \to Q)$$

$$\text{RHS} \equiv \neg P \to (P \to Q)$$

$$\equiv P \vee (P \to Q) \qquad [P \to Q \equiv \neg P \vee Q]$$

$$\text{RHS} \equiv \neg P \to (P \to Q)$$

$$\equiv P \vee (P \to Q) \qquad [P \to Q \equiv \neg P \vee Q]$$

$$\equiv P \vee (\neg P \vee Q)$$

$$\begin{aligned} \text{RHS} &\equiv \neg P \to (P \to Q) \\ &\equiv P \vee (P \to Q) && [P \to Q \equiv \neg P \vee Q] \\ &\equiv P \vee (\neg P \vee Q) \\ &\equiv (P \vee \neg P) \vee Q \end{aligned}$$

$$\text{RHS} \equiv \neg P \to (P \to Q)$$

$$\equiv P \vee (P \to Q) \qquad [P \to Q \equiv \neg P \vee Q]$$

$$\equiv P \vee (\neg P \vee Q)$$

$$\equiv (P \vee \neg P) \vee Q$$

$$\equiv T \vee Q$$

$$\text{RHS} \equiv \neg P \to (P \to Q)$$

$$\equiv P \vee (P \to Q) \qquad [P \to Q \equiv \neg P \vee Q]$$

$$\equiv P \vee (\neg P \vee Q)$$

$$\equiv (P \vee \neg P) \vee Q$$

$$\equiv T \vee Q$$

$$\equiv T \qquad ------> (2)$$

RHS $\equiv \neg P \rightarrow (P \rightarrow Q)$

$\equiv P \vee (P \rightarrow Q)$ $\qquad [P \rightarrow Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg P \vee Q)$

$\equiv (P \vee \neg P) \vee Q$

$\equiv T \vee Q$

$\equiv T$ $\qquad ------> (2)$

From eqn (1) and (2), we have LHS $\equiv$ RHS

Therefore $P \rightarrow (Q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow Q)$

Solution : (iii)

Let A $\equiv \neg(P \leftrightarrow Q)$

B $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

C $\equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

To prove : A $\equiv$ B $\equiv$ C

First we prove A $\equiv$ B

i.e., to prove $\neg(P \leftrightarrow Q) \equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$$\text{LHS} \equiv \neg(P \leftrightarrow Q)$$

$$\text{LHS} \equiv \neg(P \leftrightarrow Q)$$

$$\equiv \neg[(P \to Q) \wedge (Q \to P)]$$

$$
\begin{aligned}
\text{LHS} &\equiv \neg(P \leftrightarrow Q) \\
&\equiv \neg[(P \rightarrow Q) \wedge (Q \rightarrow P)] \\
&\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)]
\end{aligned}
$$

$$
\begin{aligned}
\text{LHS} &\equiv \neg(P \leftrightarrow Q) \\
&\equiv \neg[(P \rightarrow Q) \wedge (Q \rightarrow P)] \\
&\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)] \\
&\equiv \neg[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)]
\end{aligned}
$$

$$
\begin{aligned}
\text{LHS} &\equiv \neg(P \leftrightarrow Q) \\
&\equiv \neg[(P \to Q) \wedge (Q \to P)] \\
&\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)] \\
&\equiv \neg[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)] \\
&\equiv \neg[(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P)]
\end{aligned}
$$

$$\text{LHS} \equiv \neg(P \leftrightarrow Q)$$
$$\equiv \neg[(P \to Q) \land (Q \to P)]$$
$$\equiv \neg[(\neg P \lor Q) \land (\neg Q \lor P)]$$
$$\equiv \neg[((\neg P \lor Q) \land \neg Q) \lor ((\neg P \lor Q) \land P)]$$
$$\equiv \neg[(\neg P \land \neg Q) \lor (Q \land \neg Q) \lor (\neg P \land P) \lor (Q \land P)]$$
$$\equiv \neg[(\neg P \land \neg Q) \lor F \lor F \lor (P \land Q)]$$

$$\text{LHS} \equiv \neg(P \leftrightarrow Q)$$
$$\equiv \neg[(P \to Q) \wedge (Q \to P)]$$
$$\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)]$$
$$\equiv \neg[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)]$$
$$\equiv \neg[(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P)]$$
$$\equiv \neg[(\neg P \wedge \neg Q) \vee F \vee F \vee (P \wedge Q)]$$
$$\equiv \neg[(\neg P \wedge \neg Q) \vee (P \wedge Q)] \qquad [By\ P \vee F \equiv P]$$

$$
\begin{aligned}
\text{LHS} &\equiv \neg(P \leftrightarrow Q) \\
&\equiv \neg[(P \to Q) \wedge (Q \to P)] \\
&\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)] \\
&\equiv \neg[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)] \\
&\equiv \neg[(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P)] \\
&\equiv \neg[(\neg P \wedge \neg Q) \vee F \vee F \vee (P \wedge Q)] \\
&\equiv \neg[(\neg P \wedge \neg Q) \vee (P \wedge Q)] && [By\ P \vee F \equiv P] \\
&\equiv \neg[\neg(P \vee Q) \vee (P \wedge Q)]
\end{aligned}
$$

$$\text{LHS} \equiv \neg(P \leftrightarrow Q)$$
$$\equiv \neg[(P \to Q) \wedge (Q \to P)]$$
$$\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)]$$
$$\equiv \neg[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)]$$
$$\equiv \neg[(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P)]$$
$$\equiv \neg[(\neg P \wedge \neg Q) \vee F \vee F \vee (P \wedge Q)]$$
$$\equiv \neg[(\neg P \wedge \neg Q) \vee (P \wedge Q)] \qquad [By\ P \vee F \equiv P]$$
$$\equiv \neg[\neg(P \vee Q) \vee (P \wedge Q)]$$
$$\equiv (P \vee Q) \wedge \neg(P \wedge Q) \equiv RHS$$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

$\equiv [(F) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (F)]$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

$\equiv [(F) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (F)]$

$\equiv [(Q \wedge \neg P)] \vee [(P \wedge \neg Q)]$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

$\equiv [(F) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (F)]$

$\equiv [(Q \wedge \neg P)] \vee [(P \wedge \neg Q)]$

$\equiv (Q \wedge \neg P) \vee (P \wedge \neg Q)$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

$\equiv [(F) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (F)]$

$\equiv [(Q \wedge \neg P)] \vee [(P \wedge \neg Q)]$

$\equiv (Q \wedge \neg P) \vee (P \wedge \neg Q)$

$\equiv (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Next we prove B $\equiv$ C

i.e., to prove $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

LHS $\equiv (P \vee Q) \wedge \neg(P \wedge Q)$

$\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$

$\equiv [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\equiv [(P \wedge \neg P) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

$\equiv [(F) \vee (Q \wedge \neg P)] \vee [(P \wedge \neg Q) \vee (F)]$

$\equiv [(Q \wedge \neg P)] \vee [(P \wedge \neg Q)]$

$\equiv (Q \wedge \neg P) \vee (P \wedge \neg Q)$

$\equiv (\neg P \wedge Q) \vee (P \wedge \neg Q)$

$\equiv (P \wedge \neg Q) \vee (\neg P \wedge Q) \equiv RHS$

Therefore A $\equiv$ B $\equiv$ C

Hence

$\neg(P \leftrightarrow Q) \equiv (P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$

1. Without using truth tables, prove the following

   (i) $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

   (ii) $P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$

   (iii) $\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$

   (iv) $(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$

   (v) $(P \wedge (P \leftrightarrow Q)) \to Q \equiv T$

Solution : (i) To prove : $\neg P \rightarrow (Q \rightarrow R) \equiv Q \rightarrow (P \vee R)$

LHS $\equiv \neg P \rightarrow (Q \rightarrow R)$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg Q \vee R)$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg Q \vee R)$

$\equiv (P \vee \neg Q) \vee R$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg Q \vee R)$

$\equiv (P \vee \neg Q) \vee R$

$\equiv \neg Q \vee P \vee R$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg Q \vee R)$

$\equiv (P \vee \neg Q) \vee R$

$\equiv \neg Q \vee P \vee R$

$\equiv \neg Q \vee (P \vee R)$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg Q \vee R)$

$\equiv (P \vee \neg Q) \vee R$

$\equiv \neg Q \vee P \vee R$

$\equiv \neg Q \vee (P \vee R)$

$\equiv Q \to (P \vee R)$

Solution : (i) To prove : $\neg P \to (Q \to R) \equiv Q \to (P \vee R)$

LHS $\equiv \neg P \to (Q \to R)$

$\equiv P \vee (Q \to R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv P \vee (\neg Q \vee R)$

$\equiv (P \vee \neg Q) \vee R$

$\equiv \neg Q \vee P \vee R$

$\equiv \neg Q \vee (P \vee R)$

$\equiv Q \to (P \vee R)$

$\equiv RHS$

Solution : (ii) To prove :

$$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$$

Solution : (ii) To prove :

$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$

LHS $\equiv P \to (Q \to R)$

Solution : (ii) To prove :

$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$

LHS $\equiv P \to (Q \to R)$

$\equiv P \to (\neg Q \vee R) \qquad\qquad [P \to Q \equiv \neg P \vee Q]$

Solution : (ii) To prove :

$$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$$

LHS $\equiv P \to (Q \to R)$

$\equiv P \to (\neg Q \vee R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee R)$

Solution : (ii) To prove :

$$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$$

LHS $\equiv P \to (Q \to R)$

$\equiv P \to (\neg Q \vee R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee R)$

$\equiv (\neg P \vee \neg Q) \vee R$

Solution : (ii) To prove :

$$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$$

LHS $\equiv P \to (Q \to R)$

$\equiv P \to (\neg Q \vee R)$ $\quad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee R)$

$\equiv (\neg P \vee \neg Q) \vee R$

$\equiv \neg(P \wedge Q) \vee R$

Solution : (ii) To prove :

$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$

LHS $\equiv P \to (Q \to R)$

$\equiv P \to (\neg Q \vee R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee R)$

$\equiv (\neg P \vee \neg Q) \vee R$

$\equiv \neg(P \wedge Q) \vee R$

$\equiv (P \wedge Q) \to R$

Solution : (ii) To prove :

$$P \to (Q \to R) \equiv P \to (\neg Q \vee R) \equiv (P \wedge Q) \to R$$

LHS $\equiv P \to (Q \to R)$

$\equiv P \to (\neg Q \vee R)$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv \neg P \vee (\neg Q \vee R)$

$\equiv (\neg P \vee \neg Q) \vee R$

$\equiv \neg(P \wedge Q) \vee R$

$\equiv (P \wedge Q) \to R$

$\equiv RHS$

Solution : (iii) To prove :

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

Solution : (iii) To prove :

$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$

LHS $\equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$

Solution : (iii) To prove :

$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$

LHS $\equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$

$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$

Solution : (iii) To prove :

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

$$\text{LHS} \equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$$

$$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee \neg(Q \vee \neg Q)] \vee (P \wedge Q)$$

Solution : (iii) To prove :

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

$$\text{LHS} \equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$$

$$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee \neg(Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee (\neg Q \wedge Q)] \vee (P \wedge Q)$$

Solution : (iii) To prove :

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

$$\text{LHS} \equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$$

$$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee \neg(Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee (\neg Q \wedge Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee F] \vee (P \wedge Q)$$

Solution : (iii) To prove :

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

$$\text{LHS} \equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$$

$$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee \neg(Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee (\neg Q \wedge Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee F] \vee (P \wedge Q)$$

$$\equiv P \vee (P \wedge Q)$$

Solution : (iii) To prove :

$$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$$

$$\text{LHS} \equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$$

$$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee \neg(Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee (\neg Q \wedge Q)] \vee (P \wedge Q)$$

$$\equiv [P \vee F] \vee (P \wedge Q)$$

$$\equiv P \vee (P \wedge Q)$$

$$\equiv P \qquad \text{[By Absorption law]}$$

Solution : (iii) To prove :

$\neg((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee (P \wedge Q) \equiv P$

LHS $\equiv \neg[(\neg P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee (P \wedge Q)$

$\equiv \neg[\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$

$\equiv [P \vee \neg(Q \vee \neg Q)] \vee (P \wedge Q)$

$\equiv [P \vee (\neg Q \wedge Q)] \vee (P \wedge Q)$

$\equiv [P \vee F] \vee (P \wedge Q)$

$\equiv P \vee (P \wedge Q)$

$\equiv P$ [By Absorption law]

$\equiv RHS$

Solution : (iv) To prove :

$$(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$$

Solution : (iv) To prove :

$(P \vee Q) \rightarrow R \equiv (P \rightarrow R) \wedge (Q \rightarrow R)$

LHS $\equiv (P \vee Q) \rightarrow R$

Solution : (iv) To prove :

$$(P \vee Q) \rightarrow R \equiv (P \rightarrow R) \wedge (Q \rightarrow R)$$

LHS $\equiv (P \vee Q) \rightarrow R$

$\equiv \neg(P \vee Q) \vee R$ $\quad [P \rightarrow Q \equiv \neg P \vee Q]$

Solution : (iv) To prove :

$(P \vee Q) \rightarrow R \equiv (P \rightarrow R) \wedge (Q \rightarrow R)$

LHS $\equiv (P \vee Q) \rightarrow R$

$\equiv \neg(P \vee Q) \vee R$ $\qquad [P \rightarrow Q \equiv \neg P \vee Q]$

$\equiv (\neg P \wedge \neg Q) \vee R$

Solution : (iv) To prove :

$$(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$$

LHS $\equiv (P \vee Q) \to R$

$\equiv \neg(P \vee Q) \vee R$ $\qquad [P \to Q \equiv \neg P \vee Q]$

$\equiv (\neg P \wedge \neg Q) \vee R$

$\equiv (\neg P \vee R) \wedge (\neg Q \vee R)$

Solution : (iv) To prove :

$$(P \vee Q) \rightarrow R \equiv (P \rightarrow R) \wedge (Q \rightarrow R)$$

LHS $\equiv (P \vee Q) \rightarrow R$

$\equiv \neg(P \vee Q) \vee R$ $\quad [P \rightarrow Q \equiv \neg P \vee Q]$

$\equiv (\neg P \wedge \neg Q) \vee R$

$\equiv (\neg P \vee R) \wedge (\neg Q \vee R)$

$\equiv (P \rightarrow R) \wedge (Q \rightarrow R)$

Solution : (iv) To prove :

$$(P \vee Q) \rightarrow R \equiv (P \rightarrow R) \wedge (Q \rightarrow R)$$

$$\text{LHS} \equiv (P \vee Q) \rightarrow R$$

$$\equiv \neg(P \vee Q) \vee R \qquad [P \rightarrow Q \equiv \neg P \vee Q]$$

$$\equiv (\neg P \wedge \neg Q) \vee R$$

$$\equiv (\neg P \vee R) \wedge (\neg Q \vee R)$$

$$\equiv (P \rightarrow R) \wedge (Q \rightarrow R)$$

$$\equiv RHS$$

Solution : (v) To prove : $(P \wedge (P \leftrightarrow Q)) \rightarrow Q \equiv T$

$$
\begin{aligned}
LHS &\equiv (P \wedge (P \leftrightarrow Q)) \rightarrow Q \\
&\equiv \neg[P \wedge (P \leftrightarrow Q)] \vee Q \qquad [P \rightarrow Q \equiv \neg P \vee Q] \\
&\equiv \neg[P \wedge ((P \rightarrow Q) \wedge (Q \rightarrow P))] \vee Q \\
&\equiv \neg[(P \wedge (\neg P \vee Q)) \wedge (\neg Q \vee P)] \vee Q \\
&\equiv \neg[((P \wedge \neg P) \vee (P \wedge Q)) \wedge (\neg Q \vee P)] \vee Q \\
&\equiv \neg[((F) \vee (P \wedge Q)) \wedge (\neg Q \vee P)] \vee Q \\
&\equiv \neg[(P \wedge Q) \wedge (\neg Q \vee P)] \vee Q \\
&\equiv \neg[((P \wedge Q) \wedge \neg Q) \vee ((P \wedge Q) \wedge P)] \vee Q \\
&\equiv \neg[(P \wedge F) \vee (P \wedge Q)] \vee Q
\end{aligned}
$$

$$\equiv \neg[F \vee (P \wedge Q)] \vee Q$$

$$\equiv \neg[P \wedge Q] \vee Q$$

$$\equiv (\neg P \vee \neg Q) \vee Q$$

$$\equiv \neg P \vee (Q \vee \neg Q)$$

$$\equiv \neg P \vee (T)$$

$$\equiv T$$

$$\equiv RHS$$

1. Prove $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \equiv R$ without using truth tables

Solution : To prove :

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \equiv R$

$$
\begin{aligned}
LHS &\equiv (\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \\
&\equiv [(\neg P \wedge \neg Q) \wedge R] \vee [(Q \wedge R) \vee (P \wedge R)] && \text{[By Associ} \\
&\equiv [\neg(P \vee Q) \wedge R] \vee [(Q \vee P) \wedge R] && \text{[By De Morg} \\
&\equiv [\neg(P \vee Q) \wedge R] \vee [(P \vee Q) \wedge R] && \text{[By commut} \\
&\equiv [\neg(P \vee Q) \vee (P \vee Q)] \wedge R && \text{[By Distribu} \\
&\equiv T \wedge R && \text{[By Complem} \\
&\equiv R && \text{[By Identi}
\end{aligned}
$$

$$\equiv RHS$$

Problem-4 :Prove

$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

is a tautology

Solution : To prove :

$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

is a tautology

i.e., to prove $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \equiv T$

$$
\begin{aligned}
LHS &\equiv ((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg \\
&\equiv [(P \vee Q) \wedge \neg(\neg P \wedge \neg(Q \wedge R))] \vee [\neg(P \vee Q) \vee \neg(P \vee R)] \\
&\equiv [(P \vee Q) \wedge (P \vee (Q \wedge R))] \vee \neg[(P \vee Q) \wedge (P \vee R)] \\
&\equiv [(P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R))] \vee \neg[(P \vee Q) \wedge (P \vee R)]
\end{aligned}
$$

$$\equiv [((P \vee Q) \wedge (P \vee Q)) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$$
$$\equiv [((P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)] \equiv T$$

Hence
$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$
is a tautology

## THEORY OF INFERENCE

Inference theory is concerned with the inferring of a *conclusion* from certain hypotheses or basic assumptions, called *premises*, by applying certain principles of reasoning, called rules of Inference

## Rules of inference

Rule P :

A given premise may be introduced at any step in the derivation

Rule T :

A formula S may be introduced in a derivation, if S is tautologically implied by one or more preceding formulas in the derivation

Rule CP:

If a formula S can be derived from another formula R and a set of premises, then the statement $R \to S$ can be derived from the set of premises alone.

Note : If the conclusion is of the form $R \to S$, we will take R as an additional premise and derive S using the given premises and R

Inconsistent Premises :

A set of premises( or formulas ) $H_1, H_2, ....H_n$ is said to be *inconsistent*, if their conjunction implies a contradiction. ie., if $H_1 \wedge H_2 \wedge .... \wedge H_n \Rightarrow F$

A set of premises is said to be consistent, if it not inconsistent

Indirect method of proof

Let the premises be $H_1, H_2, ....H_n$ and conclusion be C.

In order to show that a conclusion C follows from the premises $H_1, H_2, ....H_n$ by this method, we assume that C is false and include $\neg C$ as an additional premise. If the new set of premises is inconsistent leading to a contradiction, then the assumption that $\neg C$ is true does not hold good. Hence C is true whenever $H_1, H_2, ....H_n$ is true. Thus C follows from $H_1, H_2, ....H_n$

Therefore, in order to prove $H_1 \wedge H_2 \wedge .... \wedge H_n \Rightarrow C$,
it is enough to prove that $H_1 \wedge H_2 \wedge .... \wedge H_n \wedge \neg C \Rightarrow F$

## Rules of Inference

| S.No | Rules | Name |
|---|---|---|
| 1 | $P, Q \Rightarrow P$ | Simplification |
| 2 | $P, Q \Rightarrow Q$ | Simplification |
| 3 | $P, Q \Rightarrow P \wedge Q$ | Conjunction |
| 4 | $P, P \rightarrow Q \Rightarrow Q$ | Modus Ponens |
| 5 | $\neg Q, P \rightarrow Q \Rightarrow \neg P$ | Modus Tollens |
| 6 | $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ | Hypothetical Syllogism |
| 7 | $P \vee Q, \neg P \Rightarrow Q$ | Disjunctive Syllogism |

Note : $P \rightarrow Q \equiv \neg P \vee Q$ and $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

## PROBLEMS

1. Show that the premises $P \rightarrow Q, R \rightarrow \neg Q$ and R lead to the conclusion $\neg P$

   Solution : Premises are $P \rightarrow Q, R \rightarrow \neg Q$ and R

   Conclusion is $\neg P$

| S.No | Statement | Reason |
|---|---|---|
| 1 | R | Rule P |
| 2 | $R \rightarrow \neg Q$ | Rule P |
| 3 | $\neg Q$ | Rule T, Modus Ponens, 1 & 2 |
| 4 | $P \rightarrow Q$ | Rule P |
| 5 | $\neg Q \rightarrow \neg P$ | Rule T, Contra positive, 4 |
| 6 | $\neg P$ | Rule T, Modus Ponens, 3,5 |

2. Using rules of inference, Show that $S \vee R$ is tautologically implied by $P \vee Q, P \to R, Q \to S$

Solution : Premises are $P \vee Q, P \to R, Q \to S$

Conclusion is $S \vee R$

| **S.No** | **Statement** | **Reason** |
|---|---|---|
| 1 | $P \vee Q$ | Rule P |
| 2 | $\neg P \to Q$ | Rule T, 1 |
| 3 | $Q \to S$ | Rule P |
| 4 | $\neg P \to S$ | Rule T, HS, 2,3 |
| 5 | $\neg S \to P$ | Rule T, Contra positive, 4 |
| 6 | $P \to R$ | Rule P |

| | | |
|---|---|---|
| 7 | $\neg S \to R$ | Rule T, HS,5,6 |
| 8 | $S \vee R$ | Rule T, 7 |

3. Show that $T \wedge S$ can be derived from the premises $P \to Q, Q \to \neg R, R, P \vee (T \wedge S)$

Solution: Premises are $P \to Q, Q \to \neg R, R, P \vee (T \wedge S)$

Conclusion is $T \wedge S$

| **S.No** | **Statement** | **Reason** |
|---|---|---|
| 1 | $P \to Q$ | Rule P |
| 2 | $Q \to \neg R$ | Rule p |
| 3 | $P \to \neg R$ | Rule T, HS,1,2 |
| 4 | $R \to \neg P$ | Rule T, Contra positive, 3 |
| 5 | $R$ | Rule P |
| 6 | $\neg P$ | Rule T, MP, 4,5 |

| | | |
|---|---|---|
| 7 | $P \vee (T \wedge S)$ | Rule P |
| 8 | $\neg P \to (T \wedge S)$ | Rule T, 7 |
| 9 | $T \wedge S$ | Rule T, MP,6,8 |

4. Show that $(P \to Q) \wedge (R \to S)$, $(Q \to T) \wedge (S \to U)$, $\neg(T \wedge U)$ and $P \to R \Rightarrow \neg P$

Solution: Premises are $(P \to Q) \wedge (R \to S)$, $(Q \to T) \wedge (S \to U)$, $\neg(T \wedge U)$ and $P \to R$

Conclusion is $\neg P$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $(P \to Q) \wedge (R \to S)$ | Rule P |
| 2 | $P \to Q$ | Rule T, 1 |
| 3 | $R \to S$ | Rule T, 1 |
| 4 | $(Q \to T) \wedge (S \to U)$ | Rule P |
| 5 | $Q \to T$ | Rule T, 4 |

| | | |
|---|---|---|
| 6 | $S \to U$ | Rule T, 4 |
| 7 | $P \to T$ | Rule T, HS, 2, 5 |
| 8 | $R \to U$ | Rule T, HS, 3, 6 |
| 9 | $P \to R$ | Rule P |
| 10 | $P \to U$ | Rule T, HS, 8, 9 |
| 11 | $\neg(T \wedge U)$ | Rule P |
| 12 | $\neg T \to \neg P$ | Rule T, Contra positive, 7 |
| 13 | $\neg U \to \neg P$ | Rule T, Contra positive, 10 |
| 14 | $(\neg T \vee \neg U) \to \neg P$ | Rule T, 11, 12 |
| 15 | $\neg(T \wedge U) \to \neg P$ | Rule T, 14 |
| 16 | $\neg P$ | Rule T, MP, 11, 14 |

5. Show that $(A \to B) \wedge (A \to C), \neg(B \wedge C), (D \vee A) \Rightarrow D$

Solution: Premises are $(A \to B) \wedge (A \to C), \neg(B \wedge C)$, $(D \vee A)$

Conclusion is $D$

| **S.No** | **Statement** | **Reason** |
| --- | --- | --- |

| | | |
|---|---|---|
| 1 | $(A \to B) \wedge (A \to C)$ | Rule P |
| 2 | $A \to B$ | Rule T, 1 |
| 3 | $A \to C$ | Rule T, 1 |
| 4 | $A \to (B \wedge C)$ | Rule T, 2, 3 |
| 5 | $\neg(B \wedge C) \to \neg A$ | Rule T, Contra Positive, 4 |
| 6 | $\neg(B \wedge C)$ | Rule P |
| 7 | $\neg A$ | Rule T, MP,5, 6 |
| 8 | $D \vee A$ | Rule P |
| 9 | $\neg D \to A$ | Rule T, 8 |
| 10 | $\neg A \to D$ | Rule T, Contra Positive, 9 |
| 11 | $D$ | Rule T, MP, 7, 10 |

6. Prove that the premises $A \to (B \to C), D \to (B \wedge \neg C)$ and $(A \wedge D)$ are inconsistent

Solution: Premises are $A \to (B \to C), D \to (B \wedge \neg C)$ and $(A \wedge D)$

Conclusion is $F$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $A \wedge D$ | Rule P |
| 2 | $A$ | Rule T, 1 |
| 3 | $D$ | Rule T, 1 |
| 4 | $A \to (B \to C)$ | Rule P |
| 5 | $B \to C$ | Rule T, MP, 2, 4 |

| | | |
|---|---|---|
| 6 | $\neg B \vee C$ | Rule T,5 |
| 7 | $D \rightarrow (B \wedge \neg C)$ | Rule P |
| 8 | $\neg(B \wedge \neg C) \rightarrow \neg D$ | Rule T, Contra Positive, 7 |
| 9 | $(\neg B \vee C) \rightarrow \neg D$ | Rule T, 8 |
| 10 | $\neg D$ | Rule T, MP, 6, 9 |
| 11 | $D \wedge \neg D$ | Rule T, 3, 10 |
| 12 | $F$ | Rule T, 11 |

7. Show that $P \to Q, P \to R, Q \to \neg R$ and P are inconsistent

Solution: Premises are $P \to Q, P \to R, Q \to \neg R$ and P

Conclusion is $F$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $P \to Q$ | Rule P |
| 2 | $Q \to \neg R$ | Rule P |
| 3 | $P \to \neg R$ | Rule T, HS, 1,2 |
| 4 | $P$ | Rule P |
| 5 | $\neg R$ | Rule T, MP, 3, 4 |
| 6 | $P \to R$ | Rule P |

| | | |
|---|---|---|
| 7 | $\neg R \rightarrow \neg P$ | Rule T, Contra Positive, 6 |
| 8 | $\neg P$ | Rule T, MP, 5, 7 |
| 9 | $P \wedge \neg P$ | Rule T, 4, 8 |
| 10 | $F$ | Rule T, 9 |

8. Show that B can be derived from the premises $A \to B$, $C \to B$, $D \to (A \vee C)$ and $D$ by the indirect method

<u>Solution:</u> By indirect method, we consider negation of the conclusion as an additional premise

Premises are $A \to B$, $C \to B$, $D \to (A \vee C)$, $D$ and $\neg B$

Conclusion is $F$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $A \to B$ | Rule P |
| 2 | $C \to B$ | Rule P |
| 3 | $(A \to B) \wedge (C \to B)$ | Rule T, 1, 2 |
| 4 | $(A \vee C) \to B$ | Rule T, 3 |
| 5 | $D \to (A \vee C)$ | Rule P |
| 6 | $D \to B$ | Rule T, HS, 4, 5 |
| 7 | $D$ | Rule P |
| 8 | $B$ | Rule T, MP, 6, 7 |
| 9 | $\neg B$ | Rule P |
| 10 | $B \wedge \neg B$ | Rule T, 8, 9 |
| 11 | $F$ | Rule T, 10 |

9. Derive $P \to (Q \to S)$ using the CP-rule, from the premises $P \to (Q \to R)$ and $Q \to (R \to S)$

   Solution: By CP-rule, we consider $P$ as an additional premise and the conclusion as $Q \to S$

   Premises are $P, P \to (Q \to R)$ and $Q \to (R \to S)$

   Conclusion is $Q \to S$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $P$ | Rule P |
| 2 | $P \to (Q \to R)$ | Rule P |
| 3 | $Q \to R$ | Rule T, MP, 1, 2 |
| 4 | $Q \to (R \to S)$ | Rule P |
| 5 | $\neg Q \vee (\neg R \vee S)$ | Rule T, 4 |
| 6 | $\neg R \vee (\neg Q \vee S)$ | Rule T, 5 |
| 7 | $R \to (Q \to S)$ | Rule T, 6 |
| 8 | $Q \to (Q \to S)$ | Rule , HS,3, 7 |
| 9 | $\neg Q \vee (\neg Q \vee S)$ | Rule T, 8 |
| 10 | $\neg Q \vee S$ | Rule T, 9 |
| 11 | $Q \to S$ | Rule T, 10 |

10. Show that the following premises are inconsistent
If Jack misses many classes through illness, then he fails high school.
If Jack fails high school, then he is uneducated
If Jack reads a lot of books, then he is not uneducated.
Jack misses many classes through illness and reads a lot of books.

Solution: Let P : Jack misses many classes through illness

Q : Jack fails high school

R : Jack reads a lot of books

S : Jack is uneducated

Premises are $P \to Q, Q \to S, R \to \neg S, P \wedge R$

Conclusion is $F$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $P \to Q$ | Rule P |
| 2 | $Q \to S$ | Rule P |
| 3 | $P \to S$ | Rule T, HS,1, 2 |
| 4 | $P \wedge R$ | Rule P |
| 5 | $P$ | Rule T, 4 |
| 6 | $R$ | Rule T, 4 |
| 7 | $S$ | Rule T, MP, 3, 5 |
| 8 | $R \to \neg S$ | Rule P |
| 9 | $\neg S$ | Rule T, MP, 6, 8 |
| 10 | $S \wedge \neg S$ | Rule T, 7, 9 |
| 11 | $F$ | Rule T, 10 |

11. Test the validity of the following argument. If I study, I will not fail in Mathematics. If I do not play Basket ball, then I will study. But I failed in Mathematics. Therefore I must have played Basket ball.

Solution : Let P : I will study

Q : I will not fail in Mathematics

R : I do not play Basket ball

Premises are $P \to Q, R \to P, \neg Q$

Conclusion is $\neg R$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $R \rightarrow P$ | Rule P |
| 2 | $P \rightarrow Q$ | Rule P |
| 3 | $R \rightarrow Q$ | Rule T, HS, 1, 2 |
| 4 | $\neg Q \rightarrow \neg R$ | Rule T, Contra positive, 3 |
| 5 | $\neg Q$ | Rule P |
| 6 | $\neg R$ | Rule T, MP, 4, 5 |

12. Construct an argument using rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the life saving demonstration will go on", "If the sailing race is held, then the trophy will be awarded", and "The trophy was not awarded" imply the conclusion "It rained"

Solution: Let P : It rains

Q : It is foggy

R : The Sailing race will he held

S : The lifesaving demonstration will go on

T : The trophy will be awarded

Premises are $(\neg P \vee \neg Q) \rightarrow (R \wedge S)$, $R \rightarrow T$ and $\neg T$

Conclusion is $P$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $(\neg P \vee \neg Q) \rightarrow (R \wedge S)$ | Rule P |
| 2 | $(\neg P \rightarrow (R \wedge S)) \wedge (\neg Q \rightarrow (R \wedge S))$ | Rule T, 1 |
| 3 | $(\neg P \rightarrow (R \wedge S))$ | Rule T, 2 |
| 4 | $P \vee (R \wedge S)$ | Rule T, 3 |
| 5 | $(P \vee R) \wedge (P \vee S)$ | Rule T, 4, Distributive |
| 6 | $(P \vee R)$ | Rule T, 5 |
| 7 | $\neg P \rightarrow R$ | Rule T, 6 |
| 8 | $\neg R \rightarrow P$ | Rule T, Contra positive, 7 |
| 9 | $R \rightarrow T$ | Rule P |
| 10 | $\neg T \rightarrow \neg R$ | Rule T, Contra positive, 9 |
| 11 | $\neg T$ | Rule P |
| 12 | $\neg R$ | Rule T, MP,10, 11 |
| 13 | $P$ | Rule T, MP,8, 12 |

13. Show that the hypotheses "It is not sunny this afternoon and it is colder than yesterday", "we will go swimming only if it is sunny", "if we do not go swimming, then we will take a canoe trip, and If we take a canoe trip, then we will be home by sunset leads to the conclusion that we will be home by sunset. <u>Solution :</u> Let P : It is sunny this afternoon

Q : It is colder than yesterday

R : We will go swimming

S : We will take a canoe trip

T : We will be home by sunset

Premises are $\neg P \land Q, R \rightarrow P, \neg R \rightarrow S, S \rightarrow T$

conclusion is $T$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $\neg P \wedge Q$ | Rule P |
| 2 | $\neg P$ | Rule T, 1 |
| 3 | $R \rightarrow P$ | Rule P |
| 4 | $\neg P \rightarrow \neg R$ | Rule T, Contra positive, 3 |
| 5 | $\neg R$ | Rule T, MP, 2, 4 |
| 6 | $\neg R \rightarrow S$ | Rule P |
| 7 | $S$ | Rule T, MP, 5, 6 |
| 8 | $S \rightarrow T$ | Rule P |
| 9 | $T$ | Rule T, MP, 7, 8 |

14. Show that the hypotheses ″If you send me an e-mail message, then I will finish writing the program″, ″If you do not send me e-mail message, then I will go to sleep early″, and ″If I go to sleep early then I will wake up feeling refreshed″ leads to the conclusion that, ″If I do not finish writing the program, then I will wake up feeling refreshed″

Solution : Let P : You send me an e-mail message

Q : I will finish writing the program

R : I will go to sleep early

S : I will wake up feeling refreshed

Premises are $P \to Q, \neg P \to R, R \to S$

Conclusion is $\neg Q \to S$

| S.No | Statement | Reason |
|---|---|---|
| 1 | $P \rightarrow Q$ | Rule P |
| 2 | $\neg Q \rightarrow \neg P$ | Rule T, Contra positive, 1 |
| 3 | $\neg P \rightarrow R$ | Rule P |
| 4 | $\neg Q \rightarrow R$ | Rule T, HS, 2, 3 |
| 5 | $R \rightarrow S$ | Rule P |
| 6 | $\neg Q \rightarrow S$ | Rule T, HS, 4, 5 |

## INTRODUCTION TO PROOFS

1. Direct proofs
2. Proof by contra-positive (or) Indirect proof
3. Proofs by contradiction

### Procedure for Direct proofs

Step (i) Hypothesis : First we assume that P is true

Step (ii) Analysis : Starting with the hypothesis and employing the rules/laws of logic and other known facts, infer that Q is true

Step (iii) Conclusion : $P \rightarrow Q$ is true.

## PROBLEMS

1. Give a direct proof of the statement "The square of an odd integer is an odd integer"

   <u>Solution:</u> To prove : The square of an odd integer is an odd integer

   i.e., to prove : If n is an odd integer , then $n^2$ is an odd integer

   Let P : n is an odd integer and Q : $n^2$ is an odd integer

   By assuming that "n is an odd integer" we need to prove that "$n^2$ is also an odd integer "

   Since n is an odd integer, we have $n = 2k + 1$ where k is some integer

Now, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
which is an odd integer
Hence the proof.

2. Give the direct proof of "sum of two odd integers is even"

<u>Solution:</u> To prove :"Sum of two odd integers is even"
i.e., to prove : If $n$ is odd and $m$ is odd then $n + m$ is even integer
Let P: $n$ is odd integer and $m$ is odd integer
Q: $n + m$ is even integer
By assuming that $n$ is odd integer and $m$ is odd integer we need to prove that $n + m$ is even integer
Since $n$ and $m$ are odd integer, we have $n = 2k + 1$ and $m = 2u + 1$ where k and u are some integers
Now, $n + m = (2k + 1) + (2u + 1) = 2k + 2u + 2 = 2(k + u + 1)$ which is an even integer

Hence the proof.

## Proof by Contraposition (or) Indirect proof

This method will make use of the fact that the conditional statement $P \to Q$ can be proved by showing that its contrapositive $\neg Q \to \neg P$ is true

## Procedure for Proof by contraposition (or) Indirect proof

Step (i) Hypothesis : Statement $P \to Q$ its contrapositive $\neg Q \to \neg P$ are logically equivalent

So we assume that $\neg Q$ is true

Step (ii ) Analysis : Starting with the hypothesis and employing the rules/laws of logic and other known facts, infer that $\neg P$ is true

Step (iii) Conclusion : $\neg Q \to \neg P$ is true

3. Prove that if n is an integer and 5n+2 is odd, then n is odd by contraposition method.

Solution: Let P: n is an integer and 5n+2 is odd integer

Q: n is odd integer

To prove : $P \to Q$ is true

It is enough to prove $\neg Q \to \neg P$ is true, since

$P \to Q \equiv \neg Q \to \neg P$

Let us assume that $\neg Q$ is true.

i.e., n is an even integer

So, $n = 2k$ where k is some integer

Now, $5n + 2 = 5(2k) + 2 = 10k + 2 = 2(5k + 1)$

which is an even integer

$\Rightarrow \neg P$ is true.

Hence, if $5n + 2$ is odd then n is an odd integer

4. Prove that if n is an integer and $n^3 + 5$ is odd, then n is an even by contraposition method

Solution: Let P: n is an integer and $n^3 + 5$ is odd integer

Q: n is an even integer

To prove : $P \rightarrow Q$ is true

It is enough to prove $\neg Q \rightarrow \neg P$ is true, since

$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

Let us assume that $\neg Q$ is true.

i.e., n is an odd integer

So $n = 2k + 1$ where k is some integer

$$\begin{aligned}
\text{Now, } n^3 + 5 &= (2k+1)^3 + 5 \\
&= ((2k)^3 + 3(2k)^2(1) + 3(2k)(1)^2 + (1)^3) + 5 \\
&= (8k^3 + 12k^2 + 6k + 1) + 5 \\
&= 8k^3 + 12k^2 + 6k + 6 \\
&= 2(4k^3 + 6k^2 + 3k + 3) \ \text{ which is an even integ}
\end{aligned}$$

$\Rightarrow \neg P$ is true.

Hence, if $n^3 + 5$ is odd then n is an even integer

## Procedure for Proofs by contradiction

A proof by contradiction establishes by assuming that the hypothesis P is true and that the conclusion Q is false and then, using P and $\neg Q$ as well as other axioms, definitions

The lines of argument in this method of proof of the statement $P \to Q$ are as follows

Step (i) Hypothesis : Statement $P \to Q$ is false. That is, assume that P is true and Q is false

Step (ii ) Analysis : Starting with the hypothesis that Q is false and employing the rules/laws of logic and other known facts, infer that P is false. This is contradicts the assumption that P is true

Step (iii) Conclusion : Because of the contradiction arrived in the analysis, we infer that $P \to Q$ is true

5. Prove that $\sqrt{2}$ is irrational by giving a proof using contradiction

<u>Solution :</u> Assume that $\sqrt{2}$ is not irrational number

$\Rightarrow$ $\sqrt{2}$ is rational number

Let $\sqrt{2} = \frac{a}{b}, b \neq 0$ [a and b have no common factors other than 1]

$\Rightarrow 2 = \frac{a^2}{b^2}, b \neq 0$

$\Rightarrow 2b^2 = a^2$ $\qquad - - - - - - - - - - > (1)$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow a^2$ is an even integer

$\Rightarrow a$ is an even integer

Let $a = 2k$ where k is some integer

$(1) \Rightarrow 2b^2 = a^2 = (2k)^2 = 4k^2$

$\Rightarrow b^2 = 2k^2$

$\Rightarrow b^2$ is an even integer

$\Rightarrow b$ is an even integer

Thus, we have proved that both a and b are even

Therefore, they have 2 as a common factor.

This contradicts our assumption that a and b have no common factor other than 1, $\Rightarrow$ our assumption $\sqrt{2}$ is rational is false

$$\therefore \sqrt{2} \text{ is irrational number}$$