# Statistical Inference

## Assignment 1

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2020326

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1)

$X_i \overset{iid}{\sim}$ Exponential($\lambda$)

$\therefore f(x) = \lambda e^{-\lambda x}$

$\therefore L(\theta) = \prod_{i=1}^{n} (\lambda e^{-\lambda x_i})$

$L(\theta) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$

Log likelihood function

$= \text{Log}(L(\theta))$

$\therefore \ln(L(\theta)) = \log(\lambda^n e^{-\lambda \sum x_i})$

$\ln(L(\theta)) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$

$\therefore$ In order to obtain maximum

$$\frac{d(\ln(L(\theta)))}{d\lambda} = 0$$

$$\therefore 0 = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\therefore \lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\therefore \hat{\lambda} = \frac{1}{\bar{x}}$$

To check for maximum

$$\frac{d^2(\ln(L(\theta)))}{d\lambda^2}$$

$$= -\frac{n}{\lambda^2}$$

$$\Rightarrow -n(\bar{x})^2$$

$\therefore$ As $n$, $\bar{x}$ are +ve

$$\frac{d^2(\ln(L(\theta)))}{d\lambda^2} < 0$$

Using Method of moments estimation

$$X_i \sim \text{Exponential}(\lambda)$$

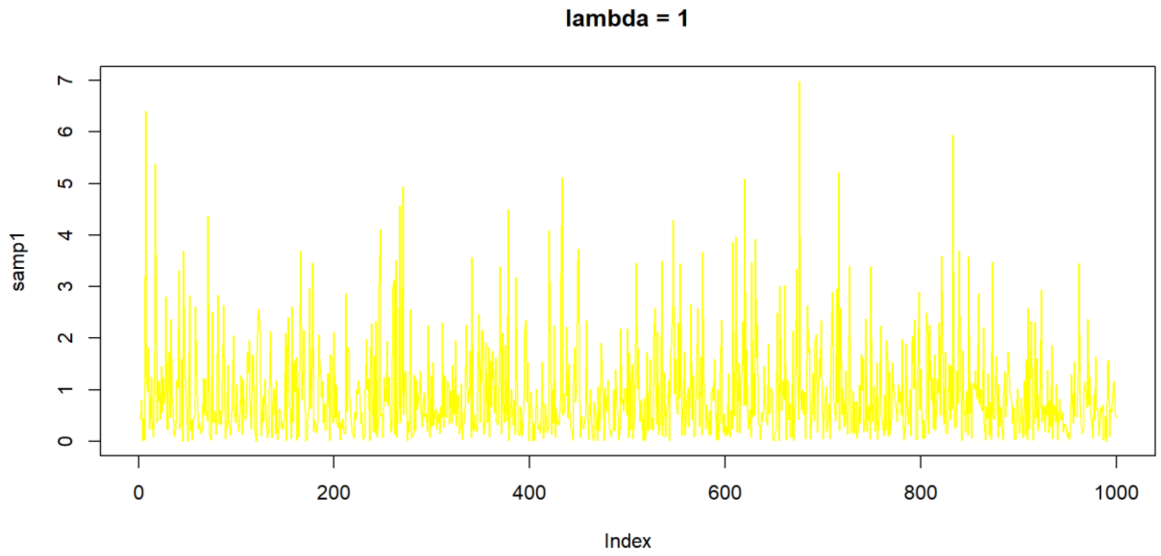
$$E[X] = \frac{1}{\lambda}$$

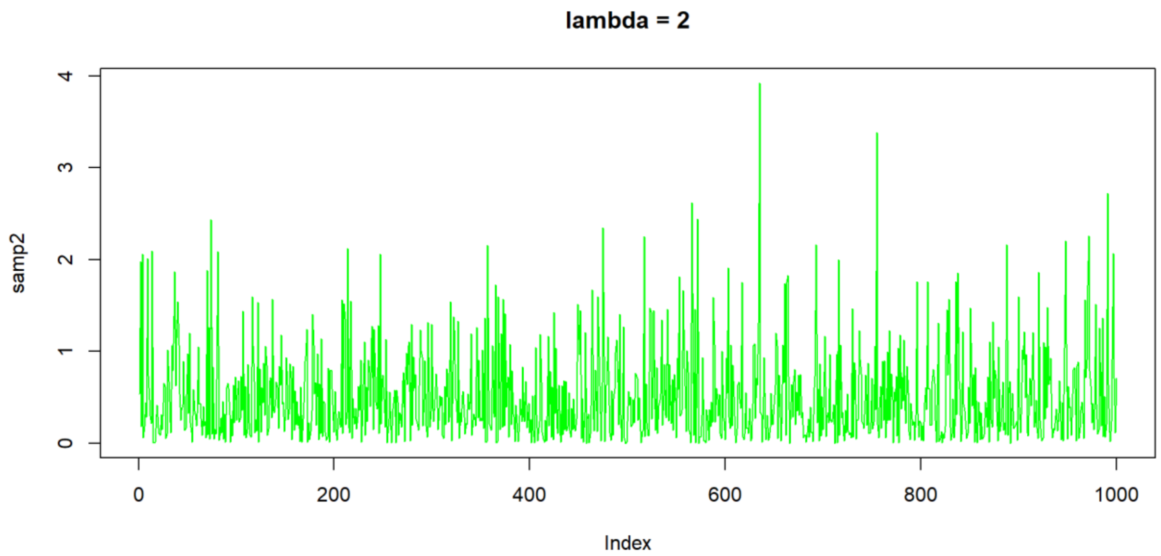
$$\therefore \frac{\sum_{i=1}^{n} x_i}{n} = \frac{1}{\lambda}$$

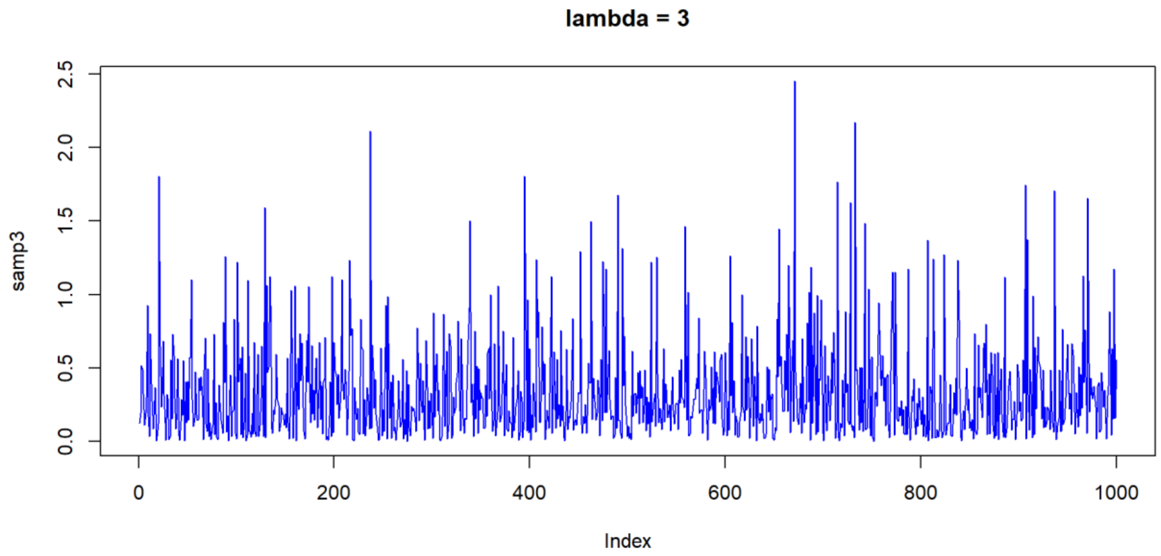
$$\therefore \lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

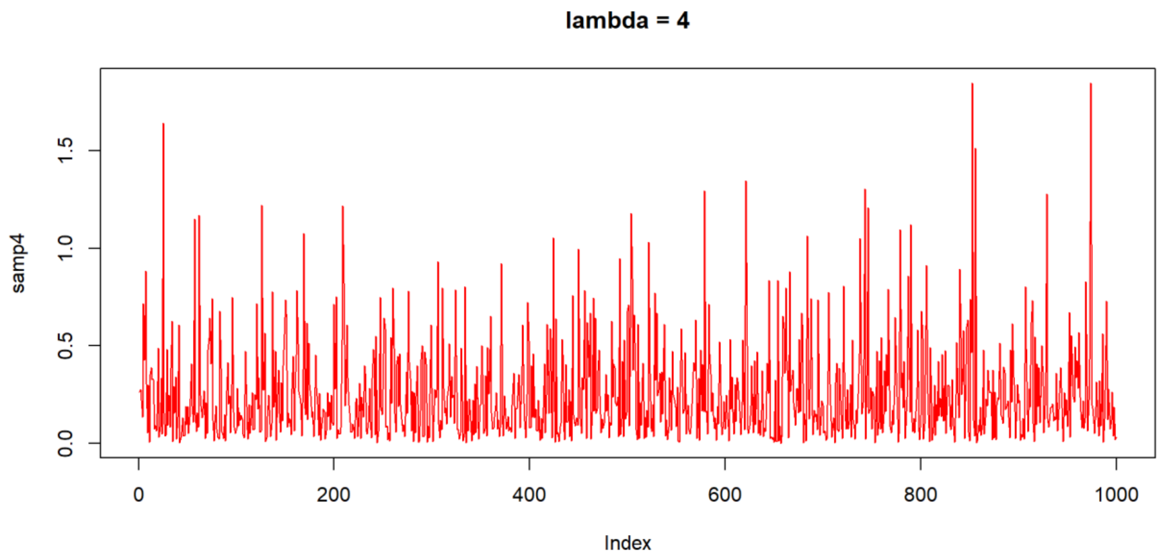
$$\therefore \hat{\lambda} = \frac{1}{\bar{X}}$$

We have simulated four random samples of size 1000 by using the rexp function with lambda = 1,2,3,4. Their graphs are as follows:


lambda = 1
samp1
Index
0
1
2
3
4
5
6
7
0
200
400
600
800
1000


lambda = 2
samp2
Index
0
1
2
3
4
0
200
400
600
800
1000




For finding the Maximum Likelihood Estimate, we use function nlminb(), in order to minimise the negative of log likelihood which is equivalent to maximising the log likelihood of the random samples.

For λ = 1:

The Maximum Likelihood Estimate (λ=1) is 1.058466. The value of log likelihood is -943.1797

The Maximum Likelihood Estimate using Method of Moments (λ =1/mean)is 1.058466. The value of log likelihood is -943.1797

The Maximum Likelihood Estimate with third set of values (λ=0.5) is 1.058466. The value of log likelihood is -943.1797

For λ = 2:

The Maximum Likelihood Estimate (λ=2) is 1.947926. The value of log likelihood is -333.2347

The Maximum Likelihood Estimate using Method of Moments(λ =1/mean) is 1.947926. The value of log likelihood is -333.2347.

The Maximum Likelihood Estimate with third set of values (λ=1.5) is 1.947926. The value of log likelihood is -333.2347.

For λ = 3:

The Maximum Likelihood Estimate (λ=3) is 2.97426. The value of log likelihood is 89.9953

The Maximum Likelihood Estimate using Method of Moments(λ =1/mean) is 2.97426. The value of log likelihood is 89.9953.

The Maximum Likelihood Estimate with third set of values(λ=2.5) is 2.97426. The values of log likelihood is 89.9953.

For λ = 4:

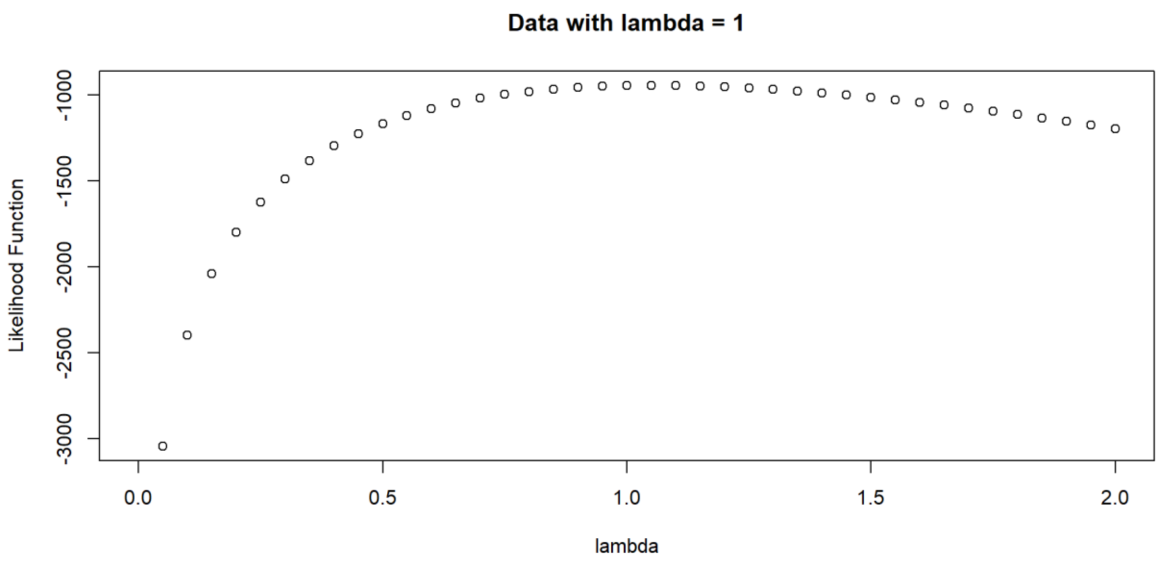
The Maximum Likelihood Estimate(λ=4) is 3.894381. The value of log likelihood is 359.5348

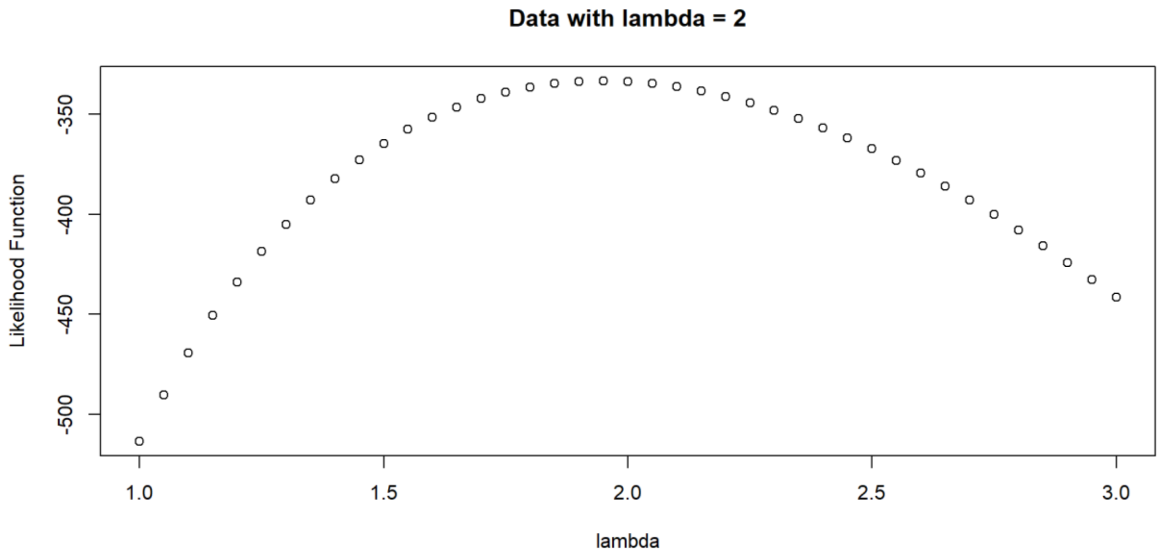
The Maximum Likelihood Estimate using Method of Moments(λ =1/mean) is 3.894381. The value of log likelihood is 359.5348

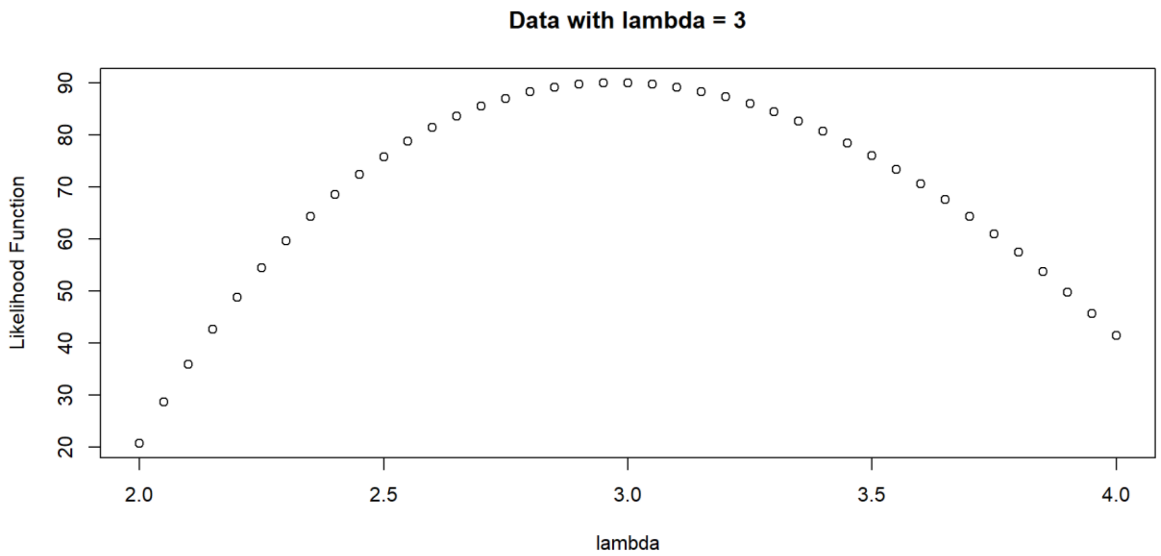
The Maximum Likelihood Estimate with third set of values(λ=3.5) is 3.89438. The value of log likelihood is 359.5348

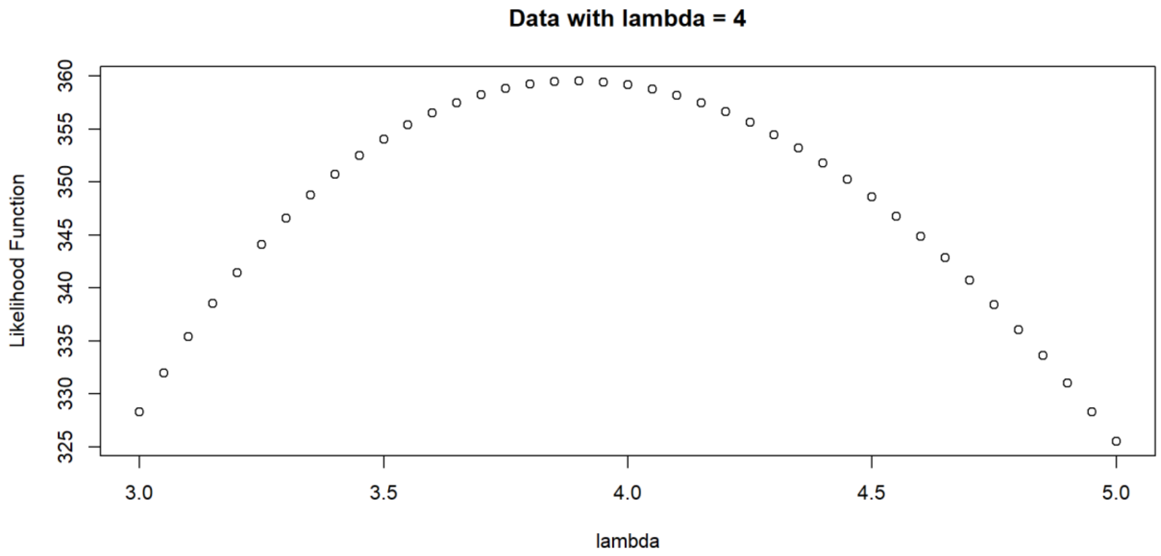
c)

These are the plot of varying λ against the log likelihood function. We can see that the log likelihood function reaches its maximum at values of λ which agrees with what we got through the use of nlminb function and through theoretical calculations.




Data with lambda = 2
Likelihood Function
-350
-400
-450
-500
1.0
1.5
2.0
2.5
3.0
lambda


Data with lambda = 3
Likelihood Function
90
80
70
60
50
40
30
20
2.0
2.5
3.0
3.5
4.0
lambda




Therefore, we can see from the graphs that the log likelihood reaches its maximum at the values that we received through the usage of nlminb function and through the theoretical calculations.

2)

27) $X \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right)$$

$$\therefore L(\theta) = \prod_{i=1}^{n} f(x)$$

$$= \frac{1}{(\sqrt{2\pi\sigma^2})^n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2\right)$$

Log likelihood function

$$\Rightarrow \log(L(\theta))$$

$$\log(L(\theta)) = -\frac{n}{2}\log(2\pi\sigma^2)$$

$$- \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$\therefore$ To obtain maximum

$$\frac{d(\log(L(\theta)))}{d\mu} = 0$$

$$\frac{d(\log(L(\theta)))}{d\sigma} = 0$$

$\therefore$ Differentiate wrt $\sigma$

$$\frac{d(\log(L(\sigma)))}{d\sigma} \Rightarrow$$

$$\Rightarrow \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\therefore \sigma^2 = \frac{\sum(x_i-\mu)^2}{n}$$

$$\therefore \sigma = \sqrt{\frac{\sum(x_i-\mu)^2}{n}}$$

For checking maximum

$$\frac{d^2(\log(L(\sigma)))}{d\sigma^2}$$

$$\Rightarrow \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum(x_i-\mu)^2$$

substitute value of $\sigma$

$$\Rightarrow \frac{n}{\frac{\sum(x_i-\mu)^2}{n}} - \frac{3}{\left(\frac{\sum(x_i-\mu)^2}{n}\right)^2}\sum(x_i-\mu)^2$$

$$= \frac{n^2}{\sum(x_i-\mu)^2} - \frac{3n^2}{\sum(x_i-\mu)^2}$$

$$= \frac{-2n^2}{\sum(x_i-\mu)^2}$$

$\therefore$ As $n^2$ and $\sum(x_i-\mu)^2$ are +ve

$$\frac{d^2(\log(L(\theta)))}{d\sigma^2} < 0$$

$\therefore$ We have verified maximum

$$\therefore \sigma_{MLE} = \sqrt{\frac{\Sigma(x_i-\mu)^2}{n}}$$

Now, we will differentiate wrt $\mu$

$$\frac{d(\log(L(\theta)))}{d\mu} \Rightarrow$$

$$\Rightarrow \frac{-1}{\sigma^2}\left(\Sigma(x_i-\mu)\right)$$

~~Substitute~~

Substitute value of $\sigma_{MLE}$

$$\Rightarrow \frac{-1}{\frac{\Sigma(x_i-\mu)^2}{n}}\left(\Sigma(x_i-\mu)\right) = 0$$

$$\therefore \Sigma x_i - n\mu = 0$$

$$\mu = \frac{\Sigma x_i}{n}$$

$$\mu = \overline{x}$$

∴ For checking maximum

$$\frac{d^2(\log(L(\sigma)))}{d\mu^2}$$

$$\Rightarrow \frac{-1}{\sigma^2}(n) = -\frac{n}{\sigma^2}$$

∴ As $n, \sigma^2$ are +ve

$$\frac{d^2(\log(L(\sigma)))}{d\mu^2} < 0$$

∴ We have verified maximum

$$\mu_{MLE} = \bar{X}$$

c) We already determined that ML estimate of $\mu$ will be 4000.04371

We need to find the ML estimate of $\exp(-\mu)$;

∴ We will use invariance property.

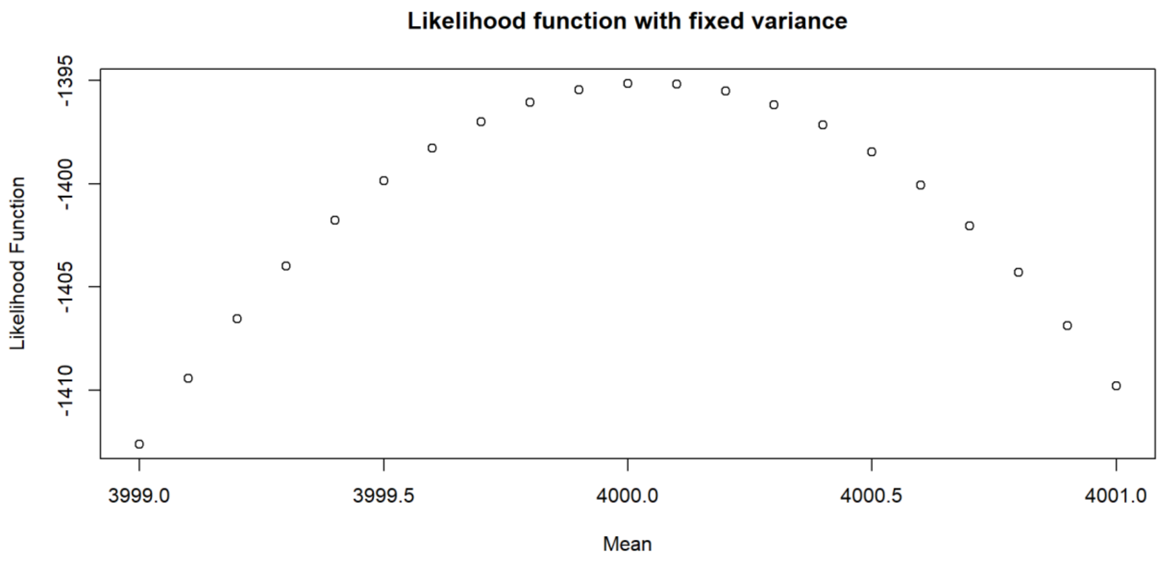
∴ The ML estimate of $\exp(-\mu)$ will be $\exp(-\hat{\mu}_{MLE})$

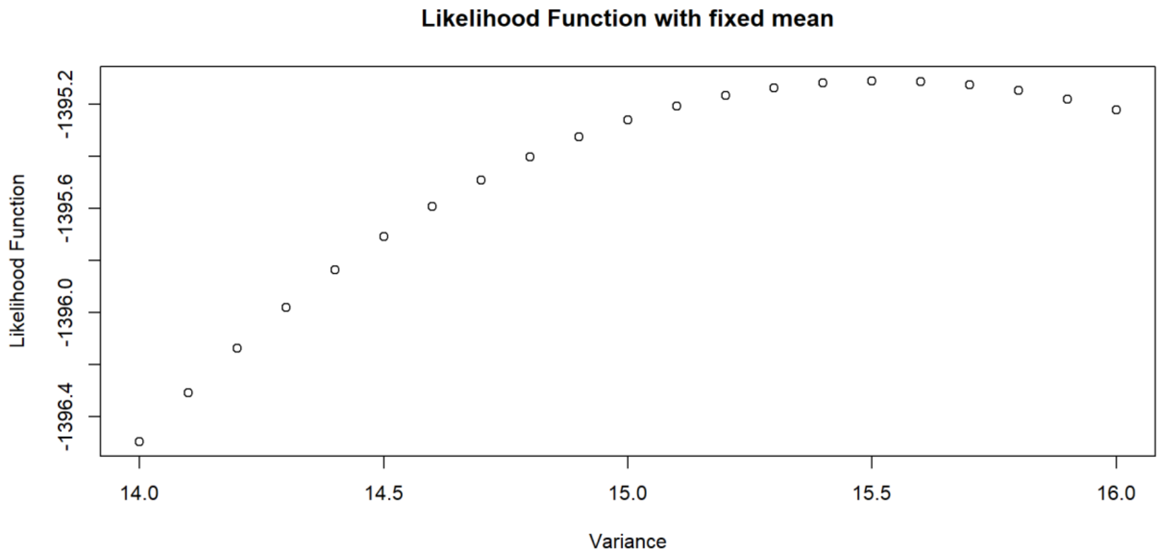
∴ Thus solved

I have successfully written a function in R that obtains the maximum likelihood estimate of the parameters. We find the Maximum likelihood estimates as mean ($\mu$) = 4000.04371 and variance ($\sigma^2$) = 15.52669. The value of the log likelihood at these values of the parameters is - 1395.109.

b)

We have plotted two curves, one with varying values of variance and a fixed mean, and the other with varying values of mean and a fixed variance against the log likelihood values. We keep the log likelihood function on the y axis, and the mean and variance on the x axis.




We can see that the likelihood function achieves its maximum at approximately mean = 4000 and variance = 15.5. Thus, we can see that the value of maximum likelihood estimators we get from the function we made gives us results that are consistent with the theoretical calculations.