GRAPH THEORY

Basic Definitions:

A graph $G = (V, E)$ consists of a nonempty set V, called the set of vertices (nodes, points) and a set E of ordered or unordered pairs of elements of V called the set of edges such that there is a mapping from the set E to the set of ordered or unordered pairs of elements of V.

Connect or join: If an edge $e \in E$ is associated with an ordered pair (u, v) or an unordered pair (u, v) where $u, v \in V$ then e is said to connect or join the nodes u and v. The edge e that connects u and v is said to be incident on each of the nodes.

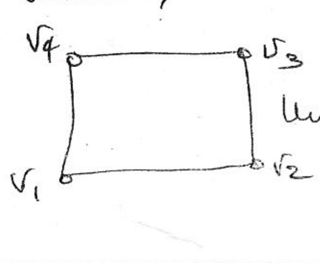
Adjacent nodes: The pair of nodes that are connected by an edge are called adjacent nodes.

Isolated node: A node of a graph which is not adjacent to any other node is called isolated node.

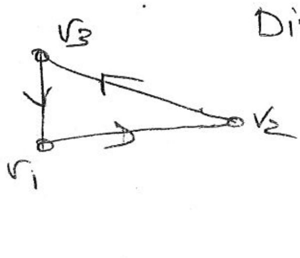
Null graph: A graph containing only isolated nodes is called a null graph.

Directed graph: If in a graph $G = (V, E)$, each edge $e \in E$ is associated with an ordered pair of vertices, then G is called is a digraph.

Undirected graph: If each edge is associated with an unordered pair of vertices, then G is called an undirected graph.



Undirected graph.



Directed graph.

Loop: An edge of a graph that joins a vertex to itself is called a loop.

Parallel edges: If, in a directed or undirected graph, certain pairs of vertices are joined by more than one edge such edges are called parallel edges.

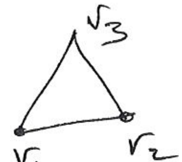
Note: In a directed graph the two possible edges between a pair of vertices which are opposite in direction are considered distinct.

Simple graph: A graph, in which there is only one edge between a pair of vertices, is called a simple graph.

Multigraph: A graph which contains some parallel edges is called a multigraph.

Pseudograph: A graph in which loops and parallel edges are allowed is called a pseudo graph.

Weighted graph: Graphs in which a number is assigned to each edge are called weighted graph.



Simple graph.

Undirected multigraph.

Directed graph with distinct edges ~~Multigraph~~.

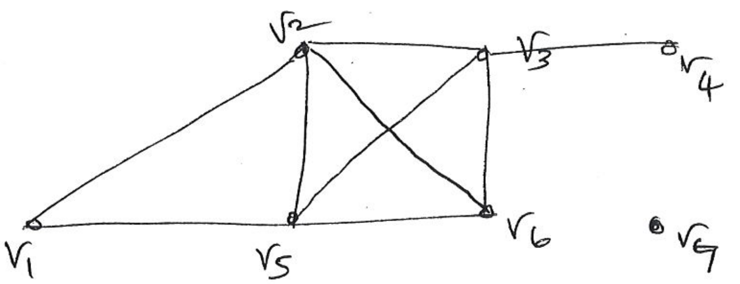
— Directed Multigraph.

Degree of vertex: Degree of a vertex in an undirected graph is the number of edges incident with it, with the exception that a loop at a vertex contributes twice to the degree of that vertex.

<u>Note:</u> Degree of a vertex is denoted by deg(V).

2. Degree of a isolated vertex is 0.

3. Degree of a vertex is 1 then it is called <u>pendant</u> vertex.



$deg(v_1) = 2$
$deg(v_2) = 4$
$deg(v_3) = 4$
$deg(v_5) = 4$

$deg(v_4) = 1$, $deg(v_6) = 3$, $deg(v_7) = 0$

v_4 is a pendant vertex and v_7 is an isolated vertex.

<u>The Handshaking theorem:</u> If $G = (V, E)$ is an undirected graph with e edges then $\sum_i deg(v_i) = 2e$.

The sum of the degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

<u>Proof:</u> Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices

$\therefore$ All the e edges contribute $(2e)$ to the sum of the degrees of the vertices, $\sum_i deg(v_i) = 2e$.

<u>Theorem:</u> The number of vertices of odd degree in an undirected graph is even.

<u>Proof:</u> Let $G = (V, E)$ be the undirected graph.

Let V_1 and V_2 be the sets of vertices of G of even and odd degrees respectively. By previous theorem.

$$2e = \sum_{v_i \in V_1} deg(v_i) + \sum_{v_j \in V_2} deg(v_j) \quad \text{--- (1)}$$

Since each degree (v_i) is even, $\sum_{v_i \in V_i} \deg(v_i)$ is even.

LHS of (1) is even we get

$\sum_{v_j \in V_2} \deg(V_j)$ is even. (diff of two even nos is even)

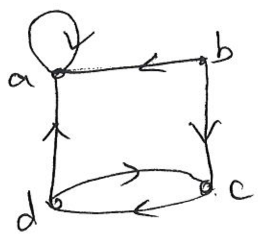
$\therefore$ The number of vertices of odd degree is even.

Indegree: In a directed graph, the number of edges with v as their terminal vertex (ie the no of edges that converge at v) is called in-degree of v and is denoted as $\deg^-(v)$

Out degree: The number of edges that emanate from v is called the out degree of v and denoted as $\deg^+(v)$.

Source: A vertex with zero indegree is called a Source.

Sink: A vertex with zero outdegree is called a sink.



$\deg^-(a) = 3$ $\quad \deg^+(a) = 1$

$\deg^-(b) = 0$ $\quad \deg^+(b) = 2$

$\deg^-(c) = 2$ $\quad \deg^+(c) = 1$

$\deg^-(d) = \underline{1}$ $\quad \deg^+(d) = \underline{2}$

$\qquad\quad 6 \qquad\qquad\qquad 6$

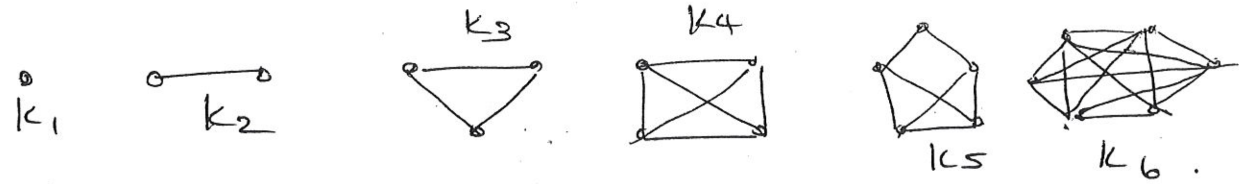
$\sum \deg^-(v) = \sum \deg^+(v) = 6$ = number of edges.

$G = (V, E)$ viz., $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = e$.

because each edge of the graph converges at one vertex and emanates from one vertex and hence contributes 1 each to sum of the indegrees and to the sum of the out degree.

Complete graph: A simple graph in which there is exactly one edge between each pair of distinct vertices, is called a complete graph.

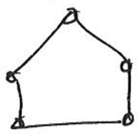
<u>Notation</u>: A complete graph on n vertices is denoted by K_n.



The number of edges in $K_n = {}^nC_2 = \frac{n(n-1)}{2}$.

<u>Regular graph</u>: If every vertex of a simple graph has same degree it is called regular graph.

Example of 2 regular graph.

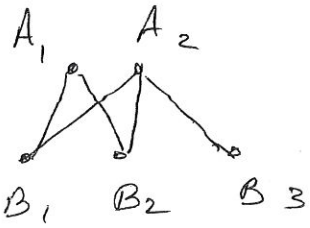


Example of 3 regular graph.

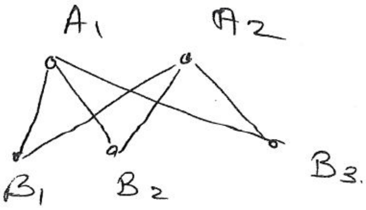
<u>Bipartite graph</u>:

If the vertex set V of a simple graph $G = (V, E)$ can be partitioned into subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2 (so that no edge connects two vertices in V_1 or two vertices in V_2) then G is called a bipartite graph.

<u>Completely bipartite graph</u>: If each vertex of V_1 is connected with every vertex of V_2 by an edge then G is called completely bipartite graph. If V_1 contains m vertices and V_2 contains n vertices then the completely bipartite graph is denoted by $K_{m,n}$.



Bipartite.



$K_{2,3}$ graph.

<u>Subgraph</u>: A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$, and $E' \subseteq E$.

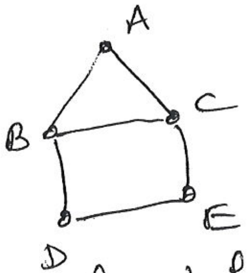
If $V' \subset V$ and $E' \subset E$ then H is called a proper subgraph of G.

If $V' = V$ H is called a <u>spanning subgraph</u> of G.

If we delete a subset U of V and all edges incident on the elements of U from a graph $G = (V, E)$ The subgraph $(G-U)$ is called a <u>vertex deleted</u> <u>subgraph</u> of G.

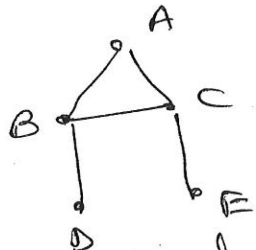
If we delete a subset F of E from a graph $G(V, E)$ then the subgraph $(G-F)$ is called an <u>edge deleted</u> subgraph of G.

A subgraph $H = (V', E')$ of $G = (V, E)$ where $V' \subseteq V$ and E' consists of only those edges that are incident on the elements of V' is called an <u>induced</u> <u>subgraph</u> of G.



Graph G.

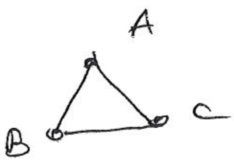
A vertex deleted subgraph of G.



An edge deleted subgraph of G.

$V = A, B, C, D, E$ $V' = A, B, C$

Induced subgraph of G may be



<u>Isomorphic graphs</u>:

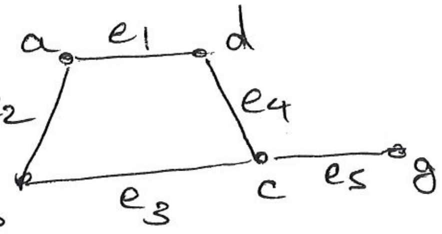
Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists a 1-1 correspondence between the vertex sets which preserves adjacency of vertices.

Invariant property of Isomorphic graphs:

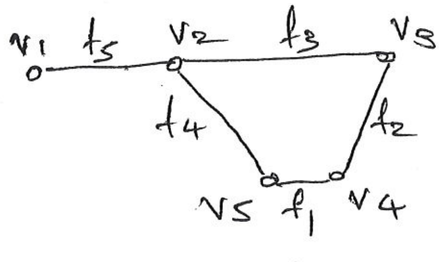
Isomorphic graphs have (i) the same number of vertices (ii) the same number of edges (iii) corresponding vertices with same degree. These three properties are called invariant properties.

Note: If any of these conditions are not satisfied then the graphs are not isomorphic.

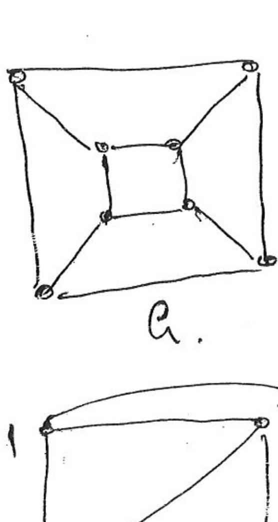
Isomorphic graphs G and H. Consider the graphs G and H given in the figure below. Although the vertices and edges in G and H are labelled differently the second diagram can be obtained from the first by renaming the vertices and drawing the edges differently. We say that these two graphs are isomorphic to each other.



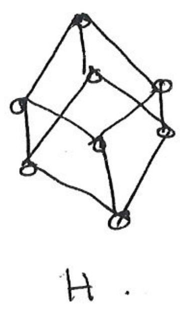
G.



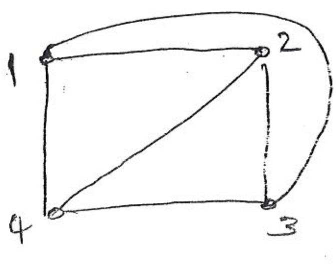
H

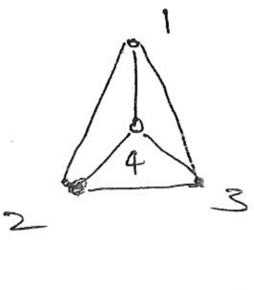


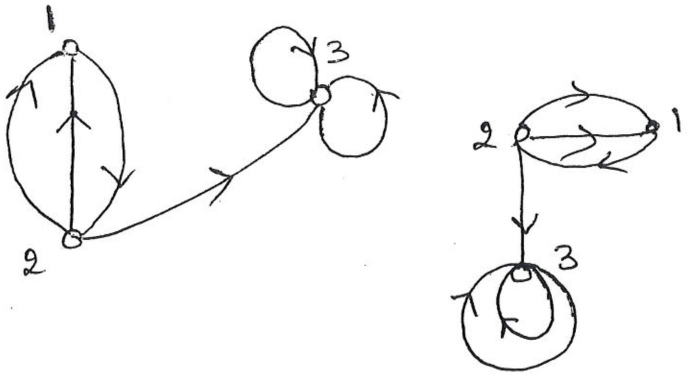
G.



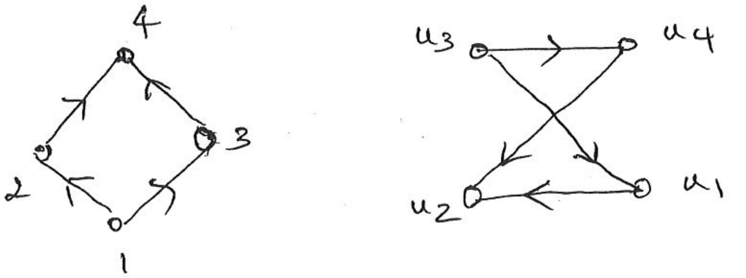
H.







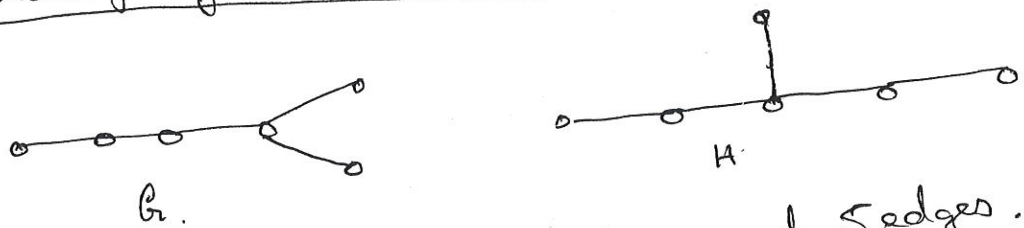
Both these graphs are isomorphic directed graphs.



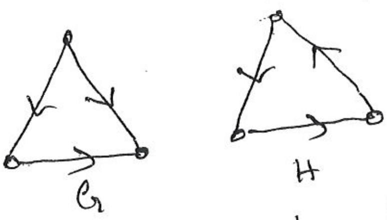
$1 \to u_3$
$2 \to u_1$
$3 \to u_4$
$4 \to u_2$.

$\therefore$ they are isomorphic

Example of non-isomorphic graphs:

1. 

Both G and H have 6 vertices and 5 edges.
3 vertices of degree 1, 2 vertices of degree 2 and one vertex of degree 3.
But in G there is a vertex which is adjacent to two pendant vertices. But in H there is no vertex which is adjacent to two pendant vertices. Hence H and G are not isomorphic.

2. 

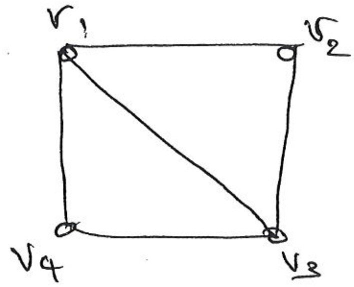
In H every vertex has indegree 1 but in G there is a vertex whose indegree is zero. $\therefore$ not isomorphic.

Matrix Representation of Graphs:

When G is a simple graph with n vertices $v_1, v_2, \ldots v_n$ the matrix $A \equiv [a_{ij}]$ where $a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge in} \\ 0 & \text{otherwise.} \end{cases}$

is called adjacency matrix of G.

<u>Ex:</u>



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

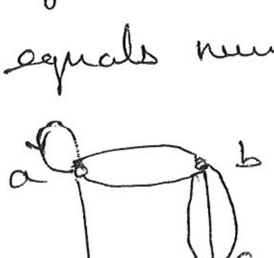
Basic properties of adjacency matrix:

1. Since a simple graph has no loops each diagonal entry of A ie $a_{ii} = 0$.
2. The adjacency matrix of simple graph is symmetric ie $a_{ij} = a_{ji}$ both of these entries are 1 when v_i and v_j are adjacent and both zero otherwise.

<u>Converse:</u> Given any symmetric zero-one matrix A which contains only 0's on its diagonal, there exists a simple graph G whose adjacency matrix is A.

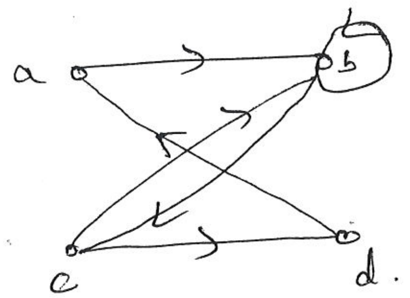
3. $\deg(v_i)$ = number of 1's in the ith row or ith coloumn.

<u>Adjacency matrix of pseudograph</u>: loop at vertex v_i is represented by a 1 in (i,i) position and the (i,j) entry equals number of edges that are incident on v_i and v_j



$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Directed simple graphs or multigraphs can be represented by adjacency matrices



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 0 & 0 \end{array}$$

<u>Definition</u>: <u>Incidence matrix</u>: If $G=(V,E)$ is an undirected graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$ then the $n \times m$ matrix $B=[b_{ij}]$ where $b_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident on } v_i \\ 0 & \text{otherwise} \end{cases}$

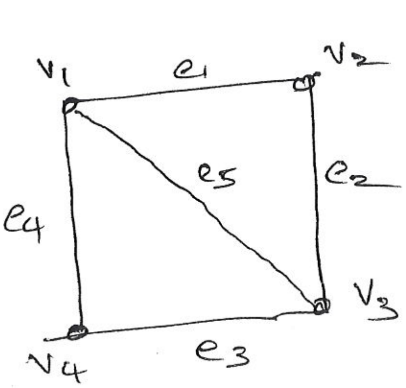
is called incidence matrix of G.

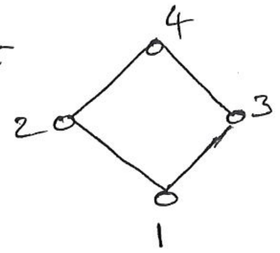
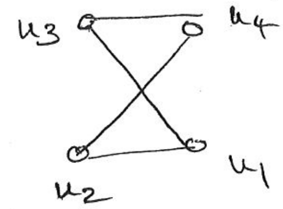
<u>Properties of incidence matrix</u>:

1. Each coloumn of B contains exactly two unit vectors
2. A row with all 0 entries corresponds to an isolated vertex.
3. A row with a single unit entry corresponds to a pendant vertex.
4. $\deg(v_i)$ is equal to number of 1's in the ith row.

<u>Example</u>:

$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline v_1 & 1 & 0 & 0 & 1 & 1 \\ v_2 & 1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 1 & 0 \end{array}$$



Show that  

<u>Sol:</u> Here $1 \to u_1 \quad 2 \to u_2 \quad 3 \to u_3 \quad 4 \to u_4$

$$\therefore \quad A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$P = I_4$.

If all the vertices of an undirected graph are each of odd degree k, show that the number of edges of the graph is multiple of k.

<u>Sol:</u> Since the no of vertices of odd degree in an undirected graph is even let it be $2n$. Let the number of edges be n_e

By Hand Shaking theorem

$$\sum_{i=1}^{2n} \deg(v_i) = 2n_e \quad \text{ie} \quad \sum_{i=1}^{2n} k = 2n_e$$

$$2nk = 2n_e \quad \text{or} \quad kn = n_e$$

$\therefore$ number of edges is a multiple of k.

Verify the hand shaking theorem for the complete graph with n vertices. Verify also the number of odd vertices in this graph is even. Also find the ratio of the number of edges to that of vertices called β index. for this graph.

In a complete graph, every pair of vertices is connected by an edge. From the n vertices of the complete graph K_n choose ${}^nC_2$ pairs of vertices and hence there are ${}^nC_2$ edges in K_n.

Also degree of the n vertices $= n-1$

$$\sum_{i=1}^{n} \deg(v_i) = n(n-1) = 2({}^nC_2)$$

Thus Hand shaking theorem is verified.

If n is even, then degree of each of these n vertice is $(n-1)$ ie odd. The number of odd degree vertices is zero.

$$\text{Beta Index} = \frac{\text{No of edges}}{\text{No of vertices}} = \frac{{}^nC_2}{n} = \frac{1}{2}(n-1).$$

Prove that the number of edges in a bipartite graph with n vertices is at most $\left(\frac{n^2}{2}\right)$.

<u>Proof</u>: Total no of vertices $= n$

Partition V into $V_1 = x$ vertices and $V_2 = n-x$ vertices

the largest number of edges can be obtained when each of $(n-x)$ vertices in V_2 is connected to each of x vertices in V_1

$\therefore$ The largest number of edges $= f(x) = x(n-x)$

To find x for which $f(x)$ is maximum.

$f'(x) = n-2x \qquad f''(x) = -2.$

$f'(x) = 0 \Rightarrow x = n/2 \qquad f''(n/2) < 0$

$f(x)$ is max when $x = n/2$.

Maximum number of edges required $= f(n/2) = \frac{n}{2}\left(n - \frac{n}{2}\right)$

$= n^2/4$

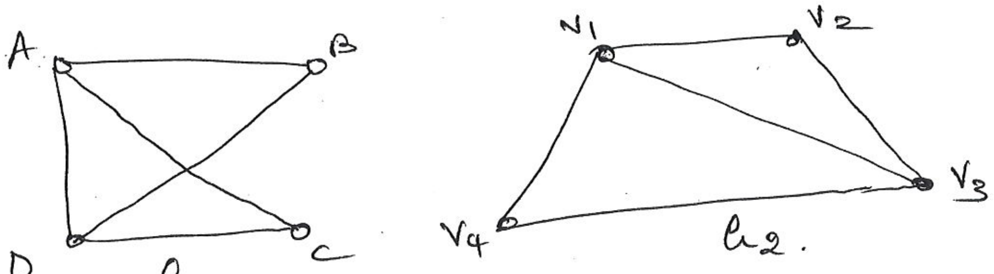
Note: Incidence matrices can also be used to represent pseudo graphs.

<u>Isomorphism and adjacency Matrices</u>:

<u>Theorem 1</u>: Two graphs are isomorphic iff their vertices can be labelled in such a way that the corresponding adjacency matrices are equal.

<u>Theorem 2</u>: Two labeled graphs G_1 and G_2 with adjacency matrices A_1 and A_2 respectively are isomorphic if there exists a permutation matrix P such that $PA_1P^T = A_2$

Establish the isomorphism of the two graphs given in the figure by considering their adjacency matrix



G_1 $\quad$ G_2.

The adjacency matrices A_1 & A_2 of G_1 & G_2 respectively are given below

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The matrices A_1 & A_2 are not same.

To find P such that $PA_1P^T = A_2$.

To find appropriate P use degree of the vertices of G_1 & G_2

$\text{Deg}(A) = 3$ and $\text{Deg}(V_1) = 3$.

$\therefore$ 1st row of I_4 is taken as 1st row of P

$\text{Deg}(D) = 3$ and $\text{Deg}(V_3) = 3$

$\therefore$ 4th vertex of G_1 corresponds to third vertex of G_2

Hence 4^{th} row of I_4 is taken as 3rd row of P

$\text{Deg}(B) = \text{Deg}(C) = 2$ and $\text{Deg}(V_2) = \text{Deg}(V_4) = 2$

ie 2nd vertex of G_1 may be taken to correspond to 2nd or 4th vertex of G_2. Accordingly the 3rd vertex of G_1 may be taken to correspond to 4th or 2nd vertex of G_2.

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or}$$

Degrees.

$A = V_1$

$D = V_3$

$B = V_2$

$C = V_4$

1st of I_4 is 1st row P

4th of I_4 is 3rd row P

2nd of I_4 is 2nd row P

3rd of I_4 is 4th row P.

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now $PA_1P^T = A_2$ $\therefore$ the two graphs are isomorphic.

Path Cycles and Connectivity:

Path: A path of length n from vertex v_0 to v_n is a sequence of the form $v_0, e_1, v_1, e_2, ---, v_{n-1}, e_n, v_n$ where $e_i = v_{i-1} v_i$ $i = 1$ to n. The vertices v_0 and v_n are called end points of the path. v_0 is the initial point. v_n is the terminal point.

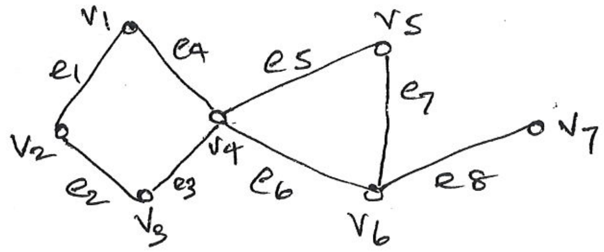
Length: The number of edges appearing in the path is its length.

Simple path: If the edges of the path are distinct then it is called simple path.

Trivial path: A path of length zero ie it contains only one vertex

Circuit or cycle: A nontrivial path is called a cycle or circuit if it starts and ends with same vertex.

Example.



A path from v_1 to v_5 is $P_1: v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_5 v_5$

It is a simple path of length 4 (no of edges)

A cycle is $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$. It is a simple circuit of length 4.

Note: A loop is not a path. It is cycle of unit length

If the length of cycle is k it is called k-cycle

A non loop edge together with its end vertices is a path

A triangle is 3 cycle.

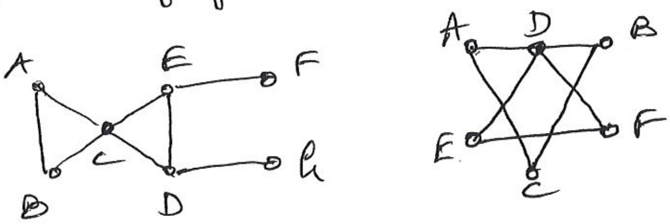
Connectedness in Undirected Graphs:

A graph is connected if there is a path between every pair of distinct vertices of the graph.

A graph that is not connected is called disconnected.

Note: 1. Any graph with isolated vertices is a disconnected graph.

2. Null graph is totally disconnected.



Both are connected graphs.

Theorem: If a graph G (either connected or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

Proof: Case 1: Let G be connected.

Let v_1 and v_2 be the only two vertices of odd degree in G. But we have already proved that the number of odd vertices is even. Clearly there is a path connecting v_1 and v_2 since G is connected.

Case 2: Let G be disconnected.

Then the components of G are connected. Hence v_1 & v_2 should belong to the same component of G. Therefore there is a path between v_1 and v_2.

Theorem: The maximum number of edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$

<u>Proof:</u> Let the number of vertices in the ith component of G be n_i ($n_i \geq 1$)

Then $n_1 + n_2 + \cdots + n_k = n$ or $\sum_{i=1}^{k} n_i = n$ — ①

$$\sum_{i=1}^{k}(n_i - 1) = n - k$$

$$\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n-k)^2 = n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i \neq j}(n_i - 1)(n_j - 1) = n^2 - 2nk + k^2 \quad \text{— ②}$$

$$\therefore \sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 - 2nk + k^2 \quad [\text{the second term in LHS} \geq 0]$$

$$\therefore \sum_{i=1}^{k}(n_i^2 + 1 - 2n_i) \leq n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 - k + 2n \quad \text{— ③}$$

The maximum number of edges in the ith component of G is $G = \frac{1}{2} n_i(n_i - 1)$

$\therefore$ Max number of edges of $G = \frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$

$$= \tfrac{1}{2}\sum_{i=1}^{k} n_i^2 - \tfrac{1}{2}n \quad \text{by ①.}$$

Max number of edges of $G \leq \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k\right] - \frac{1}{2}n$ by ③

$$\leq \tfrac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$

$$\leq \tfrac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$\leq \tfrac{1}{2}(n-k)(n-k+1).$$

<u>Note:</u> $\left[\sum(n_i - 1)\right]^2$

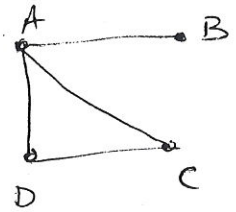
$= \left[(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)\right]^2$

$= \left[a_1 + a_2 + \cdots + a_k\right]^2 = \sum_{i}^{k} a_i^2 + 2\sum_{i \neq j} a_i a_j$

Circuits and Isomorphism:

If two graphs are isomorphic, they will contain circuits of same length k where $k > 2$.

Theorem: If A is the adjacency matrix of a graph G then the number of different paths of length r from v_i to v_j is equal to $(i-j)^{th}$ entry of A^r.



The adjacency matrix of this graph is

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Find the number of paths between B and D which are of length 4 by finding A_G^4

$$A_G^4 = \begin{bmatrix} 11 & 2 & 6 & 6 \\ 2 & 3 & 4 & 4 \\ 6 & 4 & 7 & 6 \\ 6 & 4 & 6 & 7 \end{bmatrix}$$

Now the element in $(2-4)^{th}$ entry of A_G^4 is 4. Hence there are 4 paths of length 4 from B to D in the graph G.

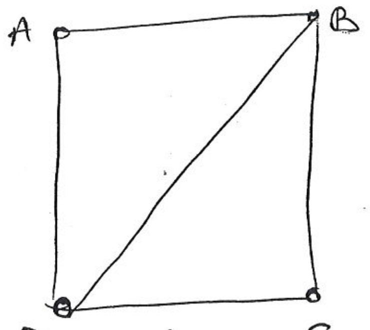
The four paths are B-A-B-A-D, B-A-C-A-D, B-A-D-A-D, B-A-D-C-D.

Eulerian and Hamiltonian Graphs:

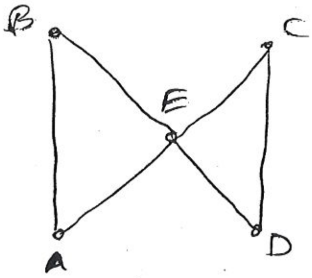
Eulerian path: A path of a graph G is called an Eulerian path if it includes each edge of G exactly once.

Eulerian circuit: A circuit of a graph G is called an Eulerian circuit, if it includes each edge of G exactly once.

A graph containing an Eulerian circuit is called an Eulerian graph.



G_1



G_2

Graph G_1 contains an Eulerian path between B and D namely B – D – C – B – A – D.

G_2 contains a Eulerian circuit A – E – C – D – E – B – A. Since it includes each of the edges exactly once.

G_2 is a Euler graph as it contains a Eulerian circuit.

<u>Note:</u> A connected graph contains an Euler circuit, if and only if each of its vertices is of even degree.

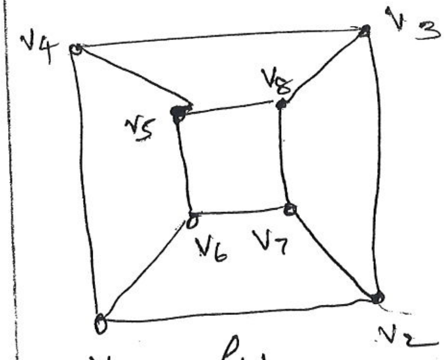
A connected graph contains an Euler path if and only if it has two vertices of odd degree.

The Euler path will have the odd degree vertices as its end points.

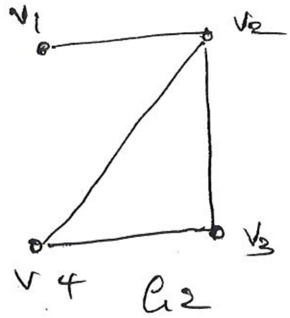
<u>Hamiltonian path</u>: A path of a graph G is called an Hamiltonian path if it includes each vertex of G exactly once

<u>Hamiltonian circuit</u>: A circuit of a graph G is called Hamiltonian circuit if it contains every vertex of G exactly once.

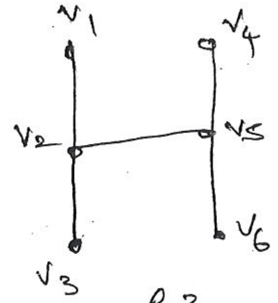
1. Identify the Hamiltonian path, Hamiltonian cycle in the following graphs.



G1.



G2



G3

(i) In G_1 : $v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_1$ is a Hamilton cycle and so G_1 is a Hamilton graph.

$v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ is a Hamilton path.

In G_2 : v_1, v_2, v_3, v_4 is a Hamilton path.

There is no Hamilton cycle since the edge $v_1 v_2$ occurs twice in every circuit. So v_2 is repeated.

In G_3 : Any circuit will have v_2, v_5 atleast twice Therefore it is not Hamilton graph. It has no Hamilton path.

2. Find the number of edges and degree of each vertex in the complete graph K_5.

K_5 has five vertices.

The number of edges $= 5C_2 = 10$

The degree of each vertex $= 5-1 = 4$.

3. Can a simple graph with 8 vertices have 40 edges?

Sol: Maximum number of edges in a simple graph $= nC_2$

$= 8C_2 = 28$

$\therefore$ NO.

(19)

4. A regular graph G has 10 edges and degree of any v is 5, find the number of vertices.

Sol: No of edges say $e = 10$.
degree of each vertex $= 5$.

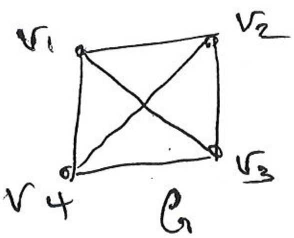
$$\sum \deg(v) = 2e$$

$$5n = 2 \times 10 \quad \text{if there are } n \text{ vertices}$$

$$\therefore \boxed{n = 4.}$$

5. Find the number of paths of length n between two different vertices in K_4 if n is (i) 2 (ii) 3

Sol: K_4 is a complete graph on four vertices.



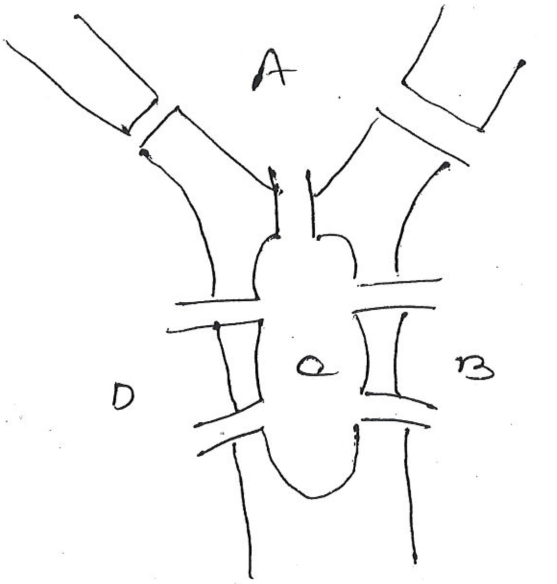
the adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$$

(i) The no of paths of length 2 between any two different vertices $= 2$.
The no of paths of length 2 from any vertex to itself $= 3$.

(ii) The no of paths of length 3 between two different vertices $= 7$
The no of paths of length 3 ~~between two d~~ from a vertex to itself is 6.

The Konigsberg Bridge problem and Eulerian graphs:

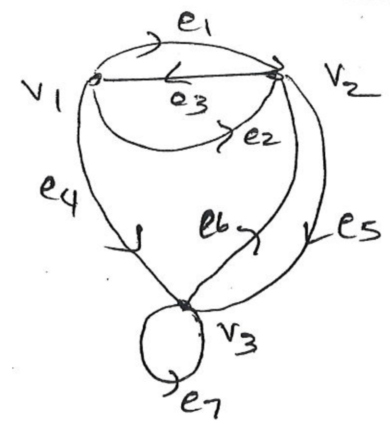


The figure shows the seven bridges connecting four land masses that made up the city. The problem states that the citizens searched invain to a walking tour that passed over each bridge exactly once. No one could find such a tour.

This led to Euler's Theorem: "No walking tour of Konisberg can be designed so that each bridge is used exactly once".

Euler's theorem: (General Case) An undirected graph is Eulerian if and only if it is connected and has either zero or two vertices with an odd degree. If no vertex has odd degree then the graph has Eulerian circuit.

Example.



The directed graph G has vertex set $V = \{v_1, v_2, v_3\}$ and edge set $E = \{e_1 = (v_1, v_2),\ e_2 = (v_1, v_2),\ e_3 = (v_2, v_1),\ e_4 = (v_1, v_3)$
$e_5 = (v_2, v_3),\ e_6 = (v_3, v_2),\ e_7 = (v_3, v_3)\}$

Since the edges e_1, e_2 start and end with same vertices

So e_1 and e_2 are called parallels.

e_7 is a loop since it starts with v_3 and ends with v_3.

e_5 & e_6 are not parallels.

Directed Multigraph: A directed graph with parallels is called a directed multigraph.

Simple directed graph: is a directed graph which has no loop and has parallels.

Mixed graph: A graph with both directed and undirected edges is called a mixed graph.

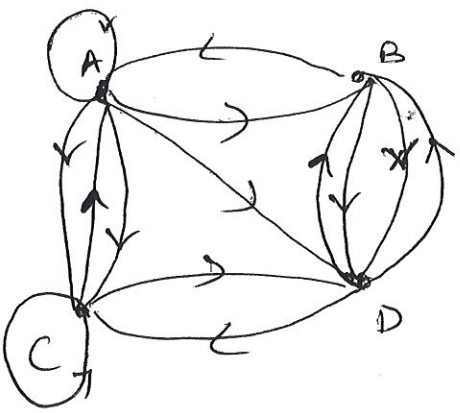
Adjacent vertices: Let $G = (V, E)$ be a directed graph and let $e = (u, v)$ be an edge. Then we say u is adjacent to v and v is adjacent to u.

In-degree: Let $G = (V, E)$ be a directed graph and let $v \in V$ the number of edges with v as terminal vertex is called indegree of the vertex v and denoted by $\deg^-(v)$.

The number of edges with v as initial vertex is called **out degree** denoted by $\deg^+(v)$.

The sum of indegree and out degree of a vertex v is called its **total degree**.

Find the in degree and out degree of each vertex. Verify the degree theorem for the following graph.



$\deg^+(A) = 5$, $\deg^+(B) = 3$, $\deg^+(C) = 3$

$\deg^+(D) = 3$

$\deg^-(A) = 3$, $\deg^-(B) = 3$, $\deg^-(C) = 4$,

$\deg^-(D) = 4$.

Sum of out degrees $= 5+3+3+3 = 14$

Sum of in degrees $= 3+3+4+4 = 14$

Number of edges $= 14$ (directed edges)

$\therefore$ No of directed edges $=$ sum of in degrees $=$ sum of out degrees

Isomorphisms of digraphs:

Two digraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are said to be **isomorphic** if there exists a bijective map $f: V_1 \to V_2$ such that $(u,v) \in E_1$ if and only if $(f(u), f(v)) \in E_2$.

We write $G_1 \cong G_2$.

Note:

1. Two isomorphic digraphs have the same number of

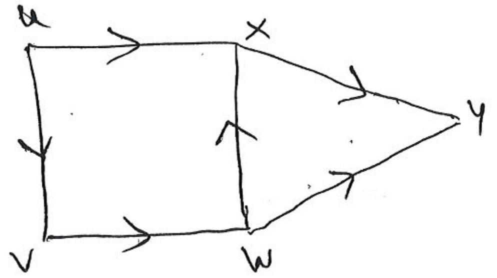
have the same number of vertices and same number of directed edges.

2. If two digraphs are isomorphic then the corresponding vertices have the same degree pairs.

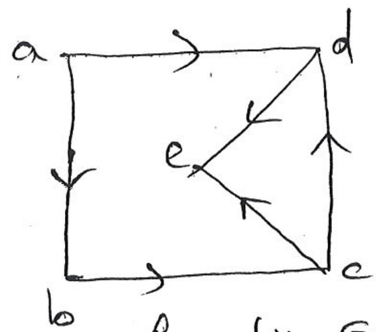
ie If f is the isomorphism then

$$(\deg^-(u), \deg^+(u)) = \{\deg^-(f(u)), \deg^+(f(u))\}$$

Test whether the graphs G_1 and G_2 are isomorphic



$G_1 = (V_1, E_1)$



$G_2 = (V_2, E_2)$

Consider the map $f: V_1 \to V_2$ given by
$f(u) = a$, $f(v) = b$, $f(w) = c$, $f(x) = d$, $f(y) = e$.
clearly f is bijective.

$(u, v) \in E_1$, $(f(u), f(v)) = (a, b) \in E_2$

$(v, w) \in E_1$, $(f(v), f(w)) = (b, c) \in E_2$

$(w, x) \in E_1$, $(f(w), f(x)) = (c, d) \in E_2$

~~$(u, x) \in E_1$, $(f(u), f(x)) = (a, d) \in E_2$~~

$(u, x) \in E_1$ $(f(u), f(x)) = (a, d) \in E_2$

$(x, y) \in E_1$ $(f(x), f(y)) = (d, e) \in E_2$

$(w, y) \in E_1$ $(f(w), f(y)) = (c, e) \in E_2$

then $G_1 \cong G_2$.

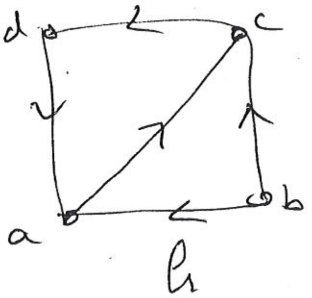
Converse of a diagraph: The converse of a directed graph $G = (V, E)$ is the directed graph obtained by reversing the direction of each edge of E denoted by G' or G^{conv}

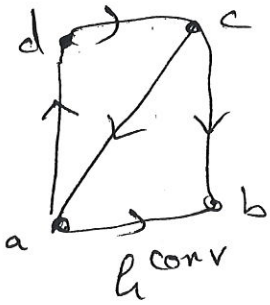
Note: It is obvious $\left(G^{conv}\right)^{conv} = G$.

2. $\deg^-_{G}(v) = \deg^+_{G'}(v)$

3. $\deg^+_{G}(v) = \deg^-_{G'}(v)$

Example:



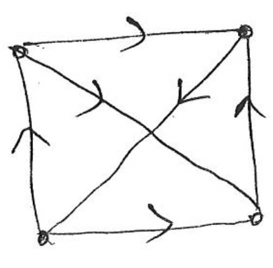


Connectedness in digraph:

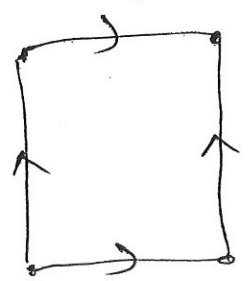
Strongly Connected: A directed graph G is said to be strongly connected if there is a path from u to v and from v to u for any pair of vertices u and v in G.

Weakly Connected: A digraph G is weakly connected if there is a path between any two vertices of the underlying undirected graph.

Strongly Connected.



Weakly Connected



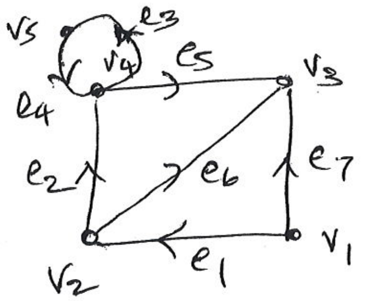
<u>Eulerian digraph</u>: A digraph containing an Eulerian circuit is called an Eulerian digraph.

<u>Incidence Matrix of digraph</u>:

Let $G = (V, E)$ be a digraph with $V = \{v_1, v_2, \dots, v_n\}$ $E = \{e_1, e_2, \dots, e_m\}$ then the incidence matrix of G is

$$B = (b_{ij}) \quad \text{where} \quad b_{ij} = \begin{cases} 1 & \text{if the } j\text{th edge is incident out of } i\text{th vertex} \\ -1 & \text{if } j\text{th edge is incident into } i\text{th vertex} \\ 0 & \text{if } j\text{th edge is not incident with } i\text{th vertex.} \end{cases}$$

Construct an incidence matrix for the graph given below



$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

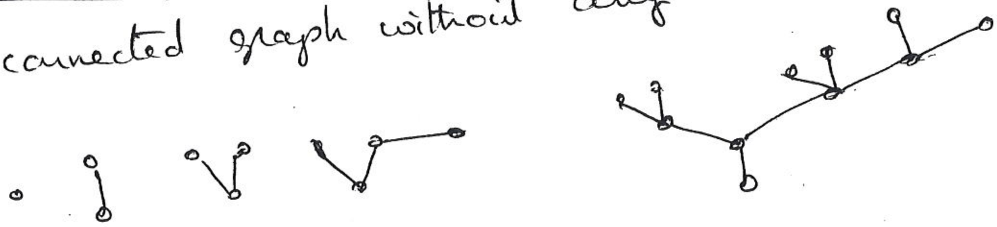
<u>Note</u>

1. the row heading are vertices and coloumn headings are the edges.
2. The number of 1s in a row gives the out degree of the corresponding vertex and the number of -1's in the row is the indegree of the vertex.
3. There will be equal number of 1s and -1s in the matrix
4. Sum of entries in each coloumn is zero.
5. If the entries -1 are replaced by 1s, the incidence matrix becomes the incidence matrix of the underlying graph.

6. The rank of the incidence matrix of a simple connected graph with n vertices is $(n-1)$.

TREES:

A connected graph without any circuits is called a tree



Examples of Trees.

Some Properties of Trees:

1. Property 1: An undirected graph is a tree, if and only if there is a unique simple path between every pair of vertices.

Proof: Given: Let the undirected graph T be a tree

Proof: By definition T is connected.

There is a simple path between any pair of vertices say v_i and v_j

If possible let there be two paths between $v_i - v_j$

One from v_i to v_j and second from v_j to v_i.

Combination of these two paths would contain a circuit. But T cannot have a circuit. Therefore there is a unique simple path between every pair of vertices in T.

Converse: Given: Let a unique path exist between every pair of vertices in graph T.
If possible let T contain a circuit. This means that there is a pair of vertices v_i and v_j between which two distinct paths exist, which is against the data. Hence proved.

Property 2: A tree with n vertices has (n-1) edges.

Proof: Proof is by Mathematical induction

$n=1$ ∴ It is a null graph.

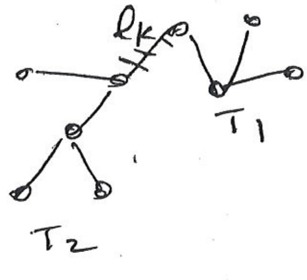
Let the property be true for all trees with less than n vertices.

Let us consider a tree T with n vertices.

Let e_k be the edge connecting the vertices v_i and v_j of T

e_k is the only path between v_i and v_j. If we delete the edge e_k, $(T-e_k)$ consists of two connected components

Say T_1 and T_2 which are connected



T does not contain any circuit, T is tree.

∴ T_1 and T_2 do not contain any circuit.

Let T_1 have r vertices then $T_2 = n-r$ vertices (T has n vertices)

∴ T_1 has $(r-1)$ edges $\quad (r<n$ by induction hypothesis$)$

T_2 has $(n-r-1)$ edges $\quad (n-r<n \quad$ " $\quad)$

∴ T has $(r-1)+(n-r+1)+1 = n-r+r-2+1$
$= n-1$ edges.

∴ a tree with n vertices has n-1 edges.

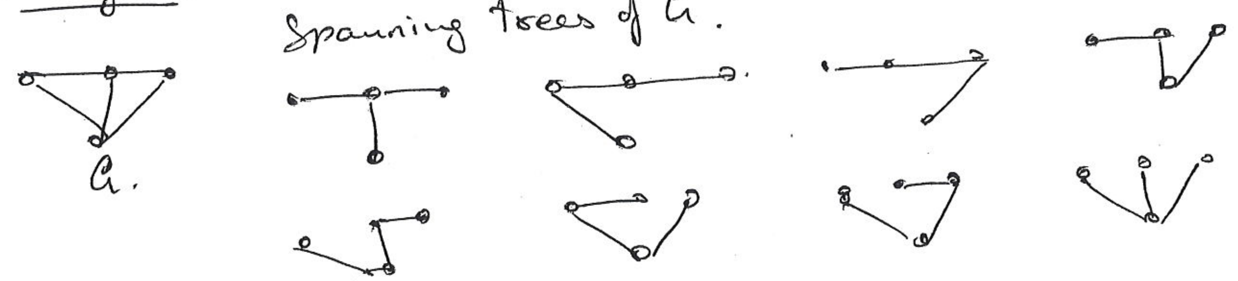
Property 3 : Any connected graph with n vertices and (n-1) edges is a tree.

Property 4 : A circuit-less graph with n vertices and (n-1) edges is a tree.

Spanning trees :

Definition : If the subgraph T of a connected graph G is a tree containing all the vertices of G, then T is called a spanning tree of G.

Example :



Minimum Spanning tree :

If G is a connected weighted graph, the spanning tree of G with the smallest total weight (sum of weights of its edges) is called the minimum spanning tree of G.

※. Kruskal Algorithm :

Step 1 : the edges of the given graph G are arranged in the order of increasing weights.

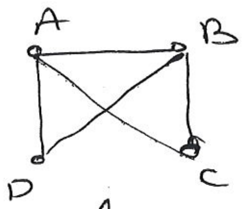
Step 2 : An edge G with minimum weight is selected as an edge of the required spanning tree.

Step 3 : Edges with minimum weight that do not form a a circuit with the already selected edges are successively added.

<u>Step 4:</u> the procedure is stopped after $(n-1)$ edges are selected.

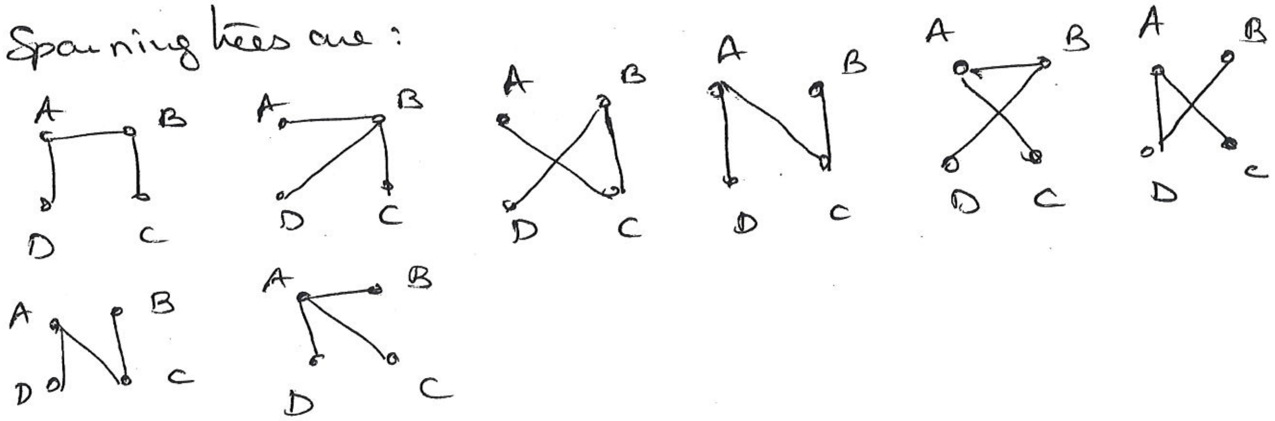
<u>Note:</u> Weight of minimum spanning tree is unique.

1. Draw all the spanning trees of the graph G

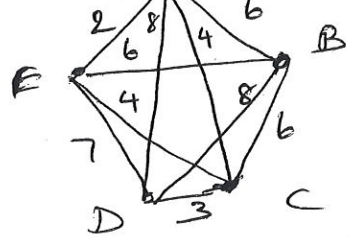


G.

<u>Sol</u> Spanning trees are:



2. Find the minimum spanning tree for the weighted graphs by using Kruskal's algorithm.



Edge	Weight.	Included or not.
A E	2	Yes
D C	3	Yes
E C	4	Yes
A C	4	No (A-E-C-A)
A B	6	Yes
B C	6	No (A-B-C-A
B E	6	—
D E	7	—
A D	8	—
B D	8	—

Since there are five vertices in the graph, we should stop the procedure for finding edges when $n-1 = 5-1 =$ 4 edges are found.

The edges of the minimum spanning tree are AE, CD, AC and AB whose length $= 15$.

The other alternatives minimum spanning trees are

(1) AE, CD, AC, BC
(2) AE, CD, AC, BE
(3) AE, CD, CE, AB
(4) AE, CD, CE, BC
(5) AE, CD, CE, BE.

13/10/11
SWF Rudhu – 10, 11, 14, 16, 71, 17
19, 23, 30, 32, 36, 46, 48, 49
56, 58, 64, 69, 73, 43 Harsh
Rahul

14/10/11 – 11, 32, 34, 56, 58, 59
61, 67, 73, 43 2
Rohit

14/10/11 – 33, 35, 40, 46, 47, 54, 66, 68,
5th hr 70, 71, ~~340~~, ~~350~~, 364, 365,

ROOTED AND BINARY TREES :

<u>Rooted tree</u> : A tree in which a particular vertex is designated as the root of the tree is called a rooted tree.

<u>Level or depth of v or height of v</u> : The length of the path from the root of a rooted tree to any any vertex v is called the level or <u>depth</u> of v.

<u>Height of the tree</u> : The maximum level of any vertex is called the depth or the height of the tree.

<u>Descendant of v</u> : Every vertex that is reachable from a given vertex v is called the descendant of v.

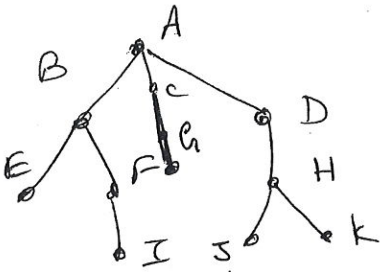
<u>Children of v</u> : All vertices that are reachable from v through a single edge are called children of v.

If a vertex v has no children then v is called a leaf or a terminal vertex or a pendant vertex.

Note: Root is said to be at level zero.

2. Root is also considered as an internal vertex.

Example:



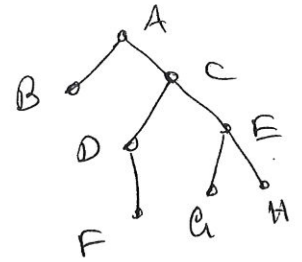
A --- 0
B C D --- 1
E F G H --- 2
I J K --- 3

A is the root of the tree It is at level 0.
The vertices B,C,D are at level 1.
E, F, G, H are at level 2.
I, J, K are at level 3.
The height of the tree is 3.
The vertices E,F,I are descendants of B.
E and F are the children of B.
J and K are the children of H.
E, I, G, J, K are leaves of the tree.
The vertices A, B, F, C, D and H are internal vertices of the tree.

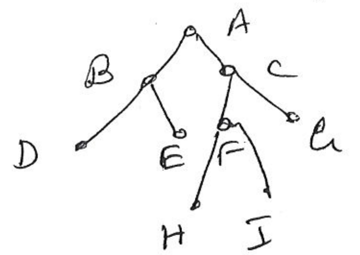
Binary Tree: If every internal vertex of a rooted tree has exactly 2 children the tree is called full binary tree.

If every internal vertex of a rooted tree has atmost 2 children the tree is called binary tree.

Binary tree.



Full Binary Tree



Properties of Binary trees:

Property 1: The number n of vertices of a full binary tree is odd and the number of pendant vertices (leaves) of the tree is equal to $\frac{(n+1)}{2}$.

Proof: In a full binary tree only one vertex is of even degree (namely 2). Each of the other $(n-1)$ vertices is of odd degree (namely 1 or 3). Number of vertices of odd degree in a graph is even. Therefore $n-1$ is even. or n is odd.

Let p be the number of pendant vertices of the full binary tree.

The number of vertices of degree $3 = n-p-1$.

Sum of the degrees of all the vertices of the tree $\Big\} = 1\times 2 + p\times 1 + (n-p-1)3$

$= 2+p+3n-3p-3$

$= 3n-2p-1$

$\therefore$ Number of edges of the tree $= (3n-2p-1)\frac{1}{2}$. — ①

But number of edges of a tree with n vertices $= n-1$. — ②

Equating ① and ② $\frac{1}{2}(3n-2p-1) = n-1$.

$p = \frac{n+1}{2}$. //

Property 2: The minimum height of a n-vertex binary tree is equal to $\lceil \log_2(n+1) - 1 \rceil$ where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof: Let h be the height of the binary tree.
The maximum level of any vertex of the tree is h.
If n_i represents the number of vertices at level i, then

$$n_0 = 1; \quad n_1 \le 2^1, \; n_2 \le 2^2, \; ---, \; n_h \le 2^h$$

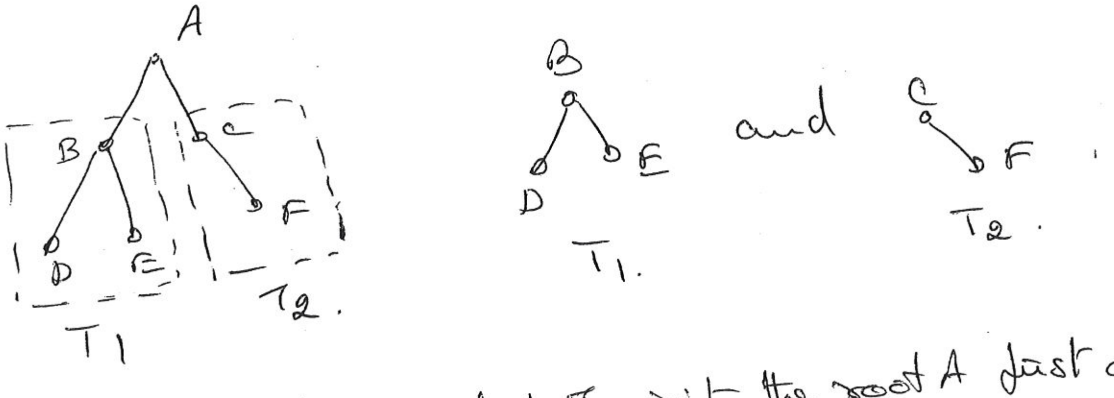
ie $n = n_0 + n_1 + -- + n_h \le 1 + 2^1 + 2^2 + -- + 2^h$

$$n \le 2^{h+1} - 1 \quad \text{ie} \quad 2^{h+1} \ge n+1$$

$$\therefore \; h+1 \ge \log_2(n+1) \quad \text{or} \quad h \ge \log_2(n+1) - 1$$

$\therefore$ Minimum value of $h = \log_2(n+1) - 1$

TREE TRAVERSAL:



The preorder traversal of T visits the root A first and then traverses T_1 and T_2 in preorder.
Preorder traversal of T_1 BDE, Preorder traversal of T_2 CF
Preorder traversal of T is ABDECF.

ii. The Inorder traversal of T traverses T_1 in the inorder first then visits the root node A and finally traverses T_2 in inorder.

Inorder traversal of T_1 is D B E, Inorder traversal of T_2 C F

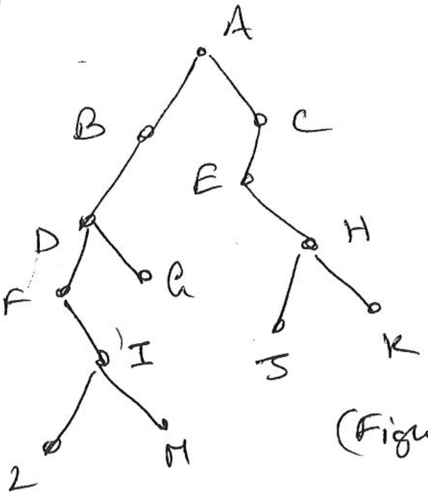
In order traversal of T D B E A C F.

(iii) The postorder traversal of T - processes T_1 then T_2 is post order

Post order traversal of T_1 - D E B. Post order traversal of T_2 F C

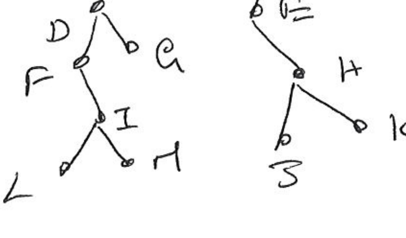
The post order traversal of T is D E B F C A.

List the order in which the vertices of the tree given in the figure below are processed using preorder, inorder and post order traversal.

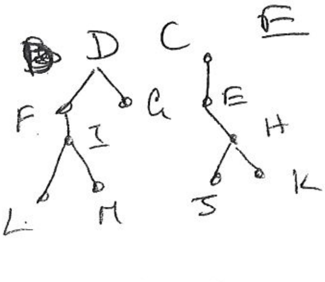


(Figure 1)

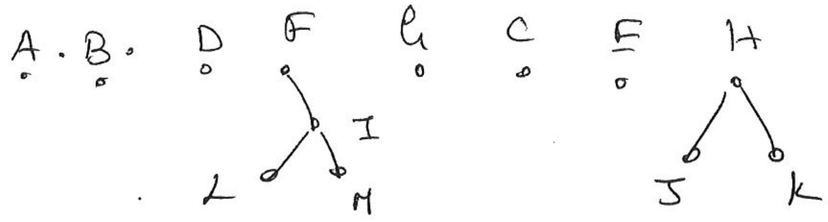
Preorder: Stage 1. A



Stage 2: A B

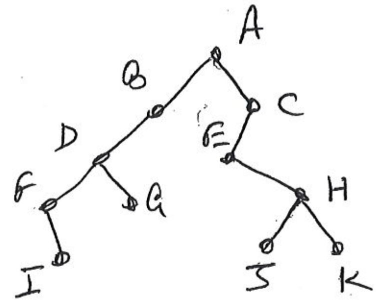


Stage 3 :



Stage 4 : A B D F I L M G C E H J K

(ii) Inorder traversal :



Stage 1 : B A C

Stage 2 : D B A E C

Stage 3 : F D G B A E H C

Stage 4 : F I D G B A E J H K C .

(iii) Post order traversal : Refer . Figure 1 .

Stage 1 : B C D A

Stage 2 : E F B C G H I D A .

Stage 3 : J K E F B C L M G H I D A

Stage 4 : J N O P K E F B C L M G H I D A .

Expression Trees :

We can represent expressions in three different ways by using binary trees .

The Infix form of an algebraic expression corresponds to the inorder traversal of the binary tree representing the expression . Gives the original expression with the operations and operands in the same position.

Infix notation is $((A+B) * (C/D))$

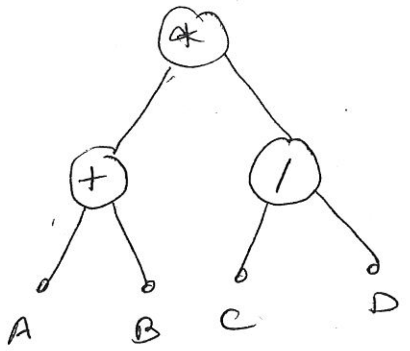


Figure 1.

Prefix Notation: the prefix form of an algebraic expression represented by a binary tree corresponds to the preorder traversal of the tree. The expression in the prefix notation is unambigous and so no parenthesis need be used in this form. It is also known as Polish notation.

Example: Prefix Notation of figure 1 $* + AB / CD$.

Postfix Notation: The postfix form of an algebraic represented by binary tree corresponds to the postorder traversal of the tree. Expressions written in postfix form are also said to be in reverse Polish notation.

Example: Postfix notation of figure 1 $AB + CD / *$

Evaluate the prefix expression $+ - \uparrow 3 2 \uparrow 2 3 / 8 - 4 2$

(i)
$$
\begin{aligned}
+ - \uparrow 3 2 \uparrow 2 3 / 8 - 4 2 &= + - \uparrow 3 2 \uparrow 2 3 / 8 (4-2) \\
&= + - \uparrow 3 2 \uparrow 2 3 / 8 (2) \\
&= + - \uparrow 3 2 \uparrow 2 3 (8/2) \\
&= + - \uparrow 3 2 \uparrow 2 3 (4). \\
&= + - \uparrow 3 2 \; 2 \uparrow 3 (4) \\
&= + - \uparrow 3 2 (8) (4) \\
&= + - \uparrow 3 2
\end{aligned}
$$

$= + - (3 \uparrow 2)\ 8\ 4$

$= + - (9)\ 8\ 4$

$= + (9-8)\ 4$

$= 1+4 = 5.$

Evaluate the postfix expression $72-3+232+-13-*/$

Sol. $72-3+232+-13-*/ = (72-)\ 3+232+-13-*/$

$= (7-2)\ 3+232+-13-*/$

$= (5+3)\ 232+-13-*/$

$= 82\ (3+2)-13-*/$

$= 8(25-)\ 13-*/$

$= 8\ (2-5)\ 13-*/$

$= 8(-3)\ (1-3)*/$

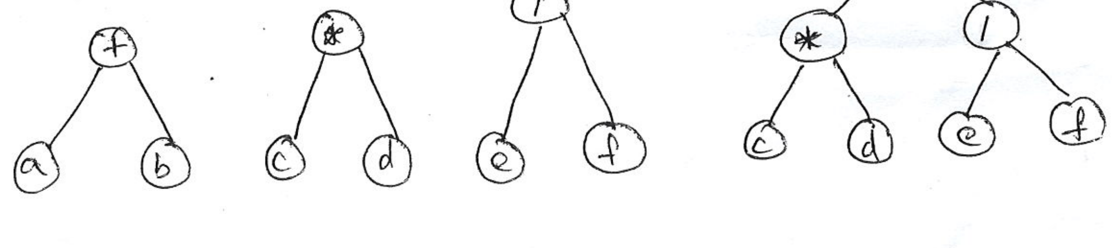
$= 8\ (-3)(-2)*/$

$= 8\ 6\ /$

$= \frac{8}{6} = \frac{4}{3}$ //.

Represent the postfix expression $ab+cd*ef/--a*$ as a binary tree and write also the corresponding infix and prefix forms.

Sol: The subtrees are drawn by considering the operations from left to right when two operands precede an operator. Accordingly we get the following subtrees in the order given and the final tree.



Infix form:

Step 1: $-, *, a$

Step 2: $+, -, -, *, a$

Step 3: $a+b, -, *, -, /, *, a$

Step 4: $((a+b) - ((c*d) - (e/f))) * a$.

Fully parenthesized form is $(((a+b) - ((c*d) - (e/f))) * a)$

1st 2nd hr. 21/10/14 4, 71, 17, 22, 25, 26, 31, 34, 39, 41, 46, 44, 50, 54, 59, 61, 75, debit

Prefix form!

Step 1: $*, -, a$

Step 2: $*, -, +, -, /, a$

Step 3: $*, -, +, a, b, -, *, /, a$.

Step 4: $*, -+ab-*cd/efa$