Scientific Computing
Homework 4
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2020326

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a) The data points are
$\{(-2, 15), (0, -1), (1, 0), (3, -2)\}$

Monomial basis:
Let the polynomial be $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

Substitute the values of x_i

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$$a_0 - 2a_1 + 4a_2 - 8a_3 = 15$$
$$a_0 = -1$$
$$a_0 + a_1 + a_2 + a_3 = 0$$
$$a_0 + 3a_1 + 9a_2 + 27a_3 = -2$$

$a_0 = -1$

$a_1 = -8/15$

$a_2 = 34/15$

$a_3 = -11/15$

$\therefore$ The required polynomial as calculated by monomial basis is

$$-1 - \frac{8}{15}x + \frac{34}{15}x^2 - \frac{11}{15}x^3$$

It can also be expressed as

$$-1 - 0.533x + 2.266x^2 - 0.733x^3$$

b) Lagrange basis

Let the required polynomial be $P(x)$

$$\therefore P(x) = y_1\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$+ y_2\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$+ y_3\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$+ y_4\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$p(x) = \frac{15(x)(x-1)(x-3)}{(-2)(-3)(-5)}$$

$$+ \frac{(-1)(x+2)(x-1)(x-3)}{(2)(-1)(-3)}$$

$$+ (0)$$

$$+ \frac{(-2)(x+2)(x)(x-1)}{(5)(3)(2)}$$

$$= \frac{-1}{30}\left(18(x)(x-1)(x-3) + 5(x-1)(x+2)(x-3) + 2(x)(x-1)(x+2)\right)$$

$$= -0.733x^3 + 2.266x^2 - 0.533x - 1$$

$\therefore$ This is the required polynomial as calculated by Lagrange basis

c) Newton basis

Let the required polynomial be $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$\therefore$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (x_2-x_1) & 0 & \cdots & 0 \\ 1 & (x_3-x_1) & (x_3-x_2)(x_2-x_1) & & 0 \\ 1 & (x_4-x_1) & (x_4-x_1)(x_4-x_2) & & (x_4-x_1)(x_4-x_2)(x_4-x_3) \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix} =$$

$\therefore$ Substitute values

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 5 & 15 & 30 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

$a_0 = 15$

$\therefore$ $a_0 + 2a_1 = -1$

$a_0 + 3a_1 + 3a_2 = 0$

$a_0 + 5a_1 + 15a_2 + 30a_3 = -2$

$a_0 = 15$

$a_1 = -8$

$a_2 = 3$

$a_3 = \frac{-22}{30}$

∴ The required polynomial is

$$15 - 8(x+2) + 3(x+2)(x) - \frac{22}{30}(x+2)(x)(x-1)$$

$$= 15 - 8x - 16 + 3x^2 + 6x - \frac{22}{30}x^3 - \frac{22}{30}x^2 + \frac{44}{30}x$$

$$= -0.733x^3 + 2.266x^2 - 0.533x - 1$$

∴ This is the required polynomial.

d Monomial basis:

$-0.733x^3 + 2.266x^2 - 0.533x - 1$

Lagrange basis:

$-0.733x^3 + 2.266x^2 - 0.533x - 1$

Newton basis:

$-0.733x^3 + 2.266x^2 - 0.533x - 1$

We can see that all the polynomials ~~have~~ are the same despite what ways we use.

2 $$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

$$\int_0^1 \sqrt{x}\log x\,dx = -\frac{4}{9}$$

Midpoint Rule:

$$\int_a^b f(t)\,dt \approx (b-a)\,f\left(\frac{a+b}{2}\right)$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} \approx (1-0)\,f\left(\frac{1}{2}\right)$$

$$\approx (1)\left(\frac{1}{1+\frac{1}{4}}\right)$$

$$\approx \frac{1}{5/4}$$

$$\int_0^1 \frac{dx}{1+x^2} \approx \frac{4}{5}$$

$$\int_0^1 \sqrt{x}\log x\,dx \approx (1-0)\,f\left(\frac{1}{2}\right)$$

$$\approx (1)\left(\sqrt{\frac{1}{2}}\log\left(\frac{1}{2}\right)\right)$$

$$\int_0^1 \sqrt{x}\log x\,dx \approx \frac{-\log 2}{\sqrt{2}}$$

Trapezoid Rule:

$$\int_a^b f(t)\,dt \approx \left(\frac{b-a}{2}\right)\left(f(a)+f(b)\right)$$

$$\int_0^1 \frac{dx}{1+x^2} \approx \left(\frac{1-0}{2}\right)\left(f(0)+f(1)\right)$$

$$\approx \left(\frac{1}{2}\right)\left(1+\frac{1}{2}\right)$$

$$\int_0^1 \frac{dx}{1+x^2} \approx \frac{3}{4}$$

$$\int_0^1 \sqrt{x}\,\log x\,dx \approx \left(\frac{1-0}{2}\right)\left(f(0)+f(1)\right)$$

$$\approx \left(\frac{1}{2}\right)\left(0+0\right)$$

$$\int_0^1 \sqrt{x}\,\log x\,dx \approx 0$$

Simpsons Rule

$$\int_a^b f(x)\,dx = \left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \left(\frac{1-0}{6}\right)\left(\frac{1}{1} + \frac{16}{5} + \frac{1}{2}\right)$$

$$= \left(\frac{1}{6}\right)\left(\frac{10+32+5}{10}\right)$$

$$= \frac{47}{60}$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{47}{60}$$

$$\int_0^1 \sqrt{x}\log x\,dx = \left(\frac{1-0}{6}\right)\left(0 + \frac{4}{\sqrt{2}}\log\left(\frac{1}{2}\right) + 0\right)$$

$$= \left(\frac{1}{6}\right)\left(\frac{-4}{\sqrt{2}}\log(2)\right)$$

$$\int_0^1 \sqrt{x}\log x\,dx = -\frac{\sqrt{2}}{3}\log 2$$

2 point Gaussian Quadrature Rule :-

For range of $(-1, 1)$, gaussian points will be $\pm \frac{1}{\sqrt{3}}$

∴ Applying change limits

$$\int_a^b f(x)\,dx = \frac{(b-a)}{(\beta-\alpha)} \sum_{i=1}^{n} w_i f\left(\frac{(b-a)x_i + a\beta - b\alpha}{\beta - \alpha} \right)$$

$\alpha = 1$

$\beta = 1$

$w_1 = 1$

$w_2 = 1$

$x_1 = +1/\sqrt{3}$

$x_2 = -1/\sqrt{3}$

$$\int \frac{dx}{1+x^2} = \frac{1}{2}\left(f\left(\frac{1+1/\sqrt{3}}{2} \right) + f\left(\frac{1-1/\sqrt{3}}{2} \right) \right)$$

$$= \frac{1}{2}\left(f\left(\frac{1+\sqrt{3}}{2\sqrt{3}} \right) + f\left(\frac{\sqrt{3}-1}{2\sqrt{3}} \right) \right)$$

$$= \frac{1}{2}\left(\frac{1}{1+\left(\frac{1+\sqrt{3}}{2\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)^2}\right)$$

$$= \frac{6}{16+2\sqrt{3}} + \frac{6}{16-2\sqrt{3}}$$

$$= \frac{192}{244}$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.78688$$

$$\int_0^1 \sqrt{x}\log x\, dx = \frac{1}{2}\left(f\left(\frac{\frac{1}{\sqrt{3}}+1}{2}\right) + f\left(\frac{1-\frac{1}{\sqrt{3}}}{2}\right)\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{1+\sqrt{3}}{2\sqrt{3}}}\log\left(\frac{1+\sqrt{3}}{2\sqrt{3}}\right) + \sqrt{\frac{\sqrt{3}-1}{2\sqrt{3}}}\log\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)\right)$$

$$= \frac{1}{2}\left(\sqrt{0.78}\log(0.78) + \sqrt{0.21}\log(0.21)\right)$$

$$= \frac{1}{2}\left(-0.2112 - 0.702\right)$$

$$= -0.4566$$

$$\therefore \int_0^1 \sqrt{x}\log x\, dx = -0.4566$$

$\therefore$ We have computed all of the required integrals

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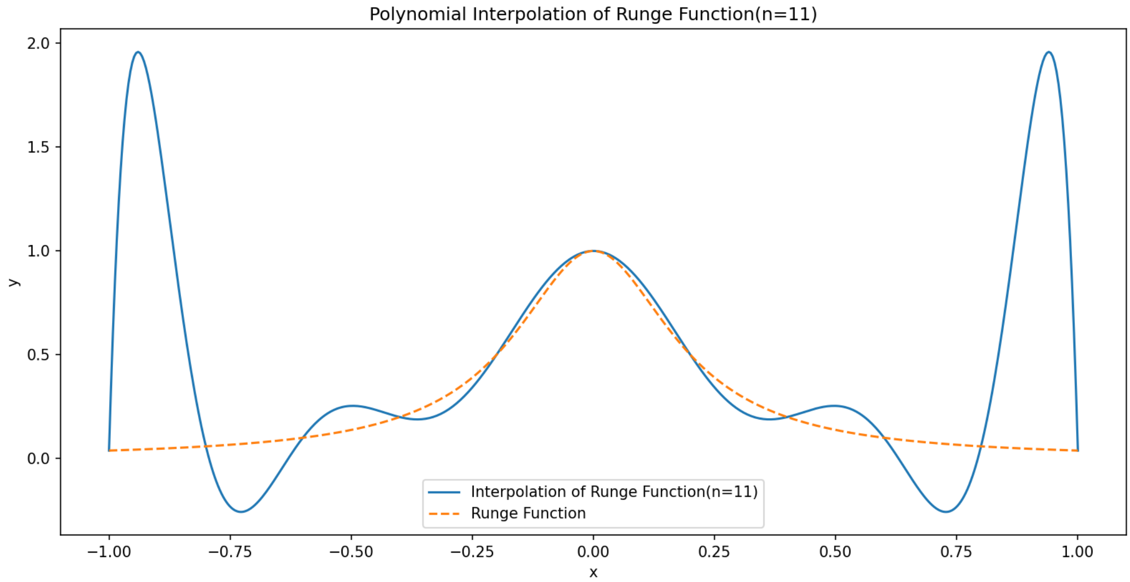
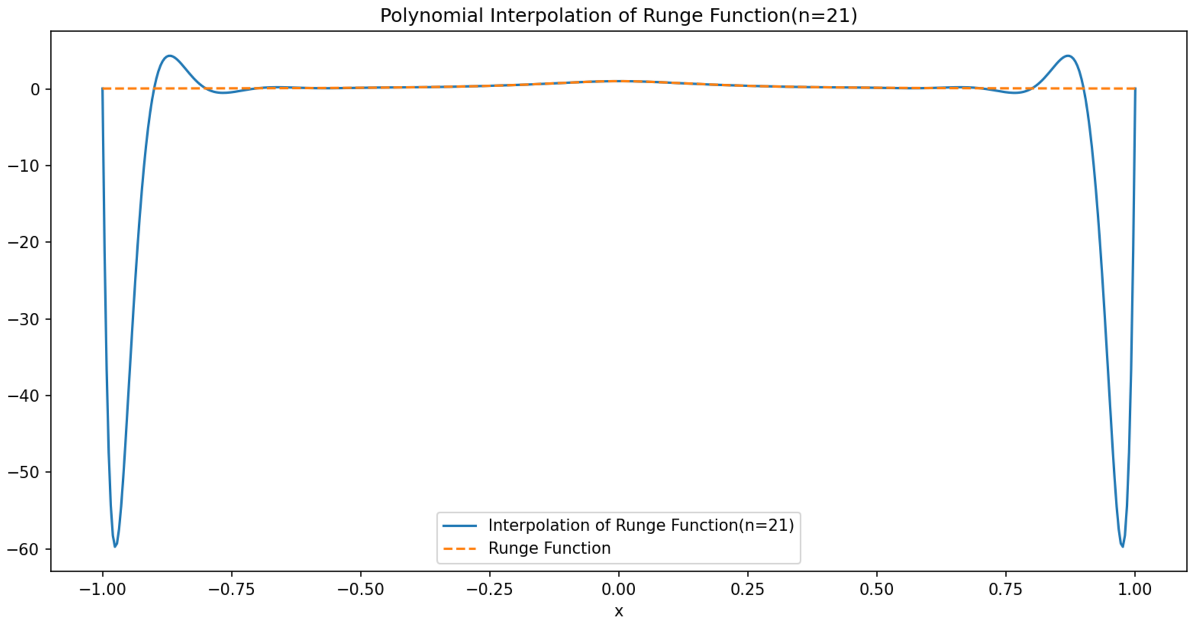
3.

a)

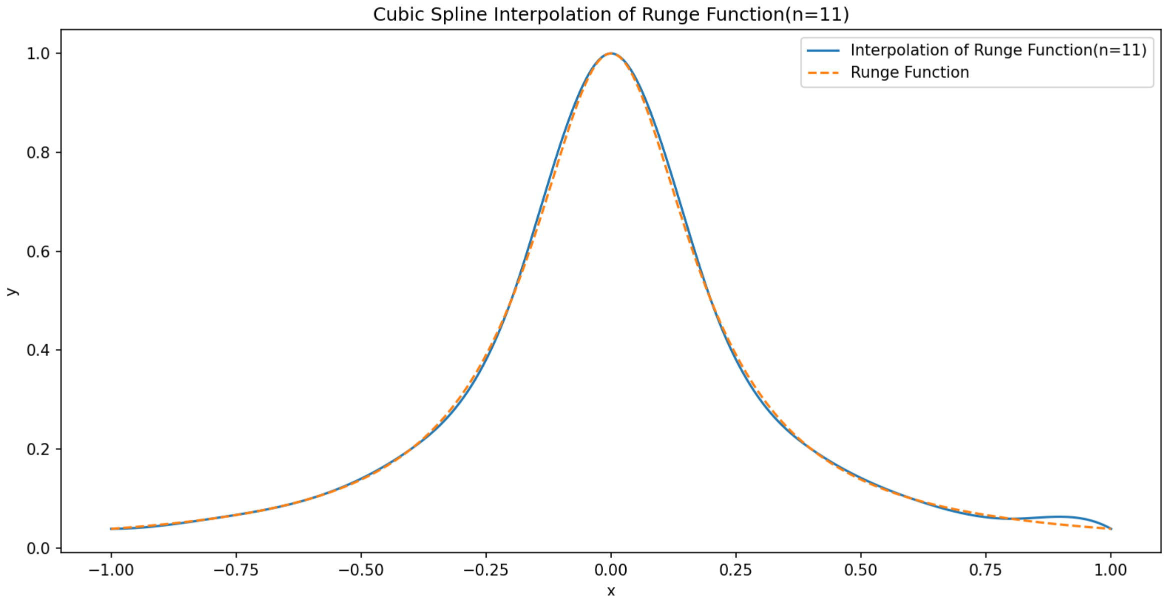


When we use 11 equidistant nodes, we can see that the polynomial interpolations interpolates the Runge function almost correctly except at the edges. This phenomenon is known as Runge's phenomenon

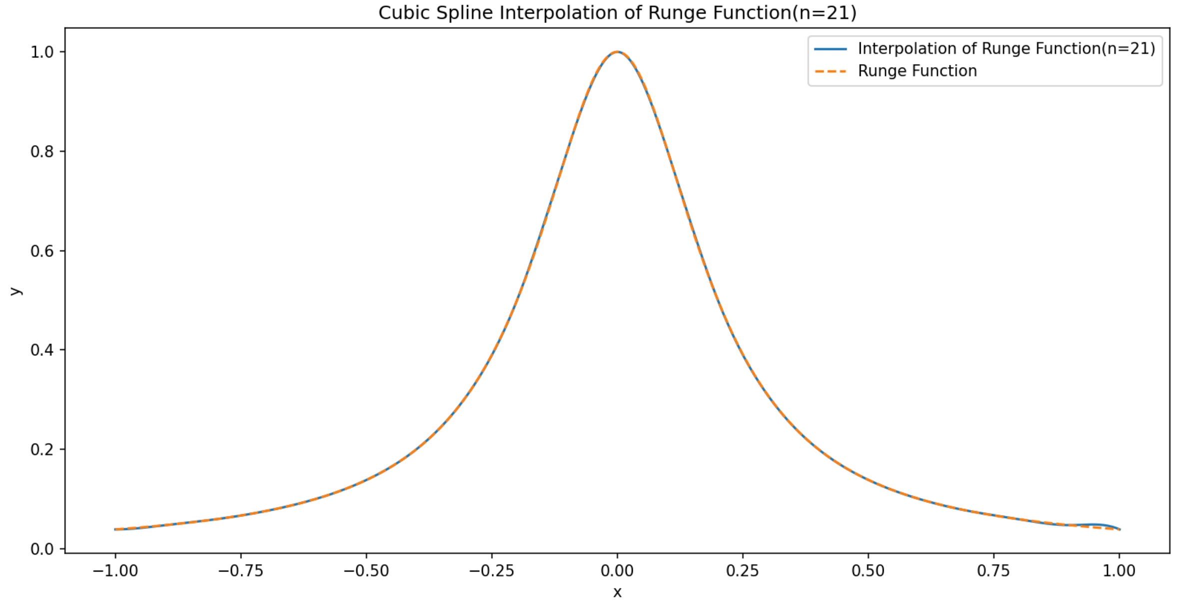


When we use 21 equidistant nodes, we can see that the polynomial interpolations interpolates the Runge function almost correctly except at the edges. This phenomenon is known as Runge's phenomenon

b)



When we use 11 equidistant nodes, we can see that cubic spline interpolation almost correctly interpolates the Runge function.



When we use 21 equidistant nodes, we can see that cubic spline interpolation almost correctly interpolates the Runge function.

4.

```
For n= 2 :
The interpolated value is 21.589657875842885
The relative error is 2.1484901032945363
For n= 4 :
The interpolated value is -9.76123404248495
The relative error is 0.4807383813951242
For n= 6 :
The interpolated value is -18.478977278578718
The relative error is 0.016986621765830726
For n= 8 :
The interpolated value is -19.340167311023777
The relative error is 0.028825508977831458
For n= 10 :
The interpolated value is -19.21212160759043
The relative error is 0.022013950220935635
For n= 12 :
The interpolated value is -19.054556800438228
The relative error is 0.013632084112519826
For n= 14 :
The interpolated value is -18.95441593250772
The relative error is 0.0083049595969341
For n= 16 :
The interpolated value is -18.895837167333262
The relative error is 0.005188785526322299
For n= 18 :
The interpolated value is -18.86138945489323
The relative error is 0.0033562943842195343
For n= 20 :
The interpolated value is -18.840550524795418
The relative error is 0.002247740227492265
```

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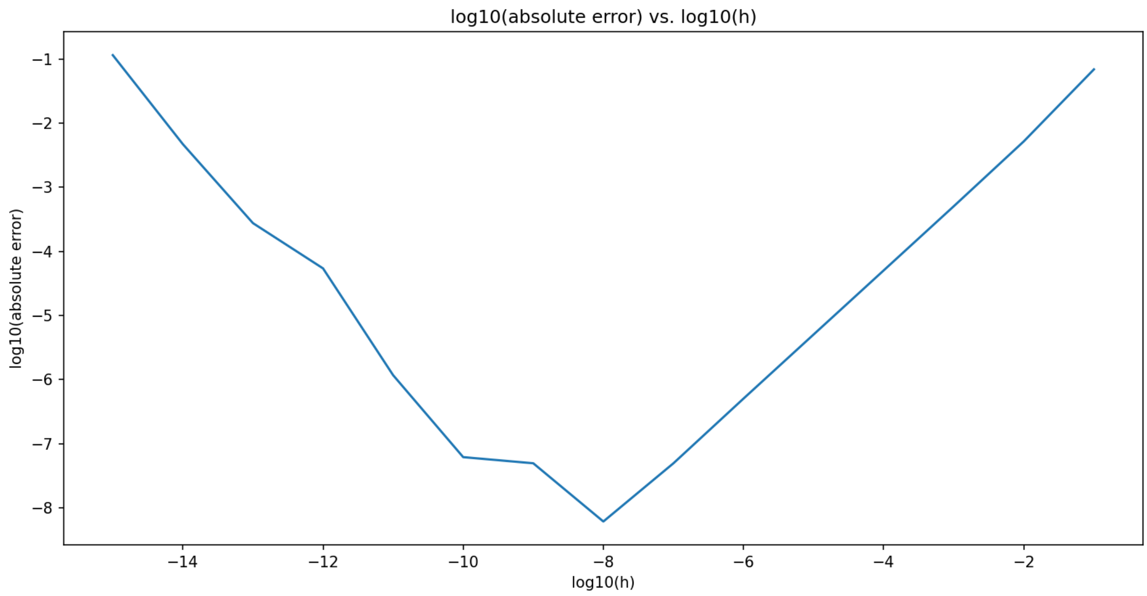
For n= 22 :
The interpolated value is -18.827512567308137
The relative error is 0.0015541690172660926
For n= 24 :
The interpolated value is -18.81908029609953
The relative error is 0.0011056033157123724
For n= 26 :
The interpolated value is -18.8134567401391
The relative error is 0.0008064508973174943
For n= 28 :
The interpolated value is -18.80960091017635
The relative error is 0.000601334976645186
For n= 30 :
The interpolated value is -18.806890752169185
The relative error is 0.00045716457489 57297
For n= 32 :
The interpolated value is -18.804943245212424
The relative error is 0.00035356439379052716
For n= 34 :
The interpolated value is -18.803515860409295
The relative error is 0.0002776327912883191
For n= 36 :
The interpolated value is -18.802451029313193
The relative error is 0.00022098770767532572
For n= 38 :
The interpolated value is -18.80164395055663
The relative error is 0.00017805409706317905
For n= 40 :
The interpolated value is -18.801023413379088
The relative error is 0.00014504380985840386
```

```
For n= 42 :
The interpolated value is -18.800540080502145
The relative error is 0.00011933228497190141
For n= 44 :
The interpolated value is -18.80015915750591
The relative error is 9.906858770490915e-05
For n= 46 :
The interpolated value is -18.79985570517074
The relative error is 8.292604363267271e-05
For n= 48 :
The interpolated value is -18.79961158078952
The relative error is 6.993952770848358e-05
For n= 50 :
The interpolated value is -18.799413404866478
The relative error is 5.9397300145894976e-05
For n= 52 :
The interpolated value is -18.799251186009784
The relative error is 5.076785578173084e-05
For n= 54 :
The interpolated value is -18.79911737605008
The relative error is 4.3649659869378575e-05
For n= 56 :
The interpolated value is -18.799006211148818
The relative error is 3.7736097474444124e-05
For n= 58 :
The interpolated value is -18.798913246012496
The relative error is 3.2790695392081886e-05
For n= 60 :
The interpolated value is -18.79883502040778
The relative error is 2.862938198180036e-05
```

```
For n= 62 :
The interpolated value is -18.798768817555782
The relative error is 2.5107634635661933e-05
For n= 64 :
The interpolated value is -18.7987124871472
The relative error is 2.2111064836170353e-05
```

We can see that the composite Gaussian Quadrature approximates the integrals well. We can see that the best relative error is about 2.2111064836170353e-05 when the interpolated value is about -18.7987124871472.

5.



We can see that the error reaches its smallest value of 10^{-8} when h is log10(h) = -8. We can see that loss decreases uptil this point and then increases after this point. This is because as 10^h keeps on getting smaller, the deviation begins to get rounded off and we begin to get incorrect results.

```
For h= 0.1 :
log10(h)= -1.0
The forward difference approximation is -0.2589437504200143
The exact value is -0.3283496313634305
The absolute error is -1.1586037290687068
For h= 0.01 :
log10(h)= -2.0
The forward difference approximation is -0.32312262868096076
The exact value is -0.3283496313634305
The absolute error is -2.2817472770245946
For h= 0.001 :
log10(h)= -3.0
The forward difference approximation is -0.3278426595997308
The exact value is -0.3283496313634305
The absolute error is -3.295016228459381
For h= 0.0001 :
log10(h)= -4.0
The forward difference approximation is -0.3282990900810301
The exact value is -0.3283496313634305
The absolute error is -4.296353742781166
For h= 1e-05 :
log10(h)= -5.0
The forward difference approximation is -0.3283445787927164
The exact value is -0.3283496313634305
The absolute error is -5.296487599529581
```

```
For h= 1e-06 :
log10(h)= -6.0
The forward difference approximation is -0.32834912611079403
The exact value is -0.3283496313634305
The absolute error is -6.296491411621148
For h= 1e-07 :
log10(h)= -7.0
The forward difference approximation is -0.32834958196836794
The exact value is -0.3283496313634305
The absolute error is -7.306316460362023
For h= 1e-08 :
log10(h)= -8.0
The forward difference approximation is -0.3283496252670659
The exact value is -0.3283496313634305
The absolute error is -8.21492906885509
For h= 1e-09 :
log10(h)= -9.0
The forward difference approximation is -0.32834968077821713
The exact value is -0.3283496313634305
The absolute error is -7.306143075363405
For h= 1e-10 :
log10(h)= -10.0
The forward difference approximation is -0.32834956975591467
The exact value is -0.3283496313634305
The absolute error is -7.21036630283281
For h= 1e-11 :
log10(h)= -11.0
The forward difference approximation is -0.32834845953289005
The exact value is -0.3283496313634305
The absolute error is -5.9311351875282865
```

```
For h= 1e-12 :
log10(h)= -12.0
The forward difference approximation is -0.3284039706841213
The exact value is -0.3283496313634305
The absolute error is -4.264885795106051
For h= 1e-13 :
log10(h)= -13.0
The forward difference approximation is -0.32862601528904634
The exact value is -0.3283496313634305
The absolute error is -3.5584872189654866
For h= 1e-14 :
log10(h)= -14.0
The forward difference approximation is -0.33306690738754696
The exact value is -0.3283496313634305
The absolute error is -2.326308710940345
For h= 1e-15 :
log10(h)= -15.0
The forward difference approximation is -0.44408920985006256
The exact value is -0.3283496313634305
The absolute error is -0.9365181036319075
```