Module - 5

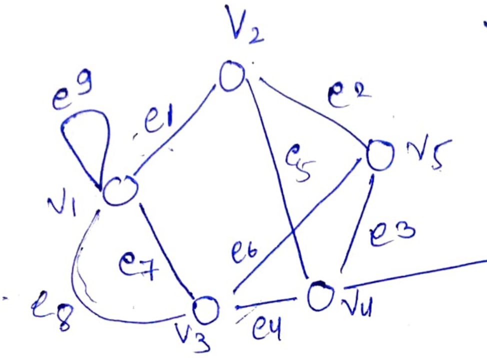
Graph Theory

Graph is an ordered pair ~~an~~

$G(V, E)$

V - set of all vertices or nodes

E - set of all edges (or links)



$V = \{V_1, V_2, V_3, V_4, V_5 \ldots\}$

$E = \{e_1, e_2, \ldots e_7\}$

$\circ V_6 \rightarrow$ isolated vertex.

$V_7 \rightarrow$ pendant vertex

e_5 is incident on $V_2 V_4$ or $V_4 V_2$.

e_7 and e_8 are parallel edges, e^9 - loop edges.

Null graph: Graph w/o edges

Multigraph: Graph with $\parallel^{el}$ edges

Pseudograph: Graph with $\parallel^{el}$ & loop edges.

Graph with atmost one end edge between pair of vertices: Simple graph

Complete graph :- (K_n)

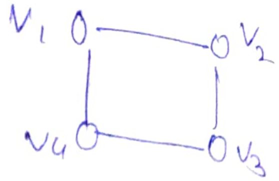
no. of vertices.

$$\boxed{n\left[\frac{n-1}{2}\right]} \rightarrow \text{no. of edges.}$$

K-Regular graph: If all the vertices in graph has k edges incident on it.

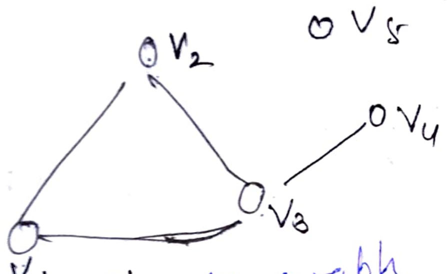
K_n has $(n-1)$ regular graph.

K- regular graph:

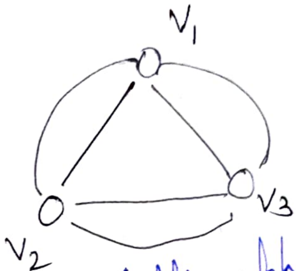


Every

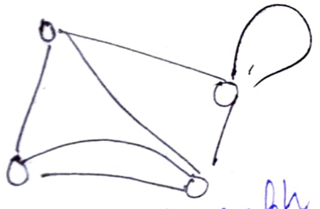
S//



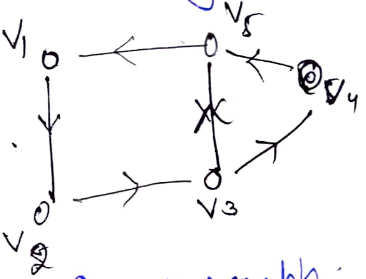
Simple graph



Multigraph.



Pseudograph



Directed graph.

No of edges incident on the vertex is called degree of vertices.

Denoted by $deg(v)$

Loops will contribute 2 to the degree(v) at a vertex

Handshaking Theorems

Sum of degrees of vertices is equal to twice the no. of edges of the graph.

Proof: Since, each edge is ~~contributing~~ incident on exactly two vertices.

Hence, the sum of degrees of vertices is equal to twice the no. of edges of graph.

The no. of vertices of odd degree in a graph is even.

Proof: By previous theorem

$$\sum_{v \in V} \deg(v) = 2 \times (\text{No. of edges}).$$

$V \begin{cases} V_1 \longrightarrow \text{set of all vertices of even degree.} \\ \cup \\ V_2 \longrightarrow \text{set of all vertices of odd degree.} \end{cases}$

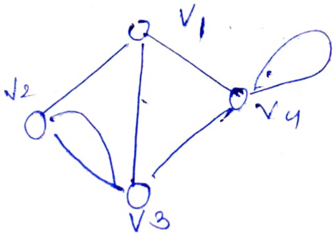
$$\sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) = 2 \times E.$$

$$\text{even} + \sum_{v \in V_2} \deg(v) = 2 \times E$$

$$\sum_{v \in V_2} \deg(v) = 2 \times E - \text{even no.}$$

$$\sum_{v \in V_2} \deg(v) = \cancel{2 \times E}\ \text{even no.}$$

Hence, there are even no. of vertices of odd ~~gre~~ degree.

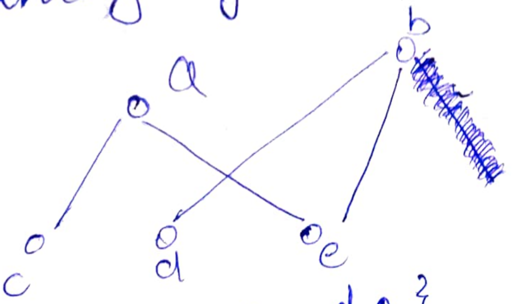


$\left.\begin{array}{l} \deg(v_1) = 3 \\ \deg(v_2) = 3 \\ \deg(v_3) = 4 \\ \deg(v_4) = 4 \end{array}\right\} 14 = 2 \times 7.$

Bipartite Graph

Graph $G(V,E)$ is said to be bipartate, if the vertex set B partition~~ed~~

If there is an edge, then edge is from V_1 to V_2
No 2 vertices of V_i $(i=1,2)$ are connected ~~to~~ through edges.



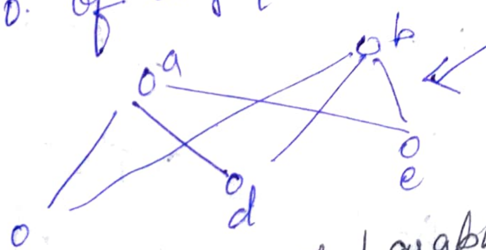
$V = \{a, b, c, d, e\}$
$V_1 = \{a, b\}$, $V_2 = \{c, d, e\}$

Completely bipartate graph.

Every pair of vertices from V_1 to V_2 has to be connected through edges.

$(K_{m,n})$

$K_{m,n}$ has 6 edges.
No. of edge of $K_{m,n} = mn$.



Degree of directed graph

Every vertex in a directed graph has in degree & outdegree

Indegree

$deg^-(v)$

No. of edges coming into the vertex v

Outdegree

$deg^+(v)$

No. of edges going out from vertex v.

Handshaking Theorem for directed graph

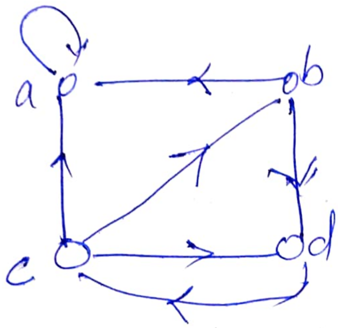
sum of indegree of vertices is equal to sum of outdegree of vertices

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v)$$

Loop at vertex v contribute 1 to indegree and 1 to outdegree.

~~Loop at a v~~

Q:



Q Find indegree and outdegree of following graph. Also ~~pro~~ verify handshaking theorem of graph.

$deg^-(a) = 3$ $\quad$ $deg^+(a) = 1$

$deg^-(b) = 1$ $\quad$ $deg^+(b) = 2$

$deg^-(c) = 1$ $\quad$ $deg^+(c) = 3$

$deg^-(d) = 2$ $\quad$ $deg^+(d) = 1$

$$\sum_{v \in V} deg^-(v) = 7 \qquad \sum_{v \in V} deg^+(v) = 7$$

Isomorphic graphs

Two graphs G_1 and G_2 are isomorphic, if there is one to one correspondence between vertices of G_1 & vertices of G_2, which preserves adjacency of ~~matrix~~ vertices.

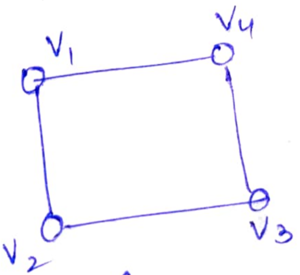
An isomorphic graph should have (i) same no. of vertices, (ii) same no. of edges, (iii) corres. vertices should have same degree.

These above are invariant prop. of isomorphic graph.

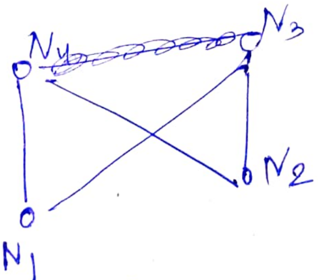
If one of the above condn failed then the graph is not isomorphic.

Example for isomorphic graphs:

(1)



G1



G1

$f: G_1 \to G_2 \Rightarrow f$ is one one and onto

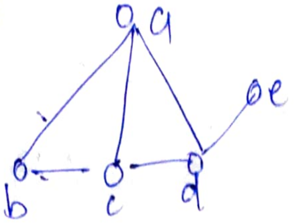
$f(V_1) = N_1$

$f(V_2) = N_3$

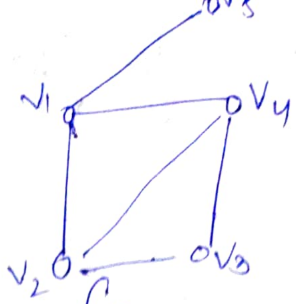
$f(V_3) = N_2$

$f(V_4) = N_4$.

(2)



G_1



G_2

$f: G_1 \to G_2 \ni$

$f(a) = v_2$

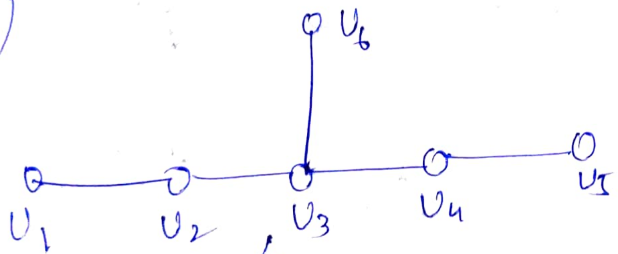
$f(b) = v_3$

$f(c) = v_4$

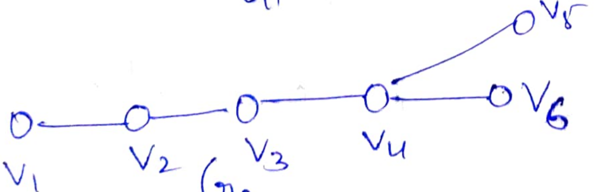
$f(d) = v_1$

$f(e) = v_5$

(3)



G_1



G_2

~~As we don't have any two adjacent~~

There exist no vertex in G_1 which is adjacent to two pendant vertices so that is can be mapped to v_4.

Matrix Representation Of Graph

Adjacency Matrix

Adjacency matrix of a simple graph with n vertices $v_1, v_2 \ldots v_n$ is $n \times n$ square matrix given by

$$A = [a_{ij}] = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Results: (I) For simple graph, there are no loops. Hence, diagonal entries are zero.

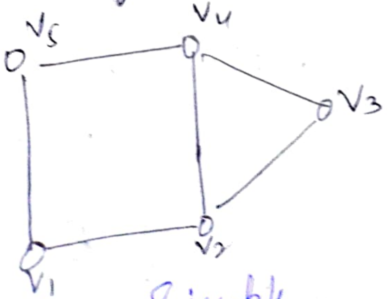
(II) Adjacency matrix is symmetric.

(III) Degree of v_i is equal to no. of 1 in i^{th} row or i^{th} column.

~~(IV)~~ Adjacency Matrix Of Pseudograph with n vertices

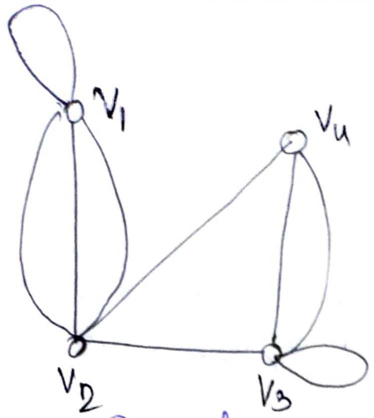
$$A = [a_{ij}] = \begin{cases} 1, & \text{If there is a loop at } v_i v_j,\ i = j \\ m, & \text{If there are m edge between } v_i \& v_j \\ 0, & \text{Otherwise} \end{cases}$$

Q Find adjacency matrix



Simple

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Pseudo graph

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrix

Let G be a graph with n vertices and m edges $v_1, v_2 \ldots v_n$; $e_1, e_2 \ldots e_m$

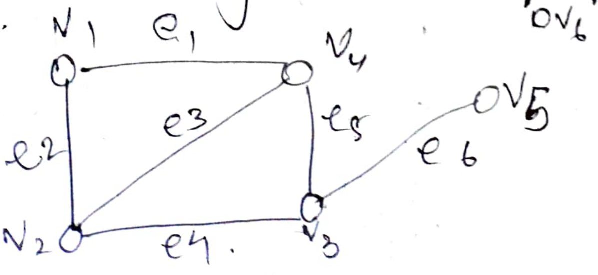
Then incidence matrix is of $n \times m$ order

$$[b_{ij}] = \begin{cases} 1, & \text{if } e_j \text{ is incident on } v_i \\ 0, & \text{otherwise} \end{cases}$$

Remark : * Each column will have exactly two ones

* Row with all vertices are zero (Isolated matrix).
* Row with single one correspond to a single vertex.
* Degree of v_j = no. of ones in j^{th} row

Q Find incidency matrix of following graph.



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Form CQ Theorems.

Theorem - 1.

2 graphs are isomorphic iff their vertices can be labelled in such a way that the corresponding adjacency matrix are equal.

Theorem - 2

2 labelled graphs G_1, G_2 with adjacency matrix A_1 & A_2, G_1 & G_2 are isomorphic iff there exist a permutation matrix P such that.

$$A_1 = PA_2P^T$$

Path, Cycle & Connectedness.

A path of length n from vertex V_0 to V_n is sequence of form $V_0\, P_1 V_1\, P_2\, V_2, \ldots V_{n-1}\, P_n V_n$

where $P_i = V_{i-1} V_i$ & V_0 is initial point and V_n is terminal point

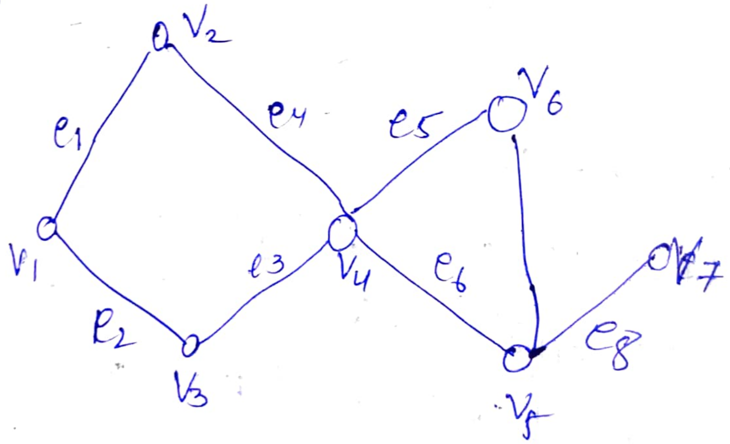
Length : The no. of edges appearing in path is the length.

* Simple Path : If edges of path are distinct then it is called simple path

* Trivial Path : A path of length 0 then there is only 1 vertex i.e each vertex individually is trivial path.

* Circuit or cycle : A non trivial path is ~~locat only~~ called a cycle or circuit if it starts and ends at same vertex.

Eg :



Find path from V_1 to V_4 of length 5

$V_1 \, e_2 \, V_3 \, e_3 \, V_4 \, e_6 \, V_5 \, e_7 \, V_6 \, e_5 \, V_4 \Rightarrow$

Find cycle of length 4

$V_1 \, e_1 \, V_2 \, e_4 \, V_4 \, e_3 \, V_3 \, e_2 \, V_1$

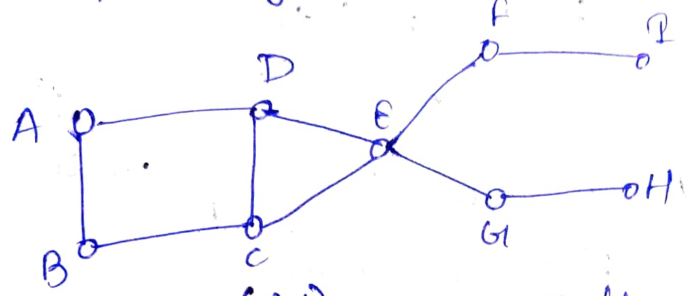
NOTE → Each loop will be called a cycle of length 1.

connectedness In Undirected graph.

A graph G is said to be connected if there exist a path b/w every pair of ~~vert~~ distinct vertices if failed then disconnected

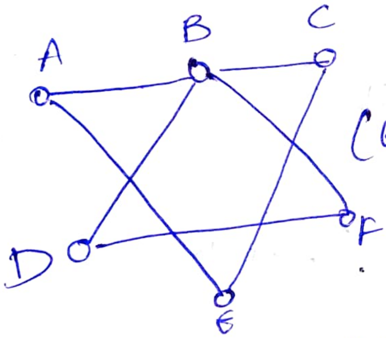
NOTE : Any graph with isolated vertex is disconnected

Null graph is totally disconnected.



(G1)

Both are connected graphs



(G2)

→ If F—G is not in G1, graph is disconnected.

Theorem1: If graph G is either connected or disconnected has exactly 2 vertices of odd degree then there is path joining these two.

PROOF : CASE 1: G is connected

Let V_1 & V_2 are vertices of odd degree.

We know that no. of vertices of odd degree in a graph is even so definitly there exist a path from V_1 to V_2.

CASE 2: G is disconnected.

If G disconnected then we have components v_1 & v_2 lie in same component hence there exist path from v_1 to v_2

Theorem 2: Let G be a graph (disconnected) with n vertices & k components then max no. of edges of G is

$$\frac{(n-k)(n-k+1)}{2}.$$

PROOF: Let n_i denotes no. of vertices in i^{th} component.

$$n_1+n_2+ \cdots +n_k=n$$

$$\sum_{i=1}^{k} n_i = n$$

$$\sum_{i=1}^{k} (n_i-1) = n-k.$$

$$\left[\sum_{i=1}^{k} (n_i-1)\right]^2 = (n-k)^2$$

$$\left[(n_1-1)+(n_2-1)+\cdots (n_k-1)\right]^2 = \sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i\neq j}(n_i-1)(n_j-1) = n^2+k^2-2kn$$

$$\sum_{i=1}^{k}(n_i-1)^2 \leq n^2+k^2-2nk.$$

$$\sum_{i=1}^{k}(n_i^2+1-2n_i) \leq n^2+k^2-2nk$$

$$\sum_{i=1}^{k} n_i^2+k-2n \leq n^2+k^2-2nk.$$

$$\sum_{i=1}^{K} n_i^2 \leq n^2 + K^2 - 2nK - K + 2n.$$

Max no. of edges in ith component

$$= \sum_{i=1}^{K} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{K} n_i^2 - \frac{n}{2}\right]$$

$$\leq \frac{1}{2}\left[n^2 + K^2 - 2nK - K + 2n - \frac{n}{2}\right.$$

$$\leq \frac{1}{2}\left[(n-K)^2 + (n-K)\right]$$

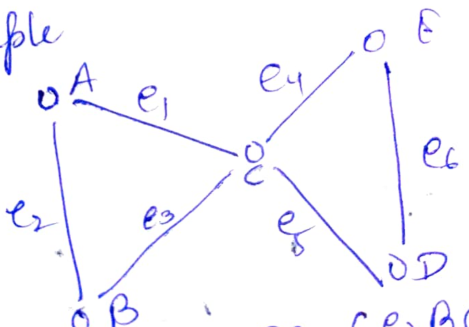
$$\leq \frac{(n-K)(n-K+1)}{2}.$$

Eulerian & Hamiltonian Graph

A path in a simple graph G which includes each edge exactly once is called ~~euler~~ eular path.

A circuit of graph G is said to be an eulerian circuit if it includes each edge ~~in~~ exactly once.

Example



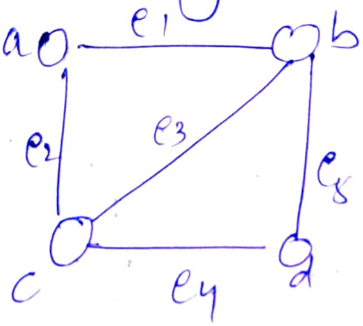
This is euler graph.

$A e_1 C e_5 D e_6 E e_4 C e_3 B e_2 A$.

A path in a graph G is said to be Hamiltonian if it includes all the vertices exactly once

A circuit in a graph G which includes all the vertices exactly once is called Hamiltonian graph.

A graph with Hamiltonian circuit is Hamiltonian Graph.

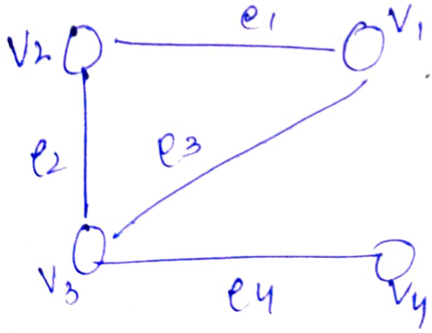


$b e_1 a e_2 c e_3 b e_5 d e_4$ → euler path.

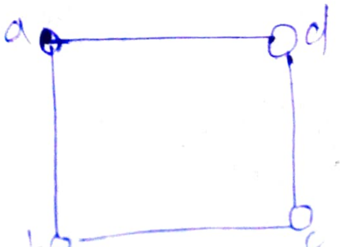
$a e_1 b e_5 d e_4 c e_2 a$ → Hamiltonian circuit

Not euler graph

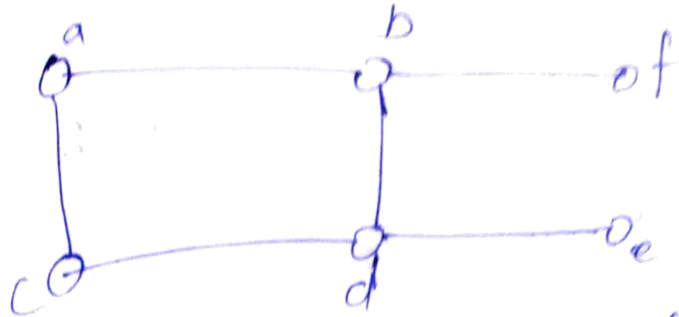
Hamiltonian



Neither eulerian nor hamiltonian



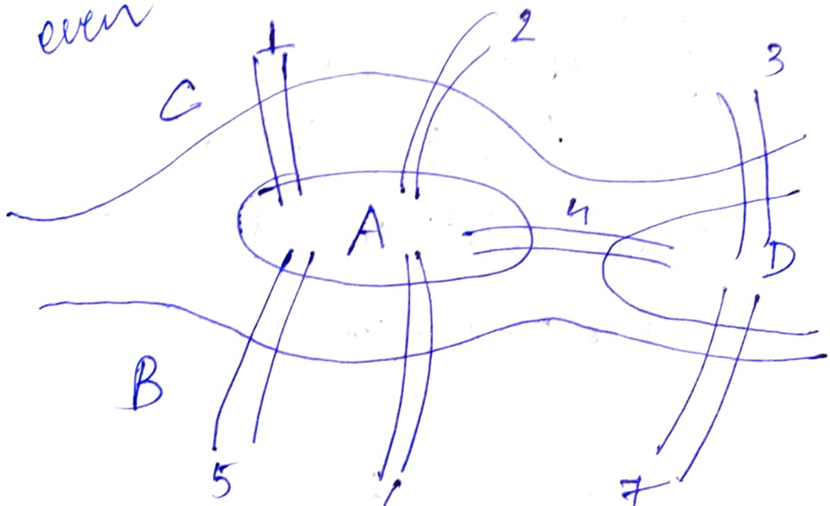
Both eulerian & Hamiltonian.



Neither eulerian nor Hamiltonian

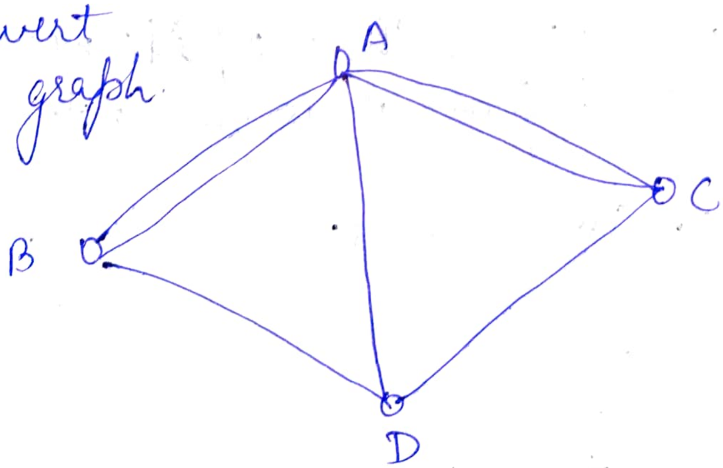
Necessary & Sufficient Condⁿ for Graph to Be Eulerian

Graph G is said to be Eulerian if and only if degree of all the vertices are even



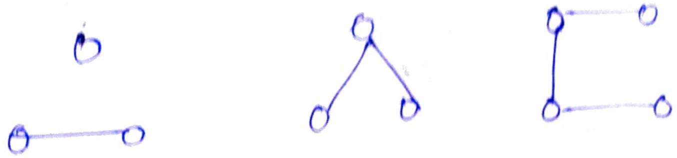
island → ver.
bridges → edges

convert to graph



Trees

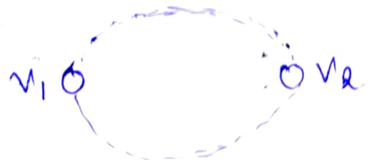
Connected graph with no circuit is called a tree.



Theorem 1: An undirected graph is a tree if and only if there is a unique path between every pair of vertices.

Assume that T is a tree.

Suppose there exist two distinct path b/w vertices v_1 and v_2.



∴ Combining these 2 path we will have circuit which implies T is not a tree.

Conversely, assume that T is not a tree.

To Prove: T is tree

~~So~~

$\Rightarrow$ ~~this is~~ there is a circuit

$\Rightarrow$ there exist two distinct paths b/w v_i & v_j.

...rem 2: Tree with n vertices has $n-1$ edges.

Basic step: $P(1) = n-1$ edges
$= 0$ edges

Assume that it is true k vertices, $k < n$

Any tree with k vertices will have $k-1$ edges.

Let e^k be the only edge connecting v_i & v_j.

Removing e^k graph will ~~have~~ be partitioned into T_1 and T_2. ~~the T_2 will have~~

r vertices $n-r$ no. of vertices

$r < n$, also $n-r < n$.

By assumption there all $r-1$ edges in T_1 and $n-r-1$ edges in T_2.

Total no. of edges for T

$= 1 + r - 1 + n - r - 1$

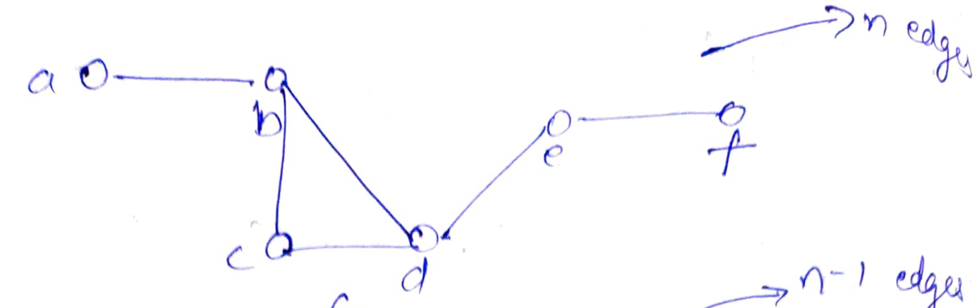
$= n - 1$ no. of edges.

Any connected graph with n

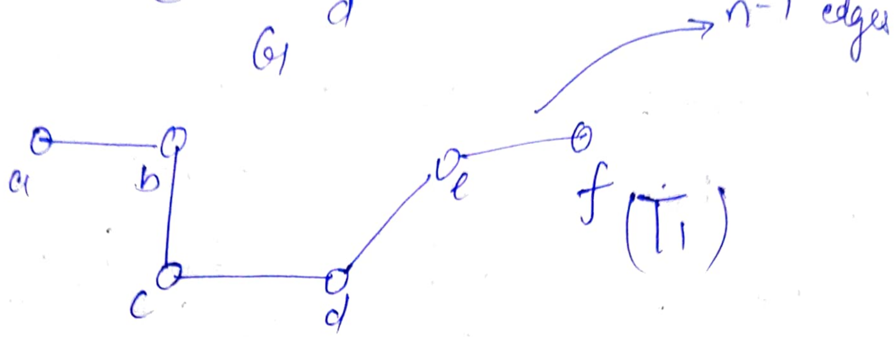
A circuitless graph with n vertices and $n-1$ edges is a tree

Spanning Tree: A subgraph of a graph G containing all vertices is a spanning tree.

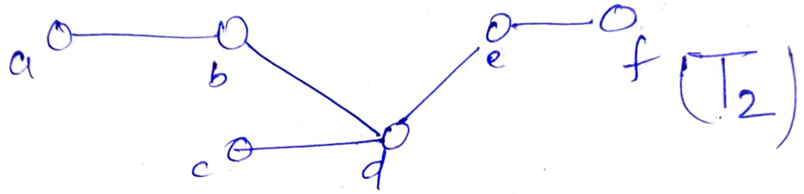
Example:



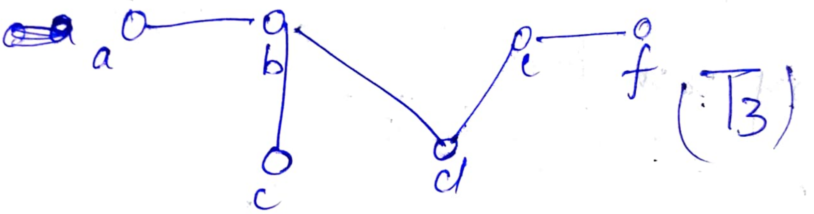
G



(T_1)



(T_2)

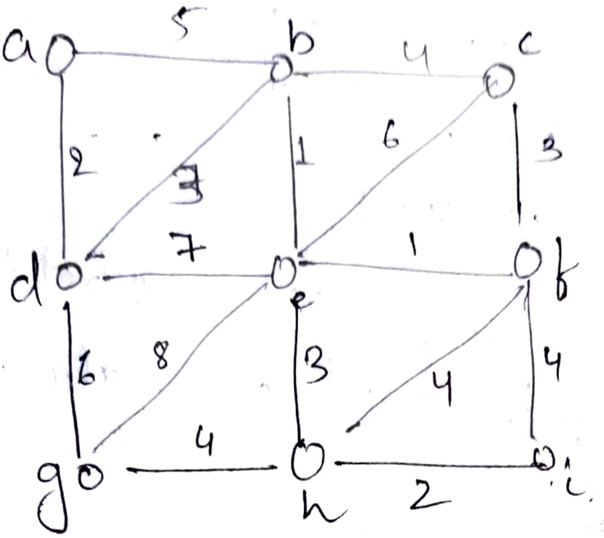


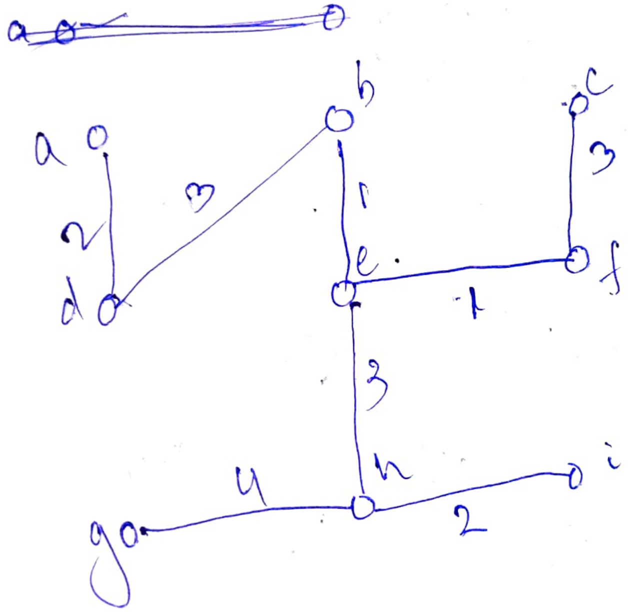
(T_3)

NOTE: 1 If G has n vertices, any spanning tree of G with n vertices has $n-1$ edges.

Minimum Spanning Tree: If G is a connected weighted graph. The spanningtree of G with total weight (sum of weight of edges) is minimum is called minimum spanning tree of G.

Kruskal's Algo. To Calculate Minimum Spanning Tree.





minimum weight = sum of all weights = 19

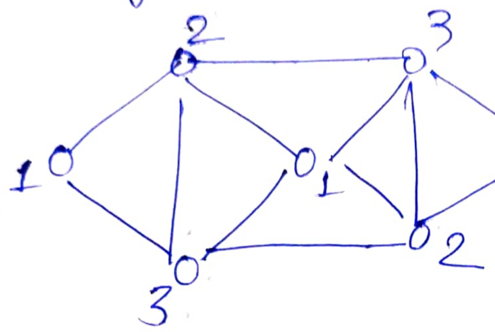
Edges	Weight	Edges	Weight
(b, e)	1	(b, c)	4
(e, f)	1	(g, h)	4
(a, d)	2	(h, f)	4
(h, i)	2	(f, i)	4
(d, b)	3	(a, b)	5
(e, h)	3	(e, c)	6
		(d, g)	6
		(d, e)	7
[illegible]	3	[illegible]	8

Graph Coloring

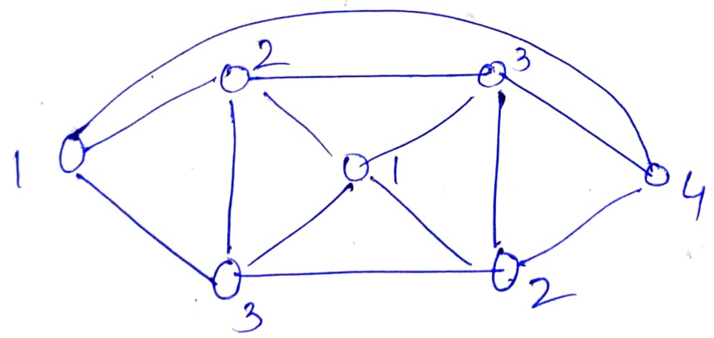
Assignment of colours to the vertices in a graph. So that no two adjacent vertices will have same color.

CHROMATIC NUMBER

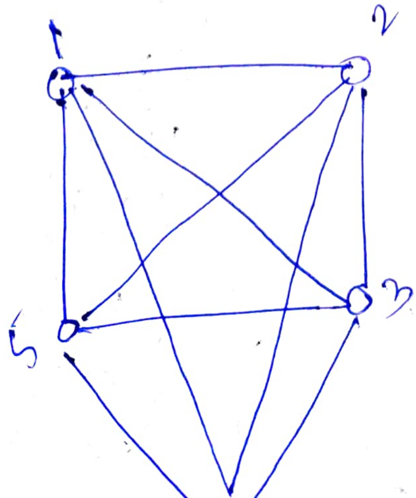
The min. no. of colors required to color the graph is called chromatic number.



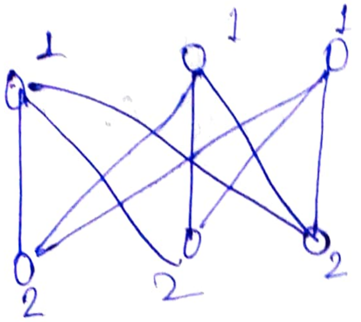
Chromatic no. = 3.



Chromatic No. = 4.

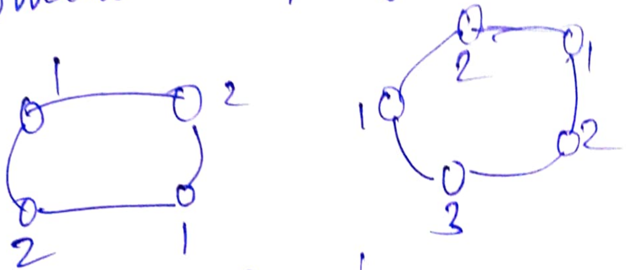


Chromatic No. for K_n is n

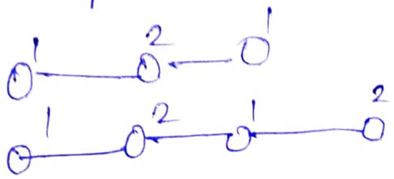


Bipartedgraph
CN = 2 } in general.

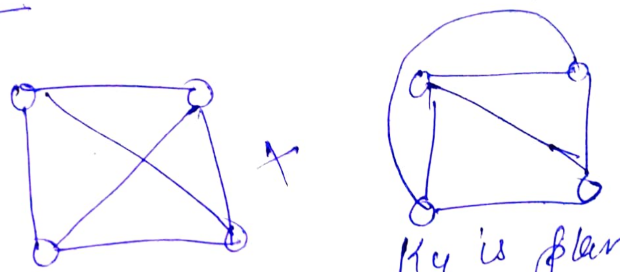
Chromatic no. for cyclic < even - 2, odd - 3.



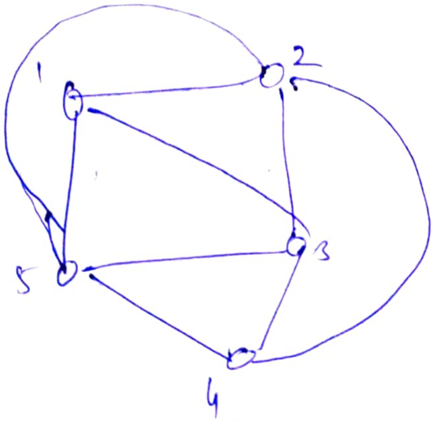
P_n



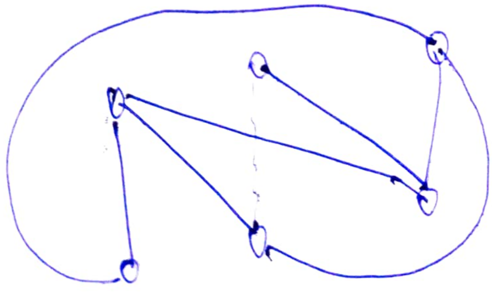
<u>Planar graph :</u>



K_4 is planar



K_5 is non planar.



$K_{3,3}$ is nonplanar.

K_5, 4 and $K_{3,3}$ — are non planar.

4 Color Theorem

Any planar graph whose C_n is not greater than 4.