Scientific Computing
Homework 2
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1

a) $x_1^2 + 2x_2^2 + 3x_3^2 + (x_1 - x_2 + x_3 - 1)^2 + (-x_1 - 4x_2 + 2)^2$

In order to minimize it, we can equate all ~~...~~ the parts of the equation to 0.
~~...~~
We can write this as a linear least squares problem by writing it as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \\ 1 & -1 & 1 \\ -1 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$\Downarrow$ $A \in R^{5\times 3}$ $\quad \downarrow$ $x \in R^3$ $\quad \downarrow$ $b \in R^5$

$\therefore$ The problem can now be expressed as $Ax \simeq b \Leftrightarrow \min_{x \in R^3} \|Ax - b\|_2^2$

b) $(-6x_2+4)^2+(-4x_1+3x_2-1)^2+(x_1+8x_2-3)^2$

In order to minimize it, we can equate all the parts of the equation to 0

We can write this as a linear least squares problem by writing it as

$$\underset{A \in \mathbb{R}^{3\times 2}}{\begin{bmatrix} 0 & -6 \\ -4 & 3 \\ 1 & 8 \end{bmatrix}} \underset{x \in \mathbb{R}^2}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{b \in \mathbb{R}^3}{\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}}$$

$\therefore$ The problem can now be expressed as $Ax \simeq b \Leftrightarrow \min_{x \in \mathbb{R}^2} \|Ax-b\|_2^2$

c) $2(-6x_2+4)^2+3(-4x_1+3x_2-1)^2+4(x_1+8x_2-3)^2$

In order to minimize it, we can equate all the parts of the equation to 0.

We can write this as a linear least squares problem by writing it as

The equation can be written as

$$(-6\sqrt{2}x_2+4\sqrt{2})^2+(-4\sqrt{3}x_1+3\sqrt{3}x_2-\sqrt{3})^2$$
$$+(2x_1+16x_2-6)^2$$

$\therefore$ Linear least squares form:

$$\underbrace{\begin{bmatrix} 0 & -6\sqrt{2} \\ -4\sqrt{3} & 3\sqrt{3} \\ 2 & 16 \end{bmatrix}}_{A\in\mathbb{R}^{3\times 2}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x\in\mathbb{R}^2} = \underbrace{\begin{bmatrix} -4\sqrt{2} \\ \sqrt{3} \\ 6 \end{bmatrix}}_{b\in\mathbb{R}^3}$$

$\therefore$ The problem can now be expressed as $Ax \simeq b \Leftrightarrow \min\limits_{x\in\mathbb{R}^2} \|Ax-b\|_2^2$

d) $x^T x + \|Bx-d\|_2^2$ $(B\in\mathbb{R}^{p\times n}\ d\in\mathbb{R}^p)$

We know that $x^T x = \|x\|_2^2$ for all $x\in\mathbb{R}^n$

$\therefore$ $\|x\|_2^2 + \|Bx-d\|_2^2$

To minimize them

$\|x\|_2^2 \Rightarrow x = 0_n \quad x \in \mathbb{R}^n, 0_n \in \mathbb{R}^n$

$\|Bx - d\|_2^2 \Rightarrow Bx = d \quad B \in \mathbb{R}^{p \times n}, x \in \mathbb{R}^n,$
$d \in \mathbb{R}^n$

$\therefore$ The linear least squares problem can be formulated as

$$\underbrace{\begin{bmatrix} B \\ I_n \end{bmatrix}}_{A \in \mathbb{R}^{(n+p) \times n}} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{x \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} d \\ 0_n \end{bmatrix}}_{b \in \mathbb{R}^{p+n}}$$

$\therefore$ Here I_n is the identity matrix of dimensions $n \times n$ and 0 is null matrix of dimension $n \times 1$

$\therefore$ The problem can be expressed as $Ax \simeq b \Leftrightarrow \min\limits_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$

e) $x^T D x + \|Bx - d\|_2^2$, $D \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{p \times n}$, $d \in \mathbb{R}^p$

D is diagonal matrix with positive diagonal elements

Let $d_1, d_2, \ldots d_n$ be the elements ~~xDx~~ on diagonal of D
Let $x_1, x_2 \ldots x_n$ be elements of x

$\therefore x^T D x = d_1 x_1^2 + d_2 x_2^2 \ldots d_n x_n^2$

$= (\sqrt{d_1}\, x_1)^2 + (\sqrt{d_2}\, x_2)^2 \ldots (\sqrt{d_n}\, x_n)^2$

$\therefore$ ~~We need~~

$\therefore$ Linear least squares for this will be

$$\begin{bmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let $D_2 = \begin{bmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{bmatrix}$

To formulate entire equations as linear least squares, we get

$$\underbrace{\begin{bmatrix} B \\ D_2 \end{bmatrix}}_{A \in \mathbb{R}^{(p+n)\times n}} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{x \in \mathbb{R}^n} = \underbrace{\begin{bmatrix} d \\ 0 \end{bmatrix}}_{b \in \mathbb{R}^{(p+n)}}$$

Here 0 is null matrix of dimensions $n \times 1$

The problem can be expressed as a $Ax \cong b \Longleftrightarrow \min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$

2 $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \simeq \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

~~To formulate it as~~

Here $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Neccessary condition for existence of minimum is

$$\nabla f_{(x^*)} = 0$$

$$f(x) = (b - Ax)^T (b - Ax)$$

$$= b^T b - b^T A x - x^T A^T b + x^T A^T A x$$

$$\therefore \nabla f = -A^T b - A^T b + (A^T A + (A^T A)^T) x$$

$\therefore$ As $\nabla f(x^*) = 0$

$$0 = -A^T b + A^T A x \qquad (A^T A = (A^T A)^T)$$

$$A^T A x = A^T b$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad x_1 + x_2 = 2$$
$$x_1 + 2x_2 = 3$$

$$\therefore \quad x_2 = 1$$
$$\therefore \quad x_1 = 1$$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ The minimum exists

We will verify it using the sufficiency condition ($A^T A$ is positive definite)

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

∴ To prove we will show

$x^T A^T A x > 0 \quad \forall x \in \mathbb{R}^n$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

$$= \begin{bmatrix} x_1 + x_2 & x_1 + 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1(x_1 + x_2) + x_2(x_1 + 2x_2)$$

$$= x_1^2 + 2x_2^2 + 2x_1x_2$$

$$= (x_1 + x_2)^2 + x_2^2$$

∴ As both values are squares, they will be positive as long as ~~x≠0~~ $x \neq 0$

∴ $A^T A$ is positive definite

The solution is a minimum through sufficiency condition

$$r = b - Ax = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \|r\|_2 = \sqrt{0 + 0 + 1^2} = 1$$

$\therefore$ Euclidean norm of minimum residual ~~rao~~ vector is 1

3

$A \in R^{m \times n}, m \geq n$

A has linearly independant columns

$\therefore$ rank $(A) = n$

a) We need to show

$$\begin{bmatrix} 1 & A \\ A^T & 0 \end{bmatrix}$$ is nonsingular

$A^T A \in R^{n \times n}$

$\text{rank}(A^T A) = \min(\text{rank}(A^T), \text{rank}(A))$
$= n$

$\therefore$ $A^T A$ is also a full rank matrix.

$\therefore$ A^TA is non singular and has an inverse.

~~Let C be~~

Let us take a matrix

$C \in \mathbb{R}^{m\times m}$ such that

$$c_i = e_i - A(A^TA)^{-1}A^T e_i \qquad (e_i \in \mathbb{R}^n)$$

and $D \in \mathbb{R}^{n\times m}$ such that

$$d_i = (A^TA)^{-1}A^T e_i \qquad (e_i \in \mathbb{R}^M)$$

Here e_i is the i^{th} standard basis

$$\therefore \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} c_i \\ d_i \end{bmatrix} = \begin{bmatrix} I c_i + A d_i \\ A^T c_i \end{bmatrix}$$

$$c_i = e_i - A(A^T A)^{-1} A^T e_i$$

$$\therefore A^T c_i = A^T e_i - (A^T A)(A^T A)^{-1} A^T e_i$$

$$= A^T e_i - A^T e_i$$

$$\therefore A^T e_i = 0$$

$$c_i + A d_i$$

$$= e_i - A(A^T A)^{-1} A^T e_i + A(A^T A)^{-1} A^T e_i$$

$$= \cancel{e_i}$$

$$\therefore c_i + A d_i = e_i$$

$$\therefore \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} c_i \\ d_i \end{bmatrix} = \begin{bmatrix} e_i \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} =$$

$$= \begin{bmatrix} IC + AD \\ A^T C \end{bmatrix}$$

$$= \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} C & A(A^TA)^{-1} \\ D & -(A^TA)^{-1} \end{bmatrix}$$

$$\therefore \begin{bmatrix} C+AD & A(A^TA)^{-1} - A(A^TA)^{-1} \\ A^TC & (A^TA)(A^TA)^{-1} \end{bmatrix}$$

$$\therefore \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\therefore \begin{bmatrix} C & A(A^TA)^{-1} \\ D & -(A^TA)^{-1} \end{bmatrix}$ is the inverse of $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$

$\therefore$ As it has an inverse, its must be nonsingular.

$\therefore$ Thus Proved

b) Forward direction:

Given: x is solution of least squares $Ax \simeq b$

$$\hat{x} = b - Ax$$
$$\hat{y} = x$$

To prove:

$\hat{x}, \hat{y}$ is solution of set of linear equations

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Proof:

From normal equation

$$A^T A x = A^T b$$

$\therefore$ For minimising residual, it must be perpendicular to A's range space

$$\therefore A^T (b - Ax) = 0$$

$$\therefore \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

$$= \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} b - Ax \\ x \end{bmatrix}$$

$$= \begin{bmatrix} (b - Ax) I + Ax \\ (b - Ax) A^T + 0 \end{bmatrix}$$

$$= \begin{bmatrix} b - Ax + Ax \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$\therefore$ Thus Proved

Backwards Direction

Given: $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$

To prove:

$$\hat{x} = b - Ax$$
$$\hat{y} = x$$

where x is solution of $Ax \simeq b$

Proof :

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$\therefore I\hat{x} + A\hat{y} = b$

$A^T\hat{x} = 0$

$\therefore \hat{x} = b - A\hat{y}$

$A^T\hat{x} = 0$

$\therefore A^T(b - A\hat{y}) = 0$

$\therefore A^Tb - A^TA\hat{y} = 0$

$\therefore A^Tb = A^TA\hat{y}$

A^TA is non singular as it is ~~of rank n~~ full rank matrix (A is also full rank matrix)

$\therefore \hat{y}$ is solution of $Ax \simeq b$

$\therefore \hat{y} = x$

$\therefore \hat{x} = b - A\hat{y}$

$= b - Ax$

$\therefore$ Thus Proved

4

a) We need to find $\|b\|_2$, and we are given $[A\ b] = QR$

The first column in $A b$ will be represented by QR_1 (R_i = i^{th} column of R)

second column by QR_2

Last column which is b will be represented by QR_{n+1}

$$\therefore b = [q_1 \dots q_{n+1}] \begin{bmatrix} R_{1,n+1} \\ \vdots \\ R_{n+1,n+1} \end{bmatrix}$$

$$\therefore b = \sum_{i=1}^{n+1} q_i R_{i,n+1}$$

If $\{e_1, e_2 \dots e_n\}$ is an orthonormal list of vectors, and $a_1, a_2 \dots a_n$ are scalars then

$\|a_1 e_1 + a_2 e_2 \dots a_n e_n\|_2 = \sqrt{|a_1|^2 + |a_2|^2 + \dots |a_n|^2}$

$\therefore \|b\|_2 = \|\sum_{i=1}^{n+1} R_{i,n+1} q_i\|_2$

$$= \sqrt{|R_{1,n+1}|^2 + |R_{2,n+1}|^2 \ldots + |R_{n+1,n+1}|^2}$$

$$= \sqrt{\sum_{i=1}^{n+1} |R_{i,n+1}|^2}$$

$$\therefore \|b\|_2 = \sqrt{\sum_{i=1}^{n+1} |R_{i,n+1}|^2}$$

$\therefore \|b\|_2$ can be computed from just the last ~~[illegible]~~ column of R

b) $\|A\hat{x}\|_2$

Let QR factorization of A be $Q_A R_A$

By normal equation

$$A^T A \hat{x} = A^T b$$

$$\therefore \hat{x} = (A^T A)^{-1} A^T b$$

$$\therefore \hat{x} = (R_A^T Q_A^T Q_A R_A)^{-1} R_A^T Q_A^T b$$

$$\therefore \hat{x} = (R_A^T R_A)^{-1} R_A^T Q_A^T b$$

Multiply both sides with A

$$A\hat{x} = A(R_A^T R_A)^{-1} R_A^T Q_A^T b$$

$$= Q_A R_A (R_A^T R_A)^{-1} R_A^T Q_A^T b$$

$$= Q_A (R_A R_A^{-1})(R_A^{-T} R_A^T) Q_A^T b$$

$$= Q_A (I)(I) Q_A^T b$$

$$= Q_A Q_A^T b$$

$$= I b$$

(Q_A is orthrogonal)

$$\therefore \|A\hat{x}\|_2 = \|b\|_2$$

$$= \sqrt{\sum_{i=1} |R_{i,n+1}|^2}$$

c) $\|b - A\hat{x}\|_2$

$$\|b\|_2 = \|A\hat{x} + b - A\hat{x}\|_2$$

$\therefore$ As $A\hat{x}$ is orthogonal to $b - A\hat{x}$

$$\therefore \|A\hat{x} + b - A\hat{x}\| = \|A\hat{x}\| + \|b - A\hat{x}\|$$

$$\therefore \|b\| = \|A\hat{x}\| + \|b - A\hat{x}\|$$

$$\therefore \|b - A\hat{x}\| = \|b\| - \|A\hat{x}\|$$

$$= 0$$

$$(\|b\|_2 = \|A\hat{x}\|_2)$$

$$\therefore \|b - A\hat{x}\|_2 = 0$$

$\therefore$ We have calculated all three values by using elements from last column of R

$\therefore$ Thus proved

5

a) $\frac{e^{\alpha t_i+\beta}}{1+e^{\alpha t_i+\beta}} \approx y_i$, where $i=1,2,\ldots m$

Also, $m=90$, $0<y_i<1$ $\forall$ $i=1,\ldots m$

$$e^{\alpha t_i+\beta} = y_i + y_i e^{\alpha t_i+\beta}$$

$$e^{\alpha t_i+\beta} = \frac{y_i}{1-y_i}$$

take Logarithm on both sides

$$\log(e^{\alpha t_i+\beta}) = \log\left(\frac{y_i}{1-y_i}\right)$$

Let $\log\left(\frac{y_i}{1-y_i}\right) = s_i$

$$\therefore \log(e^{\alpha t_i+\beta}) = s_i$$

$$\therefore \alpha t_i+\beta = s_i$$

∴ After appropriate use of Logarithm
∴ We can formulate this as a
linear least squares problem

$$\begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}$$

$$A \in R^{m \times 2} \qquad x \in R^{2 \times 1} \qquad b \in R^{m \times 1}$$

∴ Using $Ax = b$

where $A = \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}$

$x = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

$b = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}$

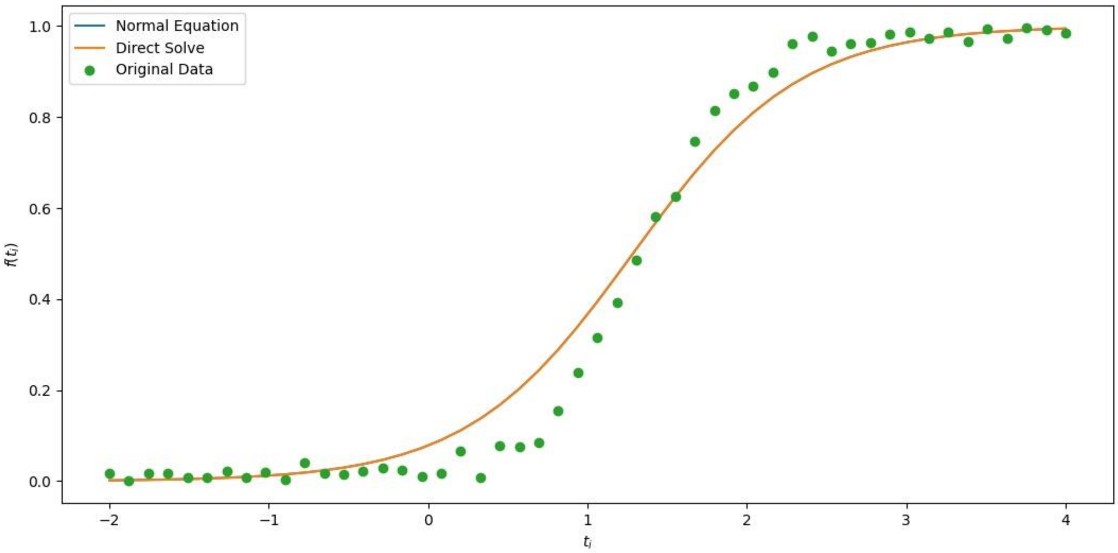
We can find values of the parameters
α, β

∴ Thus solved.

Q5:

b):

Graph:



Output:

```
The error while using the normal equation (np.linalg.solbe) is 6.980329177231976
The error while directly solving (np.linalg.lstsq) is 6.980329177231976
```

Explanation:

As we can see, the error for the normal equation, and directly solving it is exactly the same (6.980329177231976). This is because np.linalg.lstsq also internally uses the normal equation (np.linalg.solve) in order to solve the least squares system. The lines in the graph overlap as well. We can see that the lines fit the data very well.

6

a) $A = \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix}$

$b = \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$

For $Ax \cong b$

The normal equation is

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix} x = \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix}$$

$$\begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10^{-2k} \\ -10^{-2k} \end{bmatrix}$$

$$\therefore \; x_1(1+10^{-2k}) + x_2 = 10^{-2k}$$

$$x_1 + x_2(1+10^{-2k}) = -10^{-2k}$$

$$10^{-2k} x_1 - 10^{-2k} x_2 = 2 \times 10^{-2k}$$

$$\therefore \; x_1 = x_2 + 2$$

$\therefore x_1 = 1$
$x_2 = -1$

$\therefore x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the solution
for the given linear least squares problem

Computers ~~follow~~ have finite precision with an ϵ_m value of approximately 10^{-16}

$\therefore$ ~~consider~~

$\therefore$ For $10^{-2k} < 10^{-16}$ or $k > 8$ underflow will occur.

$A^T A = \begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix}$

$\therefore$ For finite precision, it will become

$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore$ It will become a singular matrix and we will not be able to solve $A^T A x = A^T b$ as we will get error in python code.

Q6:

b):

Output:

```
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 6 : [ 1. -1.]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 7 : [ 1. -1.]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 8 : [ 1.00000001 -1.
00000001]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 9 : [ 1.00000009 -1.
00000009]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 10 : [ 1.0000007 -1.
0000007]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 11 : [ 1.00001954 -1.
00001954]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 12 : [ 0.99990638 -0.
99990638]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 13 : [ 0.99944487 -0.
99944487]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 14 : [ 1.00988041 -1.
00988041]
QR factorization (np.linalg.qr) and traingular solver (scipy.linalg.solve_triangular) for k= 15 : [ 0.89212348 -0.
89212348]
```

This is the output we got by using QR factorisation and triangular solver. We can see that even as we increase the value of k, the output does not deviate that much from the actual output, and is very close to the actual output.

c)

output:

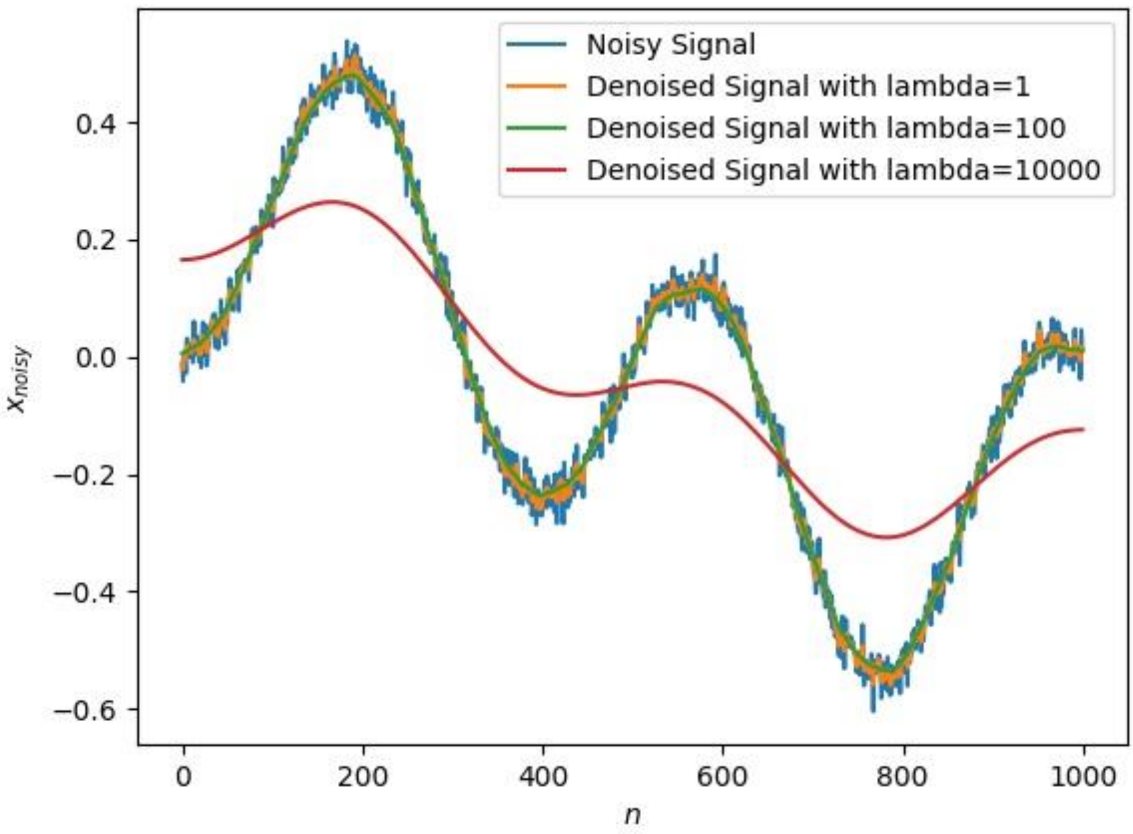
```
Normal equation (np.linalg.solve) for k= 6 : [ 0.99991111 -0.99991111]
Normal equation (np.linalg.solve) for k= 7 : [ 1.00079992 -1.00079992]
The matrix is singular
The matrix is singular
The matrix is singular
The matrix is singular
The matrix is singular
The matrix is singular
The matrix is singular
The matrix is singular
```

We can see that when k >= 8, the matrix A^TA becomes a singular matrix, and its determinant becomes 0. Theoretically, the answer should be x = (1, -1). But practically, we are getting a singular matrix for k >= 8. This is because we have a term $1 + 10^{-2k}$ in matrix, and when k >=8, 10^{-2k} becomes smaller than or equal to 10^{-16}. Because the value of 10^{-2k} drops below the value of machine epsilon, the value of $1 + 10^{-2k}$ becomes 1 because of finite precision. Because of this, the matrix becomes singular, and we do not get the actual solution of the equation.

Thus, on comparing the two methods, we can see that QR decomposition is not affected by singular matrix, and it gives us an answer that I extremely close to the original. This is because it uses the inverse of R for finding the solution, which can never be 0, as it is an upper triangular matrix.

Q7:

Output:



Observations:

- We can see from the legend, that the blue line represents the original noisy data, the orange line represents the denoised signal with λ = 1, the green line represents the denoised signal with λ = 100, and the red line represents the denoised signal with λ = 10000.
- We can see that the λ = 1 curve smoothens the data a little, but it is not very much.
- The λ = 100 curve greatly smoothens the data, and the denoised signal does not deviate from the original signal much either.
- The λ = 10000 curve deviates greatly from the original signal, and thus it cannot be used.
- Thus, we can conclude that the curve with λ = 100 does the best job of smoothing the data, as it properly smoothed the data, and the signal did not deviate from the original data either.
- Thus, we can conclude that as we increase lambda until a certain point, the signal gets smoother, while not deviating from the data either. But, after a certain point, the signal begins to deviate from the actual data, because of which it is of no use.
- Hence, for ensuring proper smoothing, we should use a λ such that the curve is properly smoothed, and it doesn't deviate rom the original data.