GRAPH THEORY

UNIT-V

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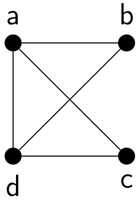
Graph

A graph G = (V,E) consists of a non-empty set V, called the set of vertices(nodes, points) and a set E of ordered or unordered pairs of elements of V, called the set of edges, such that there is a mapping from the set E to the set of ordered or unordered pairs (v_i, v_j) of elements of V

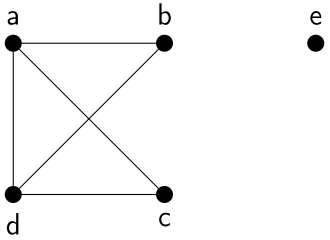
Example:

Let $V = \{a, b, c, d\}$
$E = \{(a, b), (a, c),$
(a,d), (b,d), (c,d) }



Isolated Vertex

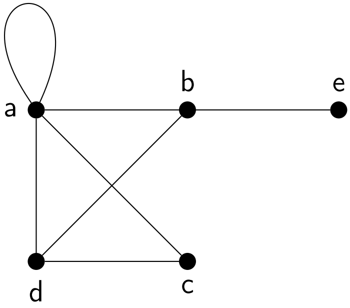
A vertex which is not connected by an edge to any other vertex is called isolated vertex



Here vertex 'e' is said to be isolated vertex

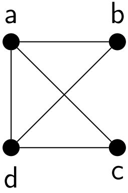
Loop

An edge of a graph that joins a vertex to itself is called a loop. The direction of a loop is not significant, as the initial and terminal vertices are the same.



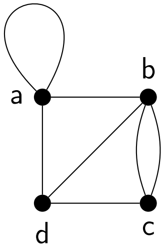
Simple graph

A graph, in which there is only one edge between a pair of vertices, is called a simple graph



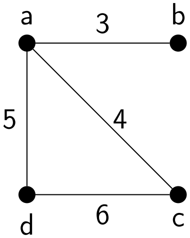
Pseudograph

A graph in which loops and parallel edges are allowed is called a pseudograph.



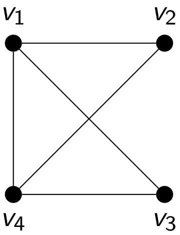
Weighted Graph

A graph in which a number is assigned to each edge is called a weighted graph



Adjacent Vertices

Two vertices are adjacent in G if v_1 and v_2 are end points of an edge



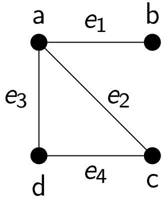
Here the vertex v_1 is adjacent to v_2, since there is an edge between them.

Similarly v_1,v_3 and v_1,v_4 are adjacent vertices

Also v_2,v_4 are adjacent vertices

Incident Edge

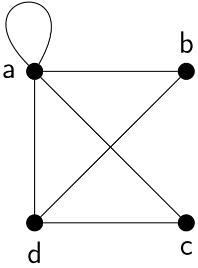
If the edge 'e' is associated with the vertices v_1 and v_2 then the edge 'e' is incident with the vertex v_1 and v_2.



In the above example, the edge e_1 is incident with a and b, the edge e_2 is incident with a and c,the edge e_3 is incident with a and d and the edge e_4 is incident with c and d,

Degree of a Vertex

It is number of edges incident with the vertex. A loop contributes two degrees to the vertex. It is denoted by deg(v)



Degree of each vertices are

$$deg(a) = 5$$

$$deg(b) = 2$$

$$deg(c) = 2 \text{ and}$$

$$deg(d) = 3$$

A vertex of degree 1 is called "Pendant vertex"

A vertex of degree 0 is called "Isolated vertex"

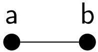
COMPLETE GRAPH

A complete graph on n vertex denoted by K_n is a simple graph that contains exactly one edge between each pair of distinct vertices

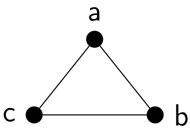
Note :

1. Degree of each vertices in K_n is $n-1$
2. Total no. of edges in K_n is $\frac{n(n-1)}{2}$

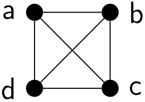
K_1



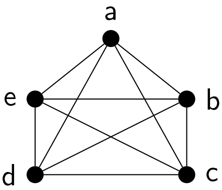
K_2



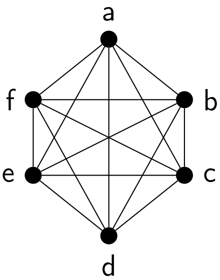
K_3



K_4



K_5



K_6

CYCLIC GRAPH

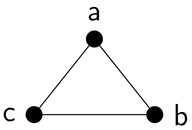
For a given set of vertices $v = \{v_1, v_2, ..., v_n\}$ of G.

Cyclic graph will contain only the following edges

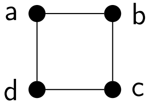
$(v_1, v_2), (v_2, v_3), (v_3, v_4), ...(v_{n-1}, v_n), (v_n, v_1)$

Cyclic graph of n vertices is denoted by C_n

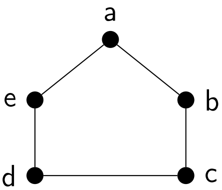
Examples:



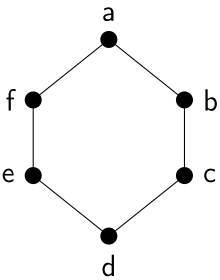
C_3



C_4



C_5



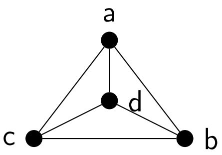
C_6

WHEEL GRAPH

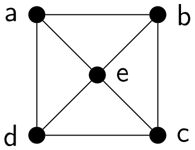
Wheel graph are got by adding an additional vertex to the cyclic graph C_n which connect every other vertex by new vertex

Wheel graph, formed from C_n, is denoted by W_n

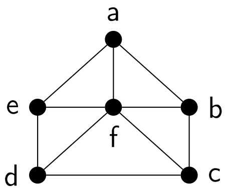
Example :



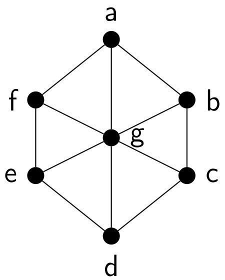
W_3



W_4



W_5



W_6

Regular Graph

If every vertex of the graph has same degree, then the graph is called regular graph

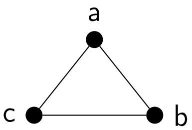
K-regular Graph

If every vertex of the graph has degree k then the graph is called k-regular graph

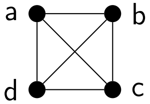
Note

1. All complete graph are regular graph
2. All cyclic graph are regular graph

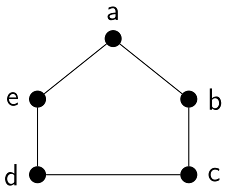
Examples for Regular Graphs



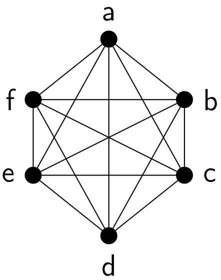
C_3 / K_3 or 2-regular graph



K_4 or 3-regular graph



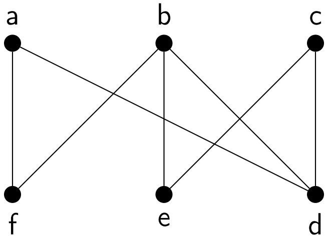
C_5 or 2-regular graph



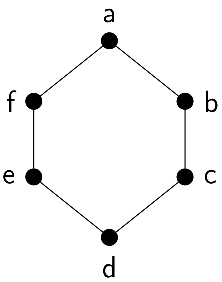
K_6 or 5-regular graph

Bi-partite Graph

A simple graph G is called a bi-partite graph, if its vertex set V can be partitioned into two disjoints sets V_1 and V_2, such that every edge in G connect a vertex in V_1 and a vertex in V_2



Example of Bi-Partite Graph



C_6

Clearly C_6 is an bi-partite graph, since vertex set can be partitioned in to two disjoints sets V_1 and V_2, such that each edge connects a vertex in V_1 and in V_2

$V_1 = \{a, c, e\}$ and

$V_2 = \{b, d, f\}$

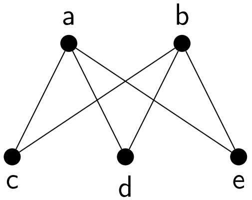
Complete Bi-partite Graph

If each vertex of V_1 is connected with every other vertex of V_2 by an edge then the bipartite graph G is called complete Bipartite graph

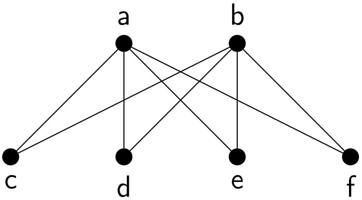
Note

If the vertex set V_1 contains m vertices and vertex set V_2 contains n vertices, then the complete bipartite graph is denoted by $K_{m,n}$

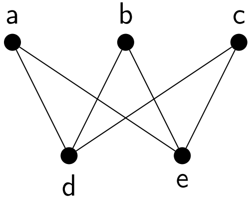
Examples :



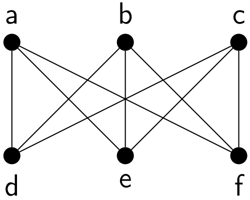
$K_{2,3}$



$K_{2,4}$



$K_{3,2}$



$K_{3,3}$

Degree Sequence of a graph

It is a sequence of $D_1, D_2, D_3, ...D_n$ where $D_n = deg(v_n)$ i.e., degrees of vertices in ascending order

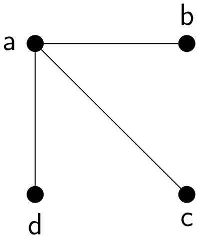
Complementary Graph

The complementary graph $\overline{G}$ of a simple graph G has the same vertices as that of G

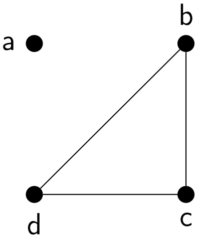
Two vertices are adjacent in $\overline{G}$ if and only if they are NOT adjacent in G

$$G \cup \overline{G} = K_n \text{ (Complete graph)}$$

Example for Complementary Graph :

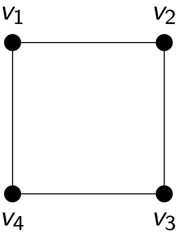


Graph G

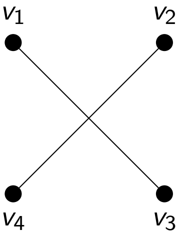


Complementary graph $\overline{G}$

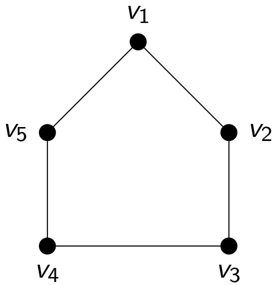
Problem 1: Find the complementary graph of



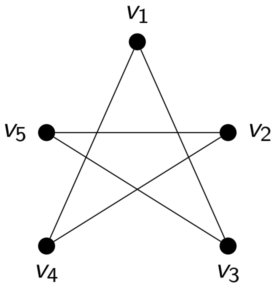
Solution: Complementary graph $\overline{G}$ is



Problem 2: Find the complementary graph of



Solution: Complementary graph $\overline{G}$ is



Matrix Representation of Graphs

Adjacency Matrix

If $G = (V, E)$ is a simple graph with n vertices $v_1, v_2, ..., v_n$, then $n \times n$ matrix A of G, defined by

$$A_G = [a_{ij}], \text{ where } a_{ij} = \begin{cases} 1 & if\,(v_i, v_j) \in E(G) \\ 0 & otherwise \end{cases}, \text{ is called the}$$

Adjacency matrix of G

Basic Properties of Adjacency matrix

1. Since a simple graph has no loops, each diagonal entry of A, viz, $a_{ii} = 0$ for $i = 1, 2, 3, ..., n$

2. The adjacency matrix of a simple graph is symmetric

3. The degree of the vertex v_i i.e., $deg(v_i)$ is sum of values in $i'th$ row or $i'th$ column

Incident Matrix

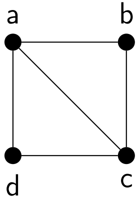
If $G = (V, E)$ is an undirected graph with n vertices $v_1, v_2, ..., v_n$ and m edges $e_1, e_2, ..., e_m$ then $n \times m$ matrix $B = [b_{ij}]$ where $b_{ij} = \begin{cases} 1 & \textit{if } e_j \text{ is incident with } v_i \\ 0 & \textit{otherwise} \end{cases}$, is called the incident matrix of G

Basic Properties of Incident Matrix

1. Each column of B contains exactly two unit entries
2. A row with all 0 entries corresponds to an isolated vertex.
3. A row with a single unit entry corresponds to a pendant vertex
4. The degree of the vertex v_i i.e., $deg(v_i)$ is sum of values in $i'th$ row

Problems:

1. Find the adjacency matrix for the graph G

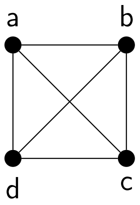


Solution :

Adjacency matrix is

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

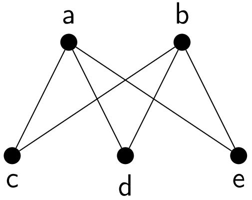
2. Find the adjacency matrix for the graph G



Solution : Adjacency matrix is

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

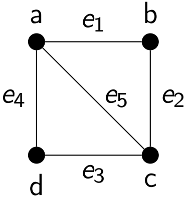
3. Find the adjacency matrix for the graph G



Solution : Adjacency matrix is

$$A_G = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

4. Find the incident matrix for the graph G

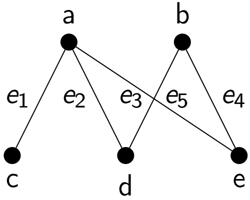


Solution : There are 4 vertices and 5 edges. So 4 rows and 5 columns

Incident matrix is

$$I_G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

5. Find the incident matrix for the graph H



Solution : There are 5 vertices and 5 edges. So 5 rows and 5 columns

Incident matrix is

$$I_H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Isomorphic Graphs

Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic, if there is a one-to-one and onto function f from V_1 to V_2 with a property that, a and b are adjacent in G_1, if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all $a, b \in V_1$
Also, such a function f is called isomorphism.

Isomorphic Graphs will have

1. Same number of vertices
2. Same number of edges
3. Same degree sequence(ascending order)
4. Corresponding vertices with same degree
5. Corresponding adjacency matrices are same

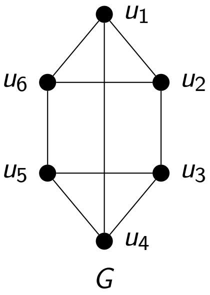
Steps to check whether two graphs are isomorphic

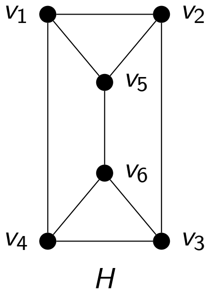
1. List the number of vertices, number of edges and degree sequences
2. If the numbers are same, then the graphs may be isomorphic
   Otherwise we can conclude that, the graphs are NOT isomorphic
3. Unfold the first/second graph in such a way that, it looks similar to the another graph
4. If the first/second graph cannot be unfolded, then they are not isomorphic

5. Identify one-to-one and onto mapping between the vertex sets of two graphs, using the graphs, one in original form and another in the unfolded form
6. Form the adjacency matrix for each graph, as per the mapping
7. If the adjacency matrices are same, then conclude that the two graphs are isomorphic

PROBLEMS

1. Check whether the following two graphs are isomorphic

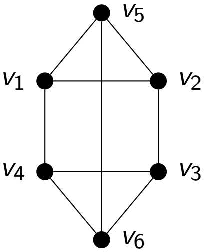




Solution:

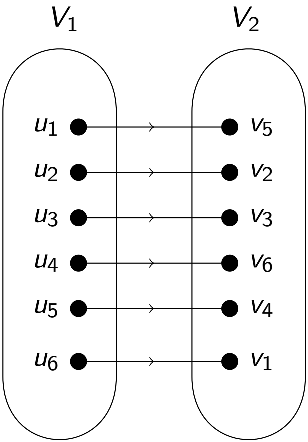
Graphs	G	H
No. of Vertices	6	6
No. of Edges	9	9
Degree Sequences	3,3,3,3,3,3	3,3,3,3,3,3

Unfolding of H



H

Mapping



Adjacency matrices are

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In A_G, vertex order is

$u_1 - u_2 - u_3 - u_4 - u_5 - u_6$

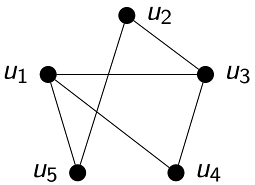
In A_H, vertex order is

$v_5 - v_2 - v_3 - v_6 - v_4 - v_1$

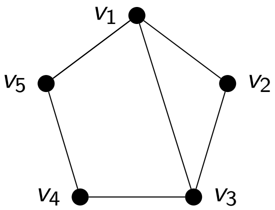
$$\therefore A_G = A_H$$

Hence G and H are isomorphic graphs

2. Check whether the following two graphs are isomorphic



G

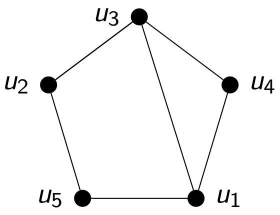


H

Solution:

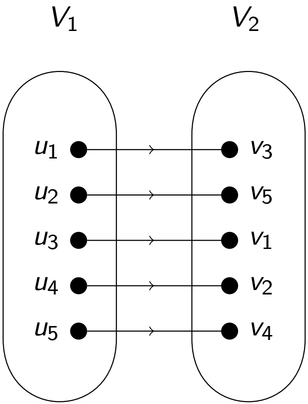
Graphs	G	H
No. of Vertices	5	5
No. of Edges	6	6
Degree Sequences	2,2,2,3,3	2,2,2,3,3

Unfolding of G



G

Mapping



Adjacency matrices are

$$A_G = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

In A_G, vertex order is

$u_1 - u_2 - u_3 - u_4 - u_5$

$$A_H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

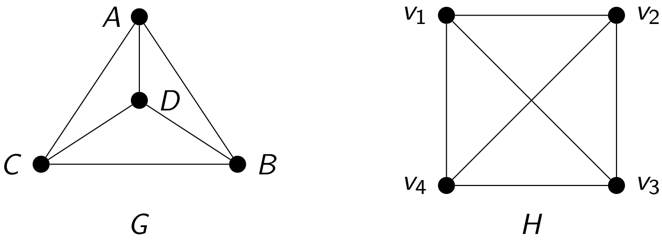
In A_H, vertex order is

$v_3 - v_5 - v_1 - v_2 - v_4$

$$\therefore A_G = A_H$$

Hence G and H are isomorphic graphs

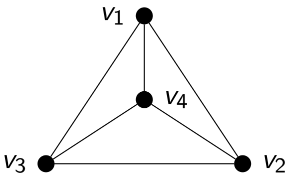
3. Check whether the following two graphs are isomorphic



Solution:

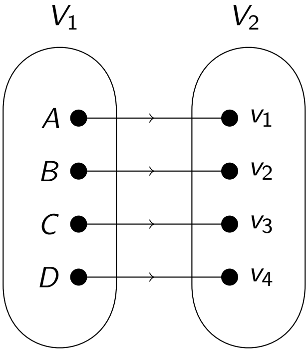
Graphs	G	H
No. of Vertices	4	4
No. of Edges	6	6
Degree Sequences	3,3,3,3	3,3,3,3

Unfolding of H



H

Mapping



Adjacency matrices are

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

In A_G, vertex order is

$A - B - C - D$

$$A_H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

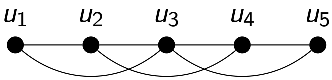
In A_H, vertex order is

$v_1 - v_2 - v_3 - v_4$

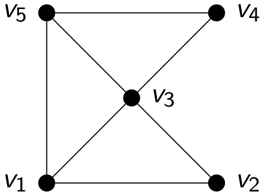
$$\therefore A_G = A_H$$

Hence G and H are isomorphic graphs

4. Check whether the following two graphs are isomorphic



G

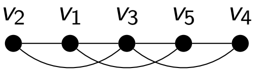


H

Solution:

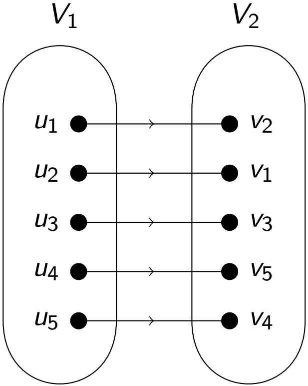
Graphs	G	H
No. of Vertices	5	5
No. of Edges	7	7
Degree Sequences	2,2,3,3,4	2,2,3,3,4

Unfolding of H



H

Mapping



Adjacency matrices are

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

In A_G, vertex order is

$u_1 - u_2 - u_3 - u_4 - u_5$

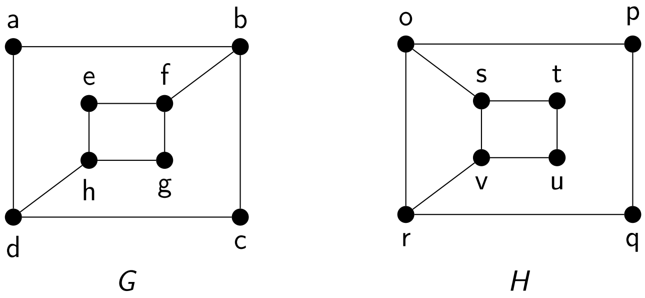
In A_H, vertex order is

$v_2 - v_1 - v_3 - v_5 - v_4$

$$\therefore A_G = A_H$$

Hence G and H are isomorphic graphs

5. Check whether the following two graphs are isomorphic



Solution:

Graphs	G	H
No. of Vertices	8	8
No. of Edges	10	10
Degree Sequences	2,2,2,2,3,3,3,3	2,2,2,2,3,3,3,3

Here $deg(h) = 3$

Adjacent vertices of h have the degree 3,2,2

and $deg(v) = 3$

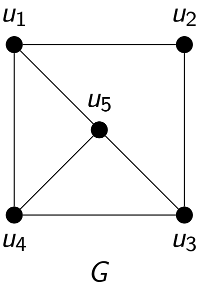
Adjacent vertices of v have the degree 3,3,2

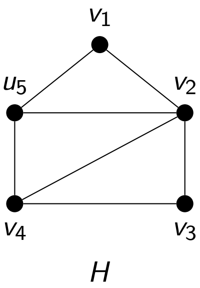
The degree sequence of adjacent vertices are different

Simply, two degree vertices are not adjacent in G whereas they are adjacent in H

$\therefore$ G and H are not isomorphic

6. Check whether the following two graphs are isomorphic or not





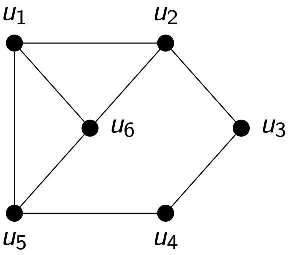
Solution:

Graphs	G	H
No. of Vertices	5	5
No. of Edges	7	7
Degree Sequences	2,3,3,3,3	2,2,3,3,4

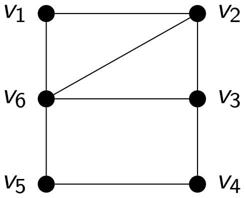
Here degree sequence of G and H are not same

$\therefore$ G and H are not isomorphic

7. Check whether the following two graphs are isomorphic



G_1



G_2

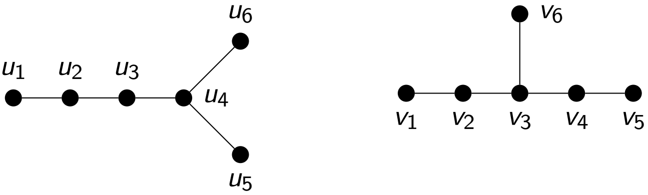
Solution:

Graphs	G_1	G_2
No. of Vertices	6	6
No. of Edges	8	8
Degree Sequences	2,2,3,3,3,3	2,2,2,3,3,4

Here degree sequence of G_1 and G_2 are not same

$\therefore$ G_1 and G_2 are not isomorphic

8. Check whether the following two graphs are isomorphic



G_1 G_2

Solution:

Graphs	G_1	G_2
No. of Vertices	6	6
No. of Edges	5	5
Degree Sequences	1,1,1,2,2,3	1,1,1,2,2,3

Here $deg(u_4) = 3$

Adjacent vertices of u_4 have the degree 1,1,2

and $deg(v_3) = 3$

Adjacent vertices of v_3 have the degree 1,2,2

The degree sequence of adjacent vertices are different

$\therefore$ G_1 and G_2 are not isomorphic

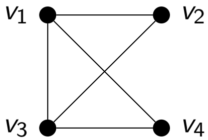
9. Examine whether the graphs G and H associated with the adjacency matrices are isomorphic or not.

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Solution: The corresponding graphs are given below :

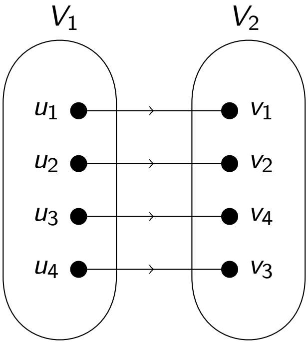
Graphs	G	H
No. of Vertices	4	4
No. of Edges	5	5
Degree Sequences	2,2,3,3	2,2,3,3

Unfolding of H



H

Mapping



Adjacency matrices, based on the mapping, are

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad A_H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

In A_G, vertex order is

$u_1 - u_2 - u_3 - u_4$

In A_H, vertex order is

$v_1 - v_2 - v_4 - v_3$

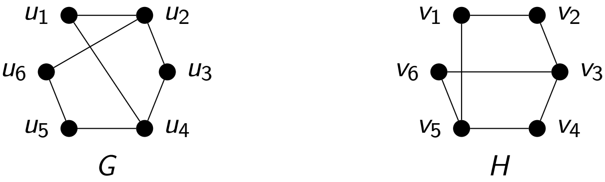
$$\therefore A_G = A_H$$

Hence G and H are isomorphic graphs

10. Examine whether the graphs G and H associated with the adjacency matrices are isomorphic or not.

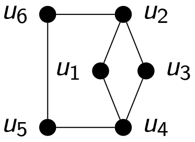
$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Solution: The corresponding graphs are given below :

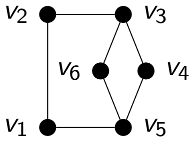


Graphs	G	H
No. of Vertices	6	6
No. of Edges	7	7
Degree Sequences	2,2,2,2,3,3	2,2,2,2,3,3

Unfolding of G & H

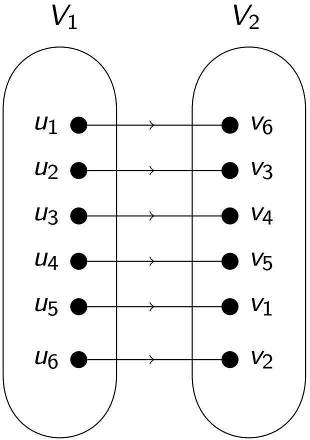


G



H

Mapping



Adjacency matrices, based on the mapping, are

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad A_H = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In A_G, vertex order is

$u_1 - u_2 - u_3 - u_4 - u_5 - u_6$

In A_H, vertex order is

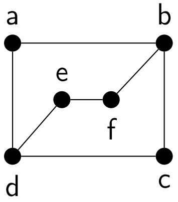
$v_6 - v_3 - v_4 - v_5 - v_1 - v_2$

$$\therefore A_G = A_H$$

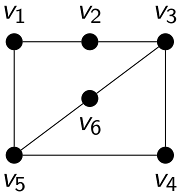
Hence G and H are isomorphic graphs

Problems for Practice

Check whether the following two graphs are isomorphic



G



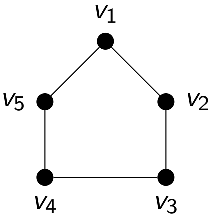
H

Self Complementary Graph

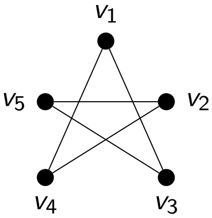
A graph G is said to be a self complementary graph if G is isomorphic to its complementary graph $\overline{G}$

i.e., if $G \approxeq \overline{G}$ then G is called self-complementary graph

Example :



G



$\overline{G}$

G is self complementary graph

Important Theorems

1. State and prove Handshaking theorem. Also prove that maximum number of edges in a simple graph G with n vertices is ${}^nC_2$ [or] $\frac{n(n-1)}{2}$

   Proof: Statement : Let $G = (V, E)$ be a graph.

   Then $\sum_{u \in V} \deg(u) = 2|E|$ where $|E|$ is number of edges in G

   proof:Consider an edge $e = (u, v)$ of G

   The edge e is incident with both end vertices u and v. The contribution of e to each of the degrees of u and v is one. Hence the total contribution of the edge e to sum of the degrees of vertices of G is two.

This is true for each and every edge of G. Also a loop contributes two to the sum of the degrees of the vertices. Hence the sum of the degrees of the vertices of G is twice the number of edges in G

Then $\sum_{u \in V} \deg(u) = 2|E|$ where $|E|$ is number of edges in G

To prove : Maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

By Handshaking theorem, $\sum\limits_{u \in V} \deg(u) = 2|E|$

$deg(u_1) + deg(u_2) + ... + deg(u_n) = 2|E|$

WKT, maximum degree of a vertex in a simple graph of n vertices is $(n-1)$

$\implies (n-1) + (n-1) + ... + (n-1) = 2|E|$

$\implies n(n-1) = 2|E|$

$\implies |E| = \frac{n(n-1)}{2}$

2. Prove that the number of vertices of odd degree in any graph is even [OR]

   Prove that undirected graph G has an even number of odd degree vertices

   Proof : Let $G = (V, E)$ be an undirected graph

   Let us split the vertex set V in to two disjoint subsets as follows

   $$V_1 = \text{Set of all odd degree vertices}$$

   $$V_2 = \text{Set of all even degree vertices}$$

   $$\therefore V = V_1 \cup V_2 \text{ and } V_1 \cap V_2 = \phi$$

By Handshaking Theorem, we have

$$\boxed{\sum_{u \in V} \deg(u) = 2|E|}$$

$$\implies \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u) = 2|E|$$

$$\implies \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u) = \text{even number} \quad --- > (1)$$

Since each element of V_2 is an even degree vertex, each term in the second summation of LHS of eqn (1) is an even number

$$\implies \sum_{u \in V_1} \deg(u) + \text{even number} = \text{even number}$$

$$\implies \sum_{u \in V_1} \deg(u) = \text{even number}$$

This is possible, only when V_1 has even number of elements.

Since V_1 consist of odd degree vertices of G, it is clear that the number of vertices of odd degree in a graph G is always even.

Hence the theorem

Congruence Modulo n

If $a - b$ is divisible by n then we can say that

" a is congruent to b (modulo n) " and

can be written mathematically as

$a \equiv b$ (mod n)

3. If G is a self complementary graph, then prove that
G has n $\equiv$ 0 or 1 (mod 4) [OR]
Show that the self complementary graph have 4n or 4n+1 vertices

Proof : Given G is a self complementary graph with n vertices

$$\text{WKT } G \cup \overline{G} = K_n$$

$$\implies |E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2} \qquad ----> (1)$$

Since G is self complementary, $G \approxeq \overline{G}$

$$\therefore |E(G)| = |E(\overline{G})|$$

Eqn (1) becomes $2\,|E(G)| = \frac{n(n-1)}{2}$

$\implies |E(G)| = \frac{n(n-1)}{4} \qquad ----> (2)$

Since $E(G)$ is a positive integer

$\implies l = \frac{n(n-1)}{4}$

$\implies 4l = n(n-1)$

$\implies n(n-1) = 4l$

$\implies$ Either n is divisible by 4 or n - 1 is divisible by 4

$\implies$ Either n = 4k or n - 1 = 4k

$\implies$ n $\equiv$ 0 (mod 4) or n - 1 $\equiv$ 0 (mod 4)

$\implies$ n $\equiv$ 0 (mod 4) or n $\equiv$ 1 (mod 4)

Path

A path from v_1 to v_n in a graph G, is a finite alternating sequence of vertices and edges, beginning and ending with vertices

Simple Path

A path with neither repeated vertex nor a repeated edge is called simple path

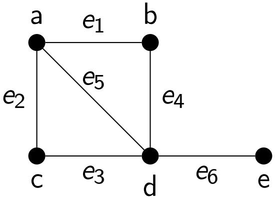
Cycle or circuit

If the initial and final vertex of the path are same, then the path is called circuit or cycle.

Simple circuit

A Cycle of a simple path is called simple circuit. In other words, the simple circuit does not contain the same edge more than once

Examples of Path and Circuit/Cycle



Path :

$a - e_1 - b - e_4 - d - e_5 - a - e_2 - c - e_3 - d - e_6 - e$

[OR] $a - b - d - a - c - d - e$

Simple Path : $a - e_1 - b - e_4 - d - e_6 - e$ [OR]

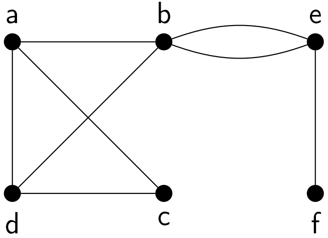
$a - b - d - e$

Simple Cycle : $a - e_1 - b - e_4 - d - e_5 - a$

Simple Cycle : $a - e_1 - b - e_4 - d - e_3 - c - e_2 - a$

Connected graph

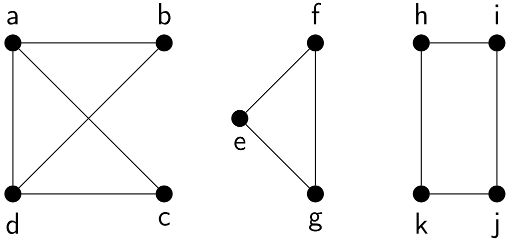
An undirected graph is said to be connected if there is a path between every pair of vertices



Note : If there is no path between a pair of vertices, then the graph G is disconnected graph

Connected components of the graph G

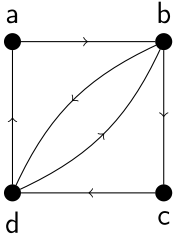
If a graph G is disconnected then the various connected path is called connected components of a graph



CONNECTEDNESS IN A DIRECTED GRAPH

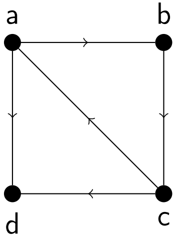
Strongly connected graph

A directed graph G is said to be strongly connected if there is a path from v_i to v_j and from v_j to v_i where v_i, v_j are any pair of vertices of the graph



Unilaterally connected graph

A simple directed graph is said to be unilaterally connected if for any pair of vertices of the graph, atleast one of the vertices of the pair is reachable from the other vertex



Weakly connected graph

A directed graph G is said to be weakly connected, if there is a path between every two vertices in the underlying undirected graph

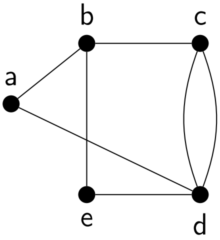
i.e., if there exist a path between every pair of vertices, after removing the direction in each edges, then the graph is weakly connected graph

Note:

1. Any strongly connected graph is unilaterally connected
2. Any unilaterally connected graph is weakly connected

Euler path

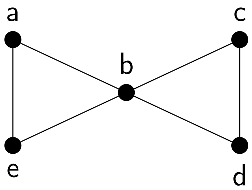
A path that contains every edge exactly once is called the Euler path

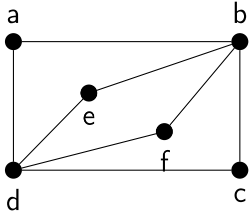


Euler Path : $b - a - d - c - d - e - b - c$

Euler circuit

If Euler path begins and ends at the same vertex, then the circuit is called Euler circuit





Note-1 : If the graph G contains Euler circuit, then the graph G is called "Eulerian"

Note-2 : All vertices have even degree

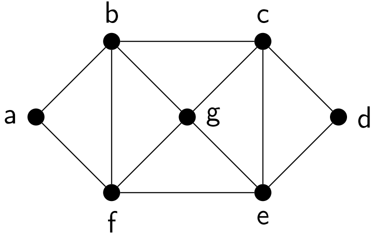
Hamiltonian path

A path that passes through every vertex exactly once is called Hamiltonian path

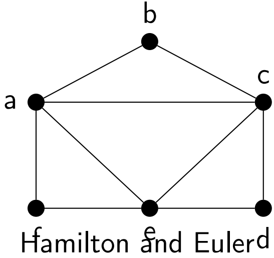
Hamiltonian circuit

A Hamiltonian path that starts and ends at the same vertex is called Hamiltonian circuit

Example for Hamilton Circuit and Euler Circuit



Hamilton and Euler



Hamilton and Euler

Note :

If a graph has Hamiltonian circuit then it is automatically has Hamiltonian path but reverse is not necessarily true

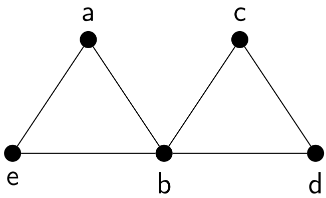
Problems

Give an example of a graph which is

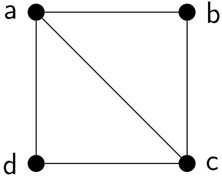
1. Eulerian but not Hamiltonian
2. Hamiltonian but not Eulerian
3. Hamiltonian and Eulerian
4. Neither Hamiltonian nor Eulerian

Solution (1) : Euler graph but not Hamiltonian

Solution (2) : Hamiltonian circuit but not Eulerian

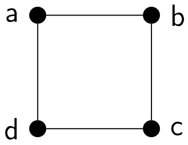


Eulerian not Hamiltonian



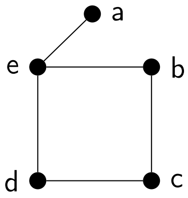
Hamiltonian not Eulerian

Solution (3) : Both Hamiltonian and Eulerian



Hamiltonian and Eulerian

Solution (4) : Neither Hamiltonian nor Eulerian



Neither Hamiltonian nor Eulerian

Theorems(Cont...)

4. Prove that maximum number of edges in a simple disconnected graph G of n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$

<u>Proof</u> : Given : G is disconnected graph with n vertices and k components

Let n_i be the number of vertices in i^{th} component,

$i = 1, 2, 3, ..., k$

Then $n_1 + n_2 + ... + n_k = n \qquad ---> (1)$

$$\implies \sum_{i=1}^{k} n_i = n$$

To prove : Maximum number of edges is $\frac{(n-k)(n-k+1)}{2}$

We know that, maximum number of edges in i^{th} component of G is $\frac{n_i(n_i-1)}{2}$

Therefore the maximum number of edges in G is

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{k} [{n_i}^2 - n_i]$$

$$= \frac{1}{2} \left[\sum_{i=1}^{k} {n_i}^2 - \sum_{i=1}^{k} n_i \right]$$

$$= \frac{1}{2} \left[\sum_{i=1}^{k} {n_i}^2 - n \right] \qquad ---> (2) \text{ [By eqn (1)]}$$

To find $\sum_{i=1}^{k} {n_i}^2$:

Equation (1) implies

$$n_1 + n_2 + ... + n_k = n$$

Subtracting k on both sides, we get

$$\Longrightarrow \ (n_1 - 1) + (n_2 - 1) + ... + (n_k - 1) = n - k$$

$$\Longrightarrow \ \sum_{i=1}^{k} (n_i - 1) = n - k$$

Squaring on both sides, we get

$$\Longrightarrow \ \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n - k)^2$$

$$\implies \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$

$$\implies \sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

$$\implies \sum_{i=1}^{k} ({n_i}^2 + 1 - 2n_i) \leq n^2 + k^2 - 2nk$$

$$\implies \sum_{i=1}^{k} {n_i}^2 + k - 2 \sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk$$

$$\implies \sum_{i=1}^{k} {n_i}^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\implies \sum_{i=1}^{k} {n_i}^2 \leq n^2 + k^2 - 2nk - k + 2n \qquad --- > (3)$$

Substituting eqn(3) in eqn (2) we get,

$\therefore$ Maximum number of edges in G is

$$
\begin{aligned}
&= \frac{1}{2}\left[\sum_{i=1}^{k} {n_i}^2 - n\right] \\
&\leq \frac{1}{2}\left[(n^2 + k^2 - 2nk - k + 2n) - n\right] \\
&= \frac{1}{2}\left[(n-k)^2 - k + n\right] \\
&= \frac{1}{2}\left[(n-k)^2 + (n-k)\right] \\
&= \frac{1}{2}\left[(n-k)(n-k+1)\right] \\
&= \frac{(n-k)(n-k+1)}{2}
\end{aligned}
$$

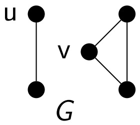
5. Prove that the complement of disconnected graph is connected

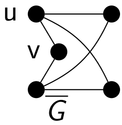
<u>Proof</u>: Let G be a disconnected graph

To prove : Complement of G is connected

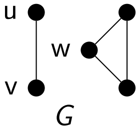
i.e., to prove that $\overline{G}$ is connected

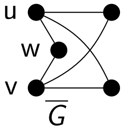
Since G is disconnected, G has more than one connected component. Let u and v be any two arbitrary vertices in G





Case (i) : Let u and v be in two different components of G. So u and v are not adjacent in G. But adjacent in $\overline{G}$. Hence there is a path between u and v in $\overline{G}$. Therefore $\overline{G}$ is connected





Case (ii) : Let u and v be in same component of G. Let w be some vertex in another component of G. This means the edges uw and vw were not in G. And this implies that, they both present in $\overline{G}$. This gives the path $u - w - v$. Therefore there is a path between u and v in $\overline{G}$. Hence $\overline{G}$ is connected

6. Show that the graph G is disconnected if and only if its vertex set V can be partitioned into two non empty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex in V_1 and other in V_2

   Proof: Let G be disconnected graph.

   Let us prove that, vertex set V can be partitioned into two non-empty subsets V_1 and V_2 such that there exists no edge in G between the vertices of V_1 and V_2

   Consider a vertex 'a' in G and let V_1 be set of all vertices that are joined by paths to a.

Since G is disconnected, V_1 does not include all the vertices of G and the remaining vertices will form a set V_2. No vertex in V_1 is joined to any other vertex in V_2 by an edge. Hence the partition exist in the vertex set V.

Conversely, let us assume that partition exists in the vertex set.

Let G be a graph with vertex set V.

To prove : G is disconnected graph

Let V_1 and V_2 be two subsets of V, such that $V_1 \cup V_2 = V$.

Consider two arbitrary vertices 'a' and 'b' of G such that 'a' in V_1 and 'b' in V_2.

No path can exist between the vertices a and b. Otherwise there would be at least one edge whose one end vertex is in V_1 and other in V_2.

Hence if the partition exist, G is not connected and therefore G is disconnected graph.

Hence the theorem

7. If all the vertices of an undirected graph are of degree k then show that the number of edges of the graph is a multiple of k.

Proof : Let $G = (V, E)$ be a simple graph with n vertices and e edges

$$\therefore |V| = n \text{ and } |E| = e$$

Consider each of the vertices of G having degree k.

By Handshaking theorem, we have

$$\boxed{\sum_{u_i \in V} \deg(u_i) = 2e} \implies \sum_{i=1}^{n} \deg(u_i) = 2e$$

$\implies deg(u_1) + deg(u_2) + deg(u_3) + ... + deg(u_n) = 2e$

$\implies$ k + k + k + ... + k (n times) = 2 e

$\implies$ nk=2e

$\implies e = \frac{nk}{2}$

Case(i) : If k is even, then nk is even

Hence $e = \frac{nk}{2}$ is an integer and multiple of k

Hence the theorem

Case (ii) If k is odd, then n is even

$\implies$ nk is even

WKT, any simple graph contains even number of odd degree vertices

$\implies$ $e = \frac{nk}{2}$ is an integer and multiple of K

Hence the theorem

8. Prove that maximum number of edges in bi-partite graph with n-vertices is $\frac{n^2}{4}$ [OR]
Prove that the number of edges in a bi-partite graph with n-vertices is atmost $\frac{n^2}{4}$

Proof: Let the vertex set can be partitioned into two subsets V_1 and V_2. Let V_1 contains x vertices, then V_2 contains $n - x$ vertices. The maximum number of edges in the graph can be obtained when each of x vertices in V_1 is connected to each of $n - x$ vertices in V_2

Therefore the maximum number of edges is given by

$f(x) = x(n - x)$ which is a function of x

Now we have to find the value of x for which $f(x)$ is maximum

By calculus $f'(x) = n - 2x$ and $f''(x) = -2$

$f'(x) = 0 \implies x = n/2$

$f''(n/2) = -2 < 0$

Hence f(x) is maximum at $x = \frac{n}{2}$

$\therefore$ Maximum number $= f\left(\frac{n}{2}\right) = \frac{n}{2}\left(n - \frac{n}{2}\right) = \frac{n^2}{4}$

Hence the proof

9. A non empty connected graph is Euler if and only if its vertices are of even degree

   Proof : Let us assume that the connected graph G is Eulerian

   To prove : Vertices of G are all of even degree

   since G is Eulerian, G contains Euler circuit which begins and ends at one vertex, say u. If we travel along the circuit, then each time we visit a vertex, we use two edges one in and one out. This is also true for the start vertex because we also ends there. Since eulerian circuit uses every edge once, each occurrence of an edge represents a contribution of 2 to its degree. Thus degree of u is even.

Conversely let us assume that the connected graph G has an even degree vertices.

To prove : G contains Euler circuit or G is Eulerian graph

Choose an arbitrary vertex 'a' in G.

Let $a = x_0$. Choose an edge (x_0, x_1) incident with x_0.

This is possible because the G is connected.

We continue building a simple path $x_0, x_1, x_2, ..., x_n$ by adding edges until we reach a vertex, for which we have included all the edges incident with vertex in that graph.

Since the degree of each vertex is even, if we enter a vertex other than 'a', there is an another edge at vertex to leave.

Therefore the path constructed should end only at a. Thus we get a circuit.

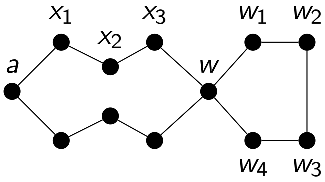
Now there are two possibilities

1.The Circuit we have constructed contains all edges of G

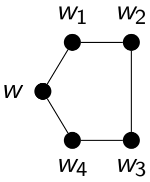
2.The Circuit we have constructed has not used all the edges

In case (1), the circuit constructed is an Eulerian circuit.

In case (2), now we construct a subgraph H of G, by deleting all the edges used and the vertices that are not incident with the remaining edges



G



H

Since G is connected, H has at least one vertex in common with the circuit that has been deleted. Let w be such vertex. Since G contains even degree vertices, each vertex in H has even degree. Since all vertices are of even degree, beginning at w, construct s simple path in H, by choosing the vertex as long as possible. This path must terminate at w. Next form a circuit in G, by incorporating circuit in H with the original circuit in G, via w. Continue this process until all the edges have been used.

This produces the Euler circuit

Hence the theorem

10. A connected graph G has an Euler path, but not Euler circuit, iff it has exactly two odd degree vertices

    Proof:Let us assume that the connected graph G has an Euler path.

    To prove : G is having two odd degree vertices

    In the Euler path, every vertex in the middle is associated with two edges and since there is only one edge associated with each end vertices of the path. These end vertices must be of odd degree and other vertices must be of even degree

Conversely, let us assume that the connected graph G has exactly two odd degree vertices

To prove : G has Euler path

Let v_i, v_j be the only odd degree vertices in the connected graph. Suppose that v_i and v_j are not adjacent. If we adjoint a new edge e_{ij} to the edge set of G, then all vertices in the enlarged graph are of even degree. Hence it contains Euler circuit. By deleting the edge e_{ij} from the Euler circuit, we can get Euler path between v_i and v_j

11. Let G be a graph with exactly two vertices of odd degree then there is a path between those two vertices.

<u>Proof</u>: Let G be a graph and u and v be two odd degree vertices. Let us consider two cases.

Case (i) : Let G be a connected graph. Since G is connected, there exists a path between every pair of vertices. Hence there is a path between u and v

Case (ii) : Let G be a disconnected graph. Let G_1 be a connected component of G such that $u \in V(G_1)$.Since G_1 is a connected graph, it must contain even number of odd degree vertices. Since u and v are the only two odd degree vertices in G, $v \in V(G_1)$. Hence there is a path between u and v