Scientific Computing
Homework 3
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1 We have a triangular matrix $A \in \mathbb{R}^{n \times n}$ and we need to prove that its eigenvalues are exactly its diagonal elements

We know that for a triangular matrix, the determinant is equivalent to the product of its diagonal elements.

$\therefore$ For $A \in \mathbb{R}^{n \times n}$

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

Let Λ be eigenvalue of A.
$\therefore$ We can find eigenvalues using formula
$\therefore \det(A - \Lambda I) = 0$

$$\therefore \prod_{i=1}^{n} (a_{ii} - \Lambda) = 0$$

(As ΛI is a diagonal matrix, it won't affect non diagonal entries of A, so det will be $\prod_{i=1}^{n} (a_{ii} - \Lambda)$)

$\therefore (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0$

$\therefore$ At least one of the values inside the bracket on the left side must be 0.

$\therefore \lambda = a_{ii}$ for at least one $i \in \{1, 2, \dots, n\}$

$\therefore$ In order to equate the equations on both sides, λ ~~must be~~ can be $a_{11}, a_{22}, \dots, a_{nn}$

$\therefore$ The eigenvalues of a triangular matrix $A \in \mathbb{R}^{n \times n}$ are exactly its diagonal elements

2. We need to show that given a nondefective matrix $A \in \mathbb{K}^{n \times n}$ its rank is equal to the number of nonzero eigenvalues of A

As A is nondefective it is a diagnolisable matrix (properties of matrix)

$\therefore$ A will be diagnolizable

$\therefore$ $A = PDP^{-1}$ (where P is invertible matrix and D is diagonal matrix)

$\therefore$ rank (A) = rank (PDP^{-1})

$\therefore$ Rank of A will be minimum of rank of P, rank of D, rank of P^{-1}.

$\therefore$ As P, P^{-1} are invertible, $n \times n$ matrices their rank is n.

$\therefore$ As D is a $n \times n$ diagonal matrix its rank will be the number of nonzero diagonal elements

$\therefore$ rank(D) $\leq n$

$\therefore$ D will have minimum rank out of P, D, P^{-1}

$\therefore$ rank(A) = rank(D)

$\therefore$ The rank of A will be equal to the number of nonzero diagonal elements in D

$\therefore$ Rank of A will be equal to the number of nonzero eigenvalues

(For Diagonal matrix, eigenvalues are the diagonal elements. Hence rank of D = number of non zero eigenvalues |

$\therefore$ ~~rank(A)~~ = A and D have same eigenvalues)

$\therefore$ For a nondefective matrix $A \in \mathbb{R}^{n \times n}$, its rank will be equal to number of non zero eigenvalues

3 For $u, v \in \mathbb{R}^n$ such that $u^T v = 1$

(b) and $A = uv^T$

we need to find eigenvalues of A.

We know that in matrix multiplication, rank of product is minimum of the rank of individual matrices

$\therefore$ rank $(A) = \min(\text{rank}(u), \text{rank}(v^T))$

$= \min(1, 1)$

$\therefore$ rank$(A) = 1$

$\therefore$ As ~~A~~ rank of A is not n, it will have 0 as an eigenvalue.

A will be diagonalizable if $uv^T \neq 0$

rank(A) = No of non zero eigenvalues

~~rank~~ Number of nonzero eigenvalues = 1

$\therefore$ Number of zero eigenvalues
$= n-1$

$\therefore$ Let $\lambda = v^T u$ be the nonzero eigenvalue of A

$\therefore$ Au
$= (uv^T)u$
$= uv^Tu$
$= u(v^Tu)$ (matrix multiplic is associative)
$= u\lambda$
$= \lambda u$ (λ is scalar)

$\therefore$ $Au = \lambda u$

$\therefore$ $\lambda = v^T u$ is the nonzero eigenvalue of A

$\therefore$ The eigenvalues of A are
$0, v^T u$

Now we will find out how many iterations does power iteration take to converge to the dominant eigenvalue eigenvector pair for this matrix

$$x_n = \lambda_1^n \left(c_1 v_1 + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1} \right)^n c_i v_i \right)$$

Here (λ_i, v_i) are the eigenvalue and the corresponding eigenvector and x_n is the estimated eigenvector at the n^{th} iteration and $c_i \in \mathbb{R}$ $\forall i \in [1, n]$

We have $\lambda_1 = 1$ and $\lambda_2, \lambda_3 \ldots \lambda_n = 0$

$$\therefore x_n = 1^n \left(c_1 v_1 + \sum_{i=1}^{n} (0)^n c_i v_i \right)$$

$$x_n = 1^n (c_1 v_1)$$

$$x_n = c_1 v_1$$

$\therefore$ As x_n is independant of n, we can see that it will converge in a single iteration

Power iteration converges to dominant eigenvalue - eigenvector pair in a single iteration.

NoteBook

4 We need to show that the eigenvalues of a real symmetric positive definite matrix are all real and strictly greater than 0

Let A be the real symmetric positive definite matrix

Let λ be eigenvalue of A and $x \neq 0$ be the corresponding eigenvector

$$\therefore Ax = \lambda x$$

Multiply both sides with $\bar{x}^T$

$$\bar{x}^T(Ax) = \bar{x}^T \lambda x$$
$$(\bar{x}^T A)x = \lambda(\bar{x}^T x)$$

$$(A^T \bar{x})^T x = \lambda(\bar{x}^T x)$$
$$(\bar{A}\bar{x})^T x = \lambda(\bar{x}^T x)$$ (As A is real symmetric $A = A^T = \bar{A}$)

$$\therefore (\bar{\lambda}\bar{x})^T x = \lambda(\bar{x}^T x)$$

$$\therefore \bar{x}^T \bar{\lambda} x = \lambda(\bar{x}^T x)$$
$$\therefore \bar{\lambda}\bar{x}^T x = \lambda(\bar{x}^T x)$$ (λ is scalar)

$$(\bar{x}^T x)(\bar{\lambda} - \lambda) = 0$$

$\therefore$ As $\bar{x}^T x \neq 0$, $\bar{\lambda} - \lambda = 0$

$\therefore \bar{\lambda} = \lambda$

$\therefore \lambda$ is real

$\therefore$ Eigenvalue of A is real.

As A is ~~ray~~ real symmetric positive definite matrix

$\therefore x^T A x > 0 \quad \forall x \in \mathbb{R}^n$

$A x = \lambda x$

where λ is eigenvalue of A and x is corresponding eigenvector

$\therefore$ Multiply by x^T on both sides

$x^T A x = x^T \lambda x$

$\therefore x^T A x = \lambda (x^T x)$

$\therefore x^T A x = \lambda \|x\|^2$

$\therefore \lambda = \dfrac{x^T A x}{\|x\|^2}$

$\therefore x^T A x > 0$ (property of positive definite matrix)

$||x||^2 > 0$ (norm of non zero matrix is always +ve)

$\therefore \lambda > 0$

$\therefore$ All the eigenvalue of a real symmetric positive definite matrix are all real and strictly greater than 0

$\therefore$ Thus Proved

5 We have $A \in \mathbb{R}^{n \times n}$ which is a symmetric and positive definite matrix and non defective

Exponential of A is

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

We need to provide an alternate expression for e^A in terms of the eigenvalues of A

As A is non defective and symmetric it is also a diagonalizable matrix

$\therefore$ $A = PDP^{-1}$ (where P is an invertible $n \times n$ matrix and D is a diagonal $n \times n$ matrix)

$\therefore$ ~~The eigenvalue~~

$\therefore$ The diagonal elements of D will be the eigenvalues

$$\therefore e^{D} := \sum_{n=0}^{\infty} \frac{1}{n!} D^{n}$$

$$\therefore e^{D} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{d_1^{n}}{n!} & & & \\ & \sum_{n=0}^{\infty} \frac{d_2^{n}}{n!} & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{d_n^{n}}{n!} \end{bmatrix}$$

$$e^{D} = \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix}$$

$$\left(\sum_{n=0}^{\infty} \frac{d_i^{n}}{k!} = e^{d_i} \right)$$

$$\therefore e^{D} = \sum_{n=0}^{\infty} \frac{1}{n!} D^{n}$$

For diagnolizable matrix A

$$A^n = P D^n P^{-1}$$

$$D^n = P^{-1} A^n P$$

$$\therefore e^D = \sum_{n=0}^{\infty} \frac{1}{n!} P^{-1} A^n P$$

$$e^D = P^{-1} \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) P$$

$$e^D = P^{-1} \left(e^A \right) P$$

$$e^A = P e^D P^{-1}$$

$$e^A = P \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix} P^{-1}$$

where $d_1, d_2, \ldots d_n$ are the diagonal elements of D

Here $P = [v_1 \; v_2 \; \ldots \; v_n]$ where ~~...~~ v_i is ith eigenvector of ~~...~~ of A

$$e^{A} = [v_1 \ldots v_n] \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(As eigenvectors of real symmetric matrix are orthogonal, P is orthogonal $P^{-1} = P^T$)

$$e^{A} = [v_1 \ldots v_n] \begin{bmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots e^{\lambda_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(As diagonal elements of diagonal matrix are equal to the eigenvalues)

Q6)

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The results from implementing normalized power iteration are:
The magnitude of the largest eigenvalue is: 11.0
The corresponding eigenvector for the largest eigenvalue is: [0.5  1.   0.75]

The results from implementing inverse iteration are:
The magnitude of the smallest eigenvalue is: 1.9999999999999996
The corresponding eigenvector for the smallest eigenvalue is: [-0.2 -0.4  1. ]

Comparison:

The results from using np.linalg.eig are:
The eigenvalues from np.linalg.eig are: [11. -2. -3.]
The eigenvectors from np.linalg.eig are: [[ 3.71390676e-01  1.82574186e-01  2.17732649e-17]
 [ 7.42781353e-01  3.65148372e-01 -5.54700196e-01]
 [ 5.57086015e-01 -9.12870929e-01  8.32050294e-01]]
```

- We can see that we have gotten the correct value of largest eigenvalue from normalised power iteration and the smallest eigenvalue magnitude from inverse iteration and the correct eigenvectors as well. The eigenvectors also have approximately the same values.
- In inverse iteration method, we have gotten the correct magnitude, but the sign is different. This is because we are using the infinite vector norm, which always gives a positive output. The eigenvector we have received has approximately the same magnitude, but it has the different sign. This does not matter because we can just multiply it by -1, which would give us the same answer.

Q7)

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The results from implementing shifted inverse iteration are:
The eigenvalue is: 2.133074475348525
The corresponding eigenvector for the eigenvalue is: [-0.60692002  1.          0.34691451]

Comparison:

The results from using np.linalg.eigh are:
The eigenvalues from np.linalg.eigh are: [0.57893339 2.13307448 7.28799214]
The eigenvectors from np.linalg.eigh are: [[-0.0431682  -0.49742503 -0.86643225]
 [-0.35073145  0.8195891  -0.45305757]
 [ 0.9354806   0.28432735 -0.20984279]]
```

We can tell that shifted inverse iteration method can compute the eigenvalue nearest to 2. The eigenvector we have received has approximately the same magnitude as the eigenvector we got from np.linalg.eigh function.

Q8)

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The results from implementing Rayleigh quotient iteration are:
The eigenvalue is: -2.000000000000002
The corresponding eigenvector for the eigenvalue is: [ 0.2  0.4 -1. ]
The rate of convergence is: 0.0009552679413376446

Comparison:

The results from using np.linalg.eig are:
The eigenvalues from np.linalg.eig are: [11. -2. -3.]
The eigenvectors from np.linalg.eig are: [[ 3.71390676e-01  1.82574186e-01  2.17732649e-17]
 [ 7.42781353e-01  3.65148372e-01 -5.54700196e-01]
 [ 5.57086015e-01 -9.12870929e-01  8.32050294e-01]]
```

- We can see that we have gotten the same eigenvalue from Rayleigh quotient method as from np.linalg.eig. The eigenvector also has approximately the same magnitude.
- The convergence rate is approximately 0.00095. It is very low, because Rayleigh quotient iteration converges at -2 and not at 11.

Q9)

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Using the matrix A from Q6:
The eigenvalues from the QR algorithm are: [11.000000000000004, -3.000000000000001, -1.9999999999999991]
The eigenvalues using np.linalg.eig are: [11. -2. -3.]

Using the matrix A from Q7:
The eigenvalues from the QR algorithm are: [7.287992138960421, 2.133074475348525, 0.5789333856910526]
The eigenvalues using np.linalg.eig are: [7.28799214 2.13307448 0.57893339]
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- We can see that the eigenvalues we got from QR algorithm and from np.linalg.eig are the same in the matrices from both the questions.
- Therefore, we can see that our algorithm is correct.